SUPPORTING THE DEVELOPMENT OF A SECOND-GRADE CLASSROOM'S CONCEPTIONS OF MULTIPLICATION THROUGH A HYPOTHETICAL LEARNING TRAJECTORY

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ABSTRACT

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The first aim of this study is to develop, test, and revise a Hypothetical Learning Trajectory and corresponding instructional sequence for teaching multiplication in second grade. The second aim is to document second graders' classroom mathematical practices that emerged related to conceptualizations of multiplication and to use those practices for making revisions to the HLT and the instructional sequence. An instructional sequence was developed in line with Realistic Mathematics Education theory to achieve these goals. This sequence was implemented and revised in a design experiment in a year with a design research perspective. Data were collected from a second-grade classroom in a public school in the Çankaya District of Ankara with the collaboration of the classroom teacher. The emergent perspective, which coordinates social and individual perspectives and supports collective learning, was used as an interpretive framework for data collection and analysis.

The data collected through the video recordings of class sessions, students' written works, and the researcher's field notes were analyzed by adapting

Toulmin's argumentation model to establish classroom mathematical practices. Five classroom mathematical practices emerged related to students' conceptualizations of multiplication concepts. The study has important implications for students, teachers, mathematics education researchers, and educational resource designers in practical and theoretical ways.

Keywords: Classroom Mathematical Practices, Multiplication, Realistic Mathematics Education, Design Research, Hypothetical Learning Trajectories

ÖZ

İLKOKUL 2. SINIF ÖĞRENCİLERİNİN DOĞAL SAYILARLA ÇARPMA İŞLEMİNİ KAVRAYIŞLARININ ÖĞRENME ROTASI İLE DESTEKLENMESİ

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Bu çalışmanın ilk amacı, ikinci sınıfta çarpma öğretimi için varsayıma dayalı bir öğrenme rotası ve buna karşılık gelen etkinlik dizisi geliştirmek, test etmek ve revize etmektir. Çalışmanın ikinci amacı ise, ikinci sınıf öğrencilerinin çarpma işlemini kavramsallaştırmaları ile ilgili ortaya çıkan sınıf içi matematiksel uygulamalarını ortaya koymak ve bu uygulamalar doğrultusunda varsayıma dayalı öğrenme rotasına ve öğretim dizisine yönelik düzenlemeler yapmaktır. Bu hedeflere ulaşmak için Gerçekçi Matematik Eğitimi teorisine uygun bir etkinlik dizisi geliştirilmiştir. Bu etkinlik dizisi, tasarım tabanlı araştırma perspektifiyle bir öğretim deneyi ile uygulanmış ve revize edilmiştir. Çalışmanın verileri, Ankara ilinin Çankaya ilçesinde bir devlet okulunda bulunan bir ikinci sınıftan, sınıf öğretmeni işbirliği ile toplanmıştır. Verilerin toplanması ve analizinde kolektif öğrenmeyi destekleyen ve sosyal ile bilişsel perspektiflerin koordinasyonuna işaret eden Gelişen Bakış Açısı, yorumlayıcı bir çerçeve olarak kullanılmıştır.

Sınıf oturumlarının video kayıtları, öğrencilerin yazılı çalışmaları ve araştırmacının alan notları aracılığıyla toplanan veriler, sınıf içi matematiksel uygulamaları oluşturmak için Toulmin'in argümantasyon modelinin uyarlaması ile analiz edilmiştir. Öğrencilerin çarpma işlemi konusunu kavramsallaştırmalarıyla ilgili beş sınıf içi matematiksel uygulama ortaya çıkmıştır. Çalışmanın öğrenciler, öğretmenler, matematik eğitimi araştırmacıları ve eğitim kaynakları tasarımcıları için hem pratik hem de teorik açıdan önemli çıkarımları bulunmaktadır.

Anahtar Kelimeler: Sınıf içi Matematiksel Uygulamalar, Çarpma, Gerçekçi Matematik Eğitimi, Tasarım Araştırması, Varsayıma Dayalı Öğrenme Rotaları

To my lovely parents

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LIST OF ABBREVIATIONS

MoNE	Ministry of National Education
NCTM	National Council of Teachers of Mathematics
HLT	Hypothetical Learning Trajectory
RME	Realistic Mathematics Education
CMP	Classroom Mathematical Practice
CLT	Classroom Learning Trajectories
TAS	Taken-as-Shared
MCF	Multiplicative Conceptual Field
METU	Middle East Technical University
ODTÜ	Orta Doğu Teknik Üniversitesi

CHAPTER I

INTRODUCTION

Multiplication is one of the basic arithmetic concepts in primary school mathematics (National Council of Teachers of Mathematics [NCTM], 2000; Ministry of National Education [MoNE], 2018). In mathematics instruction, multiplication is often introduced as the third arithmetic operation after students learn basic addition and subtraction concepts. It is constructed on addition and defined as repeated addition, in which a number of collections of the same size are put together (Anghileri, 2006; Fishbein et al., 1985; Fosnot & Dolk; 2001; Greer, 1992; Nesher, 1988, 1992; Schwartz, 1988; Vergnaud, 1983). In this way, students gain the advantage of procedural connection between operations and use repeated addition to solve multiplication problems (Anghileri, 2006; Nunes & Bryant, 1996). Therefore, repeated addition is viewed as the basic intuitive model of multiplication (Fishbein et al., 1985; Freudenthal, 1973).

As an operation, multiplication is initially introduced as repeated addition; however, it differs from addition with some special characteristics (Fosnot & Dolk; 2001; Greer, 1992; Nesher, 1988, 1992; Schwartz, 1988; Vergnaud, 1983). Addition is a unary operation that manipulates quantities with similar units. For instance, an addition problem may involve adding three to seven apples, which results in ten apples. On the other hand, multiplication is a binary operation through which two quantities with different units are manipulated (Anghileri, 2006; Barmby et al., 2009). The result is a new unit or relationship that is not immediately apparent from the factors (multiplier and multiplicand) (e.g., three bags, four apples per bag, 12 apples). Therefore, while multiplying via repeated addition, one should be able to think simultaneously across units (Lamon, 1994).

In fact, multiplication is not exclusive to the repeated addition of equal groups. There are many more uses for multiplication. Mathematical concepts are multifaceted as they have several meanings and interpretations depending on the situation (Chin & Pierce, 2019). In this sense, multiplicative situations have been analyzed, and various multiplication models have been revealed (Fishbein et al., 1985; Greer, 1992; Nesher, 1988, 1992; Schwartz, 1988; Vergnaud, 1983). According to the classification of Greer (1992), multiplication is used to solve problems related to equal groups, multiplicative comparison, rectangular area/array, and Cartesian product. An example of an equal group model is three bags of seven apples, while a rectangular array could be three rows of seven apples placed on a tray. An example for multiplicative comparison model is that Adam has three times as many apples as Eva, who has seven apples. As for the rectangular area, an example would be the continuous measuring of length by multiplication, which transforms into the measurement of the area. Finally, for example, possible combinations of three shirts and seven skirts are related to the Cartesian product model of multiplication (Greer, 1992). Therefore, various problems that are heavily dependent on instructional setting and linguistic cues can be solved using multiplication (Nesher, 1988). It can be stated that multiplication has different meanings when context changes.

Having such a wide and rich range of definitions, multiplication is given a large place in mathematics education programs as an essential concept (NCTM, 2000; MoNE, 2018). Multiplication is a crucial skill for students to become ready for life in the mathematics environment of the 21st century (NCTM, 2000), and mathematics education programs aim to develop the fundamental verbal and numerical reasoning skills students will need in everyday life (MoNE, 2018). The programs further aim to develop the necessary mathematical competence so that students can solve a series of problems encountered in daily life. For this reason, curriculum objectives related to multiplication include solving real-life problems in various contexts.

In addition to its role in finding solutions to daily problems, multiplication also builds a firm foundation for the development of further topics both conceptually and procedurally (Steel & Funnell, 2001). It is a crucial mathematical concept that has a significant role in the conceptual development of future topics. Due to their relations to fractions, measurement, ratios and rates, and proportional reasoning; whole number multiplication and division comprise a web of related concepts (De Corte & Verschaffel, 1996; Hino & Kato, 2019; Verschaffel et al., 2007). If students conceptually understand basic multiplication facts in the primary grades, they will be much better prepared for these related concepts (Wong & Evans, 2007; Vergnaud, 1988). Hence, the conceptualization of multiplication is highly important for constructing many mathematical concepts built on each other.

Moreover, multiplication as a skill (Shanty & Wijaya, 2012) is vital for all domains of mathematics. If students strive with multiplication algorithms or number combinations and fail to grasp the underlying concepts, they also have difficulty grasping the tasks related to further topics (Flores et al., 2014). In other words, many tasks in mathematical domains and a wide range of subject areas require recalling fundamental multiplication facts as a lower-order component of the overall task. Basic knowledge and abilities related to operations help students concentrate on more sophisticated tasks, like problem-solving (Kilpatrick et al., 2001). Otherwise, the student's focus during problem-solving will be on fundamental abilities rather than the task at hand, taking attention away from the learning objectives of the assignment if they lack procedural fluency and the capacity to remember knowledge from memory (Mercer & Miller, 1992). Thus, if fundamental multiplication facts are not learned in primary school, it is doubtful they will be adequately practiced in secondary school (Steel & Funnell, 2001). In other words, if students do not have a solid mathematical base in multiplication, it will affect their performance in future mathematics classes.

In line with its importance, multiplication is included in educational programs as a mathematics topic from the second grade on (MoNE, 2018; NCTM, 2000;

Olfos et al., 2021). However, the studies related to students' understanding of multiplication have shown that students have problems understanding and processing multiplication. It is seen that students cannot express a given model as multiplication, model a multiplication, evaluate the effect of "0" and "1" in multiplication, memorize the multiplication table, distinguish addition and multiplication, comprehend commutative and distributive properties in multiplication, decide on the operation to solve a given problem, and process multiplication (Ekici & Demir, 2018; Kubanç & Varol, 2017; Sidekli et al., 2013; Üçüncü, 2010). Unfortunately, students overgeneralize the rules of addition and subtraction to multiplication as well (Kubanç & Varol, 2017). Hence, they have difficulties in performing multiplication operations correctly and understanding the operation conceptually.

Most of the studies in the literature were conducted by asking the participants to pose problems to reveal their conceptual understandings or misunderstandings (Cai et al., 2015; Tichá & Hošpesová, 2013), and it was revealed that most students could not pose an appropriate problem for a given operation (Doğan & Doğan, 2019; Drake & Barlow, 2007; Graeber & Tirosh, 1990; Kılıç, 2013; Tertemiz, 2017) since they had difficulty connecting multiplication to everyday activities (Doğan & Doğan, 2019; Kılıç, 2013; Tertemiz, 2017). For this reason, in addition to learning how to do multiplication procedurally, students also need to learn the language that is connected with it and how to recognize it in real-life contexts (Calabrese et al., 2020). Studies revealed that only a small portion of students can write an appropriate problem, but only related to repeated addition (Graeber & Tirosh, 1990; Tertemiz, 2017). The reason for using the repeated addition model is generally attributed to the fact that students are predominantly exposed to the equal groups model in early education and daily life (English, 1998; Verschaffel et al., 2007). Therefore, in addition to emphasizing the relationship between multiplication and real-world experiences, it is recommended to introduce other models of multiplication, connecting them with realistic scenarios.

The specified difficulties and limitations students encounter in multiplication might stem from various factors affecting their comprehension of this operation. The studies related to multiplication revealed that these factors include scarce sources of curriculum (Remillard, 2005; Valverde et al., 2002) and representations (Bruner, 1964; Goldin & Shteingold, 2001), students' limited knowledge of fundamental concepts (Carpenter et al., 1989; Fischbein et al., 1985), and weak knowledge of teachers on the topic (Shulman, 1986). For this reason, in order to support the meaningful learning of multiplication, these factors should be taken into account. Otherwise, these factors may obstruct students' conceptual understanding of multiplication.

Research studies on students' conceptualization of multiplication have demonstrated that it is essential to provide meaningful multiplication learning. Difficulties in assisting students' learning point to the need for a deeper understanding of how to encourage students' knowledge of multiplication. Hence, the question of how to teach multiplication meaningfully arises. Clements and Sarama (2004) suggest providing students with well-sequenced instructional tasks to develop the concepts related to numbers and operations. In parallel to this suggestion, the current study aims to develop an instructional sequence composed of mathematical tasks to build a conceptual understanding of multiplication for second graders.

The Hypothetical Learning Trajectory (HLT) might be a practical framework to trace students' initial and evolving understanding of multiplication throughout a series of lessons, with an instructional emphasis on Realistic Mathematics Education (RME). The study of learning trajectories can help gain a deeper understanding into students' thinking and learning, enable to employ more efficient instructional strategies, and direct the development of better curricula and standards (Clements & Sarama, 2004; Gravemeijer, 2004; Simon, 1995; Simon & Tzur, 2004). In line with these powerful aspects of learning trajectories, it may be effective to employ the collective mathematical practices approach in order to notice students' mathematical thinking in a learning environment and portray their collective growth (Lobato & Walters, 2017).

According to this viewpoint, an HLT contains a series of mathematical practices used in the classroom together with a hypothesis regarding their progression from earlier practices (Cobb, 1999). Students participate in classroom mathematical practices by conducting social interactions. In this way, collective argumentation is developed where students and teachers make mathematical claims and then support those claims or arguments with justification (Krummheuer, 1995; Lobato & Walters, 2017). Through the argumentation process, students also reflect on conflicts, which leads to the stages of revision, retraction, and replacement. In addition to collective argumentation and its role in classroom mathematical practices (Lobato & Walters, 2017), proposing hypotheses and convincing arguments is an important part of mathematical thinking and reasoning, according to the Principles and Standards for Classroom Mathematics (NCTM, 2000). Therefore, by participating in the argumentation process in such a setting, students can understand that claims need to be supported or disproved by evidence and come to an understanding of what constitutes an acceptable argument in the mathematics classroom (NCTM, 2000). The Turkish mathematics education program (2018) emphasizes the significance of nurturing students to ask questions, verbally communicate their thinking, and produce evidence-supported statements (MoNE, 2018). Thus, allowing students to attend learning environments that would enhance their reasoning and communication skills in mathematics is crucial.

It is important to promote students' collective ways of thinking and learning through collective argumentation and to enable them to investigate informal tools and advance to more formal mathematics by negotiating, collaborating, and discussing (Gravemeijer et al., 2000). To this end, it is also essential to create an instructional sequence composed of high-demanding tasks, which ensures a collective argumentation environment and students' growth in increasingly sophisticated ways (Clements & Sarama, 2004). According to recent research, the design and implementation of instructional sequences can be influenced by the domain-specific instructional theory of Realistic Mathematics Education (Gravemeijer et al., 2003a, 2003b; Stephan & Akyüz, 2012). In a classroom context, students and the teacher participate in a collective reinvention process, where they formulate hypotheses and then defend or disprove them to develop taken-as-shared meanings (Gravemeijer et al., 2000). Therefore, it is significant to create instructional sequences that start from practical situations and proceed toward the target formal mathematics in order to promote the collective reinvention process from an RME perspective (Gravemeijer et al., 2000; Gravemeijer & Stephan, 2002).

To sum up, multiplication has a rich content, with different meanings in different contexts. With the various models it includes, multiplication is highly important for constructing many mathematical concepts built on each other and for solving problems on these concepts by processing multiplication. However, studies have revealed that students are generally capable of only repeated addition as the primitive model of multiplication, and they have problems understanding and processing multiplication and connecting multiplication with real-life situations. The challenges and restrictions students encounter when multiplying may result from a variety of factors that affect their understanding of this operation. Given these possible factors that might be the barriers to the conceptualization of multiplication, it is suggested to provide students with well-sequenced instructional tasks to develop the concepts related to multiplication. Moreover, it is recommended to use RME as an effective theory enabling students to open their eyes to the world of mathematics that naturally occurs around them in order to give meaning to the operation by connecting it with realistic scenarios.

1.1. Purposes and Research Questions of the Study

This study has multiple objectives as follows: (1) to develop, test, and revise a classroom HLT and related instructional sequence for teaching multiplication to second-graders; (2) to document students' collective development of mathematical concepts related to multiplication (i.e., documenting mathematical practices). To illustrate how the instructional sequence and the HLT can be helpful in teaching and learning multiplication, the first purpose is to develop an HLT and a local instructional theory that would go along with it. It seeks to combine the critical elements and viewpoints of HLT and RME in this process. It is aimed to construct teaching on the knowledge of students' informal methods of thinking about multiplication and those methods' developmental patterns in young children. The aspects being investigated within the context of the first purpose are, more specifically, the initial points of departure and informal tools for the teaching and learning of multiplication, as well as how students rely on their informal knowledge and informal use of tools as they try to mathematize (horizontally and vertically) these initial situations.

In line with the second purpose, how the hypothesized learning trajectory takes place in the classroom environment is investigated by documenting the classroom mathematical practices. The analysis for this purpose focuses on the potential and obstacles that the conjectured instructional sequence presents to facilitate the collective mathematization of students. To support the validity of the suggested local instructional theory for teaching multiplication and to suggest improvements to the HLT and the instructional sequence, the study tries to gather evidence from the classroom experiments conducted by using the HLT and the instructional sequence. Hence, for these purposes, the research questions that guide the study are formulated as follows:

1-What would an optimal HLT and instructional sequence for multiplication look like?

2-What are the mathematical practices as students engage in the instructional sequence for multiplication?

- What are the mathematical ideas that support the mathematical practices developed by students during the implementation of instructional sequence for multiplication?

The answer to the first research question is spread throughout the thesis. Chapter 3 explains how the HLT and instructional sequence were developed and revised throughout the implementation. In Chapter 4, classroom mathematical practices and taken-as-shared ideas, which is the answer to the second research question, are presented. Then, in Chapter 5, the actual HLT is presented in the light of Chapter 4, again related to the first research question. The next section clarifies the reasoning behind doing this study in light of the purposes and research questions.

1.2. Significance of the Study

The study aims to support the objectives and scarce resources of mathematics curricula on multiplication through real-world contexts, by applying an educational sequence created using the Realistic Mathematics Education (RME) theory. The study is planned to be carried out as design research, incorporating both instructional design and classroom-based research (Cobb et al., 2001). A crucial aspect of design research is the broad scope of processes and contexts. In this sense, the study seeks to document what resources and prior knowledge students bring to a task, how students and teachers interact, how records and inscriptions are created, how conceptions emerge and change, what resources are used, and how teaching is accomplished throughout instruction, by examining student work, video records, and classroom assessments (Confrey, 2006, p. 135). It can be stated that at this point of the study, classroom-based research comes to the fore. In the scope of the current study, it is also planned to develop, test, and refine conjectures about the learning trajectory in multiplication as the researcher collaborates with the teacher or acts as the teacher and assembles extensive evidence on what students, teachers, and researchers learn from the process. Following this part of the study, it is aimed to conduct further analysis on all the products of the process, including research reports and iterations of the tasks, materials, and instrumentation.

With the detailed documentation on the design and analysis process, this dissertation study is expected to provide theoretical and practical implications for the field of mathematics education. It may contribute to theory in terms of design research, RME theory, and the literature related to multiplication, while also contributing to the practices of multiplication instruction, collective argumentation, and instructional sequence from the perspectives of students, teachers, teacher educators, mathematics education researchers, and educational resource designers.

The current design of the study has a major impact on the HLT and related instructional sequence on multiplication specific to second-grade students. Although there are several available learning trajectories on multiplication and related concepts (e.g., Götze & Baiker, 2021; Kennedy et al., 2008; Mendes et al., 2021; Wright et al., 2014), these trajectories are too general and independent of grade level. Most of these learning trajectories focus on developing an understanding of multiplication. Unfortunately, they do not offer suggestions on how to teach multiplication, specifically in second-grade classrooms. Additionally, most of the big ideas included in the available trajectories are new for second-grade students in Turkey. Therefore, they should be adapted for these students, considering the objectives in the Turkish curriculum. Instructional activities should be adapted to the Turkish context to be clear and understandable for the students. This study has the potential to close this gap by giving teachers a thorough understanding of the central concepts and fundamental principles of multiplication as well as of the instructional tasks and resources needed to help students develop those concepts in increasingly sophisticated ways.

It is also crucial for the teacher to understand the students' ways of thinking (Butterfield et al., 2013). The idea is that instructors design different tasks according to their individual students' perspectives (Sarama & Clements, 2009), and this task differentiation enables students with different learning styles to achieve the common goals of a class (Mousley et al., 2004). It is also claimed that teachers' understanding of how students think and learn improves their instruction, which in turn improves students' math achievement (Carpenter et al., 1999). In this sense, learning trajectories offer crucial insight into instructor expertise. For instance, thanks to the design research process of this study, teachers may improve their awareness of students' learning and their ability to use didactic discourse to support students' engagement and empowerment in mathematics. Such improvements can encourage a more realistic, context-sensitive teaching and learning perspective (Kwon et al., 2013). Therefore, it can be stated that the design research process and context might offer an informative framework for blending teachers' practical knowledge with research-based techniques and producing creative design concepts to enhance learning.

In order to create a productive design through research-based techniques, the practicality of the instructional sequence should be evaluated, and the sequence should be revised if necessary and possible. In this sense, it is suggested to reveal both the classroom mathematical practices of the classroom community and the mathematical reasoning of individuals (Cobb, 2003; Stephan et al., 2003). A way to document the mathematical practices of a community is to analyze the argumentation in whole-class discussions (Cobb & Yackel, 1996). During the argumentation process, social and socio-mathematical norms play a key role in managing classroom discussions and in the evolution of taken-as-shared ideas. The present study revealed that the social and socio-mathematical norms of the classroom where the study was conducted are parallel with the norms established in previous studies (Andreasen, 2006; Gruver, 2016; Miller, 2016; Stephan, 1998; Wheeldon, 2008). These norms might enable to create an argumentative environment and observe the development of classroom mathematical practices through discussion in a classroom community. Without these norms, the actual learning trajectory would simply consist of the teacher's predetermined list of class activities (Lobato & Walters, 2017). Other instructors can also examine these norms and use them in their classrooms directly or by revising them. They can see the consequences of these norms on maintaining collective argumentation and involving students in classroom mathematical practices.

In addition to the norms of sustaining argumentation, it is widely known that prior experience, knowledge, and cultural background play important roles in developing students’ understanding (Ausubel, 1968; Hart, 1988). It is vital to comprehend how students with various knowledge and cultural backgrounds develop mathematically (Daro et al., 2011). Although the relationship between students' academic and cultural backgrounds and mathematical development is not investigated in the scope of the current study, how multiplication develops in the current case of a Turkish school setting is provided in detail. It is believed that this might improve the understanding of how mathematics develops in different environments.

The current study might contribute to the objectives of mathematics education in Turkey. One of these possible contributions is the conceptualization of the commutative property of multiplication. An array model is suggested in the literature as the most effective representation to teach the commutative property of multiplication (Greer, 1992; Outhred & Mitchelmore, 2004; van de Walle et al., 2020). According to the theory, an array is a composite of composites, which is one of the semantic structures related to multiplication (Outhred & Mitchelmore, 2004). The mathematical objectives and textbooks for second graders in Turkey show that the only representation of multiplication is equal grouping (MoNE, 2018). However, moving items into rows and columns provides more opportunities to explore the commutative rule than static pictures (Anghileri, 2006). Thus, it can be stated that the array model helps students interpret multiplication with its properties, especially the commutative property. The present study might play an essential role in introducing multiplication as an array and documenting students' reactions and conceptualizations during the implementation of the instructional sequence. Therefore, the use of the array model in the current design might be a sample practice for teachers, which might guide them to use the model in their classrooms.

Another contribution to national mathematics instruction might be the use of activities involving equal groupings and sharing to provide students with an environment where they can learn an appropriate language for multiplication by connecting with counting patterns (Anghileri, 1995). Equal grouping activities planned to be used in this study might have a crucial role in developing a multiplicative language and conceptualizing multiplication and division. Furthermore, the specific phrases that are used to handle real-world issues improve the use of more formal terminology to experience the connections between multiplication and division (Anghileri, 1995). Therefore, using composing and decomposing activities within the instructional sequence might contribute to bridging multiplication and division. These tasks might be sample practices for developing multiplication while preparing students for other related topics like measurement, fractions, rates, and proportional reasoning.

In addition, in line with didactical phenomena, teachers are responsible for using and promoting students' informal understandings of linking composite units, unitizing, and building-up strategies in mathematically relevant directions (Freudenthal, 1983; Lamon, 1995). Unfortunately, research on proportional reasoning revealed that teachers have difficulty teaching proportional reasoning and they view the topic as procedural, superficial, and isolated from other topics (Sowder et al., 1998). Thus, teachers should improve their subject matter knowledge and pedagogical content knowledge of proportional reasoning, considering the role of multiplication in understanding proportional reasoning. In parallel with this suggestion, the current study might help teachers to have an insight into students' informal knowledge and the evolution of related concepts like proportional reasoning in a classroom context and in an increasingly sophisticated way in the light of the reinvention and mathematization process through the theory of RME. With this study, the readiness and performance of second graders for other topics might be observed by documenting their mathematical practices. Moreover, it is believed that the current study might help improve teachers' awareness of students' conceptual progressions, the connections between topics, and the resources and activities they need to enhance students' mathematical development. The study has the potential to bridge the theoretical and practical issues of learning multiplication by presenting the findings on the designed instruction and students' informal ways of thinking that are fundamental skills for other concepts.

Another area where the current study can make a useful contribution is the use of problem posing tasks in the second-grade classrooms. Although the use of problem posing activities is strictly emphasized in the curriculum (MoNE, 2018; NCTM, 2000), the objectives in the second grade curriculum in Turkey do not include problem-posing in multiplication (MoNE, 2018). Turkish students are expected to pose multiplication problems in the third grade. In this sense, the current study might play an essential role in providing a sample instructional sequence enriched with problem posing tasks for second graders. Moreover, few researchers have tried to describe the dynamics of classroom instruction where students engage in problem posing activities (Cai et al., 2015). This study might fill this gap by identifying and describing how tasks should be implemented in the classroom and how classroom discussions should be managed by revealing the dynamics of classroom discussion and culture. This research might also provide an insight into the readiness and competence of second graders in problem posing by documenting their mathematical practices in multiplication.

Problem-posing activities are recommended to help students connect multiplication and real-world issues and foster deeper knowledge (English, 1997). Besides, these activities are used to encourage students to create their problems based on various contexts (NCTM, 2000). As problem posing involves cognitively challenging tasks for mathematical communication (Cai & Hwang, 2002), problem posing activities and discussions that would create a shared understanding are thought to be useful in helping students share their ideas and in gaining new insights on multiplication (Cai et al., 2015). In light of this, the study used problem posing activities in line with the theory of Realistic Mathematics Education. It is believed that the current study might provide evidence on the cognitive processes of students while they are posing problems and recording mathematical practices in order to comprehend the reasons and explanations that evolve within a classroom community.

Overall, it can be stated that the current study outlines potential contributions to educational settings from the perspectives of students, teachers, educators, and educational resource designers. In addition to educational settings, the design of the current study contributes to the literature in terms of domain-specific institutional theory. It is important to note that this study aims to create a local instructional theory for teaching multiplication, utilizing the RME theory as a model and a guideline. In the study, RME has been developed as a local instructional theory for multiplication (Gravemeijer & Stephan, 2002). Local instruction theories can be the foundation for developing a more sophisticated version of the general theory since they include newly developed examples of how RME might be worked out (Gravemeijer & Stephan, 2002). In light of this, the results of this study might contribute to the development of the RME theory since a theory in use is rebuilt and how RME can influence the creation of an HLT for multiplication is revealed.

1.3. Definition of Important Terms

In this section, definitions of the key concepts used in this study are provided conceptually and operationally to clarify terms and avoid uncertainty.

Hypothetical Learning Trajectory (HLT) was first used by Simon (1995) as a hypothesis about the direction that learning might take. In addition to various definitions (Confrey et al., 2014; National Research Council [NRC], 2007), Clements and Sarama (2004) explain hypothetical learning trajectories as "descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (p. 83). In addition to this definition, Stephan (2015) uses the term classroom learning trajectory (CLT) which refers to "the hypothesized learning route developed by a class of students as they interact with one another and a teacher rather than an individual learning trajectory which is created by an individual in a one-on-one experiment with a teacher or researcher" (p. 907). That is to say, CLT entails speculating about the mathematical concepts taken-as-shared and people's participation and contribution ways (i.e., the class's mathematical practices and the variety of students' thinking). The current study used the viewpoints related to classroom learning trajectories to define and interpret hypothetical learning trajectories. Furthermore, the hypothetical learning trajectory developed by Stephan and her colleagues (2003) was used as a framework for the HLT developed in this study. In line with this framework, the HLT table was arranged according to the following categories: big ideas, tools, imagery, activity/taken-as-shared interests, possible topics of mathematical discourse, and possible gestures and metaphors (Rasmussen et al., 2004; Stephan et al., 2003) that would support students' learning of multiplication.

Instructional sequence is a local instructional theory that should provide settings for students to model their informal mathematical activities (Gravemeijer, 1994). It includes a sequence of instructional tasks that support the learning goals of the hypothetical learning trajectory. The current study developed and implemented an instructional sequence to promote students' conceptual learning of multiplication by requiring them to mentally and externally apply the actions. In this sense, instructional tasks related to skip counting, composing equal groups, iterating equal groups, forming rectangular arrays, and representing multiplication in realistic contexts were developed and sequenced in the light of the HLT.

Realistic Mathematics Education (RME) is an instructional theory developed at the Freudenthal Institute (Cobb, 2000; Gravemeijer, 2004). In this approach, teachers apply problem-solving methods, involve students in real-life scenarios, and encourage them to apply problem-solving skills by comparing and discussing solutions in small groups and during whole-class interaction. Eventually, children apply mathematical ideas individually and collectively (van den Heuvel-Panhuizen, 2008; Freudenthal, 1983). In the current study, three heuristics—guided reinvention, didactical phenomenology, and emergent models—were used to apply RME (Gravemeijer et al., 2003a; Gravemeijer, 2004). It was intended to assist students through mathematical tasks by providing problem scenarios with the possibility to reinvent mathematics following guided reinvention. Teachers encouraged their students to mathematize situations to develop mathematics to solve difficulties. According to didactical phenomenology, real-life contexts that students experience daily were used for the mathematical task. For the emergent models heuristic, the HLT was constructed by anticipating students' informal mathematical activities shifting into a model for more formal multiplicative reasoning.

Argumentation refers to a social phenomenon in which students present justifications for their actions and adjust their intentions. It is also called "collective argumentation" (Krummheuer, 1995). In the current study, the perspective of Krummheuer (1995) was followed to support a classroom community's collective ways of reasoning and learning in a collective argumentation setting by giving them opportunities to explore informal material and advance to more formal mathematics by participating in negotiation, collaboration, and discussion processes.

Classroom mathematical practice (CMP) is the normative way of acting, communicating, and symbolizing mathematically at a given moment (Gravemeijer & Cobb, 2006). They are taken-as-shared ways of reasoning and arguing mathematically in a social learning environment (Cobb et al., 1997). In the current study, classroom mathematical practices refer to reasoning, explaining, and justifying related to mathematical ideas on multiplication that are taken-as-shared by the classroom community. They are more localized to the classroom and are formed collaboratively by students and teachers through debate; they emerge from the classroom rather than being imported from elsewhere (Stephan & Cobb, 2003). Moreover, they are specific to the mathematical topic (multiplication) at hand, not to more general social practices.

1.4. My Motivation to Conduct the Study

One year before this research, a design project aimed to develop a mathematics learning trajectory in the "numbers and operations" learning area for first-grade students based on the Realistic Mathematics Education approach (Çakıroğlu et al., 2019) was conducted with a collaborating teacher in her classroom. My advisor and I were also part of the research team on this project. For this project, the research team participated in the lessons shaped by the hypothetical learning trajectory over two semesters. The teacher and students evolved social and sociomathematical norms with the design team's help during this one-year project. The data was collected through formal and informal interviews, observations, design team meetings, tests, and field notes. Both qualitative and quantitative analysis methods were used to assess students' mathematical competencies.

The students had difficulties with addition and subtraction. However, their progress at the end of the project was terrific. This situation has motivated me. I wondered if the same improvement would be available for multiplication. I knew that many people recall how stressful it was to learn the multiplication table. The variety of challenges students could encounter while learning to multiply numbers is also well known to educators and researchers. As an eager and ongoing learner of students' learning, I have found myself conjecturing new insights related to multiplication. For that reason, I wanted to conduct a similar study for multiplication. When I shared my thoughts with the collaborating teacher, I saw that she was willing to participate in such design research for the following year with her students. Therefore, knowing the significance of designing instruction on multiplication, this dissertation was developed and conducted on the previous research as capable of prior knowledge of students and classroom norms.

CHAPTER II

LITERATURE REVIEW

The initial purpose of this study is to develop a hypothetical learning trajectory (HLT) and related instructional sequence in order to teach multiplication in the second grade, also named as ‘the purpose of development’. In connection with this goal, the second purpose is to document the classroom mathematical practices used by second graders concerning their conceptualizations of multiplication and to use those practices to update the HLT and the instructional sequence which is called ‘the purpose of documentation’ for the ease of recognition. For these purposes, this chapter is devoted to reviewing the related literature to offer a rationale for carrying out this research and justifying the theoretical viewpoint enabling the interpretation of the findings.

Before going into the details of the current study's methodology, this chapter clarifies the critical ideas by making references to the relevant literature relevant with the purposes of the study. In the first section, the conceptual analysis of the domain is explained in detail in order to provide a basis for the purpose of development and gain insight for interpreting students’ arguments for the purpose of documentation. It gives information about conceptual models related to multiplication, their place in mathematics education programs, and students' conceptual structures and methods by presenting international and local studies on multiplication. The second part reveals possible factors influencing children's conceptual construction of multiplication that should be considered while developing the design germane to the purpose of development. The third part introduces the definition of Hypothetical Learning Trajectory and previously developed HLTs in multiplication to explain the nature of the design developed for the purpose of development. Consequently, the collective mathematical practices approach and Realistic Mathematics Education as a domain-specific instructional theory are explained to clarify the classroom mathematical practices for the purpose of documentation. Subsequently, a summary of these sections is presented to outline the underlying rationale of this research at the end of this chapter.

2.1. Theoretical Background for Multiplication

In this section, theoretical approaches related to multiplication are presented to understand multiplication, its properties, and the related structures of the models. In correspondence with these approaches, the conceptual models of multiplication in mathematics education programs are explained. Then, the students' strategies to process multiplication are presented by examining the relevant literature. Finally, the empirical findings of international and national studies investigating students' knowledge of the multiplication concept are presented and discussed in terms of students' performance on multiplication tasks, difficulties, and misconceptions.

2.1.1. Historical Background of Theoretical Perspectives on Multiplicative Reasoning and Multiplication

Multiplicative reasoning involves complex definitions and quantitative reasoning (Carpenter et al., 1999). The majority of number-related concepts like fractions, percentages, ratios, rates, similarity, trigonometry, proportion, area and volume, probability, and data analysis are supported by multiplicative relationships (proportionality), as may be seen by a casual glance at the mathematics curriculum in schools (Mulligan & Watson, 1998). Multiplicative reasoning comprises the ability to think creatively and flexibly while working with a wide range of numbers, solving problems involving multiplication (or division), and successfully communicating this reasoning through written algorithms, diagrams, symbols, and written language (Siemon et al., 2005).

Theories surrounding multiplicative reasoning have evolved through the research on multiplication and division word problems revived in the early 1980s. Researchers (Bell et al., 1984; Fischbein et al., 1985; Greer, 1992; Nesher, 1988, 1992; Schwartz, 1988; Vergnaud, 1983) have conducted significant studies to assess the comprehension of ideas of multiplication and division. In these studies, the researchers worked through word problems to examine students' cognitive processes in the contexts representing the various meanings of the operation and documented several approaches related to multiplication. Before interpreting these approaches, it should be clarified that the researchers called multiplication and division word problems "multiplicative word problems." (Nesher, 1988, 1992).

This is because multiplication and division share an underlying logical structure, namely multiplicative structure. The difference between multiplication and division in the same context is what is given and what is asked. The information given in a division problem always contains the string that was the question component in the related multiplication problem (Nesher, 1992). For instance, the division version of a multiplication problem can start with, "There are a total of 30 glasses in the boxes." The information in the division problem text splits off into two directions from this point. One of these directions is the presenting problem: "I want to put ten glasses in each box. How many boxes do I use?". This problem is called quotitive division, which means dividing the sum by the group size to determine the total number of groups (sometimes termed measurement division, reflecting its conceptual links with the operations of measurement) (Greer, 1992). The other direction is to present the problem as "I have three boxes. How many glasses do I put in each box?". This problem is called partitive division, and it is common practice to divide the total by the number of groups to determine the number in each group (Greer, 1992). It is also known as equity with social connotations and is called fair sharing (van de Walle et al., 2020). In both situations, it is assumed that the glasses are distributed equally among the boxes.

The approaches associated with multiplicative cases, including multiplication and division and their relationships, are introduced in the following sections to give information related to basic situations modeled by multiplication. Nesher (1992) classified and ordered the multiplicative models under three approaches. The first approach (Fischbein et al., 1985) deals with multiplicative cases from the point of view of primitive intuitive models. The second approach employs the multiplicative conceptual field to classify multiplication situations by dimensional analysis (Schwartz, 1988; Vergnaud, 1983; 1988). The third approach is the textual approach, which focuses on the verbal formulation of the problems to construct a mathematical model of semantic analysis (Nesher, 1988; 1992). Overall, each approach employs a distinctive mode of reasoning in describing different types of multiplicative contexts. They are explained in the following sections in detail.

2.1.1.1. The First Approach: Primitive Implicit Models

The first major approach to multiplication which is from a primitive implicit models' perspective is derived from the theoretical work by Fischbein, Deri, Nello, and Marino (1985), who asserted that "each fundamental operation of arithmetic generally remains linked to an implicit, unconscious, and primitive intuitive model" (p. 4). They claimed that this constrains students' predictions of the operation required when solving multiplication problems. They hypothesized that repeated addition served as the earliest intuitive paradigm for multiplication. With this assumption, they conducted research with students between the ages of 10 and 15. They used an instrument containing 12 whole number and decimal multiplication problems to observe how students dealt with these problems. At the end of the study, they concluded that the decimal's position in a multiplication problem's structure plays a crucial role in determining the correct operation. When the multiplier is a decimal, the multiplication problem gets more challenging. And thus, making it violate the repeated addition model. The repeated addition interpretation states that 2 x 4 equals either 4 + 4 or 2 + 2 + 2 + 2 and therefore is not viewing multiplication as commutative. According to this viewpoint, any positive quantity may be used as the operand, but the operator must only be a whole number. While it is difficult to imagine taking a quantity of 0.42 g times, it is simple to imagine 3 times 0.42 = 0.42 + 0.42 + 0.42, even if the operation cannot be done. However, this interpretation causes the misconception that multiplication always "makes bigger" because the operator is a whole number. In 1985, Fischbein and his colleagues interpreted the sources of these primitive models and presented two possible explanations. First, a model represents how the related topic or operation was initially taught in school. Second, students use primitive models since people take the situations as models that they find behaviorally meaningful.

Several researchers also investigated multiplicative contexts and split them into symmetrical and asymmetrical circumstances (Bell et al., 1989). In asymmetrical situations like repeated addition, different roles are assigned to the multiplier and multiplicand, the two factors that make up the multiplication (e.g., liters times cost per liter). Bell, Greer, Mangan, and Grimison (1989) listed examples of asymmetric multiplicative situations (Figure 2.1).

Structure	**Multiplication**	**Multiplicand**	**Multiplier**
Multiple groups	3 boxes contain 4 eggs each. How many eggs are there altogether?	4 eggs/box	3 boxes
Repeated measure	A gardener needs 3 pieces of string each 4.6 meters long. How much string should he buy?	4.6 meters/piece	3 pieces
Rate	A man walked at an average speed of 4.6 miles per hour for 3.2 hours. How far did he walk?	4.6 miles/hour	3.2 hours
Change of size (same units)	A photograph is enlarged by a factor of 4.6. If the height was originally 3.2 inches, how high is the enlarged photograph?	3.2 inches	scale factor 4.6
Change of size (different units)	A model boat is made to a scale of 4.6 meters to an inch. If the model is 3.2 inches long, how long is the boat?	4.6 meters/inch	3.2 inches
Mixture (same units)	A painter makes a particular color by using 4.6 times as much red as yellow. How much red should he use with 3.2 pints of yellow?	3.2 pints	scale factor 4.6
Mixture (different units)	4.6 lbs. of powder are to be mixed per gallon of water. How many lbs. should be mixed with 3.2 gallons?	4.6 lbs./gallon	3.2 gallons

Figure 2.1. A classification of asymmetric multiplicative situations (Bell et al., 1989, p. 435).

When Figure 2.1 is examined, it is seen that the researchers also included two more columns for multiplicand and multiplier. Because of the problem structures, the multiplier has an active role in the multiplicand. For instance, in the problem related to the change of size (different units), each inch represents 4.6 meters. Since the model is 3.2 inches long, it represents 3.2 times 4.6 meters. Precisely, 4.6 meters is repeated, not 3.2 inches. When these problems are parsed, it is also seen that it is possible to write partition and quotation problems for each multiplication problem. Thus, it should be noted that the asymmetry of multiplication is significant since it causes children to associate division with two primitive models of division; quotitive and partitive (Fischbein et al., 1985).

As indicated, asymmetric situations were used to replicate earlier experiments in which the values were changed in both the multiplicand and multiplier positions. In addition to asymmetric cases, there are also symmetric cases mentioned by Greer (1992). The area and possible combinations (e.g., how many different outfits can one arrange from four shirts and three skirts?) do not fall under the repeated addition model (Nesher, 1992). In symmetrical cases, the roles of the two factors are easily interchangeable (e.g., length times width for computing area). In order to investigate these cases, Verschaffel, De Corte, and Van Collie (1988) studied symmetrical problems and found that the effect of varying numbers holds only for asymmetrical problems in which the multiplier and multiplicand have distinctively different roles. In symmetrical cases, such as finding the area of a rectangle, changing the type of multiplier did not affect the problem's difficulty. In this sense, it was hypothesized that there might be additional models of multiplication than the repeated addition model that impose constraints distinct from those discovered for the asymmetrical problems (Verschaffel et al., 1988).

2.1.1.2. The Second Approach: Dimensional Analysis

Vergnaud (1983, 1988) and Schwartz (1988) revealed another approach to research on models of multiplication. They (a) view simple word problems involving multiplication as a subset of a larger multiplicative conceptual field that includes ratio, rational numbers, vector space, and other concepts, and (b) deal with the dimensions and unit structure of these problems. Vergnaud (1983) defined three main classes of problems within multiplicative structures: isomorphism of measures, the product of measures, and multiple proportions.

Isomorphism of measures refers to viewing a multiplicative connection as a four-place relation rather than a three-place relation or a binary operation. For example, consider the following problem: “Each cat has three kittens; how many kittens would four cats have altogether?” Typically, we would treat this kind of problem as a three-place relationship among “kittens,” “cats,” and “kittens per cat.” According to Vergnaud’s analysis, this problem involves two primary dimensions: Ml (cats) and M2 (kittens), each containing two numbers, as shown in the mapping table below:

Cats (M1)	Kittens (M2)
1	3
4	?

Figure 2.2. Schematic representations of multiplicative problems used by Vergnaud (1988)

According to Figure 2.2, Vergnaud placed "1" in the table in addition to the two numbers (4 and 3). One of the measures in such problems always includes 1 to represent the basis for its ratio. The problem reflects the ratio of 1 cat to 3 kittens while stating, "Each cat has three kittens." So, "each" represents "1" to decide the ratio. In the problem, the ratio of 1 cat to 3 kittens is given, and the ratio of 4 cats to how many kittens is asked. Within each dimension, there is a scalar enlargement or decrement (between 1 & 4 or 3 & ?). Between the two dimensions, Ml and M2, a mapping function maintains a constant ratio (between 1 & 3 and 4 & ? respectively). In this type of problem, there are no restrictions on the numbers used; the quantities within each measure space may be expressed as integers, fractions, or decimals (Greer, 1992).

A product of measures refers to the Cartesian composition of two measure spaces (Ml and M2) into a third measure (M3), as in the following example: "What is the area (M3) of a rectangle of length (Ml) 5 cm and width (M2) 3 cm?" Length and width are two spaces mapped to the third space, area. This problem requires dealing with double proportions rather than with a single proportion, as in the isomorphism of measures problems. As shown in Figure 2.3, each unit area is also stated as 1 unit square.

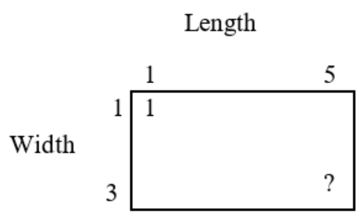


Figure 2.3. Schematic representation of product of measures.

Multiple proportions refer to a measure space (M3) being proportional to two different independent measures (M1 and M2). For example, a family of 4 people wants to spend 13 days at a resort. The cost per person per day is $35. What will the total cost of the holiday be? (Greer, 1992). This problem can be decomposed into simpler problems falling within the classes already defined in the following way, for example,

4 people x 13 days = 52 person-days

$35 per person-day x 52 person-days = $1820

Multiple proportion problems involve magnitudes that have intrinsic meaning; none of them can be reduced to the product of the others (Nesher, 1992). In addition to the approach of Vergnaud (1983, 1988), Schwartz (1988) proposed a second-dimensional analysis for multiplication word problems, which is related to the distinction between intensive (I) and extensive (E) quantities. While extensive quantity refers to a single entity derived from the environment by counting or measuring (e.g., six books), intensive quality (I) involves a ratio, which is a relationship between two extensive quantities (e.g., 12 bottles per box). According to this distinction, Schwartz categorized multiplication word problems into the following three groups:

Multiplication I x E. Such problems are the most common multiplication word problems, corresponding to Vergnaud's isomorphism of measures problems. It can also be presented as repeated addition when one of the numbers is an integer (Nesher, 1992). For instance, in the problem of four cats and three kittens for each, the intensive quantity, three kittens per cat, corresponds to one cat and three kittens in Vergnaud's mapping table. The result of an I x E multiplication problem is an E quantity of the same kind that initially appeared in the intensive quantity. All the problems given as examples of asymmetric situations (Figure 2.1) on primitive implicit models fall under Schwartz's I x E category.

Multiplication E x E. Such problems correspond to Vergnaud's product of measures problems, which are Cartesian product or area. Two extensive measures are multiplied to form a third measure. For instance, five shirts (E1) and two pairs of pants (E2) can be combined to form 10 possible outfits (E3). In the analyses of Fischbein and his colleagues (1985), such problems are considered symmetrical.

Multiplication I x I. Such problems are common in science, including the multiplication of two intensive quantities. For example, 3 km per hour times 5 hours per day makes 15 km per day. These problems are more complicated than other problems for students (Nesher, 1988).

The choice of operation becomes more evident once the students qualitatively examine the dimensions and fill the mapping table with the provided dimensions and numerical data. Hence, students can avoid falling into the numerical errors stated in the study of Fischbein and others (1985). To direct students' attention away from quantitative concerns to qualitative ones, teachers can give students tools like a mapping table. Using such tools, teachers can help students move away from the repeated addition model of multiplication and give them a more comprehensive understanding of this operation (Nesher, 1992).

On the other hand, Vergnaud's (1983; 1988) and Schwartz's (1988) approaches focus on dimensional analysis rather than textual analysis as in they do not focus on what a child must do to discriminate the crucial dimensions of the problem. However, a child's process of creating the structure of the dimensions in question from the verbal text is crucial in solving multiplication word problems. In this sense, Nesher (1988; 1992) focused on textual analysis, as explained in the following section.

2.1.1.3. The Third Approach: The Textual Approach

The textual approach presents the importance of analysis of verbal formulation of the problem (Nesher, 1988; 1992) in which, students can determine which relevant dimensions need to be considered in the solution only after reading and understanding the text. For this reason, it is crucial to investigate the textual level of the problems. Nesher (1988, 1992) examined multiplicative cases and identified three subtypes: mapping rule multiplication, multiplicative compare, and Cartesian multiplication. Similarly, Greer (1992) classified multiplication problems as equal groups, multiplicative comparison, and Cartesian product but, he also defined one more model, rectangular area/array, in addition to Nesher's multiplication models. Most researchers identified these four distinct classes of multiplication structure in their studies (English & Halford, 1995; Greer, 1992).

Mapping rule multiplication refers to repeated addition, mentioned as an implicit model and the I x E structure of Schwartz (1988). For instance, a mapping rule problem can be "There are five shelves of books in Dan's room. Dan put eight books on each shelf. How many books are there in his room?"

As in the case of an addition, a minimal multiplication problem consists of three propositions (strings) in the underlying structure. The three strings are used to define the logical requirements for a minimum, well-formed multiplication problem text. The first string declares in general terms that there are n1 Xs (shelves) for which there are Ys (books), and there is a relationship between the Xs and the Ys. The second string presents a general mapping rule: it says that there are precisely n2 Ys (8 books on each shelf) which describes a mapping between each shelf and eight books, typical of this problem. And lastly, the third string asks how many Ys (books) there are for all the Xs (shelves).

Multiplicative compare refers to "change of size with the same units" cases (Bell et al., 1989). In particular, a multiplicative compare problem can be "Dan has five marbles. Ruth has four times as many marbles as Dan. How many marbles does Ruth have?". In this problem, the first string says that there is a referent set that Dan has five marbles. The second string says that there is a specific function that maps each of Dan's marbles to four of Ruth's marbles. The third string, the question component in multiplication problems, asks how many marbles Ruth has. In here, children must understand that the problem (in a specific verbal formulation) has a direction that is not interchangeable ("Ruth has four times as many marbles as Dan" is different from "Dan has four times as many marbles as Ruth"). Moreover, the quantities of the compared objects do not have to be similar for multiplicative compare problems as Dan's marbles could be compared to Ruth's stamp collection.

Cartesian multiplication refers to E X E structure defined by Schwartz (1988). As a sample problem, a combination of clothes can be asked: "Ruth has four skirts and three blouses. How many different combinations of skirt and blouse outfits can Ruth make?" This kind of problem also consists of three strings. The first sentence represents the first two strings since it describes two independent sets of objects that are blouses and skirts and the third string is the question component, asking how many outfits can be combined with blouses and skirts. This class of problems is less known and is exercised in elementary and middle schools. It involves multiplying two different dimensions to get a third one. Since there is no mapping rule or comparison function to act as a verbal cue for multiplication, children may mistakenly believe that this sort of problem is an addition problem due to the structure of the first two strings.

Cartesian multiplication problems are symmetrical problems (Bell et al., 1989) where two factors can be replaced easily but in caution that it does not mean changing the context from "2 pairs of shorts and three shirts" to "3 pairs of shorts and two shirts" making these two cases give quite different sets of outfits (Hiebert & Behr, 1988). Symmetry refers to the interchanging multiplier and multiplicand. In this case (2 pairs of shorts and three shirts), one can start with shorts and combine each short with three shirts (Figure 2.4a), while one can start with shirts and combine each shirt with two shorts (Figure 2.4b).

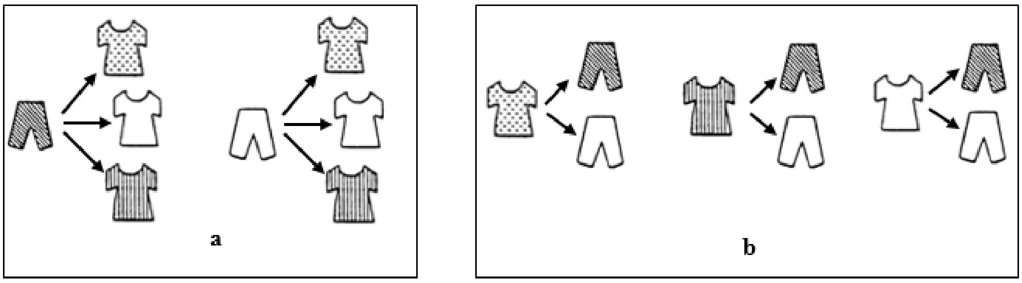


Figure 2.4. Two pictorial representations for the combination of 2 pairs of shorts and 3 shirts

According to the textual approach, problems with mapping rules should be the simplest ones. The students can understand the dynamic process where two provided quantities play and convert it to repeated addition. Teachers should be cautious and discuss the mapping rule rather than treat multiplication as just repeated addition. Otherwise, students could not conceptualize multiplication as a distinct operation. Compare problems are more demanding than mapping rule problems and that is why understanding the asymmetric relationship between the compared and the referent quantity is necessary to overcome such difficulties. The child may find it challenging to determine which direction is correct. The most complicated type is Cartesian multiplication problems where the first two strings (Ruth has four skirts and three blouses) in such problems are identical to the first two propositions in additive compare problems as both types of problems declare the numbers of two quantities. For this reason, students may treat Cartesian multiplication problems as addition problems.

Rectangular area/array contexts are symmetrical (Bell et al., 1989), where two factors can be replaced easily. Greer (1992) added this case and explained that a rectangular array has m rows and n columns. In contrast to earlier models, which called for object visualization, this sort of question does not as the students must use more abstract thinking to deal with the problem, including the rectangular model. For instance, “What is the area of a rectangle 6 cm long by 3 cm wide?” is a sample problem for this category. The format of an array draws attention to the binary input needed for multiplication (Barmby et al., 2009) where the product in the area model has a different unit from the units of two factors. In this sense, the area model should be explicit for students regarding the units. For example, in Figure 2.5 below, the area model on the left comprises three units along the left side and six units along the top. When multiplied together, the total area is 18-unit squares.

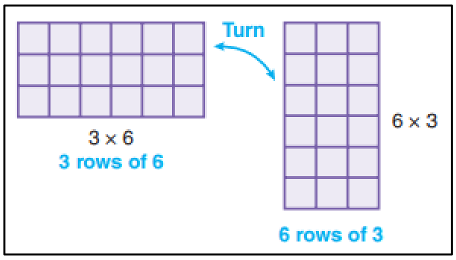


Figure 2.5. A model for the commutative property for multiplication (van de Walle et al., 2020, p.164)

As a symmetrical model, array representation facilitates the learning of commutativity. Using an array and allowing children to arrange and rearrange sets of items is one method to examine the commutative rule as a more abstract idea, as in Figure 2.5. In comparison to static images, which can often be challenging to observe in two different ways, moving objects into rows and columns will offer a better opportunity to investigate the commutative rule (as rows of one number and, at the same time, columns of another number)

(Anghileri, 2006; Barmby et al., 2009; Greer, 1992; Sowder et al., 2010; van de Walle et al., 2020).

In conclusion, several researchers analyzed multiplication word problems from different theoretical approaches. They described the primary categories of basic multiplication models in these contexts to highlight these variations, as summarized in Table 2.1 below.

Table 2.1. Models of multiplication from different theoretical perspectives

Fischbein et al. (1985)	Repeated Addition (asymmetrical)		Symmetrical situations		
Nesher (1988)	Mapping rule	Multiplicative Compare	Cartesian multiplication		
Vergnaud (1983)	Isomorphism of measures		Product of measures		Multiple proportion
Schwartz (1988)	I x E	S x E	E x E		I x I
Greer (1992)	Equal groups (Repeated addition and rate)	Multiplicative comparison	Cartesian product	Rectangular area / array	

Studies examining issues from various viewpoints are intertwined rather than presented linearly. Indeed, they all must contribute to the solution of multiplicative problems. Although, it should be kept in mind that children initially encounter a problem in its verbal formulation, and their problem-solving activity starts with this text. As a result, teaching students how to discern a mathematical structure in a text is crucial. So, textual analysis must be used in this situation to consider the full text, not just a few essential words.

The analyses of the multiplicative models revealed that it might be beneficial to use more than one model of multiplication while teaching multiplication in the classroom. Students may learn more about the commutative property using symmetrical models (Graeber & Tirosh, 1988). Thus, from a practical perspective, researchers may see the structures that a person who has acquired a particular degree of cognitive development may resolve (Nesher, 1992). In other words, solving difficulties can be thought as creating a pre-existing schema to solve different kinds of word problems. Therefore, it is recommended to arrange multiplication word problems according to their basic structure and incorporate them into math education programs consequential. In an essence, mathematics education programs in several countries are examined to reveal the models related to multiplication in the following section.

2.1.2. The Importance of Multiplication and Its Place in Mathematics Education Programs

Multiplication is one of the most important mathematical concepts since a brief look at the mathematics curriculum in schools reveals that nearly all number-related topics are supported by multiplicative relationships (proportionality) (e.g., fractions, percentages, ratio, rates, similarity, trigonometry, rates of change) (MoNE, 2018; White & Mitchelmore, 2005). For this reason, during the instruction of multiplication, the purpose is to make students multiplicative thinkers who are familiar with how multiplication works and who can solve multiplication problems with ease (MoNE, 2018; Smith & Smith, 2006). Hence, educational programs should assist all learners in grasping what operations represent and how they are connected (MoNE, 2018; NCTM, 2000).

Regarding "number and operations" in Principles and Standards for School Mathematics (2000), students should comprehend multiplication-related scenarios, including groups of objects equally. Students should start learning the fundamentals of multiplication from kindergarten through grade two. They can identify multiplication with the repeated joining (addition) of groups of equal size by working in instances where there are equal subgroups within a collection (NCTM, 2000). Regarding the scope of the current study, the objectives related to multiplication in the mathematics curriculums of Turkey and several countries were examined.

The Turkish Mathematics Education Program (2018) includes 19 objectives directly related to multiplication. Three of these objectives are in second grade, six of them in the third grade, six of them in the fourth grade, and four of them in the fifth grade. According to the Turkish Mathematics Education Program, multiplication is introduced in the second grade for the first time. Second-grade students deal with the concept of multiplication by multiplying two one-digit whole numbers, defining multiplication as the repeated addition of equal groups, and solving multiplication problems that require a one-step operation. They also experience and explain the commutative law and multiplication properties of 1 and 0. The multiplication tables until 5 (included) are created using a hundred charts in the second grade. The whole multiplication table is created in the third grade. Students are introduced the multiplicative comparison meaning of the operation in the third grade. They discover the rules of multiplication with 10 and 100. In this grade, students multiply two-digit numbers with numbers that have at most two digits and multiply three-digit numbers with one-digit numbers. Students are also expected to discover the change in the product when the factor is increased or decreased by 1 unit. After that, they work on multiplication problems that require two-step operations. Next, fourth-grade students deal with the multiplication of three-digit numbers and the rules of multiplying with 5, 10, 25, 50, 100, and 1000. Here, students are expected to estimate the results of multiplication in the fourth grade. Additionally, they multiply three numbers and explain the associative law of multiplication. In the fifth grade, students find the missing factor in multiplication operations by associating multiplication with division by constructing the inverse relationship between these two operations. Moreover, they work on estimation activities in this grade.

In addition to the Turkish Mathematics Education Program (2018), the analysis of multiplication in different countries made by Olfos, Isoda, and Estrella (2021) was taken advantage of in order to see how multiplication instruction is organized globally across national curriculum standards (programs). Singapore, Japan, Portugal, the United States, Mexico, Brazil, and Chile are the nations selected to be compared. These nations are two from Asia, one from Europe, two from North America, and two from South America consecutively. The comparison highlights the variations in different nations' multiplication programs concerning the order of the descriptions and the methods of explanation (Olfos et al., 2021).

In the first or second grade, counting by twos or fives is introduced in every country, as shown in Table 2.2 below. Except for Singapore and the USA, most nations start teaching multiplication in the second grade. In all countries, multiplication is modeled as repeated addition. In the USA, repeated addition is first introduced in second grade, but a definition of multiplication is given in third grade. Singapore starts teaching multiplication by 40 in the first grade. Mexico and Turkey do not describe multiplication as a group of groups or length based on unit length in a tape diagram. All countries except Turkey introduce multiplication with various models in addition to repeated addition. These countries use the length model or array or both in the first three grades. The Turkish mathematics education program does not include multiplication as length based on unit length in a tape diagram or rectangular array. Every country introduces students to the rectangular area in the upper grades. Only in Portugal are discussions about combinatorics.

Table 2. 2. Grade levels in which different countries introduce various models of multiplication

Country	Turkey (2018)	Chile (2012)	Mexico (2017)	Brazil (2016)	Portugal (2013)	Singapore (2012)	Japan (2017)	USA (2010)
Counting by 2s	1	1	1	1	2	1	1	2
Situation for adding equal quantities	2	2	2	2	2	1	2	2
Repeated addition	2	2	2	2	2	1	2	2
Group as a unit or group of groups (without repeated addition)	–	2	–	3	2	1	2	3
Length based on unit length (tape diagram)	–	2	–	3	2	1	2	3
Array diagram (or rectangular shape)	–	3	2	3	–	–	2	2
Area (rectangle)	4	4	4	5	3	3	4	3
Combinatorics	–	–	–	–	2	–	–	–

From the implicit model approach (Fischbein et al., 1985), the table also reveals that all countries introduce models representing both symmetry and asymmetry of multiplication. All countries except Turkey present at least one of the symmetric models (array, area, combinatorics) in the second or third grade. On the other hand, Turkish students are presented with a symmetrical model (area) in the fourth grade and are taught multiplication as repeated addition of equal groups in the second grade, then, multiplicative comparison, which is another asymmetric model, in the third grade.

The comparison in Table 2.2 demonstrates how and when multiplication is taught in each country. Different instructional methodologies and materials are used in these countries. In the light of these comparisons, various issues can be subject to research such as how the various interpretations support comprehending multiplication, is repeated addition the only way to solve a problem involving multiplication, is multiplication best explained by the utilization of equal amounts in a group, is an array model a suitable substitute for a group of groups, so on and forth. As these issues had a critical role in interpreting the instructional practices in educational documents and the schools and directing future studies, the literature on how to teach multiplication should be elucidated.

Fosnot and Dolk (2001) listed ‘big ideas’ that are the fundamental, structuring ideas of mathematics principles that define mathematical order in their books on constructing multiplication. These ideas are called "big" because they are essential to mathematics and represent significant advances in children's conceptual growth. From this perspective, they state the importance of *unitizing* in multiplication by consolidating a group as a unit. Thenceforth, they emphasize the importance of the *distributive property* of multiplication in relation to addition and subtraction. Understanding the structure of the part/whole relationships involved is necessary to realize that 9x4 can be solved by adding 4x4 and 5x4 or any combination of groups of four that sum up to nine groupings (Piaget, 1965). In this case, the nine groups comprise the entire system, with the parts being combinations of five and four groups, six and three groups, seven and two groups, eight and one group, and the inversion of these combinations. It is suggested to create an array to investigate the associative property of multiplication in a context that makes sense to students. For instance, students can be asked to find how many muffins the baker has, as in Figure 2.6 below. Students are asked questions related to the muffins that are sold and left, in which they are expected to discover that the muffins in the second and third trays are equal to the amount in the first tray.

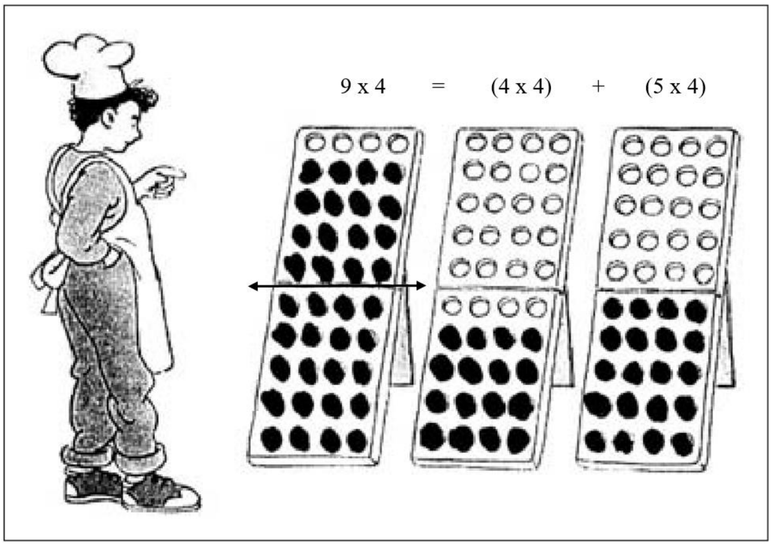


Figure 2.6. An array model for the distributive property for multiplication in a realistic context.

Moreover, understanding the commutative property—5x3=3x5—and the associative property—(2x3)x5=2x(3x5)—are also big ideas. These aspects of multiplication can be observed and investigated in a two-dimensional graph paper array (commutative property) (Figure 2.5) or three-dimensional boxes (associative property) (Figure 2.7). As an example, it is suggested to create a three-dimensional array, as in Figure 2.7, to discover the associative property for multiplication. It does not matter whether the first or last pair is multiplied first as 4 x (2 x 3) is the same as (4 x 2) x 3. In this manner, comprehending arrays—and volume, for that matter—is a significant idea in and of itself. For this reason, Fosnot and Dolk (2001) also stress the importance of understanding arrays since they suggest designing instruction for multiplication around these big ideas.

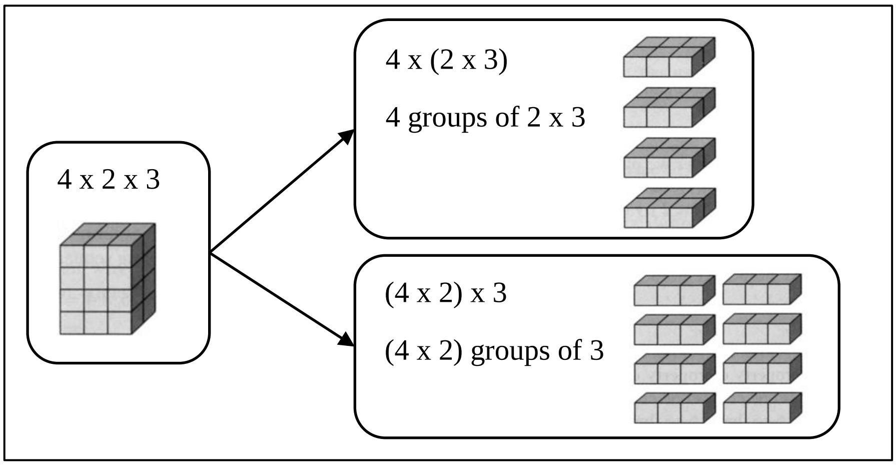


Figure 2.7. A three-dimensional array model for the associative property for multiplication

Similarly, Kennedy, Tipps, and Johnson (2008) also provide a sequence for developing concepts and skills for multiplication and division by using a framework with three columns: concepts, skills, and connections. The sequence starts with number concepts 1-100, including skills of skip counting, recognizing groups of objects, and thinking in multiples. Then, the activities followed with the numbers between 1 and 1000 related to the skills of representing numbers with base-10 materials, exchanging rules and games, and regrouping and renaming. Later, students are expected to work on numbers more significant than 1000. After dealing with the concepts related to the numbers, students work on multiplication facts, including stories and actions for joining equal-sized sets (repeated addition), arrays and area (geometric interpretation), and Cartesian combinations as well as being able to represent multiplication with materials, pictures, and number sentences. In parallel with multiplication, students work on multiplication facts, including stories and actions for repeated subtraction, division, and partitioning. In addition, students represent division with materials, pictures, and number sentences. After working on realistic mathematical tasks on multiplication and division, students are expected to deal with basic facts for multiplication and division and operations with larger numbers. During the implementation of the tasks on these concepts, the skills related to fact fluency, estimation, and the use of technology are targeted to be developed.

Another sample instructional model was obtained from the EngageNY mathematics curriculum resources provided on the webpage of the New York State Education Department (2014). These resources were developed by New York State Common Core (NYSCC) (2014) and released as free curriculum files. Among these resources, a module for foundations of multiplication and division is developed around four topics: formation of equal groups (Topic A); arrays and equal groups (Topic B); rectangular arrays as a foundation for multiplication and division (Topic C); and the meaning of even and odd numbers (Topic D).

In Topic A which is formation of equal groups, students start by creating equal groups with concrete materials, learning to manipulate a given number of objects to create equal groups (for example, they are expected to create three groups of 5 or 5 groups of 3 when given 15 objects), and moving onto pictorial representations, where they might start by circling a group of 5 stars, adding 5 more, then adding 5 again, before moving onto pictorial representations from which they calculate the sum and connect their illustrations to the relevant repeated addition equation. Students can either add stepwise to the prior addends to calculate repeated addition sums, or they can pair the addends and then add them all together (NYSCC, 2014). In Topic B, students make arrays out of the equal groups they formed in Topic A, where a row or a column is viewed as the new unit being counted. For instance, in order to create an array of 3 columns of 2 counters, or six counters, students might arrange one column of 2 counters, then another, and so on and construct several number phrases that add up to the same total as they compose and decompose arrays such as, 2+ 2+ 2 = 6 and 3+3 = 6. Students transition to the pictorial level as Topic B advances in order to express arrays and distinguish rows from columns by dividing equal groups horizontally and vertically (e.g., three columns of 2 or 2 rows of 3), then continue their work with arrays in Topic C to improve their spatial reasoning abilities in preparation for the area subject in Grade 3. They tile a rectangle using identical-sized squares without any gaps or overlaps and then count to determine how many squares make up the rectangle. Following that, pupils break down rectangles after they have composed them. By doing this, they learn that the row and column are made up of several squares or composite units that are the components of the rectangle. In this way, students relate repetitive additions to the model throughout the entire topic where they are urged to think creatively and consider how an array might be built or divided (NYSCC, 2014). In Grade 2, students do not multiply or divide; instead, this lesson builds the groundwork for the connections between the two operations. A whole can be divided into equal parts, just as equal parts can be combined to produce a whole. Finally, Topic D is all about even- and double-digit integers. They discover the subsequent explanations of even numbers.

- An even number appears as we skip-count by twos
- The number is even when all objects have been paired with one another.
- Doubles, or numbers that are twice a whole number, are even.
- An even number has the last digit of 0, 2, 4, 6, or 8.

The final section of the module explores the results of a repeated addition when two even numbers, two odd numbers, or an odd number and an even number are added (for example, 3 + 3 is even, but 3 + 3 + 3 is odd).

These (what are they in summary) sample instructional documents emphasized that students should be taught the meaning of multiplication and fundamental multiplication properties, such as the commutative property. They also state using symmetric models of multiplication as arrays (Fosnot & Dolk, 2001; Kennedy et al., 2008; NYSCC, 2014), area and Cartesian combinations (Kennedy et al., 2008) in addition to repeated addition. It is seen that, as stated in historical background and previous research, various models are suggested in educational documents considering the properties of multiplication.

Therefore, the use of multiplication models in classrooms is presented and interpreted to develop instruction for teaching multiplication in second grade.

While designing an instruction, it is crucial to know about students to understand their conceptions of the idea (Hill et al., 2008). In other words, how students understand a content domain is a fundamental factor in teaching it. To this end, students' computational strategies are presented in order to have a deeper understanding related to students' comprehension of multiplication in the following session.

2.1.3. Children's Strategies for Single-Digit Multiplication

The nature of multiplication requires a higher level of sophistication in thinking about numbers and procedures (Jacob & Willis, 2001; Schwarz, 1988; Vergnaud, 1983). Various research studies have identified the stages that children go through as they develop an understanding of multiplication concepts (Anghileri, 1989; Jacob & Willis, 2001; Kouba, 1989; Mulligan & Mitchelmore, 1997). Despite a growing amount of research, there are still significant differences between researchers' descriptions of specific strategies and terminology. In this regard, Sherin and Fuson (2005) tried to agree on the taxonomy of multiplication strategies and they presented a taxonomy that can be used to clarify the nature of the learning tasks involved in multiplication, building on the work of other scholars. Figure 2.8 gives an overview of relevant studies and highlights the taxonomic schemes from the most relevant and essential papers (Anghileri, 1989; Cooney et al., 1988; Kouba, 1989; Lefevre et al., 1996; Lemaire & Seigler, 1995; Mulligan & Mitchelmore, 1997; Siegler, 1988). As in the figure, students' solutions for multiplicative cases vary from counting all the objects by ones to hybrid counting. This section explains the literature about students' strategies for multiplication in detail.

	Count-all		Additive calculation	Count-by	Pattern-based	Learned products		Hybrid
Mulligan & Mitchelmore, 1997	Unitary counting (direct counting)	Repeated addition				Multiplicative calculation		
		Rhythmic counting	Repeated adding, additive doubling	Skip counting		Known multi-plication fact		Derived multi-plication fact
Kouba, 1989	Direct representation (only use of physical objects)		Additive	Transitional counting		Recalled number facts (includes derived facts)		
Anghileri, 1989	Unary					Known fact (binary)		
	Unitary counting	Rhythmic counting		Number pattern				
Lefevre et al., 1996			Repeated addition	Number series	9's rule 0's and 1's rules	Retrieval		Derived fact
Lemaire & Seigler, 1995; Siegler, 1988	Counting-set-of-objects	Repeated addition			(Rapid responses coded as retrieval)	Retrieval	Writing problem	
Cooney et al., 1988	Counting (not clear if this also includes count-all)				Rules (0 and 1) Other	Memory retrieval		Derived fact

Figure 2.8. Overview of strategy taxonomies from selected articles (Sherin & Fuson, 2005, p.361).

In the first strategy, *the count-all strategy*, a person carries out the calculation from 1 to the product (Anghileri, 1989; Kouba, 1989; Mulligan & Mitchelmore, 1997). Children count one-to-one in this phase (Jacob & Willis, 2003). When the operands are huge, count-all solutions can be the most time-consuming and challenging to implement correctly. Instead of this, three distinct counts must be coordinated in order to perform a count-all computation. Consider the task of multiplying 4 and 3 as an example. Rhythmic counting is counting the total made by counting 3s four times. This strategy requires enacting and coordinating the four counting sequences shown in Figure 2.9. To make it clear, (1) students need to count from 1 to 4 to keep track of the number of groups; (2) then, count from 1 to 3 four times to keep track of where we are within each group; and (3) finally, count from 1 to 12, thus keeping track of the running total.

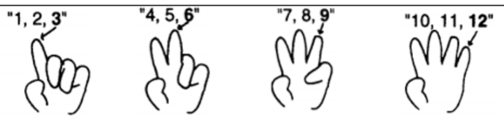
		1			2			3			4	Count of the number of groups
1	2	3	1	2	3	1	2	3	1	2	3	Count of entities in a group
1	2	3	4	5	6	7	8	9	10	11	12	Count of total
												Rhythmic counting with fingers

Figure 2.9. The three coordinated counting sequences for multiplying 4 x 3

Children in the count-all stage must understand that a collection can be counted in several ways while maintaining the same quantity. They should also be aware that even if a collection is rearranged, the quantity will not change (Jacob & Willis, 2003). These students require activities that force them to organize collections, so that skip counting is more effective than counting by ones. In this sense, students are recommended to use equal groups in arrays' rows and columns (Anghileri, 1989; Fosnot & Dolk, 2001).

In the *additive calculation strategy*, students use addition-related procedures in a more sophisticated way. Students already have resources that can serve as the foundation for quicker and simpler methods than count-all strategies since they have prior learning experiences related to addition. They use repeated addition and doubling to collect equal-size groups (Kouba, 1989; Lefevre et al., 1996; Mulligan & Mitchelmore, 1997). For instance, students multiply 3 x 5 by first adding 5 + 5 to get 10, then 10 + 5 to get 15. This strategy has features that clearly distinguish it from the strategies of count-all. Students do not present every value between 1 and 15; instead, they jump from 5 to 10 to 15. In multiplicative words, students in the phase of additive composition can identify equal groups and use the groups to count more quickly through repeated addition. Before repeatedly adding to determine how many, they might still need to spread out the items in the groupings (Jacob & Willis, 2003). They still do not understand that the groups can be tallied by themselves. They ignore the multiplier's function because they are focused on the multiplicand.

When solving simple multiplication problems, children that employ additive reasoning can arrange the objects as the problem specifies and count them. However, they might only count by twos or threes. Given that there are existing groups, in reality, they do not need to keep a count of the number. All they need to do is to concentrate on the multiplicand and count as they do not have to build the multiplier themselves (Jacob & Willis, 2003). Children's tasks change completely when tools are not made available to them because they need to keep track of how many groups there are in some way. Children may be forced to create the multiplier on their own, learn to count groups, and then, learn to count the groups and the number in each group (Jacob & Willis, 2003).

In the *count-by strategy*, students practice transitional counting and skip counting using number sequences (Anghileri, 1989; Kouba, 1989; Lefevre et al., 1996; Mulligan & Mitchelmore, 1997). In this phase, they can keep track of two things (i.e., group number and size) at once. Students can record or verbally state the multiplication fact as well as visualize the groups, but they must compute the solution using a counting sequence based on multiples of one of the factors in the issue. A pupil can draw, count with one's fingers, or utilize tally marks to keep track of the count such as counting by 4s six times while using their fingers to help, as in Figure 2.10 below.

4	8	12	16	20	24	Count of total
1	2	3	4	5	6	Number-of-groups count

Figure 2.10. The two sequences to be coordinated for multiplying 6 x 4

While students use count-by strategies, they get the advantage of number sequences. Most students who can recall the systematic order of words and their connection to numbers can learn the skills necessary for counting. Counting in multiples also requires memorizing regular word sequences with a recognizable rhythm and pattern (Anghileri, 1995). These counts provide a strong foundation upon which multiplication and division concepts can be developed when they are connected to repeated groups of objects or equal spaces on the number line. In this sense, instead of teaching the children how to do written math, they should be encouraged to think critically, look for patterns, predict outcomes, and talk about the connections that might be made (Anghileri, 2006). Hence, making students practice skip-counting and composing number sequences is crucial.

In *patter-based strategy*, parallel with count-by sequences and strategies, students use numerous patterns, including Nx1=N and the 9's patterns, that are number-specific resources (Sherin & Fuson, 2005). The 0's pattern, 1's pattern, and 10's pattern are a few of these patterns. With the help of these three subtypes of patterns, pupils can achieve specific outcomes quickly and covertly (Cooney et al., 1988; Lefevre et al., 1996). These pattern-based tactics may be challenging to discern from taught product strategies because they are linked to students' quick answers. However, as these techniques are based on a fundamentally distinct number-specific resource, it makes sense to treat them as a separate category (from learned product) (Sherin & Fuson, 2005).

Learned-products strategies are associated with an extensive collection of number-specific resources as the multiplication triad (Sherin & Fuson, 2005). Students answer the questions as they remember without counting. These strategies are called known multiplication fact, recalled number fact, or retrieval in different studies (Anghileri, 1989; Cooney et al., 1988; Kouba, 1989; Lefevre et al., 1996; Mulligan & Mitchelmore, 1997). According to Figure 2.8, Lemaire and Siegler (1995) divided the learned product category into two parts: "retrieval" and "writing problem," the latter of which is a smaller category. The only way this approach differs from retrieval is that the learner writes out the two multiplicands (for example, 8x4) before voicing the solution. Both categories are treated as learned-product strategies in the scope of the current research.

Finally, in *hybrid strategies*, students use combinations of the strategies above (Cooney et al., 1988; Kouba, 1989; Lefevre et al., 1996; Mulligan & Mitchelmore, 1997; Sherin & Fuson, 2005). In principle, many possible combinations of current methods exist to create hybrid strategies. As an example, students can multiply 6x5 by starting from 5x5=25 and then counting from 26 to 30 on their fingers. In another way, they can start from 5x5=25 but add on the last multiple of 5 using additive resources. This strategy is thus described as an instance of learned product + additive calculation.

In conclusion, an effort to get an agreement on the taxonomy of single-digit multiplication strategies is made in this section. The intention is to create a thoroughly explained plan and minimize the possibility of misinterpretation. Due to these attempts, many instances are provided and explored in depth. It has been shown that children use diverse strategies throughout their learning period. Students start by employing counting techniques, move on to strategies based on repeated addition, and then use multiplication-related properties, like commutativity and basic facts, to solve multiplication problems (Mulligan & Mitchelmore, 1997). Students' computational strategies related to multiplication are sensitively dependent on specific details of instruction (Sherin & Fuson, 2005). For this reason, these strategies should be kept in mind while developing instruction to provide an environment for students to develop their strategies and interpret their reasoning. Along with these strategies, studies related to students' conceptualization of multiplication are reviewed in the following section for detailed information.

2.1.4. Previous Research Studies on Multiplication in International and National Arena

Students build their first understanding of multiplication through practice with whole-number multiplication. They commonly discover that multiplication is repeated addition (Thompson & Saldanha, 2003). In this reasoning, multiplication usually includes groupings of objects with the same number in each group (Greer, 1992; van de Walle et al., 2020). Fischbein and his team (1985) pointed out that students start to believe that the multiplier (group number), which indicates how many times a quantity must be added, must also be a whole number. Therefore, when students face a multiplier which is any other kind of number (e.g., a decimal or a fraction), they may not be able to translate the multiplicative situation into a "times" situation resembling the repeated addition operation as it would be awkward to say "1.23 times 4". These empirical findings of the study (Fischbein et al., 1985) were replicated by many other researchers several times (e.g., De Corte et al., 1988; Graeber & Tirosh, 1990).

For instance, De Corte, Verschaffel, and Van Coillie (1988) studied how well students perform when using various multipliers (integers, decimals larger than 1, and decimals smaller than 1) and worked with 116 students from five sixth-grade classrooms in two Flemish schools. They used a survey, including 16 multiplication word problems. While eight problems had an asymmetrical structure, the other eight problems were symmetrical. For the asymmetrical structure, the rate problem type was chosen (e.g., Pete buys a rope of 5.7 meters. One meter of rope costs 14.5 Bfr. How much does he pay?). For the symmetrical structure, they preferred area problems (e.g., the dimensions of a tennis net are 0.8 and 7 meters. What is the area of that tennis net?). All symmetrical and asymmetrical problems differed only concerning the type of the multiplier or the multiplicand (either an integer, a decimal larger than 1, or a decimal smaller than 1). The study's findings are parallel to those in the study of the team of Fischbein. It is revealed that students' performances are strongly affected by the nature of the multiplier (whether it is an integer, a decimal larger than 1 or a decimal smaller than 1). On the other hand, the nature of the multiplicand has little or no effect on the problem's difficulty.

Graeber and Tirosh (1990) interviewed fourth and fifth graders using eight tasks involving multiplication and eight similar tasks involving division. For this study, only the three multiplication tasks are discussed here. In the first task, each student is shown a card with the equation 4 x 5 = 20 written on it and asked to define multiplication by identifying the terms associated with the numbers 4, 5, and 20. The second task asks students to use a 7x7 grid to show 3x4=12. In the third task, students write a word problem for the sentence 5 x 6 = 30. Graeber and Tirosh (1990) found that 65% of the students defined multiplication in terms of repeated addition whereas, twenty percent of the students either gave an incoherent definition or no definition. One of the students defined multiplication to check division and in contrast, the other students defined multiplication as being vaguely related to addition or being vaguely related to groups. In the second task, 63% of the students were successful at representing either 3 x 4 or 4 x 3 on the grid. Finally, 75% of the students successfully wrote word problems with minimal help from the interviewer. Each problem reflected either the union of equivalent groups or an example of repeated addition. The findings show that students can represent multiplication as a rectangular array but, they define multiplication as repeated addition and equal groups thus and so posing repeated addition and equal group problems.

While constructing multiplication on repeated addition of equal groups, teachers should be careful not to cause misconceptions in the future. For instance, Lo, Grant, and Flowers (2008) detected misconceptions stemming from the overgeneralization of addition strategies. They conducted a study with prospective elementary teachers and reported their challenges as they revisited whole number multiplication through a sequence of tasks. The prospective teachers were asked to develop and justify reasoning strategies for the multiplication of two-digit numbers. The following was the erroneous method that was most frequently observed: 18 x 26 = (10 x 20) + (8 x 6). When asked to explain their reasons, many participants responded that their argument was valid because the same strategy worked for addition: 18 + 26 = (10 + 20) + (8 + 6), and thus should have worked for multiplication. Moreover, another error was treating each number's increase or decrease as an independent change in the multiplication problem. Students were asked to evaluate 36 x 17 by starting with 40 x 20. Three different errors were observed. These are subtracting 3x40 and 4x17 from 800, 3x20 and 4x40 from 800, and 3x36 and 4x17 from 800. Prospective teachers treated the subtracting two operations (e.g., 3x40 and 4x17) from 800 independently. The participants did not consider that each subtraction affects the other one. Hence, it can be concluded that overgeneralization of addition (and subtraction) may cause misconceptions related to the multiplication of multi-digit numbers.

When all these studies are examined, it is apparent that there are various tools to measure the comprehension of multiplication. One of these assessment tools is problem writing (problem posing). Drake and Barlow (2007) conducted a study with forty-five sixth-grade students and asked them to write a problem that could be solved by the expression 4x8. Six types of problems were detected. Eleven percent of the students used the numbers 4, 8, and/or 32 but did not correctly represent the multiplication fact in any way (e.g., Jimmy needed help to solve this math problem: You have Tom's four tires, and you buy Rob's eight rims, and [it] equals [what?]). Interestingly, 7 percent of the sixth graders provided problems that represented division (e.g., four dogs, and all of them have puppies. There are 32 puppies in all. How many puppies did each dog have?). Thirty-one percent of the students successfully created a multiplication scenario yet failed to correctly represent four as the multiplier and eight as the multiplicand (Amy has eight fish tanks for sale. Each tank comes with four fish. How many altogether? (thirty-two)). Thirteen percent of the students created problems by accurately representing a multiplication scenario but failing to ask an appropriate question (There were four groups of rescue workers with eight people in each group. How many people were in each group?). Other students correctly represented 4x8 and have included a question that calls for product 32 (There are four rows of 8 dogs. How many dogs are there?). Through these problems written by the students, the researchers gained insight into the students' understanding of multiplication.

Parallel to this study, problem writing is also popular in Turkey to measure students' ways of thinking and depth of understanding related to the topic. For instance, Kılıç (2013) examined how elementary students performed on tasks that involved problem-posing and operations using natural numbers. For this purpose, students were given a problem-posing test that included 4 cases. The participants were 270 fifth graders and 182 fourth graders. According to the findings of the study, it can be concluded that almost half of the students in each grade had difficulty posing multiplication problems. The most common error was that students tended to pose problems requiring two different operations (e.g., division and addition) simultaneously. Besides, in that study, some issues as posing problems using different arithmetic operations, using missing data, not giving any answer, using decimal numbers, writing mathematical exercises, and posing problems for different mathematical topics have emerged.

In another study, Doğan and Doğan (2019) investigated the characteristics of the multiplication and division operations problems by fifth-grade students and the meanings the students attributed to these operations. The participants of the study consisted of 95 fifth-grade students. The data was obtained from the structured problems posed by the students for multiplication and division operations and semi-structured interviews with 12 students. The study's findings showed that the students preferred scaling (i.e., multiplicative comparison) for multiplication and equal sharing for division structures. It also became clear that students had a great deal of difficulty posing problems involving multiplication and division and most students had problems relating operations to everyday activities. Additionally, it was shown that students posed problems for other operations, used irrelevant data, were unable to create questions, and shared less or more for division. After conducting the interviews, it was determined that the challenges might have resulted from students' erroneous understandings of the operations for multiplication and division. Furthermore, it was observed that students attributed meaning to operations that were irrelevant in real life.

Similarly, Tertemiz (2017) explored the meanings associated with the mathematical problems generated by children in primary school that call for the application of mathematical operations with natural numbers. The data were gathered using a semi-structured form that included problem statements. The participants were 65 participants from the first grade, 85 from the second grade, 90 third from the third grade, and 88 fourth-grade students. The mathematical problems that the participants generated were examined. The findings revealed that regardless of grade level, most participants were significantly more successful in creating mathematical problems that called for addition and subtraction rather than multiplication and division. Moreover, it was discovered that participants evaluated multiplication and division as equal groups in terms of the meanings connected to mathematical issues.

In addition to interpreting students' understanding of multiplication through the problems they posed, other studies also explore students' performances in terms of errors and misconceptions while multiplying. In this sense, Üçüncü (2010) developed an achievement test and implemented it for 998 students in the 2nd to 5th grades. She examined how students make multiplications and how they model the operations of multiplication. In this way, she found that students have errors in expressing the given model as multiplication, modeling multiplication, evaluating the effect of "0" in multiplication, memorizing the multiplication table, distinguishing addition and multiplication, problem posing on multiplication, comprehending commutative and distributive properties in multiplication, multiplying with 10, 100, and 1000, deciding the operation for given problem and processing procedural multiplication.

In another study, Kubanç and Varol (2017) conducted a study to determine the misconceptions and mistakes that the second (n=36) and third (n=36) grade students in a primary school experienced in verbal arithmetic problems requiring multiplication, together with their reasons. While the document analysis technique, which is a qualitative research method, was used to detect errors, clinical interviews were conducted with students to determine the reasons for the errors. The data obtained were first classified according to the correct, incorrect, and blank answers the children gave to the problems. Then, the wrong answers to the questions were classified according to the four error types determined by the researcher, and descriptive analysis was performed. As a result of the research, making subtraction instead of multiplication, adding instead of multiplication, generalizing the rules of addition and subtraction to multiplication, continuing the process without shifting the digits, and incomprehension of the rule of multiplication by 0 and 1 were the most common mistakes that students exhibited in questions requiring multiplication.

Ekici and Demir (2018) also surveyed the mathematical errors the fourth graders make when using their mathematical language skills to solve story problems. They utilized an instrument containing ten problems for their work with the seven students. According to the Newman Error Analysis methodology, they completed the appropriate applications. According to the survey's results, the fourth graders struggled to read, comprehend what they read, and articulate what they read in their own words. Since they had difficulty understanding the problems, they could not identify the proper pattern to solve them. In addition, they made up their operations using the numbers they saw in the questions. As their lack of knowledge of four operations might cause them to make errors, they mainly avoided multiplication operations.

In the same grade level, Sidekli, Gökbulut, and Sayar (2013) researched the difficulties of four operations in mathematics lessons experienced by 4th-grade students in a primary school to overcome these difficulties. They developed an achievement test that consisted of 5 items for each operation with whole numbers. It was observed that the students could not answer any of the questions in the multiplication part. It was determined that the main reason for this was that the students made mistakes in the addition process and did not gain the comprehension of the multiplication process as the shortest way to the addition process. Other findings showed that the students had difficulty in the division process because they could not do the multiplication.

Gürsel (2000) conducted a study with thirty 6th, 7th, and 8th-grade students working with middle school students. He used an instrument including 64 questions related to multiplication with single-digit, two-digit, and three-digit whole numbers. He detected seven types of errors in multiplication. Two of them were related to conceptual errors, such as adding instead of multiplying and failing to understand how to multiply by 0. The other five errors were related to the multiplication procedure, so students were confused about the numbers and their digits while multiplying.

In conclusion, various studies were conducted to understand students' identification of multiplication, interpretation related to multiplication situations, multiplication strategies, errors, and misconceptions. Considering these studies, it is seen that students' conceptions of multiplication are mainly limited to repeated addition and equal groups. Their interpretation of multiplication causes various misconceptions, even for the following topics such as multiplying multi-digit numbers. For this reason, it is crucial to develop meaningful learning environments related to multiplication.

While developing effective learning environments, it is important to determine the factors which affect students' learning. In this regard, numerous research has identified potential variables that may play a significant role in influencing the development of multiplication (Goldin & Shteingold, 2001; Remillard, 2005; Tobin, 1987; Usiskin, 1985; Valverde et al., 2002). These variables could be considered when creating and carrying out educational programs (Wawan & Retnawati, 2022). Proportionately, the critical factors are presented in the following section in detail to show their relationship between conceptualization of multiplication.

2.2. Factors Affecting Development of Multiplication Concept

Research suggests that several factors may influence the multiplication knowledge the children display. Below, the most important factors found in the literature are discussed. These factors can be classified under three umbrella terms: curriculum resources and representations, students, and teacher knowledge. First, the educational resources (curriculum and textbooks) and representations (language) offered to the children may affect their performance in multiplication. Secondly, students' pre-instructional knowledge (prior knowledge, number sense, and repeated addition) may have an influence.

Finally, the teacher's knowledge can potentially support students' learning. Each factor is explained in detail.

2.2.1. Curriculum Resources and Representations

Leaning on the work of Pepin and Gueudet (2020), the phrase curriculum resources is used to include all the material resources created and used by teachers and students in their interactions with mathematics during/for teaching and learning, both within and outside the classroom. Hence, curriculum resources include curricular guidelines, textbooks, and representations.

2.2.1.1. Curriculum Guidelines

A curriculum is a way of organizing content and goals for teaching and learning in schools. It shapes our identity and future by influencing what teachers teach and, in turn, what students learn (Walker, 2003). The curriculum has a significant role in teaching mathematical content in line with the objectives of teaching mathematics (Remillard, 2005; Valverde et al., 2002). In the scope of the current study, the Turkish mathematics education program (MoNE, 2018) was examined in terms of teaching multiplication.

As explained above, Turkish students are taught multiplication as repeated addition in the second grade and as a multiplicative comparison in the third grade. These models are asymmetrical, which means they do not view the commutative property. As a symmetric model, students are presented with the area of a rectangle in fourth grade. However, this is not specified to be introduced as a model of multiplication but as a new concept. Considering the importance of teaching multiplication via both asymmetric and symmetric models in presenting all the properties of multiplication and preventing possible misconceptions (Bell et al., 1989; Fischbein et al., 1985; Verschaffel et al., 1988), the Turkish mathematic education program was found limited in terms of models used to teach multiplication compared to those in other counties and the theoretical perspectives. The limitation related to the curriculum may cause students to have difficulty and misconceptions related to multiplication. Mathematics textbooks are also examined in the next section for a deeper investigation related to curriculum resources.

2.2.1.2. Textbooks

In school mathematics, textbooks are considered a mediator between the intended curriculum and the attained curriculum (Schmidt et al., 1997). Mathematics textbooks are crucial instances of mathematics curricula since they operationalize curricular objectives for what to teach and how much emphasis is given to mathematical topics. Consequently, textbooks impact what teachers teach, how they teach, and what students learn (Tobin, 1987; Usiskin, 1985). In that prospect, mathematics textbooks used in the schools at the time of the current research were reviewed to get the general structure of multiplication instruction in Turkish classrooms.

In the second-grade mathematics textbook (Atlı et al., 2018), multiplication is presented as the repeated addition of equal groups, as in Figure 2.11. Objects are placed in groups equally, and students are asked to find the total number of objects by repeated addition and multiplication.

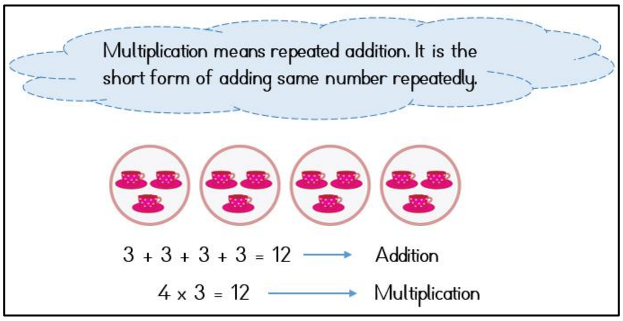


Figure 2.11. Repeated addition of equal groups in 2. Grade Mathematics textbook (Atlı et al., 2018, p.165)

As stated in the curriculum, students are also introduced to the commutative property in this book. While presenting this property, repeated addition is used as a model, as in Figure 2.12. However, 3x5 and 5x3 do not represent the same situation according to the representations of fish in the figure. As priorly noted, repeated addition does not view multiplication as commutative because of its asymmetric nature and hence making the book limited in explaining commutativity in multiplication.

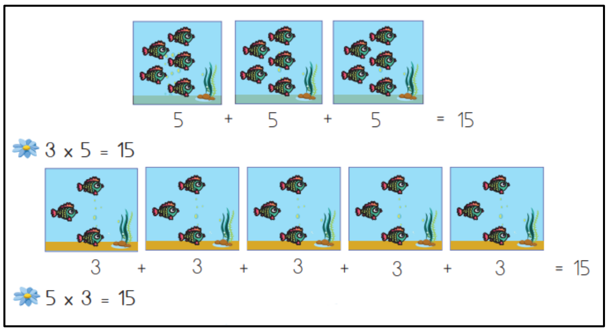


Figure 2.12. Commutative property of multiplication in 2nd Grade Mathematics textbook (Atlı et al., 2018, p.170)

In the third-grade mathematics textbook (Doğan & Gezmiş, 2018), multiplicative comparison is introduced using oranges in bags in Figure 2.13. In each step, one bag with two oranges is added. Repeated addition is used, and representation is directly explained. For instance, the oranges in the 3rd step are equal to 3 (step number) and as many as 2 (oranges in each step). Then, each step is represented as the product of step number and two. The textbook directly states what to do when students face a multiplicative comparison word problem.

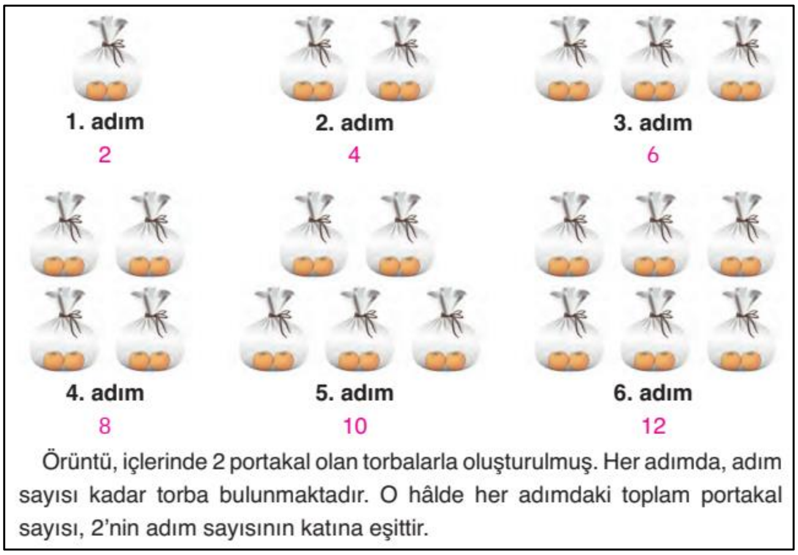
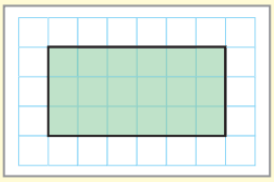


Figure 2.13. Multiplicative comparison in 3rd Grade Mathematics textbook (Doğan, & Gezmiş, 2018, p. 105).

In the fourth-grade mathematics textbook (Özçelik, 2018), the area formula of a rectangle is introduced. Students are asked to draw a rectangle, as in Figure 2.14 and are asked to answer these questions: "What are the sides' lengths (units) of the rectangle? Count the squares in the rectangle and write them down. Multiply the long and short sides of the rectangle and write it down. Specify the relationship between the numbers written down. According to this relationship, write a statement related to finding the area of a rectangle. Share this common statement with your friends". As can be seen, students are directed to find the area formula of a rectangle. It should be noted that students may observe commutativity in this activity since the order of multiplier and multiplicand is not specified. This activity can be revised, and some other questions can be included as students can be asked to voice how they multiplied the long and short sides. In this case, while some say 3x6, others say 6x3. This situation can be discussed and associated with the commutative property in multiplication.

- Kareli kâğıda, yandaki gibi bir dikdörtgen çizelim ve dikdörtgeni yeşile boyayalım.
- ➔ Dikdörtgenin uzun ve kısa kenar uzunlukları kaçar birimdir?
- Dikdörtgenin içindeki birimkareleri sayalım ve not edelim.



- Dikdörtgenin uzun ve kısa kenar uzunluklarını çarpalım ve çarpımı not edelim.
- ➔ Not ettiğiniz sayılar arasındaki ilişkiyi belirleyiniz.
- ➔ Belirlediğiniz ilişkiye göre dikdörtgenin alanını bulmaya yönelik genel bir ifade yazınız. Yazdığınız genel ifadeleri arkadaşlarınızla paylaşınız.

Figure 2.14. Activity on the area of rectangle in 4th Grade Mathematics textbook (Özçelik, 2018, p. 249).

To sum up, it is seen that the textbooks are parallel with the objectives in the Turkish mathematics education program and scarce to teach multiplication and fundamental multiplication properties.

2.2.1.3. Representations

The purpose of learning mathematics is to get familiar with the mathematical concepts and structures contained in the content being studied and determine the relationship between mathematical notions and structure (Bruner, 1964). Ostensibly, Bruner (1964) emphasizes the importance of representations and materials in mathematics education. Multiple representations are crucial in carrying students from an operational to a structural view (Sfard, 1991). These representations support the development of a concept by highlighting its various features (Goldin & Shteingold, 2001). Representations in mathematics education refer to internal and external manifestations of mathematical concepts. Internal representations, including mental images and problem-solving approaches, are the pictures we conjure up in our brains for mathematical concepts and procedures. Students' visual images illustrating their spatial conceptions are internal representations while external representations are those used to explain concepts to others through drawing, writing, and building models out of physical objects (Goldin & Shteingold, 2001). Whereas thinking about mathematical ideas involves internal representations, expressing mathematical ideas requires external representations such as spoken words, written symbols, pictures, or physical objects (Hiebert & Carpenter, 1992).

Studies about multiplication with younger children highlight the role of representation in development at an early age (Anghileri, 1989; Clark & Kamii, 1996; Steffe, 1994). According to Jerome Bruner's learning theory, teaching materials must be presented to develop students' mathematics learning, inasmuch as the students' cognitive/knowledge development stage since using materials in a structured pattern makes the students' knowledge easier to recall and lasts longer. Bruner suggests that providing concrete objects for students to handle is the first stage of development. In the second phase, students should use visuals like pictures and images of objects rather than concrete objects. Finally, in the third stage, students start manipulating symbols. Therefore, by connecting different representations (concrete and pictorial, real-world and symbolic), instructional activities progressively move from concrete to abstract so, the students should be presented with tasks to enable them to make a transition between concrete and abstract levels. Therefore, students can reflect on their conceptualization of multiplication (internal representation) via various external representations in which they capacitate students to reflect on themselves while enabling the teacher to understand the students' comprehension (Bruner, 1964). For this reason, representations are crucial factors affecting the instruction of multiplication.

2.2.1.4. Language

Another critical issue for the instruction of multiplication is language, which is a type of representation (Anghileri, 2006; 2008). To understand the multiplicative scenarios and distinguish them from those that suggest an addition, subtraction, or division operation, students need to have enough experience interpreting word problems that describe multiplicative situations. The language and meanings of the multiplicative situations and their units must also be associated with the meanings of standard multiplication notation, such as 7x5=35. Although students experience multiplicative situations in daily life, it is not easy and obvious for them to recognize the multiplication process in these situations (Calabrese et al., 2020). The difficulty of connecting multiplication in realistic situations stems from the inconsistency between the vocabulary used in multiplication operations and daily language.

In mathematics terminology, there are many words used in real life (Brumbaugh et al., 2005). However, the term "multiply" is not used within typical scenarios but in multiplication operations (Anghileri, 2006). In multiplication, a factor times a factor yields a product. The word "times" is no longer linked merely to the idea of events but is related to multiplication. "Product" represents more than something associated with a brand name or store item. "Factor" refers to something beyond an idea or point to be considered (Brumbaugh et al., 2005). The vocabulary used in the real world related to multiplication should be taught in mathematics classrooms in addition to conceptual and procedural knowledge of multiplication (Calabrese et al., 2020). Consequently, instructors are suggested to utilize problem-posing strategies to gain insight into how students conceptualize multiplication and improve their understanding of multiplication (Calabrese et al., 2020).

Educators frequently employ problem-solving techniques to teach and assess students' comprehension of multiplication. A different approach to teaching multiplication and examining student understanding is to employ problem posing or having students build word problems (Dickman 2014; Lin 2004). Posing problems provide an opportunity to transform their knowledge by using multiplicative language. For instance, Haylock and Cockburn (2013) conducted a study with children aged 9 to 11 years and asked them to write a story that goes with '9 x 3'. They revealed that many children struggle to understand multiplication in real-world terms. When given this task, very few children came up with convincing scenarios, such as "I had nine cats, and they all had three kittens." Only a tiny percentage of them appeared to have distinct mental frameworks that they could associate with multiplication. Children seemed to have no conceptual understanding of what occurs when two numbers are multiplied. If this is the case, then it is the responsibility of those who teach multiplication in the early years of primary schooling to establish a foundation of experience and to offer images that can be connected to the language and symbols of multiplication (Haylock & Cockburn, 2013).

To sum up, language as a verbal representation of multiplication is an important factor that affects students' understanding of the concept. For this reason, it is suggested to contemplate the fact that while teaching multiplication, enriching curriculum resources by linking multiplicative language and real-life via various activities such as problem posing is essential for the cause. Therefore, this section addresses the issues related to the curriculum resources. In the following sections, the issues related to the interactions of students and teachers with the curriculum resources on multiplication are presented as the factors affecting the development of multiplication.

2.2.2. Student Knowledge

In this section, the issues associated with students' competencies related to number sense and prior knowledge that constitutes a base for multiplication are addressed to ratify the critical issues in developing a conceptual understanding of multiplication. Additionally, the consequences of overgeneralization of repeated addition are propounded as an obstacle to the conceptualization of multiplication.

2.2.2.1. Students' number sense levels

Number sense is the ability to use multiple relationships among numbers and operations flexibly, using benchmarks to judge number magnitude and recognize unreasonable results (Andrews & Sayers, 2015; McIntosh et al., 1992; NCTM, 2000). In a more advanced level of number sense, students understand place value, compose and decompose whole numbers, and grasp the meanings of four operations through formal education. Furthermore, students can comprehend the commutative, associative, and distributive properties and apply these principles to solve problems (National Mathematics Advisory Panel, 2008). For example, students with developed number sense can use the distributive property (or splitting property) to find six eights by adding on five eights ($6 \times 8 = 5 \times 8 + 8 = 48$) (van den Heuvel-Panhuizen, 2008). Furthermore, they can double six eights to find twelve eights (12x8=96) and can further find that "12x80 is 96 tens, which is 960" (National Mathematics Advisory Panel, 2008). In summary, students retain some basic facts and use these facts to figure out additional or more complex information through mental strategies.

The development of number sense is crucial for mathematics education since the lack of understanding of what numerals mean causes barriers to learning mathematics (Ekenstam, 1977). It is also stated in the report of the National Mathematics Advisory Panel (2008) that poor number sense makes it challenging to learn algorithms and number facts and limits the use of strategies to verify whether solutions to problems are acceptable. Students' weak number sense might be the reason why children struggle to acquire mathematics in early grades (Jordan et al., 2010; Yılmaz, 2017). Considering this, it is crucial to focus on developing number sense in early grades as a foundation of mathematical competency in the upper grades.

In this strand, students learn various ways to express numbers, operations, and the connections between them, which helps them better understand. By learning to count in different ways, they acquire a deep understanding of the four fundamental operations and develop their computational speed and accuracy using various tools and techniques (Resnick, 1989). While designing an environment for teaching operations considering number sense, key components are suggested as an understanding of operations' effect, awareness of operations' mathematical properties, and awareness of the relationship between operations (McIntosh et al., 1992). In order to acquire these components, it is suggested to provide rich activities for making connections, exploring and discussing concepts, and ensuring an appropriate sequence of concepts (Griffin, 2004).

Therefore, students should be provided with numerous opportunities to create their number-working skills while learning multiplication.

2.2.2.2. Students' prior knowledge for multiplication

According to constructivism, children construct new knowledge by building on prior knowledge and their own experiences (von Glasersfeld, 1996). They actively participate in the instruction rather than passively absorbing the knowledge from the teacher. While students build on their existing knowledge, the teacher facilitates their construction of mathematical knowledge (Carpenter et al., 1989). As a result, misconceptions from earlier classes or a lack of understanding of earlier knowledge restrict mathematical development.

Multiplication is initially constructed on repeated addition (Fischbein et al., 1985). In this regard, students' knowledge of addition is essential as a fundamental concept for developing multiplication. Any misconception or misunderstanding related to addition automatically affects students' understanding and processing of multiplication. Moreover, as stated in the multiplication strategies, students multiply two numbers via counting strategies (Sherin & Fuson, 2005) and therefore their comprehension of number relations and counting skills play an essential role in multiplication. In this perspective, the limited prior knowledge causes problems in understanding multiplication.

2.2.2.3. Students' overgeneralization of addition

According to the famous quote by Abraham Maslow, "If the only tool you have is a hammer, you tend to see every problem as a nail." which describes a tendency to rely too heavily on an instrument that is familiar or beloved. Similarly, students may overgeneralize addition and have misconceptions about multiplication (Lesh et al., 2003; Lo et al., 2008). For instance, students may generalize the properties of addition incorrectly to multiplication (e.g., 18 x 26 = (10 x 20) + (8 x 6)), since students are familiar with addition and construct multiplication on addition (Lo et al., 2008). However, as an operation, multiplication differs from addition in several special ways due to its complexity (Downton & Sullivan, 2017; Steffe, 1994).

The addition is a unary operation that manipulates quantities with similar units. For instance, an addition problem may involve adding two to seven apples, resulting in a total of nine apples where the scenario has homogeneous units. On the other hand, multiplication is a binary operation through which two quantities with different units are manipulated (Barmby et al., 2009; Smith & Smith, 2006). For instance, a multiplication problem may involve finding the total number of cookies on four plates if there are three cookies on each plate. The scenario is more complex and abstract than the scenarios in addition and subtraction problems since the two input variables have different units, such as plates and cookies. The quantities that are multiplied together differ from one another while still being dependent on one another. In this case, the total number of cookies depends on the number of plates. This understanding reveals a critical distinction between addition and multiplication (Schoenfeld et al., 2017). Therefore, compared to addition, multiplication involves the ability to coordinate groups of units on a more abstract level (Clark & Kamii, 1996; Downton & Sullivan, 2017; Steffe, 1994).

Furthermore, it calls for flexible and effective dealing with a wide range of numbers and scenarios since multiplication is viewed with different models as context changes (Barmby et al., 2009; Greer, 1992; van de Walle et al., 2020; Steffe, 1994). The meaning of multiplication varies in terms of context. As an instance, the process of combining shirts and skirts (Cartesian product) does not reflect addition in nature. Unfortunately, many educators believe that multiplication is just an extension of addition since it is possible to solve whole-number multiplication problems using repeated addition. This situation may cause students to over-rely on addition properties while interpreting multiplication as it may also limit students' conceptualization of multiplication concepts.

2.2.3. Teacher Knowledge

According to social constructivist viewpoints, teachers in control of creating and implementing lessons that encourage students to actively engage in their learning. The teacher should assess the progress of the classroom community by keeping track of taken-as-shared ideas through dialogic discussions with the whole class, in small groups, and individually (Cobb & Yackel, 1996). However, implementing a discourse community is not as easy as explained since it requires teachers to have well-developed subject matter and pedagogical content knowledge, self-efficacy, and the ability to manage classroom discussions (Hill et al., 2008). The lack of teacher capabilities also causes problems related to students' actions and cognitions in the classroom environment.

Commensurably, it is undeniable that teachers' expertise significantly impacts the effectiveness of mathematics instruction and students' success because the teacher's knowledge directly influences what students learn (Shulman, 1986). For this reason, teachers must gain a profound grasp of the concepts to implement the mathematics education programs as expected. Even the most excellent curriculum only offers a set of tools but, the teacher brings these tools to real life in the classroom as well as determining when, how, and why to employ them (Griffin, 2004).

Furthermore, teachers' limited knowledge also limits students' development as less qualified teachers frequently focus on facts, rules, and procedures and heavily rely on their lesson plans (Shulman, 1986). For this reason, teacher knowledge related to multiplication dramatically impacts students' meaningful learning of multiplication.

Therefore, the literature related to students' conceptualization of multiplication shows that there are possible factors that can prevent the development of the concept. These factors include scarce sources of curriculum guidelines, students' limited knowledge of fundamental concepts, and weak knowledge of teachers on the topic. Since many students face one or more obstacles that prevent them from conceptualizing multiplication, these issues should be considered and removed during instruction, or else, the limitations of these issues constitute barriers to students' conceptual understanding of multiplication.

In conclusion, the meanings of multiplication, its place in mathematics lessons, students' strategies to multiply, the most common misconceptions and errors related to multiplication, and possible factors causing these problems were described up to this point. It is posited that students have serious problems with understanding and processing multiplication. Unfortunately, curriculum resources are limited in supporting students' comprehension of the concept and can be suggested to develop an alternative instructional model corresponding to the possible obstacles preventing students' conceptualization of multiplication. In order for this, the current study aims to develop a hypothetical learning trajectory leaning on the information up to this point, providing a comprehensive background. In the following section, Hypothetical Learning Trajectories are explained by concentrating on their various definitions and the contexts in which they are applied and used.

2.3. Hypothetical Learning Trajectories

As in national and international mathematics education programs, constructivist viewpoints have begun to rule the educational field, and more and more in-depth information on learning and students has become available (Simon, 1995). Worldwide, mathematics education has been considerably transformed due to these shifting perspectives on learning (National Council of Teachers of Mathematics [NCTM], 1989). However, there is a gap between constructivism theory and practice since constructivism "does not tell us how to teach mathematics" and does not enforce any specific methods of education (Simon, 1995). To make it clear, by the end of the lesson, it is anticipated that every student will have a consistent understanding of what "it" is. As they are supposed to follow the same path; the only individual differences would be that some children follow the path more slowly than others, necessitating additional time or remediation (Fosnot & Dolk, 2001). Hence, students are believed to follow a linear path. However, an important finding is that children do not all think the same way. Based on this concern, Simon (1995) suggests that teachers should create mathematics classes following applicable research results on student thinking and learning and predicted reasoning and he also proposes creating and applying Hypothetical Learning Trajectories to integrate research and educational practice.

The learning trajectory is hypothetical since we can never be sure of what students will do or whether or how they will develop new interpretations, ideas, or techniques before they work on a topic. Teachers anticipate specific approaches from their students while solving a problem. Alternatively, to put it more precisely, their expectations vary depending on the child. Simon (1995) illustrates this learning trajectory with the analogy of a sailing voyage to make it clear:

> You may initially plan the whole journey or only part of it. You set out sailing according to your plan. However, you must constantly adjust because of the conditions that you encounter. You continue to acquire knowledge about sailing, about the current conditions, and about the areas that you wish to visit. You change your plans with respect to the order of your destinations. You modify the length and nature of your visits as a result of interactions with people along the way. You add destinations that prior to the trip were unknown to you. The path that you travel is your [actual] trajectory. The path that you anticipate at any point is your “hypothetical trajectory.” (136–37)

Simon (1995) emphasizes that since the actual learning trajectory cannot be known, HLTs are purely hypothetical. He describes learning trajectories as predictions about the direction in which learning might occur and continues by recommending that they contain the learning objectives, the learning activities, and the thinking and learning in which students may engage. The learning process entails how students' thinking and knowledge will be developed through these activities, while the learning aim specifies the direction to be taken (Simon, 1995). Teachers, like sailors, require a broad strategy to direct the ideas they draft for students as well as having to adapt their overall approach to accommodate each student's potential for learning, any thoughts or doubts that may surface, and any unexpected circumstances that may develop (Simon, 1995). Like a sailor, teachers plan each step of their journey while considering the potential course of events and the circumstances brought on by the execution of earlier steps. Therefore, clarifying the vital turning points that establish the stages of a nonlinear path is the first step in creating the overall strategy for the "journey" (learning multiplication).

In another metaphor, Confrey (2006) defines a learning trajectory as a stream's flow, as in Figure 2.15 below. In this analogy, students start with their prior knowledge and move along a trajectory to new concepts. They go through landmarks and circumnavigate obstacles along the trajectory as it is confined to a domain by the stream's bounds. Landmarks represent fundamental ideas that develop and get better over time. Landmarks and obstacles are decided based on an analysis of the related literature. Respectively, the critical ideas and models related to teaching and learning multiplication, possible misconceptions, and potential barriers to the conceptualization of multiplication were examined and revealed in the previous sections. These issues were used while developing a hypothetical learning trajectory.

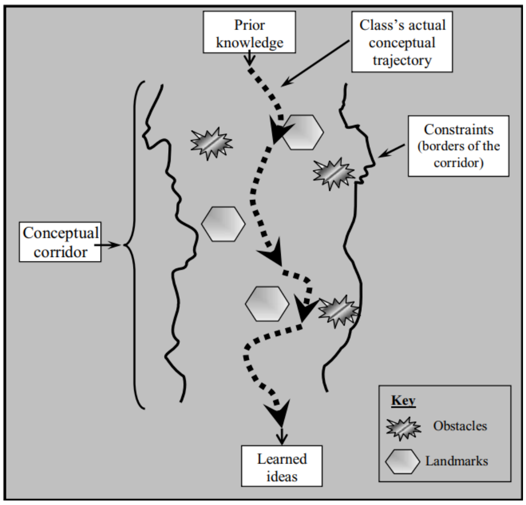


Figure 2.15. A conception of a learning trajectory within a conceptual corridor (Confrey, 2006)

Moreover, in Figure 2.15, the black dotted line represents a class' progress along the learning trajectory. Building on this metaphor, we assert that the curriculum can be used to symbolize the flow of the stream, particularly as it approaches a landmark or must contend with a barrier. As a result, two classrooms using different curricula to approach a landmark notion may be experiencing the landmark differently, impacting how they advance through the remaining portions of the trajectory. The landmarks pupils visit and the ones they skip depend on the curriculum's covering of its subject matter. In this regard, the objectives in the curriculum (MoNE, 2018) and some additional big ideas should be determined to develop a learning trajectory.

Hypothetical learning trajectories (HLTs) were initially developed to organize and describe the pedagogical reasoning required in teaching mathematics for understanding. In the corpus of research, the term "HLT" has been interpreted in several ways (Clements & Sarama, 2004; Gravemeijer, 2004; Gravemeijer et al., 2003a; Simon, 1995; Simon & Tzur, 2004). For instance, Clements and Sarama (2004) define hypothetical learning trajectories as "descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking, created with the intent of supporting children's achievement of specific goals in that mathematical domain" (p. 83). Another definition of HLT is made by the National Research Council (2007) as "descriptions of the successively more sophisticated ways of thinking about a topic that can follow one another as children learn about and investigate a topic over a broad span of time" (p. 214). Similarly, Confrey et al. (2014) explain learning trajectories as "research-based frameworks developed to document in detail the likely progressions, over long periods of time, students' reasoning about big ideas in mathematics" (p. 720).

Although there are various definitions, most of them concur that the HLT comprises three aspects: (1) the learning objectives, (2) the task-based instructional sequence to support those objectives, and (3) the projected student developmental progressions as a result of the task-based instructional sequence. The HLT is initially created as a set of instructional tasks with expectations for how the class would participate in the instruction while thinking and learning (Clements & Sarama, 2004; Gravemeijer, 2004; Gravemeijer et al., 2003a; Simon, 1995; Simon & Tzur, 2004). Confrey (2006) states that instruction should also include tasks, tools, modes of interaction, and evaluation techniques to help children shift from informal to formal knowledge.

As an example, in parallel to these components, Stephan and her colleagues (2003) developed a classroom learning trajectory (CLT) for integer addition and subtraction, which is provided in Figure 2.16. This instructional theory that emerged from five cycles of classroom-based experiments in the case of the integers, a Design Research Program is arranged in a table that is divided into five categories (Stephan et al., 2003) which are the tools, imagery, activity/taken-as-shared interests, possible topics of mathematical discourse, and possible gestures and metaphors (Rasmussen et al., 2004) that would support students' learning of integer operations. The instructional theory has been divided into six phases to outline the suggested adjustments in mathematical thinking that the teacher-researchers intend to assist in their classrooms. Each phase corresponds to hypotheses, taken-as-shared activities and goals that are anticipated that would manifest as students worked through the difficulties (Stephan & Cobb, 2013).

Phase	Tool	Imagery	Activity/Taken-as-shared Interests	Possible Topics of Mathematical Discourse	Possible gesturing and metaphors
ONE	Net Worth Statements	Assets and debts are quantities that have opposite effect on net worth	Learning finance terms	• Conceptualizing an asset as something owned, a debt as something owed • Conceptualizing a net worth as an abstract quantity (not tangible)	
TWO	Net Worth Statements (Vertical number line)	Differences in Collections of assets & Collections of debts	• Determining a person's net worth • Who is worth more?	Different strategies for finding net worths	Pay off

Figure 2.16. A portion of the CLT table for integer addition and subtraction (Stephan & Cobb, 2013, p.284).

Similarly, the learning trajectory created for the current study is a CLT that contains the same components (i.e., tools, imagery, activity/taken-as-shared interests, potential subjects for mathematical discourse, and potential gesturing and metaphors). The relevant educational sequence students would engage with is also established to assist their growth along the anticipated trajectory. There are, however, some distinctions as well. Firstly, in keeping with Simon's (1995) suggestion that learning trajectories should include learning goals, the CLT created in this work includes big ideas as learning goals. Secondly, tools and images are displayed in the same column because they are related. They are not given separately since the students used visual representations in the instructional tasks as tools such as providing students with pictures of objects in the composing and decomposing activities. Students circled these images to make equal groups and counted these equal groups repeatedly. Hence, the images related to the context of the tasks became the students' tools for solving the problems.

Up to this point, various means of HLTs have been discussed as well as having explained various definitions and applications of HLTs in this context. A review of HLTs created for multiplication or other similar disciplines is presented in the following section to reveal a rationale for the goals in the HLT and describe the convergences of the HLT created for this study in the previous studies.

2.3.1. Hypothetical Learning Trajectories in Multiplication

Mendes, Brocardo, and Oliveira (2021) developed a multiplication learning trajectory for third graders and it comprises modifications due to 10 task sequences tested in a classroom and school-specific circumstances. They also draw attention to the task contexts, which incorporate elements of multiplication learning. The HLT framework comprises three columns, including sequences of tasks, learning milestones, and contexts and numbers. Their trajectory uses five different types of tasks, including multiplication tasks where the calculation by groups is made evident, tasks whose context is related to the rectangular array, tasks with numbers in decimal representation, division tasks where multiplication is favored, revealing the relation between two operations and multiplication tasks where multiplication is selected, showing the relation between two operations. These tasks in the sequences were designed around the learning milestones, which are the consolidation of understanding of a group as a unit, the distributive property of multiplication in relation to addition and subtraction, the commutative property of multiplication, the associative property of multiplication, knowledge of the inverse relationship between multiplication and division, and comprehension of the proportional reasoning of multiplication. When it comes to context and numbers presented in the trajectory, initially, examples with multiples of 2, 3, 5, and 6 are used. The use of those multiples is then "revisited" to accommodate multiples of 4, 10, and 12. To develop the concept of inverse relationship between multiplication and division, the numerical set is limited to natural numbers, and the groups of 6, 8, and 10 are used again (sequences 7 and 9). This numerical "revisiting" is a sequential chain that repeats itself when new learning milestones are introduced. Also, the decimal 1.25 and multiples of 10 are used to introduce the proportional meaning of multiplication; these numbers were previously thought of as a reference (sequence 10) as this serves as the foundation for developing relationships with new number values.

In some other study, Götze and Baiker (2021) described a learning trajectory in four main phases: (1) direct counting, (2) rhythmic or skip counting, (3) additive thinking (possibly by saying the count-by sequence), and (4) multiplicative thinking (Anghileri, 1989; Battista, 1999; Downton & Sullivan, 2017; Larsson, 2016; Ruwisch, 1998; Siemon et al., 2005; Simon & Blume, 1994; Sherin & Fuson, 2005; Stefe, 1992; Sullivan et al., 2001; Thompson & Saldanha, 2003). In this sequence, repeated addition is initially thought to be more sophisticated than counting all or counting in multiples. However, equating repeated addition with multiplication is restrictive because this way of thinking is no longer possible beyond natural numbers (Thompson & Saldanha, 2003). In contrast, multiplicative thinking implies the identification of the various meanings of the multiplier and the multiplicand and involves the ability to coordinate bundled units more abstractly than additive thinking (Clark & Kamii, 1996; Downton & Sullivan, 2017; Larsson, 2016; Singh, 2000; Steffe, 1992).

In their learning trajectory of multiplication, consisting of six activities, Hendriana, Prahmana, and Hidayat (2019) start the instruction with games to support understanding of the multiplication concept. In these activities, students begin learning multiplication by grasping the fundamentals of addition using the term "box." For instance, 2x3 indicates that there are two boxes, each containing three items. After that, students are expected to memorize the multiplication of numbers 1, 10, 9, 2, and 5. By utilizing the finger method, singing a number song, a pattern of multiplication numbers, and other techniques, children can master the multiplication section by learning how to multiply the numbers 1, 10, 9, 2, and 5. Similarly, the trajectory includes activities related to the patterns of multiplying two same numbers, such as 3×3 and 10x10. After that, activities for using multiplication characteristics as a commutative operation and for memorizing the multiplication of numbers 8, 7, and 6 are used. Finally, teachers are suggested to evaluate the students' multiplication problems in both formal and informal forms via evaluation activities.

Wright, Stanger, Stafford, and Martland (2014) also propose a learning trajectory for early multiplication and division. This trajectory includes a list of sequential topics. The first one is counting by 2s, 5s, 10s, and 3s to improve students' ability to skip count. The second topic is multiplication, where items and groups are visible and since all the groups and the items in groups provide students, they can count by ones, skip count the groups, or use more advanced strategies. In the further topic, students are provided equal groups with covers for each group. They are told how many items there are in each group, and the cover is removed to show the items when they cannot comprehend these items. More descriptively, students are given cards containing repeated equal groups, like six groups of 2 dots. On each card, small lids are used to screen each group separately. Students are told how many dots are contained on each card. Then, they are asked to find how many dots there are altogether. Each of the six lids is lifted if the child cannot solve the task. In the following topic, students are given equal-sized groups with small lids on each and a larger lid that covers all the groups. Students are explained that the large lid has six groups of 2. Students are asked how many dots there are and how many small lids there are. As the students struggle, the large lid and the small lid are lifted, respectively. For the next step, students work on cards with dots. They are asked questions related to them with the help of lids to make them experience quotitive division (number in each group given) and partitive division (number of groups given). In other words, students can process multiplication and division using the same tools from which they can simultaneously be taught both multiplication and division. In the upcoming topic, students arrange dots and construct arrays to represent multiplication and are asked to find how many dots there are in the array. After this, they are asked to find how many rows or columns there are in the array to practice division using arrays. Then, tasks including basic multiplication facts involving 2, 10, and 5 as multipliers are used to gain fact fluency. As another topic, relational thinking is emphasized under two sub-topics: commutative and distributive principles, and multiplication and division as inverse.

Therefore, the literature review on the previously published HLTs in multiplication demonstrates that the development of multiplication goes along a route from additive to multiplicative thinking. Almost all of them include the big ideas related to skip counting, equal grouping, commutative property, and array models. With this respect, it is suggested to use physical objects or pictures to form equal groups to count the whole, solve contextual problems with materials, pictures, and number sentences, and the composing and decomposing activities to reveal the inverse relationship between multiplication and division. Unfortunately, most of the big ideas included in these trajectories are new for second-grade students in Turkey (e.g., teaching multiplication and division simultaneously and using an array). Ergo, they should be adopted for these students considering the objectives. Also, the instructional activities should be adapted to the Turkish context to be clear and comprehensible for the students in a realistic context.

Besides, it was discovered that the HLTs that were already in place mainly emphasized the progressive growth of people. In other words, none of the studies mentioned above—although they may be beneficial in informing those—describes communal development in increasingly sophisticated ways. It is suggested to focus on justification and explanation that develop within a classroom community (Cobb et al., 2001). In that respect, there is a gap related to the existence of an HLT developed within an approach of collective mathematical practices. With all said and given, this study might contribute to the literature by outlining a group of students' collective developmental path by documenting their classroom mathematical practices. In the next section, collective mathematical practices approach to learning trajectories is explained in detail.

2.3.2. Collective Mathematical Practices Approach to Learning Trajectories

There are various HLTs for a variety of mathematical ideas like verbal and object counting, early addition and subtraction, integer addition and subtraction, geometric measurement, spatial thinking, composition, and decomposition of shapes (Clements & Sarama, 2004; Gravemeijer et al., 2003a; Stephan & Akyüz, 2012). Each study related to learning trajectories (LT) addresses a different approach to LTs. Within this context, Lobato and Walters (2017) reviewed the literature systematically. They presented a taxonomy of seven approaches to LTs, including (1) cognitive levels, (2) levels of discourse, (3) schemes and operations, (4) hypothetical learning trajectory, (5) collective mathematical practices, (6) disciplinary logic and curricular coherence, and (7) observable strategies and learning performances. While most of these methods rely on the progressive development of mathematical learning at the individual level, the fifth approach to LT (i.e., Collective mathematical practices) explains the collective development of a community of learners (Lobato & Walters, 2017).

In this section, the approach related to collective mathematical practices is explained in detail with the interpretive and instructional frameworks directing the approach and its characteristics, purposes, and benefits. The trajectories in the approach related to collective mathematical practices show how a community has developed (Lobato & Walters, 2017). The emergent perspective, an interpretive framework in which individual constructions are coordinated with collective constructs (Cobb & Yackel, 1996), serves as a major source of inspiration for this line of research. First, this framework will be explained in detail.

Before going into details of the emergent perspective, the terms collective and taken-as-shared should be defined to prevent misunderstandings in this section. The term "collective" is used to describe a quality of a group rather than most students in the classroom. To clarify, Lobato and Walters (2017) use the metaphor of a married relationship where the wife is active and disorganized, and the husband is systematic and severe. They are comical together, which is a quality neither of them possesses on their own. Similarly, teachers view each class as a social group with features that set it apart from other classes and go beyond the traits of the specific students who make up the class. Additionally, the phrase “taken-as-shared” is used to underline that the claim does not belong to one individual's understanding but rather to ways of operating that no longer need justification. It shows the idea's institutionalization in the classroom community's microculture (Rasmussen & Stephan, 2008). Keeping these definitions and explanations in mind, the interpretive framework and the current approach will be more explicit.

In the interpretive framework, the two theories, namely interactionist theories which emphasize learning through collective classroom or communal processes (Bauersfeld et al., 1988) and constructivist theories which specify individual students' reorganizing their learning through these communal processes (von Glasersfeld, 1998) are coordinated (Cobb & Yackel, 1996). In other words, the emergent perspective is a social constructivist theory that combines individual and collective learning through social interactions within the classroom environment, as shown in Figure 2.17.

SOCIAL PERSPECTIVE	PSYCHOLOGICAL PERSPECTIVE
Classroom social norms	Belief about own role, others' role, and the general nature of mathematical activity in school
Socio-mathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical conceptions and activity

Figure 2.17. An interpretive framework for analyzing communal and individual mathematical activity and learning (Cobb & Yackel, 1996, p.177)

There are two theoretical viewpoints in Figure 2.17; social perspective and psychological perspective. A social perspective refers to the interactionist view of learning as a social accomplishment, whereas a psychological perspective refers to the psychological constructivist view of individual and self-regulated activities in a social context (Cobb & Yackel, 1996). From the social perspective, there are three constructs denoting three aspects of a classroom: classroom social norms, socio-mathematical norms, and classroom mathematical practice.

In addition, psychological constructs which are related to these social constructs, including their individual aspects, are listed under the psychological perspective. Therefore, in each row, there is a relationship between the aspects of the classroom microculture and individual's activities in this classroom culture (Cobb & Yackel, 1996). Briefly, according to this perspective, norms and beliefs develop as complementary (Hershkowitz & Schwarz, 1999).

Classroom social norms, as a subconstruct of social perspective, are characteristics of the classroom community and regularities in a classroom activity that the teacher and students jointly establish through negotiation (Cobb

& Yackel, 1996; Cobb et al., 2001; Stephan et al., 2003). These norms refer to taken-as-shared communication ways within the classroom. Social norms and beliefs are reflexively related such that neither exists independently of the other. In other words, students reorganize their individual beliefs about their roles, others' roles, and the general nature of the mathematical activity as they negotiate and renegotiate social norms before and during the instructional processes (Cobb & Yackel, 1996).

The interpretive framework's second aspect concerns socio-mathematical norms specific to mathematics and mathematical activities (Cobb & Yackel, 1996). Socio-mathematical norms include "a different mathematical solution, a sophisticated mathematical solution, an efficient mathematical solution, and an acceptable mathematical explanation" (Cobb & Yackel, 1996, p. 178) where students correlate teacher's and others' mathematical beliefs and values as psychological constructs for socio-mathematical norms that establish their mathematical disposition (Bowers et al., 1999; Cobb & Yackel, 1996).

The third aspect of the social perspective is classroom mathematical practices related to the mathematical development of a classroom community (Cobb & Yackel, 1996). Mathematical practices are defined as the taken-as-shared ways of reasoning and arguing mathematically while engaging in pedagogical content tools based on the connection of social and socio-mathematical norms (Cobb et al., 2001). Classroom mathematical practices emerge within the classroom discussion on situations, problems, and ways of solving them, including aspects of symbolizing and notating (Cobb et al., 1997). Viewed against the background of classroom social and socio-mathematical norms, which are more general, the mathematical practices can be seen as content-specific mathematical interpretations that become normative through social interactions within the classroom (Stephan & Cobb, 2003; Cobb & Yackel, 1996). For example, using an array model for multiplying two numbers in second grade can be a mathematical practice.

According to this perspective, there is a robust connection between societal and individual processes since the two cannot be separated since each is necessary for the other to exist. On one hand, each student's growth is examined concerning their involvement in and contribution to the newly formed collective mathematical practices. On the other hand, mathematical practices are the taken-as-shared methods by which a community develops its mathematical reasoning and argumentation (Cobb et al., 2001; Stephan, 1998). Since students contribute to the evolution of the classroom's mathematical practices as they rearrange their mathematical activities, we can say that the relationship between the individual and the social process is vital, even though practices represent what is taken to be shared learning within a community. In other words, it is believed that there is a reflexive relationship between collective mathematical practices and individual pupils' conceptual growth.

The collective mathematical practices approach to learning trajectories incorporates how most teachers see their classrooms—as collectives—which is one of its key advantages (Lobato & Walters, 2017). Teachers are aware that each class might have a unique personality and that each class's mathematical growth can be described. Cobb and Yackel (1996) describe how they transitioned from conducting individual research to working in classrooms. At first, they attempted to explain how children's conceptual arrangements changed due to their interactions with the instructor and their classmates and they soon realized that there were missing patterns in social behavior, including those relating to the responsibilities and roles of students and teachers, as well as changes in reasoning over time that at first required students to explain them but eventually became accepted as practices that did not require justification.

Additionally, researchers discovered that the development of instructional theory and design was not adequately served by individualistic psychological theories of learning (Lobato & Walters, 2017). Fundamentally, continuous instructional development efforts are influenced by students' mathematical development as it occurs in the social environment of the classroom (Cobb, 2003). To put it another way, one technique to determine if an instructional sequence is effective is to record the mathematical practices used in the classroom and the various ways students participate in and contribute to them. A large portion of the studies related to learning trajectories in the collective mathematical practices approach has been conducted within the domain-specific instructional theory of realistic mathematics education (RME) (Lobato & Walters, 2017). In the following section, this theory is explained in detail.

2.3.3. Realistic Mathematics Education (RME)

Realistic Mathematics Education (RME) is grounded in understanding mathematics as a human activity, which is about the process of mathematizing reality (Freudenthal, 1968). It was developed as an alternative method of mathematics instruction to traditional teaching methods in the Netherlands (Streefland, 1991). It is believed that no subject should be taught to students as a ready-made product (Freudenthal, 1973). Contrary to "ready-made products," Freudenthal (1973) proposes "the process of reinvention" as a teaching approach that focuses on comprehending and evaluating mathematics as a human activity, particularly as a student's activity. Students are suggested to learn mathematics by mathematizing subject matter from real contexts and their mathematical activities (Gravemeijer, 1994).

Mathematizing is related to organizing for generality, certainty, exactness, and brevity (Gravemeijer et al., 2000). Through these goals, learners show their creativity and construct models to generalize conjectures (Freudenthal, 1973). To give an instance, students analyze and discover properties of the parallelogram in such a way that one of them emerges as the foundation for the rest. Then, they formalize the definition of a parallelogram by arranging its properties. This process is called the mathematization of the parallelogram's conceptual domain. Mathematizing is crucial for mathematics education since it helps students become familiar with the related concepts and their existence in daily life. On top of that, it relates to the reinvention process, a procedure whereby learners articulate their informal understandings and insights. Thus, mathematizing while working on problems that are experientially real to learners is the core of the reinvention process (Freudenthal, 1973; Gravemeijer et al., 2000).

Mathematization is distinguished as vertical and horizontal mathematization (Treffers, 1987), as represented in Figure 2.18 below and horizontal mathematization refers to explaining and solving a problem using informal knowledge which makes problems manageable for mathematical solutions. On the other hand, vertical mathematization refers to converting informal strategies to mathematical language or an appropriate algorithm by processing fairly sophisticated mathematics. Namely, extracting the required information from a given problem and using informal strategies refers to horizontal mathematization whereas representing the problem by using symbols and representing it as an equation refers to vertical mathematization (Gravemeijer et al., 2000).

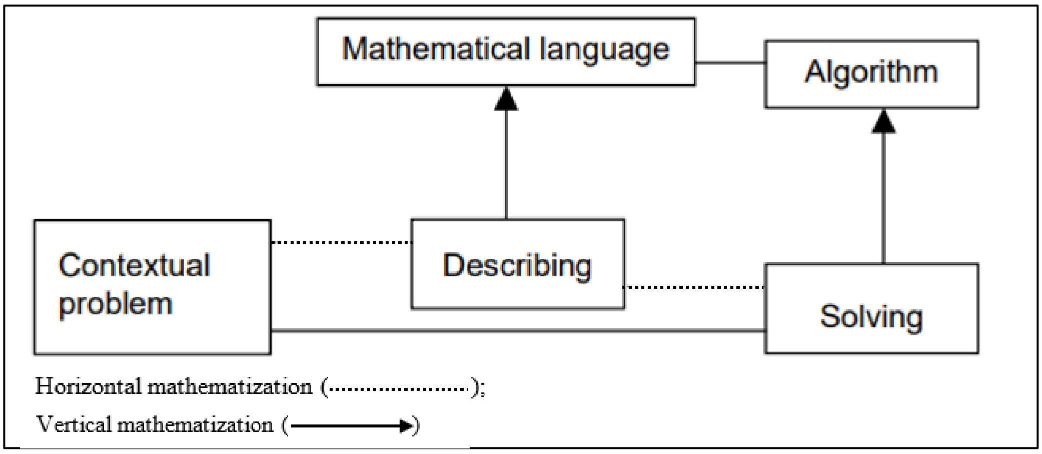


Figure 2.18. Representation of horizontal and vertical mathematization (Treffers, 1987)

Both vertical and horizontal mathematization give insight on how to reinvent through progressively mathematizing in a classroom setting. However, mathematization does not suggest explicit heuristics to direct how to design instruction in line with RME theory. In this regard, Gravemeijer and his colleagues delineated three core heuristics which are guided reinvention through progressive mathematization, didactical phenomenology, and emergent models (Gravemeijer et al., 2003a; Gravemeijer, 2004).

The first heuristic, guided reinvention, provides a solution to the problem of bridging the gap between informal and formal mathematics (Gravemeijer & Doorman, 1999). This heuristic guides instructional designers to consider the potential obstacles while developing a learning trajectory. With this insight, the designers wonder if students may acquire an understanding of the topic by taking a comparable developmental course. Another function of the reinvention principle is to reveal students' informal interpretations and solutions that may conjecture more formal mathematical reasoning. Therefrom, the process of reinvention might begin with the students' initial informal thinking (Gravemeijer, 1994). Accordingly, the developers start by exposing the history of the topic and students' informal strategies and interpretations as a source of the targeted learning trajectory. After that, they try to conjecture a tentative and revisable learning trajectory along which collective reinvention (as a process of progressive mathematization) might be supported.

The second heuristic, didactical phenomenology, is closely related to the first. The didactical phenomenology heuristic focuses explicitly on identifying specific activities in sequence to support the students' mathematical development, leading to the instructional sequence of tasks and activities (Gravemeijer, 2004; Gravemeijer & Doorman, 1999). This heuristic establishes the conditions in which students can work together to find progressively sophisticated answers to problems based in reality. These initial settings, where students collectively work on the given problem, play an essential role in promoting horizontal mathematization that encourages further vertical mathematization.

The third RME heuristic, emergent models, is related to the role of emergent models in the mathematical growth of individual students' learning and classroom community. It focuses on the informal use of students' models and supports the transition from a model of to a model for by using symbols corresponding to their reasoning styles (Gravemeijer et al., 2000). In this sense, students should be allowed to utilize and create their own models when solving problems. As an analogy, while working on a problem related to a double-decker bus, students can use a rack functioning as a model of the passengers on two decks. In further activities, students may use a rack as a model for reasoning number relations (e.g., 10+6 =9+7). This situation means that the rack's role shifted from being a model of the given scenario to functioning as a model for numerical reasoning.

2.4. Summary of the Literature

The literature review in this chapter is organized into three main sections, including theoretical background related to conceptual models and understanding of multiplication, factors affecting the development of multiplication concepts, and hypothetical learning trajectories. The studies in this chapter provide essential insights into the structures related to multiplication, students' understanding of multiplication, and hypothetical learning trajectories as an alternative framework for teaching multiplication.

Collectively, multiplication is a critical concept having a place in the mathematics education area. It has a significant role in promoting many number-related concepts like fractions, percentages, ratios, rates, similarity, trigonometry, proportion, area and volume, probability, and data analysis (Behr et al., 1992; Christou et al., 2005; Lamon, 2007; Lesh et al., 1988; Mulligan & Watson, 1998). Due to this importance, much research has been done on multiplication. The features and models related to multiplication via word problem structures have been revealed (Fischbein et al., 1985; Nesher, 1988, 1992; Schwartz, 1988; Vergnaud, 1983; 1988). Also, the studies on multiplication structures stated that symmetric and asymmetric models should be used collaboratively for meaningful learning to fulfill the properties related to multiplication (Bell et al., 1989; Graeber & Tirosh, 1988; Verschaffel et al., 1988).

The mathematics education programs were investigated and compared according to multiplication models where the analysis revealed that all the documents except those in Turkey include multiple models to teach multiplication concepts (Olfos et al., 2021). Both asymmetric and symmetric models were preferred to present multiplication and its properties. On the contrary, Turkish resources represent multiplication as an asymmetric structure (repeated addition and multiplicative comparison) (MoNE, 2018). Hence, the Turkish mathematics education program was found scarce to teach multiplication and fundamental multiplication properties. Also, studies related to students' understanding of multiplication show that students' conception of multiplication is mainly limited to repeated addition and equal groups (e.g., Kubanç & Varol, 2017; Sidekli et al., 2013; Tertemiz, 2017; Üçüncü, 2010). Besides, these students were discovered to have difficulty connecting multiplication to everyday activities (e.g., Doğan & Doğan, 2019; Kılıç, 2013; Tertemiz, 2017).

These findings show that there are problematic issues related to both teaching and learning multiplication. In addition to the limitations that stem from depending on multiplication on repeated addition, other factors also affect the conceptual development of multiplication that stem from curriculum sources and representations (Anghileri, 2006; Goldin & Shteingold, 2001; MoNE, 2018; Pepin & Gueudet, 2020; Tobin, 1987; Usiskin, 1985), students' knowledge (Ekenstam, 1977; Fischbein et al., 1985; Jordan et al., 2010; Lo et al., 2008; Yılmaz, 2017), and teacher knowledge (Griffin, 2004; Shulman, 1986). Thence, considering all the problems related to teaching and learning multiplication and possible obstacles to the development of the concept, there is a gap in the literature regarding a well-developed instructional model for teaching mathematics.

Complementary to the necessity of an alternative framework for teaching and learning, it is suggested to use a hypothetical learning trajectory (Simon, 1995) and implement it in a classroom environment to document classroom mathematical practices for testing its effectiveness in learning and refining the

HLT (Lobato & Walters, 2017; Rasmussen & Stephan, 2008). Unfortunately, the previously published HLTs in multiplication are not suitable for second-grade students in Turkey since they are not parallel with the Turkish mathematics education program (e.g., Mendes et al., 2021; Wright et al., 2014) as well as not describing communal ways of development. Therefore, it was concluded that there is a need for an HLT adapted to the Turkish context realistically to be clear and understandable for the second-grade students in a classroom to fill the gap in the literature. In order to meet the necessity of developing an HLT and related instructional sequence, RME theory was decided to be used as assigned in many studies related to learning trajectories (Lobato & Walters, 2017).

CHAPTER III

METHODOLOGY

The aims of this study are (1) to develop, test, and revise a Hypothetical Learning Trajectory and corresponding instructional sequence for teaching multiplication in second grade, and (2) to document second graders' classroom mathematical practices that emerged through a five-week instructional sequence about multiplication. To achieve these goals, this study seeks the answer to the following research questions:

1- What would an optimal HLT and instructional sequence for multiplication look like?
2- What are the mathematical practices as students engage in the instructional sequence for multiplication?
    - What are the mathematical ideas that support the mathematical practices developed by students during the implementation of instructional sequence for multiplication?

In order to answer these research questions, design research was conducted to provide a precise and deep understanding of an optimal HLT and instructional sequence for multiplication and classroom mathematical practices through a collective learning environment of second graders within the context of multiplication. In this chapter, the methodological approach of the study, the context and participants of the study, data collection procedures, data collection tools, and data analysis methods are explained in detail. Under the data collection procedure section, the development, implementation, and modification procedures related to the initial hypothetical learning trajectory are documented in detail to answer the first research question. Finally, at the end of the chapter, the issues regarding trustworthiness, the role of the researcher, the limitations of the study, and ethical considerations are provided.

3.1. Design of the Study

The purposes of this study are to develop a Hypothetical Learning Trajectory and corresponding instructional sequence for teaching multiplication in second grade, and to document second graders' classroom mathematical practices that emerged as the teacher and the students interact around the instructional sequence. These purposes of the study require researchers and practitioners to create progressively feasible and effective interventions with the enhanced articulation of principles that underlie their effectiveness by carefully analyzing progressive approximations of ideal interventions in the targeted environments. Therefore, the design research approach was adopted in this study as it intends to develop new theories, artifacts, and practices that account for and potentially influence learning and teaching in naturalistic situations (Barab & Squire, 2004). In this section, brief information about design research, and the reasons to employ design research are presented by connecting it to the current study.

Design experiments, in their most basic form, require both "engineering" specific kinds of learning and carefully examining those forms of learning within the context defined by the means of support as in the current study (Cobb et al., 2003). This designed context is put to the test and revised, and the subsequent iterations serve a similar purpose as systematic variation in experiments. In this respect, design research methodologies met the purpose of this study. That is to say, it was planned to develop, test and revise a Hypothetical Learning Trajectory on multiplication in second grade. In order to test the HLT and related instructional sequence, it was decided to implement the instructional sequence in the real classroom environment by handling the classroom complexity, which is a hallmark of educational environments.

The term “complexity” is related to the tasks presented to students to solve, the types of discourse that are encouraged, the norms of participation that are established, the tools/materials that are provided, and the teacher’s practices to orchestrate relationships between these various aspects in the classroom (Cobb et al., 2003). In this sense, it was targeted to understand the learning ecology by designing its elements and anticipating how they interact to enhance learning through the hypothesized learning trajectory on multiplication. Therefore, it was planned to develop the multiplication tasks with the collaborating teacher by taking her practices to implement these tasks into account. Related tools and materials were selected or created to provide for students. Moreover, guiding questions to encourage classroom discourse during the implementation of the instruction were determined. Finally, it was planned to implement all these constructs in the classroom environment with the help of social and sociomathematical norms that had already been established by the classroom community. Considering these components of a classroom complexity, design research was thought to be an effective approach for this study, because it takes a variety of factors, including the study's goals and nature, into account.

Design research is carried out in a variety of settings that differ both in nature and scope (Cobb et al., 2003). In this sense, the type of the current design research is a classroom teaching experiment whose purpose is to investigate and develop effective methods to assist students in learning a specific topic by a research team collaborating with the teacher (Cobb et al., 2003). As for a classroom teaching experiment, the research team collaborated with the teacher who takes part in the research team to take the responsibility of teaching multiplication in this study. Therefore, the team consisted of three people; a professor of Mathematics education, the researcher and the collaborating teacher. The team members planned to carry out this design research consisting of a cyclical process of ongoing analysis of student reasoning on multiplication tasks and revision of instructional tasks and the possible paths that students’ learning might take to understand multiplication.

In addition to all of the foregoing, design research has several features. First, as a common feature, design research is theory-oriented that aims to develop theories about learning and tools that can be used to support them (Cobb, et al., 2003; Design-Based Research Collective, 2003; McKenney & Reeves, 2012; van den Akker et al., 2006; Wang & Hannafin, 2005). In this sense, the purpose of this study is to look at how the teaching and learning process of multiplication is planned and implemented in the light of the ideas, theories, and findings in the related literature. Therefore, it was believed that the design team would develop an instruction theory and instructional means (e.g. tools, and accompanying learning activities) that would support second-grade students' learning of multiplication. Consequently, the current study was constructed as a means of doing formative research in order to evaluate and modify educational designs related to the instruction of multiplication based on theoretical principles gathered from previous studies.

Second, according to the interventionist nature of design research, the process begins with the identification of significant educational problems requiring comprehensive solutions that are acceptable for scientific investigation as well as the understanding of their causes (Cobb, et al., 2003; McKenney & Reeves, 2012; van den Akker et al., 2006). From this point of view, the current study started with critical problems related to teaching and learning multiplication. As suggested in design research, this study required a research team to participate in the creative activity of producing solutions based on current scientific information, empirical testing, and project participants' craft wisdom (McKenney & Reeves, 2012). Therefore, this team was expected to design an intervention for teaching multiplication in second grade by documenting the design procedure and the outcomes of the intervention by conducting retrospective analyses.

Third, design research emphasizes the importance of collaboration between the participants and the researchers (Cobb, et al., 2003; McKenney & Reeves, 2012; Wang & Hannafin, 2005). Direct theory application without the interaction of practitioners is usually not possible due to the dynamic and complicated relationship between theory and practice; hence, researchers and practitioners should work collaboratively (Wang & Hannafin, 2005). That is to say, practitioners should be involved in the design processes and work together with the researchers. In the current study, the teacher of the second-grade classroom that had been chosen to conduct the teaching experiment was included in the design team. She attended in reviewing the literature, developing the HLT and related instructional sequence on multiplication, implementing the instructional sequence in the classroom, observing students' learning, and evaluating the design of the instruction. She also contributed to the research and research team with her pedagogical content knowledge. She shared her experiences with the participating students, knowledge of students, and knowledge of multiplication considering her teaching experiences with the design team. The design was shaped in line with her guidance. Moreover, the team members also contributed to the teacher, especially in terms of orchestrating classroom discourse. Lest the teacher tended to give the answer and the solution strategy to the students directly, she was trained to use guiding questions to maintain a discussion environment and facilitate students to generate their formal knowledge related to multiplication collaboratively.

Fourth, in line with the iterative process of design research, ideas and interventions grow throughout the time as a result of several iterations of study, development, testing, and refinement (Design-Based Research Collective, 2003; McKenney & Reeves, 2012). Through the cycles of invention and revision, new hypotheses are produced and tested while old ones are tested and refuted (Cobb, et al., 2003). In this study, a longer macro-cycle and daily mini-cycles were used considering design study approach. The design research helped us in enacting and refining the instruction, observing the effects of the refinements on students' learning, and examining both confirmed resolution of the issues as well as additional learning challenges through daily mini-cycles. That is to say, after each lesson the teacher and the researcher were having a short conversation about what went as planned and what did not. If the lesson had not been as envisioned by the team, the tasks for the following day were revised considering the issues that emerged in the previous lesson. For instance, the task related to composing an array by drawing did not work as was expected and students were all confused. After that lesson, the design team decided that it was too early for the array concept to be introduced through visuals and pictures to the students, thus concrete materials were developed for the students to compose arrays and discuss the rows and columns. Through this iterative process, the instruction of multiplication was revised and refined by an understanding of students' learning processes of multiplication. Therefore, the team simultaneously made adjustments to their hypothetical learning trajectory and related instructional sequence while implementing the multiplication tasks through daily mini-cycles.

As stated before, the current study targets to develop, test, and revise a Hypothetical Learning Trajectory and corresponding instructional sequence for teaching multiplication in second grade, and to document second graders' classroom mathematical practices that emerged within the whole class discourse over the course of several weeks of instruction. This twofold purpose is met with the nature of design research whose goal is to establish ideas regarding domain-specific learning as well as tools to promote that learning (Bakker & van Eerde, 2015). In the light of design research, it was planned to create educational products as an HLT related to teaching the domain of multiplication in second-grade classrooms and to build theoretical understandings of how this HLT might be useful in education as a result of this work. Hence, it was aimed to establish a local instruction theory that encompasses both students' learning processes of multiplication as well as the instructional tasks and instruments created to support their learning of this concept. In this sense, it was decided to use enactments related to the multiplication task sequence in a second-grade classroom to generate information that can be applied to educational practices. By doing so, it was aimed to fill the gap in the literature and theory related to knowledge on designing a learning environment about multiplication by taking second grade students' thinking into consideration, and to improve policy and practice related to the domain of multiplication.

In conclusion, the purposes and features of design research and the reasons for choosing design research as the most appropriate methodology for the current study were explained in this section. After deciding to use a design research methodology, the study was organized around the three phases of design research; preparing for the experiment, experimenting in the classroom, and conducting retrospective analyses (Gravemeijer & Cobb, 2006). In the following subsections, brief information about how to employ the phases of design research in this study is presented.

3.2. Context and Participants of the Study

This study was conducted in a second-grade classroom in a primary school consisting of 23 students and a primary school teacher within the Çankaya district in Ankara, Turkey. The study was conducted for the topic of multiplication. For that reason, the instruction was developed around the objectives related to multiplication in this grade level. The Turkish mathematics education program includes three objectives for multiplication in second grade as in Table 3.1.

Table 3. 1. Objectives related to multiplication in second grade (MoNE, 2018)

Objective #	Objectives
M.2.1.4.1.	Students will be able to explain multiplication as repeated addition.
M.2.1.4.2.	Students will be able to do multiplication with whole numbers.
M.2.1.4.3.	Students will be able to solve multiplication problems with whole numbers.

The Turkish Middle School Curriculum (2018) devotes 20 class hours for the development of multiplication and 16 class hours for the development of division in second grade (MoNE, 2018). Although the instructional sequence took more time than required in the mathematics curricula, the teacher spent just 5 more class hours on the instruction of division after the design experiment since students had already been engaged in partitioning activities in the instructional sequence for multiplication. Therefore, it was tolerated that the instructional sequence took approximately 5 weeks.

The classroom was selected based on the purposive sampling strategy. The criterion in choosing the classroom was the teacher's willingness to take part in the study. Besides, she was experienced in attending classroom teaching experiments and collaborating with the research team since she had taken part in a design research a year before with her students. The teacher and her students had already set norms and were familiar with collaborative learning environments with ambition, responsibility, and readiness.

The design experiment of the study was conducted in this teacher's classroom that included 8 girls and 15 boys starting in February 2018. Students' developmental age levels according to pretest scores on the Number Knowledge Test are provided in Table 3.2 below. There was one boy who was on the spectrum for autism. All the students knew that and helped him during the teaching experiment. One other student was in the level of 5-6 years. The teacher also stated that this student had learning disabilities, but was not diagnosed for it. Four of the students were known to perform below their grade level. On the contrary, three students were known to perform above the second-grade level. The remaining (N = 14) students were known to perform at their age level.

Table 3. 2. Demographics of students according to the Number Knowledge Test

# of Students	Developmental Age Score (Chronological Age Equivalents)	Grade Level Equivalent
1	3-4 years	Preschool
1	5-6 years	K-1
4	6-7 years	1-2
14	7-8 years	2-3
1	8-9 years	3-4
2	9-10 years	4-5

When the teaching experiment started, students had just learned addition and subtraction. It was the first time that they learned about the multiplication concept. Pre- and post-tests were given to all students before and after the experiment. The same test was used for both the pretest and the posttest. Moreover, 10 students were chosen for pre-and post-interviews based on the teacher's recommendation and their responses to the pre-test.

The collaborating teacher was a primary teacher for 23 years. It was her fourth time teaching in second-grade classroom. She followed the training that was given by the research team prior to the implementation process, and she was responsible for applying the instructional sequence for multiplication in the classroom, directing classroom argumentations on the tasks, and encouraging students to share and discuss their works and ideas. She orchestrated classroom activities as whole-class and pair argumentations that were all consistent with the educational sequence and HLT. Krummheuer (1995) suggests that the teacher “should try to push the communication as close as possible towards a point of breakdown” (p. 263) to promote the justifications, clarifications, and evaluation of arguments. In line with this suggestion, the teacher also encouraged students to engage in discussion and directed the classroom debates to ensure that students understood the context as it was intended. Moreover, in design studies, the teacher cannot simply declare specific criteria for what types of answers are acceptable and expect students to follow those recommendations. Instead, when the teacher and students participate in discussions, socio-mathematical norms are constantly negotiated and redefined (Gravemeijer & Cobb, 2006). In the current study, the norms were sustained and students were facilitated to participate in the implementation of the instructional sequence actively.

The teacher was also responsible for the formative evaluation of the design as a member of the design team. She was involved in all the phases of developing, implementing, and analyzing the design. Her professional experiments and student knowledge played a significant role in developing tasks, selecting the tools, interpreting students' reactions and gestures, understanding their reasoning, and analyzing the discussion sessions in the classroom.

3.3. Data Collection Procedures

As aforementioned, a design research study entails three phases: "1) preparing for the experiment, (2) experimenting in the classroom, and (3) conducting retrospective analyses" (Gravemeijer & Cobb, 2006, p. 19). In the following sections, the procedures followed, and the steps taken in each of these phases within the context of this study are described in detail.

3.3.1. Phase one-Preparing for the experiment

The critical issue from a design research perspective is to clarify the study's theoretical goal (Gravemeijer & Cobb, 2006). In other words, the design should be started by explaining how to set learning goals, also known as instructional endpoints, and instructional beginning points. For this purpose, it is suggested to analyze the study's needs and context, conduct literature research, and develop a conceptual or theoretical framework (Plomp, 2013). More specifically, a review of the literature as well as (previous and/or current) initiatives that have addressed questions similar to those addressed in the study is the critical attempt to create a framework and the intervention's first blueprint as a result (Plomp, 2013). Therefore, the primary work of this study was the review of literature in order to identify and clarify learning objectives related to multiplication for the second graders.

Initially, a research team was built to work together and specify the learning goals through a literature review. The team comprised the advisor (expert), the researcher, and the teacher. It was crucial to include the teacher in the team since one of the most important features of educational design research is practitioner involvement (Nieveen & Folmer, 2013). While anticipating the evolution of students' thinking and understanding in order to plan revisable instructional activities (Gravemeijer & Cobb, 2006), the teacher's knowledge of her students and teaching experiences played a significant role. All of the working and discussion sessions of the team members were audio recorded.

The team started the work with an examination of the Turkish Middle School Mathematics Curriculum (MoNE, 2018) in order to reveal the scope of the program in the means of multiplication. There are three main objectives for multiplication in the second grade in the program (Table 3.1). The first one is explaining multiplication as repeated addition. This objective is also supported by an additional explanation of working with concrete materials. The second one is performing multiplication with whole numbers. This objective includes identifying the symbolic representation (x) of multiplication, multiplying the numbers up to 10 with 1,2,3,4 and 5, noticing that the change in the orders of multiplier and multiplicand does not change the product (commutativity), creating multiplication tables until 5 (included) by using hundred charts and explaining the effect of 1 and 0 in multiplication (identity and zero properties). Finally, the third objective was to solve multiplication problems including one-step operations (MoNE, 2018). The team also examined the objectives related to multiplication in the third grade since the instruction in the second grade plays a crucial role as it is a prerequisite for the following year to be built on. In the third grade, the comparison meaning of multiplication is presented to students. After, they create whole multiplication tables and explain the effect of change in multiplier on the product considering multiplication tables until 5s. Students multiply with two-digit numbers. Finally, they also experience problem posing in this grade level (MoNE, 2018).

When these objectives were listed, a discussion started in the research team about how to start the instruction. According to the order of the objectives in the curriculum, it was expected to start with providing concrete materials in equal groups and direct students to repeated addition. However, it did not feel right to the members. The excerpt taken from the related part of the audio recordings of the team meeting is given below:

> Researcher: I am not sure about starting with repeated addition. There must be some other concepts that are prerequisites for repeated addition. I mean what should these students know to be able to proceed with repeated addition? Imagine a scenario involving "3+3+3+3". How do they solve it?
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> Teacher: By counting 3 by 3. When we ask them to use repeated addition symbolically to add equal groups, they skip count the repeated number. Actually, there is an objective related to skip counting with 2,3,4,5, and 10 in the second-grade mathematics program, but in the first semester. We will implement the developed instructional sequence in the second semester. I am not sure whether they will remember or forget skip counting.
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> The professor: Skip counting should be noted as a prerequisite for the instruction of multiplication. Let's look at the literature to have an answer to our concern about starting with directly repeated addition.

After this discussion, various mathematics education programs, previously developed learning trajectories, and related literature were examined by the team. It was seen that skip counting was introduced prior to additive thinking in the other learning trajectories related to multiplication (Götze & Baiker, 2021; Kennedy et al., 2008; Kling & Bay-Williams, 2015; Wright et al., 2014). Moreover, the related literature suggests that students' counting in multiples (for example, 2, 4, 6, 8, or 3, 6, 9, 12, etc.) helps them to learn consistent sequences of words with a rhythm and pattern that can become extremely familiar. By doing so, students can link these counts to repeated groupings of objects (Anghileri, 1995; 2008). Hence, this knowledge can be expanded to improve understanding of multiplication facts (Anghileri, 2008; Hulbert et al., 2017). Therefore, the initial learning goal was determined as skip counting in the hypothetical learning trajectory in this study. In order to support this goal, activities related to counting forward and backward, and the orders of the numbers were developed. Students were provided a hundred charts and were encouraged to use their fingers. In order to support students' reasoning, they were asked to question the relationship between the order of numbers in a number sequence.

In addition to the Turkish Middle School Mathematics Curriculum (MoNE, 2018), various mathematics teaching programs were also examined. Similar to

Turkey, situations for adding equal quantities are provided and repeated addition meaning of multiplication is presented in second-grade classrooms in Chile, Mexico, Brazil, Portugal, Japan, and the USA (Olfos et al., 2021). Furthermore, in addition to the activities related to the repeated addition of equal quantities, students also work on the activities related to groups as a unit or group of groups without repeated addition in Chile, Portugal, and Japan. That is to say, students focus on composing equal groups and count them as units which is essential for multiplicative reasoning. In this sense, New York State Common Core Mathematics Curriculum (NYSCC, 2014) provides various activities related to the formation of equal groups before presenting repeated addition to find the total number of objects. It is suggested to give students concrete objects and pictures of objects to form equal groups and specify how many groups there are and how many objects there are in each group (NYSCC, 2014). After reviewing the literature about equal grouping as the initial model for multiplication in instruction (Anghileri, 1989; Greer, 1992; Izsák, 2005; Schoenfeld et al.,2017), the research team decided to spend some time to compose equal groups in order to gain students' recognizing groups of objects and thinking in multiples. Therefore, equal grouping of given objects or pictures of objects was determined as another learning goal in this study to develop an understanding of a group as a unit.

While discussing giving place to the idea of equal grouping, another issue emerged. The team pointed out the importance of language that should be used while envisioning the tasks related to grouping objects and pictures equally. It was known that although students experience multiplicative situations in daily life, it is not easy and obvious for them to recognize the multiplication process in these situations because of the inconsistency between the vocabulary used in multiplication operations and the daily language (Anghileri 2000; Calabrese et al., 2020). The arguments of the team members related to this issue are given below:

> Teacher: I imagine equal grouping tasks. Assume that we give students some objects in groups and ask the number of groups, the number of objects in each group and total number of objects. Which terms will we use? It is important. We should also specify the interrogative sentence.
>
> Researcher: In these tasks we cannot use the term "multiply". We will start with daily language related to multiplication like "each," "times," "sets," or "groups". For example, there will be 3 groups and four items in each group. They should call this situation as "3 groups of 4", "3 sets of 4" or "4 taken 3 times". We should state these wordings to be able to make them aware of the language used in everyday conversations.
>
> Teacher: You are right. On the following day, they will use 3x4 to represent these groups. Unfortunately, the term "multiply" is not used within typical scenarios of daily life but in multiplication operations. We should note that we must be careful while shifting from daily language to formal multiplication language. We should state the relationship and give these terms intertwined.

Hence, in addition to conceptual and procedural knowledge of multiplication, the team noted that multiplication-related vocabulary used in the real world should be taught in mathematics classrooms. Moreover, the team stated that daily language should be supported by the formal language of multiplication. Therefore, developing an appropriate language for multiplication was noted as the general goal to be considered during the instruction.

After developing skip counting skills, working in equal groups, and associating appropriate language, the team thought that the students might be ready for repeated addition meaning of multiplication which is the first objective of the Turkish mathematics education program. It was decided that students should use an operation to be able to find the total number of objects in equal groups. At that point, students might use addition which is the most appropriate operation they have already leant. Although there is a conceptual distinction between multiplication and addition, it is suggested that both operations have a procedural link (Nunes & Bryant, 1996). Most of the studies support using repeated addition as a way of introducing multiplication procedurally and making connections between the existing and new knowledge (Anghileri, 1989; Götze & Baiker, 2021; De Corte & Verschaffel, 1996; Fosnot, 2007; Fosnot & Dolk, 2001; Haylock, 2010; Kennedy et al., 2008; Mulligan, 1992; Nunes & Bryant, 1996;

NYSCC, 2014; Squire et al., 2004). The team members decided to represent repeated addition in order to shift from the idea of equal grouping to a procedural idea of repeated addition. Team members highlighted the importance of using repeated addition to solve multiplication sums since multiplication is more distributive compared to addition. Hence, multiplication as repeated addition is defined as another learning goal in the learning trajectory for multiplication.

So far, the learning goals related to grouping things into groups, counting these groups rhythmically, skip counting, and using repeated addition as a calculation are explained. The team members decided to introduce multiplication symbolically by associating with the equal groups in repeated addition. They argued that students must have adequate experience placing objects into groups in order to comprehend the role of equal groups in multiplicative circumstances and to generate motivation for multiplying equal groups rather than counting all of the objects in the issue. It was planned to introduce multiplication sign, multiplier, multiplicand, and product in an operation. At that point, it was noted that the role of each component in a number sentence should be stated. In other words, the multiplier indicates the number of groups, and the multiplicand indicates the number of objects in each group, yielding the total amount as the product. For that reason, the team defined performing multiplication symbolically as another learning goal in this learning trajectory.

As it can be seen, while deciding on the learning goals, the team followed the mathematics education program (MoNE, 2018) and detected the limitations related to the objectives and gaps between the objectives in the program. In the program, it is stated that students will be able to notice that the change in the orders of multiplier and multiplicand does not change the product which means commutativity. However, it is not sufficient to know multiplication as equal groups or repeated addition (Thompson & Saldanha, 2003) since the repeated addition model for multiplication is asymmetrical (Greer, 1992); and the factors have different roles, which makes commutativity covert (Lo et al., 2008). At that point, a symmetrical model should be used. In mathematics education programs in Mexico, Japan, the USA, Chile and Brazil, array diagrams or rectangular shapes are used in teaching multiplication (Olfos et al., 2021). The researcher and the professor had already known that array model is suggested as a symmetrical model of multiplication to facilitate the learning of commutativity (Schliemann et al., 1998), and they had to convince the teacher of the necessity of this model. The related part of the team meeting is given below:

> Teacher: Why don't we give some examples like 3x4 and 4x3 and ask to answer them. Then, we compare and contrast these two operations. They realize that the product remains the same when we change multiplier and multiplicand. Why do we use another model? By the way, I do not know the array model that you mentioned.
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> Researcher: It is also a way to teach. However, it remains unclear why the product does not change. In three bags of four apples, three (bags) is the multiplier and four (apples) the multiplicand; it is not evident that four bags of three apples would be as many. Imagine the bags and apples. The representations are different in these two situations. Or can you show that 2x4 and 4x2 give the same result considering 2+2+2+2 and 4+4?
>
> Teacher: You are right. I have never taught from this point of view. Up to now, I have directly given that multiplication is commutative. What about this new model? Can you show me?

After this conversation, the array model and commutativity were introduced to the teacher. She was convinced that array was an appropriate model for discovering the commutative property of multiplication. Therefore, array models were included in the HLT as a definition of multiplication as an array. It was planned to develop an environment to arrange objects in arrays with rows and columns by using counters on squared papers. Consequently, the team anticipated that students can infer why multiplication is commutative by using arrays. Hence, the commutative property via array representation of multiplication was noted down as an important learning goal for this learning trajectory.

It is inevitable to mention the multiplication table under the topic of multiplication. As stated in the mathematics education program, second-grade students are expected to create multiplication tables up to 5 (including 5) by using hundred charts and operation tables. The responsibility of the construction of the multiplication table was planned to be given to the students. Students can build these tables by applying what they have learned. They can use various representations like drawing groups or arrays and skip count. Furthermore, they can find each product by adding the multiplicand to the previous product in the table, so they will not need to add or count from the beginning to find the next product in the table. The team members thought students might generate various strategies and develop mathematical ideas while working on multiplication tables. Therefore, creating multiplication table was stated as another learning goal for this learning trajectory for multiplication.

Finally, under the objective related to multiplying numbers, the mathematics education program clarifies that second-grade students will be able to explain the effect of 1 and 0 in multiplication (MoNE, 2018). That is, it is indicated to identify identity property which states that when you multiply a number by 1, the result is the same as the original, and zero property which states that when you multiply a number by 0, the result is always 0. Students frequently have difficulty understanding factors of 0 and, to a lesser extent, 1. This is because the students are presented a collection of operations for them to see the pattern and rules for factors of 0 and 1, but they are not provided with the reason behind it (van de Walle et al., 2020). The team members decided to make up a number of interesting word problems involving 0 or 1, and discuss the results. They thought that these problems can help students reason these properties. Moreover, they also believed that modeling the multiplication with 1 and 0 by equal groups or arrays would be an enjoyable activity. Under the topic of multiplication, Therefore, they planned to include identity and zero property as two learning goals in the learning trajectory for multiplication.

In addition to specifying learning goals, it is also necessary to decide on an instructional design theory because a classroom teaching experiment was chosen as the type of Design Research methodology for this study. For that reason, the domain-specific theory of Realistic Mathematics Education was chosen as a theoretical framework to design instructional activities.

3.3.1.1. Realistic Mathematics Education (RME)

Realistic Mathematics Education (RME) was used to design and implement the HLT and the instructional activities. The team followed three RME heuristics that are domain-specific to mathematics in order to support students' progressive mathematization through the mathematical ideas specified for the HLT. These heuristics are guided reinvention, didactical phenomenology, and emergent models (Gravemeijer et al., 2003a; Gravemeijer, 2004) as explained in Chapter 2 above.

In the first heuristic of guided reinvention through progressive mathematization, the team discussed students' prior knowledge to construct a bridge between informal and formal mathematics. For instance, the instructional sequence initially started with skip counting activities to develop students' counting skills that is prerequisite of multiplication strategies (see the activities in pages 1–4 in Appendix E). Moreover, the instructional sequence was developed to help students solve the contextual problems in the light of the bridge between informal and formal knowledge by highlighting the role of the guided reinvention principle. In this precedent, the students had the chance to experience the processes of independently learning the repeated addition and multiplication concepts that were part of the RME-based instructional sequence. For instance, the students had to work through a number of the contextual problems that prompted them to perceive the concept of multiplication via equal grouping that covers a form before they could do so. First, they focused on context-based problems that could be handled by applying their informal knowledge of skip counting (see contextual problems in pages 5–18 in Appendix E). The students dealt instinctively with the idea of repeated addition at this point by counting the equal groups via skip counting. They then solved the context-related equal grouping problems by using repeated addition symbolically (see contextual problems in pages 19–23 in Appendix E). These problems were developed for students to employ symbolic representations of the context while developing counting methods. After that, students dealt with problems to apply multiplication formally by connecting with repeated addition (see contextual problems in pages 24–31 in Appendix E). Finally, the students used the counters, dots and small square units to form rectangular array that helped them comprehend the notion of multiplication through area model (see contextual problems in pages 32–44 in Appendix E). Therefore, in line with the guided reinvention heuristic, the classroom teaching experiment conducted for this study encouraged students to mathematize situations to develop the mathematics they needed to solve problems.

Based on the principle of didactical phenomenology, it is implied that all contextual problems in RME-based instruction should be created using phenomena that have meaning for the students. For the RME-based instructional sequence on multiplication, the team members created daily context problems that are experientially real to the students (such as balls in boxes, bottles on shelves, apples eaten every day, and oranges on trees). Furthermore, as mentioned in the preceding section, the settings in the contextual problems must be significant and offer a chance for the students to interpret them mathematically. In other words, didactical phenomenology looks for contextual problems for which a situation-specific strategy can be generalized and situations that result in similar problem-solving techniques that can serve as the foundation for vertical mathematization (Gravemeijer, 1994; 1999). In this sense, the problems were developed to enable students to make connections between skip counting, repeated addition, and multiplication, respectively. To this end, the problems were designed around realistic contexts that are meaningful to the students. Furthermore, problem-posing activities where the tasks were presented without a context, and students were encouraged to create one were also included in the instructional sequence (e.g., activities on pages 22-23 in Appendix E)

The third RME heuristic emphasizes the importance of the teaching tools in building mathematical practices and participating in the reinvention process. In other words, the tools should support students' development of mathematical practices throughout the instructional sequence, from models of their informal mathematical activity to models for formal mathematical reasoning (Gravemeijer et al., 2003; de Beer et al, 2017). This change is seen as a means for pupils to build mathematical relationships and activities with the model. As a result, rather than creating a model, the goal is to reify the process of mathematical activity and reasoning (Gravemeijer, 1999). With this respect, various tools, including tangible items, images, graphs, symbols, and notations in the educational sequence, can be employed. In the current study, the team envisioned shifting students' informal mathematical activities into a model for more formal multiplicative reasoning. The team members decided to use concrete objects and pictures of the objects to help students model their reasoning via these representations as seen in Figures 3.1, 3.2, 3.4 and 3.4. For instance, when students draw equal groups to be able to count all the objects, they may use these groups to interpret a multiplication operation via skip counting. That is to say, in the scenarios of multiplication problems, students may use the boxes, plates, and trees language in a problem involving objects like toys, cookies, and oranges in equal sizes. The context of the problem provides students with models to be used in problem-solving. At the formal level, students may begin to talk about repeated addition and multiplication strategies rather than using the groups like boxes, plates, and trees as a model. Hence, the model was no longer required for students to reason mathematically. Furthermore, it was decided to provide blank areas in the activity sheets to give students spaces to reflect their imagery. Through this way, it was thought that students' connections and transitions can be understood. Therefore, while developing the instructional sequence and implementing in the classroom, it was considered to assist students to model their own informal mathematical processes with the goal of eventually developing a model for more formal multiplication.

To sum up, while developing the HLT and instructional sequence tasks implemented in this study, three heuristics of RME (i.e., guided reinvention, didactical phenomenology, emergent models) were used. The instruction constructed on the aforementioned learning goals was shaped and enriched in line with RME theory. Therefore, a hypothetical learning trajectory was developed as explained in the following section.
section.

3.3.1.2. Hypothetical Learning Trajectory (HLT)

As explained in detail above, learning goals related to multiplication were determined in the light of the studies in the literature, previously developed learning trajectories, and the mathematics education program. These goals were interpreted and sequenced through principles of Realistic Mathematics Education (RME). Based on the information from this variety of sources and the theoretical framework, a hypothetical learning trajectory for multiplication was planned to be developed. While designing the HLT and instructional sequence, the means of support defined by Cobb (2003) was found important to be considered. He discusses four means of support as instructional tasks: the tools students use, the nature of the classroom discourse, and the classroom activity structure in teaching experiments.

Instructional tasks refer to the activities, in which students engage in order to strengthen their reasoning skills. These activities should be designed to help students develop their conceptual understanding in the form of problem situations that are challenging for them (Cobb, 2003). Hence, instructional activities were developed with the aim of evoking students' attention and creating a discussion to push conceptual development even further. The tools and imagery were chosen carefully in the development process of the activities so that it could enhance the permanence of knowledge. This was done to support the students to reason with tools and imagery as they structure their multiplication concepts through emergent models. During the implementation of the activities, it was suggested to set an environment for the classroom discourse based on norms to be explained in the following part. In this sense, possible discussion topics were included in the HLT. Finally, activity structure refers to how the classroom is organized (Cobb, 2003). In the line of this means of support, students were planned to work individually on the instructional activities and then share their reasoning with others during classroom discourse. Therefore, the design research team decided to add the components of “activities”, “tools/imagery” and “possible topics of mathematical discourse” to the HLT table.

Finally, in addition to the big ideas related to multiplication (i.e., means of support and RME), HLT frameworks used for the studies in the literature were examined (e.g., Gravemeijer et al., 2003a; Rasmussen et al., 2004; Stephan & Akyüz, 2012; Stephan & Cobb, 2003). Possible gestures and metaphors were also included in the HLT. Therefore, an initial HLT for multiplication in second grade was developed as given in Figure 3.1

Big Idea	Tools/Imagery	Activity/Taken-as-shared Interests	Possible topics of Mathematical Discourse	Possible Gesturing and Metaphors
Build up to skip-count	-Hundred Chart -Fingers	-Keeping adding the same number each time to the previous number to skip count.	-How to find the order of a given number in a number sequence?	-Actions with fingers -Keep track of numbers with fingers
Reasoning Number Sequence	-Hundred Chart -Fingers	-Making a connection between the orders of two numbers in a number sequence	-Should you count from the beginning to find a number in a number sequence?	-Keep track of numbers with fingers

Figure 3. 1. Phase 1 of the initial HLT for the multiplication instructional sequence

Big Idea	Tools/Imagery	Activity/Taken-as-shared Interests	Possible topics of Mathematical Discourse	Possible Gesturing and Metaphors
Additive Composition	-Pictorial representation of objects -Circles that are drawn to represent groups -Concrete objects -Repeated addition (symbolically)	-Drawing equal groups to reorganize the objects -Recognizing the ease of counting objects in equal groups -Focusing on repetition of group size	-A collection can be rearranged or counted in a different way and the quantity does not change	- Laying out the items as described in the task -Counting fingers
Many-to-one correspondence (Composite Unit)	-Composite unit -Pictures of counting groups	-Counting groups instead of objects -Focusing on the multiplier	-The roles of multiplier and multiplicand while skip counting.	-Actions with fingers -Pointing groups -Keep track of numbers with fingers

Figure 3. 2. Phase 2 of the initial HLT for the multiplication instructional sequence

Big Idea	**Tools/Imagery**	**Activity/Taken-as-shared Interests**	**Possible topics of Mathematical Discourse**	**Possible Gesturing and Metaphors**
Formalizing multiplication	-Multiplication symbol (X) -Multiplicative language (multiplier, multiplicand, product, times, multiply) - Pictorial representation -Symbolic representation	-Defining multiplier and multiplicand in equal group representation	-What are the roles of multiplier and multiplicand? What if their places are changed?	-pointing with fingers to match equal groups with the numbers in multiplication.
Modeling multiplication as equal groups	-Repeated addition -Pictorial representation -Symbolic representation	-Connecting pictorial and symbolic representation	-How to connect pictorial and symbolic representation	-pointing with fingers to match equal groups with the numbers in repeated addition.
Representing multiplication as array	-Array (row/column) -Pictorial representation -Symbolic representation	-Interpreting multiplier and multiplicand on array model	-How to connect array and symbolic representation of multiplication	-Pointing with fingers to match row/columns with multiplier/multiplicand
Reasoning multiplication in array	-Array (row/column) -Pictorial representation -Symbolic representation	-Reasoning multiplication with array -Equipartitioning array depending on commutativity	-Why does the product remain same when the places of multiplier and multiplicand change?	

Figure 3. 3. Phase 3 of the initial HLT for the multiplication instructional sequence

Big Idea	Tools/Imagery	Activity/Taken-as-shared Interests	Possible topics of Mathematical Discourse	Possible Gesturing and Metaphors
Posing multiplication problems	-Real-life problems -Familiar contexts	-Placing numbers (multiplier/multiplicand) in familiar multiplicative contexts	-How to decide that it is a multiplication problem.	
Posing multiplication problems as repeated addition	-Real-life problems -Equal groups -Concrete groupable objects as multiplicand -Closed groups as multiplier	-Assigning multiplier and multiplicand to represent as equal groups (repeated addition problem)	-What is the place of numbers in repeated addition in the related story	
Structuring multiplicative language	-Real-life problems -Formal language of multiplication.	-Connecting formal multiplication with informal multiplicative language in real life.	-What are the keywords to use in the context?	

Figure 3. 4. Phase 4 of the initial HLT for the multiplication instructional sequence

Big ideas in the HLT table were broken into four phases in order to specify the shifts in students' mathematical reasoning along this HLT. The first phase of the learning trajectory included the goal of having students skip counting and also develop number sequences considering guided reinvention heuristic of RME theory. Instructional activities related to skip counting forward, finding the order of numbers in a number sequence, finding the number whose order is known in a number sequence, using informal tools like fingers effectively, and reasoning the relationship between two numbers in a number sequence were developed. Students were provided the hundredth chart each for 2, 3, 4, and 5 to skip count. For instance, after students colored the numbers while counting by 2s, they were asked two types of questions related to the order of numbers as "What is the 7th number while you count by 2s starting at 2?" and "What is the order of 12 while counting by 2s starting at 2?". It was planned to gain prerequisite knowledge of skip counting for multiplication to be able to reinvent the repeated addition and multiplication concepts on skip counting strategies.

The second phase of the learning trajectory included the goal of having students use additive composition and many-to-one correspondence (Figure 3.2). Equal grouping activities were developed for students to use skip counting in order to count a collection of objects. For this purpose, students are provided pictures of objects so that they can form equal groups and count the objects in these groups through skip counting as in Figure 3.5 below. To direct the students to look for different arrangements of the objects, they are asked whether they can count the objects differently. In this phase, students use skip counting to find the total number of objects rather than using an operation symbolically. With this respect, students are encouraged to use math drawings to form equal groups and find the total number of objects by using these groups as model for counting through the third heuristic of RME theory. Therefore, students are expected to gain an understanding of counting objects in equal groups via skip counting as a meaning of multiplication. At the end of each task, students represent the given situation in the form of "… times …makes …". For instance, for the eggs given in Figure 3.5 below, students are expected to conclude that 8 times 4 makes 32.

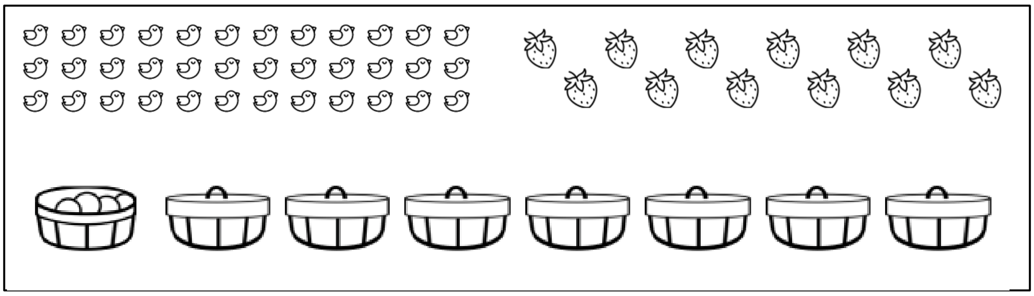


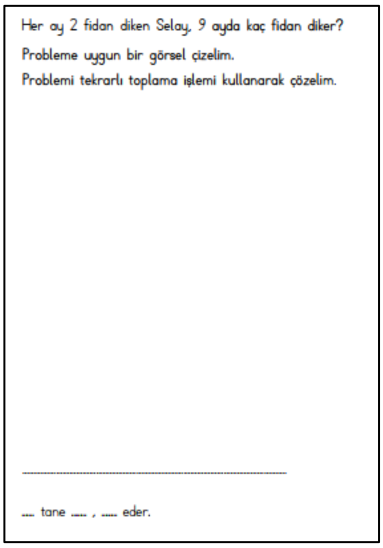
Figure 3. 5. Sample pictures from equal grouping activities

In this phase, students are asked to use number sentences to find the total number of objects. They are expected to write repeated addition as an operation to find the sum of objects in equal groups. The aim of this activity is to use visuals related to equal groups as the concrete

examples of formal mathematics to be taught for repeated addition considering emergent modeling in RME. It is also intended that equal group models are used to re-invent more formal mathematics in the RME approach. With this purpose, students are asked a verbal question to discuss and find a taken-as-shared idea of finding the number of objects in equal groups by formulating in repeated addition. After discussing this idea, students are presented with activities as given Figure 3.6 below. Templates for repeated addition are provided on the activity sheets in order to be satisfied all the students can make a connection between equal group representation and repeated addition. That is to say, group size is repeatedly added as much as the group number.

Figure 3. 6. Sample activities related to representing equal groups via repeated addition

In the following activities, in order to strengthen students' understanding of the connections between multiple representations as pictorial, symbolic, and real-life representations, the forms of the tasks are changed. The first type of task related to the second phase (Figure 3.7) includes problems that require representing the problem via drawing visuals and solving it via repeated addition. As it is seen, space is provided for students to draw equal groups considering the context of the problem. It is intended to use equal group representations as models for repeated addition in line with the RME approach. At the end of each problem, students represent the given problem in the form of "… times …makes …" as in all activities.


Her ay 2 fidan diken Selay, 9 ayda kaç fidan diker?
Probleme uygun bir görsel çizelim.
Problemi tekrarlı toplama işlemi kullanarak çözelim.

..... tane ..... , ... eder.

Figure 3. 7. Sample activity related to problem solving via repeated addition

The second type of task related to the second phase in the HLT requires students to pose repeated addition problems for the given pictures and number sentences as in Figure 3.8 below. For instance, students are given a picture of bottles on shelves and write a problem for this picture and solve it by repeated addition. Moreover, students are asked to pose a problem for the given number sentence to connect it with real-life representation. Students are asked to draw visuals related to their original problems to use as a model for in explaining and justifying their reasoning while posing the problem. It is intended that models grounded in the contexts are created by students and used to re-invent more formal mathematics in the RME approach.

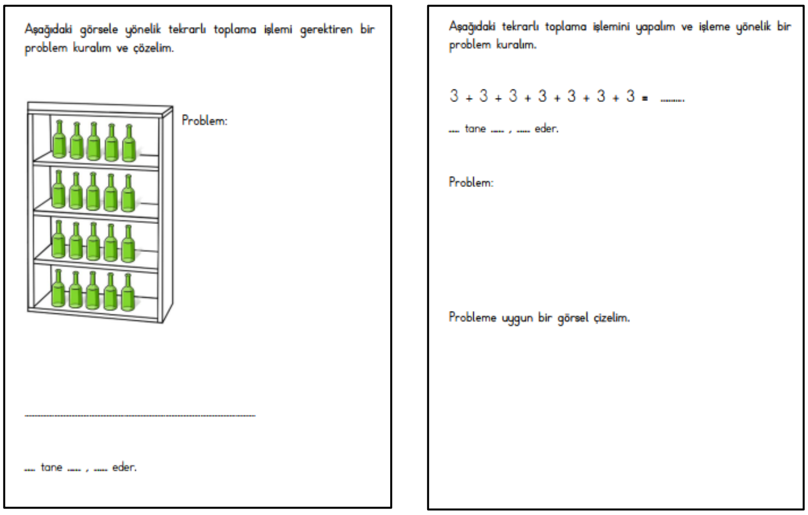


Figure 3. 8. Sample activities related to posing repeated addition problems for a given picture and number sentence

The third phase of the learning trajectory included the goal of having students formalize multiplication (Figure 3.3). In this phase, students are introduced to multiplication symbolically and analyze components and properties of multiplication operation by modeling with equal groups and arrays. Students formalize multiplication as a number sentence symbolically. The first number is defined as the multiplier while the second one is defined as the multiplicand. Students match the number sentence with the given equal group representation so that the multiplier represents the group number while the multiplicand represents the group size as shown in Figure 3.9. In these activities related to modeling multiplication as equal groups; the tasks are developed for students to represent the given problem via pictorial and symbolic representation and solve. As in the Figure 3.9, the visual directions in the tasks are removed step by step. That is to say, in the first task, the leaves (groups) and the ladybugs (elements) are given visually in addition to the problem contexts. In the second task, the plates (groups) are given, but the cookies on the plates are not given. Finally, in the third task, neither the trees nor the oranges are given. Students are asked to draw them. The explanations related to the components of the multiplication operation are also removed in the third task. The purpose of changing the structure of visuals in the tasks is to reduce the directions and enable students to make reasoning related to the given tasks. Furthermore, it is asked to solve given problem via repeated addition in addition to multiplication. Therefore, students are expected to use visuals and repeated addition as a model for higher-level mathematical reasoning related to multiplication considering the RME approach.

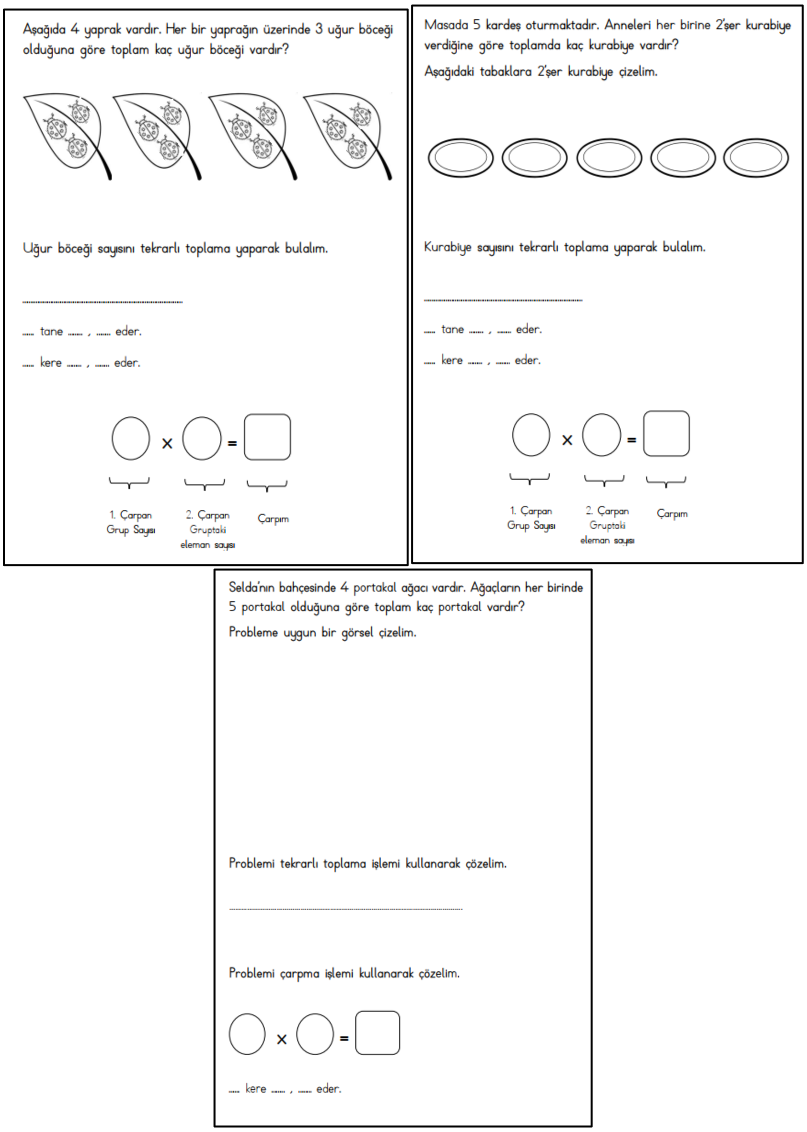


Figure 3. 9. Sample activities for formalizing multiplication in given equal groups

In this phase, multiplication is also presented in the array model. At the beginning of the activities, students are explained the terms of column and row in order to maintain a language to use and understand each other. As in Figure 3.10 below, pencils are asked to be placed in the cells given on the right to obtain a rectangular shape. Students are made sure that there are an equal number of objects in each row or column. The relationship between the arrays and multiplication is discussed. In the given task, rows are defined as groups while the columns are presented as the group size. For the first task related to array, it was decided to make students focus on rows to be sure that all the students can use the terminology related to arrays and multiplication. It was planned that students assign the number of rows as the multiplier while they assign the number of columns as multiplicand.

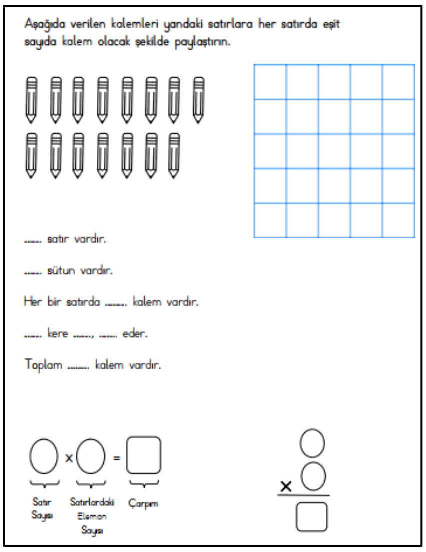


Figure 3. 10. The first task on introduction to array representation in the third phase

After being sure that students can make connections between arrays and multiplication, the tasks related to representing given arrays as multiplication symbolically and representing given number sentences of multiplication as arrays are presented to the students as given in Figure 3.11 below. The purpose is to make students gain conceptual understanding and fluency in modeling multiplication with array representation. In these activities, students are asked to explain how they show multiplier and multiplicand on the arrays. It is emphasized for students to circle the groups to be able to see whether they group the dots in rows or columns. A discussion environment is maintained between the ones who group the dots in rows horizontally and the ones who group the dots in columns vertically. Then, students are expected to conclude that they can name both rows and columns as multipliers as they wish.

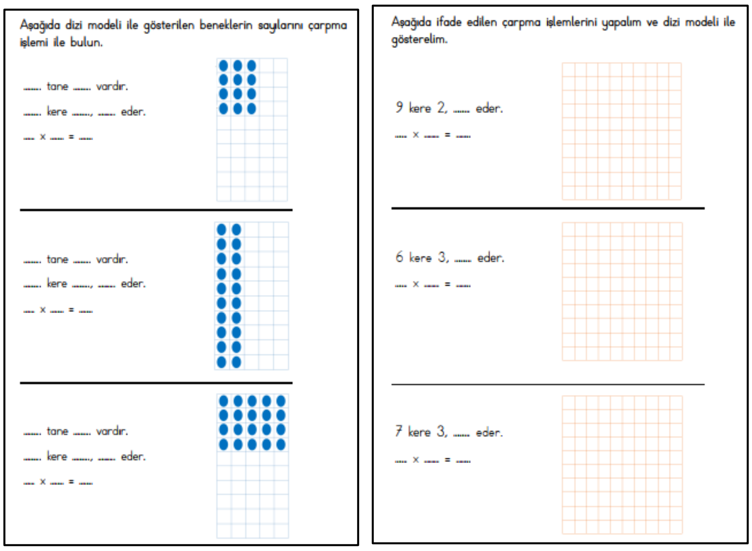


Figure 3. 11. Sample tasks for modeling multiplication on array model

In the following activities, students are asked to use arrays as a model for higher-level mathematical reasoning. That is to say, students are asked to model given problems on arrays to support their solutions via multiplication as in Figure 3.12 below. For this purpose, spaces are provided in the activity sheets for students to work on modeling their reasoning via arrays. Students are expected to show the consistency between the operations that they used and the models they made. In other words, students should show the multiplier and the multiplicand in line with the context of the problem, and the number sentence they used.

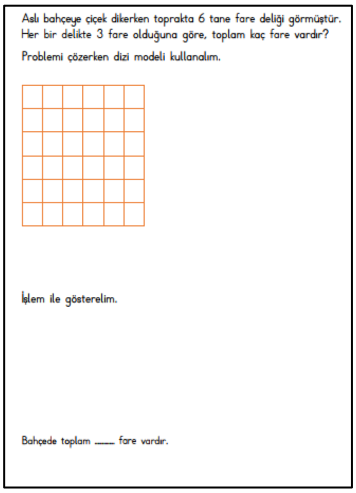


Figure 3. 12. Sample task for problem-solving by modeling multiplication on array model

At the end of this phase, students are expected to make reasoning of multiplication in array models. It is crucial since students are expected to interpret commutativity in multiplication by using arrays. It is envisioned that students can observe commutativity until this phase. However, to be able to represent commutativity, it is important to meet them with arrays. In this sense, an activity related to commutativity was developed as in Figure 3.13 below. The purpose is to make students realize that the difference is the direction of grouping dots since the multiplier and the multiplicand are replaced.

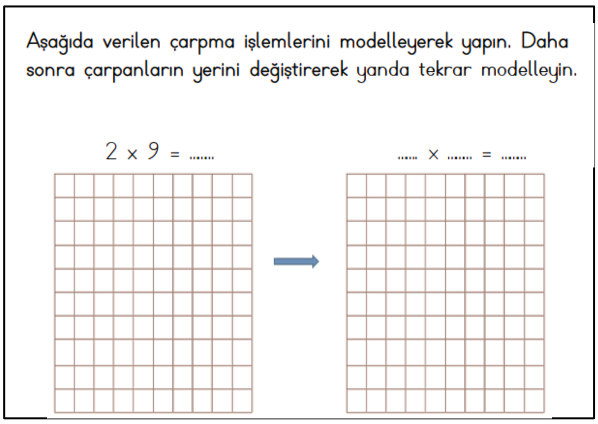


Figure 3. 13. Sample task for commutativity in multiplication via modeling with array model

The fourth phase of the HLT is related to representing multiplication in real-life contexts (Figure 3.4). This phase included the goals of having students represent multiplication as repeated addition in real-life contexts and using multiplicative language appropriately. Students are expected to match real phenomena including repeated addition of equal groups with multiplication. Up to this phase, students work on activities given in real-life contexts. Students represent the given contexts with the symbolic representation of multiplication which is a number sentence in the form of "…x…=…". Moreover, students are asked to represent the given pictures via the language of equal grouping. As it is explained in the second phase above, students group given objects equally and define these objects by using the words like "groups" and "each". For instance, in the activities given in Figure 3.14 below, statements are given below the pictures for students to fill. Students analyze and express the objects in groups considering the number of groups and group sizes.

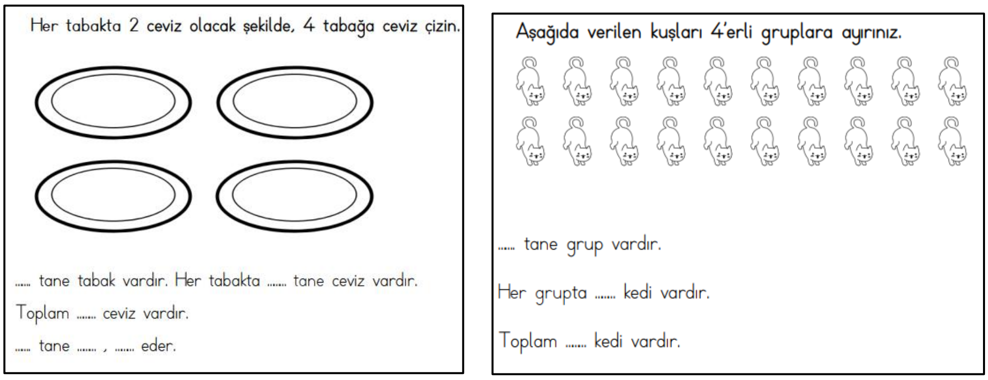


Figure 3. 14. Sample activities related to representing objects with equal group language

In the following activities, students are asked to pose problems to see their understanding of multiplication by observing how they interpret multiplication considering real-life representation. Until these activities, students were presented tasks developed in realistic contexts considering didactical phenomenology heuristic of RME theory. The stories of the tasks were selected to be meaningful for the students. At that point, it was decided to ask students to develop their own stories to be able to observe and measure their ability to connect mathematics with real-life situations. Students are expected to set their groups as group numbers and group size in order to write a story reflecting multiplication as repeated addition in a real-life context. For instance, in order to observe students' understanding related to zero property in multiplication, they are asked to pose problems for given number sentences as in Figure 3.15. As seen in this figure, two types of tasks are developed one where zero is the multiplicand and the other where zero is the multiplier. Students are expected to reflect their knowledge related to zero property considering the roles of multiplier and multiplicand. After students pose problems, they should be asked how they can be sure that this is a multiplication problem in order to get detailed information related to their thinking. Students' problems are discussed considering whether they are mathematically correct and contextually realistic.

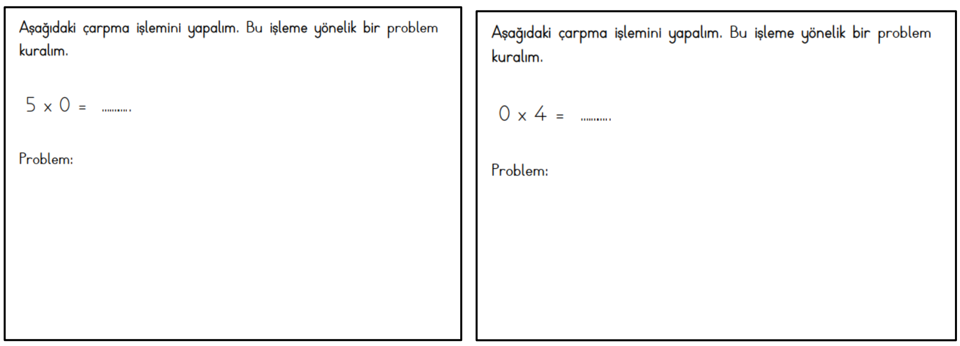


Figure 3. 15. Sample activities related to posing multiplication problems for given number sentences related to zero property.

In this phase, students are also supposed to use multiplicative language while representing the given symbolic representation of multiplication in a real-life context. Everyday words like 'grouping' and 'each' have early associations with the processes that will become multiplication. Hence, students are expected to use this terminology while building realistic contexts which represent multiplicative meaning. Structuring multiplicative language is a critical idea that is developed from the beginning of the HLT to the end. At the end of the intervention, students are expected to connect formal multiplication with informal multiplicative language in real life. That is to say, this idea is not targeted to be developed via a few activities, but via the whole instructional sequence.

To sum up, in the light of the HLT for multiplication, an instructional sequence was developed in order to support students to develop their understanding related to the big ideas. The activities were developed according to the heuristics of RME theory. That is to say, the essential mathematical ideas that were described in the HLT were embedded in realistic tasks in order to promote students' processes of mathematization and reinvention. These tasks were ordered to be experientially real for students. Moreover, they were enriched by considering the use of models as students' conceptualizations progressed from informal to formal mathematical activities as given in HLT. Finally, the HLT and instructional sequence were developed to serve four phases as explained above by providing sample tasks. All of the activities in the instructional sequence are available in Appendix X. After all the preparations related to the development of HLT and the instructional sequence were done, the design experiment started.

3.3.2. Phase 2. Design experiment

A teaching experiment entails a cycle of designing instructional sequences, putting those sequences to the test in a classroom setting, and assessing the learning shown in Figure 3.16. Based on the analysis, the sequence is revised, and the procedure is repeated (Gravemeijer et al., 2003b). In order to obtain insight into the quality of the interventions and design principles, as well as to make decisions related to revisions, empirical data is required. Therefore, formative evaluation is an important part of the design experiment (Nieveen & Folmer, 2013). The formative assessment results provide the foundation for both of a design research study's outputs: enhancing the intervention to a high-quality, completed intervention, and sharpening the underlying tentative design principles to a final set of design principles. Hence, the second phase consists of actually conducting the design experiment. The microcycles of the design experiment serve as the foundation for developing the local teaching theory, which includes ongoing studies of students' individual and collective activities, as well as classroom social characteristics (Gravemeijer & Cobb, 2006). Therefore, after completing the first phase, this study was conducted in a macrocycle which is comprised of many microcycles to develop a stronger local instructional theory. This design experiment which took place over five weeks revealed how the hypothesized instructional sequence functions in the complicated environment of a classroom.

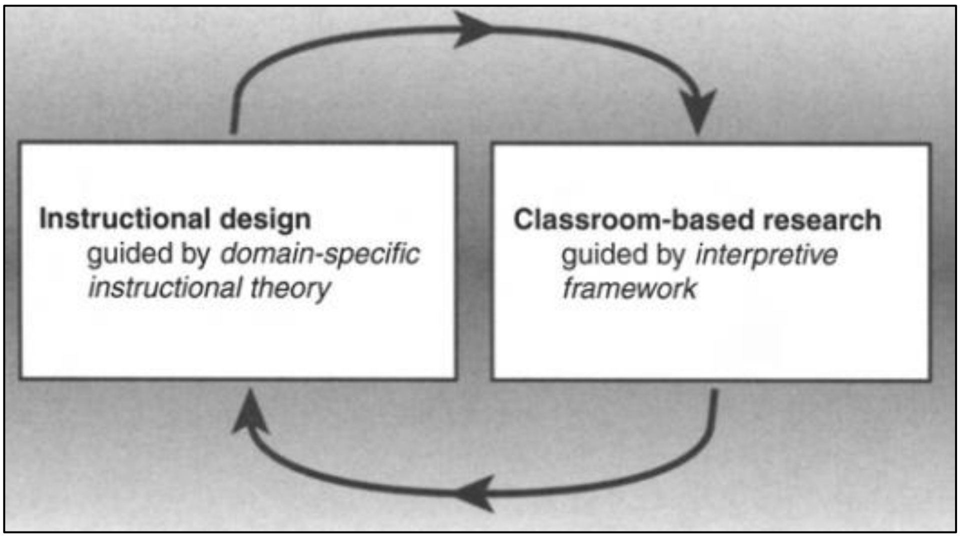


Figure 3. 16. Phases of the Design Cycle (Stephan, 2003, p.29)

As in Figure 3.16 above, the hypothetical learning trajectory and related instructional sequence for multiplication was developed by the domain-specific instructional theory of Realistic Mathematics Education. Through the classroom-based research, this instructional sequence was implemented in a classroom of second graders. During the implementation, considering collective reinvention, the design team paid attention to creating a classroom environment where students engage in whole-class conversations. That is to say, students engaged in the processes of conjecturing, explaining, and justifying, which is referred to as collective mathematizing (Gravemeijer et al., 2000). As stated by NCTM (2000), students were promoted to make conjectures and construct arguments and also respond to others' arguments while engaging in the instructional sequence of multiplication. Therefore, the design team focused on providing students with learning environments that would support their way of argumentation. In this sense, the broader aspects of the social environment in which students participated are documented in the following section considering the interpretive framework.

3.3.2.1. Interpretive Framework

The interpretative framework based on an emergent perspective was employed to make sense of students' learning while we are in a classroom and organize individual and collective mathematical learning analyses in this study (Cobb & Yackel, 1996). As explained previously, the social perspective includes three subconstructs as classroom social norms, sociomathematical norms, and classroom mathematical practices. The classroom social norms had already been established in the classroom in which the current study was conducted. These norms are explaining and justifying solutions, listening to their friends and interpreting others' solutions as well as indicating agreement or disagreement, and questioning.

The social norm that students should explain solutions and justify their reasoning had already been negotiated by the classroom community before this study began, because the collaborating teacher and students joined another teaching experiment the year before and they experienced and sustained these social norms. During the implementation phase, students were asked to explain and defend their reasoning by describing how they chose the methods of problem-solving they would employ and the solution they came up with. They were given problem-solving situations to encourage mathematical thinking in realistic contexts. Whole class discussions were held with the intention of involving students and inspiring them to explain their ideas clearly through argumentation. Interactions through argumentation gave students the chance to evaluate their strategies, defend their decisions, and impart knowledge to others.

The other social norm, which is established prior to the instructional sequence for multiplication, is listening to their friends and interpreting others' solutions, and expressing agreement or disagreement. The goal of the students' interpretation of others' strategies was to help them compare and contrast their own thinking with that of others and to provide visual or verbal interpretations. First and foremost, they had a duty to pay attention and carefully follow the discussion. The teacher asked students whether they understand the strategies of others and whether they agree with the others' methods. The teacher also asked why they agreed or disagreed with them. When they stated that they did not concur with the answer of someone, they were asked to explain their reasoning and convince this student that his/her answer is wrong. For instance, Esra posed an addition problem when they were asked to pose a multiplication problem. The others explained to Esra that her problem requires addition, not multiplication. Moreover, they helped her to revise her problem. That is to say, this social norm is related to not only interpreting the solutions of others but also helping them to revise their solutions if necessary.

The final social norm is questioning. Prior to the implementation of the instructional sequence for multiplication, students were familiar with questioning by the teacher, not by the others. During the implementation of this study, students started asking questions or requesting clarification when something is unclear. Students were aware that they were in a collective learning environment and responsible for their own learning. For that reason, they took responsibility for interpreting others' methods. Consequently, to be able to interpret, they had to understand their methods. When they could not understand, they asked questions like "What do you mean?", "Can you repeat it?", and "How did you find it?". These questions led to further explanation and clarification. Therefore, questioning to understand the mathematical processes and building their own meaning was established as a social norm in the participating classroom in this study.

The second aspect of the interpretive framework concerns sociomathematical norms that were specific to mathematics and mathematical activities (Cobb & Yackel, 1996). During this teaching experiment, students were mainly expected to offer an acceptable mathematical solution as the key concern of sociomathematical norms. The teacher encouraged students to share different solutions via different representations, explanations, and justifications for their solutions with mathematical reasoning, provide conjectures and establish claims and warrants to feed the argumentation process.

Sociomathematical norms were established as they related to criteria for what counted as acceptable mathematical solutions to the problems in the instructional sequence. Remembering the part of the social norms from earlier, the students were required to justify and explain their mathematical procedures in class. Students' explanations were counted as acceptable if they focused on both the mathematics and what the numbers in the calculations meant in the image qualified. That is to say, students did not only multiply two numbers but also explained and justified the roles of these numbers as multipliers and multiplicand during the implementation.

The other sociomathematical norm is related to sharing different solutions. Actually, students were used to sharing and discussing different solutions in the classroom as they did in the previous project. During the implementation of the instructional sequence for multiplication, the teacher frequently asked if other students had a different solution or a different solution technique. When the students were working individually, the teacher strolled about the classroom looking for these different solutions and solution methodologies. If students did not volunteer, the teacher asked those who solved in a different way. After a while, it was observed that students were eager to find a different solution and share with others to see their reflections. It was also remarkable that the students were reflecting positively by appreciating the different solutions. Therefore, sharing different solutions was established as a sociomathematical norm in the classroom.

The third aspect of social perspective is classroom mathematical practices which is related to the second purpose of the current study. This study was conducted as a design research whereas classroom mathematical practices and learning trajectory of the classroom community were examined and taken into consideration. The classroom mathematical practices emerged in this study are documented with taken-as-shared ideas in the section of Findings.

To sum up, in this study, a local instructional theory including students' reasoning multiplication was conjectured using RME as a design theory for the development of the instructional tasks was developed as explained in the first phase. Then the developed instructional sequence was implemented in the classroom for five weeks including 26 class hours (i.e., 1 hour each day). The experiment of the study took place in the second-grade class of the collaborating teacher. The classroom events that took place as students and the teacher interacted around these activities were analyzed in line with the theoretical lenses of the Emergent Perspective and Realistic Mathematics Education during the experiment. These analyses revealed the need for daily or weekly revisions to the instructional sequence as micro-cycles. These revisions are explained in the following section.

3.3.2.2. Revisions to the instructional sequence during and after the experiment

As it is explained, the second phase is a classroom teaching experiment where the research team tests and revises their initial hypotheses about how the anticipated sequence will really be realized. In line with the inherently iterative nature of Design Research, daily minicycles are implemented during the second phase at the micro level (Gravemeijer et al., 2003b). As shown in Figure 3.17, the researchers make recommendations about daily changes of the educational activities using informal evaluations of students' mathematical thinking. These daily analyses of students' mathematical activities are therefore used to shape the instructional sequence. These analyses are also conducted against the backdrop of a hypothesized learning trajectory. Furthermore, changes to the sequence influence students' next mathematical task (Gravemeijer et al., 2003b).

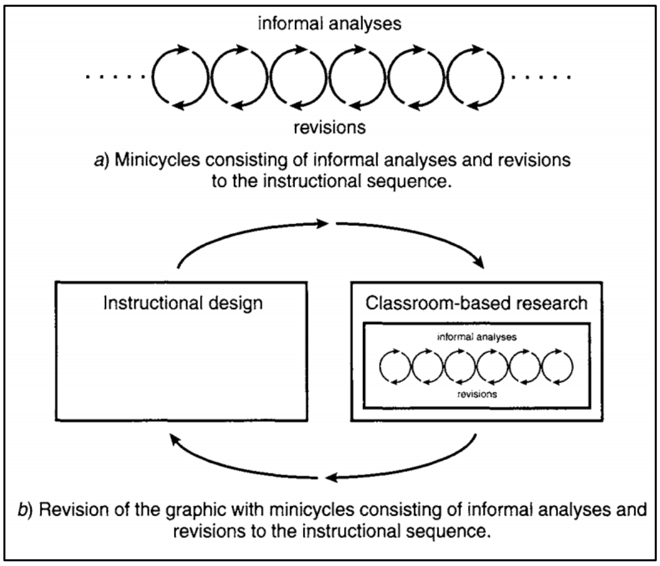


Figure 3. 17. Minicycles in Design Research (Gravemeijer et al., 2003b, p.115)

Considering the ongoing analysis of classroom sessions for modifying and refining the conjectures, classroom-based analyses came to the fore to interpret the initial instructional sequence for multiplication. In keeping with the nature of design research, the team conducted debriefing sessions after each minicycle of the experiment in order to assess the learning of students as it took place in normative ways. Through this process, the team continued ongoing analysis to improve the HLT and the instructional sequence to make a stronger case for the next day. The activities were revised and improved to assist students in transitioning from informal to formal reasoning while also incorporating tool use. For instance, in the first week of the experiment, three types of activities on equal grouping related to the second, third and fourth types of tasks were implemented in the classroom. However, it was observed that students could not distinguish three types of equal grouping tasks and they were confused. For that reason, the team decided to spend more time on each type of activity. As it is stated to gain students the ability of multiplication with 2, 3, 4, and 5, all the activities were developed for the numbers of 2, 3, 4, and 5 since the objectives related to multiplication in second grade are restricted to multiples of these numbers. Sample activities are given below in Figure 3.18.

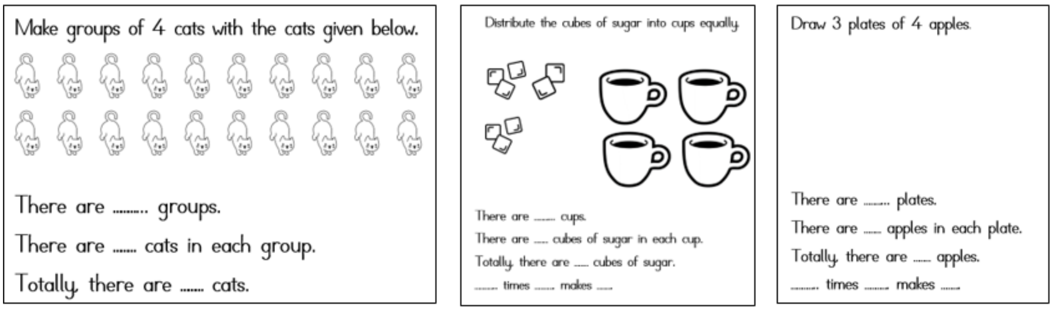


Figure 3. 18. Equal grouping activities for number 4 about division as measurement, division as partition, and multiplication as equal groups respectively.

During the implementation, it was seen that some of the students were counting all the objects one by one to find the total number of objects. The purpose was to encourage students to use skip counting instead of counting each object. For this purpose, the team decided to provide closed groups whose elements are not visible. The elements in the first group were visible since the group was open, while the ones in closed groups were invisible as in Figure 3.19. Furthermore, templates of the related operations were provided in the first version of the activities as in Figure 3.19. These templates were removed not to direct students and limit their thinking about writing number sentence for the given picture or problem. All the activities were revised in this respect.

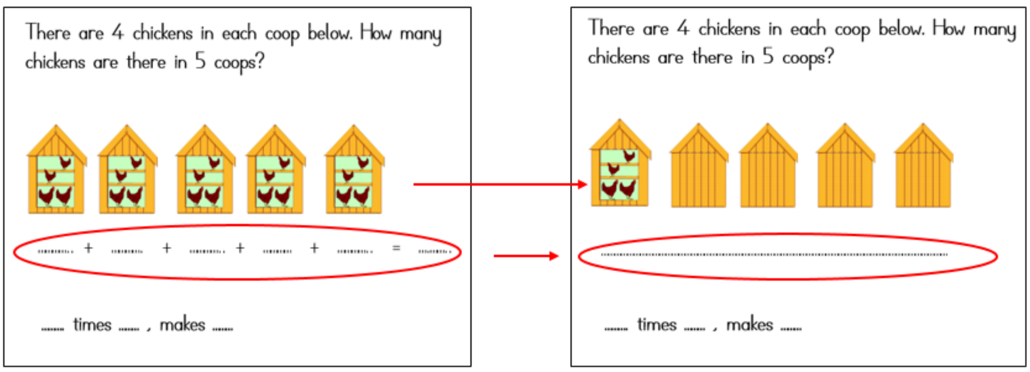
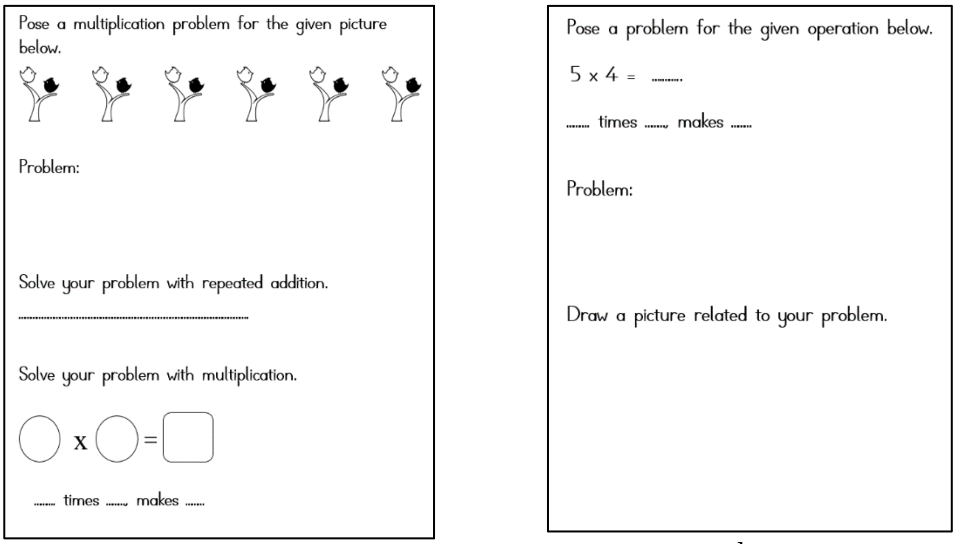


Figure 3. 19. Revisions on a sample activity

In addition to presenting instructional tasks in the form of realistic problems, students were also expected to pose their own realistic problems in this study. However, the team members were not sure whether students were ready for writing contextually realistic problems as problem posing activities that require higher-order thinking skills on the topic. They decided to observe students and involve problem-posing activities when they were ready. In this sense, students were presented with an activity to see their conceptualization of related issues and capability of posing problems. This first activity was presented as a picture in which there were four shelves and 5 bottles on each shelf. Students were asked to pose a repeated addition problem according to this picture. During this activity, students presented their problems and discussed the appropriateness, representativeness, and components of these problems considering the picture. It was observed that students developed mathematically rich arguments while posing problems. In addition to its power in terms of the conceptual development of students, it was also observed that students were highly motivated and enjoyed it. Therefore, after this lesson, the team members set a meeting and decided to include activities on problem posing in the learning trajectory. They developed problem posing tasks in line with both RME and Bruner's Theory of Learning (1964), which emphasize that instructional activities progressively move from concrete to abstract and connect different representations (concrete and pictorial, real-world and symbolic). Thus, students were asked to write word problems based on given pictures and multiplication operations. In this sense, problem posing tasks were classified as translating and comprehending quantitative information (Christou et al., 2005). Translating tasks required students to pose an appropriate problem from graphs, diagrams, or tables while comprehending tasks required students to pose problems from given mathematical equations or operations as in Figure 3.20 given below. In this respect, problem posing tasks were developed for each big idea state in the HLT for multiplication.



a b

Figure 3. 20. Sample (a) translating and (b) comprehending tasks

Furthermore, the tasks related to composing array were complicated for students. There were many terms which were new for students like column, row and array. Another limitation of the tasks was that they were presented in activity sheet requiring drawing. Therefore, activities failed in helping students to see an array as the collection of equal groups. After the lesson, the team members discussed on the limitations of the tasks and further implementations. They decided to develop concrete materials to compose array. Students were provided counters to construct arrays as in Figure 3.21. They worked on concrete tools to construct array effectively, however these materials were limited in representing their works to others during discussion session. In this sense, students were provided reusable squared cards to draw arrays on as in Figure 3.21. They were asked to use these card to show their work while discussing and reasoning the given tasks.

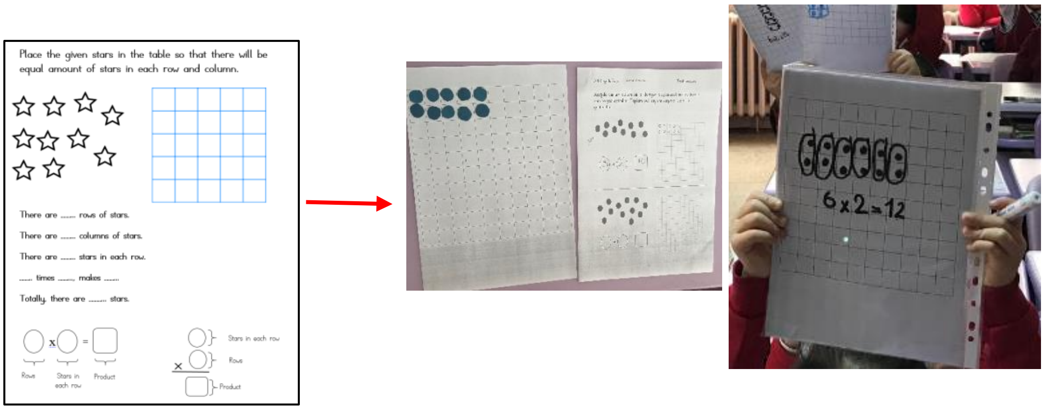


Figure 3. 21. Revisions on the task of introduction to array

At the beginning of the experiment, activities were planned to be reflected on the board via projector. Unfortunately, the projector had been broken at the time of implementation. For that reason, the materials designed on power point presentation were cancelled and concrete materials were developed instead. For instance, in addition to concrete objects and squared cards, a large squared table (array panel) and sticky counters were designed for the teacher and students to be able to compose arrays that can be seen by everyone as in Figure 3.22.

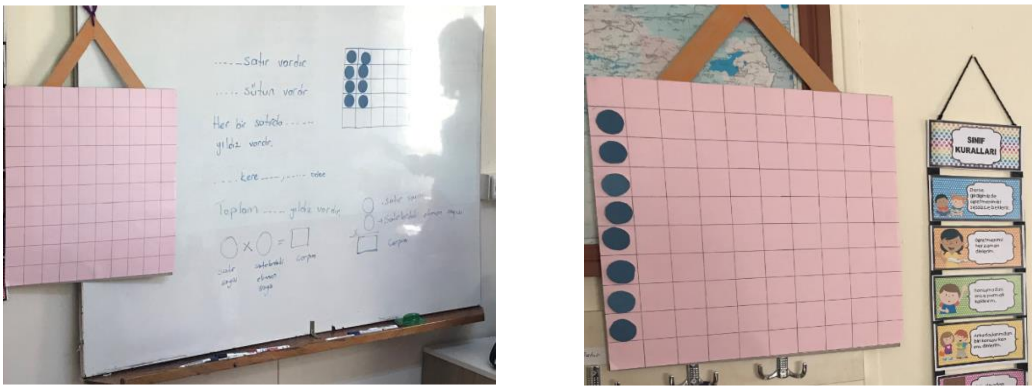


Figure 3. 22. Array panel and counters to stick on it

In conclusion, during the implementation, various revisions were made in the instructional sequence in order to increase the construct validity and practicality of the experiment. In terms of construct validity which refers to the logical design of the intervention (Plomp, 2013), invisible equal groups were used;

problem posing activities were developed for each big idea; operation templates were removed. On the other hand, in terms of practicality which refers to the usability of the intervention in the settings for which it has been designed and developed (Plomp, 2013), concrete materials of counters, array panel, and squared cards were developed.

3.3.3. Phase 3. The retrospective analysis

After the design experiment was completed, a retrospective analysis of the entire data was conducted. Retrospective analysis is recommended based on many factors (Cobb, 2003; Cobb et al., 2001; Gravemeijer, 2004). The first of these is that the methodology should make it possible to record the collective mathematical learning of the students. In this study, classroom videos were analyzed and students' classroom mathematical practices were documented. In this way, the study also enabled the documentation of individual students' growth as community members.

The second criterion for conducting a retrospective analysis is that the results should provide feedback that can be used to improve the educational design. That is to say, researchers can build an instructional sequence comprised of beneficial activities during the retrospective analysis phase of the teaching experiment (Gravemeijer, 2004). Then, the teaching sequence is redesigned based on the findings of this retrospective research (Gravemeijer et al., 2003b; Steffe & Thompson, 2000). In this study, a retrospective analysis was conducted on the entire data after the design experiment was implemented. The documented classroom mathematical practices led to the process of making final revisions for the best case HLT and the instructional sequence (local instruction theory). In the Conclusion and Discussion section, the final revisions are explained.

3.4. Data Collection Tools

Design research usually triangulates numerous sources and types of data to connect intended and unexpected results to enactment processes. Methods that document enactment processes provide crucial evidence for establishing warrants for claims about why certain outcomes occurred (Design-Based Research Collective, 2003). Moreover, triangulating data from multiple sources promotes improving the reliability of findings and measures through ongoing analyses over the cycles of enactment, and using standardized measures or instruments (Design-Based Research Collective, 2003). In this regard, multiple data collection tools such as video recordings of classroom sessions, pre-and post-tests, pre-and post-interviews, students' written works, the researcher's field notes, and audio recordings of the daily debriefing and research team meetings were used in this study. The relationships between the research questions and the data collection tools are displayed in Table 3.3 below. Each of these data sources is explained clearly in the next subsections.

Table 3. 3. The relation between the research questions and the data collection tools

The Research Question	Data Collection Tool
What would an optimal HLT and instructional sequence for multiplication look like?	Pre- and Post-tests Pre- and post-interviews Procedural fluency tests Written works Audio recordings of debriefing and research team meetings Researcher's field notes
What are the mathematical practices as students engage in the instructional sequence for multiplication?	Video recordings of classroom sessions Written works Researcher's field notes

3.4.1. Video recordings of classroom sessions

The main data source to document classroom mathematical practices is the records of classroom sessions. Each classroom implementation in this design research was videotaped. One camera was used to capture as many of the classroom activities as possible. Two voice recorders were also used in case the camera did not record all the sound in the classroom. During whole-class discussions, one camera and additional voice recorders were on to capture the process of implementing the instructional sequence, the teacher's instructional activities, students' mathematical reasoning and learning, social and sociomathematical norms, culture and environment of the classroom, formal/informal tools and interactions among them. Finally, all the records were transcribed to be analyzed.

3.4.2. Pre- and Post-tests

Students must be provided numerous opportunities to create their own strategies while dealing with numbers to construct their own computation techniques, give more meaning to operations and gain confidence in standard algorithms (van de Walle et al., 2020). By doing so, students can improve their skills related to counting, recognizing number patterns, comparing numbers, and estimating, thus number sense (Berch, 2005). Students can deepen their number sense as they employ operations and diverse solution strategies for operations. For instance, counting in twos, threes, or fives is frequently underestimated in the classroom, yet such patterns of numbers are common among children and are critical for establishing number sense (Anghileri, 2006). In this regard, the current study is crucial in developing students' number sense by supporting students with counting activities enriched with RME theory and introducing multiplication considering its various meanings and relationship with other operations. Therefore, in order to interpret students' learning and subsequently enhance and revise the design, the Number Knowledge Test was used as pre-and post-test.

The Number Knowledge Test (NKT) which was translated into Turkish by Çakıroğlu and his team (Çakıroğlu et al., 2019) was developed by Okamoto and Case (1996) to measure the intuitive knowledge of numbers that the average child has available at the age-levels of 4, 6, 8 and 10 years. This test helps to assess students' procedural and conceptual knowledge of whole numbers, comprehension of magnitude, counting abilities, and basic arithmetic operations. The NKT analyzes various aspects of a student's numerical proficiency, including the application of numbers to basic arithmetic concepts and operations, unlike single proficiency assessments that assess discrete skills and abilities in numerical proficiency. Each item was evaluated out of 1 point. For all two-part items, students were expected to answer both (a) and (b) must be answered correctly to earn a point. The test is divided into four levels, each with a different level of difficulty and analysis. Sample items for each level are provided in Table 3.4 below. Therefore, the test was used to determine students' levels of number knowledge before and after the design experiment.

Table 3. 4. Sample Items for the Levels of the Number Knowledge Test

Levels	**# of Items**	**Sample Items**	**Decision**
Preliminary	1	Let's see if you can count from 1 to 10	
Level 0	5	(Show stacks of counters, 5 vs. 2, same color.) Which pile has more?	Go to Level 1 if 3 or more are correct
Level 1	9	What number comes two numbers after 7?	Go to Level 2 if 5 or more are correct
Level 2	9	Which number is closer to 21: 25 or 18?	Go to Level 3 if 5 or more are correct
Level 3	7	Which difference is smaller, the difference between 99 and 92 or the difference between 25 and 11?	

Pretest was used to understand students' initial level of number sense in order to reveal their intuitive ways of reasoning on numbers and number relations and build the instruction on those. On the other hand, the post-test was used to obtain insight into students' learning after the experiment to examine how their reasoning processes had changed.

3.4.3. Pre- and post-interviews

In addition to the Number Knowledge Test, an interview protocol was developed by the research team in order to conduct before and after the implementation. The purpose of this interview was to investigate students' performance on multiplication tasks each of which was developed in a different semantic structure of multiplication. Totally 6 multiplication problems were developed by Mulligan and Mitchelmore (1997) in various semantic structures which are equivalent groups, rate, comparison, array, and Cartesian product (Figure 3.23).

1- There are 2 tables in the classroom and 4 children are seated at each table. How many children are there altogether? (Equivalent Groups)
2- Peter had 2 drinks at lunchtime every day for 3 days. How many drinks did he have altogether? (Equivalent Groups)
3- If you need 5 cents to buy 1 sticker, how much money do you need to buy 2 stickers? (Rate)
4- There are 4 lines of children with 3 children in each line. How many children are there altogether? (Array)
5- John has 3 books, and Sue has 4 times as many. How many books does Sue have? (Comparison)
6- You can buy chicken chips or plain chips in small, medium, or large packets. How many different choices can you make? (Cartesian product)

Figure 3. 23. Problem set for multiplication (Mulligan & Mitchelmore, 1997).

After the implementation of the Number Knowledge Test as the pretest, interviews were conducted with 10 students who comprise approximately half of the students in the classroom. These students were chosen according to the teacher's advice considering students' communication abilities and willingness to work with the researcher. Interviews session was held after the pretest in order to examine students' intuitive knowledge related to multiplication and informal ways of reasoning and tools. Each student was interviewed separately by the researcher. The paper with the questions on it was given to the student. Students were asked to read and explain the problem. Then, they were given time to solve the problem. Students were asked to explain the way they thought and solved the problem. The interview sessions were video-recorded. The pre-interviews helped the team to see students' conceptualization and interpretation of the multiplicative situations presented in real-life contexts. Students' early understanding of multiplication was taken into consideration in developing the instructional sequence. Moreover, post-interviews were conducted again after the post-test to gain insight into students' knowledge of semantic structures of multiplication after the experiment in order to see changes in their understanding of multiplicative situations and using tools if exist. Students' performance and reflection were examined in order to refine the instructional sequence.

3.4.4. Procedural fluency tests

Knowledge of processes, as well as when and how to utilize them effectively, and the ability to conduct them flexibly, accurately, efficiently, and appropriately, is referred to as procedural fluency (NRC, 2001; van de Walle et al., 2020). Gaining procedural fluency is crucial for students since students have difficulty in understanding mathematical concepts or solving mathematics problems if they lack procedural fluency (NRC, 2001). In this regard, it is aimed to improve students' procedural fluency in multiplication by supporting their conceptual understanding of multiplication and the meaning of multiplication considering daily experiences by the instructional sequence. In order to assess the effectiveness of the implementation, procedural fluency tests were used as the measurement tool. These tests included a collection of multiplication operations (see Appendix F). Students were applied to each multiplication fluency test after learning symbolic multiplication every three days. Totally 7 tests were used and students were given 2 minutes for each. The number of operations that were answered correctly was evaluated by the research team. Moreover, the mean of the students' scores was interpreted through the following tests. In this sense, students' performances on these tests were considered for daily revisions to develop students' procedural fluency in addition to conceptual understanding in multiplication.

3.4.5. Written works

During the classroom implementations, students were provided activity sheets related to the tasks in the instructional sequence. After each lesson, students' written works were scanned to keep as a copy and the original sheets were given back to the students. These written works were examined after each implementation in order to evaluate the implementation regarding students' performance and revise the further implementations. Moreover, they played an important role in providing a clue during the analysis of video records for triangulation. This data was used for trustworthiness issues in the study. That is to say, they helped to understand students' reasoning and arguments during classroom discussions and clarified students' contributions to the classroom mathematical practices in detail.

While watching and analyzing the classroom videos, sometimes students' arguments were unclear to the researchers. During such times, they turned back to the written works of the students to understand their explanations. These written works enlightened and supported their claims. The researchers got the advantage of these documents to be able to make comments related to the students and interpret their arguments. Furthermore, the findings of the study were enriched with sample works of the students as in the section of Findings.

3.4.6. Audio recordings of debriefing and research team meetings

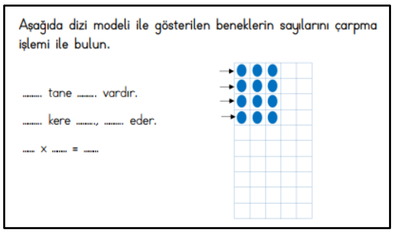
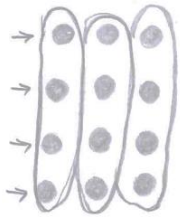
The researcher conducted short debriefing sessions with the collaborating teacher before, during, and immediately after each classroom session in order to establish a shared understanding of what was happening in the classroom. Before the implementation, they negotiated the purpose of the following lesson, follow-up questions, and envision related to students' progression. Moreover, the teacher and the researcher met after each lesson in order to discuss what went well and what should have been improved. Thus, the essential revisions related to the HLT and the instructional sequence were discussed. In addition to these debriefing sessions, the research team also met weekly in order to decide on modifications and revisions considering the week's results and the goal of implementation for the next week. All team meetings were audiotaped.

Furthermore, the team continued these meetings after the classroom teaching experiment. They discussed the analyses of classroom mathematical practices together in order to verify or refute the argumentation schemes. Finally, they revised the implemented HLT through retrospective analyses and got the final version of the HLT for multiplication in second grade. This procedure is explained in the section of Conclusion and Discussion.

3.4.7. Researcher's field notes

In this study, field notes were recorded to determine whether the teaching and learning processes were enacted in class as intended. In educational research, these usually mean the detailed notes researchers take in the educational setting (classroom or school) as they observe what is going on or as they interview their informants. They are the researchers' written account of what they hear, see, experience, and think in the course of collecting and reflecting on their data see (Fraenkel et al., 2012). In this sense, during the implementation of the instructional sequence, the researcher recorded what she observed in order to document the class activities. The researcher used an informal language like writing a diary just to give feedback to herself and also design team. For instance, for the activity related to array, the researcher made comments given in the second column of Table 3.5.

Table 3. 5. The example of the field notes from the 18th lesson

Task	Observations & Comments
	We used arrows to direct students to group the dots vertically. We expected students to count row. However, some students grouped the dots as columns  like this: We directed students to think in rows. However, we also approved the students who think in columns. Because it is also possible to group the dots horizontally. Thus, why did we direct them to rows through arrows? I think, we shouldn't show the way of grouping the dots. We should leave them free. Therefore, we can make the ones who think vertically and those who think horizontally share their solutions and discuss. We should discuss on my observation as the design team. *The activity took 20 minutes. It is more than expected. *Two students realized and told the researcher that it does not matter how to circle the dots. He said that the product is same for both vertical and horizontal grouping.

The duration of the activity, the mathematical activity that was going place, and the notes regarding the discourse were all written down in the field notes. The goal was to record as much of the classroom discussion as possible to support the video recordings for triangulation. Moreover, these field notes were presented to the team as a starting point for formative evaluation. In the light of these notes, the team made refinements in the instruction when it was needed.

3.5. Data Analysis

The purpose of this section is to describe a method for monitoring and documenting students' collective activity which refers to the classroom community's normative ways of reasoning. While methodologies to document learning of individuals are available in the literature in detail, the methodologies to describe the intellectual activities of classroom cultures are so limited (Rasmussen & Stephan, 2008). In this respect, design research has theoretical and pragmatic concerns on documenting collective ways of reasoning that progress as students involve in mathematical activities. In order to reveal students' normative ways of reasoning, a large amount of data like video and audio records of classroom sessions and audio records of meetings with research team was collected through variety of data collection tools in this study. While these data were being continuously evaluated as part of the ongoing analysis process, the retrospective analysis was processed to understand the taken-as-shared ways of learning of a classroom community and document the mathematical practices.

As it is stated previously, the research question seeks for classroom mathematical practices that emerged as second grade students engaged in the instructional sequence for multiplication. In order to answer the question, classroom discussions and students' arguments were planned to be analyzed as the retrospective analysis. With this purpose**,** the classroom videos were transcribed and organized properly as the starting point. To be able to analyze classroom discourse sessions, it was planned to use an argumentation model. Eventually, Toulmin's argumentation model adapted by Rasmussen and Stephan (2008) in order to analyze classroom argumentation and their three-phase method to document taken-as-shared ideas and mathematical practices were used. This process of analysis method includes three phases. In the first phase, the videotapes of every class session were watched, and the instances in which a claim is made were noted, and the whole class discussions were coded according to Toulmin's argumentation model. Therefore, it is important to explain this model at first.

Each phase is unique and requires different actions in themselves. In the first phase, the videotapes of each classroom implementation were transcribed to be analyzed through a retrospective analysis process in order to understand the taken-as-shared ideas that emerged during classroom implementations and document the mathematical practices. Classroom discussions were coded according to adaptation of Toulmin's argumentation model (1958, 2003) serving as an analytic tool to examine the structure of the arguments. At that point, it is crucial to describe and clarify this model.

Toulmin's basic argumentation model has three components: claim (C), data (D), and warrant (W) (2003) as in Figure 3.24. The first component, a claim, is the conclusion of the discussions offered as correct by the learners. In another saying, a claim is the students' statement or opinion, which serves as the principal expression of the argument and stemmed from the data. To reach the claim, students use the data which serves as the foundation of the argument. Hence, the data which grants the argument some validity justifies the main point of view. The connection between the data and the claim is set with the warrant by explaining and justifying why the data are considered to support the claim. Moreover, the arrow in the Figure 3.24. represents the direction of evolution of the claim taking the step from one to the other. The warrant is placed immediately below the arrow as the explanation of the relationship between data and claim (Toulmin, 2003).

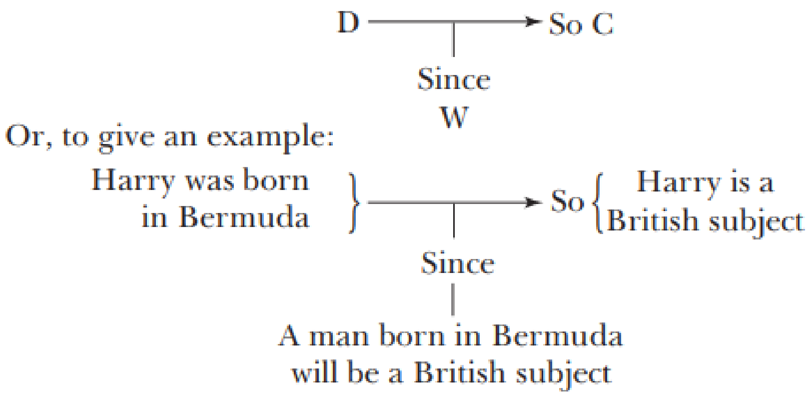


Figure 3. 24. Schematic representation of Toulmin's basic model and an exemplary argument (Toulmin, 2003, p. 92).

In addition to the basic model, Toulmin explains more detailed components for more complex arguments. These are qualifier (Q), rebuttal (R), and backing (B) (Toulmin, 2003). A qualifier refers to the strength conferred by the warrant, while a rebuttal (R) denotes conditions in which the warrant's general authority would have to be overruled. Moreover, backing supports the authority and currency of a warrant (Toulmin, 2003). The places of all components (D, C, W, Q, R and B) in an argumentation scheme are represented in Figure 3.25 below.

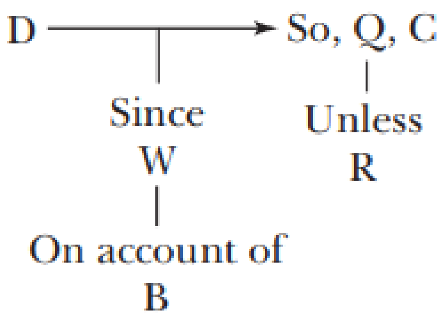


Figure 3. 25. Schematic representation of Toulmin's extensive model of argumentation (Toulmin, 2003, p. 97).

As in Figure 3.25, a qualifier is inserted in front of the claim since it shows the degree of force which the data confer on the claim to advantage the warrant. On the other hand, a rebuttal is inserted below the qualifier since it may refute the claim. Finally, a backing is placed under the warrant as funding. A sample argumentation including all the components of an argument is given in Figure 3.26 below.

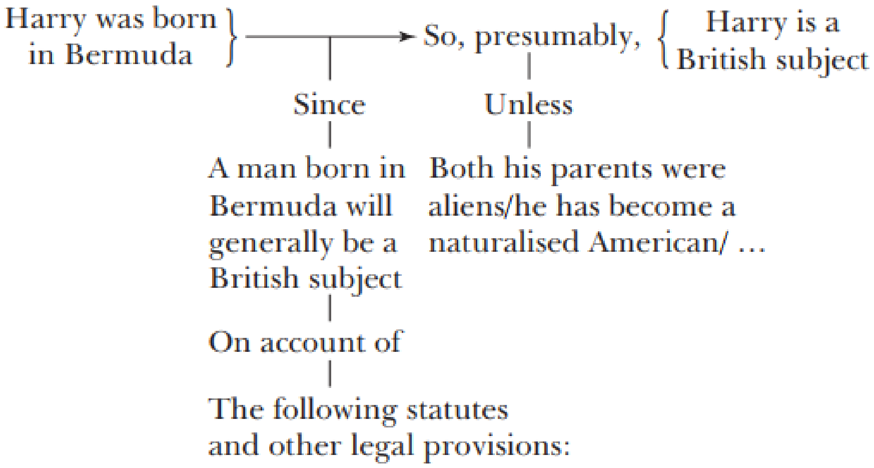


Figure 3. 26. Sample argument with all elements of an argument (Toulmin, 2003, p. 97).

Despite the fact that Toulmin (1958) provides this argumentation model for the area of law and for individual argumentation processes, he emphasizes this model can be used in other areas since some aspects of it are field independent. Depending on this idea, Krummheuer (1995) adapted Toulmin's model to investigate collective argumentation in classroom settings. This adapted model has two critical features. One of these features is that the improvement of an argumentation theory is specified for mathematics education. The other one is that the reasoning is expanded from an individual to a collective level. That is to say, Krummheuer focused on the social dimensions of the argumentation process in the mathematics classrooms, viewing it as a social phenomenon in which students provide justifications for their reasoning and make changes to their arguments (Krummheuer, 1995). This model of argumentation involves four components of conclusions (or claims), data, warrants, and backing as in Figure 3.27 below.

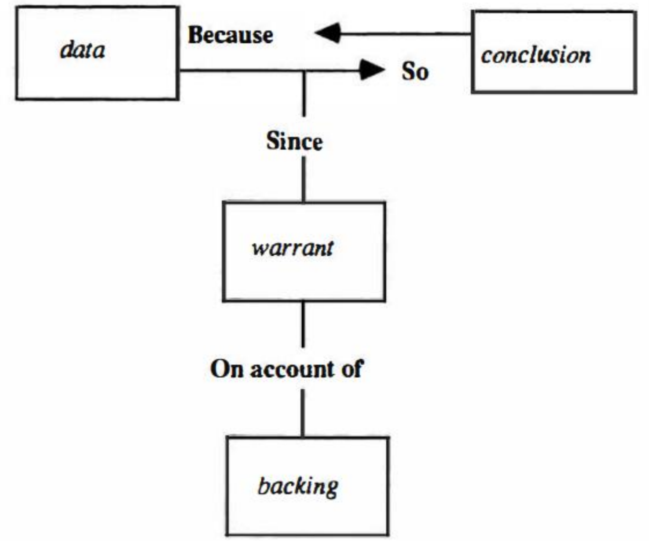


Figure 3. 27. Schematic representation of the model of argumentation in mathematics education (Krummheuer, 1995, p. 248).

As seen in Figure 3.27, the conclusion and the data in the adjacent boxes have arrows moving from one to the other to indicate the relationship between them. The warrant is displayed below the conclusion and data, providing justification for the connection between the pair above. Moreover, it should be noted that the term "since" clarifies how the warrant might be interpreted in the context of the argument (Whitenack & Knipping, 2002). Finally, the backing is contained in the box underneath the warrant, indicating the role of the backing in providing additional support for the backing. The term "on account of" exemplifies how support can be identified during a debate.

The model of a sample argument is provided in Figure 3.28 below. According to this argument, suppose that during a whole-class discussion students are asked to explain their solution for 4x4.

They explain that 4x4=16 because 8+8 is 16 considering two sets of fours as eight. The first part of this statement (4x4=16) is the conclusion (claim), that is, a mathematical claim while the second part of the statement (8+8=16) is the data, the initial information that supports or grounds their conclusion. If the connection between data and conclusion is not clear to others and they question how the data support the conclusion, the warrant is presented. In the given example, students explain that 4x4 represents four sets of fours meaning there are two more sets of fours. The other explanations of the students below about showing their fingers offer additional information (backing) to support the data (Krummheuer, 1995). The interesting part of argumentation is that the backing is like a general claim of the students and does not require extra support since students' backing is what they believe to be mathematically correct (Whitenack & Knipping, 2002).

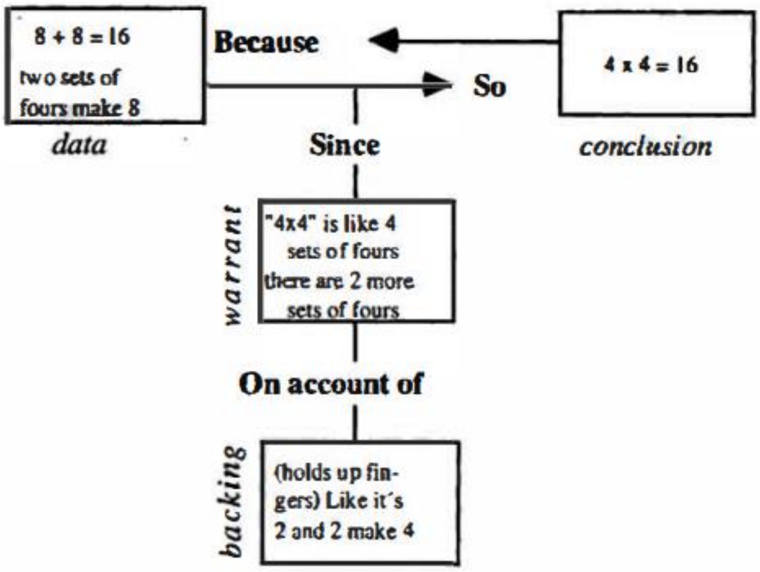


Figure 3. 28. Schematic representation of argument from class example (Krummheuer, 1995, p. 245).

Parallel to Krummheuer's (1995) approach, the videotapes of each classroom implementation in this study were watched, and discussions were coded according to the components of the argumentation model. Claim, data, and warrants were specified for each argumentation scheme. All these inferences were discussed by the research team systematically until agreeing to maintain reliability. Therefore, an argumentation log ordering all the argumentation schemes in succession across all whole-class discussions was obtained.

In the second phase, the obtained argumentation log was analyzed through the following lessons to determine whether the mathematical thinking became the groups' taken-as-shared ideas (Rasmussen & Stephan, 2008). To be able to decide on this, Rasmussen and Stephan (2008) defined two criteria:

1 When the backings and/or warrants for an argumentation no longer appear in students' explanations (i.e., they become implied rather than stated or called for explicitly, no member of the community challenges the argumentation, and/or if the argumentation is contested and the student's challenge is rejected), we consider that the mathematical idea expressed in the core of the argument stands as self-evident.

2 When any of the four parts of an argument (the data, warrant, claim, or backing) shifts position (i.e., function) within subsequent arguments and is unchallenged (or, if contested, challenges are rejected), the mathematical idea functions as if it were shared. For example, when students use a previously justified claim as unchallenged justification (the data, warrant, or backing) for future arguments, we would conclude that the mathematical idea expressed in the claim has become a part of the group's normative ways of reasoning (p. 200).

During the analysis, these two instances were looked for in order to determine if an idea became taken-as-shared thus forming a mathematical practice. Rasmussen and Stephan (2008) propose the researchers a mathematical ideas chart for each class session in order to identify classroom mathematical practices effectively. These mathematical ideas chart includes three columns: "(a) a column for the ideas that now function as if shared, (b) a column of the mathematical ideas that were discussed and that we want to keep an eye on to see if they function subsequently as if they were shared, (c) a third column of additional comments" (p. 200). There is an example represented in Table 3.6 produced for the current study below. This table is only one of the charts that were created for each day of the experiment. On subsequent days, the entire set of tables was utilized to see if the concepts in the second column (i.e., keep an eye on) shifted to the first column (i.e., taken-as-shared). The results about the mathematical ideas being taken-as-shared were reached as a result of comparing the elements of the mathematical ideas chart. Finally, fifteen taken-as-shared ideas were identified over the five-week instruction.

Table 3. 6. Sample mathematical ideas chart

Ideas that function as-if-shared	*Ideas to keep-an-eye-on*	*Additional comments*
Reasoning the effect of change in the order of numbers skip counted	Some of the students apply counting on strategy instead of counting all the groups or number from the beginning of counting process while others cannot use their previous knowledge, but count all counting process.	It is related with reasoning the effect of change in the first number (multiplier) in a multiplication operation

In the third phase of the analysis, the mathematical ideas charts were collected, and taken-as-shared ideas were listed to be organized according to common mathematical activities serving as classroom mathematical practices (Rasmussen & Stephan, 2008). Classroom mathematical practices were characterized as this general mathematical activity. For instance, it was observed that the students started skip counting by using their fingers fluently and effectively to find the ordered or grouped numbers. Later on, they shifted from counting all the numbers in a counting sequence to find the ordered numbers to counting on the number whose order they know. Then, they made connections between different place of two numbers in a number sequence. They developed a strategy to find the order of numbers in a number sequence by reasoning the order of previously established numbers. Therefore, these two mathematical ideas were put together and organized around the common activity of reasoning with fingers to skip count as they emerged and became taken-as-shared. Thus, the first mathematical practice in this study was called “using skip counting to find the total number of groupable objects”. Finally, fifteen taken-as-shared ideas that were identified at the end of the second phase were classified and five classroom mathematical practices, including the one mentioned above, were documented in this study. All of these taken-as-shared ideas and classroom mathematical practices were explained in detail in the section of Findings.

3.6. Trustworthiness

One of the main purposes of researchers is to consider validity and reliability issues for all kinds of studies. In short, validity refers to whether or not we are measuring what we plan to measure while reliability refers to the researcher's independence (Bakker & van Eerde, 2015). These concepts of validity and reliability in qualitative research differ slightly from those in quantitative research, as they do in the current study. In a qualitative study, these terms are replaced respectively with credibility, transferability, dependability, and confirmability (Lincoln & Guba, 1985). They are discussed in this section considering how they were addressed in this study.

First, credibility, which is related to internal validity, is defined as "belief in the 'truth' of an inquiry's conclusions for the respondents with whom and in the context in which the inquiry was conducted" (Lincoln & Guba, 1985, p. 218). That is to say, credibility is about how well the findings of the study match the reality (Merriam, 2009). It also refers to the quality of data collections and arguments (Bakker & van Eerde, 2012). Four different types of methods were employed to ensure the internal validity of the current investigation. One of them is prolonged interaction engagement, which calls for spending a lot of time with participants in order to accurately comprehend their behaviors and discourses

(Creswell & Miller, 2000; Lincoln & Guba, 1985; Merriam, 2009). This method helps understanding the culture of the participants, detect misinformation given by distortions either of the self or of the respondents, and create trust. For the current study, it can be said that the researcher was familiar with the students and the classroom teacher. One year before the current study, another design research was conducted in this classroom when the students were first graders. The researcher spent 2 semesters with the students for that project. The researcher was familiar with the students' cognitive and affective levels, their capabilities, characters, and their reactions since she was with them from the time they just started school. The students and the teacher were familiar with the researcher too. They were used to her presence in the classroom while she was setting the camera, walking around, and asking questions. It can be claimed that their accepting the presence of the researcher in the classroom helped to reduce the observer effect which means the impact that an observer's presence might have on a subject's behavior (Fraenkel et al., 2012). As a result, prolonged engagement enabled both the researcher and the participants to get used to and understand each other. This relationship helped them to implement the instructional sequence in the classroom effectively and interpret the arguments in the classroom accurately.

Another method for establishing credibility is peer debriefing (Creswell & Miller, 2000; Lincoln & Guba, 1985; Merriam, 2009). In the current study, members of the research team were involved in all the stages of the study. While the teacher was the facilitator in the classroom, the first researcher was also in the classroom and took field notes while participating in discussion sessions. After the lessons, they discussed what worked and what did not work in the classroom and also the field notes. They also shared and discussed their decisions with the second researcher who is the professor of Mathematics Education. Moreover, a detailed peer examination was held with the involvement of team members by scanning some of the raw data and assessing whether the findings were plausible based on the data.

The other method for establishing credibility is triangulation which is the most well-known method of ensuring a study's internal validity (Creswell & Miller, 2000; Lincoln & Guba, 1985; Merriam, 2009). A variety of triangulations are supported by the literature, including multiple methods, multiple sources of data, multiple investigators, and multiple theories (Denzin, 1978). In the current study, data triangulation was employed to assure validating and cross-checking findings. Various tools such as interviews, observation, and field notes were used to collect data in order to develop and test the conjectures of the study. Additionally, the professor in the design team closely monitored the process of data collection and interpretation during the team meetings and provided her insightful thoughts on methodological and analytical concerns. In addition to data triangulation, investigator triangulation was also used in the current study. All the members of the team joined in the analyses of the big data collected through these tools to maintain credibility. They discussed the argumentation schemes, and following it, they either came to a consensus on them or came up with new argumentation schemes as a matter of reliability. At this point, it should be stated that the investigator triangulation decreased the observer bias, which refers to the potential for particular traits or viewpoints of observers to influence what they see (Fraenkel et al., 2012).

While collecting data, the researcher’s position is also taken into consideration to ensure credibility (Merriam, 2009). Being involved in a design research team has several benefits. For instance, it may offer more accurate information about the complexity of the problem at hand, and more intensive discussions about the intervention's requirements (Nieveen & Folmer, 2013). In line with this characteristic of design research, the researcher had an interventionist role in the study. As a result, she was responsible for recording and observing teaching-learning sessions, taking notes, interacting with students and the teacher, leading argumentations in cooperation with the teacher, and intervening in the flow of instruction when students required additional answers or had some questions.

Indeed, the researcher was an active participant in all phases of the study. This active involvement in preparing for the experiment, implementing the experiment, and retrospective analysis enabled the researcher to grasp the study as a whole and to analyze the study based on her own experiences in the field. While this is considered to be a valuable experience for the researcher, there is a contrasting idea suggesting that the researcher may become overly 'connected' to the design, resulting in a less objective approach to problems and responses from respondents (Nieveen & Folmer, 2013). Thus, this may cause being biased during the evaluation; however, various ways of triangulation were used in the current study to eliminate the probable bias of the researcher.

In this study, the researcher attempted to read up on various types of qualitative research, as well as significant problems affecting research quality. Moreover, it should be underscored that the researcher was not the only one who designed the HLT and the instructional sequence and collected and analyzed the data. As previously explained, all the members were involved in all the steps of the study. Thus, the past experience and disposition of the researcher were not the concern for the current study. Moreover, as the researchers conducted another study in the previous year, they were familiar with the nature of classroom teaching experiments and unbiased in each part of the study.

Second, transferability, which is related to external validity, refers to the applicability of the design experiment in other settings (Bakker & van Eerde, 2015; McKenney & Reeves, 2012; Merriam, 2009). Transferability is established by providing a detailed account of the environment, period, and people involved in it, so that others who might be interested in applying the findings elsewhere can assess the likelihood of doing so (Merriam, 2009). In this sense, thick descriptions of the study's methodology and findings are provided. For instance, data collection tools and procedures are explained comprehensively in detail in the related sections above. Although the participants, classroom, and school are particular to this study, their characteristics are also delineated clearly for further research. Moreover, interpretive frameworks, the context of the study, revised

HLT and instructional sequence, and the way of analysis are provided in the method section in detail in order to assist the researchers in estimating the extent to which the stated scenario can be transferred to their own studies. Transcripts from the classroom and team meetings are provided in order to make the context and the theoretical claims clear and comprehensible. In this sense, the limitations are also mentioned to reveal the threats to generalizability. Indeed, the current study is presented in detail considering each step.

Third, dependability, which is related to reliability, refers to the consistency between the data and the inferences drawn from the data (Merriam, 2009). Merriam (2009) offered four suggestions for boosting a qualitative research study's dependability: audit trials, peer review, triangulation, and researcher position. Peer review, triangulation, and the researcher position were attained in this study. Two members of the team coded and interpreted the data together through negotiating. Moreover, randomly selected portions of the data were examined by a colleague to sustain dependability. All of these ideas were previously covered in order to guarantee the study's credibility. In a qualitative study, an audit trail is followed to explain how data were gathered, categories were created, and choices were made throughout the investigation in detail. The researcher keeps a research journal or makes memos on the study process as it is being carried out in order to build this trail (Merriam, 2009). In this sense, the researcher took notes during the teaching experiment and shared these notes with team members in daily and weekly meetings. These field notes and records of the meetings were kept as advised by Merriam (2009). The transcripts of these meetings are also provided in the related section of the methodology chapter.

Similarly, confirmability, which is related to internal reliability, refers to the consistency in the study. Since the results of qualitative investigations may vary from person to person, this issue is viewed as troublesome (Merriam, 2009). In this sense, an audit trail is suggested to maintain confirmability (Merriam, 2009; Patton, 2002). To preserve the confirmability, as explained above, a full collection of documentation is provided. The data gathering and analysis procedure are explained in this context, and operational and technical details are documented. Moreover, all these techniques used for credibility and dependability also supported and contributed to the confirmability of the study.

3.7. Role of the researcher

One advantage of the design research methodology is that it encourages the researcher to take on a variety of roles (Zhang et al., 2013). In this structure, the researcher was intimately involved in all aspects of the design and evaluation of the HLT and the instructional sequence. This means that she had several roles in this study.

In other words, the researcher -as a member of the design team- served as a designer, implementer, and evaluator in the process of developing and evaluating the intervention. Moreover, she took on responsibilities like observer and facilitator in classroom observation. She attended all the classroom sessions to observe the students and the teacher and took notes. During the implementation, she facilitated students besides the teacher and assisted the teacher to apply the intervention as it should be. Thus, the researcher endeavored to gain deeper insights into the strong and weak sides of the intervention.

In addition to obtaining a lot of information about the intervention, the researcher gained various new competencies from her responsibilities and deepened her understanding of the implementation. The researcher was ready to perform the additional roles of designer, advisor, and facilitator while maintaining focus on her primary duty as a researcher. She was tolerant with the frequently unavoidable blurred lines between her roles and kept the research design flexible because the project process required it. Moreover, she permitted the study to be impacted by the needs and wishes of the partners when a collaboration that is typically ongoing for a long time is present.

3.8. Limitations

In this section, information related to the limitations of this study is provided. To begin with, the findings of the current study are less generalizable with the other contexts since it was conducted as qualitative research. On the other hand, the cyclical nature of design research allows for some generalizability. Thus, the generalizability of the study can be increased by developing and using the cycles with other second graders from other schools.

Furthermore, the evolution of mathematical practices was analyzed by using Toulmin's argumentation model (1958). This model helps to document collaborative, taken-as-shared mathematical ideas, not individual learning. In this sense, it should be stated that individual student learning was used to investigate whole-class interactions, despite the fact that individual student learning analysis is beyond the scope of this study.

Also, another limitation would be that the study was conducted with only one macrocycle. It would be appropriate to conduct a pilot study to get more accurate findings prior to the main study. However, design research had been conducted with the collaborating teacher and her classroom instead, which eventually played the role of a pilot study. Consequently, the instructional sequence of the study was carefully planned over a long period of time by consulting with other mathematics teachers, and researchers and asking for their feedback.

3.9. Ethical Considerations

Before starting the design experiment, the compulsory permissions were taken from Middle East Technical University Human Subjects Ethics Committee (see Appendix A) and the Head of Elementary Mathematics Education program of the university. After the approval of the related departments in the university, official permissions were taken from the Ministry of National Education (see Appendix B) since the study was planned to be conducted in a public school.

After getting the necessary permissions, the collaborating teacher and the principal of the school were informed about the nature of the design experiment. They were ensured that their identities would be kept private. To keep their rights, the informed consent form was used (see Appendix C).

Moreover, students were informed about the lessons and data collection procedures. In order to get permission from the parents of the students, a classroom meeting was held with parents, collaborating classroom teacher, and the researcher. Parents were informed about the nature of the design, the purpose of the study, the data collection procedure, and confidentiality. They consented for their children's participation in the study since they were satisfied with the design experiment conducted in the previous year. In order to collect data via audio and video recordings, the parent permission form was used (see Appendix D).

CHAPTER IV

FINDINGS

The first purpose of this study is to develop, test, and revise a Hypothetical Learning Trajectory for teaching multiplication in second grade. For this purpose, an HLT was developed and tested through a classroom teaching experiment. The HLT was revised through daily implementations. These procedures and the initial HLT are provided in Chapter 3 in detail. Furthermore, the HLT was revised in the light of the classroom mathematical practices analyses. The revised HLT is also given in Chapter 5.

The second purpose is to document second graders' classroom mathematical practices (CMPs) that emerged through a five-week instructional sequence about multiplication. This chapter is devoted to the second purpose to explain the findings related to classroom mathematical practices extracted by Toulmin's model of argumentation. In this sense, mathematical ideas' charts were evaluated to examine students' mathematical ideas that became taken-as-shared and classroom mathematical practices over the course of 26 class periods.

Collective argumentations and activities that evolve in the classroom were analyzed through a three-phase approach (Rasmussen & Stephan, 2008). Particularly, students' ways of reasoning with informal knowledge, using intuitive models related to multiplicative reasoning, communicating, connecting to realistic situations, and mathematization during social interactions within the classroom are examined and documented. Several taken-as-shared ideas (TAS) and five mathematical practices were obtained over five-week instruction.

Five mathematical practices arose throughout the course of the 26 days of the classroom teaching experiment. The method by which these practices were developed was highly complicated because the mathematical ideas they contained emerged in a network-like fashion rather than linearly. In other words, while the instructional sequence invited students to create a web of ideas, there was not always a clear distinction between practices. For example, during the same class period, more than a particular mathematical idea emerged and contributed to a different practice. Due to this, the classroom mathematical practices overlapped, as shown in Figure 4.1 below.

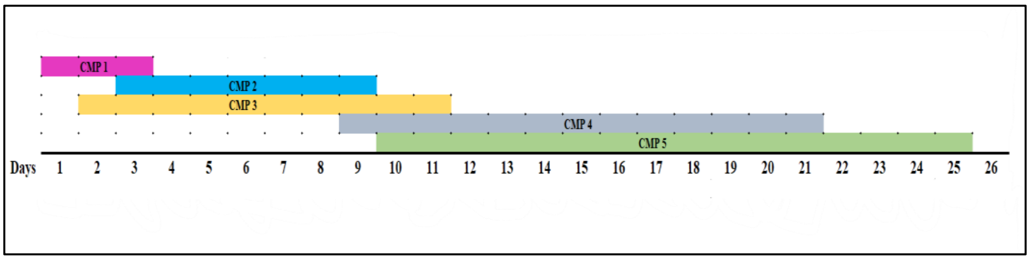


Figure 4. 1. Overlapping classroom mathematical practices.

According to the figure, there are bars for each classroom mathematical practice. The position of the bars shows the days when the particular ideas of the related practice were first initiated and when these ideas were taken-as-shared. For instance, on the third day of the teaching experiment, an aspect of the first mathematical practice was taken as shared while an aspect of the second mathematical practice was first initiated. On the same day, an aspect of the third mathematical practice was in the process of becoming taken-as-shared. Because of this complexity, these five mathematical practices are covered separately in order to make the chapter easier to read.

These taken-as-shared ideas and mathematical practices (Table 4.1) are described in the following part in detail. For each taken-as-shared idea, Toulmin’s Analysis scheme was provided. In addition, the researcher also produced illustrations of the students' justifications and reasoning in order to make the mathematical ideas more clear. All the illustrations are the researcher’s works.

Table 4. 1. Five classroom mathematical practices and taken-as-shared ideas supporting these practices

CMP 1: Reasoning with fingers to skip count Idea 1: Skip counting by using ordinal aspects of fingers Idea 2: Finding the order of numbers in a number sequence by reasoning the order of previously established numbers
CMP 2: Partitioning objects into equal groups to add them repeatedly Idea 3: Reorganizing collections by using math drawings in order to use skip counting to find the total number of objects. Idea 4: Equipartitioning collections into equal-sized groups through building up strategies Idea 5: Halving to reproduce equal-sized groups Idea 6: Distributing left over numbers (remainder) to equal groups by conserving equality
CMP 3: Iterating linked units by using pictures and fingers Idea 7: Skip counting the groups (composite units) in order to iterate Idea 8: Assigning hands (5 or 10 fingers) as composite units to iterate Idea 9: Double matching collection of equal groups to iterate on via pictorial representation
CMP 4: Analyzing components and properties of multiplication operation by modeling with equal groups and arrays Idea 10: Interpreting the meaning of multiplier and multiplicand in equal group representation Idea 11: Connecting repeated addition and multiplication operations by interpreting multiplier and multiplicand Idea 12: Reasoning the effect of change in multiplier on the product Idea 13: Analyzing arrays by interpreting rows and columns as multipliers and multiplicands Idea 14: Reasoning commutative property by using arrays
CMP 5: Writing contextually realistic problems by coordinating the relationship among multiplicative representations Idea 15: Analyzing the multiplier and multiplicand to pose multiplication problems on known contexts Idea 16: Interpreting the multiplier and multiplicand to pose multiplication problems as repeated addition Idea 17: Interpreting the multiplier and multiplicand to pose multiplication problems as rate Idea 18: Focusing on structure and keywords in the problems to conceptualize multiplication

4.1. Classroom Mathematical Practice 1: Reasoning with fingers to skip count

The first mathematical practice emerged as students engaged in the instructional sequence conducted in light of the designed HLT: reasoning with fingers to skip count. In this practice, students skip counted using their fingers to represent the order of verbal count, construct and recite number sequences, and establish a relationship between the two numbers in the sequence by comparing their places (order). Specifically, two ideas became taken-as-shared as the students engaged in counting and finding the number of objects, especially in the first two weeks of the instruction:

- Skip counting by using ordinal aspects of fingers
- Finding order of numbers in a number sequence reasoning the order of previously established numbers

The importance of skip counting is stated as to be built upon to strengthen understanding of multiplication facts. For that reason, students begin to skip counting even in kindergarten and become familiar with the rhythm and order of these series of numbers (Schoenfeld et al., 2017). In the mathematics education program in Turkey, students count by 1s, 5s and 10s in the first grade, by 2s, 3s, 4s, 5s, and 10s in the second grade, and by 6s, 7s, 8s and 9s in the third grade. For that reason, the instructional sequence started with skip counting exercises related to 2s, 3s, 4s and 5s. That is to say, students were asked to count by 2s (within 20), 3s (within 30), 4s (within 40), and 5s (within 50). These activities aimed to promote their ability and fluency in skip-counting to gain their skip counting ability and make them ready for the following activities and reasoning. While engaging in these activities, follow-up questions were posed by the teacher, and students developed various strategies, justifications, and reasoning. It was observed that students accompanied verbal counting by their fingers. They used their fingers as representatives of ordinal numbers.

4.1.1. Idea 1: Skip counting by using ordinal aspects of fingers

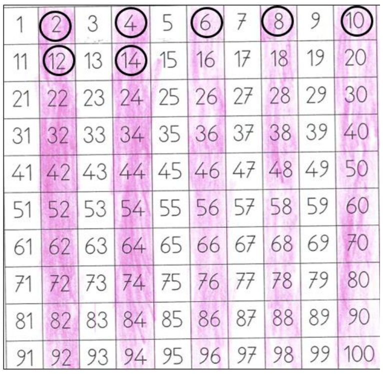
The first mathematical idea included in the first mathematical practice emerged at the beginning of instruction while students engaged in skip counting activities. On the first day, students were provided a hundred charts for each skip counted numbers of 2, 3, 4 and 5 separately. They were asked to skip count by 2s, 3s, 4s and 5s and color the numbers in these number sequences. After finishing, they counted loudly (verbal counting) and checked whether they colored right or wrong. Following that, students were asked questions related to orders of numbers in a number sequence like "what is the 6th number when you count on by 3s starting at 3". While answering this type of question, students counted their fingers. Below is a dialogue showing how students used their fingers:

> Teacher: What is the 7th number you count on by 2s starting at 2?
> Zehra:14
> Teacher: How did you find it?
> Gökhan: I counted on by 2s with my fingers until the seventh finger. 2,4,6,8,10,12,14. I found 14.
> Birce: Each finger represents 2. Counting seven fingers, I can find the seventh finger as 14.

As can be seen, students used their fingers to represent the orders of numbers. At first, they closed their hands. While they counted by 2s orally, they opened and showed one finger for each number they voiced. They counted by 2s until opening the seventh finger. When they voiced the seventh finger, they stopped. They gave this final (seventh) number as the answer. As another way, some of the students used the hundred chart to find the answer as below:

> Mahmut: Teacher, can we use the hundred chart? We colored the hundred chart (See Figure 4.2 below). We can count the seventh colored cell and find 14.
> Teacher: Sure, you can use it. If you have filled the chart correctly, you can find the answer on your chart. Otherwise, you may find the wrong answer.
> (A few students agreed with Mahmut. They stated that they counted the cells on the chart)
> Zehra: Mahmut, I think you should use your fingers. You cannot find this chart all the time. However, your fingers are always with you. It is more practical to count with fingers.

(Most of the students agreed with Zehra since they found the seventh number by counting their fingers)



1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 4. 2. Mahmut's way of finding the seventh number on hundred chart

As can be understood from the dialogue, Zehra commented on Mahmut's solution and directed him to count his fingers instead of a hundred charts. Moreover, most of the students approved Zehra' explanation. In the following question, the teacher and the first researcher observed that the ones who had used the hundred chart were trying to count their fingers. They followed how finger counting was taken-as-shared by the classroom community daily. Students helped each other to their fingers to skip count. Their counting methods for 2s, 3, 4, and 5s are illustrated below (Figure 4.3).

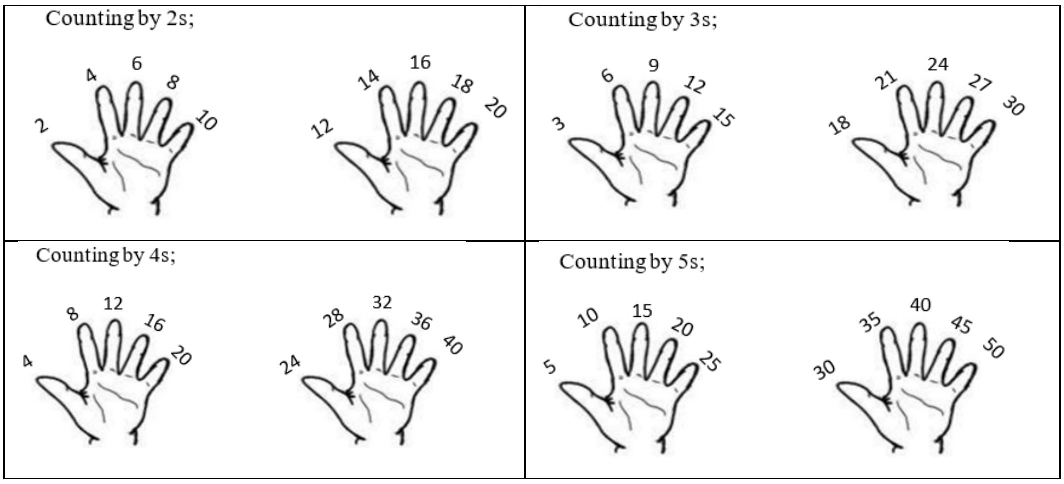


Figure 4. 3. Skip counting with fingers

On the second day, the teacher asked each student to answer the questions in the booklet and explained to the class that he was observing each student's reasoning and interaction within the classroom. As shown below, students helped each other count through the verbal chain of multiple numbers.

Classroom: Kadir, it's your turn.
Kadir: I couldn't do it (What is the 6th number when you count on by 4s starting at 4)
Ajda: It's easy. You can do it.
Ilker: You will count your fingers.
Meliha: Kadir, start with 4 and count by 4s. Tell us the sixth number.

In the other questions, it was observed that most of the students were using their fingers while counting. This reasoning was noted down to keep whether it will be taken as shared in the classroom community in the following days.

On the second day, the form of the questions also changed. Instead of asking just the nth number while skip counting, the order of a given number while skip counting was asked. An example related to this type of question and classroom dialogue is given below:

Teacher: What is the order of 15 while counting on by 3s starting at 3?
Doğan: We should count 15 fingers by 3s. It is 45.
Ali: I am confused. Teacher, can you ask again?
Teacher: What is the order of 15 while counting on by 3s starting at 3?
Hakan: We won't count 15 fingers.
Birce: 3, 6, 9, 12, 15 (counts by 3s and opens one finger for each number). 15 is the 5th number because I found 15 in the 5th finger. We used the same method, actually.
Hakan: It is the reverse of the previous questions. We should count until 15.
Ali: It is given to counting by 3s. We will count by 3s but until 15.

As can be deduced from the dialogue, Hakan realized that the form of the question was more complicated and reversed than the previous ones. However, the roles of the fingers and the numbers voiced did not change. Birce used her fingers as the order of the numbers counted gain. She opened a finger for each number as before. When a Toulmin analysis was conducted on this dialogue, Birce's solution was considered as data to the claim "15 is the 5th number". The explanation of Hakan and Ali about counting by 3s by using fingers was regarded as a warrant to Birce's data and claim. This interpretation is presented with Toulmin's analysis scheme presented in Figure 4.4 below:

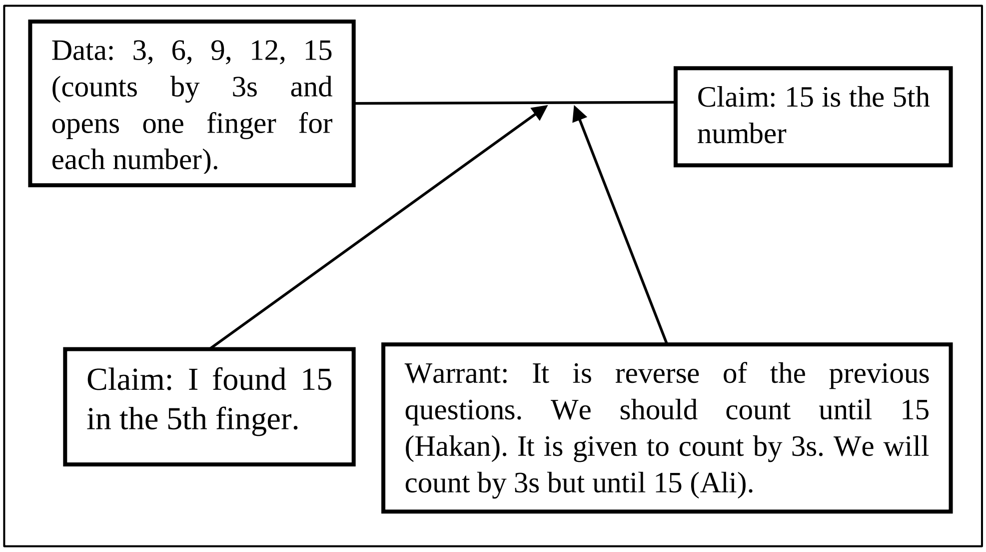


Figure 4. 4. Toulmin's Analysis scheme regarding skip counting fluently by using fingers

Students were asked questions directly related to skip counting, and the order of numbers skip counted in the first two days. Such questions aimed to prepare students for the following activities and encourage them for faster and flexible counting. As can be deduced, students used their fingers to follow the order of numbers skip counted. The mathematical idea related to skip counting with fingers fluently was noted to keep an eye on to see if it will be shared in the following lessons.

In the following lessons, it was observed that students were using their fingers to find the total number of objects asked in the problems. After reading and understanding the problem, students decided on the role of the numbers in the problem as skip counted number and the ordinal number. For instance, in the following activity, students were given repeated addition operation "3 + 3 + 3 + 3 + 3 + 3 + 3 =…" to solve. Students decided that the fingers represent the number of threes. They showed seven fingers and counted them by 3s to find the total sum (see Figure 4.5).

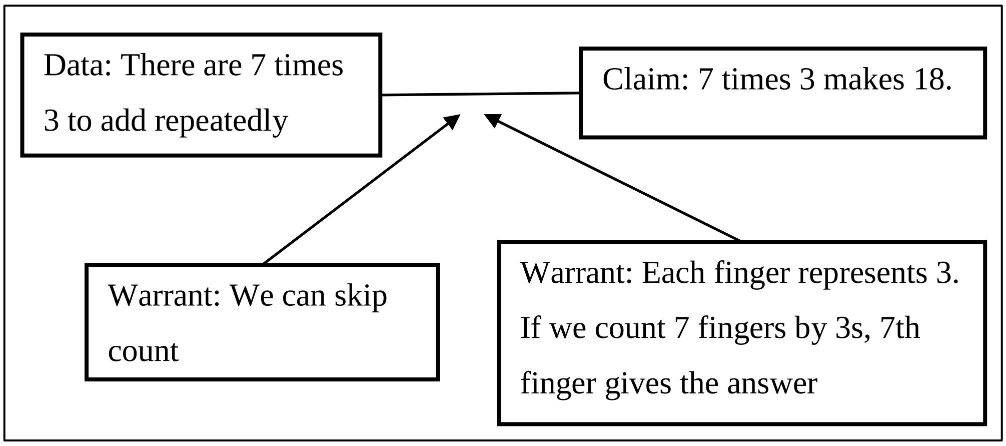


Figure 4. 5. Toulmin’s Analysis scheme regarding skip counting fluently by using fingers to solve repeated addition.

In the following days, students did not question the idea of skip counting fingers. Students' grade-level fluency standard of skip-counting had been promoted with the help of the sequence of activities over the learning trajectory. Moreover, argumentation analyses showed that this idea became taken-as-shared since students no longer had to justify the role of fingers and the number skip counted. Furthermore, students used this claim as data in their arguments related to the following ideas mentioned below. Students focused on the reasoning procedures of the activities, not on the counting procedure.

4.1.2. Idea 2. Finding the order of numbers in a number sequence by reasoning the order of previously established numbers

In the first lesson, students practiced skip counting, as explained above. Students were asked questions related to the numbers and their place (order) in a number sequence. They developed the idea of skip counting with fingers. They used their fingers to represent the order of voiced numbers while skip counting. During working on the sequenced activities, they improved this idea as the data for a further argument. It was observed that students also used their fingers in a more advanced way. For instance, on the first day, after answering the question "What is the 7th number when you count by 4s starting at 4?" students were asked, "What is the 8th number when you count by 4s starting at 4?". While some students started counting from 4 until the 8th finger (32), the others just added 4 to 28. Those students used the previous question's answer to find the solution instead of counting eight fingers starting from 4 again. As illustrated below (Figure 4.6), they explained the difference between 7 and 8 as 1 representing one more finger. Then, they added 4 to 28 since one finger means counting one more 4.

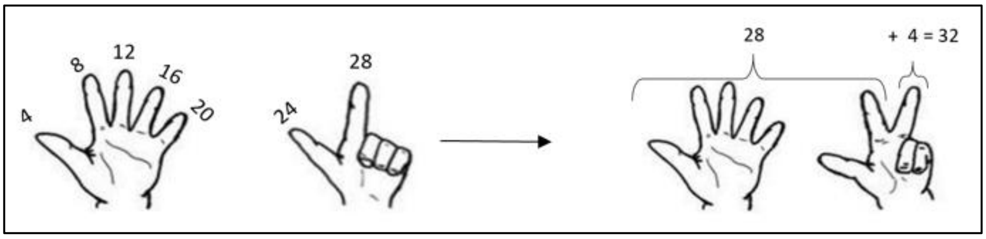


Figure 4. 6. Illustration of finding 8^{th} number by connecting with 7^{th} number while counting 4s

They focused on the order of counted numbers in two questions and compared them. Then, they interpreted the difference between the ordinal numbers and the effect of the difference. This interpretation is presented in Toulmin's analysis scheme presented in Figure 4.7 below:

Data: 7^{th} number when counting by 4s is 28.

Claim: 8^{th} number when counting by 4s is 32.

Warrant: Since the 7th number is 28, we should count one more finger to find the 8th number. Then we should add one more 4 to 28.

Figure 4. 7. Toulmin's Analysis scheme regarding the order of numbers skip counted

As can be deduced from the figures and argumentation scheme, the previous idea (TAS 1) regarding skip counting fluently by using fingers was taken as shared since it shifted place from claim to data, warrants and backing related to this idea

(TAS 2). That is to say, students found the seventh number by counting seven fingers by 4s and claimed that it is 12. Then they used this claim as the data for the further argument and built on this data to find the eighth number. Students' interpretations related to the order of the numbers were observed even in the following lessons. In the second lesson, it was decided to ask such questions and observe their reasoning in this regard. It was seen that students interpreted the effect of fingers on the result by making a connection between previous questions, as can be seen below:

> Teacher: What is the 6th number when you count by 3s starting at 3?
> Esra: 3, 6 ,9,12,15,18 (she counted 6 fingers as TAS 1)
> Teacher: What is the 7th number when you count by 3s, starting at 3?
> Gökhan: 3,6,9,12,15,18,21,24 (he counted 8 fingers as TAS 1)
> Ajda: 24? How did you find 24? You counted one finger extra since you were fast.
> Gökhan: If I counted one extra, I should subtract 3 from 24. It makes 21.
> Birce: Actually, there is no need to count seven fingers. We know that 6th number is 18. For the 7th number, we should count one more finger. After 18, it makes 21.

In this exercise, two different reasoning ways (warrants) emerged by Gökhan and Birce (Figure 4.8). Gökhan made a connection between the 8th and 7th fingers. Since he counted eight fingers, he decided to subtract one finger to find 7th finger. Hence, he made an operation by thinking backward. On the other hand, Birce pursued an operation by thinking forward. She knew that the six fingers make 18. Then she added one more 3 to 18 to find the 7th number. Both students accepted that each finger represents 3 (TAS 1) and compared the positions of the fingers. They used the values they knew by comparing them with the new situation.

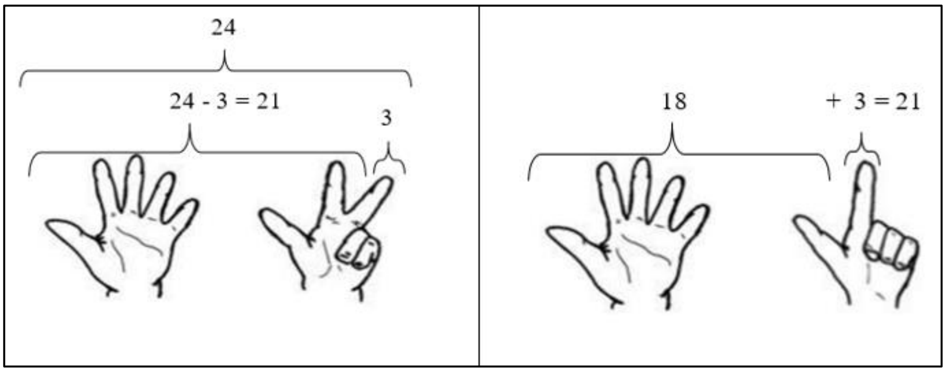


Figure 4. 8. Finger counting strategies of Gökhan and Birce respectively.

In the same way, the teacher continued to ask new questions and let students make connections between them;

> Teacher: What is the 10th number when you count by 3s?
> Esra: 3, 6 ,9,12,15,18,21, 24, 27, 30 (she counted 10 fingers as TAS 1)
> Teacher: Then, tell me the 33rd number.
> Eyüp: It takes too much to count 33 fingers.
> Teacher: Is there another way other than counting that many fingers?
> Ali: 10th number is 30. We can use it.
> Teacher: ıf 10 fingers make 30, we should add two groups of ten fingers, too.
> Ajda: It makes 90. If we count 3 threes, it becomes 99.

Only a few students could interpret the ordinal number of 33 as 3 tens and 3 threes in this question. The teacher helped them to calculate and reason. It was seen that the ordinal number was high for students to manage mentally without formal information of proportional reasoning. For that reason, the research team in the classroom decided to use lower values since it was confusing for the others who did not grasp this idea. The idea had not been taken-as-shared but examined for the following questions and activities.

The teacher asked questions with small numbers to involve all the students into classroom discussion. She asked questions related to counting by 5s, as can be seen below:

> Teacher: What is the 7th number when you count by 5s starting at 5?
> Mahmut: 5, 10, 15, 20, 25, 30, 35 (she counted 7 fingers)
> Teacher: What is the 9th number when you count by 5s?

Karan: We know that 7th number is 35. For the 9th number, we should count two more fingers. 35 plus 10 makes 45.
Egemen: Why did you add 10?
Karan: We should count two more fingers. 5-10 (he pointed two fingers)
Ajda: I found 45 by adding 20 to 25.
Ali: 20 and 25? How?
Ajda: I know that 5th number is 25. For the 9th number, I count four more fingers. Four fingers make 20. Then, 25 +20 makes 45.
Doğan: Cool.
Ali: I found the most interesting strategy.
Teacher: Why do you think that it is the most interesting strategy?
Ali: Because, I did not use addition. I used subtraction.
Zehra: How?
Ali: Actually it is very easy. I know that 10th number is 50. For the 9th number, we should count backward one finger. It means that I will get a five from 50. It makes 45.

In this exercise, again two different directions emerged as counting forward and backward. While some of the students counted on known facts, some of them counted backward from known facts. To make it clear, the strategies of Karan, Ajda and Ali to find the number in a number sequence reasoning the order of previously established numbers are illustrated below (Figure 4.9)

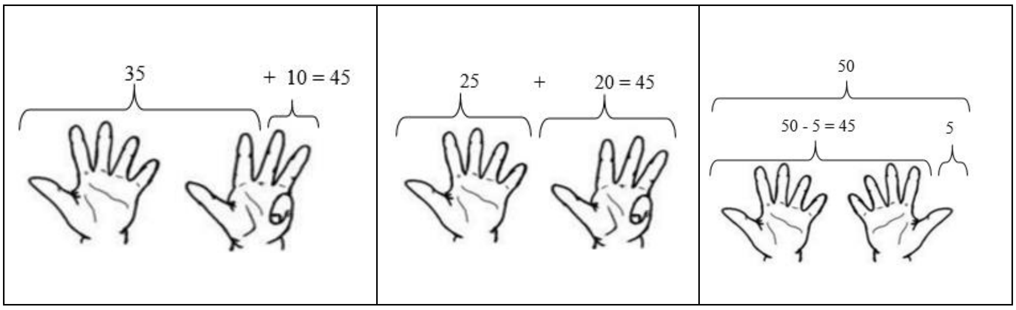


Figure 4. 9. Finger counting strategies of Karan, Ajda and Ali respectively.

In this idea, students compared the ordinal numbers (fingers) and commented on the numbers in a number sequence by reasoning the difference between the orders of numbers. They found the numbers in a number sequence by constructing on previously established numbers in this sequence. In the in followowing days, students did not question how to compare the ordinal numbers to make a connection between numbers in a number sequence. Hence, warrants were dropped off. Furthermore, students extended this idea to another taken-as-shared idea related to the effect of change in multiplier on product

(TAS12). Therefore, it was decided that the current idea was taken-as-shared by the students.

4.2. Classroom Mathematical Practice 2: Partitioning objects into equal groups to add them repeatedly

The second mathematical practice that emerged as students engaged in the instructional sequence was about students' thinking to form equal groups by partitioning the given number of objects. In this practice, students focused on forming equal groups which enable them to count easily and practically by skip counting. They grouped objects through reorganizing, equipartitioning, and halving by using pictures of the objects or drawing these objects in equal groups. Specifically, four ideas became taken-as-shared as the students engaged in this practice, especially during the first two weeks of the instruction:

- Reorganizing collections by using math drawings in order to use skip counting to find the total number of objects.
- Equipartitioning collections into equal-sized groups through building up strategies
- Halving to reproduce equal-sized groups
- Distributing left over numbers (remainder) to equal groups by conserving equality

The second mathematical practice is critical since it emerged as the foundation of other mathematical topics like division, fraction, and proportional reasoning. Conjecturing this evolution, composing activities were presented with decomposing activities. That is to say, students were given a collection of objects and asked how many objects there were. While they developed various ways of counting, the teacher encouraged them to count differently. Hence, students formed various equal groups to skip count them.

4.2.1. Idea 3. Reorganizing collections by using math drawings in order to use skip counting to find the total number of objects

In the third lesson, students were given pictures of some objects. They were asked to find the total number of these objects. They immediately started counting them. When they were asked to share their answers, they explained their thinking by using the given drawings. It was seen that students tended to divide the objects into equal groups to use skip counting. To state this idea and discuss with the classroom community, students were asked to generate different counting strategies for finding the total number. Students circled the objects in the given picture and made equal groups. They focused on the number of the groups and the number of elements in each group. Then, they skip counted the objects in these groups. For instance, students were given pictures of 16 balls and asked how many balls there were. Students' reasoning and drawings (Figure 4.10) are given below:

> Teacher: How many balls are there?
> Halil: I found 16. There are two rows of balls. I circled the 2 balls; one from top row and one from bottom row. I made 8 groups of 2 balls. Then, I skip counted them by 2s.
> (9 students agreed with him and stated that they used the same strategy, too)
> Teacher: Is there anyone who counted by ones?
> (3 students held their hands)
> Ali: We can count by ones. However, it will be easier to count them as groups.
> Teacher: Is there a different solution?
> Zehra: I found 16 but in a different way. I circled each row as a group. I got 2 groups of 8. I added 8 to 8.
> Egemen: We can make groups of 3.
> Gökhan: I tried it. But, we cannot put all of the balls into groups of trees (He drew circles as shown in Figure 10 below). We can get five groups of 3 and one ball as extra. We can use skip counting like 3, 6, 9, 12,15. Then, we should add 1 to 15 and find 16. We can find the total number but cannot use only skip counting. We should also consider the remaining balls.
> Birce: Moreover, we don't say 16 while counting by 3s. (this explanation was noted to be kept an eye on. In the following lessons, this idea became taken-as-shared as TAS 4)
> Eyüp: What about 6? Can we count by 6?
> Doğan: Let's try (he drew circles to show). If we circle six balls to make a group, we get four balls as left over. It is possible to find it by counting by six and then adding 4. However, it may not be obvious. If we could equally put all

the objects in the groups, it would be easier to find the total number of things by skip counting.

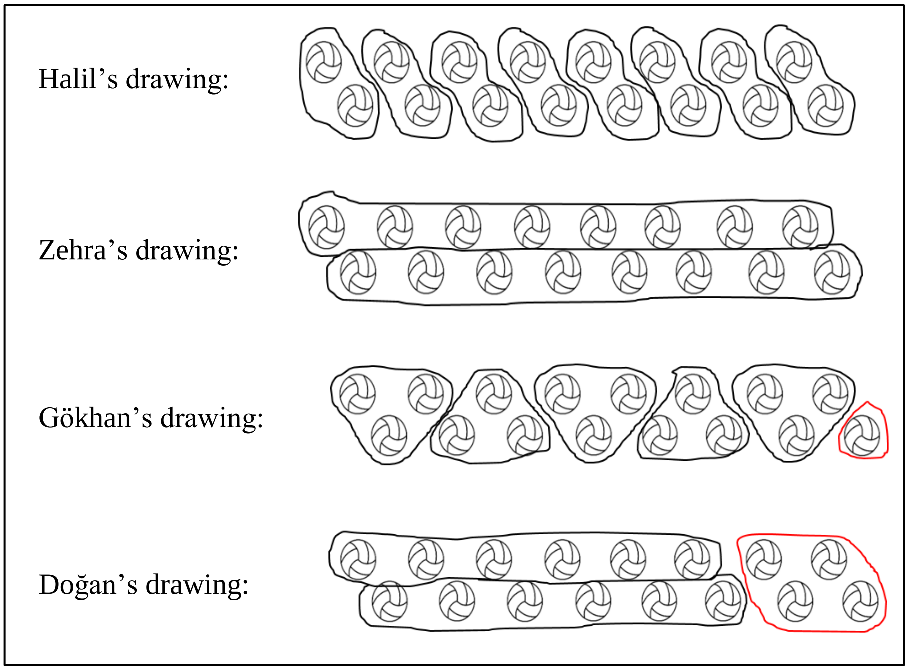


Figure 4. 10. Students' drawings on the given picture of balls

As can be deduced from the dialogue, students were just asked to find the total number of objects. They were not directed to use equal groups. Intuitively, they focused on forming equal groups in the given picture. They circled and skip counted the groups by touching on each group. In this sense, two different strategies emerged equal groups and equal groups with remaining balls. The ones who could divide the balls into equal groups without remaining directly used skip counting (see the drawings of Halil and Zehra in Figure 4.10).

On the other hand, the ones who formed equal groups with the remaining made an extra calculation (addition) (see the drawings of Gökhan and Doğan in Figure 4.10). They skip counted the equal groups and found the number of balls in these groups. Then, they added the remaining balls to this number to find the total number of the balls. Both drawing strategies with/without remaining served for students to make practical counting instead of counting by ones.

In the fourth lesson, they were given pictures of 12 strawberries and asked how many strawberries there were. Students were given the picture of strawberries. They worked on this representation and formed groups as in Figure 4.11 below:

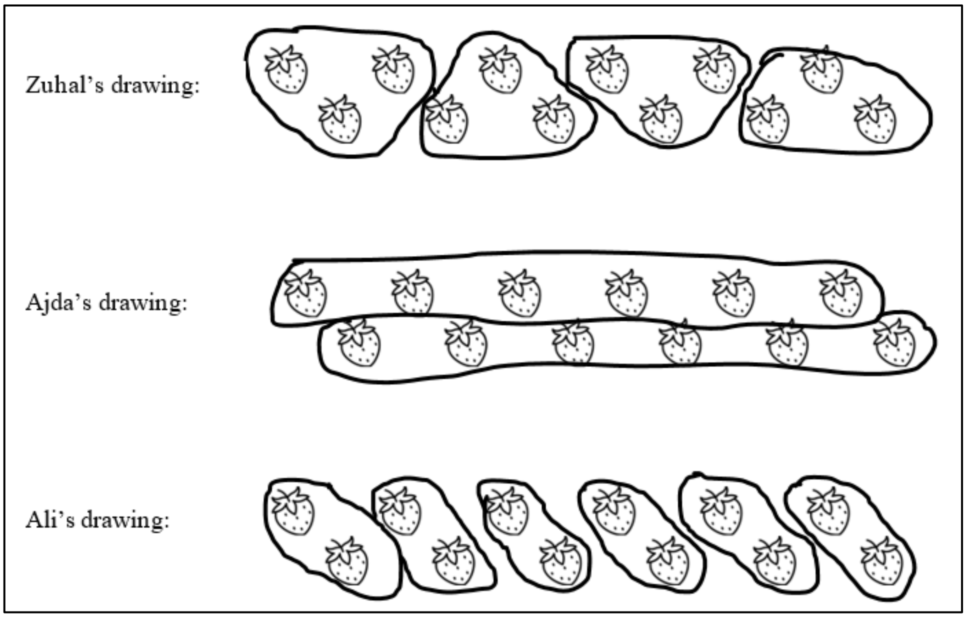


Figure 4. 11. Students' drawings on the given picture of strawberries

As shown in Figure 4.11, students divided all the strawberries into equal groups without remainder. They used these groups to skip count and found the total number of strawberries. Moreover, some of them discovered that reversing the numbers of groups and elements in each group results in forming equal groups. That is to say, the objects grouped as six sets of two elements can be grouped as two sets of six elements. It means that some of the students intuitively felt the commutative property of multiplication. For that reason, this valuable sense was noted to be observed in the following activities.

Students' arguments are presented in the argumentation analysis scheme in Figure 4.12 below.

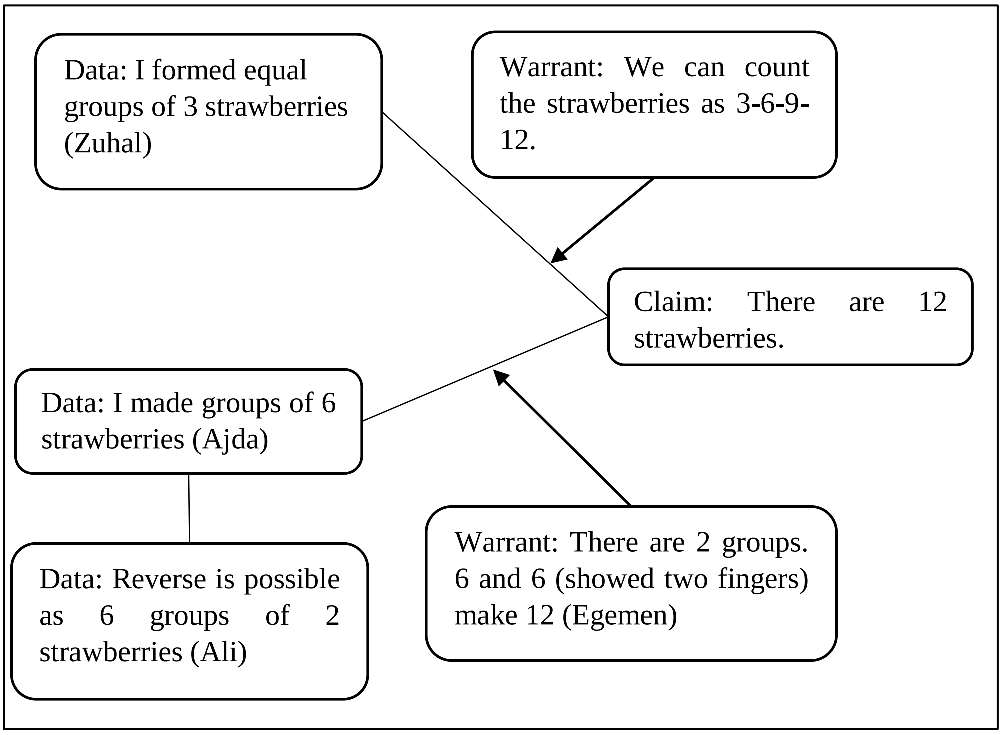


Figure 4. 12. Toulmin's Analysis scheme regarding the usage of math drawings.

As can be seen in Figure 4.12, students recognized that they could use the groups to count more effectively via skip counting and repeated addition. In this idea, it was observed that students take the objects in the groups out before they skip count or constantly add to figure out how many. They had not yet grasped the concept of counting groups. They focused on items in each group (multiplicand). They were not aware of the multiplier's role.

In the fifth lesson, students were given a picture with 18 bees. They were asked how many bees there were. Students' drawings (Figure 4.13) and dialogue related to these drawings are given below:

> Teacher: How many bees are there?
> Zuhal: I found 18. I grouped the bees into nine groups and 2 bees in each group. Then, I counted by 2s.
> Birce: If it can be counted by 2, we can count by 9, too. I divided them into groups of 9 bees. Now, I have two groups. 9+9 makes 18.
> Ali: I drew groups of 3 bees. Since there are three rows, I picked a bee from each row. It made six groups and three bees for each group.
> Ajda: I used the rows, too. I circled each row as a group. Therefore, I found three groups with six bees in each.

Melek: I drew circles with five bees each since it is easy to count by 5. I had three groups which include 5,10,15 bees. There are three more bees. 15+3 makes 18.
Zehra: What about counting by 8? I want to try it. (She circled two groups of eight bees). We can count by eight as 8,16. There are two remaining bees. If we add two bees to 16, it makes 18 bees.
Hakan: I want to try 4. (He did not draw groups. He counted his fingers as 4,8,12,16,20). I didn't say 18, which is between 16 and 20. We cannot perfectly group 18 bees as fours. There will be two remaining bees.

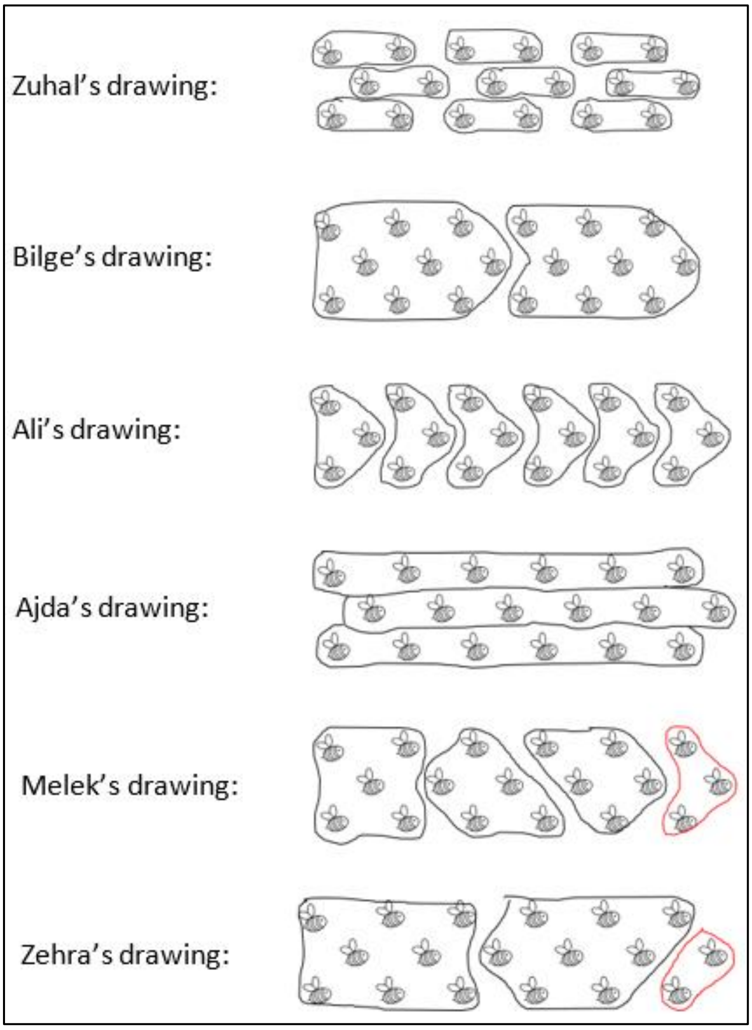


Figure 4. 13. Students' drawings on the given picture of bees

It can be understood from the dialogue and the drawings that students tended to group the objects to count them quickly. They grouped them as the number of elements they felt comfortable counting. For instance, Zuhal circled the bees by 2s. Birce took advantage of Zuhal's work since she knew that the reverse grouping is also valid (warrant) by replacing the group number and group size.

Furthermore, Ali and Ajda considered the placement of the bees in the picture and grouped these bees according to their sequencing (warrant).

On the other hand, Melek and Zehra used grouping with remainders. They used skip counting and the addition of remainders. Hence, all of these students used skip counting (claim) instead of counting by ones to find the number of objects in the picture. While making this claim, they used their drawings as data to provide them with equal groups to skip-count. They explained the procedures and reasoning they developed as a warrant to support the claim. Finally, all the students accepted that some objects could be counted as groups to find the total number practically. As long as the groups are equal, it is possible to use skip counting by considering the remainder if it exists. It was observed that this idea was taken-as-shared by the students. That is to say, students used drawing to construct their units and interpret the total number of objects in terms of these units.

Moreover, while working on the pictures, students developed another idea. As seen in Hakan's explanation in the dialogue above, he did not draw and count equal groups. He tried to skip count until he said 18. He realized that it is not possible to say 18. He concluded that 18 could not be divided into groups of 4 without a remainder. Hence, he stated divisibility intuitively, although he had not been introduced to the concept of divisibility. This idea was noted to be observed in the following lesson. Arguments related to this idea are examined below (TAS 4).

4.2.2. Idea 4: Equipartitioning collections into equal-sized groups through building up strategies

In the first week, students were given a collection of objects and asked to find how many objects there were. Through engaging in these activities, students began to use more efficient strategies to count multiple objects, which involves breaking numbers down equally (equipartitioning) and putting them back together (skip counting). The concept of equipartitioning developed in parallel with that of skip counting. Students used skip counting to equipartition the objects. For instance, in the fifth lesson, students were given a picture of 18 bees and asked how many bees there were. Students' drawings and their explanations to support their claims are presented above. They created equal groups and composed them to find the total number of objects. While discussing their work, they were asked to group given objects differently and find the total number. It was observed that students considered divisibility without knowing about divisibility. As it was stated at the end of TAS 3, verbal questions were asked to students to reveal their finger counting method considering divisibility. Classroom discussion related to such verbal questions is given in the dialogue below:

> Teacher: You know that there are 18 bees here. Assume that this picture is not given, and you don't have a chance to draw or use a pencil. How would you group 18 bees equally without remaining?
> Ajda: I can place them into groups of 3. Because, while counting by 3s, we say 18.
> Esra: Really? It would be so difficult to count 18 fingers by 3s.
> Birce: No, Esra. Not 18 fingers. We will count our fingers by 3s until saying 18. Let me show you. 3,6,9,12,15,18 (she opened a finger for each number). Can you see? I said 18 on the 6th finger. It means 18 bees can be shared into six groups of 3 bees.
> Ali: We say 18 while counting 6 by 6. 6, 12, 18. Three groups of 6 bees. The reverse of Ajda's groups.
> Teacher: What else?
> Eyüp: What about 5? 5,10,15,20.
> Meliha: You did not say 18. We cannot perfectly group 18 bees as fives.
> Teacher: Why do you use skip counting? Do you think that this method always works?
> Doğan: If objects are placed as equal groups, we can find the total number of the object by skip counting. For that reason, if we can find the total number by skip counting, it means that there are equal groups.

Students used skip counting to be able to distribute objects into equal groups. They moved through the abstract level since they did not use pictures, or drawings but counted fingers. They supported their claims by skip counting their fingers until they pronounced the given number. For instance, Birce counted by 3s until saying 18. She opened one finger for each number she voiced. She said

18 on the 6th finger. Then, she concluded that 18 bees could be distributed into six groups; each including three bees. In this method, skip-counted numbers represented the number of objects in each group, while fingers represented the groups. In fact, they developed this method to divide the given number into its divisors without knowing what they were exactly doing.

Moreover, reversing the number of groups and elements in each group emerged in this lesson too. Ali used Ajda's groups and came up with a new grouping. He needed to use skip-counting to support this claim. The researchers noted his way of getting new groups by reversing the numbers. It showed that students started developing the knowledge of commutativity intuitively. Hence, they used skip counting to show that reversing group number and group size results in obtaining equal groups. That is to say, students claimed various equal grouping strategies. Students' reasoning related to equipartitioning objects via skip counting is presented in the argumentation analysis scheme in Figure 4.14 below:

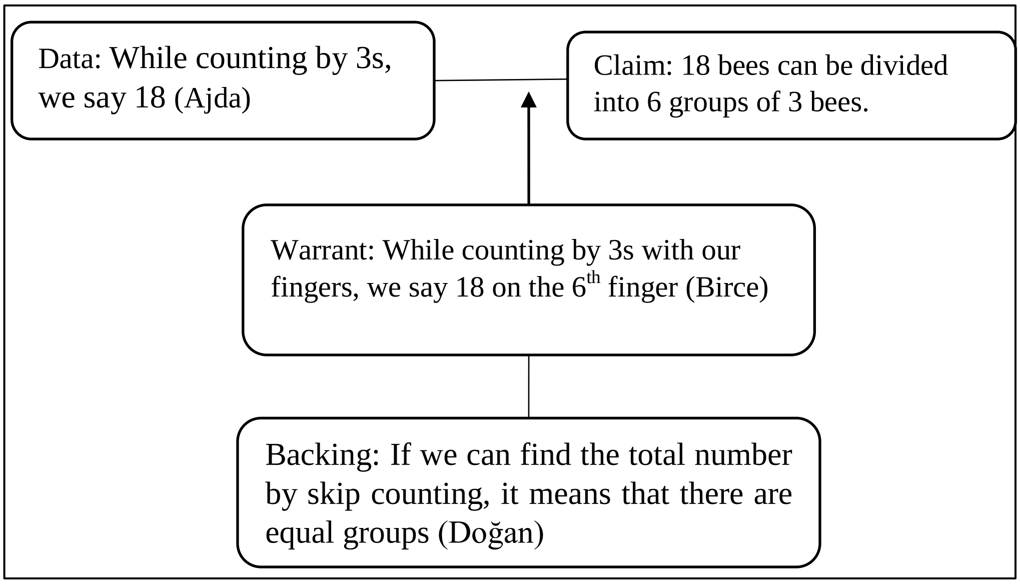


Figure 4. 14. Toulmin's Analysis scheme regarding equipartitioning objects by using skip counting

Most of the students accepted skip counting as sharing method. They were asked to explain and discuss the generalizability of this method. Doğan, explained this way by considering composing equal groups to find the total number of objects in these groups. In the previous activities, they counted fingers representing the groups to find the total number. They also used the same method to find the number of fingers (groups). That is to say; they skip counted until saying the total number. Then, they decided that the number of counted fingers represented the number of groups. Hence, students introduced their claims of equipartitioning objects by iterating the group size.

On the fourth day, students were given a picture of nine flowers and three vases to place flowers in the vases equally. Students used given pictures or skip counting to distribute the flowers. After that, they were directed with follow up questions, as in the given dialogue below:

> Teacher: Can you place given flowers into given vases equally?
> Karan: We can. There are three rows of flowers. I put flowers in each row into a vase. There will be three flowers in each vase (Figure 4.15).
> Ali: There are nine flowers. I counted until nine with my fingers. I said nine on the 3rd finger. There will be 3 flowers in each vase.
> Teacher: If one of the vases is broken, can you equally place the flowers into the other vases?
> Birce: We cannot. If we take three flowers from the broken vase, we should equally place them into two vases. If we divide them into two vases, there will be a remaining flower (she showed her three fingers and closed two of them to state they were placed in two vases).
> Teacher: Do you agree with her?
> Halil: I agree. I shared nine flowers into 2. I counted by 2s as 2,4,6,8 (he counted his four fingers). We cannot share equally. There will be four flowers in each vase and one flower remaining.

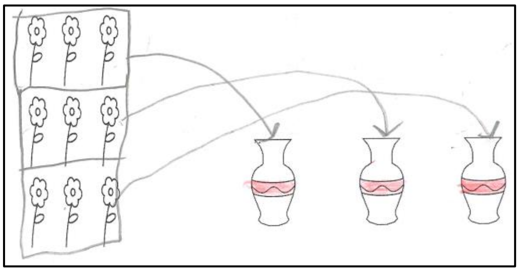


Figure 4. 15. Karan’s drawings on the given picture to share flowers.

As can be deduced from the dialogue above, Ali used skip counting until the total number. After that, students were asked to consider the case that one of the vases was broken. In this case, Halil shared nine flowers in two vases via counting by 2s. It was found remarkable. He could have answered this question like Birce, since there were few flowers, and given pictures of flowers and vases made the answer visible. However, Halil preferred skip counting to equipartition, which means he internalized this idea. Moreover, it should be pointed out that Birce also used a build-up strategy by adding the flowers in the broken vase on the other vases. Their arguments are presented in the argumentation analysis scheme in Figure 4.16 below:

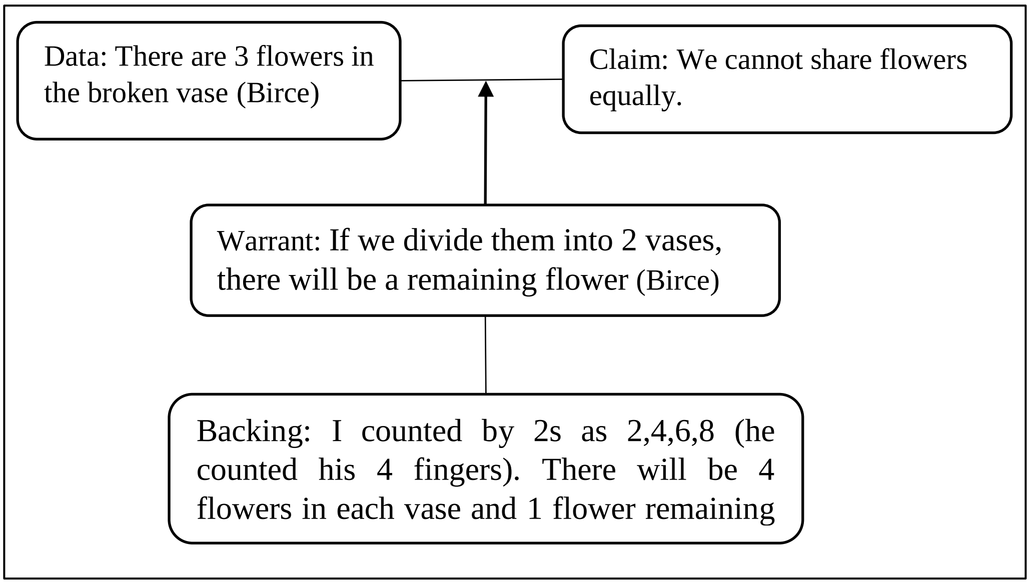


Figure 4. 16. Toulmin’s Analysis scheme regarding equipartitioning objects by build-up strategies using skip counting.

On the fifth day, students were given a picture of 36 birds and asked how many birds there were. Students used math drawings to form equal groups and counted these groups to find the total number practically. There were different grouping strategies. Students divided the birds into equal groups with or without remaining. The teacher asked verbal questions to encourage students to develop mental strategies and discuss in the classroom, as it can be seen in the dialogue below:

Teacher: Assume that you have 36 birds. You want to put them in cages four by 4. How many cages do you need?
Melek: 9 cages. If we count by 4 with our fingers, we say 36 on the 9th finger.
Teacher: Is there another way to place birds in the cages equally?
Gökhan: We can place them two by 2. We can count 36 by twos.
Teacher: Do you need more cages or fewer cages?
Gökhan: Fewer.
Classroom: More.
Birce: Since we will place fewer birds in each cage, we need more cages.
Ajda: This is division.
Zehra: We divide the birds into cages. This is a division problem.
Teacher: We haven't called it up to now, but it is division. What else? How can you place the birds?
Birce: 3 by 3. We say 36 while counting by 3. It makes 12 cages of 3 birds.

As can be seen in the dialogue above, students used skip counting to add the birds repeatedly until they had 36 to equipartition the birds. After a while, the students discovered that the operation they had performed was division. Although they had not been introduced to multiplication (symbolically) and division concepts, they realized that the questions posed to them were division problems. Students were asked these problems using the verbs "distribute, share, place," not "divide." However, they realized that they were dividing the objects. While finding equal groups, they found the divisors of 36. They proposed their claims related to equipartitioning given collections. They supported these claims by proposing warrants of skip counting.

On the sixth day, students were given a picture of 15 balloons and five children. They were asked to share these balloons with children equally. Classroom dialogue related to this activity is given below:

Teacher: As you can see in the picture, there are 15 balloons. How many balloons does each child take if we share these balloons with five children?
Ilker: 3 balloons. I counted by 5s until 15. (He showed three fingers).
Teacher: Assume that there are three children instead of 5 children; how many balloons does each child take?
Hakan: 5 ballons. The reverse of the previous.
Zuhal: We can count by 3s. We count five fingers until 15.
Ajda: There is one more way. We should take six balloons given to two children in the previous question. I shared six balloons with three children. 3,6 (she counted her two fingers). It means two more balloons will be given to three children. There will be five balloons of each (Figure 4.17)

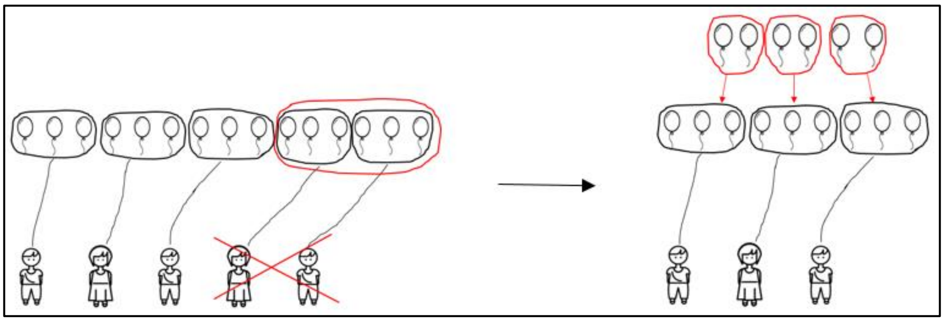


Figure 4. 17. Illustration of Ajda’s explanation

As in the problem of the broken vase, alternative problems were asked by making the given problem more complicated. Students used the same strategies as in the dialogue, which became taken-as-shared by the classroom. Students adapted to this idea of equipartitioning objects by regrouping the objects through build-up strategies. They could have adapted it to even more complicated situations. In the following lessons, it was observed that students used this idea without proposing warrants since it had already been taken-as-shared. To satisfy their claims, they did not need to explain the relationship between composing and decomposing or finger counting procedures.

4.2.3. Idea 5: Halving to reproduce equal-sized groups

In the first week, students worked on counting multiple objects by dividing them into equal groups. The students developed another idea of halving the objects during these activities. This idea started with dividing given pictures into halves getting the advantage of an alignment of the objects. The first argument related to this idea was observed in the activity in which there are pictures of 16 balls. As explained above, students were encouraged to find the total number of balls using different strategies. In this lesson, Karan developed the idea of halving as given below:

> Zehra: I found 16 but in a different way. I circled each row as a group. I got two groups of 8. I added 8 to 8.
> Zuhal: I used the same method. I counted eight fingers on eight and found 16.
> Karan: I found a new solution. If we divide Zehra's drawing into two, we get the group of four (He drew an imaginary vertical line on Zehra's drawing with his

finger) (Figure 4.18). It makes four groups of four. I can count by 4s and find 16.
(2 students agreed with him and stated that they used the same strategy, too)

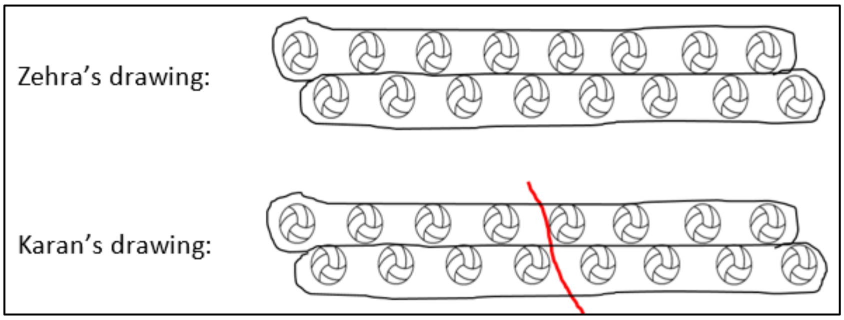


Figure 4. 18. Students' halving strategies on the given pictures of balls

It was noted that halving might be conceptualized and internalized by others, and it might be a taken-as-shared idea through argumentation in the classroom. For instance, students were given pictures of 12 strawberries and asked to calculate how many strawberries there were. Students' reasoning related to the halving strategy is given in the dialogue below:

Zehra: I circled each row as a group. I got two groups of 6. I added 6 to 6.
Egemen: I found a new solution. If we divide Zehra's groups into two groups, we get four groups of three. I can count three by three and find 12 (Figure 4.19).

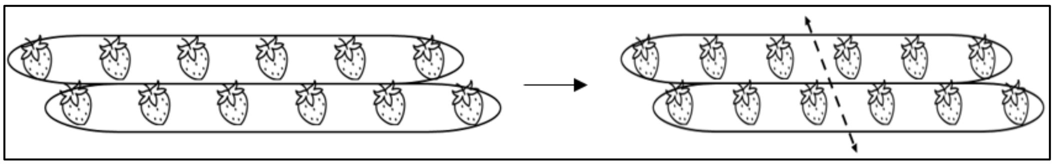


Figure 4. 19. Illustration of halving strategies of Zehra and Egemen.

As illustrated above (Figure 4.19) to make it clear, students used given visual representations to form equal groups. They broke the visuals in half as an act of subdivision. Even they applied multiple subdivisions to create new groups. After the first subdivision, the halves were used to find the total number of objects by addition. On the other hand, after the second subdivision, the quarters were used to find the total number of objects by skip counting. In the following days, students developed this idea by moving towards an abstract level without using visuals. During the discussion on the activity related to counting 36 birds, students shared their strategies to equipartition the birds. They used build-up strategies to decide on the groups and the number of birds in each group (TAS 4). Moreover, there were some students used halving to reproduce equal-sized groups. These students used skip counting as a warrant to support their claims, as in the excerpt below.

> Zehra: We can decompose 36 into 30 and 6. Then, 30 birds and six birds can be divided into two. Half of 30 is 15. I put 15 birds in each two groups. Then, I share six birds into these groups as three by 3. It means that there are 18 birds in each group.
> Hakan: Zehra claimed that 36 birds could be shared as 18 birds. I know that half of 18 is 9. If we divide the groups into two, there will be four groups of 9 birds.
> Birce: You are right. We can find 36 by counting 9, 18, 27, and 36.

Students explained their reasoning verbally without drawing. To make it understandable, their reasoning is illustrated below in Figures 4.20 and 4.21 according to their explanations. They applied multiple subdivisions so that they found quarters of the whole.

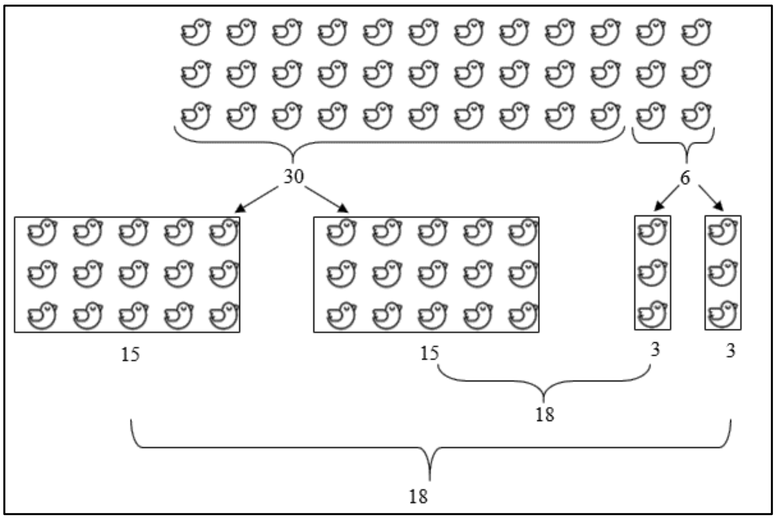


Figure 4. 20. Illustration of halving strategies of Zehra.

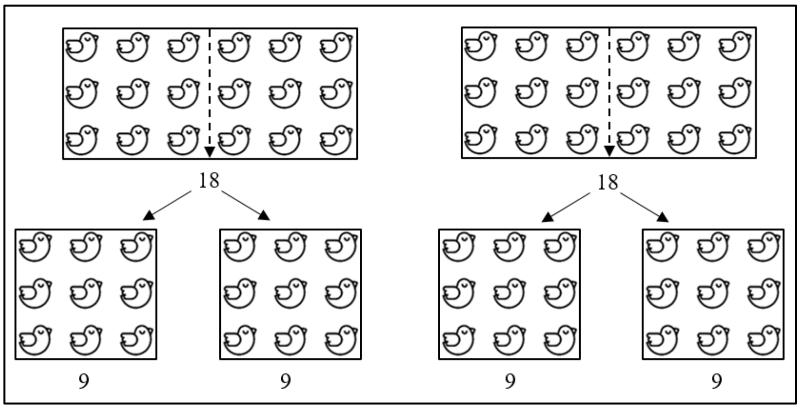


Figure 4. 21. Illustration of halving strategies of Hakan.

As seen in the dialogue and the illustrations, students halved the groups to regroup the objects by halving the group's size and doubling the number of groups. The argument of Hakan is presented in the argumentation analysis scheme in Figure 4.22 below to make it clear:

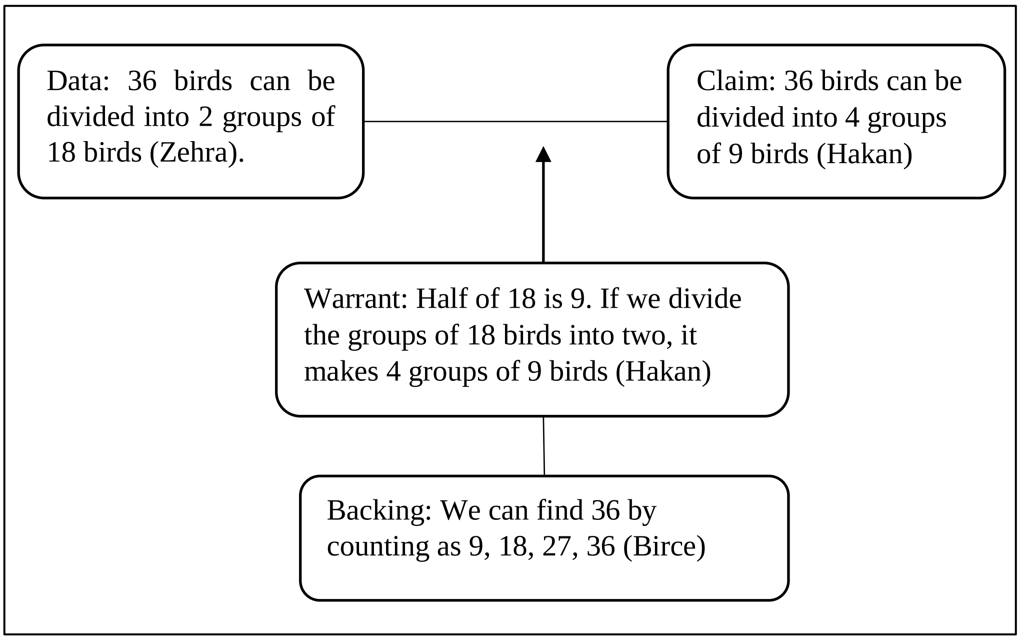


Figure 4. 22. Toulmin's Analysis scheme regarding halving to reproduce equal-sized groups

Therefore, students supported their claims with the previous idea of equipartitioning collections into equal-sized groups through building-up strategies. This situation revealed that a previously justified claim shifted place to become warrant. Moreover, students did not challenge the idea of halving the collections on the following days. Students divided the groups into two as long as they had an even number of objects. That is to say, they used halving to regroup the objects. Hence, it ass evident that the idea of halving to reproduce equal-sized groups was taken-as-shared. At that point, students split the groups of objects into equal parts whose denominators are two (fraction) intuitively.

4.2.4. Idea 6: Distributing left over numbers (remainder) to equal groups by conserving equality

During the activities related to regrouping objects, students developed grouping strategies by making connections between different groups. When students were asked to group the objects differently, they made reasoning considering previously formed groups to form new groups. During such activities, students tried various numbers as group size and sometimes they couldn't partition the objects without remainder. At that point, teacher encouraged them to question these conditions. While discussing on these groups with left overs, students developed another idea related to distributing the left over. For instance, as explained above in the sections related to TAS 3, students shared 18 objects into equal groups. Some of the students built a bridge between different groups. Students' explanations related to this idea are selected and given in the dialogue below:

> Teacher: How can you share 18 bees equally?
> Ali: We say 18 while counting 6 by 6. 6, 12, 18. 3 groups of 6 bees.
> Zuhal: Maybe, we can divide them into groups of 8 bees.
> Zehra: I tried it. It is not possible. While counting by 8, we say 8 and 16. We have two groups of 8 and 2 extra bees. But we can share the remaining 2 bees one by one to these groups with 8 bees. In this case, there will be 2 groups with 9 bees in each (Figure 4.23).
> Esra: I couldn't understand.
> (Zehra used the picture of bees and showed the groups by drawing on the booklet).
> Ajda: We say 36 while counting by 9s.
> Eyüp: Can we share as 5 bees? 5-10-15-20. No, we cannot.
> Melek: I grouped as fives. It makes 3 groups of 5. And there will be 3 remaining bees.

Karan: It makes 5, 5, 5, and 3 bees. I have an idea. If we share 3 bees, it makes 3 groups of 6. But Ali found it before.
Teacher: Ali found by skip counting. Your strategy is different. Can you explain again?
Karan: If we make groups of 5, it makes 4 groups of 5, 5, 5 and 3 bees. Actually, it makes 3 equal groups of 5 bees and 3 bees as remaining. If we distribute 3 bees one by one to three groups of 5 bees, it becomes 4 groups of 6 bees (Figure 4.23).
(Karan explained his friends by drawing on the board)
Ali: Karan didn't give up. Placed the remaining bees too.

As can be inferred from the dialogue, Zehra and Karan used the answers of others as data and made new claims related to grouping 18 bees equally. They explained their claims by drawing on the booklet and the board, as illustrated in Figure 4.23. Zehra used the wrong claim of Zuhal. She refuted the claim by using skip counting (rebuttal) as presented in the argumentation analysis scheme in Figure 4.24 below. Then, she used this wrong claim as data to make a new claim. She used the equal groups with the remaining bees and shared the remaining bees with equal groups by ensuring equality. She presented warrants to explain the equality of the groups. In the same way, Karan used the claim of Eyüp as data and made a new claim. He provided justification to others as warrant by drawing on the board. The reasoning of both Zehra and Karan was dependent on the claims of others, although they were wrong. They refuted these claims and took advantage of these claims to produce new and correct claims. that is to say, they interpreted the left over bees and distributed them on the groups by conserving equality.

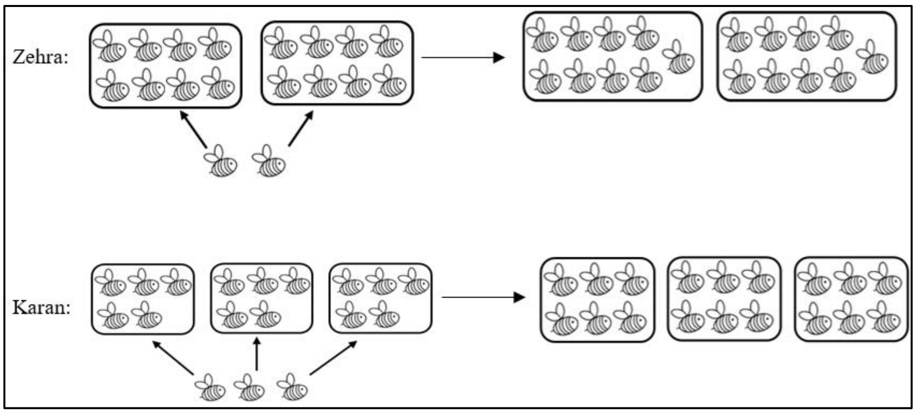


Figure 4. 23. Illustration of claimes explained by Zehra and Karan.

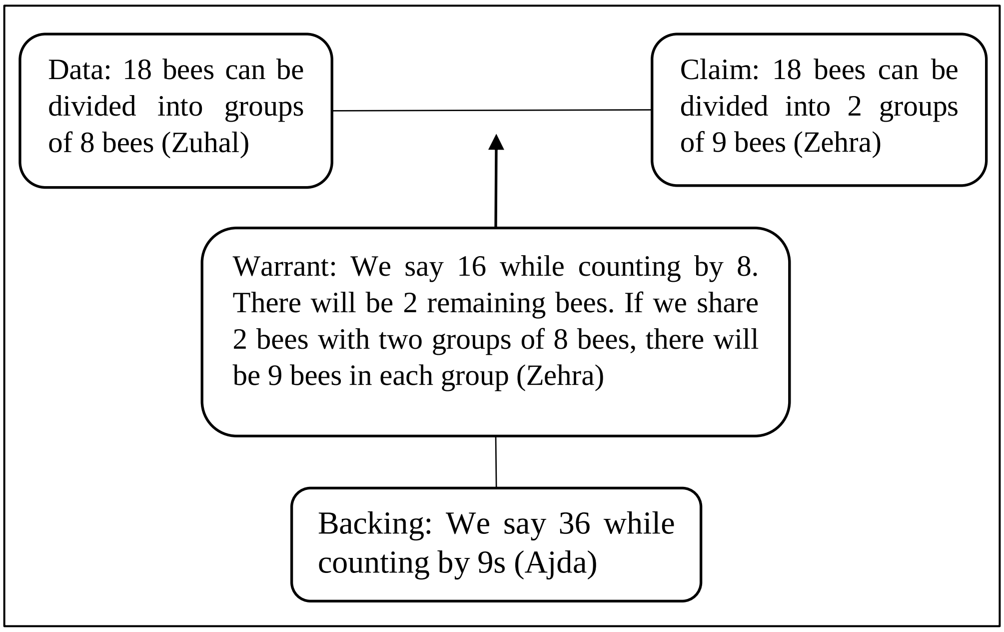


Figure 4. 24. Toulmin's Analysis scheme regarding distributing left over bees on equal groups by conserving equality

In the fifth lesson, students were asked to group 36 birds equally, as explained before in the TAS 4 and 5. Different than the claims and explanations in the dialogues given above, the claims related to this idea (TAS 6) are provided in the dialogue below:

> Teacher: How can you group 36 birds equally?
> Simge: I distribute them 10 by 10.
> Ali: Simge, you can distribute 30 birds as 10 by 10. What about 6 birds? They will be out of groups.
> Teacher: Can you help Simge?
> Doğan: I want to help. I am thinking. We can group as 10-10-10 and 6. There are 3 groups of 10 birds. Let's try to put 6 birds into three groups equally. It means that we should add 2 more birds in these groups. It makes 12 birds in each group.
> Teacher: Lets count by 12 and check your answer. 12, 24, 36. Well done.

Students explained their reasoning verbally without drawing. To make it understandable, their reasoning is illustrated below in the Figure 4.25 in the light of their explanations.

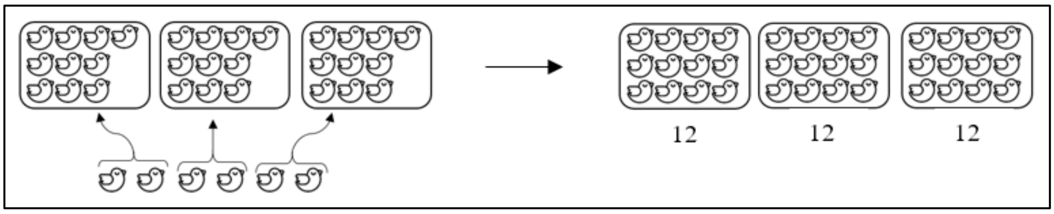


Figure 4. 25. Illustration of Doğan's reasoning

As it can be deduced from the dialogue of students and illustration of their reasoning, students developed an idea of making a connection between different grouping strategies. Students used previous claims and developed them to pose new claims. Students were encouraged to feel confident and think flexibly. When someone made a wrong claim, the teacher asked the others for help to correct the wrong claim. Students created a relationship with numbers and talked about them. They did not use given pictures or math drawings to show whether their claim was true. They computed mentally and thought fluently about numbers. They used the claims of others as data, as can be seen in Toulmin's analysis scheme regarding this idea presented in Figure 4.26 below.

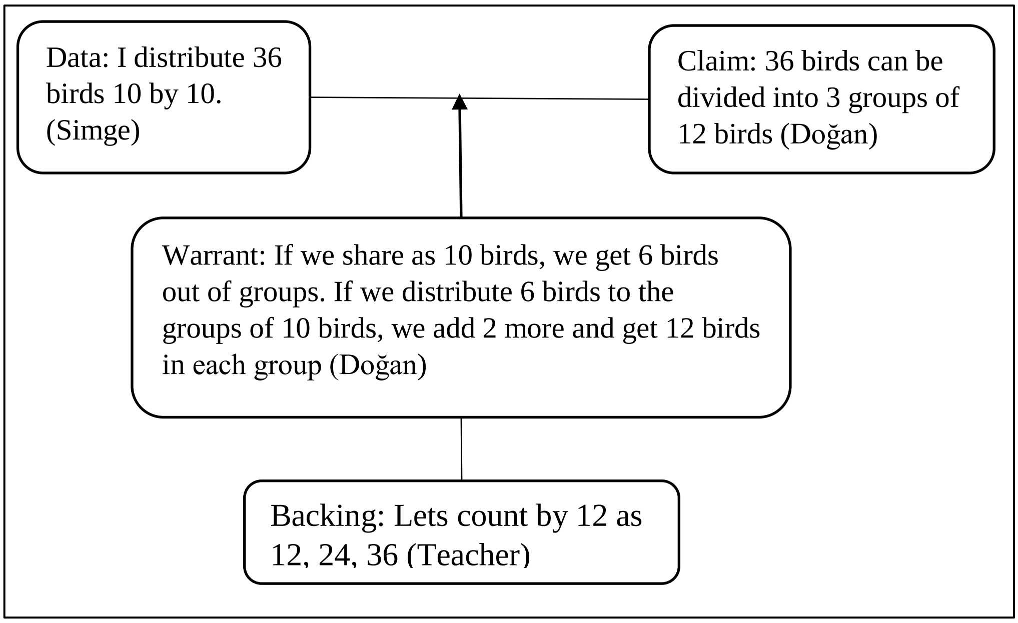


Figure 4. 26. Toulmin's Analysis scheme regarding distributing left over birds on equal groups by conserving equality

Students provided warrants to explain the relationship between the data and the claim. They used the remaining birds, which are out of equal groups. They distributed these remaining birds into equal groups by sustaining the equality of these groups. To support their warrants, they used skip counting and showed whether the objects were equipartitioned. In the following days, students internalize this idea to equalize the groups by distributing the left over objects. No one questioned how to regroup the objects by using left overs in other activities. They did not provided explanations to connect data and claim. That is to say, they did not use warrants anymore, which means that the idea became taken as shared. Moreover, they used these claims as data in the following lessons. Hence, the idea of distributing the left over objects was taken-as-shared so that students administered it in the following lessons without questioning.

4.3. Classroom Mathematical Practice 3: Iterating linked units by using pictures and fingers

Students had partitioned objects into equal groups to add them repeatedly in the second practice. The third practice is about composing these groups. That is to say, the third mathematical practice that emerged as students engaged in the instructional sequence was about students' reasoning about composing groups of objects by using pictures and fingers. In this practice, students use their fingers and pictures to iterate composite units and make sense of covariation by double-matching on pictures. Specifically, three ideas became taken-as-shared as the students engaged in this practice, especially on the first two weeks of the instruction:

- Skip counting the groups (composite units) in order to iterate
- Assigning hands (5 or 10 fingers) as composite units to iterate
- Double matching collection of equal groups to iterate on via pictorial representation

The third mathematical practice is critical since it is about processing multiplication operations. When students' ways of reasoning while composing objects were examined, two critical abilities of students were detected as unitizing and iterating. Students started to collect objects by iterating units through skip counting. In the following lessons, students assigned unit of units (composite unit) and extended their counting by operating with this linked composite unit. At that point, follow-up questions posed to students without changing the realistic context of the tasks helped students to question, examine relations between different groupings and extend their build-up strategies through forming composite units.

4.3.1. Idea 7: Skip counting the groups (composite units) in order to iterate

In the following lessons, students were asked problems for independent practice. They were expected to apply their real-world experiences and the day's learning. It was observed that students used their fingers to find the total number of objects asked in the problems. After reading and understanding the problem, students decided on the role of the numbers in the problem as skip counted number and the ordinal number. For instance, in the ninth lesson, students were asked to solve the problem of "Selay plants two trees every month. How many trees does Selay plant in 9 months?". Students worked on the problem by themselves at first. While they were solving the problem, it was observed that students were engaging with their fingers, and their lips were moving rhythmically that meant they were counting their fingers. Then, they shared and discussed their solutions with others as in the dialogues given below:

> Esra: I found 18. I drew two trees for each month and counted them (see Figure 4.27a).
> Teacher: Can you explain how you counted?
> Esra: 1, 2, 3, …., 18. (She counted all trees by ones).
> Doğan: Esra, there is an easy counting method. You can count by twos.
> (Esra tried to count by twos on her drawing. However, she confused the trees and could not count)
> Hakan: I grouped trees 2 by 2 (see Figure 4.27b). It helped me to count. I have nine groups. It means that I will count nine fingers. I counted my nine fingers as

2,4,6,8,1012,14,16,18. On the ninth finger, I said 18. It means that there are 18 trees.
Halil: I used my fingers to represent the months as well.
Teacher: How did you count your fingers?
Halil: I counted by twos, skip count.
Teacher: Why did you count by 2s instead of by 1s?
Karan: Counting by ones is a waste of time. To count by ones, we have to draw all the trees. Instead of spending time drawing, I imagined my fingers as months. It is given that there must be two trees each month. Therefore, I realized that I should count my nine fingers by 2s.

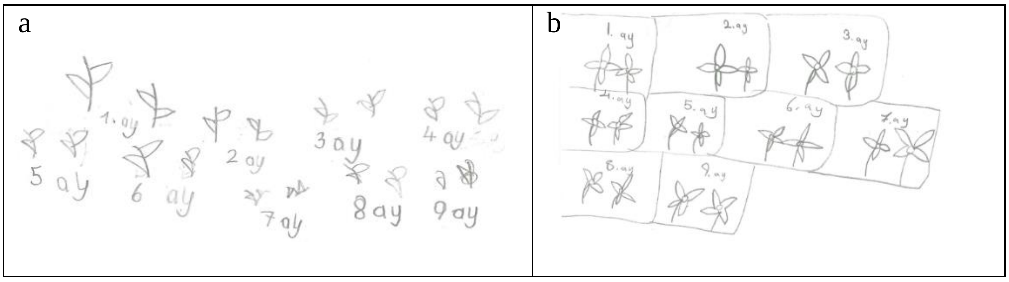


Figure 4. 27. Drawings of Esra (a) and Hakan (b)

As observed, students said that they used skip counting with fingers. Most of the students concentrated on months as equal groups and realized that they could skip count their fingers to find the total number of trees. Some of them discovered this situation after drawing trees for each month. Even those who drew trees used their fingers instead of counting the groups on their drawings. They named each finger as a month or group. This interpretation is revealed in Toulmin's analysis scheme presented in Figure 4.28 below:

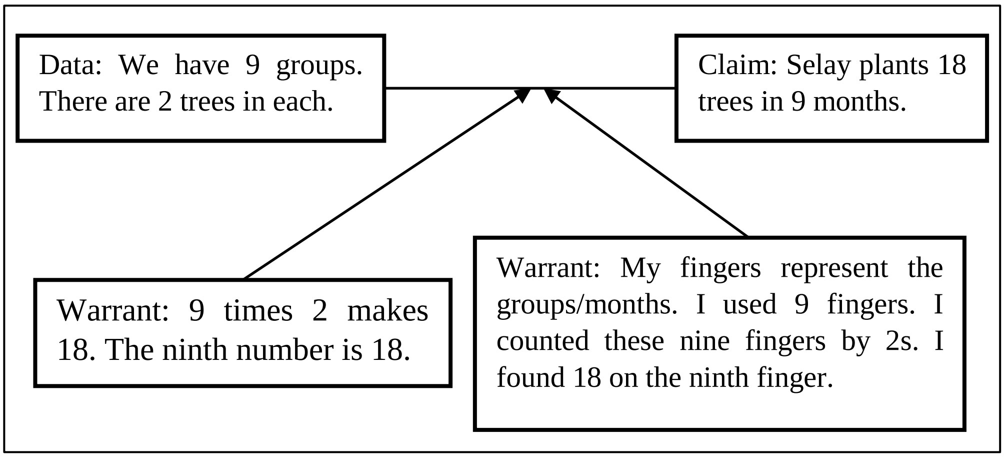


Figure 4. 28. Toulmin’s Analysis scheme regarding skip counting fluently by using fingers to solve problem

In the following activities, students approached the multiplication problems as equal group problems. Hence, they specified the groups and elements in each group via drawings and verbal explanations. After a while, students did not question how to decide on the groups. They just stated how many fingers they counted and how they counted. Students used directly skip counting to solve the problems by conceptualizing the groups as units to iterate. To be more clear, students assigned each finger as a unit which is critical didactical activity in development of proportional reasoning. Therefore, students no longer had to justify the components of equal groups. It was seen that warrants were dropped off day by day.

In addition to classroom activities taking place in the instructional sequence, it was observed that students used this idea where it is necessary in daily situations they experienced. One day, students were wasting wet wipes in the classroom and the teacher warned them:

> Teacher: Please don't waste wet wipes. 3 people can share one wet wipe.
> Birce: Then, we need 8 wipes.
> Teacher: How did you decide?
> Birce: There are 24 students in this classroom. I counted 3 by 3 until 24 with my fingers. I reached to 24 on the 8th finger. This means 8 wet wipes will be enough for all of us.

In this situation, Birce used skip counting to find the answer of "if we need a wet wipe for 3 students, then how many wet wipes do we need for 24 students". It was more than skip counting. She used one wet wipe as a composite unit which represent the pieces for each three students. Then she took it as one thing and iterated it by keeping track of how many times she will count until 24. This situation took our attention since it was related to creating composite unites and iterating by skip counting. Students internalized the idea and started using it in daily life. It was observed that students used skip counting to find the total number of objects as long as they are presented in equal groups. In the following tasks, the warrants were dropped off since students did not question each other anymore to provide warrant.

4.3.2. Idea 8: Assigning hands (5 or 10 fingers) as composite units to iterate

As it is explained above, in order to support students' fluency in formal multiplication, students worked on skip counting exercises. They were posed problems related to the order of numbers which are skip counted. In the flow of the lesson, the form of the questions also changed. Instead of asking just the nth number while skip counting, the order of a given number was also asked. For instance, students were asked to find the order of 44 while counting on by 4s starting at 4. Most of the students used their fingers for skip counting as in the previous ideas (TAS 1&2). Surprisingly, Doğan developed composite units to iterate informally by using his hand and fingers. Classroom dialogue related to this question is given below:

> Teacher: What is the order of 44 while counting on by 4s starting at 4?
> Zehra: We can count until 44. On the 11th finger, we say 44.
> Teacher: Do you agree with Zehra?
> Doğan: I found 11, but in a different way. I did not count all the fingers. I know that five fingers make 20 if we count as 4,8,12,16,20 (TAS1). After that, there is no need to count. If I include one more hand (5 fingers), it makes 40. To reach 44, I need a four which means one finger. I have two hands and a finger which means 11 fingers (Figure 4.29).
> Birce: I liked this solution.
> Teacher: Is there a different solution that you want to share? What do you remember about counting by 4?
> Egemen: 10th number is 40.
> Ali: For 44, we need one more 4, which is represented by a finger. It makes 10+1=11.

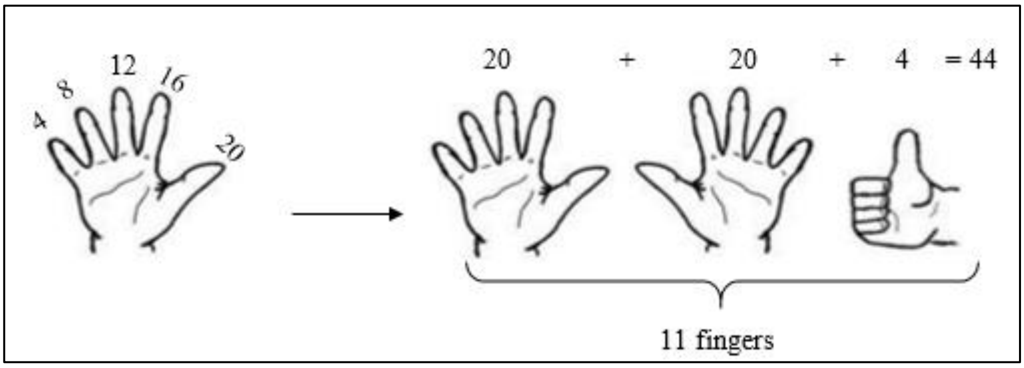


Figure 4. 29. Illustration of Doğan' using his hand as a composite unit

As can be followed from the dialogue, students used the claims established in the first idea (TAS 1). Another striking point in this dialogue is that Doğan assigned each hand (5 fingers) as composite unit to iterate. He took each hand (five fingers) as a composite unit representing 20. He iterated this unit to reach 40. Then he added one more unit (one finger) representing 4. He intrinsically developed the ability of unitizing and composite units by using the claims of previous questions as data. As he did, others also developed reasoning strategies not to count all the numbers starting from 4. They revealed the relationship between the order of the numbers in the previous and current questions as explained with Toulmin's analysis scheme presented in Figure 4.30 below:

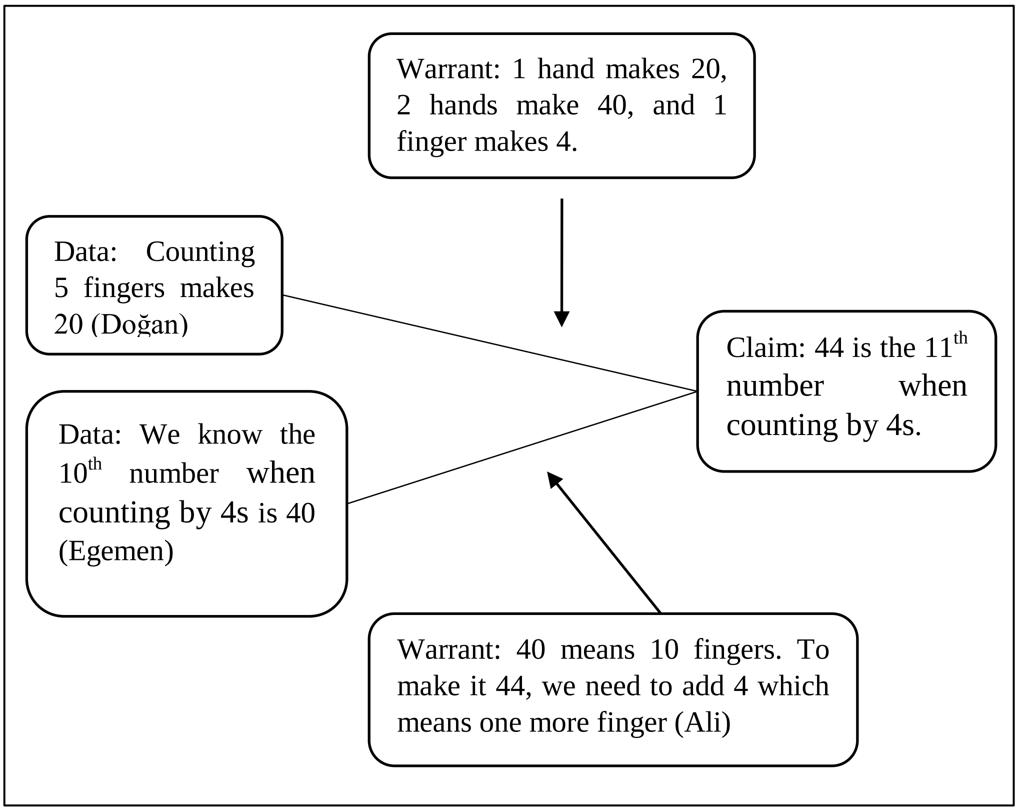


Figure 4. 30. Toulmin’s Analysis scheme regarding to find the order of numbers skip counted

Asking questions related to skip counting helped students to make a connection between known and unknown values. They convinced each other that it was a waste of time to count from the beginning. They made reasoning (warrant) to find the answer by choosing the most appropriate knowledge as the data. In the dialog below, there are samples related to reasoning the effect of change in the order of numbers skip counted during the fluency question period:

> Teacher: What is the 10th number when you count by 5s?
> Yalçın: 5,10,15,20,25,30,35,40,45,50 (He counted 10 fingers; TAS 1)
> Hakan: Yalçın, you don't need to count all of them. We know that one hand makes 25 (he raised one hand and counted five fingers as 5,10,15,20,25). 10th number means two hands (he raised two hands). Then we add 25 to 25, which makes 50.
> Teacher: What is the 12th number when you count by 5s?
> Zuhal: If the 10th number is 50, we should count two more 5s on 50. (She counted two fingers by showing) 55, 60. The 12th number is 60 (TAS 2).
> Teacher: What is the 20th number?
> Ömür: If the 10th number is 50 (Data), we should add one more 50 for the 20th number. 50+50 makes 100.
> Ali: If one 10 makes 50 (he showed his two hands), then two 10s make 100. The 20^{th} number is 100.
> Teacher: What is the order of 40 while counting on by 5s?
> Zehra: 5,10,15,20,25,30,35,40 (she counted her fingers until 40). 8th number (TAS 1).
> Teacher: What is the order of 55 while counting on by 5s?
> Burak: 40 is the 8th number (Data). We should count on 40 until 55 as 45,50, 55 (He counted three fingers). If we count three more fives on 40, it makes 55 (Warrant). 8+3 makes 11 (TAS 2).
> Teacher: What is the 20th number?
> Melek: If 11^{th} number is 55, 15^{th} number is 75 (TAS 2).
> Birce: If one hand makes 25, three hands make 75.
> Eyüp: Three hands?
> Melek: 15 fingers mean three hands.

As it can be seen in the dialogue, teacher directed questions related to finding orders of numbers in a number sequence while skip counting. They were reasoning the order of previously established numbers as in the TAS 2. There were many arguments related to TAS 2 as they are specified in parenthesis in the dialogue. It was remarkable to observe students' development through the taken-as-shared ideas and classroom mathematical practices. During the classroom discussions that some of them moved forward in their reasoning so that they were using their hands as composite units. As it is seen that Yalçın found 10^{th} number by counting ten fingers which is the basic strategy (TAS 1). On the other hand, Hakan preferred a more practical strategy considering each hand as representation of 25. That is to say, used his hand as a composite unit which is a composition of five fingers. Then, he iterated this unit as it is interpreted with

Toulmin's analysis scheme supported with illustrations in Figure 4.31 below. In a similar way, Birce used each hand as Hakan did and iterated her hand three times to find the 15th number as it is interpreted with Toulmin's analysis scheme supported with illustrations in Figure 4.32 below. It was also observed that think in terms of hands was being followed by others. That is to say, when Eyüp asked why Birce counted three hands, Melek explained Birce' reasoning as 15 fingers mean three hands. At that point, design team realized that other students could share the idea of determining each hand as composite unit.

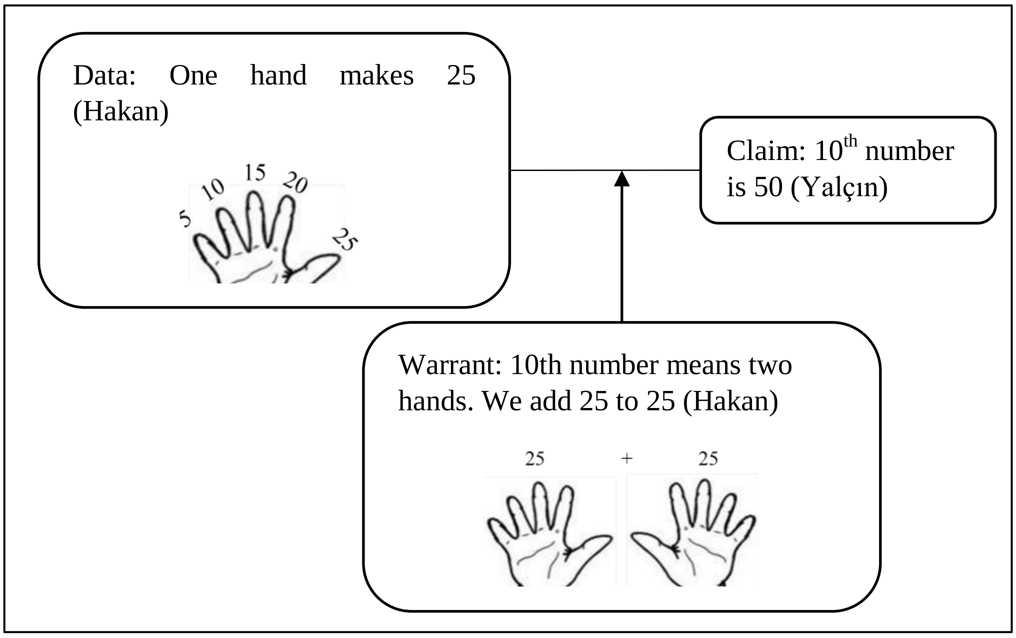


Figure 4. 31. Toulmin's Analysis scheme regarding assigning one hand (5 fingers) as composite units to iterate (two times)

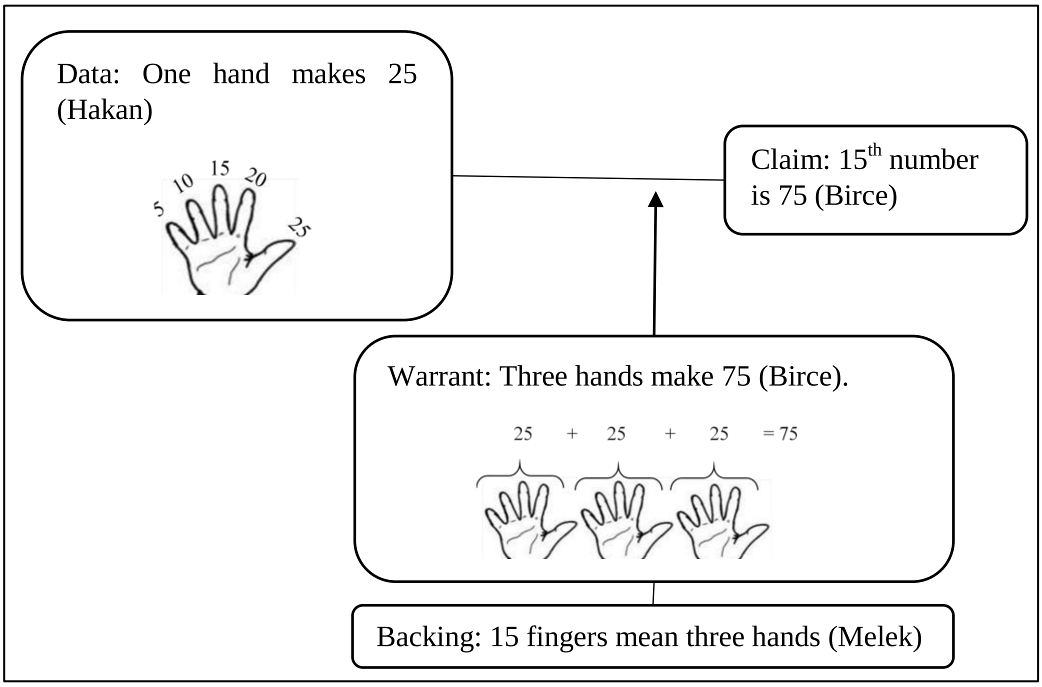


Figure 4. 32. Toulmin's Analysis scheme regarding assigning one hand (5 fingers) as composite units to iterate (three times)

Moreover, while some of them iterated each hand as a composite unit, some iterated two hands as composite units. When the teacher asked the 20th number, Ali used Hakan' claim (two hands make 50) as data and established a new claim. Ali claimed that 20 fingers make four hands and two hands should be iterated two times. Thus, he iterated 50 (two hands) two times and found 100 as it is interpreted with Toulmin's analysis scheme supported with illustrations in Figure 4.33 below.

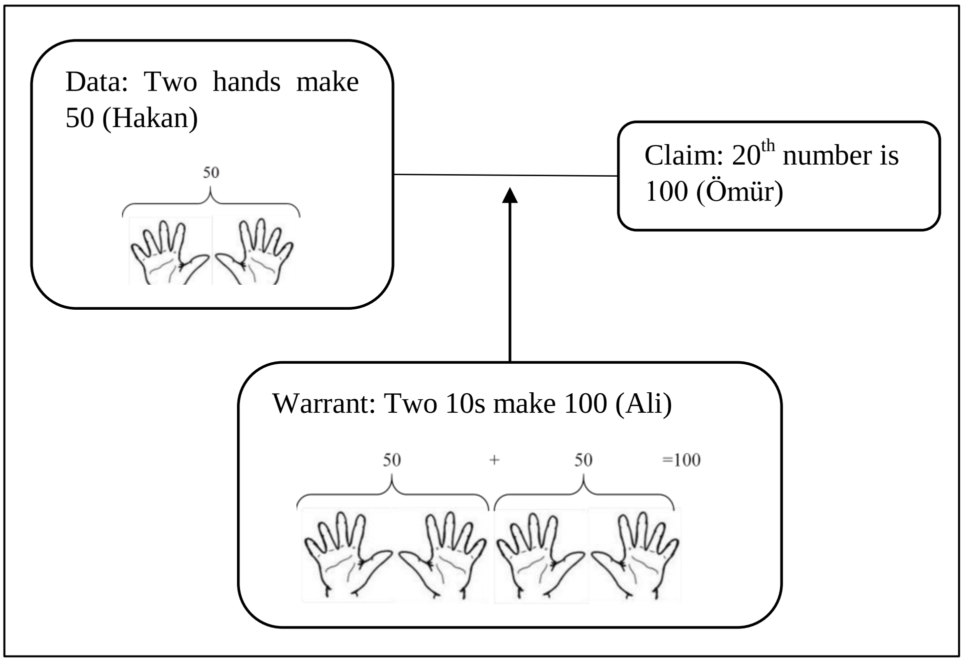


Figure 4. 33. Toulmin's Analysis scheme regarding assigning two hands (10 fingers) as composite units to iterate (two times)

At the beginning of the sixth lesson, students were asked such questions to check students' fluency and reasoning related to skip counting. Sample dialogue from this exercise is given below:

> Teacher: There are 22 students in this classroom. If everyone holds one hand, how many fingers do we see?
> Eyüp: If we count by 5, it makes 110.
> Birce: 2 students sit at each desk. There are 11 desks. Counting by 10, there are 110 at 11 desks.
> Hakan: 10 fingers on two hands and 10 toes on two feet. All of them are 440.

As deduced from the dialogue, students enriched their ability to skip counting. They made claims about the effect of change in the order of numbers skip counted. They used their previous claims to construct composite units to iterate and answer the questions related to larger numbers. It was observed that they could develop flexible and creative ways to manage numbers and counting. After a while, they explained their claims without presenting warrant/backing. Since warrants were dropped off, it was concluded that the idea was taken-as-shared. Moreover, there is also evidence in the following days that this idea shifted place from claim to data and was repeatedly used as data for different claims. It was revealed that the more students engaged in activities related to counting, skip counting, matching and grouping, the more they developed multiplicative thinking. In this way, students' strategies of assigning composite units developed and became more sophisticated as it can be seen in TAS 9.

4.3.3. Idea 9: Double matching collection of equal groups to iterate on via pictorial representation

In the ninth lesson, during the classroom discussions, the idea of double matching collection of equal groups to iterate on via pictorial representation which is related to covariational reasoning emerged. Students took the advantage of pictures of equally grouped objects to double match them.

For instance, students were given a picture of 5 coops and 4 chickens in each coop. In the picture, chickens on the first coop were given while the other coops were closed. The reason for hiding the chickens in the other 4 coops was to avoid counting all chicken by ones since it might have been easier for some students to count 20 chickens by ones. Instead, they were desired to gain abstract counting without seeing chickens but knowing that there were 4 chickens in each coops. Therefore, students were explained that there were 5 coops as in the picture and 4 chickens in each coop. Students were asked to find total number of chickens and answer follow-up questions as in the dialogue below:

> Teacher: How many chickens are there in 5 coops?
> Mahmut: 20 chickens. I counted the coops by 4s.
> Teacher: Is there anyone who used a different method?
> Birce: I counted my five fingers by 4s.
> (Some of the students stated that they counted their fingers while the others counted the coops as illustrated in Figure 4.34)
> Teacher: If there were 10 coops, how many chickens would be in them?
> Birce: 20+20 makes 40 chickens.
> Burak: Why did you add 20 to 20?
> Birce: There are 20 chicken in 5 coops. To get 10 coops, we should add 5 more coops. Actually, we double the numbers. If we add 5 more coops, we also add 20 chickens. For that reason, I added 20 to 20.

(Teacher also explained the same method by drawing as illustrated below in Figure 4.35 to make it clear).
Researcher: If there were 15 coops, how many chickens would be in them?
Doğan: Birce found that there would be 40 chickens in 10 coops. For 15 coops, we should add 5 more coops which means adding 20 chickens. 40+20 makes 60 chickens.
(Teacher explained the same method by drawing on the board).
Teacher: Assume that there are 80 chickens. How many coops would you build for these chickens?
Melek: Can we use your drawing on the board? Otherwise, it will be hard for me to draw.
Teacher: Sure, you can.
Birce: Teacher, I used your drawing and found 20 coops.
Zuhal: How can we use this drawing? (She turned back and asked the others). I am confused.
Ilker: Zuhal, there are 15 coops and 60 chickens on the board. To get 80 chickens, we should add 20 more chickens. We know that there are 20 chickens in 5 coops. It means that we should build 5 more coops.
Birce: Finally, we need 15+5 coops.
(Teacher explained the same method by drawing on the board).

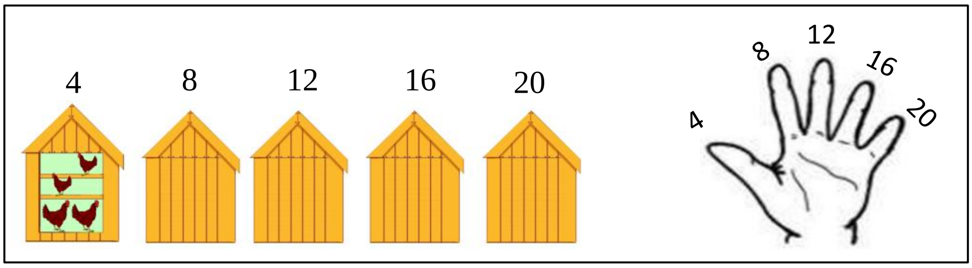


Figure 4. 34. Illustration of counting chickens in 5 beehives.

As Mahmut explained, most students found the chickens in 5 coops by counting by 4s. These students traced the coops to count 5 times 4 (TAS 7). On the other hand, as Birce explained, some of them traced their fingers instead of the picture to count 5 times 4. The answer to this question played a crucial role in making reasoning for the following questions. Students considered 5 coops as composite units and iterated to answer the questions. They made reasoning about the relationship between the coops and the relationship between chickens in all cases as illustrated in Figure 4.35. Students asked the teacher to help and support their claims by doubling the pictures of coops on the board. These pictures made composite units more clear and helped them to construct a double-matching procedure.

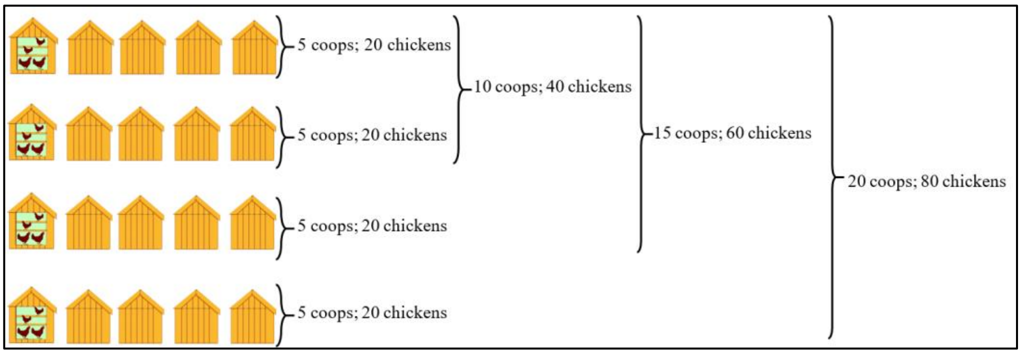


Figure 4. 35. Illustration of explanations emerged during classroom discussion

When students were asked to find the chickens in 10 coops, Birce interpreted the relationship between 5 coops of chickens and 10 coops of chickens. She stated that 5 more coops should be included to get 10 coops which means doubling both coops and chickens. Then she claimed that there were 40 (20+20) chickens. When they were asked to find the chickens in 15 coops, Doğan provided a relation between 15 coops and 10 coops by considering the claims of Mahmut (there are 20 chickens in 5 coops) and Birce (there are 40 chickens in 10 coops). Doğan claimed that there are 60 chickens in 15 coops by adding 5 coops to 10 coops. He operated with the composite unit as Birce did. Students realized that adding 5 coops means adding 20 chickens. They interpreted the change in the number of coops and reflection to the number of chickens. To see whether students could build the same relationship in reverse of this situation, students were given the number of chickens and the number of coops were asked. They offered to use the pictures to reason the relationship between known and unknown situations. They knew that there were 60 chickens in 15 coops. Ilker explained that there must be 20 more chickens to get 80 chickens. He expressed that 20 chickens require 5 coops, as in the first question. He added 5 coops and claimed that there needs to be 20 coops for 80 chickens. The students' explanations were also interpreted according to Toulmin’s analysis scheme presented in Figure 4.36 below.

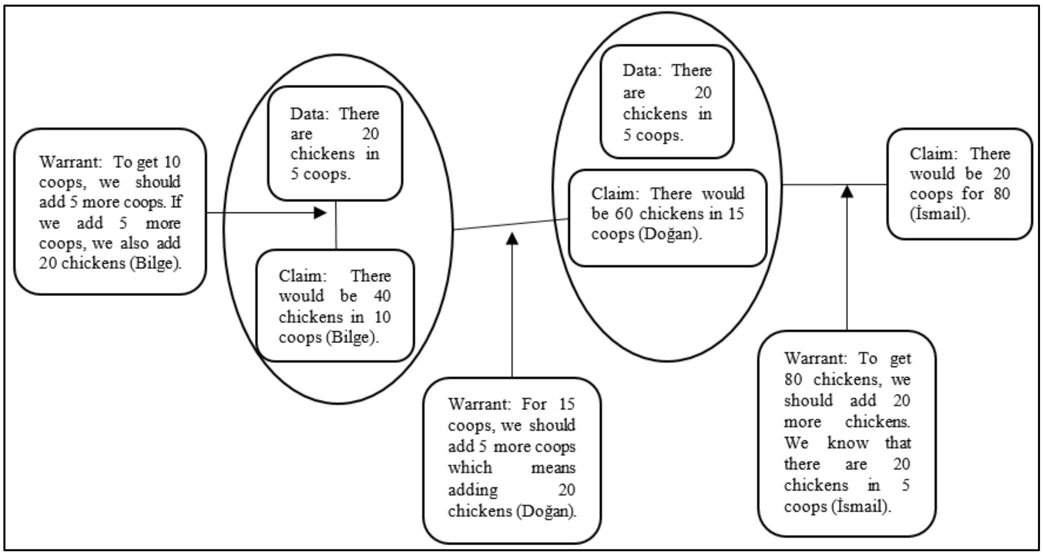


Figure 4. 36. Toulmin's Analysis scheme regarding students' reasoning related to chicken-coop task

In the eleventh day, another activity related to introducing multiplication symbolically was presented to students. They were asked to find the number of ladybugs on the leaves using multiplication (Figure 4.37). They found that there are 12 ladybugs on 4 leaves which is explained below in TAS 10. The focus of the current idea is the answers of students to follow-up questions as in the dialogue below:

> Teacher: If there were 8 leaves, how many ladybugs would be on them?
> Meliha: Four more leaves.
> Ajda: If we double the leaves, we also double the ladybugs.
> Ali: 24 ladybugs.
> Teacher: How do you show it via multiplication operation?
> Birce: 8x3 is equal to 24
> Teacher: Assume that there are 48 ladybugs. How many leaves would you need for these ladybugs?
> Doğan: I don't know how to count by 12s.
> Teacher: Why do you want to count by 12s.
> Doğan: If I count by 12s until 48, I can add four leaves for them.
> Birce: I found something. 24+24 makes 48. Can we use it?
> Hakan: We can double the picture.
> Ajda: You are right; we should double 8 leaves. 16 leaves.
> (Teacher drew on the board students' reasoning as in Figure 4.37).

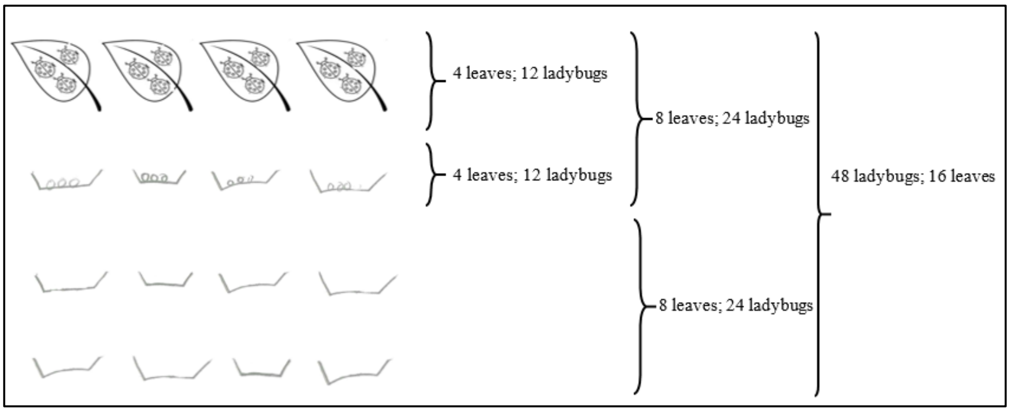


Figure 4. 37. Drawings related to double matching of leaves and ladybugs

As it can be seen in the dialogue and the drawing related to the process of double matching leaves and ladybugs, students developed a relationship between the numbers in the question and those they have already known which are the claims of previous questions. For instance, when students were asked to find the ladybugs on 8 leaves, they used their previous claim that there are 12 ladybugs on 4 leaves as it is interpreted with Toulmin's analysis scheme in Figure 4.38. In this argument, students discovered the double relationship between the leaves, then they doubled the ladybugs, too. In contrast, the second question, students discovered the double relationship between the ladybugs, then they doubled the leaves, too.

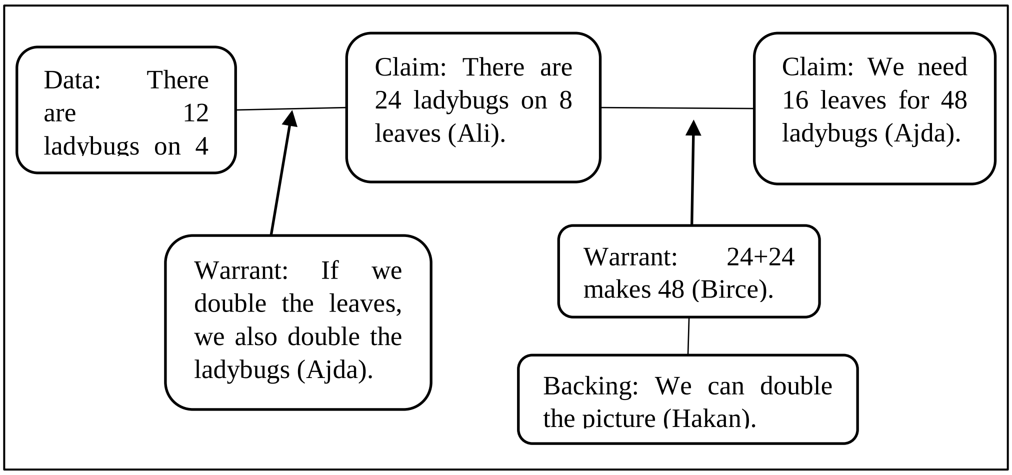


Figure 4. 38. Toulmin’s Analysis scheme regarding students’ reasoning related to ladybug-leave task

Therefore, students interpreted the ratio between the groups to compare the objects in these groups or vice versa. They worked simultaneously with both components (group number and group size) in a double-matching process (covariance). This idea was emerged frequently in the classroom by students to develop multiplicative reasoning in the following tasks. After a while, they used this idea as both data and warrants for further claims. This helped us conclude that this idea became a taken-as-shared idea for the students. For that reason, they were not questioned anymore to make them provide a warrant.

4.4. Classroom Mathematical Practice 4: Analyzing components and properties of multiplication operation by modeling with equal groups and arrays

The fourth practice that emerged as students engaged in the instructional sequence refers to the meaning of multiplication. In this practice, students focused on actions (i.e., "groups of," "set of," "building arrays") that relate to repeated addition and multiplication concepts. Here, the goal was for students to define equal groups as a starting point to represent given multiplicative situations. That is to say, students developed an understanding related to the components of multiplication (multiplier, multiplicand and product) by making connection between equal group representation, array models, and repeated addition.

4.4.1. Idea 10: Interpreting the meaning of multiplier and multiplicand in equal group representation

Activities in the learning sequence were constructed on real-life situations from the beginning of this teaching experiment. The target was to familiarize students with the contexts and feel free to develop their strategies. The tasks including equal groups were presented in the classroom. Students developed various ways of reasoning such as skip counting and repeated addition. After a while, students were introduced to multiplication symbolically. The relationship between the components of multiplication operation and equal group representation through pictures and drawings was interpreted. That is to say, while the group number is defined as the multiplier, group size is defined as the multiplicand. On the eleventh day, students were presented an activity related to multiplication operation. Students were asked to find the number of ladybugs on the leaves using multiplication (Figure 4.39). Students interpreted the given picture in the activity as equal groups, as can be deduced from the dialogue below:

> Melek: There are four leaves and three ladybugs on each leaf.
> Egemen: Leaves are given as groups in the picture. It means four groups and 3 ladybugs in each group.
> Hakan: 4 x 3 gives the number of ladybugs.
> Ali: Hakan is right. It makes 4 times 3. The first number in the operation represents the number of groups while the second number represents the number of elements in each group.

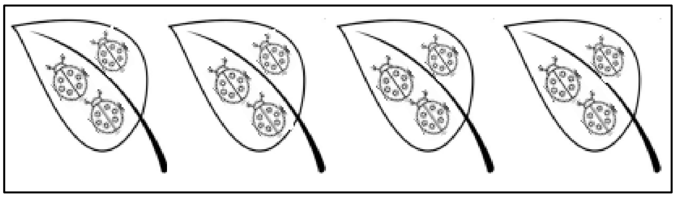


Figure 4. 39. Picture relayed to ladybugs on the leaves.

Students wrote multiplication operation in the order they read equal groups like “4 groups of 3” and “4 times 3”. Firstly, they decide on the groups, then they interpreted the multiplication operation as multiplier and multiplicand to place the number of groups and group size respectively. Students could make a connection between the groups and the multiplication operation. They reflected equal groups to multiplication.

On the same day, students were presented a problem: "A mother wants to give 2 cookies to each child. If there are 5 children, how many cookies should the mother have?". It was stated that they should use an operation. Students took their time to work on the problem as they wished. The teacher and the researcher walked around the desks and watched what they were doing. It was observed that students were comparing their drawings related to cookies with their mates.

After they expressed that they finished, classroom discussion started. Students used their drawings to satisfy their solutions. Most of the students drew cookies and explained their drawings as equal groups. They drew 2 cookies for each child. They circled cookies two by two and stated that each group would be given to a child. After they discussed the groups, they explained the operations they used (Figure 4.40). Students who used multiplication claimed that there are 5 times 2 which means 5x2. They used the group of two cookies as composite units and iterated these units through multiplication.

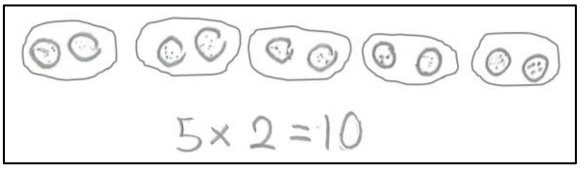


Figure 4. 40. Drawings connected to repeated addition and multiplication

Equal groups in multiplication operations are not evident as much as they are in pictures and repeated addition operations. In order to see their interpretation related of multiplication, students were asked to pose problems. For instance, on the thirteenth day, students were asked to pose a problem for the operation 5x4. In this activity, students focused on the meaning of multiplication operation as "…. times …". That is to say; students interpreted 5x4 as "five times four" where five stands for the number of groups (multiplier) and four stands for the size of the group (multiplicand). Some of the students supported their interpretations and claims by drawing equal groups related to their problems. These students shared their reasoning in the classroom discussion as in the dialogue below:

Teacher: Did you pose a problem? Who wants to share with us?
Karan: There are 5 bird cages in a pet shop. If there are 4 birds in each cage, how many birds are there in the pet shop?
Teacher: Do you think that this problem is appropriate for 5x4?
Karan: 5x4 means 5 times 4. I need five groups. I thought of the cages as groups. I drew 5 cages (He showed his drawing given in Figure 4.41). I need 4 objects in each group. I drew 4 birds in each cage. Therefore, I believe that I posed a correct problem.
Egemen: Group number and group size are correct. Then, the problem is appropriate for the operation.
Melek: I agree with Karan. I thought as he did.

Teacher: Melek, can you share yours?
Melek: I have 5 knitting yarns and 4 needles on each yarn. How many needles do I have?
Eyüp: How many yarns did you say?
Melek: 5 yarns (showed her drawing in Figure 4.42). I represented the groups with 5 yarns and 4 needles for each yarn.

As Karan and Melek explained in the dialogue, some of the students used drawing to clarify their interpretation of equal groups needed for the given operation. They used the drawing as warrants to support their claims. They emphasized the groups and elements in each group in their problems by using these drawings. Hence, they used a visual representation of the given operation as warrants while they used real-life representation as to the claim of the posed problem. These visual representations played an essential role in connecting operations and the components of problems in the tasks related to problem-solving and posing. During these activities, students analyzed the operations and the contexts of the problems to define equal groups. Students did not use drawing in the following lessons to visualize the given multiplicative situations or discuss the givens by asking explanations. They analyzed the given multiplicative situations as groups and elements of each group without using another representation.

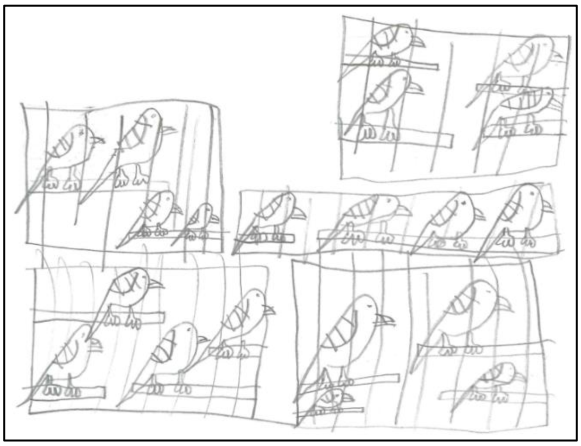


Figure 4. 41. Karan’s drawing related to his problem for 5x4.

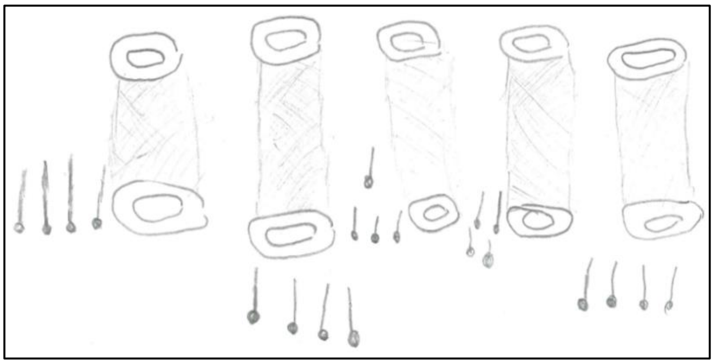


Figure 4. 42. Melek’s drawing related to her problem for 5x4.

In the same lesson, there were a few students who posed problems for 4x5 instead of 5x4. At that point, their drawing played a crucial role to make their thinking explicit. Sample dialogue from classroom discussion is given below

> Eyüp: There were four fish ponds. I saw five fish in each fish pond. How many fish did I see?
> Zehra: Wait a minute. I am not sure whether it fits 5x4.
> Eyüp: I drew it (Showing his booklet). There are four equal groups with an equal number of fish in each group (Figure 4.43).
> Birce: Eyüp, this drawing represents 4x5. We have 5x4 to pose a problem which means five times four. The results are equal, but operations are different. You should draw five fish ponds and four fish in each. You did the opposite.

Figure 4. 43. Eyüp’s drawing from his booklet

As in the excerpt, students denoted the multiplier as the number of groups and the multiplicand as the group size. They discussed the components of equal groups in detail by using visual representations as warrants to support their ideas as given in the Toulmin's analysis scheme presented in Figure 4.44 below.

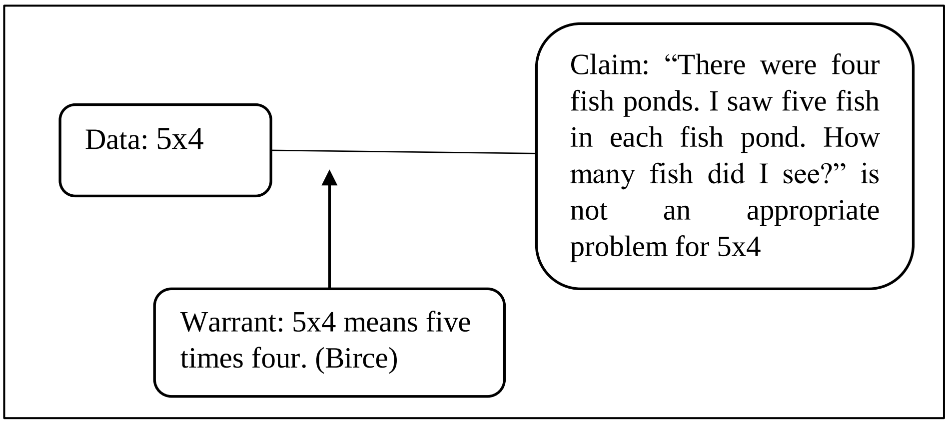


Figure 4. 44. Toulmin's Analysis scheme interpreting the meaning of multiplier and multiplicand in equal group representation of 5x4.

It should be noted that students started talking about commutativity although they did not learn about it formally. Birce explained that they find the same product with 5x4 and 4x5. However, they did not question or, try to prove the reason for this situation. The researcher noted this to question and observe in the following days.

In the same lesson, students were asked a story problem which is "Planting Tree" which is "Selay plants two trees every month. How many trees does Selay plant in 9 months?". As described above, students drew trees and wrote number sentences to connect these drawings. They interpreted the months as equal groups and drew figures to represent trees. These drawings helped them to see the equal groups visually. Then they concluded that there are 9 groups with 2 trees. Therefore, they wrote 9 times 2 and collected them by using multiplication as they explained below:

> Melek: Firstly, I drew potted plants (She showed them on her booklet as shown in Figure 4.45). I drew 9 pots and 2 trees in each pot. For that reason, I multiplied 9 and 2.
> Emin: I drew nine sets of twos. I wrote on them "1st month, 2nd month …" (He showed on her booklet as shown in Figure 4.46). I drew 2 flowers in each.
> Zuhal: You drew equal groups.
> Birce: I drew, too. (She showed on her booklet as in Figure 4.47). I drew balls representing the trees. Look at my drawing. I found the sum of the number of trees by multiplication.
> Teacher: Why did you draw?

Karan: It is enjoyable. I like painting. Moreover, it helped me to see the groups.
Egemen: We can see the groups directly. The number of months is the multiplier; the number of trees is the multiplicand.
Birce: I drew to make my solution understandable.
Halil: I used skip counting but drew groups, too. I saw how many fingers I should count clearly on the drawing (TAS 7).

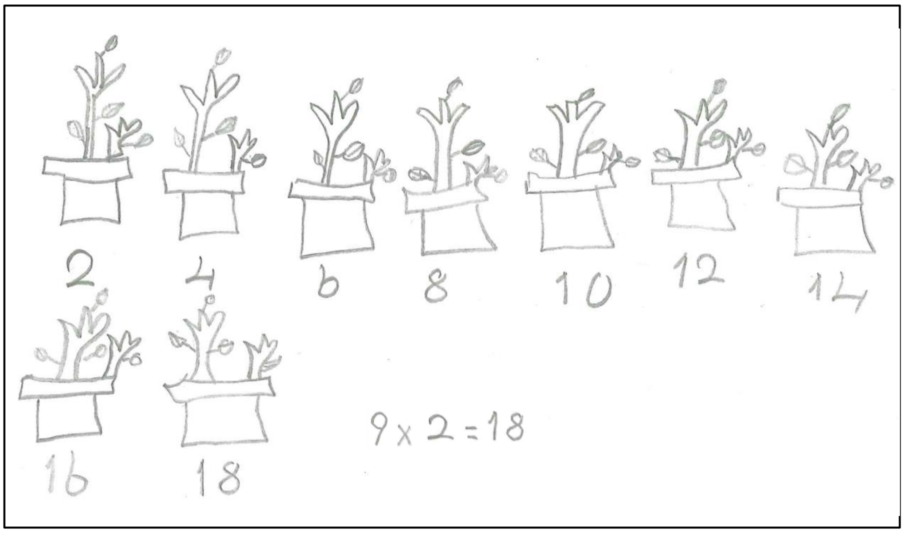


Figure 4. 45. Melek’s drawing related to “Planting Tree” problem

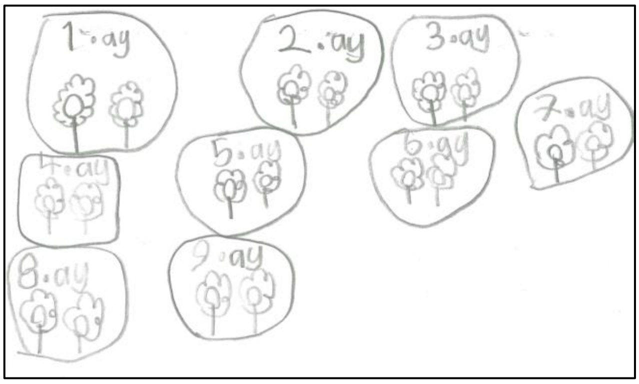


Figure 4. 46. Emin’s drawing related to “Planting Tree” problem

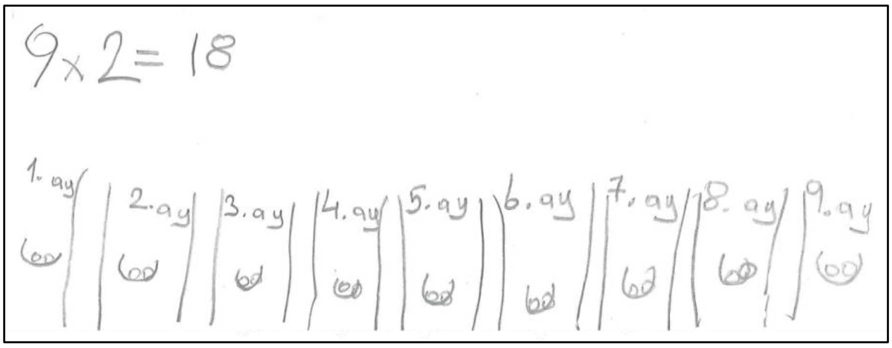


Figure 4. 47. Birce’s drawing related to “Planting Tree” problem

As they explained, students drew pictures of trees to represent the problem visually. They applied this representation to make the problem clear and visible for themselves. They matched the terms (months and trees) in the problems with the terms of multiplier and multiplicand. It helped them see equal groups and decide on the operation to make calculations. Students used multiplication since there were equal groups on their drawings. They supported their claims that they can find the total number of trees with multiplication by using the visuals they drew, as in the Toulmin's analysis scheme presented in Figure 4.48 below:

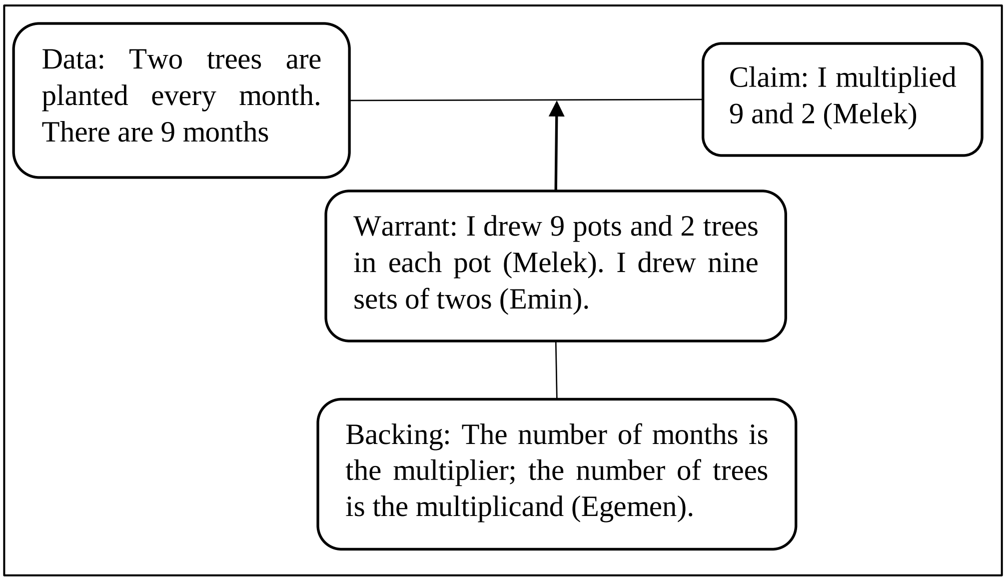


Figure 4. 48. Toulmin’s Analysis scheme regarding interpreting the meaning of multiplier and multiplicand in equal group representation of months and trees.

In the following activities, classroom discussions were observed on how students interpret given situations as equal groups. Students did not question how to decide on the multiplier and multiplicand in the visuals of given tasks. For instance, on the fourteenth day, a picture of jars of candies was given and students were asked to pose a problem related to the picture as in the Figure 4.49.

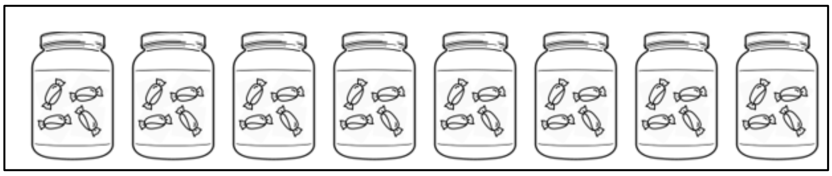


Figure 4. 49. Picture of the jars of candies given on the fourteenth day.

Classroom discussion showed that students directly focused on posing problems related to 8x4. No one questioned why the problems were posed for 8x4. Argumentation analyses showed that this idea became taken-as-shared since students no longer had to explain the components of equal groups to justify why they used 8x4. That is to say, they did not need to specify the number of jars as the multiplier and candies as the multiplicand. It has been seen that warrants were dropped off day since that day.

4.4.2. Idea 11: Connecting repeated addition and multiplication operations by interpreting multiplier and multiplicand

Students discovered that multiplicative situations comprise making relationships between the number of groups (multiplier) and the size of the groups (multiplicand). Initially, students used drawings in their booklets to show how they had modeled groups of objects having the same number in each group. They used pictures in tasks to circle groups with equal amounts of objects if a picture was given. Hence, they could skip-count these groups to find the total number of objects. After a while, students used repeated addition to find the total number of objects. They started the tasks by looking for regularity in repeated reasoning since they recognized that they were repeatedly adding the same number (multiplicand). For instance, in the ninth lesson, students were given pictures of frogs placed on 3 lily pads 2 by 2. Students were asked how many frogs there were. They counted frogs by 2s and said 6. Then, they were asked to use an operation to satisfy their claims. Students had been used to skip count the given objects, not using number sentences to represent this procedure. Through the discussion in the classroom, they could decide to write number sentence representing repeated addition symbolically as in Figure 4.50.

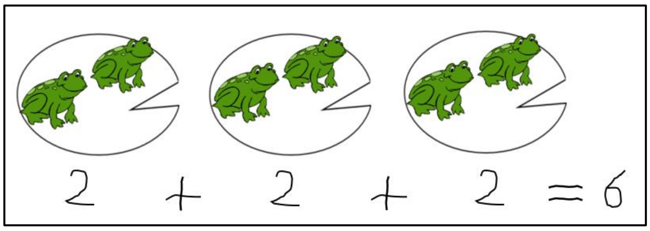


Figure 4. 50. Using repeated addition to find the total number of objects in equal groups.

As it is explained in TAS 10, students were introduced to multiplication on the eleventh day. After that day, they were expected to use multiplication to find the total number of objects in equal groups. It was observed that students tended to use both repeated addition and multiplication. For instance, in the twelfth day, students were given a task including a problem about oranges on the trees. The problem was "There are 4 orange trees in Selda's garden. If there are 5 oranges in each tree, how many oranges are there in the garden?". Students shared their solution with the others as in the dialogue below:

Teacher: How can you find the answer?
Classroom: By addition (They said altogether)
Teacher: Will you add 4 by 4?
Classroom: No. 5 by 5.
Teacher: Can someone explain?
Hakan: I drew 4 trees and 5 oranges on each tree (see Figure 4.51).
Zehra: We should find the sum of oranges in all the trees.
Zuhal: I used repeated addition as 5+5+5+5 (see Figure 4.51).
Birce: I used multiplication as 4x5.
Yalçın: There are 4 groups which means 4 times 5.
Ali: It doesn't matter. We can use both repeated addition and multiplication.

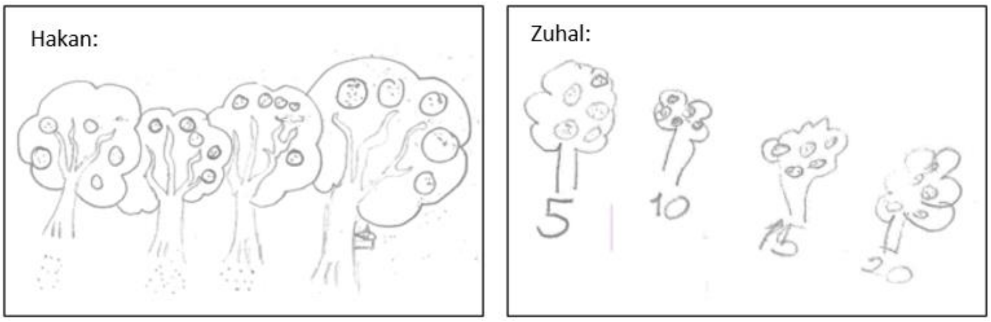


Figure 4. 51. Sample drawings by Hakan and Zuhal.

As noticed in the dialogue clearly, students claimed that they could use both operations of repeated addition and multiplication. For both operations, they focused on specifying equal groups to be able to write number sentences. While they interpreted equal groups as multiplier and multiplicand for multiplication, they added multiplicand repeatedly for repeated addition. To be able to see their conception of the relationship between repeated addition and multiplication, students were given a picture (Figure 4.52).

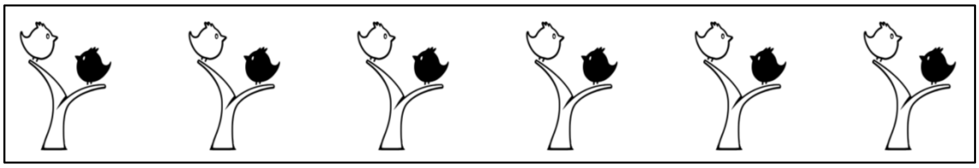


Figure 4. 52. Picture of birds on trees used on the twelfth day.

Students were asked to find the total number of birds by both repeated addition and multiplication (Figure 4.53) and explain how they write the number sentences.

$$2+2+2+2+2+2=12$$

$$6 \times 2 = \boxed{12}$$

Figure 4. 53. Zehra’s solution for birds on trees.

Students shared their solutions with the others as in the dialogue below:

Esra: We are given “6 times 2” in the picture. We will multiply 6 with 2.
Teacher: What about repeated addition?
Halil: It is clear that we will write 6 times to and collect them.
Teacher: When we look at the multiplication, we see both 6 and 2. However, in the repeated addition we use only 2. Where is 6 in repeated addition?
Hakan: We use 6, too.
Teacher: I cannot see it in the number sentence.
Doğan: We use it to decide how many times we should write 2.
Teacher: Can you show me the multiplier and multiplicand in repeated addition?
Ajda: The multiplicand is the number we write, and the multiplier is the hidden number.
Doğan: We write multiplicand repeatedly as much as the multiplier.

As it can be seen in the dialogue, students could represent given situations in number sentences of repeated addition and multiplication. The teacher asked follow-up questions to see whether students are aware of the relationship between two operations. Students could interpret the roles of multiplier and multiplicand in repeated addition. To be sure whether this idea became taken-as-shared, the researcher and the teacher decided to continue observing the reasoning of the students in the following lessons.

On the thirteenth day, students were given number sentences related to repeated addition and asked to convert them to multiplication. For instance, they were asked to write “3+3+3+3+3+3+3” as a multiplication sentence. They connected repeated addition and multiplication by interpreting multiplier and multiplicand, as in the Toulmin's analysis scheme presented in Figure 4.54 below:

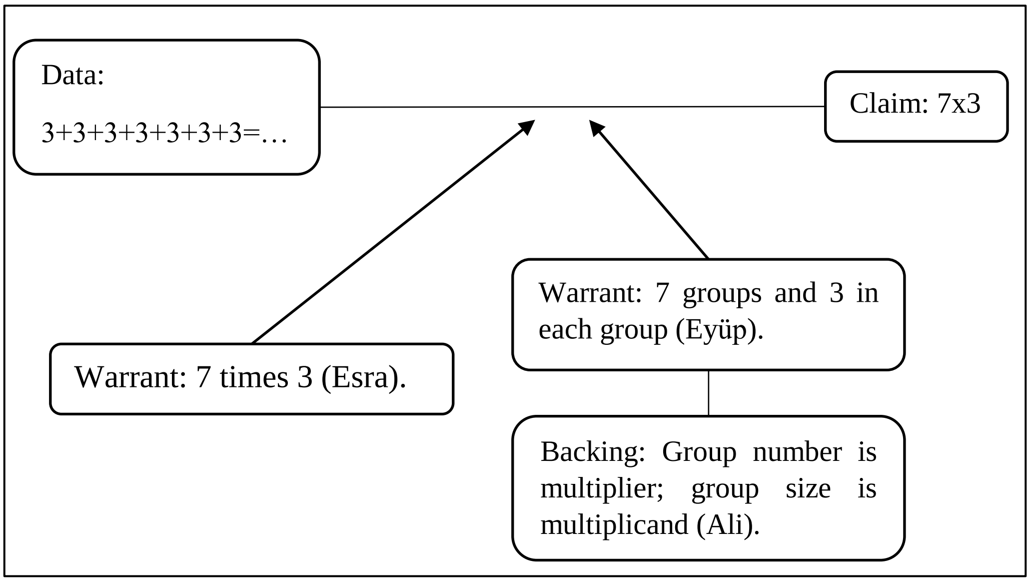


Figure 4. 54. Toulmin's Analysis scheme regarding connecting repeated addition and multiplication operations by interpreting multiplier and multiplicand.

Students discussed the components of repeated addition as multiplier and multiplicand to be able to write it in the form of multiplication. In the following activities, classroom discussions were observed to see how students connect repeated addition and multiplication. Argumentation analyses showed that this idea became taken-as-shared since students no longer had to justify how they make connection between the data and claim. It was seen that warrants were dropped off day by day.

4.4.3. Idea 12: Reasoning the effect of change in multiplier on the product

Starting from the third day, students engaged in the tasks related to equal grouping. They were asked to find the total number of objects which were given in equal groups or grouped equally by the students. While engaging in such activities, students discussed follow-up questions verbally as planned by the researchers before implementing the HLT. Students made reasoning by considering the effect of change in the number of equal groups. In other words, students connected what is known and what is asked. For instance, on the seventh day, students discussed the total number of marbles where there were 5 plates and 5 marbles in each plate. Students found that there were 25 marbles by using skip counting. After that, students were directed follow-up questions as in the dialogue below:

> Teacher: How many marbles would be if there were 6 plates of marbles?
> Meliha: There would be 30 marbles.
> Teacher: How did you find?
> Meliha: I counted 6 fingers by 5s.
> Hakan: I didn't count all the fingers. I know that there are 25 marbles in 5 plates. We add 1 plate to make them 6 plates. Since there are 5 marbles in each plate, we add 5 marbles to 25 marbles. 25+5 makes 30 (Figure 4.55).
> Teacher: Did you understand Hakan' explanation? Do you agree with him?
> Birce: I can't believe that I could not discover this method. I counted 6 fingers. Thanks Hakan.
> Teacher: Next question is coming. How many marbles would be if there were 7 plates of marbles?
> Halil: 35 marbles. I counted by 5s.
> Esra: I used skip counting, too.
> Zehra: I used Hakan' method. We have 5 plates of marbles. We know that there are 25 marbles. If we add 2 more plates, it makes 7 plates. Adding two plates means that we add 10 marbles to 25 marbles. We have 35 marbles in 7 plates (Figure 4.55).

As can be inferred from the dialogue, two students used the number of marbles in 5 plates as data for the following claims. They constructed their claims on 25 marbles in 5 plates by making sense of multiplicative reasoning. When they were asked to find the number of marbles on 6 plates, Hakan made reasoning by considering the plates added. Instead of counting 6 fingers and claiming that there would be 30 marbles in 6 plates as most of the students did, he applied what he had already known. He compared the previous situation with the new situation. He discovered that the difference is a plate with 5 marbles in it. He explained this situation as warrant to find the number of marbles by adding 5 to 25 marbles. In the same way, Zehra approached the questions related to the marbles in 7 plates by analyzing the difference between the situation of 5 plates. She focused on the marbles in 2 plates added to 5 plates to make them 7 plates. She provided warrant that adding 10 to 25 marbles gives the answer since adding 2 plates means adding 10 marbles. Hence, both Hakan and Zehra approached the question as a more advanced version of the previous one while the others approached it as a new situation as shown in Figure 4.55 below.

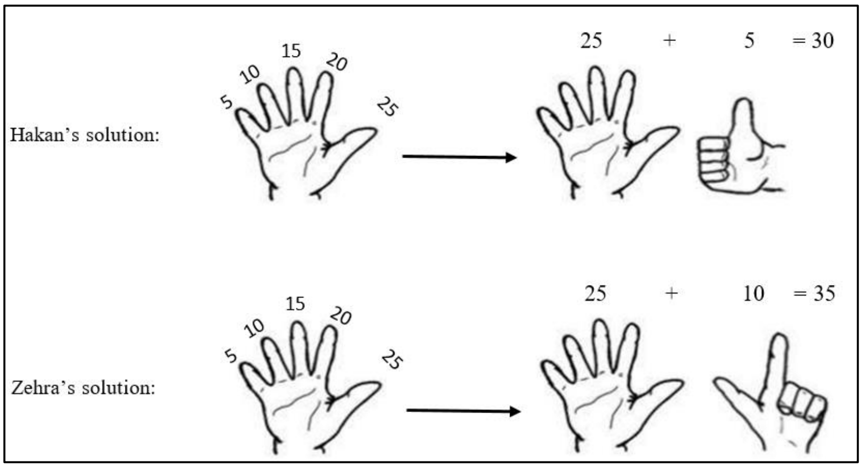


Figure 4. 55. Illustration of Hakan's and Zehra's reasoning

Students' reasoning related to the change in group number took attention of the teacher and the researcher. This situation was noted to be observed in the following lessons. In the ninth lesson, tasks including pictures of objects in equal groups were presented to students to find the total amount of objects. After they completed the tasks and discussed them in the classroom, the teacher asked follow-up questions using the same context as asked above. In the first task, there were 5 beehives and 3 bees on each hive. Students counted by 3s and found the total amount of bees. After they stated that there were 15 bees on 5 hives, the teacher asked follow-up questions as in dialogue below:

> Teacher: If I add 2 more beehives, what can you say about the number of bees?
> Ajda: There are 15 bees on 5 beehives. I counted by 3s on 15 (Figure 4.56).
> Teacher: What did you find?
> Ajda: 15, 18, 21. There would be 21 bees.
> Ali: I found 21 bees, too. However, I didn't count by 3s.
> Doğan: I counted 7 fingers as by 3s. Ali, how could you find without counting by 3s?
> Ali: We add 2 more beehives. It means that we will add 6 more bees. 15+6 makes 21 (Figure 4.56).
> Ajda: Even more practical than my solution.

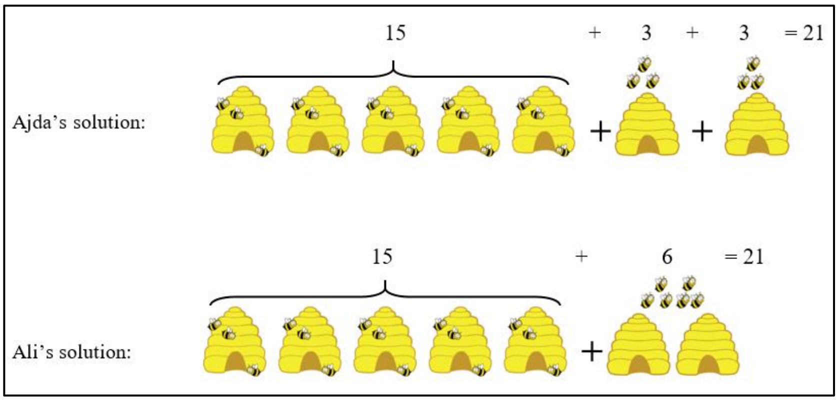


Figure 4. 56. Illustration of Ajda's and Ali's reasoning

As seen in the dialogue, students used the counting-on strategy instead of counting by starting from the first hive. Students kept the number of bees on 5 hives in mind and used this number as the initial number to count on. While counting, they considered the hives as a composite unit of 3 bees. They kept track of how many times hives were added, adding 3 bees with each hive. As can be understood from Figure 4.56, Ajda iterated the hives and added 3 to 15 for each hive added. On the other hand, Ali did not add the bees separately. He computed the bees on two hives first. Then, he added the total number of bees on new hives to 15 bees. Hence, they made their claims by reasoning the change of hives as the warrant.

As it can be seen in the excerpts above students focused on the change in group numbers. The team interpreted this idea as a clue for students' reasoning for the change of multiplier since group numbers represent the multipliers in multiplication. So, the team members wondered whether students would make this reasoning for multiplication in the following lessons. For that reason, they asked follow-up questions to observe students' thinking through classroom discussions. For instance, as it is explained above, a picture of jars of candies was given and students were asked to pose a problem related to the picture on the fourteenth day (Figure 4.57). There were 8 jars and 4 candies in each jar. They found that there were 32 candies via 8x4. Then, the teacher asked what-if questions to students as in the dialogue given below:

Teacher: What if there were 7 jars of candies?
Eyüp: It makes 7x4. We can count by 4s.
Halil: Or, in order to go from 8 to 7 (he showed his 8 fingers, then closed one finger), we can subtract 4 from 32. It is 28.
Esra: Subtraction is more practical than counting.
Teacher: What if there were 11 jars of candies?
Mahmut: 11x4
Egemen: We can count on 32.
Hakan: You will count three times 4 on 32?
Egemen: Yes, it makes 44.
Ali: I counted on 40.
Teacher: Can you explain Ali:
Ali: I know that 10x4 makes 40. For 11, I counted one more 4 on 40.

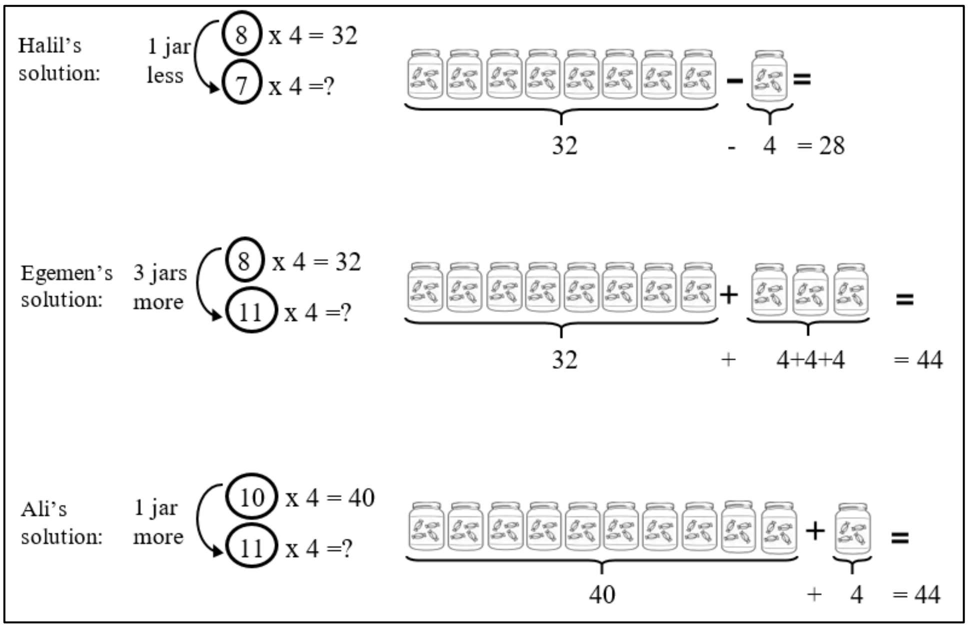


Figure 4. 57. Illustration of students' reasoning for the jar of candies.

As can be deduced from the dialogue, students used the known facts to make reasoning related to the multiplier. To make it clear, the researcher illustrated students' solutions as in the Figure 4.57 above. While they used 8x4 to find 7x4, they used 8x4 and 10x4 to find 11x4. To be able to do that they interpreted the change in the multiplier. They added to or subtracted from the product of known fact to be able to decide on the new product. Halil subtracted 4 from 32 once since the difference between 8 and 7 is one. On the other hand, Egemen added 4 on 32 three times since he interpreted the difference between 8 and 11 as increasing the number of jars. In a similar way, Ali added 4 on 40 once since he started with 10x4 and interpreted the difference between 10 and 11. That is to say, students interpreted the difference in multiplier to be able to reach the new product as in the Toulmin's analysis scheme presented in Figure 4.58 below.

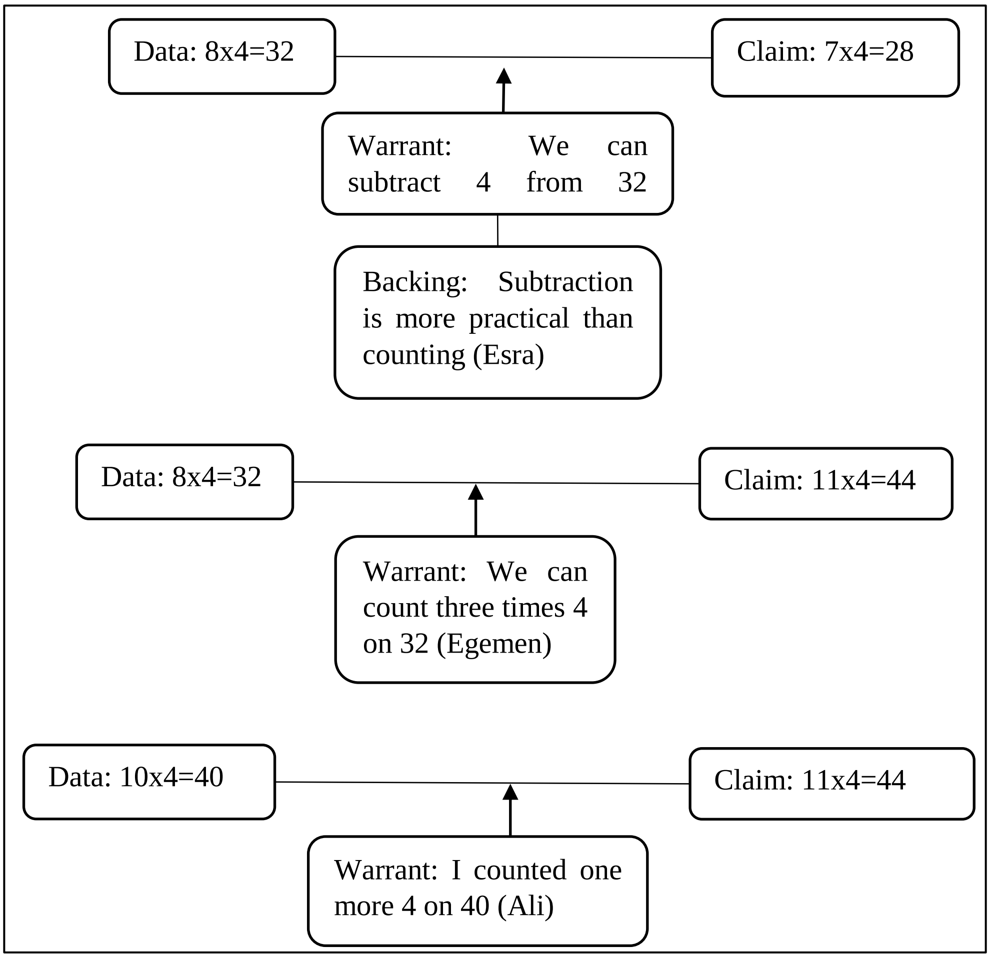


Figure 4. 58. Toulmin's Analysis scheme regarding reasoning the effect of change in multiplier on the product.

As in this lesson, students interpreted the difference between the multipliers of two operations and make a connection between the products of two operations. This idea was emerged frequently in the classroom discussions in the following lessons while multiplying two numbers. After a while, students just stated that they connected the given operation with an operation they have already known. They did not provide a warrant since the others did not ask. That is to say, the warrants were dropped off in the further lessons. This helped us conclude that this idea became a taken-as-shared idea for the students. For that reason, they were not questioned anymore to make them provide a warrant.

4.4.4. Idea 13: Analyzing arrays by interpreting rows and columns as multipliers and multiplicands

In the first week, students worked on the tasks related to counting objects in pictures to decide how many objects there were. During these activities, ideas about partitioning objects into equal groups and iterating linked units emerged. Students supported these ideas by providing warrants as explained above in the sections of CMP 2 and CMP 3. While they were explaining their reasoning related to their claims, it attracted attention that students took advantage of the arrangement of objects. The majority of the students considered how the objects were organized linearly. They interpreted the rows and lines on which objects were properly aligned. Sample drawings and students' interpretations are provided below in Figure 4.59. Their perceptions regarding the order of objects showed that students were tended to align objects by organizing them. Moreover, they used the term "row" which is used in array representation, to define the groups. It was noted down as a clue for students' sense of array representation informally.

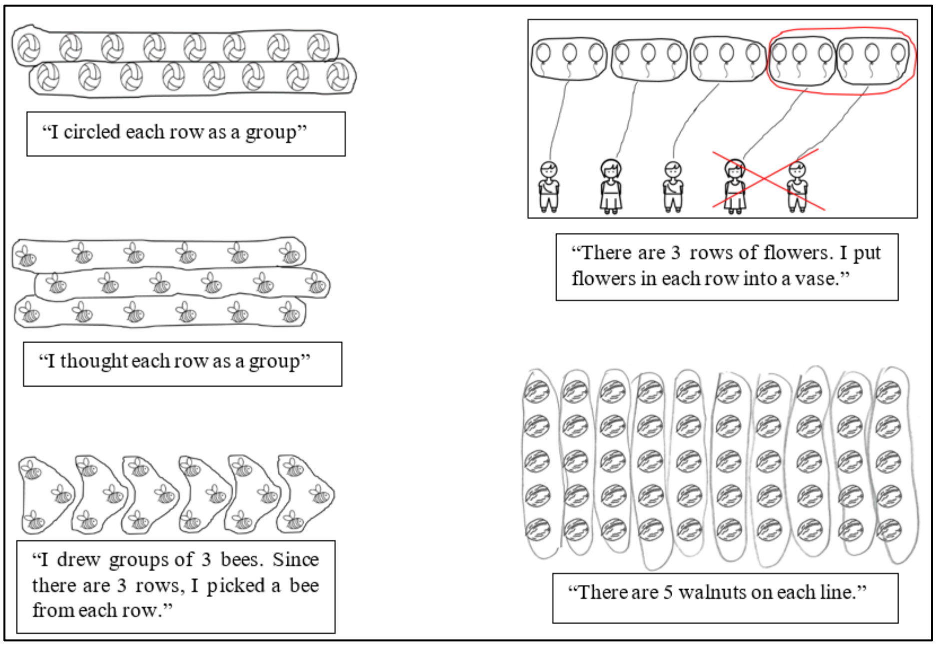


Figure 4. 59. Students' intuitive explanations related to array representation

In the following lessons, students were encouraged to use multiple modes of representation to reflect their thinking of multiplicative relationships and envision equal-sized groups. Some of the students intrinsically made productive use of the array as a multiplicative model involving the same number of objects in each row and column. On the eleventh day, students interpreted the number sentence of "3+3+3+3+3+3" as "six times three" where six stands for the number of groups (multiplier) and three stands for the size of the group (multiplicand), as in the dialogue given below:

> Karan: I drew six shelves and aligned apples in them. Each shelf represents a group of 3 apples (Figure 4.60).
> Melek: I drew a table and arranged balls on this table (Figure 4.60).
> Doğan: You made three groups with six balls in each. It represents "6+6+6". The result does not change, but it does not reflect the given operation either.
> Ali: Doğan, how do you know her groups? Maybe she grouped vertically.
> Birce: Instead of grouping rows, we can group columns. Columns have equal size of objects too.

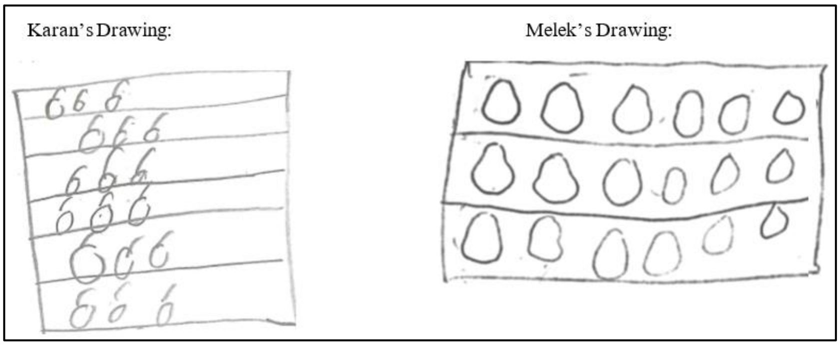


Figure 4. 60. Array representations for ""3+3+3+3+3+3"

As in the excerpt, students started arranging objects in rows and columns to get equal numbers of objects in each row and column. Students were not directed to specify groups through rows or columns. They were free to form equal groups as they wished but respected the given multiplicative situation. Hence, they modeled given operations on the array with rows and columns representing the two inputs of a binary operation. They used these models to define equal groups through collective discourse. Students constructed a meaningful mental model of the array that intuitively reflects important mathematical ideas related to multiplication. This model defined equal groups, made sense of multiplicative situations, and structured their solutions' thinking and approaches. However, it was observed that students had difficulty aligning the objects in their representation perfectly. For that reason, they were provided a squared base in the activity sheet.

Students were provided a squared base in activity sheets to observe whether this idea would become taken as shared during the conversation in the following days. They were expected to specify rows and columns as aligned parallel to the sides of the rectangle by filling unit squares in a rectangle. On the sixteenth day, students were given a picture of 15 pencils and asked to write a multiplication operation related to that picture. While some of the students circled the given pencils to form equal groups, some of them used the squared area to align pencils in rows or columns, as can be deduced from the dialogue and Figure 4.61 below:

> Melek: I think we can find 15 by multiplying 3 and 5 because 3 times 5 makes 15.
> Teacher: How did you decide that?
> Melek: We need equal groups. I placed the pencils in squares. I tried to make them equal in each row. I could place them equally on 3 rows.
> Ali: Rows represent equal groups.
> (Students who used the squared base, in the same way, stated that they agree with Melek)
> Halil: I placed them on rows too. However, I used 5 rows as the inverse of yours. I drew 3 pencils in each row.
> (Students who used the squared base, in the same way, stated that they agree with Halil)
> Zuhal: What is your claim?
> Halil: My claim is that 5 times 3 makes 15.
> Esra: I claimed the same operation, but I did not used the squared base. I circled the pencils to form equal groups. Can you show how you drew? Next time, I will try to make like you did.
> Teacher: Is there another claim about finding 15 pencils due to multiplication operation?
> Doğan: No, we can find 15 in the verbal chain while skip counting by 3s and 5s.
> Birce: I tried to move pencils to get new equal groups, but I couldn't. Because rectangular shape of the pencils became deformed which means that the groups are not equal anymore.

Some of the students used the squared base to form equal groups in rectangular borders, as Melek and Halil explained in detail. They claimed that 15 could be found as the result of 3x5 and 5x3 as illustrated in the Figure 4.61. They placed the pencils in the squared cells and specified the rows as groups. They used the equal groups in these representations to write a multiplication operation whose first number represents the number of groups (number of rows) and the second number represents pencils in each row.

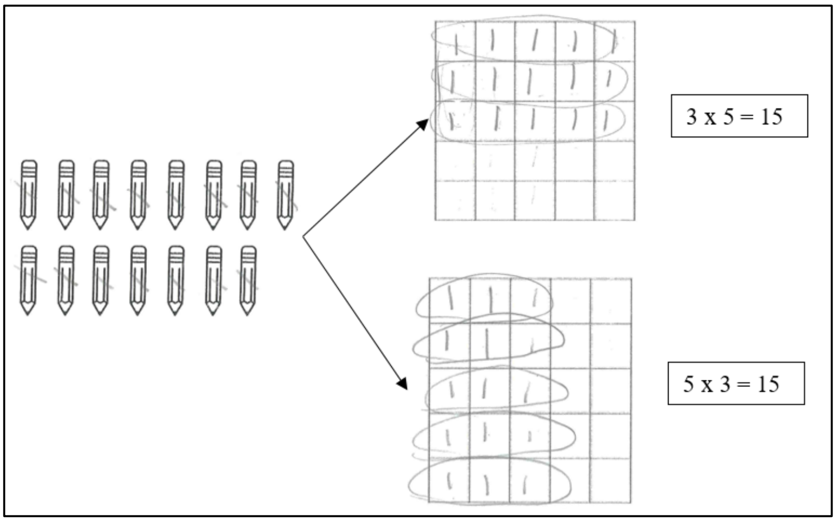


Figure 4. 61. Sample array representations from students written works.

As summarized by Toulmin's analysis scheme in Figure 4.62 below, students administered to array representation to form equal groups which carry meaning of multiplication. Students intuitively made sense of composite units (rows) in array representation. In Figure 4.61 above, students circled the pencils in each row representing composite units. Their drawings and representations played a role in defining equal groups to support their claims. They were also aware of the rectangular nature of the array model, which requires equal amounts of objects in each row and column. They made reasoning that rectangular border of an array is deformed if the groups do not include equal amounts of objects.

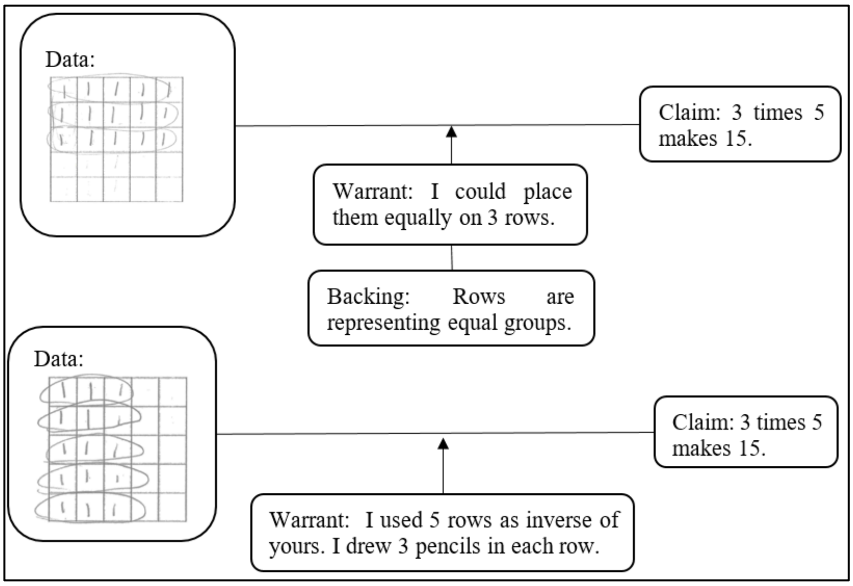


Figure 4. 62. Toulmin's Analysis scheme regarding composing arrays from rows to illustrate equal groups

Each student was provided personal reusable squared sheets to show equal groups in the following days. Moreover, a large squared panel and circular stickers were prepared to form rectangular arrays to share with others (Figure 4.63). These tools helped students share their reasoning easily and make the groups visible and understandable for others.

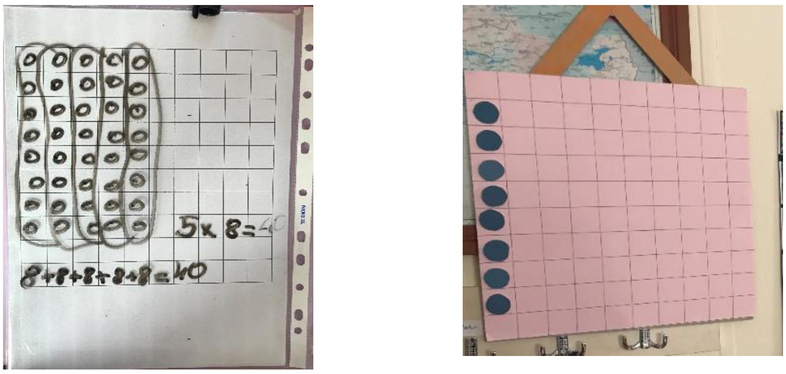
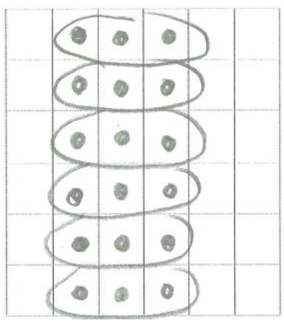


Figure 4. 63. Personal squared sheets and squared panel.

On the nineteenth day, a problem-solving task was presented in the classroom. Students were asked to work on the problem: "Aslı saw 6 mouse holes in the garden. If there are 3 mice in each hole, how many mice are there in 6 holes?". While students were working on the problem, the researcher and teacher walked around the desks and watched students. It was seen that most of the students got the advantage of the squared base to form rectangular arrays to represent mice and holes. While some of them used columns as composite units, others used the rows as composite unite representing equal groups as in Figure 4.64. Students discussed on the problem by using their arrays as in the dialogue below:

> Esra: I drew in squares and found 18 mouse holes.
> Zuhal: I multiplied 6 and 3 since there are 6 times 3 in the array I composed.
> Birce: Can you show your array?
> Zuhal: I placed 3 dots in each row (showing her booklet). There are 6 rows. I grouped each row. Consequently, I counted by 3s and found 18.
> Teacher: What do dots and groups represent in the array?
> Zuhal: Dots mean mice. My groups mean mouse holes.
> Ajda: I drew in the same way.
> Teacher: Did you draw dots to represent mice as Zuhal did?
> Ajda: Same. Since there are 3 mice in each hole, I drew dots 3 by 3 and skip counted.
> Meliha: I counted by 3 as you did since there are equal amounts of dots in each row.
> Ilker: Not only rows, but also columns should include equal number of dots.
> Birce: I counted as you did. But, I circled my groups differently.
> Egemen: How did you do Birce? Can you show yours?
> (Birce showed her booklet in which she circled columns as groups)
> Burak: She drew as I did. I considered the columns as groups too.
> Eyüp: I always use the columns as groups.
> Teacher: Does it matter to use rows or columns?
> Hakan: I think, it doesn't matter. We all found the same result by multiplying 6 and 3.
> Teacher: Why?
> Kadir: We can skip-count both vertically and horizontally.
> Zehra: We can form equal groups by rows and columns since the array is rectangular-shaped.



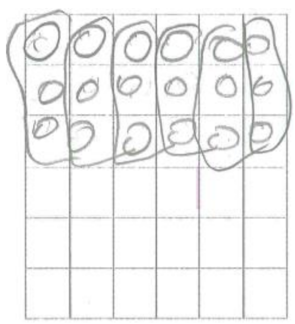


Figure 4. 64. Sample arrays to form equal groups by rows and columns

Students claimed that there are 18 mice in 6 holes by processing the operation of "6x3". They supported their claims with the array representations they drew, summarized in Toulmin's analysis scheme in Figure 4.65 below. Students administered these array models to specify the number of groups and elements in each group to decide on the operation to make a claim related to the answer to the problem. Two types of representation emerged as warrant to their claims. While some used array vertically to group columns, the others used array horizontally to group rows. They also discussed that the directions of groups do not matter since the array model is rectangular, which means including equal columns and equal rows. For that reason, they did not question the direction of grouping. They were aware that it is possible to count the columns as groups, too. Therefore, they used the givens in the problem as data and composed arrays in the light of these given numbers by considering the context of the problem. These array models played a crucial role in supporting their claims related to the number of mice. Arrays helped them visualize the equal groups of mice and generate a multiplication operation that is 6 times 3. These models made the multiplier and multiplicand more clear.

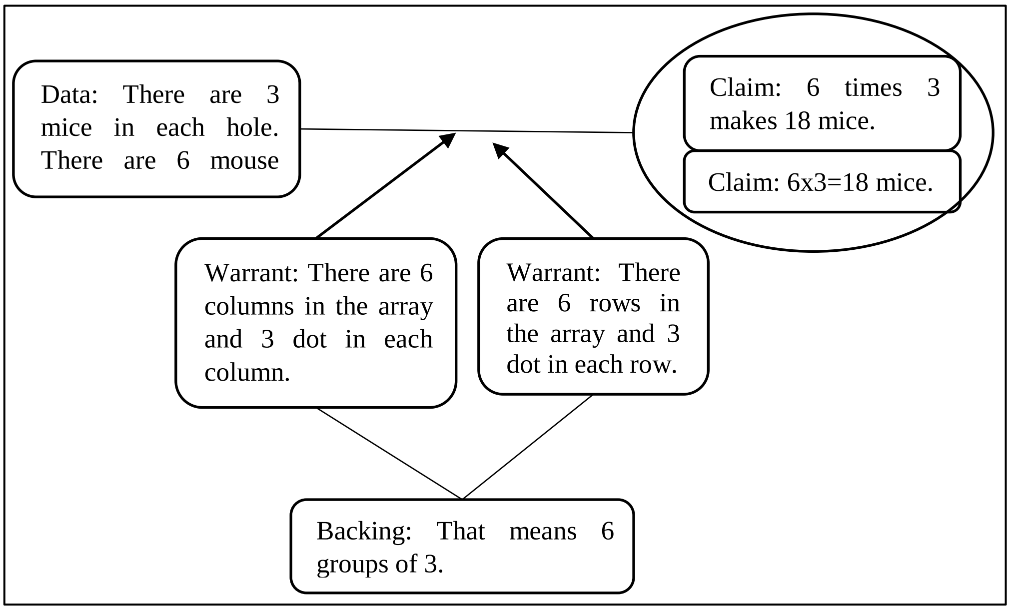


Figure 4. 65. Toulmin's Analysis scheme regarding composing arrays from rows and columns to illustrate equal groups

In the following lessons, students made reasoning by using array models to specify equal groups. For instance, on the 22nd day, students were asked to pose a problem related to the multiplicative operation of "9x4". As shown in Figure 4.66 below, students constructed rectangular arrays drawing dots in squares, and counted the number of dots on each side to find the total number of dots. Students structured the array as a combination of composite units. That helped them skip counting of these composite units to find the total number of dots.

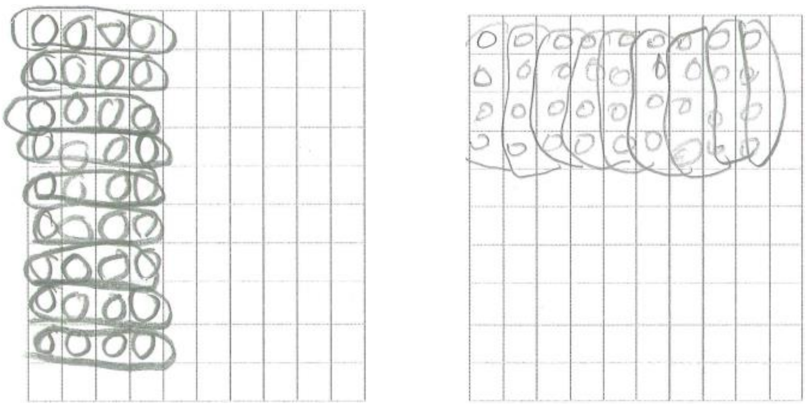


Figure 4. 66. Sample arrays to represent “9x4”

Students arranged the dots to represent "9x4" and identified equal groups. While some of them specified equal groups horizontally (row), others specified vertically (column) by circling. Students were not directed to form groups along rows or columns. They engaged on rectangular arrays through classroom discourse and decided that the direction of grouping does not matter as long as they formed equal groups. Hence, students developed array representation by abstraction from pictorial representation collectively.

As Steffe (1992) stated, students recognized and produced composite units corresponding to equivalent groups as essential understandings of multiplication. In the following lessons, it was observed that students did not question how to decide on the groups by using an array. They checked whether the groups defined on the array represent the given multiplicative situation. Students used this idea as data in their arguments in the following lessons without the need for backings to become taken-as-shared.

4.4.5. Idea 14: Reasoning commutative property by using arrays

In the light of instructional sequence and students' reasoning, the research team focused on reasoning the commutative property of multiplication by interpreting multiplicative situations and representations. In the previous parts, it is seen that some of the students made sense of the commutative property of multiplication. For instance, some of them discovered that reversing the numbers of groups and elements in each group on the fourth day can form new equal groups. That is to say, the objects grouped as 6 sets of 2 elements can also be grouped as 2 sets of 6 elements. In the same way, students used others' claims related to equal grouping to reverse and get new equal groups in the fifth and sixth days. It means that some of the students intuitively understood the commutative property of multiplication. For that reason, this valuable sense was noted to be observed in the following activities.

In the eighth lesson, the teacher wrote a phrase on the board as "5 times 3". Then asked what it means "5 times 3". All the students quickly answered as 15. The teacher completed the phrase as "5 times 3 makes 15". Then asked students to pose a problem about it. Students were given 2 minutes to think about it. Then classroom discussion started as in the dialogue below:

> Doğan: I have 5 boxes. There are 3 toys in each. I put them together. They were 15.
> Teacher: How can you turn it into a problem? You can ask "How many toys are there totally".
> Zehra: I have 3 boxes. There are 5 pencils in each. I counted them by 5s and found 15.
> Teacher: To make it a problem, you can ask total number of pencils instead of calculating it.
> Birce: Friends came together to the park. They composed 5 groups. There were 3 kids in each group. How many kids are there totally?
> (Various problems were posed. Most of the students used the context of boxes with an equal number of objects in them)

Teacher: Let’s look at the problems posed by Doğan and Zehra. (She wrote the problems on the board as in Figure 4.67). Are these problems same?

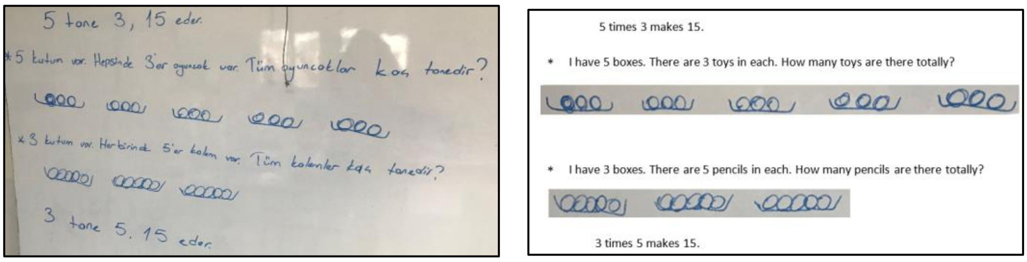


Figure 4. 67. Two types of problem discussed on the board (original version) and its’s translated version.

Classroom: Not same, but similar.
Teacher: What is the similarity? What is the difference?
Eyüp: There are boxes in both problems.
Zuhal: There are 5 and 3.
Zehra: In the first problem posed by Doğan, we multiply 5 and 3. In the second problem, we multiply 3 and 5. Both make 15.
Teacher: Is there “5 times 3” in Doğan’s problem?
Classroom: Yes.
Teacher: Is Zehra’s problem convenient?
Classroom: Yes.
Teacher: Can you show me “5 times 3” in Zehra’s problem?

(At that point, the classroom split into two as the ones who think Zehra’s problem is convenient and the ones who think Zehra’s problem is not convenient. The ones supporting Zehra claimed that she also used 3 and 5 in the problem and found 15 as a result. On the other hand, the others claimed that the problem does not include “5 times 3” although the answer is correct.)

Zehra: My problem represents “3 times 5”. However, the results are the same.
Teacher: There is an idea in the classroom that the appearances of the groups are different. However, the answers to both problems are the same. Why?
Birce: The number of boxes decreased, but objects in each box increased. Thus, there are still 15 objects.
Teacher: 2 boxes are removed you say. What if we remove 3 boxes? Can we keep total number of objects as 15 again?
Birce: No. I couldn’t build relationship between the amount of decrease and increase.
Zehra: Places of the numbers changed. They are replaced.
Teacher: Why did total number remain same when the numbers were replaced? (They tried to make interpret the change in the number of boxes and objects in each box.)

> Halil: Result does not change when the numbers are replaced, because it is a rule.
> Teacher: Which rule? How did you get this rule?
> Hakan: Because reasons (also known as because is why).

It was striking that most of the students accepted the idea that reversing the group number and group size does not affect the total number of objects that can be called commutative property in multiplicative situations. Although students had not experienced any operation related to repeated addition and multiplication, they discovered this property in grouping activities. They supported their claims by counting the groups to find the total amounts. Some of them emphasized that they have already known that since they experienced it in the previous lessons. Students proposed different reasoning to be able to support their claims. They convinced each other that “5 times 3” and “3 times 5” are different in meaning but the same in the total. They supported their claims with the warrant that replacing the numbers does not affect the result. They accepted this warrant as a fact or property as in the dialogue. Students avoided specifying the reasons since they did not know why it was always true. They could not explain why the result does not change when the numbers are replaced. However, they knew that it always works.

In the following lessons, it was seen that the number of students who used this “rule” in their arguments was increasing. Although they could not prove why it always works, they could show by giving examples. Hence, they accepted this sense of commutativity as taken-as-shared idea day by day. For instance, as it was mentioned in the part of TAS 6, students were asked to pose a problem related to “5x4”. Students were expected to write a story including 5 groups with 4 objects each. During this activity, Eyüp posed a problem, and others reflected on it as in the dialogue below:

> Eyüp: There are four fish ponds. I saw five fish in each fish pond. How many fish did I see?
> Zehra: Wait a minute. I am not sure whether it fits 5x4.

> Birce: Eyüp, this drawing represents 4x5. We have 5x4 to pose a problem which means five times four. The results are equal, but operations are different. You should draw five fish ponds and four fish in each. You did the opposite.

As in the excerpt, students intuitively discovered the commutative property of multiplication. They knew that Eyüp's problem gives the same answer since he just replaced the numbers, which is related to the idea of making sense of commutative property. They reached this knowledge as a result of their experiences. However, they could not prove or explain by providing a reason for this property. For that reason, the research team developed exercises related to composing arrays to give them a chance to see this property visually on arrays. On the seventeenth day, students were given counters and asked to compose arrays on squared paper. Then they were asked to show multiplication operations on the array symbolically. As in Figure 4.68, students were provided concrete counters, activity sheets, and personal reusable squared papers to hold and show others their arrays.

Figure 4. 68. Sample array activity and students' works

Various representations emerged. While some of them circled the counters vertically to specify equal groups in multiplication, some circled horizontally. Arrays were also varied in terms of commutativity. For instance, while some of them made 6 groups of 2 counters with 12 counters, others made 2 groups of 6 counters. That is to say, for multiplication operations, including 2 and 6, four types of arrays were composed by the students as illustrated in Figure 4.69. Students shared their models and multiplication operation with the others. During the classroom discussion, students analyzed these array models and discussed whether the representations and related operations were consistent or not.

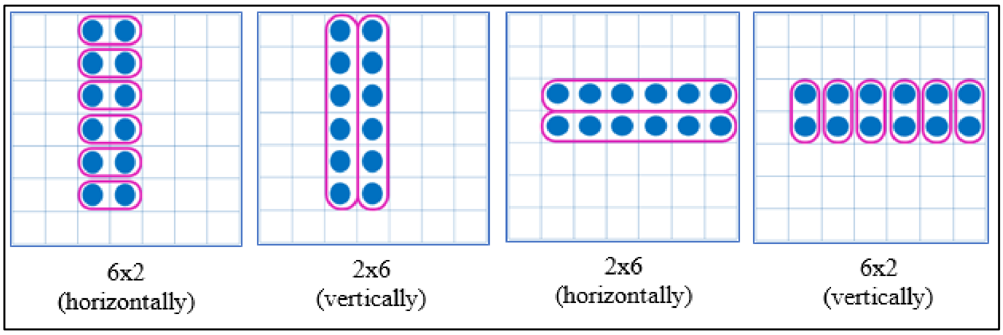


Figure 4. 69. Four types of array models for multiplication of 2 and 6 which are illustrated by the researcher.

Students focused on the groups on presented arrays to decide on the multiplier and multiplicand to check the written operation. Some of them composed array but did not specify the groups. In such situations, students asked each other to circle the groups on the arrays to see the groups and compare them with the operation. That is to say; they asked each other to specify the directions of the groups as vertical (columns) or horizontal (rows). While analyzing the arrays, the ones who wrote the same operations compared their arrays and groups. They realized that they could write the same operations although the directions of groups were different (Figure 4.69). Students experienced that they could compose various arrays as long as they were rectangular. They shared various types of arrays and operations. However, they did not specify commutativity.

After this lesson, it was seen that all the students were capable of composing an array and interpreting the array's components as groups and elements in each group as it was targeted. After being convinced that students conceptualized array models, the research team decided students to interpret array models in detail. With this purpose, in the following days (18th and 20th days), students were given array models and asked to write a multiplication operation for given array and specify it as "…times… makes …" as in Figure 4.70. Students were given readymade arrays not to distract them by drawing and composing. Thus, students had much more time to work and discuss arrays.

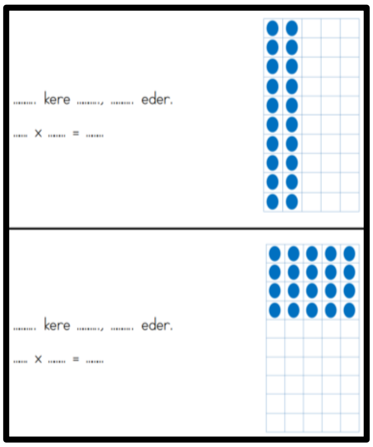


Figure 4. 70. Sample items related to array activities

The dialogue related to the discussion of the items in Figure 4.70 is given below:

Doğan: I wrote 10x2 (The ones who wrote the same operation approved Doğan)
Birce: I wrote as 2x10. You might have grouped rows.
Doğan: Did you group columns?
Birce: Yes.
Ali: I grouped columns, too.
Teacher: Does it matter to groups rows or columns?
Hakan: No, just the places of numbers in the operation change.
Teacher: What about the following array?
Halil: 5x4 makes 20 (The ones who wrote the same operation approved Halil)
Doğan: I wrote the reverse as 4x5. (Birce approved him, too). Birce, did you group rows this time? You used columns before.
Birce: Sometimes I group rows, sometimes columns. As I wish.
Teacher: Is it crucial?
Ali: We can group as we wish. The result does not change.
Teacher: Why doesn't it change?
Doğan: It is a rule.
Hakan: We didn't touch the counters. We didn't add or remove any counter.
Teacher: Can you show this rule on arrays?
Ajda: Some of us used columns, some of us used rows. We all found 20.
Birce: I got it. We just change our perspective. Changing direction doesn't change the result.
Teacher: Can you see the rule here? As you said, replacing the multipliers doesn't affect the result. We call it as commutative property of multiplication. We couldn't see it in the pictures we drew in previous lessons. However, you could easily observe on an array.

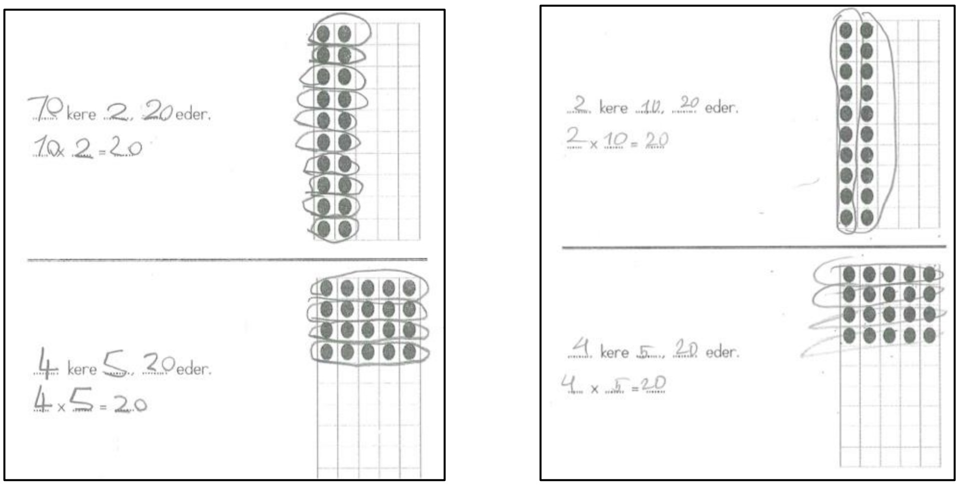


Figure 4. 71. Works of Doğan, and Birce respectively.

As the students discussed it, students made reasoning about grouping the counters in the arrays and writing a multiplication operation related to these groups. They again stated that they could group the counters vertically or horizontally as they had always done. As it can be seen in Figure 4.71, Doğan preferred grouping the rows while Birce grouped the columns. They wrote reverse operations but found the same results. They emphasized that they could group the counters as they wished since the result was not affected. The crucial part of this discussion is that students made reasoning about the commutativity property of multiplication on array representation. They were asked why the direction of grouping does not affect the result to make them provide reasoning about their claims. They could clearly explain on the array models that they did not add or remove any counter in the array. They just changed their perceptions that while some focused on the columns, others focused on the rows. The models remained the same. After explaining their claims, the teacher rephrased students' claims by expressing that it was called the commutative property of multiplication. Toulmin's analysis scheme presented in Figure 4.72 below summarizes students' claims and supports.

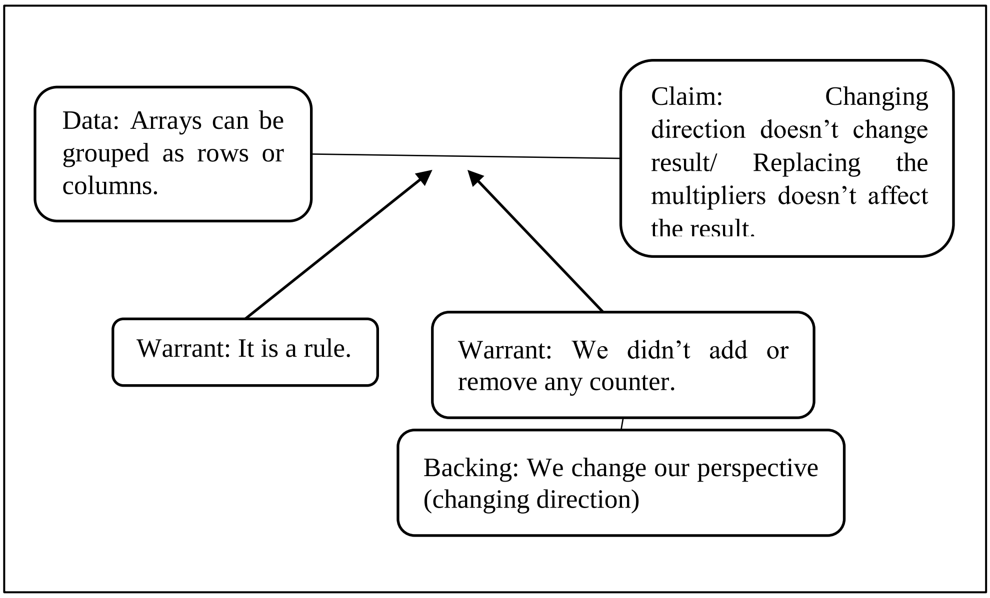


Figure 4. 72. Toulmin's Analysis scheme regarding reasoning commutative property by using arrays.

On the 22th day, students were given an array model consisting of 7 rows and 3 columns and asked to pose a multiplication problem related to this array. Many problems were posed in various contexts. Students shared their problems with others to get their opinions. They focused on equal groups in the array model, multiplication operation, and context of the problem simultaneously. They interpreted whether the number of equal groups is consistent in these 3 types of representations. Students accepted the idea related to commutativity on array models as taken-as-shared idea so that they did not question it anymore, as it can be clearly seen in the dialogue below:

> Eyüp: I am reading my problem. I have 3 boxes. There are 7 balls in each box. How many boxes do I have? (Figure 4.73) I solved by multiplying 3 and 7.
> Birce: Did you group the columns?
> Eyüp: Yes. Columns are my boxes.
> Doğan. It is good.
> Melek: Can I read mine? I buy 3 marbles in each day. How many marbles do I buy in 7 days? I multiplied 7 and 3 since I grouped the rows (Figure 4.73).
> Hakan: Correct for your array.
> (All the students shared their problems)
> Teacher: I see two types of operation in the classroom. 3x7 and 7x3.
> Zehra: It is too normal. There are two types of grouping; rows and columns.
> Teacher: Is it acceptable?
> Hakan: Sure, it is commutative property.

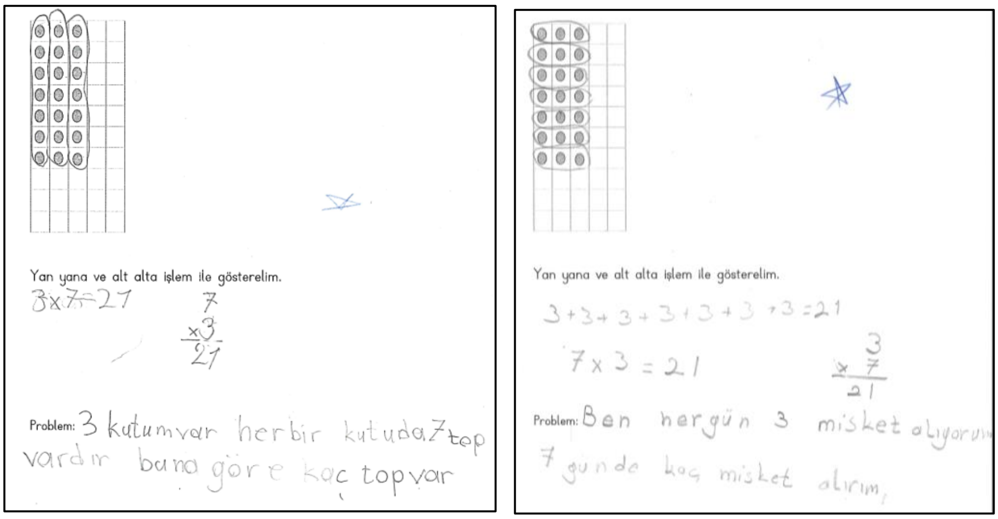


Figure 4. 73. Works of Eyüp, and Melek respectively.

As observed in this lesson, students conceptualized the idea of commutativity by using array models. Students did not question the ones who used reverse operations or reverse grouping in their problem contexts. They knew that there were two ways to group objects in array. They did not ask for reasoning about this issue which means that this idea became taken-as-shared by students.

4.5. Classroom Mathematical Practice 5: Writing contextually realistic problems by coordinating the relationship among multiplicative representations

The fifth practice emerged as students engaged in the instructional sequence involved writing problem context for a given situation. Students were asked to write realistic word problems based on given pictures, models, repeated addition, and multiplication operations. It was expected to support students to make connections between given representations and real-life representations. Students engaged in problem posing tasks individually. Then, they shared in whole-class discussions and questioned the appropriateness of the problems posed. Students embedded the equal groups they defined as Practice 4 in multiplication situations. The context of the problem was formed within words where the group number and group size are made explicit in a real-life story. Students shifted from sharing strategy talk to a more demanding procedural emphasis on the action in the problem structure during the classroom discourses. Four mathematical ideas became taken as-shared as students engaged with problem-posing tasks:

- Analyzing the multiplier and multiplicand to pose multiplication problems on known contexts
- Interpreting the multiplier and multiplicand to pose multiplication problems as repeated addition
- Interpreting the multiplier and multiplicand to pose multiplication problems as rate
- Focusing on structure and keywords in the problems to conceptualize multiplication

4.5.1. Idea 15: Analyzing the multiplier and multiplicand to pose multiplication problems on known contexts

In this instructional sequence, activities are developed in real-life contexts in the light of RME theory to make students feel familiar with the tasks. Students worked on the tasks, including various realistic multiplicative situations. According to the sequence of activities, after students practiced problem-solving activities, they were introduced to problem-posing activities produced from multiplicative situations. When they started problem-posing tasks, it was observed that students modified their experiences from problem-solving activities to new situations to pose a problem. Students reformulated already solved problems or problems posed by others in previous lessons to generate a new problem by changing what is given. In other words, students used the same contexts for different numbers. For instance, on the eighth day, students were asked to pose a problem for the phrase "5 times 3". It was seen that students used the contexts and pictures from the previous tasks like fruits in plates, toys in boxes, objects in baskets as in Figure 4.74.

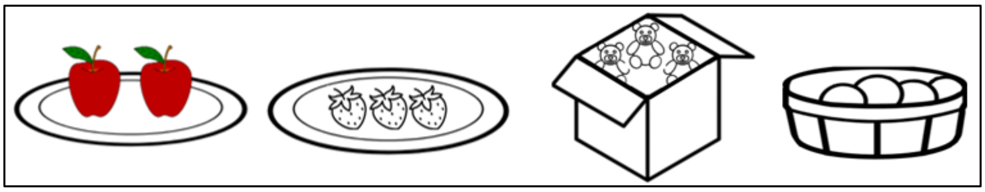


Figure 4. 74. Sample visuals used in the tasks to represent equal groups

To pose a problem, students analyzed the phrase "5 times 3" as 5 groups and 3 objects in each group. Then, they selected context to modify for this phrase. Some of the modified problems posed by students are given in the dialogue below:

> Doğan: I have 5 boxes. There are 3 toys in each. I put them together. They were 15.
> Teacher: How can you turn it to a problem? You can ask "How many toys are there totally".
> Zehra: I have 3 boxes. There are 5 pencils in each. I counted them by 5s and found 15.
> Teacher: To make it a problem, you can ask total number of pencils instead of calculating it.
> Eyüp: There are 3 oranges in 5 boxes. I added all oranges to find how many oranges there are.
> Teacher: You have 5 boxes and 3 oranges in each. How many oranges are there in these boxes? You should make an interrogative sentence that asks a question.
> Ajda: There are 5 plates. There are 3 apples in each plate. How many apples are there?

As in the dialogue, students preferred boxes and plates to place objects in them. That is to say; they used the context they had experienced before. It was observed that once one student used boxes as groups, others started using boxes by changing the objects in them. As in the example above, after Doğan had used toys in boxes, others changed them into pencils and oranges. It was the first time; they had asked to pose a problem. They could write stories for their problems by modifying previous problems. However, they had difficulty making an interrogative sentence at the end of the problem. They explained the equal groups and elements as the context of the problem and found the total number of objects. The teacher facilitated them to make a sentence at the end to emphasize that it was a problem. Students' tendency to take advantage of previously known contexts was observed in the following lessons.

On the eleventh day, students were asked to pose a problem related to repeated addition of "3+3+3+3+3+3+3". Students generated their problems and discussed these problems with others. During the discussion session, various ways of reasoning emerged. As one of these ways, students attempted modifying previously solved and posed problems to generate a new problem for the given operation as in the dialogue below:

> Karan: There are 7 shelves. There are 3 apples in each shelf. How many apples are there?
> Teacher: Did you listen to Karan? What do you think about his problem?
> Zehra: It is convenient for the given operation. As we did yesterday, 7 shelves are the groups.
> Eyüp: It is a good problem since we worked on a task related to shelves and bottles in the last lesson.
> Doğan: I want to read mine. There are 3 pencils in each plate. If I have 7 plates, haw many pencils do I have?
> Birce: Did you put pencils in plate? Isn't it weird?
> Hakan: Do you think that Doğan's problem is weird? I put toys in the plates.
> (Everyone laughed loudly)
> Teacher: Approximately half of you put objects in plates. Why did you prefer plates?
> Esra: You asked us problems about plates previously.
> Yalçın: We put fruits in the plates and represented with multiplication. For that reason, I used plates and put marbles in them.

Students had worked on a task that included a picture of bottles on shelves to pose a problem the day before. As it can be seen in the dialogue, Karan used the same context since it was examined, discussed, and accepted as a relevant problem. The others also supported his claim by summarizing the previous activity. In the same way, Doğan used the well-known context of plates with objects in. Birce criticized him since putting pencils in plates was nonsense. However, others supported Doğan by sharing the weird objects they had put in plates in their problems. Although the contexts were weird, they were realistic in the light of RME since they could imagine the story in the problem. Hence, students used pre-used and analyzed stories to get new problems by changing the numbers. They supported their claims by specifying their previous experiences and showing the relevance of their problems, as summarized in the argumentation scheme in Figure 4.75 below.

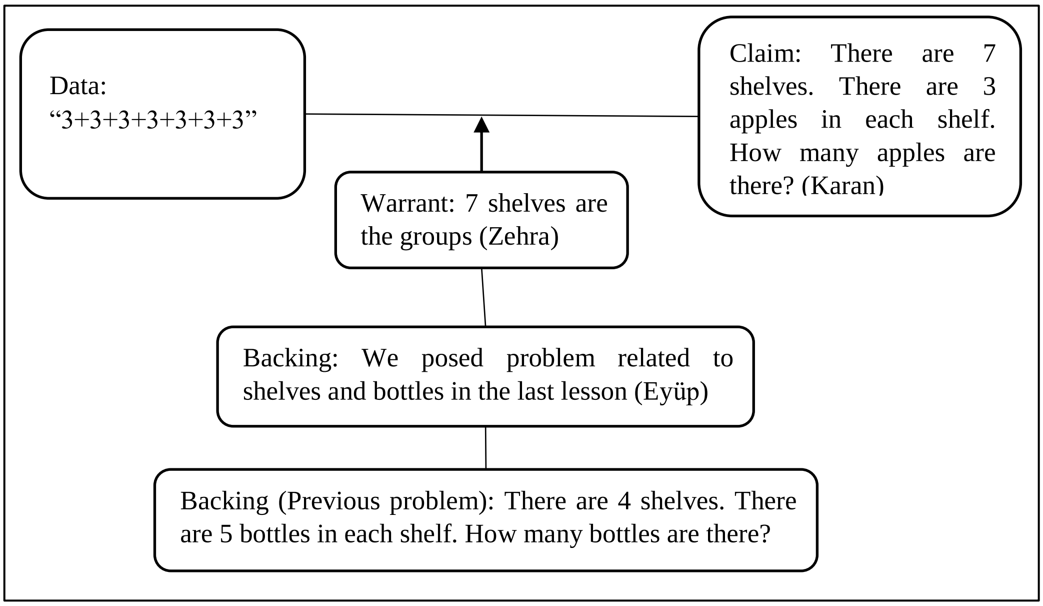


Figure 4. 75. Toulmin's Analysis scheme regarding modifying the shelf-bottle problem to pose a new problem for repeated addition operation.

On the thirteenth day, students were given a multiplication operation to pose a problem. They were asked to pose problems according to "5x4=……" and the conversation continued as follows:

> Meliha: There will be five groups and four things in each group.
> Egemen: I do four pushups each day. How many pushups do I do in five days?
> Zehra: Egemen posed such a problem before. He likes doing pushups.
> Zuhal: I posed the same problem too. I remembered the problem Egemen wrote yesterday. I changed the numbers in his problem and wrote this problem.
> Teacher: Is it valid for the given operation?
> Zuhal: I named the groups as days and the things in each group as pushups. It must be convenient.
> Doğan: Egemen's previous problem was right; this is also true.

Argumentation dialogue shows that students equated multiplicative situations with contexts like drinking milk daily, doing pushups, apples on plates. They constructed problems on given operations or pictures through these contexts. In the excerpt, although the students were not asked to change the givens on a specific problem, they used the stories used in the previous lessons and replaced the numbers in those problems and the numbers in the given operation. This approach provided making conjecture and producing new perspectives about problem outcomes. Students used well-accepted contexts to pose new problems on given operations without judging since the context and structure of the previous lessons' problems were discussed and accepted as taken-as-shared ideas. They focused on interpreting consistency between numbers of groups and elements on the operation and the numbers in the problem, as shown in the argumentation scheme in Figure 4.76 below.

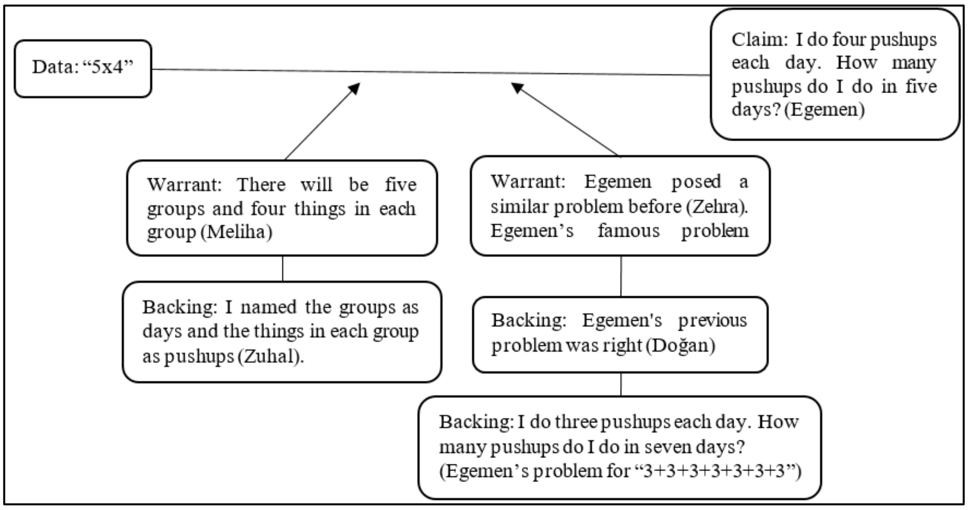


Figure 4. 76. Toulmin's Analysis scheme regarding modifying pushup problem to pose a new problem for the given multiplication operation.

As in the given dialogue, students memorized the previously posed problems. Furthermore, they called some problems with the person's name who used it first. For instance, they called the daily pushup problem Egemen's problem. The following days, they used the same context and supported their claim by emphasizing that this is Egemen's problem and previously accepted as an equal-group.

Students were asked to pose a problem for a given array model on the twenty-second day. The given array model was ungrouped to make them feel free to group the array as they wished. While some of them grouped the columns (3x7), others grouped the rows (7x3). As it was explained in the previous section (TAS 11), students did not focus and discuss the grouping procedure. They focused on the context and consistency between the groups in the problem and array. Some of the students used a known context by adapting it according to the given array, as shown in the argumentation scheme in Figure 4.77 below.

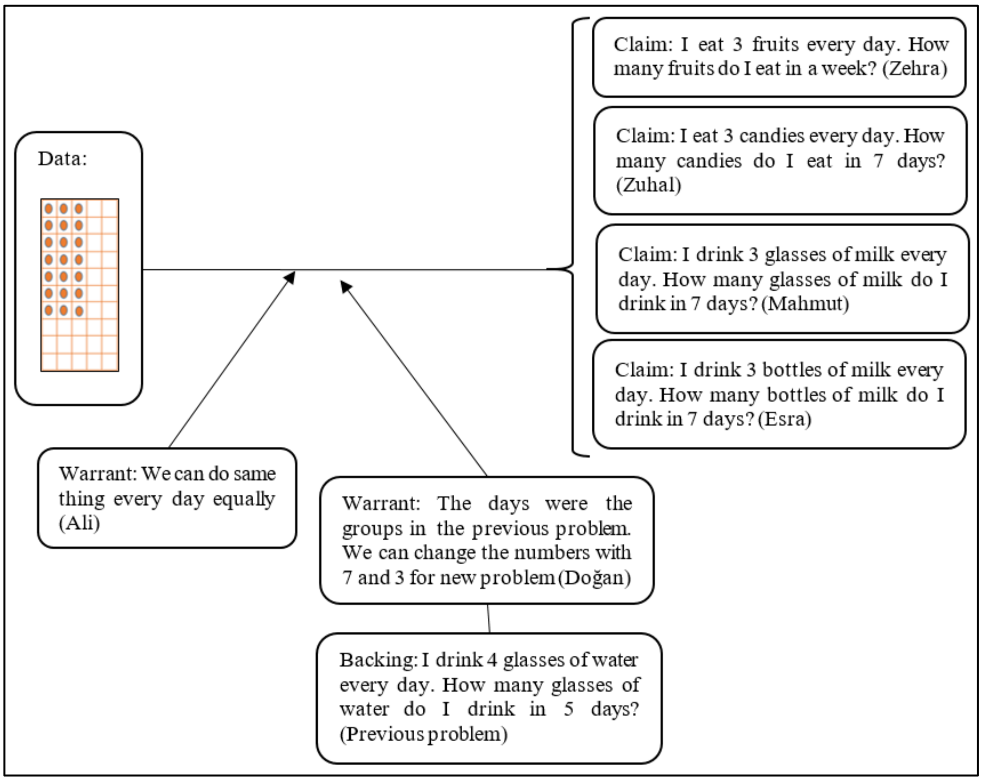


Figure 4. 77. Toulmin's Analysis scheme regarding modifying daily routine problems to pose a new problem for a given array model.

Students used the problem related to the daily routine, which is drinking water every day. Students changed water with daily activities like drinking milk, eating candy or fruit. They supported their claims by reminding previously solved or posed problems. They also stated that they do the same thing every day to represent equal groups. This issue is also explained in the following parts (TAS 14).

Students were introduced to various tasks during problem-posing activities to pose problems for a given picture, operation, or array model. In all types of activities, students provided various explanations to their claims. One explanation was related to modifying known problems for new situations to generate another problem. Students chose known contexts to be on the safe side.

It was the most used reasoning to make a claim. After a while, no one discussed such problems since they had already discussed and accepted these stories. It was seen that the idea of modifying known stories for the new situations was taken-as-shared idea for the classroom.

4.5.2. Idea 16: Interpreting the multiplier and multiplicand to pose multiplication problems as repeated addition

As stated before, students met with problem-posing tasks on the eighth day. After that day, students were provided activities related to problem posing and problem-solving. They shared and reflected on their problems and discussed them. While students talked about the problems posed, it was observed that they started to show their creativity on the story of the problems posed so that contexts of the problems varied and departed further from traditional word problems found in textbooks. After analyzing these problem contexts, two types of problems emerged as repeated addition problem including concrete groups of objects (TAS 16) and rate problems including abstract groups of actions (TAS 17).

In the first type of problem, students generated groups in which objects can be placed. As in the eighth lesson in which students had faced with problem-posing activities first time, students preferred mats like boxes and plates to put objects in them. They also selected concrete objects like an orange, apple, strawberry, pencil, ball, and toy that can be grouped on mats. It took the research team's attention, and this idea was noted to be observed for the following lessons. Students were given a picture/model or an operation to pose a problem. In such situations, students initially interpreted the given representation in the task as multiplier and multiplicand. Then, wrote a story problem about the multiplier and multiplicand. They started the story by specifying the group number (multiplier). After that, they explained how many items there are in each group (multiplicand).

On the tenth day, students were asked to pose a problem related to given picture. Students generated their problems considering shelves as groups. During this lesson, students were not just dependent on the given picture. Some of them changed the picture to pose their problem as summarized in the argumentation scheme in Figure 4.78 below.

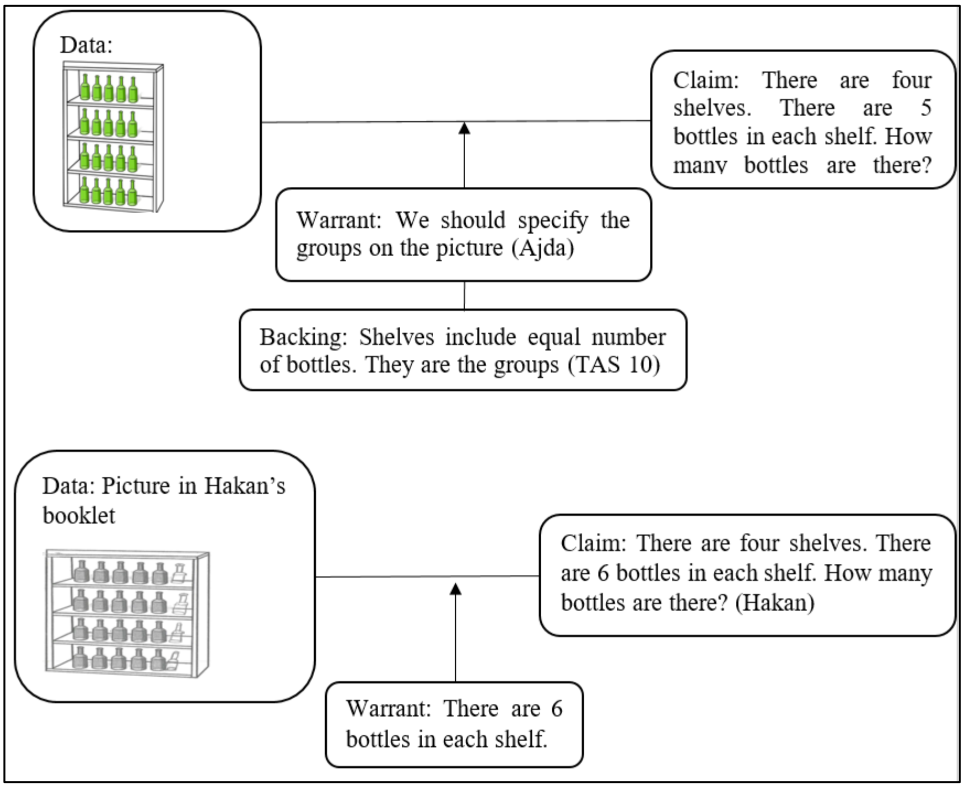


Figure 4. 78. Toulmin's Analysis scheme regarding constructing problems on bottle-shelf picture

As seen in Figure 4.78, students defined shelves as equal groups in both pictures. Then, they posed repeated addition problem related to the shelves and bottles in each shelf. They supported their problems by specifying equal group representation of multiplication. In the following lessons, students created contexts that include concrete groups like closed boxes, bags, baskets as units. They developed an understanding that any unit can be counted like five-fours through skip counting. At that point, while a student posed a problem, others divided the problem into components as a multiplier (number of groups) and a multiplicand (group size) to state the model of the problem as repeated addition of equal sets. Students were used to generating equal groups and finding the total number by adding the number of elements in each group as times as the number of groups. Therefore, they tended to decide on the groups and elements placed in these groups at first.

On the twenty-second day, students worked on two types of problem posing tasks. In the first one, students were given an array model with seven rows and three columns. They were asked to pose problems related to the given model. Sample problems from this lesson and their interpretations are summarized in the argumentation scheme in Figure 4.79 below.

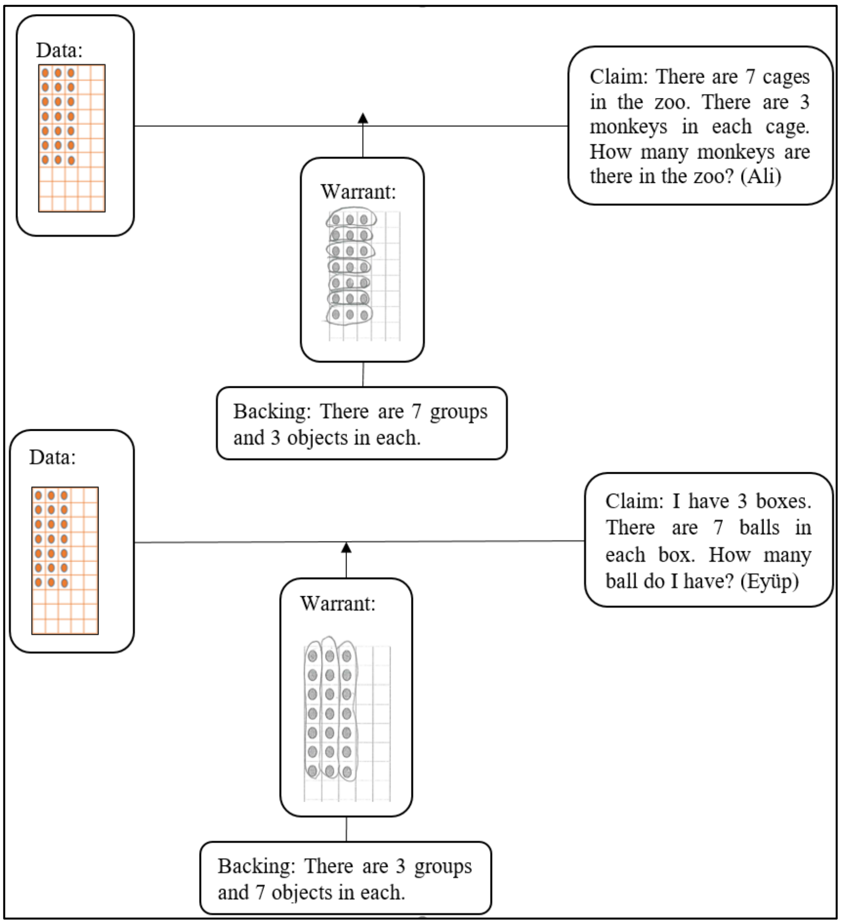


Figure 4. 79. Toulmin's Analysis scheme regarding constructing repeated addition problems on array model

Students grouped the objects in the array model as rows or columns. They used their representations related to the groups in the array as warrants to their claims. The others evaluated the problems by comparing them with the groups in the representation. In this lesson, a second type of problem posing task was used. Students were given the operation of “9x4” and asked to pose problems for this operation (Figure 4.80).

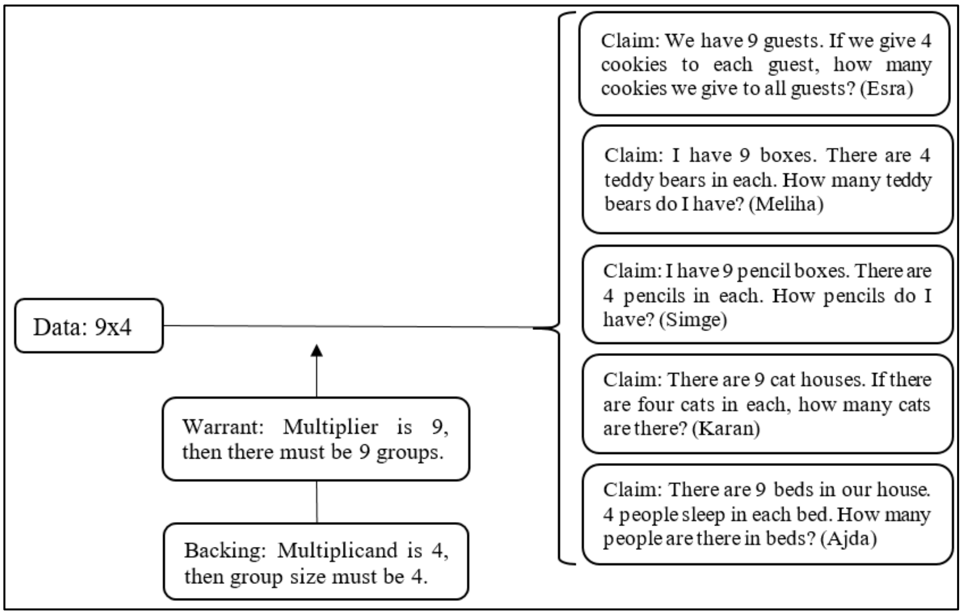


Figure 4. 80. Toulmin's Analysis scheme regarding constructing repeated addition problems on 9x4

As it is seen in the scheme above, students interpreted the multiplier and multiplicand in the given operation. They decided on the number of groups and group size with respect to the multiplier and multiplicand. After that, they used these numbers and generated repeated addition problems. They started the problem by specifying the number of groups which represents how many times the unit (multiplicand) is iterated. After a while, students did not propose any warrant since the others did not ask. Dropping off of warrants in students’ answers showed that the idea of interpreting the multiplier and multiplicand to place in a real-life context for a repeated addition problem was taken-as-shared.

4.5.3. Idea 17: Interpreting the multiplier and multiplicand to pose multiplication problems as rate

As stated above, in addition to writing problems on concrete objects, writing problems on actions and daily routines emerged in the classroom as another reasoning for problem-posing on multiplication. While most of the classroom tended to name the groups as closed shapes, some of the students called these groups the period like an hour, day, week, and month. The first example of this type of problem emerged on the tenth day by Melek. While most of the students used the context of bottles on shelves, she enriched the context with action. She posed a problem: "How many bottles of water does my brother who drinks 5 bottles of water every day drink in 4 days?". It was surprising for both researchers and students. Although there was a readymade context in the picture as “there are 4 shelves and 5 bottles on each shelf”, Melek generated a problem including actions. She discussed her problem by defining equal groups in the picture as in the argumentation scheme in Figure 4.81 below. This abstract context was found remarkable by the research team. It was noted to be observed in the following lessons to be observed.

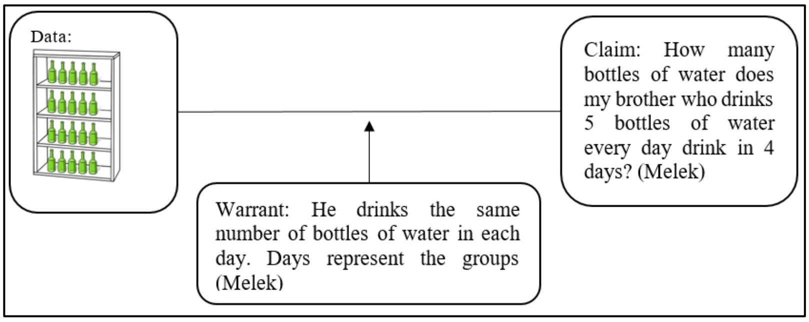


Figure 4. 81. Toulmin's Analysis scheme regarding constructing problem on bottle-shelf picture including regular actions

In the eleventh lesson, which is explained in the previous sections, some of the students posed problems about daily routines and various actions. It was seen that some of the students posed rate problems for equal groups. Students focused on the unit. They started the problem by specifying the unit (multiplicand).

Sample dialogue related to the discussion of these problems in the classroom is given below:

> (Students were asked to pose problems for 3+3+3+3+3+3+3)
> Egemen: I do 3 pushups every day. How many pushups do I do in 7 days?
> Teacher: What do you think about Egemen's problem?
> Esra: What are the groups?
> Egemen: Days are the groups. I do 3 pushups each day as I drew (He showed his drawing given in Figure 4.82).
> Zuhal: I used the days as groups, too.
> Teacher: Can you read?
> Zuhal: I eat 3 strawberries every day. How many strawberries do I eat in 7 days? I placed strawberries in days (Showed her drawing in Figure 4.82).
> Teacher: Is there anyone who uses the days as groups?
> Halil: I wrote drinking 3 glasses of water.

As can be deduced from the dialogue, students used frequency of actions as multiplicand and the number of days as multiplier for the rate problem. Students supported their claims by specifying the groups and elements. Moreover, students made drawings to visualize their reasoning and represent groups as in Figure 4.82.

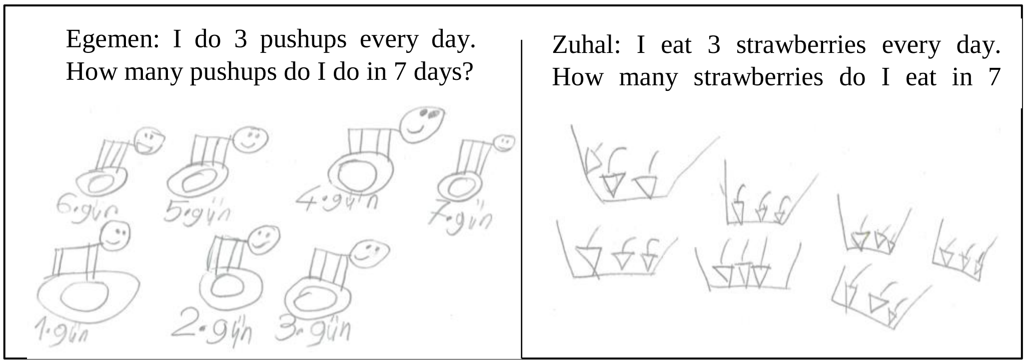


Figure 4. 82. Drawings of Egemen and Zuhal related to their problems

On the thirteenth day, students continued to use actions to be grouped for rate problems they posed. Students used TAS 10 as data to specify the number of groups and elements in each group. After that, they generated problems related to these numbers. While they prefer constant objects placed somewhere in the light of TAS 16, they preferred objects specified by a verb like visited forests, eaten food, bought products, and done work in TAS 17. Students shared their problems and supported their claims by using visual representations in Figures 4.83. Students preferred time intervals like days and months as the groups. They limited the actions in a time scale so that each action happens in the same amount of time.

We visit 4 forests in each month. How many forests do we visit in 5 months? (Ali)

I drink 4 glasses of milk every day. How many glasses of milk do I drink in 5 days? (Ilker)

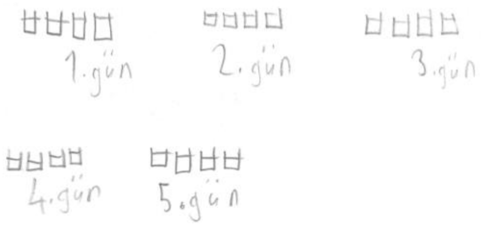


Figure 4. 83. Drawings of Ali and Ilker related to their problems on 5x4

For instance, students were asked to pose a problem for the operation "9x4" on the twenty-second day. They discussed the problems and ideas related to abstract groups as in the argumentation scheme (Figure 4.84) and dialogue below:

> Zehra: Here, we can consider each group as days. So, my problem is, "I run four laps in our playground every day. In 9 days, how many laps do I run?"
> Kadir: Where is the groups?
> Doğan: I think it is OK.
> Teacher: How is it OK? Can you explain more specifically? What should be in a problem to be OK?
> Halil: There must be nine groups. Nine days can be counted as nine groups.
> Ilker: What can we put in these groups?
> Ali: She also runs each day equally. We can solve this problem by multiplying nine times 4.
> Teacher: You say that the number of tours you run is the number of objects in each group?

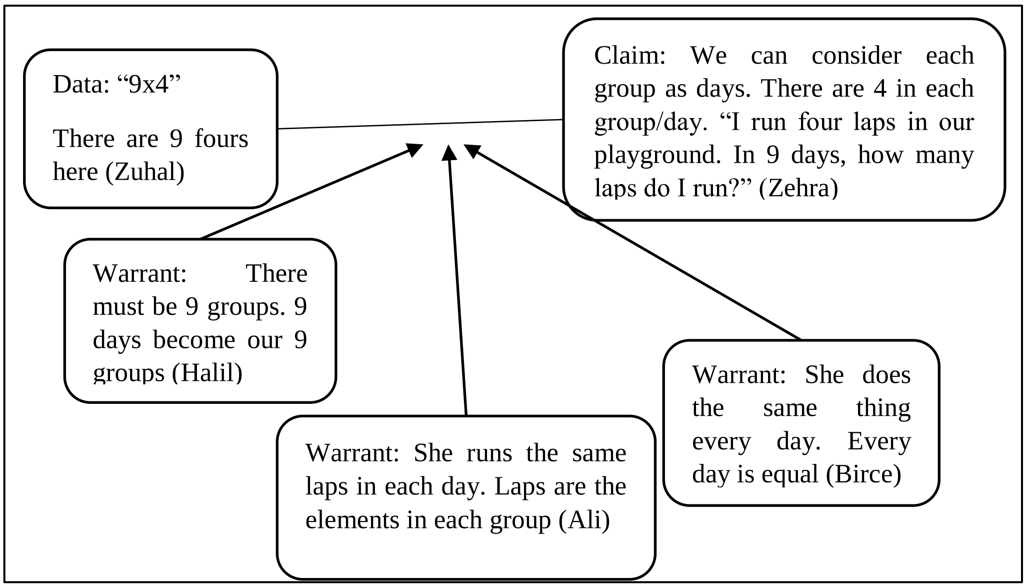


Figure 4. 84. Toulmin Analysis scheme regarding constructing problem on abstract groups of actions

During this lesson, the teacher asked questions for justification. Although students were aware of what they had done, they had difficulty explaining how they thought sometimes. The teacher restated the statement to clarify, apply appropriate language, and involve more students in such times. In this excerpt, students formed their groups as nine days and four actions each. Instead of choosing tangible or concrete objects to place in the groups, students started thinking about abstract items like daily routines, actions, verbs, and activities to place in each group. Students agreed with this type of problem, considering groups as a period at the end of a couple of lessons. Students developed shared ideas on this issue and reflected on this in the following lessons.

Furthermore, most of the students used such contexts for multiplication with zero, which is hard to conceptualize most of the time. For instance, students were asked to pose a problem for the operation "5x0" on the twenty-fifth day. They discussed the problems and ideas related to abstract groups as in the dialogue:

> Melek: I see zero dogs in the street each day. How many dogs do I see in 5 days?

Hakan: Problems about dogs and streets were constructed before (TAS 12). It is Ok.
Burak: I want to share mine. I drink tea without sugar every day. How many sugar cubes do I use for my tea in 5 days?
Birce: Clever. You do not use sugar for tea. There is zero sugar in each day. It means that the answer is zero.

Here, the idea is posing rate problems considering actions in each time interval like day, week, and months. In the following lesson, students did not question such contexts, so that the warrants were not offered. That is to say, students developed the rate meaning of multiplication. They could conceptualize the equal group representation in rate problems. Therefore, in the following lessons, no other students challenged constructing rate problems so it was concluded to be taken-as-shared at this point in instruction.

4.5.4. Idea 18: Focusing on structure and keywords in the problems to conceptualize multiplication

While posing problems on a given operation, students focused not only on the context of the problem but also on the grammar of problem text. They paid serious attention to the concept and language of problems posed to make inferences in light of multiplication (equal grouping). They questioned the meaning of the problems by looking for and expressing regularity in multiplicative reasoning. In the first lesson regarding problem posing (8th day), students wrote problems related to "5 times 3", they shared and discussed their problems as in the dialogue below:

Esra: I had 5 eggs. 3 of them were broken. How many eggs do I have now?
Ali: Yours is subtraction problem.
Teacher: How can you help Esra?
Doğan: I have 5 boxes. There are 3 toys in each. I put them together. They were 15.
Teacher: How can you turn it to a problem? You can ask "How many toys are there totally".
Zehra: I have 3 boxes. There are 5 pencils in each. I counted them by 5s and found 15.
Teacher: To make it a problem, you can ask total number of pencils instead of calculating it.

Eyüp: I have 3 oranges in 5 boxes. I added all the oranges to find total number.
Burak: I had 5 glasses. 3 of them were broken. How many glasses do I have now?
Classroom: Wrong! It is subtraction problem.
Teacher: Let's start with 5 glasses and help Burak to correct his problem.
Ajda: We need equal groups.
Doğan: I have 5 glasses. I put 3 ice cubes in each glass.
Teacher: How many ice cubes do we need?

As viewed in the excerpt above, various difficulties emerged in making sentences to write a problem. The initial difficulty was about making an interrogative sentence. Students could give the meaning of equal grouping. They knew that they should process repeated addition and find the total number of objects. However, they had difficulty in making an interrogative sentence. They stated adding 5 times 3 to find the total number. The teacher supported students to make interrogative sentences that ask a question. Another difficulty was that students tended to pose problems related to addition and subtraction that they were familiar with. As in the except, when students posed a problem for wrong operation, others were asked to help to correct the problem. For instance, Burak posed a subtraction problem. Doğan enriched his context to make it repeated addition problem. As Ajda emphasized, Doğan gave the meaning of equal groups. At that point, it was seen that students focused on the meaning of the given statement to be able to give the same meaning in the problem text.

On the tenth day, students were asked to pose a problem related to the picture of bottles on shelves. Students asked their original problems to others to get feedback from the others. Others evaluated these problems by solving and checking the consistency between the given picture. An excerpt related to the discussion of this lesson is given below:

Karan: There are 4 shelves. There are 5 bottles in each shelf. How many bottles are there?
Simge: I have 5 bottles. I buy 4 more bottles.
Doğan: More?
Yalçın: Simge, be careful. You wrote an addition problem. It must be a repeated addition.
Teacher: Can you help her?

Eyüp: I have 5 bottles. I have 5 bottles more. I have 5 bottles more. I have 5 bottles more.
Melek: Eyüp, if you have 4 shelves, it is good.
Egemen: I have 4 fives of bottles. How many bottles do I have?
Esra: I have 5 bottles in each shelf. How many bottles do I have if I have 4 more shelves?
Doğan: There is something wrong about your problem.
Ajda: Esra says "more". It requires addition.
Teacher: Can you help Esra?
Gökhan: She shouldn't use the word "more". If this word is removed, the problem will be fixed.

As can be gathered from the dialogue, students used the word "more" while posing a problem. Students stated that using the word "more" gives additional meaning. As given in Figure 4.85, Gökhan claimed that removing "more" from the problem makes the problem relevant. It was a correct claim. One more claim showed that it was possible to make a correct claim with the word "more". Eyüp used the word "more" without losing the meaning of repeated addition. He wrote a sentence for each group as "having 5 more". He repeated this sentence to give the meaning of repeated addition. In both types of claims, students stated the meaning given by the words.

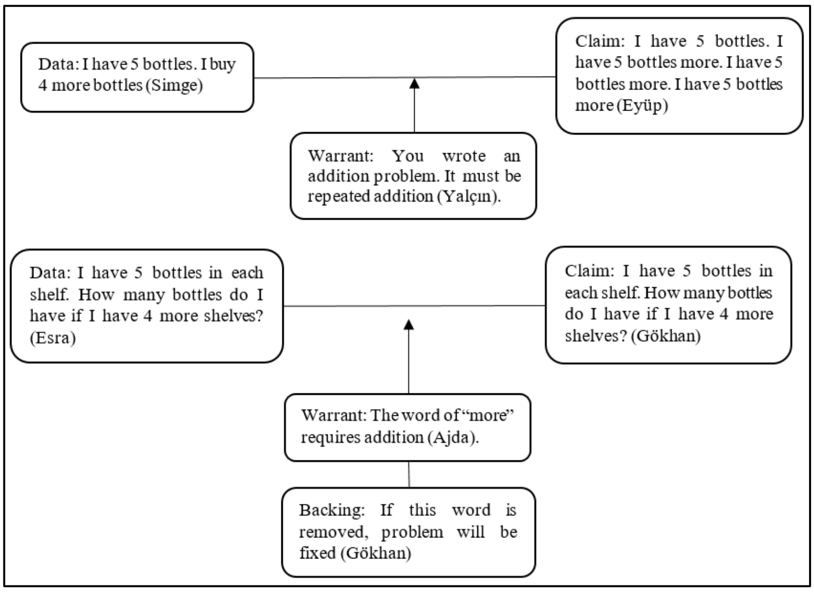


Figure 4. 85. Toulmin's Analysis scheme regarding structure and keywords in the problems on the picture of bottles on shelves.

In the same way, students interpreted the problems posed for "8x4" by reasoning the structure and keywords in the problems to give a meaning of multiplication on the fourteenth day. During this lesson, it was observed that students did not have difficulty making interrogative sentences. They had capable of ending their context with a question. Students made reasoning on the words giving multiplicative meaning. They discovered how the words in the text and their meanings are essential, as in the dialogue below:

> (Students were asked to pose problems according to "8x4=….")
> Mahmut: There are four bottles of milk in each box. If I buy eight boxes more, how many milk bottles do I have?
> Doğan: I could not understand. Can you read again?
> Mahmut: There are four bottles of milk in each box. If I buy eight boxes more, how many milk bottles do I have?
> Birce: 8 boxes more? I could not understand.
> Teacher: What is wrong? Why are you confused?
> Ali: He buys eight boxes more. It means that he has already had some other bottles at home. We should also know the numbers of these bottles at home.
> Doğan: He buys eight boxes on what? What will we add it on?
> (Students were all confused)
> Teacher: What do you suggest to Mahmut? How can we revise the problem to make it more understandable?
> Gökhan: It is confusing to use "more" in this problem. It seems like an addition. Not multiplication. Instead of using "more," we can say, "If I buy eight boxes, how many bottles of milk do I have?"

As in the dialogue, students interpreted the meaning of the problems posed. As in the tenth day, the word "more" was used and caused discussion. Students claimed that this word gives the meaning of joining. The word "more" means joining a problem that involves the initial amount, added amount, and resulting number. Since students knew that, they looked for the initial number in the problems posed with the word "more". Ali claimed that there must be some bottles to be able to add some more on. Students not only discussed the limitations in these problems posed but also revised them to make correct claims as in the scheme below (Figure 4.86)

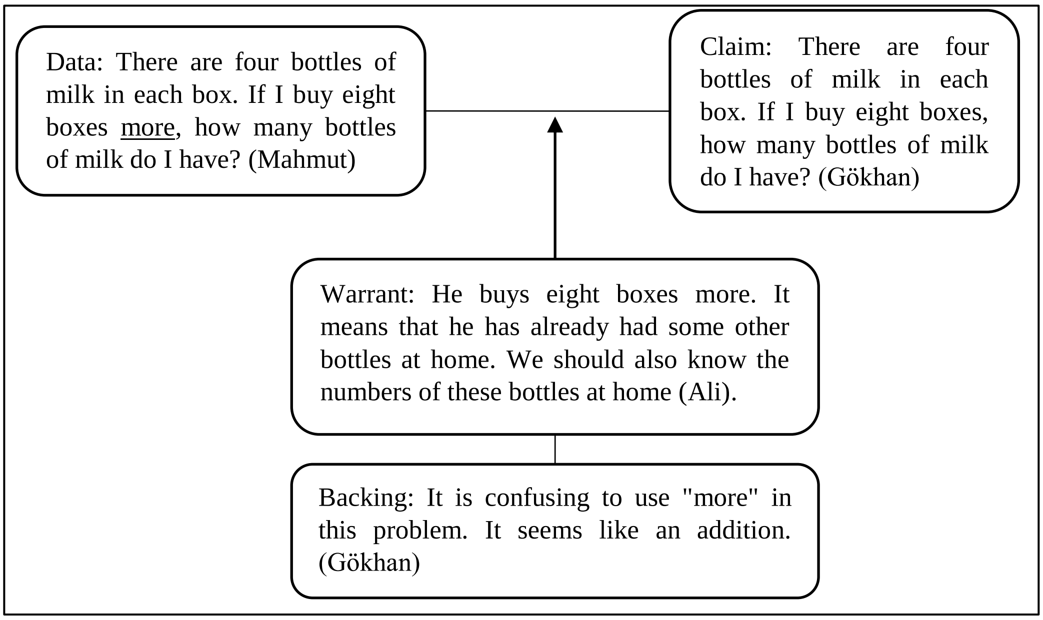


Figure 4. 86. Toulmin's Analysis scheme regarding structure and keywords in the problems on "8x4"

Students focused on the wording to pose a significant problem. Since students had worked on addition problems and the word "more" frequently until they started the multiplication concept, they tended to pose problems related to one-step addition and use the word "more." After practicing problem-solving and later posing activities on multiplication day by day, they started judging the wording in the story of the problem and the meaning under the sentences. Students focused on the action in the problem, so that it should require adding numbers in each group repeatedly or multiplying the numbers of groups and the elements in each group through skip counting. It was also surprising that students did not overgeneralize this situation. The teacher hesitated whether students considered using the word "more" as always wrong for repeated addition/multiplication problems. The posing problem was a relatively high-level activity for the students in second grade. Therefore, they were not asked to discuss whether using the word "more" is always wrong or not. It would have been confusing for them. However, surprisingly they showed that they did not overgeneralize this situation. In one of the lessons, Zuhal posed a problem for "4+4+4=…". She said, "One day, my mom gave me four pencils. My dad gave me four more pencils. My grandmother gave me four more pencils. How many pencils do I have now?" All the students accepted this as a well-written problem. They did not focus on the word "more" but the meaning so that she was given equal numbers of pencils, i.e., the same addend to add over and over. Students reached the idea of defining equal groups and reaching the whole by collecting all the groups. Students used this idea as data in their arguments on following days to claim on problems without the need for backings, which indicates it had become taken-as-shared. Students analyzed and revealed the problems in the story and revised the story to make it contextually rich and mathematically meaningful through stating the meaning and choosing appropriate wording.

In conclusion, this chapter is devoted to the presentation of the taken-as-shared ideas that were established throughout the instructional sequence for multiplication. The classroom mathematical practices were identified to illustrate how the social and psychological environments impacted students' learning of multiplication. Individual students and the classroom community both contributed to these practices as given in Toulmin's Analysis schemes. While individual students contributed to the class by introducing new concepts and contributing more proof, such as facts, warrants, backings, or challenges, the classroom community helped the development of their mathematical knowledge by detecting misunderstandings of concepts offered and revising their thinking to perceive the concepts correctly. Therefore, five classroom mathematical practices emerged in terms of students' taken-as-shared ways of thinking and communicating by using mathematical language. These practices are linked to the revisions for HLT that occurred throughout the course of the twenty-seven class hours. These changes can be applied to future studies to improve the teaching of multiplication concepts.

Finally, by documenting classroom mathematical practices, it was possible to study how the second-grade classroom community developed its multiplication and multiplicative thinking in normative and progressively sophisticated ways (CMPs). A focus was placed on how second graders' collective ways of reasoning developed using both informal and formal tools (such as models, images, gestures, and metaphors) and how this reasoning was supported in ever more complex ways using an RME perspective. These findings are discussed in light of the literature and several revisions for the HLT and the instructional sequence are interpreted and evaluated in the following chapter.

4.6. Summary of Findings

By performing design research, an instructional sequence was created under the direction of a hypothetical learning trajectory in multiplication intended for second-graders. Through the analysis of second graders' classroom discussions that emerged during the implementation of this instructional sequence, to evaluate their communal ways of development in multiplication, the classroom mathematical practices were analyzed by using the adaptation of Toulmin's (1958) argumentation model and the three-phase methodology of Rasmussen and Stephan (2008). Therefore, classroom mathematical practices were documented as collective learning activities, mostly including whole-class discussions. At the end of the analysis, eighteen taken-as-shared ideas (TAS) and five mathematical practices were obtained over the five-week instruction.

The first classroom mathematical practice emerged regarding reasoning with fingers to skip count. Two taken-as-shared ideas supported this practice: skip counting by using ordinal aspects of fingers and finding the order of numbers in a number sequence by reasoning from the order of previously established numbers. Therefore, students used skip counting with their fingers to build and recite number sequences and built a relationship between two numbers in the sequence according to their order.

The second classroom mathematical practice emerged as partitioning objects into equal groups to add them repeatedly. This practice was about students' attempts to divide the specified number of objects into equal groups. Students concentrated on creating equal groups to skip count quickly and effectively in this practice. Using images of the objects or by representing the objects in equal groups, they grouped the objects by rearranging, equipartitioning, and halving them. Furthermore, students interpreted the remainder to equalize the groups.

The third classroom mathematical practice emerged in terms of iterating linked units using pictures and fingers. This practice was about composing the groups obtained by the second classroom mathematical practice. In this practice, students used their fingers, hands, and pictures to assign and iterate composite units and make sense of covariation by double-matching pictures.

The fourth classroom mathematical practice emerged in terms of analyzing components and properties of multiplication by modeling with equal groups and arrays. Students concentrated on actions (such as "groups of," "set of," and "creating arrays") that are related to repeated addition and multiplication concepts during this practice. The objective, in this case, was for the students to define equal groups as a foundation for representing specific multiplicative situations. To put it another way, students gained knowledge of the multiplier, multiplicand, and product of multiplication by connecting the equal group representation, array models, and repeated addition.

The fifth classroom mathematical practice emerged regarding writing contextually realistic problems by coordinating the relationship among multiplicative representations. This practice emerged as students engaged in problem-posing tasks. The equal groups that the students specified as Practice 4 were incorporated into multiplication scenarios. The problem's context was created by the words used in a real-life scenario where the group size and group number were made clear. Students moved from discussing strategies informally to placing a more rigorous procedural emphasis on the action in the problem structure.

CHAPTER V

CONCLUSIONS AND DISCUSSION

In this study, a classroom Hypothetical Learning Trajectory (HLT) accompanied by an instructional sequence was constructed, tested, and refined to develop an effective local instructional theory for multiplication teaching. The growth of the second-grade classroom community's multiplication in normative and progressively complex ways was examined through the documentation of classroom mathematical practices (i.e., CMPs). The development of the second graders' collective ways of reasoning with formal and informal tools and how this reasoning is supported in more complex ways with an RME approach were carefully investigated. Consequently, the CMP analysis suggested modifications for the instructional sequence and the HLT.

The present chapter summarizes the research's conclusions concerning the goals related to developing an HLT and documenting classroom mathematical practices and evaluates them within the frame of the existing literature under two main sections: (1) Development of multiplication in the social context (classroom mathematical practices); and (2) Revisions to the instructional sequence and the HLT. The following sections present implications for practice and recommendations for future investigations.

5.1. Development of multiplication in the social context

A hypothetical learning trajectory regarding multiplication for second graders was developed based on design research. This trajectory guided the creation of the instructional process, which was later implemented in the classroom environment. During this classroom teaching experiment, second graders' discussions in the classroom were recorded and analyzed to reveal students' taken-as-shared ways of thinking.

The first classroom mathematical practice emerged as students engaged in the instructional sequence conducted within the framework of the designed HLT: reasoning with fingers to skip count. Skip counting is suggested as a foundation to increase comprehension of multiplication facts. Some studies argue that students should be strongly encouraged to use their fingers and manage numbers in multiplicative situations coordinately (Anghileri, 1989; 1995; Sherin & Fuson, 2005). Students' skip-counting reasoning and use of fingers can be considered as valuable practices for learning multiplication. In the present study, this way of reasoning and justification was not dictated but instead was collectively and independently discovered by students. For instance, the students were provided with hundreds charts, but they preferred ready-made materials (fingers) that were always available and accessible. They even persuaded the ones who had used hundreds charts to use their fingers instead. The reason why the students chose finger counting might be because they are used to doing addition and subtraction with their fingers. Since they are used to counting by tracking their fingers, they might have used their fingers to count the numbers in a number sequence. Thus, in this practice, the students built and recited number sequences, skip-counted using their fingers to indicate the verbal count order, and identified a relationship between two numbers in the sequence by comparing their locations (order).

In fact, fingers are crucial links between practical and mental processes, allowing abstraction to increase understanding in many cases (Anghileri, 2008). In this sense, TAS 2 (finding orders of numbers in a number sequence by reasoning the order of previously established numbers) under the first CMP can be argued to be a higher-order thinking skill for students. While skip counting, the students used their fingers as representatives of ordinal numbers. Using questions prepared by the design team, the teacher asked the students to find a number in a number sequence while skip counting (e.g., what is the sixth number while counting by threes). After the students counted their six fingers and found 18, the teacher asked follow-up questions like “What is the 7th number while counting by 3s?". The numbers 6 and 7 were purposefully selected to see whether the students could make a connection between these two numbers. It was observed that the students used 18 (the 6th number) and added three more to find the 7th number instead of counting from 3 to 21. They could conceptualize that while the order (multiplier) increases by 1, the number in the sequence increases by 3 (multiplicand). This reasoning is important since it is related to the role of the multiplier, which is yet to be taught in the 3rd grade. Thus, this taken-as-shared idea is one of the striking findings of the current study, showing that students can develop such reasoning in a well-planned classroom discourse. The follow-up questions in the section on possible topics of mathematical discourse in the HLT might have played a crucial role in developing students’ higher-order thinking skills.

The second mathematical practice that emerged as the students engaged in the instructional sequence was partitioning objects into equal groups to add them repeatedly, which is about students’ thinking to form equal groups by partitioning the given objects. In this practice, the students concentrated on creating equal groups to skip count them quickly and effectively. The students informally developed the division concept, which is another striking finding. As students learn more effective methods of multiplying, they start to adopt partitioning strategies (Baek, 1998). They can switch between multiplication and division more efficiently by using the inverse relationship (Jacob & Willis, 2003). Thus, in this study, the teacher asked questions about composing and decomposing while the students were working on equal grouping activities. For example, the students were given pictures of 16 balls on the third day. They were asked to find the number of balls. It was observed that the students tended to count the objects in multiples. After they stated how they counted the balls, the teacher asked, “Can you count these balls in a different way?” This question triggered students to find another way to distribute the objects into equal groups. They developed division strategies to be able to decide on equal groups and looked for the numbers that could be skip-counted until 16 (quotitive division).

The number they chose was the size of the groups, and they tried to find group numbers by skip-counting these groups. It was surprising that the students used skip counting for division. They intuitively discovered the inverse relationship between division and multiplication as stated in the literature (Jacob & Willis, 2003; Kennedy et al., 2008; Kouba & Franklin, 1993; Wright et al., 2014).

Similarly, discussion questions such as "Can you count these balls in a different way?" "Is there another way?", and "Are these the only strategies you can use?" played a huge role in developing concepts other than multiplication. When the students were asked to find all possible ways to skip count the given objects, they focused on finding the divisors of the given number of objects. For instance, on the fifth day, the students worked on 36 birds and tried to find ways to group these birds equally. They looked for the numbers that can be skip-counted until 36 (quotitive division). In fact, they developed this method to divide the given number into its divisors without knowing formally about divisibility. As they learned to skip count fluently with their fingers (TAS 1), they used this idea to connect other topics informally with higher-order thinking.

These discussion questions might also have led the students to discover the commutative and identity properties of multiplication. When asked to find all possible ways to count the given objects, they used the commutativity property. For instance, when a student said, “I can group 16 objects as eight groups of 2”, another student said, “or two groups of 8”. The design team observed this specific thinking strategy in the classroom many times. The students called this strategy “reversing”. They discussed this fact and accepted it as true but could not explain why it is always true. They represented and linked two cases using objects and pictures. However, they could not show the relationship between them. This finding coincides with the literature stating that equal-group representation is not efficient and capable of showing commutativity (Greer, 1992; Schliemann et al., 1998).

In addition to the commutativity property, the students also discovered the identity property. They realized that they could group the objects into a single group (e.g., 1 group of 16) or distribute each object into a group (e.g., 16 groups of 1). In other words, they discovered that each number could be partitioned as a single group with all objects or as groups with a single object. It is noteworthy that the students used commutativity to make this inference since these two situations are “reverses” of each other. Therefore, the students intuitively discovered both the commutativity and identity properties of multiplication. However, multiplication had to be formally introduced to them, which might be due to the tools and imageries (e.g., unit cubes and object pictures) used in the instructional tasks. Thus, we need to focus on mathematical models and their creation in order to talk about mathematics as mathematizing. These models frequently start as learners' simple representations of situations or problems (Fosnot & Dolk, 2001). Then, the more connections students make between and within these situational models, the more universal they become. In other words, physical modeling established at these increasingly complex stages of representation provides a solid foundation for investigating the links to multiplication. It might have stimulated the students to make arguments about equal grouping activities and develop an understanding of the properties of multiplication and interconnected concepts.

Another remarkable finding under the second practice is the ideas strongly related to proportional reasoning. Studies show that students can think proportionally in tasks involving simple multiplicative reasoning or fair sharing early in elementary school (Resnick & Singer, 1993; Boyer & Levine, 2012; Vanluydt et al., 2020). For this reason, well-structured didactical situations containing relationships, partitioning, and unitizing, which are the fundamental mathematical components of proportional reasoning, should be established during early arithmetic instruction (Kaput & West, 1994; Lamon, 1995; Steffe, 1994). Kaput and West (1994) highlighted the importance of creating learning environments for students to have an informal and conceptual background on proportion. In the present study, the nature of the mathematical tasks might have enabled students to think proportionally. Using the pictures of the objects or showing the objects in equal groups, the students grouped the objects by reorganizing, equipartitioning, and halving, which are three taken-as-shared ideas under this practice. These ideas are crucial since they help maintain the relationship between multiplicative and proportional reasoning.

The design team had predicted the first three taken-as-shared ideas (TAS 3, TAS 4, and TAS 5) in the second practice while developing activities and possible discourse questions in the light of the literature. However, the fourth idea (TAS 6) was surprising for the team members, as the students could manage the remainder while partitioning the objects. This might be the conclusion of the RME theory, which is employed for the development and implementation of the design. Most division problems in the real world involve remainders that must be handled appropriately. Children must deal with various situations where the setting influences the remainder in different ways (Fosnot & Dolk, 2001). They should do this to view mathematics as mathematizing by addressing problems in familiar contexts. They also develop mathematical thinking methods in their own lives. They avoid doing things that make no sense to them and treat leftover numbers accordingly. The realistic nature of the instructional tasks enabled the students in our study to handle the leftovers rationally.

In the second practice, the students partitioned the objects equally to obtain units and added the units repeatedly. After this practice, they iterated these units using pictures and fingers as the third classroom mathematical practice. The third practice is about managing multiplication operations first; however, it is related to more than multiplication. When the students' reasoning processes while putting the items together were evaluated, unitizing and iterating were found to be crucial skills for the growth of proportional thinking. The students iterated units as they began to accumulate things through skip counting. In the following sessions, they were given a unit of units (a composite unit) to solve proportional situations, enhancing their counting skills. Later, they were asked follow-up questions well-adjusted to the problem's realistic context. These questions enabled them to consider the relationships between various groupings and broaden their build-up method by creating composite units. As a result, the students might have developed the idea of double-matching collection. This idea is related to understanding covariation, which is crucial for proportional reasoning (Lamon, 1994). From a cognitive point of view, multiplicative structures were extended to new domains of experience so that proportional thinking could be grasped and conceptualized deeply and flexibly. Thus, the current study promoted early proportional reasoning since multiplicative thinking is regarded as the core of proportional reasoning (Behr et al., 1992; Lamon, 2007).

After the students engaged in partitioning and iterating practices, the fourth classroom mathematical practice on the analysis of components and properties of multiplication operations by modeling with equal groups and arrays emerged. In this practice, the students used the provided representations as multiplier and multiplicand while concentrating on group number and group size. Repeated addition is one of the models used by students in multiplicative situations. Despite the conceptual difference between addition and multiplication, the two operations have a procedural connection (Nunes & Bryant, 1996). Repeated addition should not be the foundation for teaching multiplication, but should be viewed as a computing procedure during instruction (Park & Nunes, 2001). In this regard, the current study's findings clearly have practical significance. The students developed an idea that required connecting repeated addition and multiplication operations by interpreting multiplier and multiplicand (TAS 11). The physical tools or pictures of the objects provided in the tasks might have played a crucial role in developing this idea. The students used these representations to form groups and benefited from these models to show these groups via the symbolic representations of repeated addition and multiplication. Moreover, they made drawings to represent equal-sized groups in the problem-solving activities. At the higher level, they used the imagery of repeated addition and analyzed these models, which might have enabled them to take the first steps of building a connection between a multiplier and a multiplicand within the concept of multiplication.

In analyzing the models as the multiplier and the multiplicand, the students focused on determining units suitable for the given multiplicative situation. They tried to construct a reference unit and then reinterpret the given situation in terms of that unit (norming) (Lamon, 1994). They determined the unit to count both items and groups all at once as part of the unitizing process (Fosnot & Dolk, 2001). They interpreted each unit as the multiplicand and the number of units as the multiplier. For instance, they decomposed the given arrays into rows and defined each row as a unit (multiplicand), where the number of columns represented the number of units (multiplicand). It should be noted that the students also used the partitioning practice to distribute the given objects into units (CMP 2) and to iterate these units (CMP 3). Hence, it can be concluded that the fourth classroom mathematical practice was constructed on previous practices and connected them through modeling.

Under the fourth practice, another striking finding of the study is that the students built upon their array models to show multiplicative relationships. Outhread and Mitchelmore (2004) point out the difficulty of seeing the structural similarities between discrete arrays and arrays as a grid of contiguous squares, which may not allow students to connect an array of squares with multiplication. Contrary to this view, the students in the current study established the relationship between the discrete groups and array models. Instead of drawing the actual objects given in the problem text (e.g., apples, baskets), they drew circles to represent them. Surprisingly, they tended to arrange the circles to form rectangular areas. At this point, squared papers were given to the students to reveal the row-column relationship more clearly. The students determined their groups according to the rows or columns on the array models they created on the square backgrounds. Thus, they built their intuitive models and used the array model as a valuable tool to show numerous properties of multiplication.

Arrays are recommended as they are considered to be highly practical in representing the commutative property of multiplication (Anghileri, 2006). They are critically important in developing multiplicative thinking (Hurst, 2015). In the second grade, multiplication is interpreted as repeated addition (MoNE, 2018). However, repeated addition does not view multiplication as commutative. For this reason, arrays are used to enhance students' early learning process of commutativity. In this study, the students divided their arrays into equal groups to link the array model to multiplication. While some students perceived the rows as groups, others perceived the columns as groups. They discussed and discovered that changing the orientation of the array results in swapping the numbers of groups and objects in each group (TAS 14). Therefore, swapping the two multiplied numbers did not change the total. To ensure that children make sense of multiplication and think critically, they should be provided with an environment that would allow them to use different levels of representation (e.g., physically covering a rectangle with unit squares and drawing rectangular arrays on squared paper). As the present study could provide such an environment, the students might have conceptualized the quantities involved and the relationship between multiplication, division, and commutativity.

Another unique taken-as-shared idea that emerged under the fourth classroom mathematical practice is reasoning the effect of change in multiplier on the product (TAS 12). As in TAS 2, the students interpreted the change in the product when the multiplier changed. Unlike TAS 2, they made this interpretation through the symbolic representation of multiplication in real-life contexts. This interpretation is a sophisticated way of thinking since it is one of the 3rd grade curriculum objectives. Moreover, this reasoning is crucial for creating a multiplication table, which is constructed using the pattern "the product increases by the multiplier" for each row (Isoda & Olfos, 2021). Thus, students can conclude that if the multiplier increases by 1, the product increases by the multiplicand while creating the multiplication table.

This idea is also significant because it reflects students' conceptualization of the distributive property of multiplication. According to Carpenter, Franke, and Levi (2003), students can acquire multiplication facts by connecting the challenging or complex ones to the ones they already know by using the distributive property (e.g., 3×8=2×8+8, and 9×6=10×6-6). In the same way, van den Heuvel-Panhuizen (2008) suggests splitting to analyze the given operation (6x3) as known (5x3) and the difference (1x3). These explanations and interpretations shed light on another aspect of the twelfth idea (TAS 12). In the cases related to TAS 12, the students brought the distributive property to the forefront by intuitively embedding it in the algorithms. The activities related to finding the multiples of a given number might have encouraged the students to interpret the multiplier and find the product by using known facts (e.g., Figure 4.57). In this sense, it can be claimed that the current design triggered the students to discover the distributive law, which is the objective of the sixth grade. Thus, this idea can be interpreted as a foundation for the distributive law.

The final classroom mathematical practice is writing contextually realistic problems by coordinating the relationship among multiplicative representations (CMP 5), which is undoubtedly another striking finding of this research. Although the mathematics education program (MoNE, 2018) does not include problem posing in the second grade, the students' ability to pose problems was investigated in reference to the literature. Before dealing with the problem posing tasks, the students were introduced to the multiplication concept. They were encouraged to deal with the problem-solving tasks proposed with real-life examples and concrete materials in the light of RME. Working on such tasks might have enhanced the students' performance in posing realistic, solvable, appropriate, and clear problems. The problem posing tasks served as an assessment tool for improving students' engagement and capabilities (Kwek, 2015). It was observed that the students conceptually understood multiplication and problem solving and were able to connect them with problem posing.

When the equal group problems of the students were examined, they were distinguished as repeated addition and rate problems, as Greer (1992) specified. The students posed repeated addition problems by interpreting the multiplier and multiplicand to place them in a real-life context (TAS 16). The structure of the problems posed by the students was in the form of mapping rule multiplication according to the textual approach (Nesher, 1988; 1992). They presented problems with a minimal underlying structure consisting of three strings. They started by stating that the first two strings were “group number” and “group size”. This might have helped the other students in the classroom to construct a mapping between each group and the objects in the groups. Then, they asked the question (third string) that sought the number of objects in all the groups. Their interpretations of the textual structure of the problems can be inferred from the classroom dialogues presented in the section related to TAS 16 in Chapter 4.

Another critical and striking finding during this classroom mathematical practice was interpreting the multiplier and multiplicand to pose multiplication problems as rate (TAS 17). It should be noted that equal-group problems differ slightly in that they are repeated addition and rate problems (Greer, 1992). The students in the present study posed repeated addition problems similar to those stated in the mathematics education textbooks, as expected at the beginning of the classroom teaching experiment. Surprisingly, it was observed that after a couple of weeks they developed a connection between the given symbolic and visual representations and rate problems. Rather than choosing the concrete or tangible objects in groups, they began to think about abstract items such as daily routines, actions, verbs, and activities in each group. They preferred time intervals such as days and months as groups and began to solve problems by expressing what happened. They limited the actions to a time scale so that each action happens in the same amount of time. For example, if there are three guests one day, there will be three more guests the next day. The students did not pose problems like "One day, three guests came; next week, three guests came; and next month, three guests came. How many guests came in total?". It is like thinking that the plates must be the same size when apples are being served. This shows that the students kept not only actions equal but also time intervals equal. In other words, they specified an action happening at each time interval (unit rate). They constructed their problems based on the unit they determined. Problem-solving tasks like planting trees might have triggered the formation of the unit problems concept in students. The variation of the tasks in the instructional sequence might have played an essential role in developing students' interpretation of multiplicative situations in real life.

Although studies on multiplication report that it is hard to connect multiplication-related mathematical vocabulary with real-world scenarios (Anghileri, 2006; Calabrese et al., 2020), the students in this study wrote entirely original and diverse stories for their problems. It is important to note that students should be taught vocabulary carrying the meaning of multiplication in both mathematics classrooms and daily routines in addition to the procedure of multiplication. The present study revealed that the students presented a variety of realistic multiplication scenarios. They could connect multiplication operations with real-life situations and write their multiplication problems thanks to the realistic mathematics education and the argumentation process during the instruction. As a result, they developed multiplicative language by focusing on the structure and keywords of the problems to conceptualize multiplication (TAS 18). Two types of problems emerged during problem posing sessions: addition and multiplication. The students focused on some keywords to pose multiplication problems. They helped each other to convert the addition problems into multiplication problems by changing keywords. Their capability of problem posing might have been supported with RME, which uses realistic contexts and connections with real-life situations. As the students understood the operation, they used meaningful contexts.

Furthermore, the students developed not only multiplicative language but also the zero property of multiplication through the problem-posing tasks. They were asked to pose problems for the given operation. As expected, they posed problems like "I see zero dogs in the street each day. How many dogs do I see in

5 days?". Such problems are acceptable but meaningfully unreasonable. It would have been better to say, "I don't see any dogs in the street each day" instead of "I see zero dogs in the street each day". Surprisingly, the students started creating problems involving logical contexts suitable for zero property through instruction. For instance, on the twenty-fifth day, Burak posed a problem: "I drink tea without sugar every day. How many sugar cubes do I use for my tea in 5 days?". This problem triggered the other students, and they started developing such contexts. It is thought that the more students engaged in multiplicative tasks developed with the RME theory, the more significant problems they posed. According to the RME, problems are grounded in realistic scenarios that help students imagine themselves in the situation or the role presented by the problem context (Gravemeijer et al., 2000). With didactic phenomenology, the students might have found the problems in the current study feasible and meaningful. Thus, engaging in these problems and making mathematics accessible to the students might have promoted their mathematical development.

So far, five mathematical practices and related taken-as-shared ideas that emerged over 26 days of classroom instruction experiment have been discussed. Although these practices were presented separately, it should be emphasized that the classroom mathematical practices overlapped. They did not emerge linearly since more than one mathematical idea emerged and contributed to a separate practice over the same class hour. For instance, the second and third mathematical practices emerged nearly at the same time since the students processed partitioning and iterating on the same tasks. They partitioned objects to get equal-sized units and iterated them to find the whole. The problem contexts and classroom discourse may have been crucial in this regard. The students were asked follow-up questions about the given context in order to encourage them to think and discuss. Then, during discussion sessions, more than one mathematical idea, each relating to different mathematical practices, was developed by the classroom community.

As explained in detail, the current design was developed in line with the RME heuristics. The tasks were designed in a realistic context, considering the transition from a model of situated informal mathematical strategies used by students to a model for more formal mathematical reasoning (Gravemeijer & Doorman, 1999). Students' transition from "model of" to "model for" while constituting a new mathematical reality (Gravemeijer et al., 2000) was followed and encouraged. When we examine the classroom mathematical practices and taken-as-shared ideas, this shift from "model of" to "model for" is clearly seen. For instance, in the tree-planting problem, the students first drew flowers, trees, balls, or plants in 9 groups of 2 objects to represent the trees planted in 9 mounts (Figures 4.44; 4.45; 4.46). As they reasoned with symbolic representation, these drawings might have become models for higher-level mathematical reasoning. The students used these drawings to decide on the multiplier and multiplicand. Thus, it can be claimed that they moved from "model of" to "model for" by using the contexts of the tasks, the pictures given in the tasks, and their original drawings as expected.

Moreover, the imageries of the students also played a crucial role. The students reflected their immediate understandings of the situations on their imageries. They did not adhere to a uniform understanding or representation, but developed this representation in the process. Anghileri (2008) suggests that students be encouraged to make inferences using models and imagery relevant to the tasks they engage with. Thus, it can be argued that the students used situation-specific imagery as a model of the given context and as a model for higher-level mathematical reasoning. For instance, they made drawings to decide on the group size and group number and on how to write a repeated addition sentence and explain the groups as "… times…, makes..." as in Figure 5.1. In such cases, students' models of given situations served as models for higher-level representation or interpretation, like multiplication.

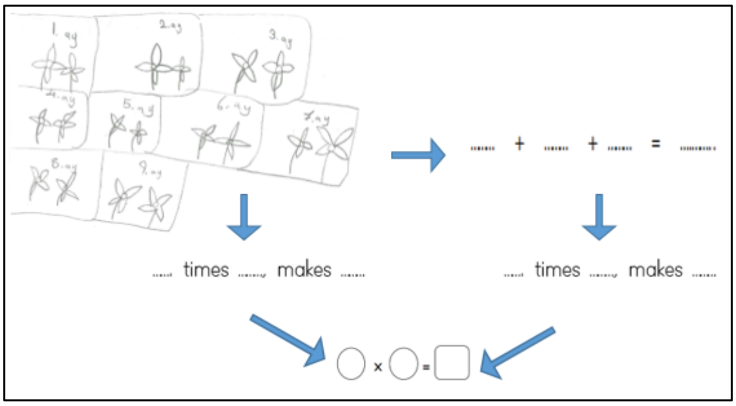


Figure 5. 1. The transition from model of to model for

Finally, the classroom culture, where the mathematical practices and taken-as-shared ideas emerged, should also be discussed. In this context, the norms that play a significant role in developing mathematical practices come to the fore. At the beginning of the teaching experiment, social and socio-mathematical norms had already been established to create a classroom environment filled with communication, think-pair-share, discourse, and openness to mistakes. During the discussion sessions, the students shared their solutions, listened to each other, discussed their ideas, and indicated their agreement or disagreement. Moreover, socio-mathematical norms had also been set to find different solutions and representations, to provide conjectures, and to justify mathematical reasoning. Other students listened carefully and discussed those explanations by offering mathematical reasoning. The students felt free and safe to share their ideas and negotiate. Therefore, it should be emphasized that norms might have played a crucial role in analyzing the social interactions that resulted in taken-as-shared ideas.

5.2. Development of the Instructional Sequence and the HLT

In this research, we aimed to design a hypothetical learning trajectory and related instructional sequence to introduce multiplication to 2nd grade students. The design phases were carefully followed in order to address the first research question. Each phase was explained in detail to show the validity, practicality, and effectiveness of the HLT and the instructional sequence for multiplication.

The first phase of design research (preparing for the experiment) is related to the validity of the developed design. The HLT and instructional sequence components should be based on state-of-the-art knowledge (content validity). All components should be consistently linked to each other (construct validity). It is valid if the product meets these requirements (Nieveen, 1999). This study examined various educational programs, sample learning pathways, and related literature to ensure content validity. Then, a framework was blueprinted for the intervention. The big ideas and objectives were assigned, and the sequence of activities was determined. These processes were documented in detail in Chapter 3. During this process, the design team (a professor of Mathematics Education, a teacher, and the researcher) conducted meetings and asked two more experts for their opinions, after which they agreed that the content validity requirements were met.

The second phase of design research (design experiment) is related to the construct validity and practicality of the developed design. In this phase, a sequence of prototypes is tried out and revised based on formative evaluations to meet construct validity (Nieveen, 1999). Moreover, a formative review is conducted using expert opinions to determine to what extent the users (teachers and students) consider the intervention appealing and usable in normal conditions (i.e., anticipated practicality) (Nieveen, 1999). In this sense, formative assessment was implemented during the instructional sequence in light of classroom observation, whole-class discussion, students' written works, and the researcher's field notes. The team members discussed the validity and practicality of these data collection methods at after-lesson meetings and weekly research team meetings. In this sense, daily revisions were made in the HLT and instructional sequence, as explained in Chapter 3.

As we spent more than a week on skip counting and equal grouping activities in the first week of the implementation, the teacher was concerned about the duration of the design. She discussed the practicality of the design with the team since she thought the first week was a waste of time. The professor and the researcher assured her they would finish the teaching experiment on time since students had gained the prerequisite knowledge. The teacher was satisfied the following week because the students were informally multiplying and dividing during their engagement in equal grouping activities. In total, the classroom teaching experiment took 26 class hours to develop multiplication skills, although the Turkish Middle School Curriculum (2018) devoted 20 class hours to teaching multiplication. After the experiment, the teacher introduced the division concept, to which the mathematics curriculum devotes 16 class hours in second grade (MoNE, 2018). The team also tracked the teacher's time on the division lessons. She spent just five hours on the development of division since students had already grasped the concept while engaging in the instructional sequence for multiplication. In the end, the teacher spent 31 hours teaching multiplication and division, whereas these concepts are given 36 hours in the Turkish Middle School Curriculum (2018). Thus, it can be suggested that the current design is practical enough to be used in Turkish second-grade classrooms.

In order to address both the first and second research questions, the third phase of design research (the retrospective analysis) is devoted to practicality and effectiveness. The consistency of the intervention's experiences and outcomes with the design objectives is related to the intervention's practicability and effectiveness (Nieveen, 1999). In order to examine whether the HLT and instructional sequence met the criteria for effectiveness, retrospective analyses of classroom videos—where the impact of the HLT and instructional sequence on the students and the teacher could be observed—were first processed and then documented. The following section explains how these practices allowed the team members to make the necessary revisions to the HLT.

5.2.1. Suggested Revisions to the Instructional Sequence and the HLT

The CMP analysis did not only document the collective growth of the classroom community as described in Chapter 4 but also provided a retrospective outline of the mathematical content that arose over five weeks of implementation. Moreover, it unfolded the necessary revisions for the Hypothetical Learning Trajectory as well as the content and sequence of the instructional activities for future uses. Several big ideas are suggested for the HLT to enrich the content for teaching multiplication and better reflect the theory of RME (content validity). These additional ideas should be included in the HLT by taking care of the logical order of these ideas. During the implementation of the initial HLT, these ideas emerged as a result of the classroom discourse through follow-up questions. Then, it is suggested that the HLT be revised in line with these unexpected but valuable ideas to make it more practical and usable for the reader. Finally, it is believed that the revised HLT will be more effective in developing multiplication and the related concepts as stated in the multiplicative conceptual field theory. Therefore, the big ideas and phases of the initial HLT are suggested to be modified and revised as in Figure 5.2 through 5.6.

Big Idea	Tools/Imagery	Activity/Taken-as-shared Interests	Possible topics of Mathematical Discourse	Possible Gesturing and Metaphors
1-Build up to skip-count	-Counting on to skip count -Counting fingers -Hundred Chart	-Keeping adding the same number each time to the previous number to skip count.	-How to find the order of a given number in a number sequence?	-Actions with fingers -Keep track of numbers with fingers
2-Reasoning Number Sequence	-The order of the numbers in a number sequence -Finger counting	-Making a connection between the orders of two numbers in a number sequence	-Should you count from the beginning to find a number in a number sequence?	-Keep track of numbers with fingers

Figure 5. 2. Phase 1 of the revised HLT for the multiplication instructional sequence

Big Idea	Tools/Imagery	Activity/Taken-as-shared Interests	Possible topics of Mathematical Discourse	Possible Gesturing and Metaphors
3-Additive Composition	-Counting on -Pictorial representation of objects -Grouping pictures by circling -Concrete objects -Repeated addition	-Drawing equal groups to reorganize the objects -Recognizing the ease of counting objects in equal groups -Focusing on repetition of group size	-A collection can be rearranged or counted in a different way and the quantity does not change	- Laying out the items as described in the task -Counting fingers
4-Equipartitioning	-Intuitive knowledge of unitizing -Build up strategies -Pictorial representation of objects -Grouping pictures by circling -Informal language of multiplication.	-Sharing collections equally by using skip counting - Reasoning the inverse relationship between multiplication and division intuitively.	-Questioning whether skip counting is used to form equal groups.	- Laying out the items as described in the task -Circling the pictorial representations of objects -Counting fingers
5-Halving	-Pictorial representation of objects -Informal proportional reasoning -Between comparison	-Dividing collections into two. -Regrouping collections by dividing each group into two	-What can be said about the group size and group number after halving?	-Making horizontal and vertical gestures to show how to halve the object
6-Sharing the remainder	-Number relations -Intuitive knowledge of divisibility -Part-whole relationship	-Sharing the remainder equally	-How to use the unequal groups to generate equal groups.	-Actions with fingers -Keep track of numbers with fingers -Making hand gestures

Figure 5. 3. Phase 2 of the revised HLT for the multiplication instructional sequence

Big Idea	Tools/Imagery	Activity/Taken-as-shared Interests	Possible topics of Mathematical Discourse	Possible Gesturing and Metaphors
7-Many-to-one correspondence (Composite Unit)	-Composite unit -Counting groups	-Counting groups instead of objects -Focusing on the multiplier	-The roles of multiplier and multiplicand while skip counting.	-Actions with fingers -Pointing groups -Keep track of numbers with fingers
8-Linking composite units	-Hands as benchmarks -Build up strategy -Informal proportional reasoning	-Conceptualizing each hand as a collection of groups rather than objects.	-How to reason this (proportional) reasoning.	-Actions with fingers and hands
9-Doubling	-Pictorial representation of objects -Informal proportional reasoning -Between comparison	-Doubling to build up by keeping track of quantities	-What is the relationship between the number of chickens and coops?	-Making hand gestures to double or iterate the given pictures

Figure 5. 4. Phase 3 of the revised HLT for the multiplication instructional sequence

Big Idea	Tools/Imagery	Activity/Taken-as-shared Interests	Possible topics of Mathematical Discourse	Possible Gesturing and Metaphors
10-Formalizing multiplication	-Multiplication symbol (X) -Multiplicative language (multiplier, multiplicand, product, times, multiply) - Pictorial representation -Symbolic representation	-Defining multiplier and multiplicand in equal group representation	-What are the roles of multiplier and multiplicand? What if their places are changed?	-pointing with fingers to match equal groups with the numbers in multiplication.
11-Modeling multiplication as equal groups	-Repeated addition -Pictorial representation -Symbolic representation	-Connecting pictorial and symbolic representation	-How to connect pictorial and symbolic representation	-pointing with fingers to match equal groups with the numbers in repeated addition.
12-Interpreting the role of multiplier	-Build up strategy -Between comparison	-Making a connection between the orders of two numbers in a number sequence	-What is the effect of change in multiplier on the product?	-Keep track of groups with fingers
13-Representing multiplication as array	-Array (row/column) -Pictorial representation -Symbolic representation	-Interpreting multiplier and multiplicand on array model	-How to connect array and symbolic representation of multiplication	-Pointing with fingers to match row/columns with multiplier/multiplicand
14-Reasoning multiplication in array	-Array (row/column) -Pictorial representation -Symbolic representation	-Reasoning multiplication with array -Equipartitioning array depending on commutativity	-Why does the product remain same when the places of multiplier and multiplicand change?	

Figure 5. 5. Phase 4 of the revised HLT for the multiplication instructional sequence

Big Idea	Tools/Imagery	Activity/Taken-as-shared Interests	Possible topics of Mathematical Discourse	Possible Gesturing and Metaphors
15-Posing multiplication problems on known contexts	-Real-life problems -Familiar contexts	-Placing numbers (multiplier/multiplicand) in familiar multiplicative contexts	-How to decide that it is a multiplication problem.	
16-Posing multiplication problems as repeated addition	-Real-life problems -Equal groups -Concrete groupable objects as multiplicand -Closed groups as multiplier	-Assigning multiplier and multiplicand to represent as equal groups (repeated addition problem)	-What is the place of numbers in repeated addition in the related story	
17- Posing multiplication problems as rate	-Real-life problems -Daily routines as multiplicand -Time interval as multiplier -Informal unit rate	-Assigning multiplier and multiplicand to represent as equal groups (rate problem)	-How to decide on the context	
18-Structuring multiplicative language	-Real-life problems -Formal language of multiplication.	-Connecting formal multiplication with informal multiplicative language in real life.	-What are the key words to use in the context?	

Figure 5. 6. Phase 5 of the revised HLT for the multiplication instructional sequence

A structured and interconnected network of mathematical concepts and abilities must be built to generate significant mathematical knowledge (National Mathematics Advisory Panel, 2008). According to this notion, the multiplicative conceptual field (MCF) was described as a network of connected subjects constructed by division and multiplication with whole numbers (Vergnaud, 1994). MCF covers measurement, fractions, rates, and proportional reasoning topics in relation to multiplication and division. As a result, from a cognitive perspective, multiplicative structures should be broadened to include new domains of experience in order to understand and conceptualize concepts clearly and flexibly. According to this claim, whole number multiplication significantly encourages the early conceptual growth of related concepts. Keeping this in mind, the design team noted mathematical discourse topics that could trigger students' thinking related to connected concepts and provided related follow-up questions to the teacher. However, big ideas or learning goals related to these connected concepts, such as division and proportional reasoning, were not included in the HLT or related instructional sequence. Unexpectedly, taken-as-shared ideas and classroom mathematical practices related to these concepts emerged during the classroom discourse. In line with these ideas, the HLT should be revised, and these ideas should be included.

Students developed ideas related to equipartitioning while participating in grouping activities. As explained before, students used skip counting to equipartition the collection of objects. This idea emerged through discussing all the possibilities for the equipartition of given objects. The purpose of this discussion was to encourage students to share their ways of grouping. Surprisingly, they developed the strategy of skip counting to divide objects equally. This way of thinking was noted as a big idea for equipartitioning. Hence, it should be included in the revised HLT as a big idea.

In addition to equipartitioning, students also developed an idea to evaluate the remainder. In the development of this idea, classroom discourse plays a significant role. When a student partitioned the given objects with remainders, the others discussed this partitioning procedure to distribute leftover numbers (remainders) to equal groups by conserving equality when possible. The social norms (e.g., interpreting others' solutions) and sociomathematical norms (e.g., sharing different solutions) also contributed to the classroom discourse, and students helped each other equipartition by distributing the remainder. Thus, the sixth taken-as-shared idea related to interpreting the remainder emerged. This idea was found valuable and was included in the HLT as another big idea.

Having analyzed these big ideas related to equipartitioning (TAS 4) and having interpreted the remainder (TAS 6), the team realized that a new practice emerged while equally partitioning the objects into equal groups to add them repeatedly (CMP 2). As these ideas take an important place in the HLT, the team decided to include a new phase related to the division comprising these ideas. The team also discussed the place of this phase in the HLT. As found in classroom mathematical practice analyses, they placed it before the phase related to iterating the units (multiplication). In the flow of the instruction, students attempted to divide objects into equal groups to be able to skip count them. Moreover, the ideas related to division became taken-as-shared before the ideas related to iterating units. Therefore, the ideas related to partitioning were placed in the second phase of the HLT.

Likewise, the ideas related to division emerged during the teaching experiment, as did those related to proportional reasoning, like halving and doubling. These ideas reflect the effectiveness of the design on early proportional reasoning. As stated by the theory of the multiplicative conceptual field (Vergnaud, 1994), the ideas related to proportional reasoning have a crucial role in reflecting the connection between multiplication and proportional reasoning. Moreover, it was surprising that students developed covariational reasoning, which is crucial for proportional thinking (Lamon, 1994). Using pictures of the objects allowed students to reason in a way that allowed them to understand covariation by double-matching. Thus, the big ideas should be included in the revised HLT, and the pictorial representations should be included in the "tools/imagery" column in the HLT table.

As stated before, students made arguments related to the role of the multiplier on the product (TAS 12) in line with the third-grade objectives. The roots of this reasoning go back to the tasks related to skip counting and number sequences (TAS 2). Students make connections between the numbers in a number sequence and their ordinal numbers. Through the teaching experiment, students developed this idea to interpret the change in the multiplier. The research team planned to discuss the interpretation of the change in multiplier on the product to observe students' conceptualization of equal groups. Surprisingly, students developed this idea and used it to solve problems. Thus, the design team decided to include this idea in the revised HLT.

The final revision to the HLT is related to rate problems, which require focusing on the unit rate. Rate problems may be more challenging for young learners since they include a rate rather than several countable objects. Indeed, the students in this study posed rate problems for a given visual or symbolic representation, although they had been asked to solve only one rate problem in the instructional sequence. Posing such questions should be considered because they are conceptualized in a manner similar to equal-group problems (van de Walle et al., 2020). In this regard, it was decided to include more rate problems and make students discuss them. Thus, the HLT was revised, and a big idea related to the rate problem was included.

To sum up, the initial HLT was revised in light of the findings obtained from classroom implementation. The initial and revised HLTs are compared in Figure 5.7. As shown in the figure, the conjectured ideas shown with arrows were included in the revised HLT since they emerged in the classroom. Other than these, seven more ideas highlighted in the figure were observed in the classroom and included in the revised HLT. Overall, the HLT was composed of eighteen big ideas. As a part of the three-phase method, these ideas were organized according to common mathematical activities that served as classroom mathematical practices. When the new ideas were included, Phase 2 in the initial HLT was split into two, as Phase 2 related to partitioning and Phase 3 related to iterating units in the revised HLT.

The Initial HLT for multiplication	
Phases	Big Ideas
Phase 1	1-Build up to skip-count
	2-Reasoning Number Sequence
Phase 2	3-Additive Composition
	4-Many-to-one correspondence (Composite Unit)
Phase 3	5-Formalizing multiplication
	6-Modeling multiplication as equal groups
	7-Representing multiplication as array
	8-Reasoning multiplication in array
Phase 4	9- Posing multiplication problems
	10-Posing multiplication problems as repeated addition
	11-Structuring multiplicative language

The Revised HLT for multiplication	
Big Ideas	Phases
1-Build up to skip-count	Phase 1
2-Reasoning Number Sequence	
3-Additive Composition	Phase 2
4-Equipartitioning	
5-Halving	
6-Sharing the remainder	
7-Many-to-one correspondence (Composite Unit)	Phase 3
8-Linking composite units	
9-Doubling	
10-Formalizing multiplication	Phase 4
11-Modeling multiplication as equal groups	
12-Interpreting the role of multiplier	
13-Representing multiplication as array	
14-Reasoning multiplication in array	
15-Posing multiplication problems on known contexts	Phase 5
16-Posing multiplication problems as repeated addition	
17- Posing multiplication problems as rate	
18-Structuring multiplicative language	

Figure 5. 7. Comparison of the initial and revised HLT for the multiplication

5.3. Implications of the Study

In line with the purposes of this study, it was documented how specific instructional activities support second-grade students' understanding of the multiplication concept from a social perspective. The implemented HLT was assessed, and a revised HLT was suggested as a possible direction for future research. Hence, grounded on the findings of this study, possible implications for people in the mathematics education field are stated in this part of the study.

In the current study, all the activities were developed in realistic contexts where students identified the specific mathematics in a general context, schematized through drawing and using imageries; formulated and visualized the problem in different ways like equal groups and arrays; discovered relations between group number and group size; discovered regularities like equality in each group; transferred a real-world issue to a mathematical problem via problem posing, and transferred a real-world problem to a known mathematical model via problem-solving in the sense of horizontal mathematization. Meanwhile, in this design, students represented the objects in equal groups via mathematical formulas of repeated addition and multiplication, proved regularities in each operation, refined and adjusted models for the given contexts, used different models of equal groups and arrays by connecting with repeated addition and multiplication, and generalized the rules and properties of multiplication in the sense of vertical mathematization. Therefore, this study indicated that a well-designed instructional sequence through RME theory has the potential to facilitate mathematization in the classroom. In this sense, teachers can design their lessons in light of Realistic Mathematics Education as in the current study. Likewise, teacher educators can incorporate those into their lessons to aid preservice teachers in building their subject matter and pedagogical content knowledge so they can use it in their future lessons.

Notably, the study's design and findings might contribute to theory and practice. In the name of the theory, the study's design enhances the existing knowledge of instructional theory, particularly multiplication. This study, which uses the RME theory as a model and a guideline, intends to develop a local instructional theory for teaching multiplication. Since local instruction theories include recently developed instances of how RME might be worked out, they can serve as the basis for creating a more complex version of the general theory (Gravemeijer & Stephan, 2002). Due to the reconstruction of an existing theory and the revelation of how RME can affect the development of an HLT for multiplication, the findings of this study may help advance the RME theory in teaching multiplication. Moreover, the norms, tools (concrete materials and pictures), models (equal groups and arrays), and activities (problem posing) explained in this study can be propounded to inform the theory specific to multiplication instruction. Therefore, the structure and nature of the developed local instructional theory might enlighten the RME theory for teaching multiplication.

In the name of the practice, the study's findings reveal what contents should be taught under the topic of multiplication. In addition to the objectives for second graders in the mathematics education program (MoNE, 2018), skip counting, modeling via arrays, and problem-posing were included in the instructions. Skip counting is placed mathematics education program (MoNE, 2018) for the previous semester. However, in the current design, it was included at the beginning of the instructional sequence to maintain students' capability of counting strategies, a prerequisite for multiplying two numbers. Utilizing number sequences, students could keep track of both the group size and group number at the same time. In this sense, the current study's findings might support the idea that skip counting might be included in the mathematics education program right before teaching multiplication rather than one semester before.

In the current study, both skip counting and equal grouping activities enriched students' number sense abilities. Skip-counting activities gradually lay the foundation for comprehension to grow facts (Ogletree et al., 1970), and number composition and decomposition in equal groups improve students' number reasoning (Anghileri, 2008). Bearing the fact that the arguments of the students were interpreted in terms of procedural and conceptual knowledge of whole numbers, comprehension of magnitude, counting abilities, and basic arithmetic operations, the activities in the instructional sequence contributed to the students' number sense abilities. From this point of view, teachers can benefit from the activities implemented in the teaching experiment to develop number sense while teaching multiplication.

Another crucial concept that was given a prominent place in the current design is the array representation of multiplication. Students worked on the tasks by modeling the given context via both equal groups and array models. Array models helped students put forth the symmetric nature of multiplication and its properties (commutative, distributive, and identity properties). In this sense, the design and findings of the current study in terms of defining multiplication as an array have several implications. The activities given in the instructional sequence can be used as sample activities to be applied in lessons by teachers, textbook writers, and curriculum developers. At the same time, the findings and argumentation schemes presented in the study can be a guideline for those who want to use an array representation of multiplication in the mathematics education area.

In addition to skip counting and array models, problem-posing is another crucial concept in this study. In the mathematics education program (MoNE, 2018), problem posing on multiplication is given place in third grade. In the current design, problem posing activities were implemented for second graders. These activities improved students' association of multiplication with daily life and allowed us to observe how students interpret multiplication. The students' problems showed how they give meaning to multiplication and connect it to routine activities they experience in real life. Therefore, the findings related to using problem posing activities and students' performance can inform those who might benefit from this study. While curriculum designers can incorporate problem posing on multiplication also in the second grade to strengthen the education, math teachers can apply such activities more often in their classes to teach multiplication.

The findings showed that students could develop an understanding of partitioning objects into equal groups to add them repeatedly (Practice 2). Students were shown composing and decomposing activities to conjecture this progression so they could use division informally. This practice shows the possibility of teaching multiplication and division together as stated in the literature. In this sense, teachers can integrate the activities supporting the evaluation of both operations simultaneously in the instructional sequence into their lessons by investigating the relationship between multiplication and division. Curriculum developers can also use these findings to revise the objectives related to multiplication and division in a nested way.

The findings also revealed that multiplication is the foundation of other mathematical topics like proportional reasoning. According to previously conducted studies, students can think proportionally in tasks involving simple multiplicative reasoning or fair sharing early in elementary school (e.g., Resnick & Singer, 1993; Boyer & Levine, 2012; Vanluydt et al., 2020). It is critical to figure out what students know informally so that effective instruction can be constructed on that intuitive knowledge and methods (Kaput & West, 1994; Lamon, 1994). As stated in the literature, the findings revealed that young children have well-developed counting, matching, partitioning, and making sense of one-to-many and many-to-many correspondence abilities. For instance, in the current study, students composed their units and iterated them to find the product; consequently, they developed early proportional reasoning. The practices and ideas taken-as-shared can be used by those who want to study the concepts of proportional reasoning. Teachers can also benefit from the findings of this study to get an idea of students' prior knowledge before teaching further concepts.

To conclude, the instructional sequence and the associated hypothetical learning trajectory could aid in the development of multiplication in a second-grade classroom community in an argumentative classroom setting. Teachers can, therefore, readily incorporate these into their second-grade lessons on multiplication. At that point, transferability which is related to the applicability of the design experiment in various contexts and is associated with external validity (Bakker & van Eerde, 2015; McKenney & Reeves, 2012; Merriam, 2009), comes to the fore. For others interested in transferring the findings elsewhere to assess the likelihood of doing so, transferability is established by providing a full account of the environment, time period, and participants in this study. The study's methodology and conclusions are given in-depth, taking each stage into account with thorough explanations. Consequently, teachers can easily integrate the instructional sequence into their multiplication courses.

Finally, all the taken-as-ideas and mathematical practices related to multiplication emerged through the classroom discourse. While managing the discussions on the tasks, social and sociomathematical norms played a crucial role, as seen in the dialogues given in the study. In this sense, teachers who want to develop the collective growth of the classroom community can benefit from the norms mentioned in this study to set their norms in their classrooms and to manage classroom discourse with the guidance of these norms.

5.4. Recommendations for Further Research

This study shows how a second-grade classroom community created fundamental concepts of multiplication, how those concepts were taken-as-shared as the children interacted with the instructional sequence and HLT, and how this development was encouraged. In addition, some other questions are raised at the end of the study. Therefore, in this section, further research is recommended to find answers to the questions that this study raised.

A vital understanding of the collective growth of a school community's multiplication skills was provided by the documentation of the mathematical procedures used in the classroom. Specifically, students developed ideas and shared them in the classroom through argumentation, justifying for their claims. As a result of this sharing process, eighteen distinct, taken-as-shared ideas evolved in the classroom when the classroom discussions were analyzed by Toulmin's argumentation model (1958), which was used to analyze the evolution of mathematical practices. These ideas might be detailed and related in many aspects. It can be challenging to distinguish some of them. However, each idea is unique for the related mathematical practices, so they cannot be merged but presented separately. In this sense, it can be suggested to interpret the study's data through another framework or perspective to search for more general ideas.

Furthermore, individual student learning was used to investigate whole-class interactions and document the evolution of taken-as-shared mathematical ideas, despite the fact that individual student learning analysis is beyond the scope of this study. Individual students' knowledge and progress may need to be adequately addressed by observing and recording the learning in an entire class. Therefore, it would be interesting to research how different students develop individually, specifically how students benefit from and contribute to this collective development. To gain an in-depth understanding of the learning and growth that occurred in the classroom, subsequent research that examines each student's individual development of multiplication through pre- and post-design is recommended. Thus, quantifying the level of learning and growth through pre- and post-testing would aid in understanding the learning and development in the classroom community.

Moreover, carrying out a longitudinal study would aid in examining the community of students' long-term retention and learning. The results of this study demonstrated how multiplicative thinking evolves with other ideas like division and proportional reasoning. In this regard, an important research area could be how multiplication develops inside the multiplicative conceptual field. Hence, the evolution of concepts like division, fractions, and proportional reasoning, which are parts of MCF, is suggested to be observed and examined through a longitudinal study.

Similarly, further design research is suggested to develop another HLT and related instructional sequence for multiplication in the third grade. In the case of the current study, the students intuitively comprehended objectives related to the third grade. For that reason, the team concluded that students have fewer objectives to learn in the next year. In this regard, it can be suggested to design the objectives and lessons for the following year for the participants of this study.

From another perspective, the current study can be complemented and extended by reanalyzing the same data for the gestures of the students and their role in establishing taken-as-shared ideas. Focusing on the function of gesturing in the collective development of meaning can be an important yet often neglected component of mathematics learning (Rasmussen et al., 2004). In this sense, it is suggested to conduct further research to illustrate a methodological approach for empirically investigating the function of gesturing in the evolution of the taken-as-shared ways of reasoning in multiplication.

In addition to the students' understanding of multiplication, the teaching practice of the participating teacher also improved. Since the current study focused on students' communal learning through classroom teaching experiments, in further research, it can be suggested to focus on the teacher by conducting "dual design" research in which the goal of supporting the teacher's learning is addressed in a parallel design research project (Gravemeijer & van Eerde, 2009). Such a design can be developed to support the learning of students and their teacher within the same study. The development of the participating teacher's expertise in scaffolding students' development can be traced.

Furthermore, professional development programs on constructing RME-based instructions considering further topics, setting norms, and maintaining negotiation in the classroom can be developed for teachers in order to improve their subject matter knowledge and pedagogical content knowledge. The current study offers a methodological tool to improve interventions and build more appropriate theories for particular contexts. Consequently, this study may provide a feasible new professional development strategy to give instructors these crucial teaching tools. Hence, professional development programs can be prepared and implemented.

Finally, the mathematical practices that emerged in the classroom recommended some changes to the HLT and the teaching sequence. A few tasks were explicitly requested to be added to the sequence, while a few were moved out. Some revisions were also made to the HLT's phases and parts. Therefore, a later design experiment, or Design Experiment 2, can shed light on whether those improvements would be implemented in a subsequent experiment and, more specifically, on whether they would support the teaching and learning of multiplication in a classroom community. Consequently, it is recommended to conduct a follow-up study in which the revised instructional sequence would be tested to create a more practical and better case for teaching sequence.

Furthermore, it should be noted that the current study was carried out with the teacher and her students, all of whom had prior experience with design research. Accordingly, the classroom norms had already been established. In this regard, those who wish to conduct such design research are advised to conduct a pilot study, which may be useful in preparing participants for the main study in terms of social and sociomathematical norms, as well as ensuring the internal validity (credibility) of an investigation through prolonged interaction engagement.

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APPENDICES

A. METU HUMAN SUBJECTS ETHICS COMMITTEE APPROVAL

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
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22 OCAK 2019

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof.Dr. Mine Işıksal BOSTAN

Danışmanlığını yaptığınız Semanur KANDİL'in **"İlkokul 2. Sınıf Öğrencilerinin Doğal Sayılarla Çarpma İşlemini Anlamalarına Yönelik Öğrenme Rotası Geliştirilmesi"** başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **028-ODTÜ-2019** protokol numarası ile onaylanmıştır.

Saygılarımla bilgilerinize sunarım.

Prof. Dr. Tülin GENÇÖZ
Başkan

Prof. Dr. Ayhan SOL
Üye

Prof. Dr. Ayhan Gürbüz DEMİR
Üye

Prof.Dr. Yaşar KONDAKÇI
Üye

Doç. Dr. Emre SELÇUK
Üye

Doç. Dr. Pınar KAYGAN
Üye

Dr. Öğr. Üyesi Ali Emre TURGUT
Üye

B. OFFICIAL PERMISSIONS TAKEN FROM THE MINISTRY OF NATIONAL EDUCATION

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

Sayı : 14588481-605.99-E.3413964 15.02.2019
Konu : Araştırma İzni

ORTA DOĞU TEKNİK ÜNİVERSİTESİNE
(Öğrenci İşleri Daire Başkanlığı)

İlgi: a) MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2017/25 nolu Genelgesi.
b) 11/02/2019 Tarihli ve E.94 sayılı yazınız.

Üniversiteniz İlköğretim Anabilim Dalı Doktora Programı öğrencisi Semanur KANDİL'in **"İlkokul 2. Sınıf Öğrencilerinin Doğal Sayılarla Çarpma İşlemini Anlamalarına Yönelik Öğrenme Rotası Geliştirilmesi"** konulu tez çalışması kapsamında uygulama talebi Müdürlüğümüzce uygun görülmüş ve uygulamanın yapılacağı İlçe Milli Eğitim Müdürlüğüne bilgi verilmiştir.

Görüşme formunun (6 sayfa) araştırmacı tarafından uygulama yapılacak sayıda çoğaltılması ve çalışmanın bitiminde bir örneğinin (cd ortamında) Müdürlüğümüz Strateji Geliştirme Şubesine gönderilmesini rica ederim.

Turan AKPINAR
Vali a.
Milli Eğitim Müdürü

Güvenli Elektronik İmzalı
Aslı ile Aynıdır.
15/02/2019

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e-posta

Bilgi için: Ayşe ARDA
Tel:
Faks:

Bu evrak güvenli elektronik imza ile imzalanmıştır. https://evraksorgu.meb.gov.tr adresinden c5eb-e413-3444-ae24-2a97 kodu ile teyit edilebilir.

C. CONSENT FORM

ARAŞTIRMAYA GÖNÜLLÜ KATILIM FORMU

Bu çalışma, Orta Doğu Teknik Üniversitesi Eğitim Fakültesi Matematik ve Fen Bilimleri Eğitimi araştırma görevlisi Semanur Kandil ve Orta Doğu Teknik Üniversitesi Eğitim Fakültesi Matematik ve Fen Bilimleri Eğitimi Bölümü Öğretim Üyesi Prof. Dr. Mine Işıksal Bostan tarafından doktora tezi kapsamında yapılmıştır. Bu form sizi araştırma koşulları hakkında bilgilendirmek için hazırlanmıştır.

Bu çalışmanın amacı öğrenme rotaları yaklaşımına dayanan 2. sınıf çarpma işlemi konusu ile ilgili bir öğretim modülünün geliştirilmesi, bu modülün uygulanması, bu süreçte öğrencilerin ve öğretmenlerin deneyimlerinin araştırılması ve sonuç olarak hazırlanan öğretim modülünün düzenlenerek son haline getirilmesidir.

Çalışma kapsamında siz matematik öğretmenimizden Matematik Dersi Öğretim Programı'nda öngörülen kazanımlar ve ders saatleri doğrultusunda öğretim sürecini kapsayacak şekilde çarpma işlemi konusu ile ilgili öğretim materyallerinin geliştirilmesine katkıda bulunmanız ve işbirlikçi olarak geliştirilen bu materyalleri 2. sınıf öğrencileri ile gerçekleştirdiğiniz derslerde kullanmanız istenecektir. Bu amaçla sizden ses kaydı alınmak üzere birebir görüşmeler ve video kaydı alınmak üzere derslerinizin gözlemlenmesi talep edilecektir.

Bu çalışmaya katılmak tamamen gönüllülük esasına dayalıdır. Herhangi bir yaptırıma veya cezaya maruz kalmadan çalışmaya katılmayı reddedebilir veya istediğiniz aşamada çalışmayı bırakabilirsiniz. Araştırma esnasında cevap vermek istemediğiniz sorular olursa cevap vermeme hakkına sahipsiniz.

Araştırmaya katılanlardan toplanan veriler tamamen gizli tutulacak, veriler ve kimlik bilgileri herhangi bir şekilde eşleştirilmeyecektir. Katılımcıların isimleri bağımsız bir listede toplanacaktır. Ayrıca toplanan verilere sadece araştırmacılar ulaşabilecektir. Bu araştırmanın sonuçları bilimsel ve profesyonel yayınlarda veya eğitim amaçlı kullanılabilir, fakat katılımcıların kimliği gizli tutulacaktır. Çalışma fiziksel veya ruhsal sağlığınıza zarar verecek hiçbir uygulama içermemektedir.

Bu cismanin sonucunda uluslararası ve ulusal alanda 2. sınıf çarpma işlemi konusunun öğretiminde kullanılabilecek öğrenme rotaları yaklaşımıma dayanan bir öğretim modülünü geliştirilecek olmasından dolayı ulaşabildiğimiz katılımcı sayısı bizim için büyük önem taşımaktadır.

Çalışma hakkında daha fazla bilgi almak için aşağıda bilgileri verilen kişilerle iletişime geçebilirsiniz.

Arş. Gör. Semanur Kandil	Prof. Dr. Mine Işıksal Bostan
ODTÜ Eğitim Fakültesi	ODTÜ Eğitim Fakültesi
Mat. ve Fen Bilimleri Eğitimi Böl.	Mat. ve Fen Bilimleri Eğitimi Böl.
Ofis Tel	Ofis Tel:
e-mail:	e-mail:

Yukarıdaki bilgileri okudum ve bu çalışmaya tamamen gönüllü olarak katılıyorum.

(Formu doldurup imzaladıktan sonra uygulayıcıya geri veriniz)

Ad-Soyad	Tarih	İmza

D. PARENT PERMISSION FORM/VELİ ONAY FORMU

Veli Onay Formu

Sevgili Veli,

Bu çalışma Orta Doğu Teknik Üniversitesi öğretim üyeleri Prof. Dr. Mine Işıksal-Bostan ve Arş. Gör. Semanur Kandil tarafından Ankara İl Milli Eğitim Müdürlüğünün onayı ile yürütülmektedir.

Bu çalışmanın amacı nedir? Çalışmanın amacı ilkokul 2. sınıf öğrencilerinin matematik dersinde "çarpma işlemi" ünitesini daha güçlü bir temel kazanarak öğrenmeleri için öğrenme programı ve ders materyalleri geliştirmektir.

Çocuğunuzun katılımcı olarak ne yapmasını istiyoruz? Çalışma sırasında çocuğunuz MEB tarafından belirlenen program kapsamında matematik derslerine öğretmeni ile devam edecek, ek olarak ekibimiz tarafından geliştirilmiş müfredata uygun öğrenme aktiviteleri yürütülecektir. Bu aktiviteler sırasında proje ekibimizden bir arkadaşımız öğretmene yardımcı olmak ve etkinliklerin etkili bir şekilde yürütüldüğünü gözlemlemek için sınıfta bulunacaktır.

Çalışma kapsamında sınıftan alacağımız bilgiler tamamen gizli tutulacak ve sadece proje ekibi tarafından gizlilik ilkesine sadık kalarak değerlendirilecektir.

Katılım sırasında herhangi bir nedenden ötürü çocuğunuzun çalışmadan ayrılmasını isterseniz proje çalışanlarını veya öğretmenleri bilgilendirmeniz yeterli olacaktır.

Çalışma hakkında daha fazla bilgi almak için doktora tezini yürüten Arş. Gör. Semanur KANDİL ile iletişim kurabilirsiniz.

Yukarıdaki bilgileri okudum ve çocuğumun bu çalışmada yer almasını onaylıyorum.

Velinin adı-soyadı: ______________________________ Bugünün Tarihi:______________

Çocuğun adı soyadı:____________________________

E. THE INSTRUCTIONAL SEQUENCE/ETKİNLİK DİZİSİ

Yüzlük kartta ikişer ritmik sayalım. Söylediğimiz sayıları boyayalım.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

2'den başlayarak ikişer sayarken söylediğiniz 2. sayı kaçtır? ....

2'den başlayarak ikişer sayarken söylediğiniz 5. sayı kaçtır? ....

2'den başlayarak ikişer sayarken söylediğiniz 7. sayı kaçtır? ....

2'den başlayarak ikişer sayarken söylediğiniz 8. sayı kaçtır? ....

2'den başlayarak ikişer sayarken söylediğiniz 9. sayı kaçtır? ....

2'den başlayarak ikişer sayarken söylediğiniz 10. sayı kaçtır? ....

1

Yüzlük kartta üçer ritmik sayalım. Söylediğimiz sayıları boyayalım.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

3'ten başlayarak üçer sayarken söylediğiniz 2. sayı kaçtır? ....

3'ten başlayarak üçer sayarken söylediğiniz 5. sayı kaçtır? ....

3'ten başlayarak üçer sayarken söylediğiniz 7. sayı kaçtır? ....

3'ten başlayarak üçer sayarken söylediğiniz 8. sayı kaçtır? ....

3'ten başlayarak üçer sayarken söylediğiniz 9. sayı kaçtır? ....

3'ten başlayarak üçer sayarken söylediğiniz 10. sayı kaçtır? ....

2

Yüzlük kartta dörder ritmik sayalım. Söylediğimiz sayıları boyayalım.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

4'ten başlayarak dörder sayarken söylediğiniz 2. sayı kaçtır? ....

4'ten başlayarak dörder sayarken söylediğiniz 5. sayı kaçtır? ....

4'ten başlayarak dörder sayarken söylediğiniz 7. sayı kaçtır? ....

4'ten başlayarak dörder sayarken söylediğiniz 8. sayı kaçtır? ....

4'ten başlayarak dörder sayarken söylediğiniz 9. sayı kaçtır? ....

4'ten başlayarak dörder sayarken söylediğiniz 10. sayı kaçtır? ....

Yüzlük kartta beşer ritmik sayalım. Söylediğimiz sayıları boyayalım.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

5'ten başlayarak beşer sayarken söylediğiniz 2. sayı kaçtır? ....

5'ten başlayarak beşer sayarken söylediğiniz 5. sayı kaçtır? ....

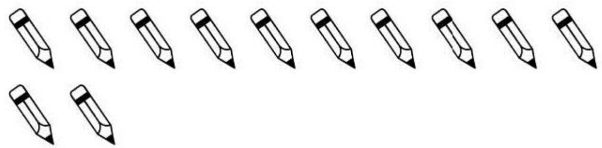
5'ten başlayarak beşer sayarken söylediğiniz 7. sayı kaçtır? ....

5'ten başlayarak beşer sayarken söylediğiniz 8. sayı kaçtır? ....

5'ten başlayarak beşer sayarken söylediğiniz 9. sayı kaçtır? ......

5'ten başlayarak beşer sayarken söylediğiniz 10. sayı kaçtır? ....

Aşağıda verilen kalemleri 2'şerli gruplara ayırınız.

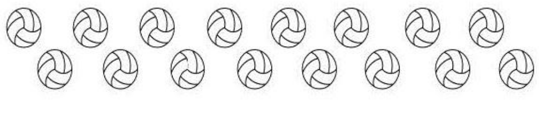


...... tane grup vardır.

Her grupta .... kalem vardır.

Toplam .... kalem vardır.

Aşağıda verilen topları 2'şerli gruplara ayırınız.

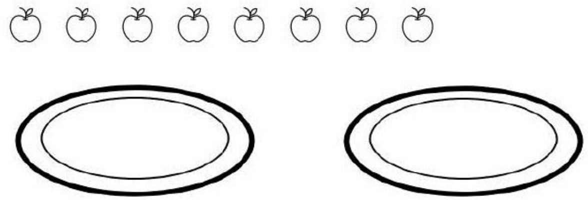


.... tane grup vardır.

Her grupta .... top vardır.

Toplam .... top vardır.

Aşağıda verilen elmaları 2 tabağa eşit olarak paylaştırınız.

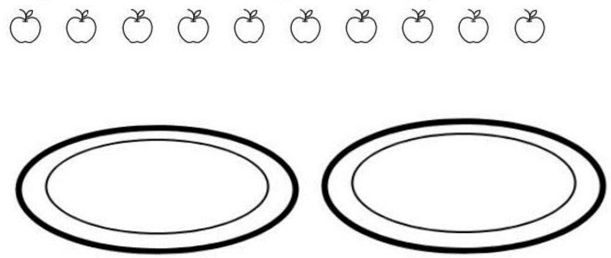


...... tane tabak vardır. Her tabakta .... tane elma vardır.

Toplam .... elma vardır.

..... tane .... , ... eder.

Aşağıda verilen elmaları 2 tabağa eşit olarak paylaştırınız.

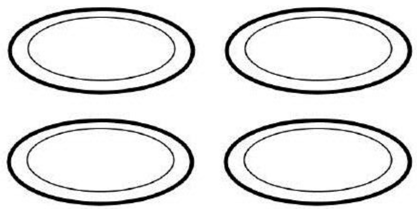


...... tane tabak vardır. Her tabakta .... tane elma vardır.

Toplam .... elma vardır.

..... tane .... , ... eder.

Her tabakta 2 ceviz olacak şekilde, 4 tabağa ceviz çizin.

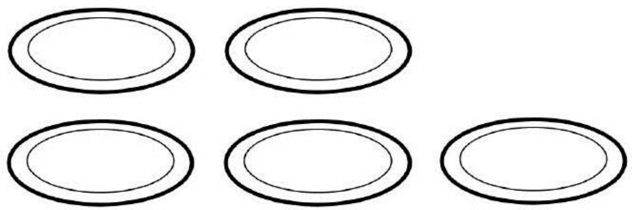


...... tane tabak vardır. Her tabakta .... tane ceviz vardır.

Toplam .... ceviz vardır.

...... tane .... , ... eder.

Her tabakta 2 ceviz olacak şekilde, 5 tabağa ceviz çizin.

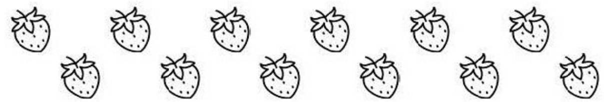


...... tane tabak vardır. Her tabakta .... tane ceviz vardır.

Toplam .... ceviz vardır.

...... tane .... , ... eder.

Aşağıda verilen çilekleri 3'erli gruplara ayırınız.

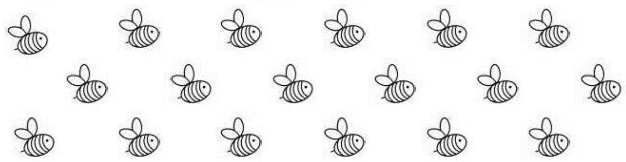


...... tane grup vardır.

Her grupta .... çilek vardır.

Toplam .... çilek vardır.

Aşağıda verilen arıları 3'erli gruplara ayırınız.

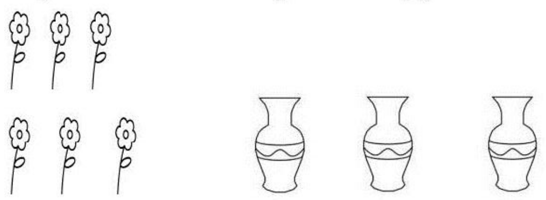


...... tane grup vardır.

Her grupta .... arı vardır.

Toplam .... arı vardır.

Aşağıda verilen çiçekleri 3 vazoya eşit olarak paylaştırınız.

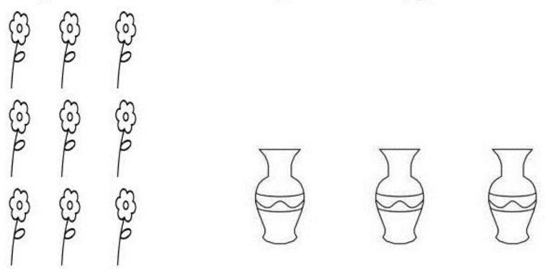


...... tane vazo vardır. Her vazoda .... tane çiçek vardır.

Toplam .... çiçek vardır.

...... tane .... , ... eder.

Aşağıda verilen çiçekleri 3 vazoya eşit olarak paylaştırınız.



...... tane vazo vardır. Her vazoda ...... tane çiçek vardır.

Toplam ...... çiçek vardır.

...... tane ...... , ... eder.

9

- Her tabakta 3 elma olacak şekilde, 6 tabağa elma çizin.

...... tane tabak vardır. Her tabakta .... tane elma vardır.

Toplam .... elma vardır.

...... tane .... , ... eder.

- Her tabakta 3 elma olacak şekilde, 7 tabağa elma çizin.

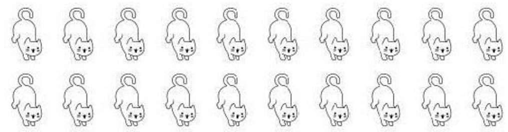
...... tane tabak vardır. Her tabakta .... tane elma vardır.

Toplam .... elma vardır.

...... tane .... , ... eder.

10

Aşağıda verilen kedileri 4'erli gruplara ayırınız.

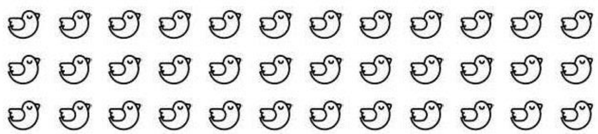


...... tane grup vardır.

Her grupta .... kedi vardır.

Toplam .... kedi vardır.

Aşağıda verilen kuşları 4'erli gruplara ayırınız.

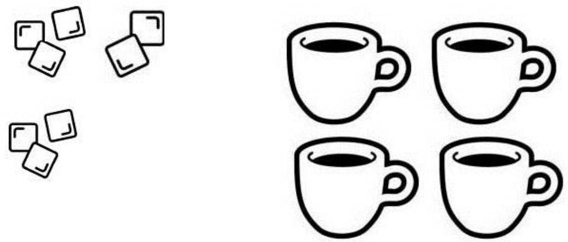


...... tane grup vardır.

Her grupta .... kuş vardır.

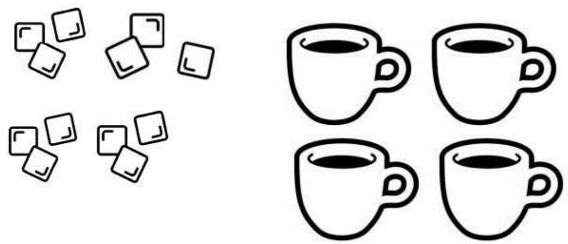
Toplam .... kuş vardır.

- Aşağıda verilen küp şekerleri 4 bardağa eşit olarak paylaştırınız.



...... tane bardak vardır. Her bardakta .... tane şeker vardır.
Toplam .... şeker vardır.
...... tane .... , ... eder.

- Aşağıda verilen küp şekerleri 4 bardağa eşit olarak paylaştırınız.



...... tane bardak vardır. Her bardakta .... tane şeker vardır.
Toplam .... şeker vardır.
...... tane .... , ... eder.

- Her tabakta 4 elma olacak şekilde, 3 tabağa elma çizin.

...... tane tabak vardır. Her tabakta ...... tane elma vardır.

Toplam .... elma vardır.

...... tane .... , ... eder.

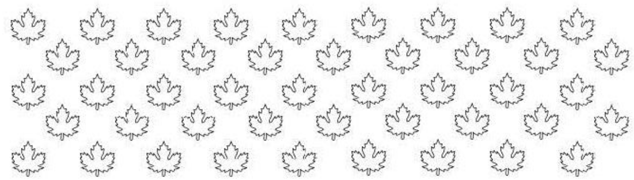
- Her tabakta 4 elma olacak şekilde, 4 tabağa elma çizin.

...... tane tabak vardır. Her tabakta ...... elma vardır.

Toplam .... elma vardır.

...... tane .... , ... eder.

Aşağıda verilen yaprakları 5'erli gruplara ayırınız.



..... tane grup vardır.

Her grupta .... yaprak vardır.

Toplam .... yaprak vardır.

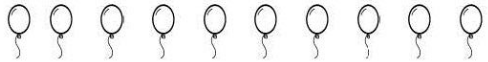
Aşağıda verilen cevizleri 5'erli gruplara ayırınız.

..... tane grup vardır.

Her grupta .... ceviz vardır.

Toplam .... ceviz vardır.

- Aşağıda verilen balonları 5 çocuğa eşit olarak paylaştırınız.

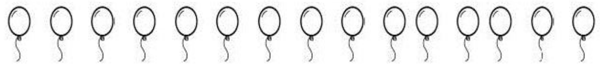


...... tane çocuk vardır. Her çocukta .... tane balon vardır.

Toplam .... balon vardır.

...... tane .... , ... eder.

- Aşağıda verilen balonları 5 çocuğa eşit olarak paylaştırınız.



...... tane çocuk vardır. Her çocukta .... tane balon vardır.

Toplam .... balon vardır.

...... tane .... , ... eder.

- Her tabakta 5 bilye olacak şekilde, 5 tabağa bilye çizin.

...... tane tabak vardır. Her tabakta .... tane bilye vardır.

Toplam .... bilye vardır.

...... tane .... , ... eder.

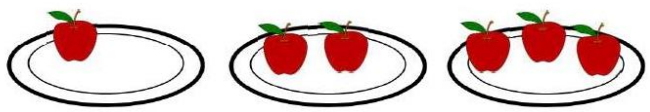
- Her tabakta 5 bilye olacak şekilde, 6 tabağa bilye çizin.

...... tane tabak vardır. Her tabakta .... tane bilye vardır.

Toplam .... bilye vardır.

...... tane .... , ... eder.

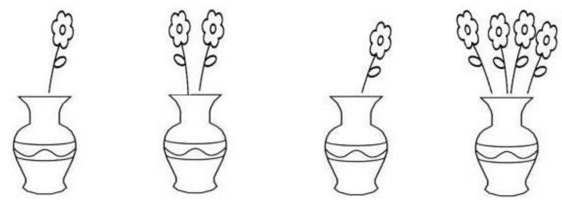
- Ozan'a 6 elma verilmiş ve 3 tabağa eşit bir şekilde paylaştırması istenmişti. Ozan elmaları aşağıdaki gibi paylaştırmıştır.
  Eğer Ozan 'ın yanlış paylaştırdığını düşünüyorsanız düzeltiniz.



...... tane tabak vardır. Her tabakta eşit sayıda .... elma vardır.

Toplam .... elma vardır.

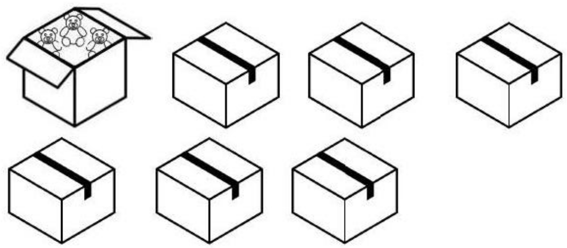
- Emine'den 8 çiçeği 4 vazoya eşit bir şekilde paylaştırması istenmişti.
  Emine çiçekleri aşağıdaki gibi paylaştırmıştır.
  Eğer Emine'nin yanlış paylaştırdığını düşünüyorsanız düzeltiniz.



...... tane vazo vardır. Her vazoda eşit sayıda .... çiçek vardır.

Toplam .... çiçek vardır.

- Aşağıda verilen kutuların her birinde 3 ayıcık olduğuna göre, tüm kutularda toplamda kaç ayıcık vardır?

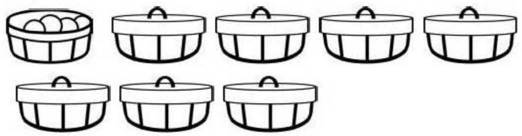


Toplam .... ayıcık vardır.

...... kutu .... , ... eder.

...... tane .... , ... eder.

- Aşağıda verilen sepetlerin her birinde 4 yumurta olduğuna göre, tüm sepetlerde toplamda kaç yumurta vardır?

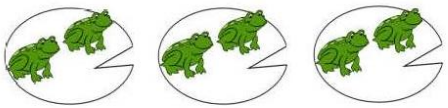


Toplam .... yumurta vardır.

...... sepet .... , ... eder.

...... tane .... , ... eder.

Aşağıdaki yaprakların üzerinde toplam kaç kurbağa olduğunu bulalım.



.... = ...

...... tane .... , ... eder.

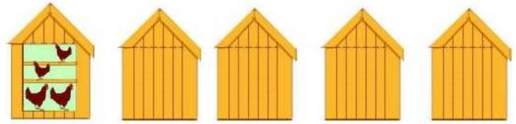
Aşağıdaki kovanların üzerinde toplam kaç arı olduğunu bulalım.

....

...... tane .... , ... eder.

Aşağıdaki kümeslerin her birinde 4 tavuk vardır.

Kümeslerde toplam kaç tavuk olduğunu bulalım.



....

...... tane .... , ... eder.

Aşağıdaki tencerelerin her birinde 5 havuç vardır.

Tencerelerde toplam kaç havuç olduğunu bulalım.

....

.... tane .... , ... eder.

Her ay 2 fidan diken Selay, 9 ayda kaç fidan diker?

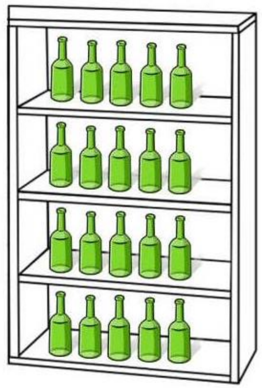
Probleme uygun bir görsel çizelim.

Problemi tekrarlı toplama işlemi kullanarak çözelim.

....

...... tane .... , ... eder.

Aşağıdaki görsele yönelik tekrarlı toplama işlemi gerektiren bir problem kuralım ve çözelim.



Problem:

....

...... tane .... , ... eder.

Aşağıdaki tekrarlı toplama işlemini yapalım ve işleme yönelik bir problem kuralım.

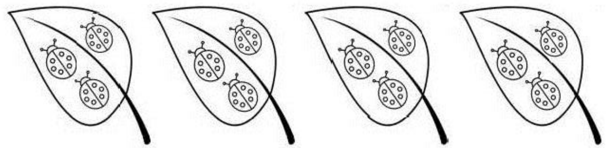
3 + 3 + 3 + 3 + 3 + 3 + 3 = ...

...... tane .... , ... eder.

Problem:

Probleme uygun bir görsel çizelim.

Aşağıda 4 yaprak vardır. Her bir yaprağın üzerinde 3 uğur böceği olduğuna göre toplam kaç uğur böceği vardır?



Uğur böceği sayısını tekrarlı toplama yaparak bulalım.

....

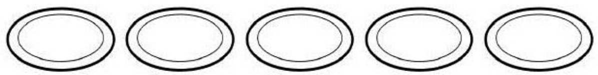
...... tane .... , ... eder.

...... kere .... , ... eder.

$\underbrace{\bigcirc}_{\text{1. Çarpan Grup Sayısı}} \times \underbrace{\bigcirc}_{\text{2. Çarpan Gruptaki eleman sayısı}} = \underbrace{\square}_{\text{Çarpım}}$

Masada 5 kardeş oturmaktadır. Anneleri her birine 2'şer kurabiye verdiğine göre toplamda kaç kurabiye vardır?

Aşağıdaki tabaklara 2'şer kurabiye çizelim.



Kurabiye sayısını tekrarlı toplama yaparak bulalım.

....

...... tane .... , ... eder.

...... kere .... , ... eder.

○ x ○ = □

1. Çarpan Grup Sayısı	2. Çarpan Gruptaki eleman sayısı	Çarpım

Selda'nın bahçesinde 4 portakal ağacı vardır. Ağaçların her birinde 5 portakal olduğuna göre toplam kaç portakal vardır?

Probleme uygun bir görsel çizelim.

Problemi tekrarlı toplama işlemi kullanarak çözelim.

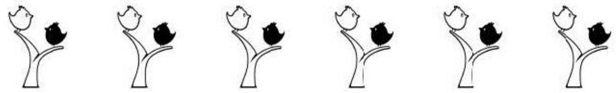
....

Problemi çarpma işlemi kullanarak çözelim.

○ x ○ = □

...... kere .... , ... eder.

Aşağıdaki görsele yönelik çarpma işlemi gerektiren bir problem kuralım.



Problem:

Problemi tekrarlı toplama işlemi kullanarak çözelim.

....

Problemi çarpma işlemi kullanarak çözelim.

$\bigcirc \times \bigcirc = \square$

...... tane .... , ... eder.

Aşağıdaki çarpma işlemini yapalım ve bu işleme yönelik bir problem kuralım.

5 x 4 = ...

...... kere .... , ... eder.

Problem:

Probleme uygun bir görsel çizelim.

Aşağıda ifade edilen nesne sayılarını çarpma işlemi ile çözelim.

9 tane 2, ... eder

...... x .... = ...

7 tane 5, ... eder

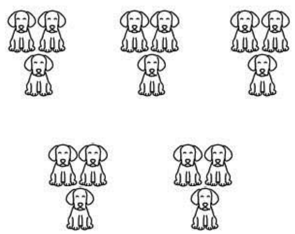
...... x .... = ...

4 + 4 + 4 + 4 + 4 =

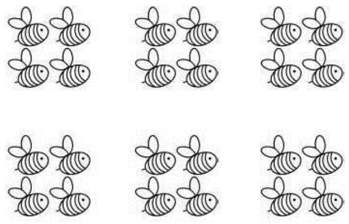
...... x .... = ...

3 + 3 + 3 + 3 + 3 + 3 + 3 =

...... x .... = ...

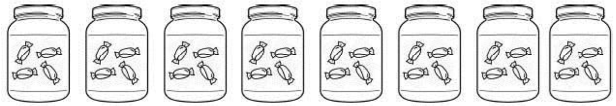


...... x .... = ...



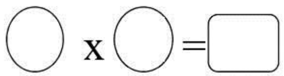
...... x .... = ...

Aşağıdaki görsele yönelik çarpma işlemi gerektiren bir problem kuralım ve çözelim.



Problem:

Hikâyeyi çarpma işlemi kullanarak ifade edelim.

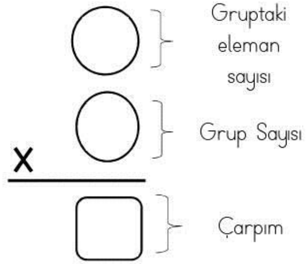


...... tane .... , ... eder.

...... kere .... , ... eder.

...... çarpı .... eşittir .... eder.

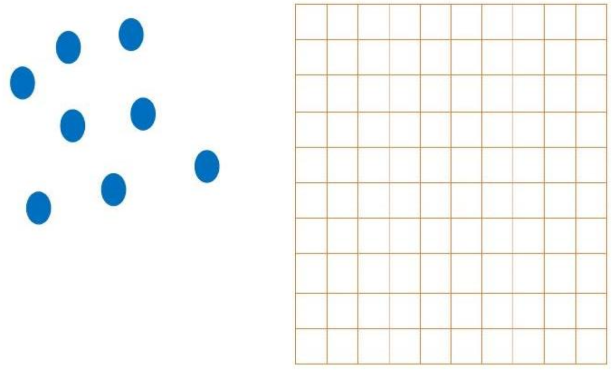
...... ile ....'ün çarpımı .... 'dir.



1.	$6 \times 5 = \square$	$\begin{array}{r} 5 \\ \times\ 6 \\ \hline \square \end{array}$	...... kere .... , ... eder.
2.	$7 \times 4 = \square$	$\begin{array}{r} \bigcirc \\ \times\ \bigcirc \\ \hline \square \end{array}$	...... kere .... , ... eder.
3.	$9 \times 3 = \square$	$\begin{array}{r} \bigcirc \\ \times\ \bigcirc \\ \hline \square \end{array}$	...... kere .... , ... eder.
4.	$8 \times 2 = \square$	$\begin{array}{r} \bigcirc \\ \times\ \bigcirc \\ \hline \square \end{array}$	...... kere .... , ... eder.

Aşağıda verilen 8 pulu bir dikdörtgen oluşturacak şekilde kareli zemine yerleştirelim.

Toplam pul sayısını çarpma işlemi ile gösterelim.



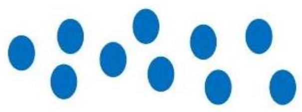
$\bigcirc \times \bigcirc = \square$

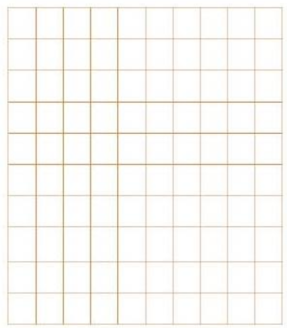
.... satır vardır.

.... sütun vardır.

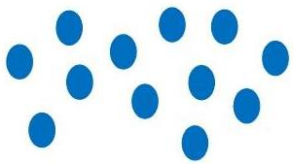
Toplam .... kalem vardır.

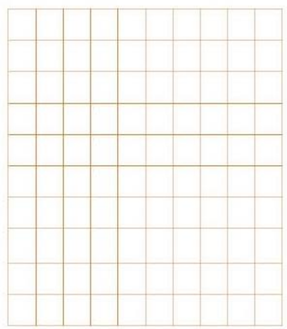
Aşağıda verilen pulları bir dikdörtgen oluşturacak şekilde kareli zemine yerleştirelim. Toplam pul sayısını çarpma işlemi ile gösterelim.





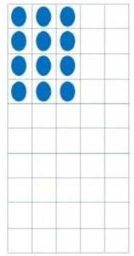
x =





x =

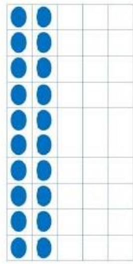
Aşağıda dizi modeli ile gösterilen beneklerin sayılarını çarpma işlemi ile bulun.



.... tane .... vardır.

.... kere ...., ... eder.

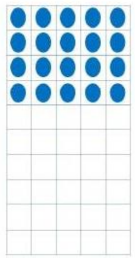
.... x .... = ...



.... tane .... vardır.

.... kere ...., ... eder.

.... x .... = ...



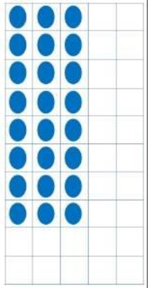
.... tane .... vardır.

.... kere ...., ... eder.

.... x .... = ...

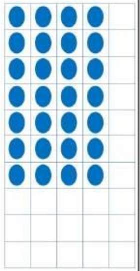
...... kere ......, ... eder

...... x ...... = ...



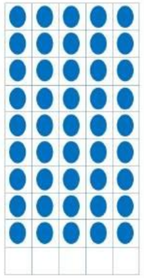
...... kere ......, ... eder

...... x ...... = ...



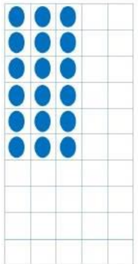
...... kere ......, ... eder

...... x ...... = ...



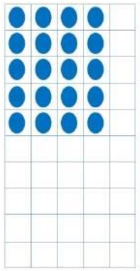
...... kere ......, ... eder

...... x ...... = ...



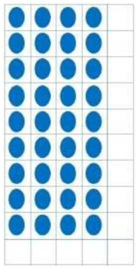
...... kere ......, ... eder

...... x ...... = ...



...... kere ......, ... eder

...... x ...... = ...



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Aşağıda ifade edilen çarpma işlemlerini yapalım ve dizi modeli ile gösterelim.

9 kere 2, ... eder.

...... x ...... = ...

6 kere 3, ... eder.

...... x ...... = ...

7 kere 3, ... eder.

...... x ...... = ...

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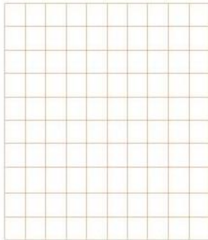
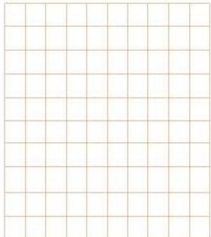
8 kere 4, ... eder.

...... x .... = ...



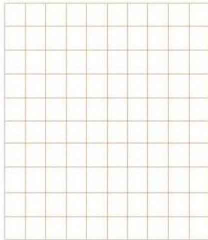
9 kere 4, ... eder.

...... x .... = ...



8 kere 5, ... eder.

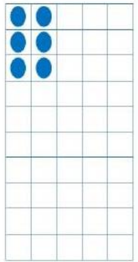
...... x .... = ...



Aşağıda verilen çarpma işlemlerine yönelik modeller eksik verilmiştir.
Bu modelleri tamamlayalım

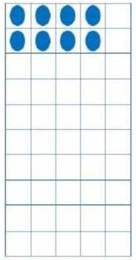
7 kere 2, ... eder.

...... x .... = ...



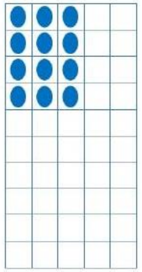
6 kere 4, ... eder.

...... x .... = ...



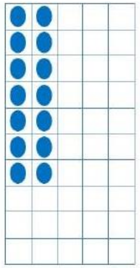
8 kere 3, ... eder.

...... x .... = ...



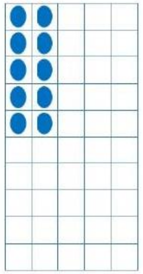
7 kere 3, ... eder.

...... x .... = ...



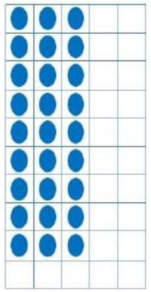
5 kere 4, ... eder.

...... x .... = ...



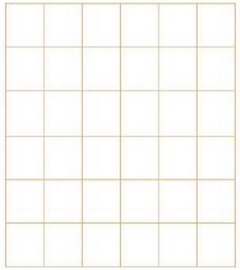
9 kere 5, ... eder.

...... x .... = ...



Aslı bahçeye çiçek dikerken toprakta 6 tane fare deliği görmüştür. Her bir delikte 3 fare olduğuna göre, toplam kaç fare vardır?

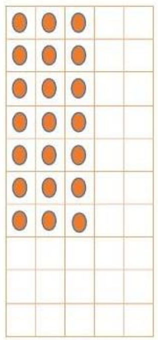
Problemi çözerken dizi modeli kullanalım.



İşlem ile gösterelim.

Bahçede toplam .... fare vardır.

Aşağıda dizi modeli ile verile çarpma işlemini yapalım ve bu işleme yönelik bir problem kuralım.



Yan yana ve alt alta işlem ile gösterelim.

Problem:

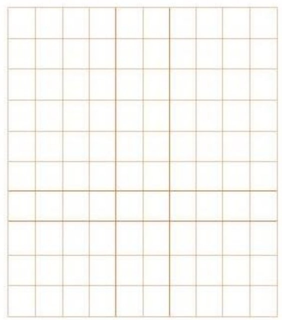
Aşağıdaki çarpma işlemini yan yana ve alt alta yapalım. Bu işleme yönelik bir problem kuralım.

$9 \times 4 =$ ....

$$\begin{array}{r} \text{.......} \\ \times \ \text{.......} \\ \hline \text{.......} \end{array}$$

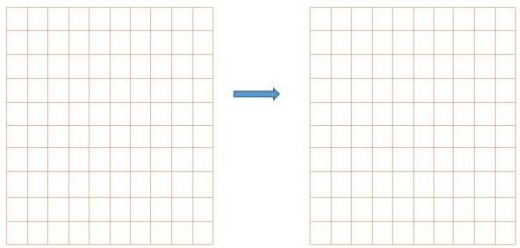
Problem:

Problemi çözerken dizi modeli kullanalım.

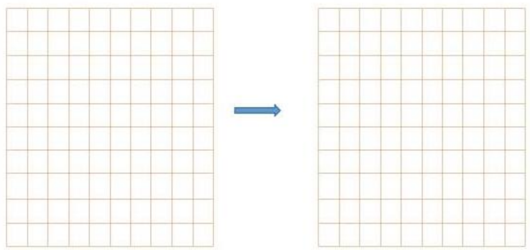


Aşağıda verilen çarpma işlemlerini modelleyerek yapın. Daha sonra çarpanların yerini değiştirerek yanda tekrar modelleyin.

2 x 4 = …… …… x …… = ……

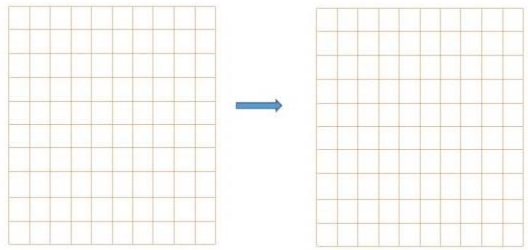


7 x 2 = …… …… x …… = ……

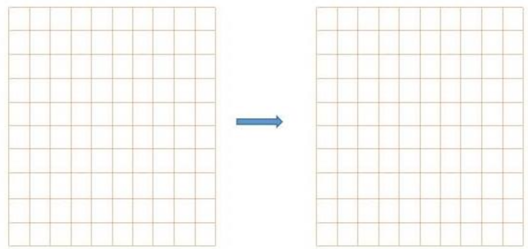


Aşağıda verilen çarpma işlemlerini modelleyerek yapın. Daha sonra çarpanların yerini değiştirerek yanda tekrar modelleyin.

2 x 9 = …… …… x …… = ……



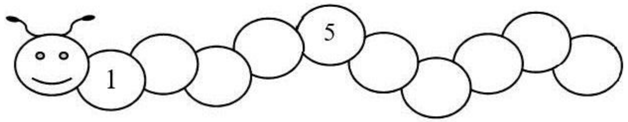
3 x 7 = …… …… x …… = ……



Aşağıda verilen tekrarlı toplama ve çarpma işlemlerini yaparak birlerin çarpım tablosunu oluşturalım.

1 =	1 x 1 = ...
1 + 1 =	2 x 1 = ...
1 + 1 + 1 =	3 x 1 = ...
1 + 1 + 1 + 1 =	4 x 1 = ...
1 + 1 + 1 + 1 + 1 =	5 x 1 = ...
1 + 1 + 1 + 1 + 1 + 1 =	6 x 1 = ...
1 + 1 + 1 + 1 + 1 + 1 + 1 =	7 x 1 = ...
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 =	8 x 1 = ...
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 =	9 x 1 = ...
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 =	10 x 1 = ...

Birer ritmik sayalım.



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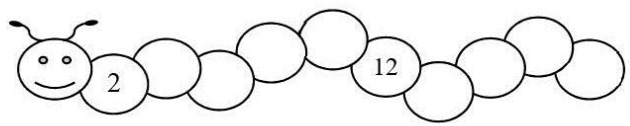
Aşağıda verilen tekrarlı toplama ve çarpma işlemlerini yaparak ikilerin çarpım tablosunu oluşturalım.

2 =	1 x 2 = ...
2 + 2 =	2 x 2 = ...
2 + 2 + 2 =	3 x 2 = ...
2 + 2 + 2 + 2 =	4 x 2 = ...
2 + 2 + 2 + 2 + 2 =	5 x 2 = ...
2 + 2 + 2 + 2 + 2 + 2 =	6 x 2 = ...
2 + 2 + 2 + 2 + 2 + 2 + 2 =	7 x 2 = ...
2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 =	8 x 2 = ...
2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 =	9 x 2 = ...
2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 =	10 x 2 = ...

İkişer ritmik sayalım.



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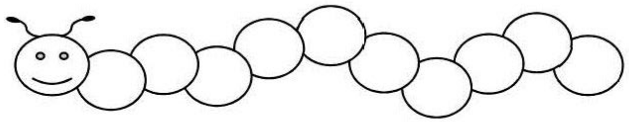
Aşağıda verilen tekrarlı toplama ve çarpma işlemlerini yaparak üçlerin çarpım tablosunu oluşturalım.

3 =	1 x 3 = ...
3 + 3 =	2 x 3 = ...
3 + 3 + 3 =	3 x 3 = ...
3 + 3 + 3 + 3 =	4 x 3 = ...
3 + 3 + 3 + 3 + 3 =	5 x 3 = ...
3 + 3 + 3 + 3 + 3 + 3 =	6 x 3 = ...
3 + 3 + 3 + 3 + 3 + 3 + 3 =	7 x 3 = ...
3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 =	8 x 3 = ...
3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 =	9 x 3 = ...
3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 =	10 x 3 = ...

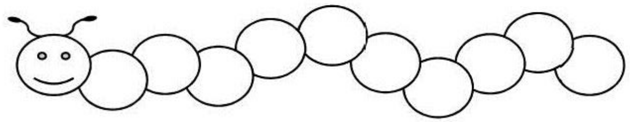
Üçer ritmik sayalım.



Aşağıda verilen tekrarlı toplama ve çarpma işlemlerini yaparak dörtlerin çarpım tablosunu oluşturalım.

4 =	1 x 4 = ...
4 + 4 =	2 x 4 = ...
4 + 4 + 4 =	3 x 4 = ...
4 + 4 + 4 + 4 =	4 x 4 = ...
4 + 4 + 4 + 4 + 4 =	5 x 4 = ...
4 + 4 + 4 + 4 + 4 + 4 =	6 x 4 = ...
4 + 4 + 4 + 4 + 4 + 4 + 4 =	7 x 4 = ...
4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 =	8 x 4 = ...
4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 =	9 x 4 = ...
4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 =	10 x 4 = ...

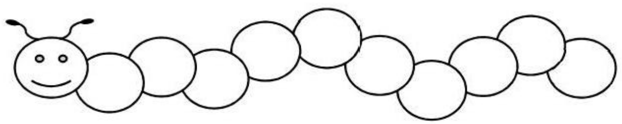
Dörder ritmik sayalım.



Aşağıda verilen tekrarlı toplama ve çarpma işlemlerini yaparak beşlerin çarpım tablosunu oluşturalım.

5 =	1 x 5 = ...
5 + 5 =	2 x 5 = ...
5 + 5 + 5 =	3 x 5 = ...
5 + 5 + 5 + 5 =	4 x 5 = ...
5 + 5 + 5 + 5 + 5 =	5 x 5 = ...
5 + 5 + 5 + 5 + 5 + 5 =	6 x 5 = ...
5 + 5 + 5 + 5 + 5 + 5 + 5 =	7 x 5 = ...
5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 =	8 x 5 = ...
5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 =	9 x 5 = ...
5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5 =	10 x 5 = ...

Beşer ritmik sayalım.



Sınıfta 3 sıra ve her bir sırada 4 öğrenci vardır. Sınıfta toplamda kaç öğrenci vardır?

.... x .... = ...

$$\begin{array}{r} \text{.......} \\ \times \ \text{.......} \\ \hline \text{.......} \end{array}$$

Her bir çiçekte 2 kelebek vardır. 8 çiçekte toplam kaç kelebek vardır?

Aşağıdaki çarpma işlemlerini modelleyerek yan yana ve alt alta yapın.

1 x 5 = ...

```
   ........
x  ........
-----------
   ........
```

8 x 1 = ...

```
   ........
x  ........
-----------
   ........
```

Bir sayının 1 ile çarpımı sayının .... eşittir.

Her gün 1 bardak şekersiz kahve içen Arif, 6 günde kaç şeker tüketmiş olur?

Problemi çözerken dizi modeli kullanalım.

İşlem ile gösterelim.

Arif toplam .... şeker tüketmiştir.

Aşağıdaki çarpma işlemlerini modelleyerek yan yana ve alt alta yapın.

0 x 4 = ...

....
× ....
....

7 x 0 = ...

....
× ....
....

Bir sayının sıfır (0) ile çarpımı her zaman .... 'dır.

Aşağıdaki çarpma işlemini yapalım. Bu işleme yönelik bir problem kuralım.

5 x 0 = ...

Problem:

Aşağıdaki çarpma işlemini yapalım. Bu işleme yönelik bir problem kuralım.

0 x 4 = ...

Problem:

PROBLEMLER

- Bir tepside 3 bardak olduğuna göre, 6 tepside toplam kaç bardak vardır?

- Parkta oynayan 10 çocuk vardır. Her bir çocuğun 2 oyuncağı olduğuna göre, toplam kaç oyuncak vardır?

- Bir sporcu her gün 4 yumurta yediğine göre, 9 günde toplam kaç yumurta yer?

- Dolapta 7 limon vardır. Her bir limonda 4 çekirdek olduğuna göre, toplamda kaç çekirdek vardır?

- Aslı her gün bir şişe su içtiğine göre, 8 günde kaç şişe su içer?

- Bir çikolata 2 lira olduğuna göre, 5 çikolata kaç liradır?

- Her bir kerevizde 0 gram yağ olduğuna göre, 6 tane kerevizde toplam kaç gram yağ vardır?

- Ali her gün 2 resim yapmaktadır. Ali 7 günde kaç resim yapar?

- Annem manavdan 8 torba portakal aldı. Her bir torbada 5 portakal olduğuna göre, toplamda kaç portakal vardır?

F. SAMPLE FLUENCY TEST

Ad Soyad: Tarih:

Akıcılık Çarpma-3

5×3	4×2	3×2	5×8	3×1
3×6	0×8	1×9	4×6	5×5
2×7	3×3	10×1	2×9	4×10
5×7	3×4	2×5	10×0	5×6

1

$$\begin{array}{r} 4 \\ \times\ 4 \\ \hline \end{array} \quad \begin{array}{r} 3 \\ \times\ 5 \\ \hline \end{array} \quad \begin{array}{r} 7 \\ \times\ 0 \\ \hline \end{array} \quad \begin{array}{r} 2 \\ \times\ 2 \\ \hline \end{array} \quad \begin{array}{r} 5 \\ \times\ 4 \\ \hline \end{array}$$

$$\begin{array}{r} 2 \\ \times\ 10 \\ \hline \end{array} \quad \begin{array}{r} 3 \\ \times\ 8 \\ \hline \end{array} \quad \begin{array}{r} 4 \\ \times\ 3 \\ \hline \end{array} \quad \begin{array}{r} 5 \\ \times\ 2 \\ \hline \end{array} \quad \begin{array}{r} 4 \\ \times\ 9 \\ \hline \end{array}$$

$$\begin{array}{r} 6 \\ \times\ 1 \\ \hline \end{array} \quad \begin{array}{r} 2 \\ \times\ 4 \\ \hline \end{array} \quad \begin{array}{r} 3 \\ \times\ 7 \\ \hline \end{array} \quad \begin{array}{r} 4 \\ \times\ 5 \\ \hline \end{array} \quad \begin{array}{r} 5 \\ \times\ 10 \\ \hline \end{array}$$

$$\begin{array}{r} 2 \\ \times\ 8 \\ \hline \end{array} \quad \begin{array}{r} 1 \\ \times\ 8 \\ \hline \end{array} \quad \begin{array}{r} 3 \\ \times\ 10 \\ \hline \end{array} \quad \begin{array}{r} 4 \\ \times\ 7 \\ \hline \end{array} \quad \begin{array}{r} 2 \\ \times\ 3 \\ \hline \end{array}$$

$$\begin{array}{r} 4 \\ \times\ 8 \\ \hline \end{array} \quad \begin{array}{r} 2 \\ \times\ 6 \\ \hline \end{array} \quad \begin{array}{r} 1 \\ \times\ 10 \\ \hline \end{array} \quad \begin{array}{r} 5 \\ \times\ 9 \\ \hline \end{array} \quad \begin{array}{r} 3 \\ \times\ 9 \\ \hline \end{array}$$

G. TURKISH SUMMARY / TÜRKÇE ÖZET

İLKOKUL 2. SINIF ÖĞRENCİLERİNİN DOĞAL SAYILARLA ÇARPMA İŞLEMİNİ KAVRAYIŞLARININ ÖĞRENME ROTASI İLE DESTEKLENMESİ

1.Giriş

Çarpma, birçok kavramın geliştirilmesinde önemli rolü olan, ilkokul matematiğindeki temel aritmetik kavramlardan biridir ((Amerikan) Ulusal Matematik Öğretmenleri Konseyi [NCTM], 2000; Millî Eğitim Bakanlığı [MEB], 2018). İlköğretim sınıflarında temel çarpma işlemini kavramsal olarak anlamlandırabilen öğrencilerin üst düzey matematik becerilerine ve çok basamaklı çarpma, bölme, kesirler, ondalık gösterim ve oran-orantı gibi çarpımsal düşünme gerektiren kavramlara çok daha iyi hazırlanmış olacakları ifade edilmektedir (Wong ve Evans, 2007; Vergnaud, 1988). Bu nedenle, çarpmanın kavramsallaştırılması, birbiri üzerine inşa edilmiş birçok matematiksel kavramın inşası için oldukça önem taşımaktadır. Birçok kavramın çarpma üzerine kurulması gibi, çarpma işlemi de çoğunlukla toplama işlemi üzerine kurulmaktadır (Anghileri, 2006). Bu strateji, öğrencilerin toplama kavramları üzerine temel çarpma bilgilerini oluşturmalarını ve işlemler arasındaki bağlantıdan yararlanarak çarpma işlemi gerektiren durumlarda tekrarlı toplamaya başvurmalarını sağlamaktadır (Nunes ve Bryant, 1996).

Önceden öğrenilen ve sağlam temellere dayanan bilgilerin kullanımı öğrencilere daha kolay geldiğinden, çarpma gerektiren durumları tekrarlı toplama ile göstermeyi tercih etmektedirler. Diğer bir ifade ile, alan/dizi modeli gibi simetrik modelleri tercih etmemektedirler (Larsson vd., 2017). Matematik derslerinde öğrencilere tanıtılan modelleri ortaya çıkarmak için ulusal ve uluslararası matematik eğitim programlarına bakıldığında, birçok ülkede asimetrik modellerin yanı sıra simetrik modellerin de kullanıldığı görülmektedir (Olfos vd., 2021). Ancak Türkiye'de öğrenciler çarpma işlemini ikinci sınıfta tekrarlı toplama ve üçüncü sınıfta ise kat kavramı ile ilişkilendirirler. Kullanılan bu asimetrik modeller açısından bakıldığında, Türkiye'de matematik müfredatı ve ders kitaplarının çarpma işleminin öğretimi açısından sınırlı olduğu görülmektedir. Çünkü asimetrik modeller çarpmanın tüm özelliklerini ortaya koymada yetersiz kalmaktadır. Örneğin, öğrencilerin çarpmayı tekrarlı toplama veya eş gruplar üzerine kurgulamaları, dağılma özelliğini açıklama adına fayda sağlasa da (Fosnot ve Dolk, 2001; Larsson vd., 2017; Mendes vd., 2021; Wright vd., 2014), değişme özelliğini açıklamada yetersiz kalmaktadır (Bell vd., 1989, Fischbein vd., 1985; Verschaffel vd., 1988).

Çarpma öğretimi ile ilgili müfredat dokümanları ve ders kitaplarının araştırılmasının yanı sıra, bu dokümanların öğrencilerin öğrenmeleri üzerindeki etkisini görmek için öğrencilerin çarpmayı kavramsallaştırmalarına yönelik çalışmaların da incelenmesi önerilmektedir. Bu doğrultuda, öğrencilerin çarpma işlemini nasıl tanımladıklarını, çarpma içeren durumları nasıl yorumladıklarını, çarpma işlemlerini nasıl yaptıklarını, çarpma stratejilerini nasıl kullandıklarını, ne tür hatalar yaptıklarını ve hangi kavram yanılgılarına sahip olduklarını anlamak için çok sayıda araştırma yapılmıştır (örn. De Corte vd., 1988; Doğan ve Doğan, 2019; Mulligan ve Mitchelmore, 1997; Sherin ve Fuson, 2005; Tertemiz, 2017; Thompson ve Saldanha, 2003). Bu çalışmalar, öğrencilerin çarpma konusundaki anlayışlarının öncelikle tekrarlı toplama ve eşit gruplamalarla sınırlı olduğunu göstermektedir. Hatta bu durumun daha sonra öğretilen konulara dair öğrencilerde kavram yanılgılarına sebep olduğu belirtilmektedir.

Sonuç olarak, öğrencilerin çarpmayı ve özelliklerini anlamlandırmasına yönelik sorunlar ile müfredat ve ders kitaplarındaki kısıtlılıklar göz önünde bulundurulduğunda, çarpma işleminin nasıl öğretileceği sorusu ortaya çıkmaktadır. Bu bağlamda, çeşitli modellerle (simetrik ve asimetrik) zenginleştirilmiş alternatif bir çerçevenin ve gerçekçi matematik eğitimi ile desteklenen matematiksel etkinliklerin faydalı olabileceğini düşünülmüştür. Bu amaçla, alternatif öğretim çerçevesi olarak tanımlanan varsayıma dayalı öğrenme rotalarının oluşturulması ve uygulanması önerilmektedir (Simon,1995). Alan yazında çeşitli matematiksel konularda çok sayıda öğrenme rotaları oluşturulmuştur (örneğin, Bowers vd., 1999; Gravemeijer vd., 2003a; Stephan ve Akyüz, 2012; Wright vd., 2014). Öğrenme rotalarına dair çok farklı yaklaşımlar vardır. Bu yaklaşımların çoğu, bireysel düzeyde matematik öğreniminin aşamalı gelişimini vurgulasa da, mevcut çalışma, kolektif matematiksel uygulamalar yaklaşımını kullanan bir öğrenci topluluğunun gelişimini tasvir etmeye odaklanmaktadır (Lobato ve Walters, 2017). Bu bakış açısına göre, bir öğrenme rotası, sınıfta kullanılan bir dizi matematiksel uygulamayı ve bunların daha önceki uygulamalar üzerine ilerlemelerine ilişkin bir hipotezi içermektedir (Cobb, 1999). Bu yaklaşıma göre, sınıf içi matematiksel uygulamalar, belirli matematiksel fikirleri incelerken oluşturulmuş tartışma, akıl yürütme ve temsil etmenin ortaklaşa akıl yürütme şekli olarak tanımlanmaktadır (Cobb vd., 2001).

Öğrenciler, sosyal etkileşimler yürüterek sınıf içi matematiksel uygulamalara katılmaktadırlar. Bu anlamda, öğrencilerin ve öğretmenlerin matematiksel iddialarda bulundukları ve ardından bu argümanları gerekçelerle destekledikleri kolektif argümantasyon geliştirilmektedir (Krummheuer, 1995; Lobato ve Walters, 2017). Argümantasyon süreci boyunca öğrenciler aynı zamanda gözden geçirme, geri çekme ve değiştirme aşamalarına yol açan çatışmalar üzerinde derinlemesine düşünürler. Bu bağlamda, kolektif bir tartışma ortamına ve öğrencilerin giderek kavramsal açıdan gelişmesine yardımcı olmak için bilişsel anlamda iyi yapılandırılmış bir etkinlik dizisi oluşturmak gerekmektedir (Clements ve Sarama, 2004). Son araştırmalara göre, öğretim dizilerinin tasarım ve uygulama aşamalarında genellikle Gerçekçi Matematik Eğitimi (GME) teorisinden faydalanıldığı görülmüştür (Gravemeijer vd., 2003a, 2003b; Stephan ve Akyüz, 2012).

Bir sınıf ortamında, öğrenciler ve öğretmen, ortaklaşa bir akıl yürütmeye varmak için, hipotezler oluşturur ve daha sonra onları savunarak veya çürüterek kolektif bir yeniden keşfetme sürecine katılırlar (Gravemeijer vd., 2000). GME perspektifinden bakıldığında toplu bir yeniden keşfetme sürecini teşvik etmek için pratik durumlardan başlayan ve hedeflenen formel matematiğe doğru ilerleyen etkinlik dizileri oluşturmak önem taşımaktadır. Bu etkinlik dizisinin öğrencilere sunacağı deneyimler ve öğrenme fırsatları aracılığıyla öğrenciler hedeflenen matematiksel fikre ulaşmaktadırlar (Gravemeijer vd., 2000). Tüm bunlar göz önüne alındığında, çarpma işlemine yönelik GME temelli bir varsayıma dayalı öğrenme rotası ve ilişkili etkinlik dizisinin geliştirilerek uygulanması ve ortaya çıkan sınıf içi matematiksel uygulamaların toplu argümantasyon yoluyla belgelenmesi hedeflenmiştir.

1.1. Araştırmanın Amaçları ve Araştırma Soruları

Bu çalışmanın birden fazla amacı bulunmaktadır. Bunlardan biri ikinci sınıflara yönelik çarpma öğretimi için bir varsayıma dayalı öğrenme rotası ve etkinlik dizisi geliştirmek, test etmek ve düzenlemektir. Daha açık bir ifadeyle, etkinlik dizisinin ve varsayıma dayalı öğrenme rotasının çarpmayı öğretme ve öğrenmede nasıl yardımcı olabileceğini gösteren bir yerel öğretim teorisi ortaya koymak hedeflenmiştir. Diğer amaç ise öğrencilerin çarpma işlemine yönelik fikrini ve matematiksel kavramların ortaklaşa gelişimini sınıf içi matematiksel uygulamalar analiziyle ortaya koymaktır. Bu amaç doğrultusunda , sınıf içi matematiksel uygulamalar ortaya koyularak, varsayıma dayalı öğrenme rotasının sınıf ortamında nasıl gerçekleştiği araştırılmıştır. Bu amaca yönelik analiz, öğrencilerin toplu matematikleştirmesini kolaylaştırmak için varsayıma dayalı öğrenme rotasının sunduğu potansiyellere ve engellere odaklanmıştır. Çarpma öğretimi için önerilen yerel öğretim teorisinin geçerliliğini desteklemek, öğrenme rotasında ve etkinlik dizisinde iyileştirmeler önermek ve sınıf içi matematiksel uygulamaları ortaya koymak için, varsayıma dayalı öğrenme rotası ve etkinlik dizisi kullanılarak yürütülen sınıf öğretim deneylerinden veri toplanması hedeflenmiştir. Neticede bu amaçlar doğrultusunda çalışmaya yön veren araştırma soruları şu şekilde sunulmuştur:

3- Çarpma işleminin öğretimi için ideal bir varsayıma dayalı öğrenme rotası ve ilgili etkinlik dizisi nasıl olmalıdır?
4- Çarpma işleminin öğretimine yönelik etkinlik dizisi ile yapılan öğretim sürecinde hangi matematiksel uygulamalar ortaya çıkmıştır?
    - Çarpma işleminin öğretimine yönelik etkinlik dizisi ile yapılan öğretim sürecinde hangi ortaklaşa akıl yürütmeler matematiksel uygulamaları desteklemiştir?

Bu amaçlar ve araştırma soruları ışığında bu çalışmayı yapmanın önemi bir sonraki bölümde açıklanmaktadır.

1.2. Çalışmanın Önemi

Bu çalışma, Gerçekçi Matematik Eğitimi (GME) teorisi doğrultusunda oluşturulan bir öğretim dizisi kullanılarak gerçek yaşam bağlamları aracılığıyla çarpmaya yönelik müfredat hedeflerini yerine getirmek ve sınırlı kaynakları desteklemek için geliştirilmiştir. Bu amaçla hem öğretim tasarımı hem de sınıf temelli araştırmayı içeren bir tasarım araştırması olarak yürütülmüştür (Cobb vd., 2001). Tasarım ve analiz süreci detaylı bir şekilde ortaya koyularak, bu tez çalışmasının matematik eğitimi alanında teorik ve pratik uygulamalar sağlaması beklenmektedir. Daha açık bir ifade ile, GME teorisi ve çarpma ile ilgili alan yazına ve teoriye katkı sağlayabilir, aynı zamanda çarpma öğretimi, toplu tartışma ve öğretim dizisi uygulamalarına öğrenci, öğretmen, öğretmen eğitimcisi, matematik eğitimi araştırmacıları ve eğitim kaynakları tasarımcıları perspektifinden katkı sağlayabilir.

Çarpma ve ilgili kavramların öğretimine yönelik birkaç mevcut öğrenme rotası olmasına rağmen (örneğin, Götze ve Baiker, 2021; Kennedy vd., 2008; Mendes vd., 2021; Wright vd., 2014), bu rotaların çok genel oldukları ve sınıf seviyelerinden bağımsız olarak ele alındıkları görülmüştür. Daha açık bir ifade ile bu öğrenme rotalarının çoğu, genel anlamda bir çarpma anlayışı geliştirmeye odaklanmaktadır. İkinci sınıf özelinde çarpma işleminin nasıl öğretileceğine dair bilgi içermemektedir. Ayrıca, bu rotalarda yer alan fikirlerin çoğunun, Türkiye'de ikinci sınıf düzeyi için çok yeni olduğu görülmüştür. Bu nedenle bu etkinlik dizilerinin Türk öğrenciler ve matematik öğretim programında yer alan kazanımlar göz önünde bulundurularak adapte edilmesi gerektiği sonucuna varılmıştır. Öğretim etkinliklerinin, öğrenciler için açık ve anlaşılır olacak şekilde Türkçe bağlamlara uyarlanması gerektiğine karar verilmiştir. Bu amaçla, mevcut çalışmanın, öğrencilere bu kavramları giderek daha karmaşık ve gelişmiş şekilde öğrenmeleri için ihtiyaç duydukları kaynak ve ders etkinlikleri sunmanın yanı sıra, öğretmenlere çarpmanın temel kavramları ve ilkeleri hakkında kapsamlı bir anlayış vererek bu boşluğu kapatma potansiyeline sahip olduğu söylenebilir.

Öğrencilerin çarpma ve günlük yaşam durumları arasında bağlantı kurmalarına ve daha derin bilgileri geliştirmelerine yardımcı olmak için bu tasarıma problem kurma etkinlikleri de dâhil edilmiştir (English, 1997b). Problem kurma, matematiksel iletişim için bilişsel olarak zorlayıcı görevleri içerdiğinden (Cai ve Hwang, 2002), ortak bir anlayış yaratacak problem kurma etkinliklerinin ve tartışmaların, öğrencilerin fikirlerini tanımlamalarına ve çarpma hakkında yeni anlayışlar kazanmalarına yardımcı olduğu düşünülmektedir (Cai vd., 2015). Bunun ışığında, bir sınıf topluluğu içinde matematiksel uygulamalar geliştirirken sebep ve açıklamaları anlamada öğrencilerin problem kurarken ortaya koydukları bilişsel süreçlerin bize yol gösterebileceği düşünülmüştür. Müfredatta problem kurmanın kullanımına kesin olarak vurgu yapılmasına rağmen (MEB, 2018; NCTM, 2000), ders kitaplarının, etkinliklerin ve materyallerin nasıl seçileceği açıkça belirtilmemektedir (Cai vd., 2015). Ayrıca Türkiye'de ikinci sınıf kazanımları çarpma işleminde problem kurmayı içermemektedir (MEB, 2018). Türk öğrencilerin üçüncü sınıfta çarpma problemi kurmaları beklenmektedir. Bu anlamda, mevcut çalışma, problem kurma etkinlikleriyle zenginleştirilmiş örnek bir etkinlik dizisi sağlamanın yanı sıra, ikinci sınıfta problem kurma etkinliklerini seçme veya geliştirmeye yönelik yol göstermektedir. Elde edilen bulgular, ikinci sınıf öğrencilerinin çarpma işlemindeki matematiksel uygulamalarını belgeleyerek, onların problem kurmadaki hazır bulunuşluklarını ve yeterliliklerini ortaya çıkarmıştır. Çok az araştırmacı, öğrencilerin problem kurma etkinlikleriyle meşgul olduğu sınıf öğretiminin dinamiklerini tanımlamaya çalışmıştır (Cai vd., 2015). Bu çalışmada sınıfta etkinliklerin nasıl uygulanacağı, sınıf tartışması ve kültürünün dinamiklerini ortaya çıkararak tartışma ortamlarının nasıl yürütülmesi gerektiği açıklanarak, bu boşluğun doldurulacağı düşünülmüştür. Ayrıca, bu araştırmanın, yapılandırılmış öğretimin sonuçları ve öğrencilerin çarpma problemleri hakkında informel düşünme yollarını sunarak problem kurmanın teorik ve pratik zorluklarını birleştirme konusunda yol gösterebileceği söylenebilir.

Problem kurma etkinliklerine ek olarak öğretmenlerden, eşit gruplama ve paylaşmayı içeren etkinliklerde kullanılan dilin çeşitliliğine ve bu dilin sayma kalıplarıyla nasıl bağlantılı olabileceğine odaklanma konusunda öğrencilere rehberlik etmeleri beklenmektedir (Anghileri, 1995). Bu nedenle, eşit gruplama etkinlikleri, çarpımsal dil geliştirmede ve çarpma ile bölmeyi kavramsallaştırmada önem taşımaktadır. Örneğin, çocukların gerçek hayattaki nesne ve durumları ele almalarına yardımcı olan belirli ifade ve yöntemler geliştirerek, daha matematiksel bir dil kullanmaları ve çarpma ile bölme arasında bağlantı kurabilmeleri önerilmektedir (Anghileri, 1995). Bu nedenle öğrencilerin sayıları oluşturma ve ayrıştırma becerileri önem kazanmaktadır. Çarpımsal akıl yürütme, sayıların eşit gruplara ayrılması ve tekrar birleştirilmesinde çarpanlar ve katları kullanmayı içermektedir (Smith ve Smith, 2006). Bu doğrultuda etkinlik dizisi, çarpma ve bölme arasında köprü oluşturabilecek şekilde gruplara ayırma etkinlikleriyle zenginleştirilmiştir. Bu etkinlikler, öğrencileri ölçme, kesirler, oranlar ve orantısal akıl yürütme gibi diğer ilgili konulara hazırlarken çarpmayı geliştirmek için örnek bir uygulama teşkil edebilir.

Benzer şekilde, öğretici deneyimler dışında, öğretmenler, öğrencilerin zaman içinde matematiksel olarak ilgili yönlerde birleşik birimleri bağlama, birimleştirme ve üzerine kurma stratejileri konusundaki informel anlayışlarını kullanmaktan ve geliştirmekten sorumludur (Freudenthal, 1983; Lamon, 1995). Ne yazık ki orantısal akıl yürütme üzerine yapılan araştırmalar, öğretmenlerin orantısal akıl yürütmeyi öğretmede zorluk yaşadıklarını ve konuyu işlemsel, yüzeysel ve diğer konulardan soyutlanmış olarak tanıttıklarını ortaya koymuştur (Sowder vd., 1998). Bu bağlamda, orantısal akıl yürütmeyi anlamada çarpmanın rolünü göz önünde bulundurarak, öğretmenlerin orantısal akıl yürütme konusundaki alan bilgilerini ve pedagojik alan bilgilerini geliştirmeleri gerekmektedir. Bu öneriye paralel olarak mevcut çalışma, GME aracılığıyla yeniden keşfetme ve matematikleştirme sürecinin ışığında, öğretmen ve öğrencilerin informel bilgilerini ve orantısal akıl yürütme gibi ilgili kavramların sınıf bağlamında giderek daha gelişmiş bir şekilde evrimini anlamalarına yardımcı olabilir. Bu çalışma ile ikinci sınıf öğrencilerinin diğer konulara yönelik hazır bulunuşlukları ve performansları matematiksel uygulamaları belgelenerek gözlemlenebilir ve ortaya çıkarılabilir. Ayrıca, mevcut çalışmanın öğretmenlerin öğrencilerin kavramsal ilerlemeleri, konular arasındaki bağlantılar ve öğrencilerin matematiksel gelişimini geliştirmek için ihtiyaç duydukları kaynaklar ve etkinlikler konusunda farkındalıklarını artırmaya yardımcı olacağına inanılmaktadır. Bu nedenle, bu çalışma, tasarlanan öğretimin çıktılarını ve diğer kavramlar için temel becerileri oluşturan öğrencilerin informel düşünme biçimlerini ortaya koyarak, çarpmayı öğrenmenin teorik ve pratik yönleri arasında köprü kurmaya yardımcı olabilir.

Diğer bir konu, çarpmanın değişme özelliğinin kavramsallaştırılmasıdır. Alan yazında çarpma işleminin değişme özelliğini öğretmek için en etkili temsil olarak bir dizi modeli önerilmektedir (Greer, 1992; Outhred ve Mitchelmore, 2004; Van de Walle vd., 2013). Alan yazında dizi modeli, çarpma ile ilgili anlamsal yapılardan biri olan birleşik birimlerin bileşimi (composite of composites) olarak ifade edilmektedir (Outhred ve Mitchelmore, 2004). Türkiye'de ikinci sınıf öğrencilerine yönelik matematik kazanımları ve ders kitapları çarpmayı yalnızca eşit gruplama ve bu grupların tekrarlı toplamı olarak göstermektedir (MEB, 2018). Oysa öğeleri satır ve sütunlara taşımak, değişme özelliğini keşfetmek için eş gruplamaya dair görsellerden daha fazla fırsat sağlamaktadır (Anghileri, 2006). Bu bağlamda, öğrencilerin çarpma işlemini özellikle değişme özelliği olmak üzere tüm yönleriyle yorumlamaları için, dizi modeli mevcut tasarıma dâhil edilmiştir. Bu model, çalışma süresince çarpma işleminin bir dizi olarak tanıtılmasında ve etkinlik dizisinin uygulanması sırasında öğrencilerin tepkilerini ve kavramsallaştırmasını ortaya koymada önemli bir rol oynamıştır. Bu açıdan, öğretmenlerin derslerini tasarlarken mevcut çalışmanın çıktılarından faydalanabileceği söylenebilir.

Son olarak, etkinlik dizisinin uygulanabilirliğini değerlendirmek ve mümkünse revize ederek daha iyi bir forma ulaştırmak için hem sınıf topluluğunun sınıf içi matematiksel uygulamalarının hem de bireylerin matematiksel akıl yürütmelerinin ortaya çıkarılması önerilmektedir (Cobb, 2003; Stephan vd., 2004). Bir grubun matematiksel uygulamalarını belgelemenin yolu, tüm sınıf tartışmalarında argümantasyonu analiz etmektir (Cobb ve Yackel, 1996). Argümantasyon süreci sırasında, sosyal ve sosyo-matematiksel normlar, sınıf tartışmasını yönetmede ve paylaşılan fikirlerin evriminde önemli bir rol oynamaktadır. Katılımcı sınıfın sosyal ve sosyo-matematiksel normları bu çalışma öncesinde katıldıkları bir proje çerçevesinde kurulmuştur. Bu normlar, ekibin tartışmacı bir ortam yaratmasını ve bir sınıf topluluğunda tartışma yoluyla sınıf içi matematiksel uygulamalarının gelişimini gözlemlemesini sağlamıştır. Bu bağlamda diğer eğitimciler bu normları inceleyebilir, doğrudan veya revize ederek sınıflarında kullanabilir ve toplu tartışmayı sürdürme ve öğrencileri sınıf içi matematiksel uygulamalara dahil etme konusundaki sonuçlarını görebilirler.

2. Alan Yazın

Çarpımsal akıl yürütme, farklı sayı kümeleriyle çalışırken, çarpma (veya bölme) içeren problemleri çözerken ve bu akıl yürütmeyi yazılı algoritmalar, diyagramlar, semboller ve yazılı dil aracılığıyla başarılı bir şekilde ifade ederken yaratıcı ve esnek bir şekilde düşünme yeteneği içermektedir (Siemon vd., 2005). Çarpımsal akıl yürütmeye dair teoriler, 1980'lerin başında çarpma ve bölme işlemi içeren problemler üzerine yapılan araştırmalar yoluyla gelişmiştir. Bu çalışmalar ve teorik yaklaşımlar ışığında çarpmaya dair çeşitli modeller ortaya atılmıştır (Bell vd., 1981; 1984; Fishbein vd., 1985; Greer, 1992; Nesher, 1988, 1992; Schwartz, 1988; Vergnaud, 1983). Tüm modelleri kapsayan ve en yaygın kullanılan sınıflandırma Greer (1992) tarafından yapılmıştır. Bunlar asimetrik olan tekrarlı toplama ve kat modelleri ile simetrik olan alan/dizi ve Kartezyen çarpım modelleridir. Çarpma problemlerinin analizi ile sınıfta çarpma işlemi öğretilirken birden fazla çarpma modeli kullanılmasının faydalı olabileceği ortaya koyulmuştur (Bell vd.., 1989; Downton ve Sullivan 2017; Fishbein vd., 1985; Graeber ve Tirosh, 1988; Lesh vd.., 2003; Lo vd.., 2008; Steffe 1994; Verschaffel vd.., 1988)

Çarpmanın doğası, sayılar ve işlemler hakkında üst düzey bilişsel düşünmeyi gerektirmektedir (Davydov, 1992; Jacob ve Willis, 2001; Schwarz, 1988; Vergnaud, 1983). Çocukların çarpma kavramına ilişkin geliştirdikleri anlayışları incelemek adına çeşitli çalışmalar yürütülmüş, (Anghileri, 1989; Jacob ve Willis, 2001; Kouba, 1989; Mulligan ve Mitchelmore, 1997; Mulligan ve Wright, 2000) ve öğrencilerin çarpma yaparken kullandıkları stratejiler belirlenmiştir (Sherin ve Fuson, 2005). Bunlar basitten daha üst düzey düşünme gerektiren stratejilere doğru tümünü sayma (count-all), toplamaya dayalı düşünme (additive calculation), ritmik sayma (count-by), örüntü tabanlı düşünme (pattern-based), öğrenilmiş durumlar (learned product) ve karma yöntemler (hybrid) şeklinde sıralanmıştır (Anghileri, 1989; Cooney vd.., 1988; Kouba, 1989; Lefevre vd.., 1996; Lemaire ve Seigler, 1995; Mulligan ve Mitchelmore, 1997; Siegler, 1988). Öğrencilerin çarpma ile ilgili hesaplama stratejileri geliştirmelerinde, çarpma öğretimi önemli bir rol oynamaktadır (Sherin ve Fuson, 2005).

Birçok öğrenci, çarpmayı kavramsallaştırmalarını engelleyen bir veya daha fazla hususla karşılaşmaktadır. Bunlar temel sayı duyusu ve ön bilgi eksikliği, toplamanın aşırı genellenmesi, çoklu temsil ve çarpımsal dil eksikliği ve öğretmenlerin konuyla ilgili bilgi eksiklikleri olarak belirlenmiştir. Sayı duyusu, sayılar ve işlemler arasındaki ilişkileri esnek bir şekilde kullanma, sayı büyüklüğünü yorumlamak için referans ölçütleri kullanma, işlemlerin değişme, birleşme ve dağılma özelliğini kavrama ve kullanma becerileriyle ilgilidir (Andrews ve Sayers, 2015; McIntosh vd., 1992; NCTM, 2000). Örneğin, sayı duyusu gelişmiş öğrenciler, sekiz kere yediyi (8x7=56) bulmak için dört kere yedinin (4x7=28) iki katını alabilirler ve ayrıca "8x70 ifadesinin 56 onluk yani 560" ettiğini bulabilirler (National Mathematics Advisory Panel, 2008). Bu tür ilişkiler kurarak, sayılar ve işlemler arası bağıntılarla çarpmaya dair stratejiler geliştirebilmek için gelişmiş bir sayı duyusu gerekmektedir. Aksi takdirde öğrencilerin ilişkisel anlamada sorun yaşayabileceği ifade edilmektedir

Eğitim programlarının üzerine kurgulanmış olduğu yapılandırmacı yaklaşıma göre öğrenciler, önceki bilgileri ve kendi deneyimlerini temel alarak yeni bilgileri yapılandırmaktadırlar (von Glasersfeld, 1996). Var olan bilgiler üzerine yeni kavramları inşa ederken, öğrencilerin önceki sınıflardaki kavram yanılgıları veya önceki bilgileri anlamamış olmaları onların matematiksel gelişimini kısıtlamaktadır (Carpenter vd., 1989). Buna paralel olarak, yapılandırmacı bir bakış açısıyla, çarpma işlemi başlangıçta tekrarlı toplama üzerine kurulmaktadır (Fischbein vd., 1985). Bu sebeple, öğrencilerin toplama bilgisi, çarpmanın gelişimi için önem taşımaktadır. Toplamayla ilgili herhangi bir yanlış anlama, öğrencilerin çarpma işlemini anlamalarını ve kullanmalarını otomatik olarak etkilemektedir.

Çarpma işlemini toplama işlemi üzerine kurarken dikkat edilmesi gereken bir diğer husus ise öğrencilerin toplamayı aşırı genelleme eğilimleridir (Lesh vd., 2003; Lo vd., 2008). Örneğin, öğrenciler toplamaya aşina olduklarından ve toplama üzerine çarpmayı yapılandırdıklarından, toplamanın tüm özelliklerini çarpma işlemine genelleyebilirler (örn. 18 x 26 = (10 x 20) + (8 x 6)). Oysaki çarpma işlemi toplama işleminden farklıdır ve daha karmaşık bir yapı içermektedir (Downton ve Sullivan 2017; Steffe 1994). Toplama işlemi aynı birime sahip çokluklar içermektedir. Diğer taraftan çarpma işlemi farklı birimlere sahip iki niceliğin manipüle edildiği bir işlemdir (Barmby vd., 2009; Smith ve Smith 2006). Çarpılan nicelikler birbirinden farklı olmakla birlikte birbirine bağımlıdır. Bu anlayış, toplama ve çarpma arasındaki önemli ayrımı ortaya koymaktadır (Schoenfeld vd., 2017). Toplamaya kıyasla çarpma, grupları ve elemanları daha soyut bir düzeyde koordine etme yeteneğini içermektedir (Clark ve Kamii, 1996; Downton ve Sullivan 2017; Steffe 1994). Bu nedenle, toplama üzerine kurulan bir öğretimde, toplamanın aşırı genellenmesinden kaçınılması gerektiği vurgulanmaktadır. Ne yazık ki, birçok eğitimci, tekrarlı toplama kullanarak tam sayılarla çarpma problemlerini çözmek mümkün olduğundan, çarpmanın sadece toplamanın bir uzantısı olduğuna inanmaktadırlar. Bu durum öğrencilerin çarpmayı ve özelliklerini kavramsallaştırmasını sınırlandırmaktadır. Matematik eğitiminde temsiller ve materyaller büyük rol oynamaktadır. Çoklu temsiller, öğrencileri işlemsel bir bakış açısından yapısal bir bakış açısına taşımaktadır (Sfard, 1991). Farklı temsiller, bir kavramın çeşitli özelliklerini vurgulayarak gelişimini desteklemektedir (Goldin ve Shteingold, 2001). Jerome Bruner'in öğrenme teorisine göre, matematik öğrenimini geliştirmek için, öğrencilerin bilişsel/bilgi gelişim aşamaları dikkate alınarak öğretim materyalleri sunulmalıdır. Farklı temsiller (somut ve resimsel, gerçek dünya ve sembolik) birbirine bağlanarak, öğretim etkinlikleri aşamalı olarak somuttan soyuta doğru geçişe hizmet etmelidir (Bruner, 1964). Böylece öğrenciler konuları kavrayışlarını temsiller yoluyla aktarırken, öğretmenler de öğrencilerin öğrenmelerine dair bilgi sahibi olabilirler. Bu sebeple, temsillere dair sınırlılıklar, öğrencilerin öğrenmelerine dair de sınırlılıklara sebep olmaktadır (Anghileri, 1989; Bruner, 1964; Clark ve Kamii, 1996; Steffe, 1994).

Farklı bir temsil yöntemi olarak sözel dil kullanımı çarpma öğretimindeki bir diğer kritik konu olarak karşımıza çıkmaktadır (Anghileri 1997; 2000). Çarpmaya dair bağlamları anlamak ve bunları toplama, çıkarma veya bölme işlemi gerektirenlerden ayırt etmek için, öğrencilerin çarpma durumlarını tanımlayan sözcük problemlerini yorumlama konusunda yeterli deneyime sahip olmaları gerekmektedir. Öğrenciler günlük yaşamda çarpma durumlarını deneyimleseler de bu durumlarda çarpma işlemini kolay ve açık bir şekilde fark edememektedirler (Calabrese vd., 2020). Çarpma işlemlerini gerçekçi durumlarla ilişkilendirmenin zorluğu, çarpma işlemlerinde kullanılan sözcükler ile günlük dil arasındaki tutarsızlıktan kaynaklanmaktadır. Örneğin, çarpma problemlerinde ve günlük konuşmalarda, insanlar çarpımsal anlamı temsil eden "her bir, kez, defa, kere, grup" gibi kelimeler kullanmaktadırlar. Öte yandan “çarpma” terimi tipik günlük senaryolarda değil çarpma işlemlerinde kullanılmaktadır (Anghileri 2000). Bu nedenle çarpmanın kavramsal ve işlemsel bilgisine ek olarak, çarpma ile ilgili gerçek hayatta kullanılan sözcüklerin de matematik sınıflarında öğretilmesi gerekmektedir (Calabrese vd., 2020).

Çarpma öğretiminde bir diğer kritik unsur olarak öğretmen bilgisi vurgulanmaktadır Öğretmenlerin sınırlı bilgileri doğrudan öğrencilerin gelişimini de sınırlandırmaktadır (Shulman, 1986). Öğretmenin bilgisi öğrencilerin ne öğrendiğini doğrudan etkilediğinden, öğretmenin uzmanlığının matematik öğretiminin etkililiği ve öğrencilerin başarısı üzerinde önemli bir etkisi olduğu ortaya koyulmuştur (Shulman, 1986). Bu nedenle öğretmenlerin matematik eğitim programlarını beklendiği gibi uygulayabilmeleri için kavramları derinlemesine anlamaları gerekmektedir. En mükemmel müfredat bile yalnızca bir dizi kazanım sunmaktadır. Bu kazanımların sınıfta ne zaman, nasıl ve neden hayata geçirileceğine karar verecek olan öğretmendir (Griffin, 2004). Bu açıdan öğretmenin çarpma ile ilgili bilgisinin öğrencilerin çarpma işlemini anlamlı bir şekilde öğrenmelerinde büyük etkisi bulunmaktadır. Bu anlamda, mevcut çalışmanın pragmatik bir amacı olarak, öğretmen bu gelişimsel araştırmanın her aşamasına dahil edilmiş ve bu da öğrencilerin kendi çarpma bilgilerini, modellerini ve özelliklerini geliştirmelerine yardımcı olmuştur.

Özetle, öğrencilerin çarpmayı kavramsallaştırmalarına ilişkin alan yazın gözden geçirilmiştir. Öğrencilerin kullandıkları çarpma stratejileri, olası kavram yanılgıları ve öğrencilerin çarpmayı anlamalarına engel olabilecek konular tartışılmıştır. Tüm bu hususlar göz önünde bulundurularak hem öğretme hem de öğrenme ile ilgili tüm sınırlılıkları karşılayan ve tüm sınıfın katılımıyla sınıf içi matematiksel uygulamaların geliştirilmesini teşvik eden bir varsayıma dayalı öğrenme rotası ve etkinlik dizisi üzerinde çalışılmasına karar verilmiştir.

3. Yöntem

Bu çalışmada birinci amaç, ikinci sınıfta çarpma öğretimine yönelik bir varsayıma dayalı öğrenme rotası ve buna paralel etkinlik dizisinin geliştirilmesi, test edilmesi ve düzenlenmesidir. İkinci amaç ise öğrencilerin çarpmaya yönelik ortaklaşa akıl yürütmelerini ve sınıf içi matematiksel uygulamalarını ortaya koymaktır. Bu amaçlar doğrultusunda tasarım tabanlı araştırma yöntemi kullanılmıştır (Cobb vd.., 2003; Gravemeijer ve Cobb, 2006; McKenney ve Reeves, 2012; van den Akker vd., 2006). Bu yöntemde araştırmacılar çalışmayı katılımcılarla beraber yürüterek sürece dâhil olurlar (Cobb vd., 2003). Araştırma ve süreçteki müdahaleler tasarlanır ve uygulamaya koyulur. Uygulama süresince gerek görülen değişikliklerle tasarımda iyileştirmeler yapılır. Uygulamanın sonunda geriye dönük analizlerle tasarım gözden geçirilir ve geliştirilerek tekrar uygulanır. Bu tekrar uygulayıp düzenleme döngüsü uygulamanın yeterince geliştiği fikrine ulaşana kadar devam ettirilir (Cobb vd., 2003). Bu değişikliklere ve iyileştirmelere dair döngüsel süreç göz önünde bulundurulduğunda, tasarım tabanlı araştırma yönteminin bu çalışma ile örtüştüğüne karar verilmiştir. Tasarım tabanlı araştırma modeli, uygulama için hazırlık, sınıf içi uygulama ve geriye dönük analizler olmak üzere üç aşamadan oluşmaktadır (Gravemeijer ve Cobb, 2006). Mevcut çalışma söz konusu üç aşama doğrultusunda yürütülmüştür.

3.1. Çalışmanın Bağlamı ve Katılımcılar

Bu çalışma Ankara ilinin Çankaya ilçesinde Milli Eğitim Bakanlığına bağlı bir devlet okulunda çalışmakta olan 23 yıllık bir sınıf öğretmeni ve onun 2. sınıf öğrencileriyle yürütülmüştür. Öğretmen, araştırmacı ve danışmandan oluşan tasarım ekibine dâhil edilmiştir. Bu üç kişilik ekip, 2. sınıf öğrencileri için çarpma işleminin öğrenimine yönelik bir varsayıma dayalı öğrenme rotası ve bununla ilişkili bir etkinlik dizisi geliştirmişlerdir. Bu tasarımlarını 2018-2019 öğretim yılında katılımcı öğretmenin sınıfında 5 hafta boyunca gerçekleştirilen öğretim kapsamında uygulamışlardır. Uygulama boyunca öğrencilerden fikirlerini paylaşmaları ve tartışmaları beklenmiştir.

Tasarımın hazırlık ve sınıf içi uygulama aşamaları boyunca tasarım ekibinin bir üyesi olan katılımcı öğretmen aktif rol almıştır. Öğretmen, öğrencilere dair bilgisi, müfredat bilgisi ve öğretime dair deneyimleriyle hazırlık aşamasında varsayıma dayalı öğrenme rotası ve etkinlik dizisinin geliştirilmesi amacıyla gerçekleştirilen toplantılara katkı sağlamış ve yol gösterici olmuştur. Sınıf içi uygulama aşamasında, öğretmen ile araştırmacı dersten önce dersin amacı ve muhtemel öğrenci düşünüşleri ile ilgili görüşmeler yapmışlardır. Ders esnasında, işbirliği içinde sosyal ve sosyo-matematiksel normlar ışığında argümantasyon süreçlerini yönetmişlerdir. Ders sonrasında ise, tasarım ekibi dersin güçlü ve zayıf yönlerine dair görüşlerini paylaşarak bir sonraki uygulamaya yönelik düzenlemelere karar verilmiştir.

3.2. Veri Toplama Süreci

Veri toplama süreci, tasarım deneyi aşamaları ışığında ilerlemiştir. İlk aşama olan uygulama için hazırlık kapsamında öğretmen, araştırmacı ve profesörden (danışman) oluşan tasarım ekibi ortaklaşa çalışarak çarpma işlemine dair öğretim programında yer alan kazanımları, ders kitaplarını, alan yazında var olan öğrenme rotalarını ve çarpma işlemi üzerine yapılmış çalışmaları incelemişlerdir. Bu bilgiler ışığında dört bölümden oluşan bir varsayıma dayalı öğrenme rotası oluşturmuşlardır.

Varsayıma dayalı öğrenme rotasının ilk bölümü, öğrencilerin ritmik sayma ve sayı dizileri oluşturmalarını içermektedir. İleri doğru ritmik sayma, bir sayı dizisindeki sayıların sırasını bulma, bir sayı dizisinde sırası bilinen sayıyı bulma, parmakların etkili bir şekilde kullanılması ve bir sayı dizisinde, iki sayı arasında ilişki kurma ile ilgili öğretim etkinlikleri geliştirilmiştir. Ritmik saymaları için 2, 3, 4 ve 5'in her biri için yüzlük tablo kullanılmıştır. Örneğin, öğrenciler sayıları 2'şerli sayarken söyledikleri sayıları yüzlük tabloda boyamışlardır. Ardından, sayıların sırasına ilişkin iki tipte sorular sorulmuştur. Bu soru tipleri, “2'den başlayarak 2'şer sayarken söylediğiniz 7. sayı kaçtır?” ve “2'den başlayarak 2'şer sayarken 12'yi kaçıncı sırada söylersiniz?” şeklindedir. Bu bölüm ile çarpma işlemi için ritmik sayma ön bilgisinin kazandırılması planlanmıştır.

Öğrenme rotasının ikinci bölümü, çoklukları bir birim olarak görmeye ve bu birimler üzerinden toplamsal düşünmeye dair hedefler içermektedir. Nesneleri tekrarlı saymak için eş gruplama etkinlikleri geliştirilmiştir. Bu amaçla, öğrencilere somut nesneler ya da etkinlik kâğıdı üzerinde görseller sunulmuştur. Nesnelerin farklı düzenlemelerini bulmaya yönlendirmek için öğrencilere farklı şekilde sayıp sayamayacakları sorulmuştur. Bu aşamada öğrencilerin, bir işlemi sembolik olarak yapmak yerine toplam nesne sayısını bulmak için ritmik saymaları beklenmiştir. Bu doğrultuda öğrencilerin matematik çizimlerini kullanarak eşit gruplar oluşturmaları ve toplam nesne sayısını bulmaları teşvik edilmiştir. Bu nedenle öğrencilerin çarpmanın bir anlamı olarak ritmik sayma yoluyla nesneleri eşit gruplar halinde sayma anlayışı kazanmaları beklenmiştir. Öğrencilerden her görevin sonunda oluşan durumu ifade etmek için “…kere… , … eder” ifadesini kullanmaları beklenmiştir. İlerleyen günlerde, öğrencilerden toplam nesne sayısını bulmak için sayı cümleleri kullanmaları istenmiştir. Eşit gruplardaki nesnelerin toplamını bulma işlemi olarak tekrarlı toplama yapmaları beklenmiştir. Ayrıca bu aşamada problem çözme ve problem kurmaya dair etkinlikleri kullanılmıştır. Problem kurma etkinliklerinde, günlük yaşam durumları ile sembolik ve görsel temsilleri ilişkilendirmeyi geliştirmek adına verilen görsele ya da işleme dair problem kurma etkinliklerine yer verilmiştir.

Öğrenme rotasının üçüncü aşamasında, öğrencilere sembolik olarak çarpma işlemi tanıtılıp ve eşit gruplar ve dizilerle modelleme yaparak çarpma işleminin bileşenleri ve özellikleri üzerinde durulmuştur. Çarpma işleminde birinci ve ikinci çarpanın rolleri eş gruplar gösterimi ile ilişkilendirilmiştir. Ayrıca verilen problemin çarpma işlemine ek olarak tekrarlı toplama ile çözülmesi istenmiştir. Böylece öğrencilerin GME yaklaşımına göre çarpma ile ilgili üst düzey matematiksel akıl yürütme için görselleri ve tekrarlı toplamayı model olarak kullanmaları beklenmiştir. Bu aşamada dizi modeli de kullanılmıştır. Etkinliklerin başında öğrencilere birbirlerini anlayabilecekleri ve kullanacakları bir dilin sağlanması için sütun ve satır terimleri anlatılmıştır. Dizi ve çarpma arasındaki ilişki tartışılmıştır. Bu aşamada çarpma işlemini dizi ile modelleme ya da dizi modelini çarpma ile ifade etmeye dair etkinlikler kullanılmıştır. Öğrencilere diziler ile çarpmayı modellemede kavramsal anlayış ve akıcılık kazandırmak hedeflenmiştir. Bu etkinliklerde öğrencilerden çarpanları diziler üzerinde nasıl gösterdiklerini açıklamaları istenmiştir. Satırlardaki noktaları yatay olarak gruplayanlar ile dikey olarak sütunlardaki noktaları gruplayanlar arasında bir tartışma ortamı sağlanmıştır. Daha sonra öğrencilerden hem satırları hem de sütunları istedikleri gibi çarpan olarak adlandırabilecekleri sonucuna varmaları beklenmiştir. Böylece, çarpmada yer değiştirme özelliğinin yorumlanması hedeflenmiştir.

Öğrenme rotasının dördüncü aşaması ise, çarpmayı gerçek yaşam bağlamlarıyla ifade etmekle ilgilidir. Sembolik ve görsel temsillere göre problem kurma etkinlikleri kullanılmıştır. Öğrencilerin çarpma işlemini gerçek yaşam bağlamlarında tekrarlı toplama ile temsil etmeleri ve çarpımsal dili uygun şekilde kullanmaları beklenmiştir. Çarpımsal dilin yapılandırılması, öğrenme rotasının başlangıcından sonuna kadar geliştirilen kritik bir fikir olmuştur. Uygulamanın sonunda öğrencilerden formel çarpma işlemini gerçek hayattaki informel çarpma dili ile ilişkilendirmeleri beklenmiştir. Yani bu fikrin birkaç etkinlikle değil, tüm öğretim dizisiyle geliştirilmesi hedeflenmiştir.

Özetle, bu varsayıma dayalı öğrenme rotası sınıf içinde çarpmaya dair akıl yürütmelerin ortaklaşa geliştirilmesine yönelik bir çerçeve sunmuştur. Öğrenme rotası doğrultusunda geliştirilen etkinlik dizisinin sınıfta nasıl uygulandığı, çarpmaya dair akıl yürütmeyi nasıl geliştirdiği ve ne tür değişikliklere ve düzenlemelere ihtiyacı olduğu sınıf içi matematiksel uygulamalar analizi ile ortaya koyulmuştur. Bu analize dair bilgiler bir sonraki bölümde açıklanmıştır.

3.3. Veri Analizi

Bu çalışmada varsayıma dayalı öğrenme rotası çerçevesinde geliştirilen etkinlik dizisinin sınıfta uygulanması beş hafta ve 26 saat sürmüştür. Bu sürece dair veriler ders öncesinde ve sonrasında öğretmenle araştırmacının görüşmelerine dair ses kayıtları, derslerin video kayıtları, öğrenci ürünleri ve araştırmacının alan notlarından oluşmuştur. Görüşme kayıtları doğrultusunda öğrenme rotasına ve etkinlik dizisine dair değişiklikler günlük düzenlemeler kapsamında yapılmıştır. Derslerin video kayıtları, öğrenci ürünleri ve araştırmacının alan notları geriye dönük analizler için kullanılmıştır. Bu kapsamda sınıf videoları incelenerek öğrencilerin çarpmaya dair akıl yürütmelerinin ortaklaşa gelişimi analiz edilmiştir. Bu amaçla sınıf videoları deşifre edilerek üç aşamadan oluşan Sınıf içi Matematiksel Uygulamalar Analizine (Rasmussen ve Stephan, 2008) hazır hale getirilmiştir. Birinci aşamada, Toulmin'in (1958) argümantasyon modeli kullanılarak sınıf tartışmaları veri, iddia, gerekçe ve destek olarak kodlanmıştır. Böylece tüm günlere dair argümantasyon şemaları elde edilmiştir. İkinci aşamada, bu şemalar incelenmiş ve gün geçtikçe sınıfça kabul gören ve ortaklaşa akıl yürütmeye (taken-as-shared) dönüşen fikirler belirlenmiştir. Bunun belirlenmesi için 2 ölçüt kullanılmıştır (Rasmussen ve Stephan, 2008). Bunlar, bir argüman için yapılan açıklamalarda artık destek ve/veya gerekçelerin kullanılmaması ve bir argümanda herhangi bir bileşenin (veri, iddia, gerekçe, destek) daha sonraki bir argümanda başka bir değişkenin yerine geçmesi şeklindedir. Bu şekilde tüm ortaklaşa kabul gören fikirler ortaya koyulmuştur. Ardından üçüncü aşama olarak, bu fikirler bir araya getirilerek belirli matematiksel uygulamalar etrafında sınıflandırılmıştır. Bu yöntem doğrultusunda, sınıfça ortaklaşa akıl yürütmelerin gelişiminin düzenli bir şekilde sunulmuş hali olarak sınıf içi matematiksel uygulamalar elde edilmiştir.

4. Bulgular, Tartışma ve Öneriler

Bu çalışmada birinci amaç, ikinci sınıfta çarpma öğretimine yönelik bir varsayıma dayalı öğrenme rotası ve buna paralel etkinlik dizisinin geliştirilmesi, test edilmesi ve düzenlenmesidir. İkinci amaç ise öğrencilerin çarpmaya yönelik ortaklaşa akıl yürütmelerini ve sınıf içi matematiksel uygulamalarını ortaya koymaktır. Bu amaçlar doğrultusunda yapılan bu çalışmanın bulguları bu bölümde bir bütün olarak sunulmuştur. Yöntem bölümünde bahsedildiği gibi oluşturulan varsayıma dayalı öğrenme rotası beş hafta (25 ders saati) boyunca katılımcı öğretmen ve onun 2.sınıf öğrencileriyle uygulanmıştır. Bu uygulamanın analizi sonucunda, beş sınıf içi matematiksel uygulama ortaya çıkmıştır (bkz. Tablo 4.1)

Tablo 4.1. Sınıf içi matematiksel uygulamalar ve bu uygulamaları destekleyen ortaklaşa akıl yürütmeler

Uygulama 1: Ritmik sayma için parmaklarla akıl yürütme
Fikir 1: Parmakların sıra sayılarını kullanarak ritmik sayma
Fikir 2: Bir sayı dizisindeki sayıların sırasını, sırası bilinen sayılarla akıl yürüterek bulma
Uygulama 2: Nesneleri daha sonra tekrarlı toplayabilmek için eşit gruplara ayırma
Fikir 3: Toplam nesne sayısını bulmada ritmik saymayı kullanmak için matematik çizimlerini kullanarak çoklukları yeniden düzenleme
Fikir 4: Üzerine ekleme stratejileri ile çoklukları eşit büyüklükteki gruplara bölme
Fikir 5: Yeniden eşit büyüklükteki grupları oluşturmak için yarıya bölme
Fikir 6: Artan sayıları (kalanı) eşitliği koruyacak şekilde gruplara dağıtma
Uygulama 3: Resimleri ve parmakları kullanarak bağlı birimleri yineleme
Fikir 7: Yineleme için grupları (birleşik birimler) ritmik sayma
Fikir 8: Yineleme için elleri (5 veya 10 parmak) birleşik birimler olarak kullanma
Fikir 9: Yineleme için resim temsillerini kullanarak eş gruplanan çoklukları çift eşleştirme
Uygulama 4: Eş gruplar ve diziler ile modelleyerek çarpma işleminin bileşenlerini ve özelliklerini analiz etme
Fikir 10: Eş grup gösteriminde birinci ve ikinci çarpanın anlamını yorumlama
Fikir 11: Birinci ve ikinci çarpanı yorumlayarak tekrarlı toplama ve çarpma işlemlerini ilişkilendirme
Fikir 12: Birinci çarpandaki değişimin çarpım üzerindeki etkisini muhakeme etme
Fikir 13: Satırları ve sütunları birinci ve ikinci çarpan şeklinde yorumlayarak dizileri analiz etme
Fikir 14: Dizileri kullanarak değişme özelliği hakkında akıl yürütme

Uygulama 5: Çarpımsal temsiller arasındaki ilişkiyi düzenleyerek bağlamsal olarak gerçekçi problemler yazma
Fikir 15: Bilinen bir problem bağlamına yerleştirmek için birinci ve ikinci çarpanı analiz etme
Fikir 16: Gerçek yaşam bağlamında bir tekrarlı toplama problemine yerleştirmek için birinci ve ikinci çarpanı yorumlama
Fikir 17: Gerçek yaşam bağlamında bir oran problemine yerleştirmek için birinci ve ikinci çarpanı yorumlama
Fikir 18: Çarpmayı kavramsallaştırmak için problemlerde yapı ve anahtar kelimelere odaklanmak

Tabloda yer alan matematiksel uygulamalar ve ortaklaşa geliştirilen matematiksel fikirler, geliştirilen öğrenme rotası ve etkinlik dizisinin öğrencilerin çarpmaya dair bilgilerini ve akıl yürütmelerini geliştirmedeki etkisini göstermektedir. İlk matematiksel uygulama ritmik sayarken parmakları kullanmayı içermektedir. Çarpma durumlarının anlaşılmasında ritmik saymanın ne kadar önemli olduğu açıkça belirtilmektedir (Schoenfeld vd., 2017). Bu nedenle öğretim dizisine 2şer (20 içinde), 3er (30 içinde), 4er (40 içinde) ve 5er (50 içinde) ritmik sayma çalışmaları ile başlanmıştır. Bu etkinliklerle öğrencilerin ritmik sayma becerilerini ve akıcılıklarını artırılması ve sonraki etkinliklere ve akıl yürütmeye hazır hale getirilmesi amaçlanmıştır. Uygulama süresince öğretmen tarafından öğrencilere yönlendirilen sorular ışığında sınıf tartışmaları gerçekleşmiş ve öğrencilerin çeşitli stratejiler, gerekçeler ve akıl yürütmeler geliştirdikleri görülmüştür. Öğrencilerin sözel olarak ritmik sayarken parmaklarıyla saymayı takip ettikleri gözlemlenmiştir. Öğrenciler ritmik sayarken sayıların sıra sayılarının temsilcisi olarak parmaklarını kullanmıştır. Bu açıdan öğrencilerin içsel olarak geliştirdikleri ve sınıf tartışmaları doğrultusunda herkesçe kabul gören ritmik sayarken koordineli olarak parmakları kullanmaları alan yazınla paralellik göstermektedir (Anghileri, 1995; Sherin ve Fuson, 2005).

İkinci matematiksel uygulama, öğrencilerin tekrarlı toplanabilecek eş grupları elde etmek için paylaştırma yapmalarıyla ilgilidir. Öğrencilerin çarpmayı öğrendikçe bölmeye dair de stratejileri geliştirmeye başladıkları öne sürülmektedir (Baek, 1998). Öğrenciler çarpma ve bölme arasındaki ters ilişkiyi kullanarak bu işlemler arasında daha kolay geçiş yapabilmektedirler (Jacob ve Willis, 2003). Bu doğrultuda öğrenciler eş gruplama etkinlikleri üzerinde çalışırken öğretmen “Daha farklı nasıl gruplardınız?” sorusu ile sınıf tartışmalarını başlatmış, öğrencileri yeni stratejiler geliştirmeleri için teşvik etmiştir. Böylece, öğrenciler informel olarak bölme stratejileri geliştirmişlerdir. Alan yazında belirtildiği gibi bölme ve çarpma arasındaki ters ilişkiyi sezgisel olarak keşfetmişlerdir (Jacob ve Willis, 2003; Kennedy vd., 2008; Kouba ve Franklin, 1993; Wright vd., 2014). Bu tartışma soruları ayrıca öğrencileri çarpmanın değişme ve etkisiz eleman özelliklerini keşfetmeye yönlendirmiştir.

İkinci uygulama kapsamında dikkat çeken bir diğer bulgu ise orantısal akıl yürütme ile güçlü bir şekilde ilişkili olan fikirlerdir. Araştırma çalışmaları, öğrencilerin ilkokul yıllarının başlarında çarpımsal akıl yürütme veya adil paylaşım içeren görevlerde orantısal düşünebildiklerini göstermektedir (Resnick ve Singer, 1993; Boyer ve Levine, 2012; Vanluydt vd., 2020). Bu doğrultuda Kaput ve West (1994), öğrencilerin orantı konusunda informel ve kavramsal bir alt yapıya sahip olmaları için öğrenme ortamları yaratmanın önemini vurgulamıştır. Bu bilgilere paralel olarak, mevcut çalışmada matematiksel etkinliklerin doğası, öğrencilerin orantısal düşünmesini desteklemiş olabilir. Öğretim dizisi öğrencilerin informel düşünme biçimlerini ve orantısal akıl yürütmeye dair sezgisel bilgilerini düzenlemelerine yardımcı olmuş olabilir.

Üçüncü uygulama, ilk başta çarpma işlem yapmakla ilgili gibi görünse de, sadece çarpma ile ilgili değildir. Öğrencilerin nesneleri bir araya getirirken ki akıl yürütme süreçleri değerlendirildiğinde, orantısal düşünmenin gelişimi için gerekli olan birimleştirme ve yineleme becerilerinin geliştirildiği görülmüştür. Hatta orantısal akıl yürütme için çok önemli olan kovaryasyonel düşünmeye dair çift eşlemeli toplama (Lamon, 1994) fikrini geliştirmişlerdir. Bilişsel bir bakış açısından, orantısal düşünmenin derinlemesine ve esnek bir şekilde kavranabilmesi ve kavramsallaştırılabilmesi için çarpımsal yapıların yeni deneyim alanlarına genişletildiği gözlemlenmiştir. Böylece, çarpımsal düşünme orantısal akıl yürütmenin özü olarak kabul edildiğinden, mevcut tasarımın erken orantısal akıl yürütmeyi desteklediği iddia edilebilir (Behr vd., 1992; Lamon, 2007).

Dördüncü uygulamada, genel anlamda öğrenciler birinci ve ikinci çarpanı yorumlayarak çarpma işlemini çeşitli modellerle ifade etmişlerdir. Bu uygulama, öğrencilerin alan yazında belirtildiği gibi çeşitli temsiller arasında bağlantı kurmaları açısından önemlidir (Anghileri, 2006; Fosnot ve Dolk, 2001; Sowder vd., 2010; Young-Loveridge, 2005). Bruner (1964) de matematik öğretiminde materyallerin ve temsillerin önemli rolünün altını çizmiştir. Bu bağlamda etkinliklerde nesnelerin resimleri sunulmuş, öğrencilerden çizim yapmaları ve verilen durumu sembolik olarak temsil etmeleri istenmiştir. Öğrenciler bu temsiller arasında geçişler yapmıştır. Bu durum, öğrencilerin verilen etkinlikleri modelleme yoluyla birinci ve ikinci çarpan olarak yorumlamalarını sağlamış olabilir.

Dördüncü uygulamada, öğrenciler eş grupların tekrarlı toplanmasının yanı sıra dizi modeli de kullanmışlardır. Akıl yürütmelerini diziler yoluyla aktararak, ilerleyen aşamalarda çarpmanın değişme özelliğini de dizi modeli ile yorumlamışlardır. Değişme özelliği çarpımsal düşünme ve temsilinin geliştirilmesinde kritik öneme sahiptir (Hurst, 2015). Çarpmanın değişme özelliğinin temsilinde oldukça pratik olduğu ileri sürülen dizilerin kullanımı önerilmektedir (Anghileri, 2006). Bu açıdan ortaya çıkan ortaklaşa akıl yürütmelerin alan yazında dizi modelinin etkisine dair vurgulanan fikirlerle paralellik gösterdiği görülmüştür.

Beşinci uygulama, öğrencilerin verilen çarpımsal durumlara uygun gerçekçi problemler kurmalarıyla ilgilidir. Matematik öğretim programında (MEB, 2018) ikinci sınıfta problem kurma kazanımı yer almamasına rağmen, alan yazından hareketle bu dizisine eklenmiştir. Problem kurma etkinlikleri, aynı zamanda öğrencilerin katılımını, yeteneklerini ve gelişimini aydınlatmak için bir değerlendirme aracı (Kwek, 2015) olarak bu çalışmaya hizmet etmiştir. Öğrenciler kavramsal olarak çarpma ve problem çözmeden yola çıkarak problem kurmaya dair anlayışlar geliştirmişlerdir. Bu alanda yapılan çalışmalar, çarpma ile ilgili matematiksel kelimeleri gerçek yaşam senaryolarıyla birleştirmenin zor olduğunu belirtse de (Anghileri, 2006; Calabrese vd., 2020), bu katılımcı öğrenciler çarpma problemleri için oldukça özgün ve çeşitli hikâyeler yazmışlardır.

Sınıf içi matematiksel uygulamalar analizi yalnızca sınıfın toplu öğrenmesini belgelemekle kalmamış, aynı zamanda beş haftalık uygulamadan sonra ortaya çıkan matematiksel içerikle öğrenme rotası ve etkinlik dizisine dair gerekli düzenlemeleri de ortaya koymuştur. Çarpma öğretiminin içeriğini zenginleştirmek ve GME teorisini daha iyi yansıtmak için öğrenme rotasına yeni matematiksel fikirler dahil edilmiştir. Bu ek fikirler mantıksal sırasına dikkat edilerek öğrenme rotasına eklenmiştir. Böylece revize edilen öğrenme rotasının çarpma ve ilgili kavramları geliştirmede daha etkili olacağına inanılmaktadır.

Mevcut tasarım GME'nin temel ilkeleri doğrultusunda geliştirilmiştir. Etkinlikler öğrencilerin informel matematiksel stratejilerinin yer aldığı bir modelden daha formel matematiksel muhakeme için bir modele kadar olan süreç göz önünde bulundurularak gerçekçi bir bağlamda tasarlanmıştır (Gravemeijer ve Doorman, 1999). Öğrencilerin yeni bir matematiksel gerçeklik (Gravemeijer vd., 2000) oluştururken modelden modele geçişleri takip edilmiş ve teşvik edilmiştir. Sınıf içi matematiksel uygulamalara ve ortaklaşa akıl yürütmeler sonucu ortaya çıkan fikirlere baktığımızda modelden modele olan bu değişim açıkça görülmektedir. Ayrıca, gelişen modeller ilkesi doğrultusunda, öğrencilerin imgeleri de çok önemli bir rol oynamıştır. Öğrenciler, durumlara ilişkin anlık anlayışlarını imgelerine yansıtmışlardır. Thompson (1996), öğrencilerin meşgul oldukları görevlerle ilgili modeller ve imgeler kullanarak akıl yürütmeye teşvik edilmelerini önermektedir. Bu bağlamda öğrencilerin verilen bağlamın modeli ve üst düzey matematiksel muhakeme için model olarak duruma özel imgeleri kullandıkları söylenebilir.

Son olarak matematiksel uygulamaların ve fikirlerin ortaya çıktığı sınıf kültürü de tartışılmalıdır. Bu konuda matematiksel uygulamaların geliştirilmesinde önemli rol oynayan normlar ön plana çıkmaktadır. Öğretim deneyinin başında etkili iletişim kurarak öğrencilerin düşüncelerini paylaşabileceği bir sınıf ortamı yaratmak için sosyal ve sosyo-matematiksel normlar çoktan kurulmuştu. Böylece, tartışma oturumları sırasında öğrenciler çözümlerini paylaşmış, birbirlerini dinlemiş ve fikirlerini tartışmışlardır. Ayrıca, farklı çözümler ve temsiller bulmak, varsayımlarda bulunmak ve matematiksel akıl yürütmeyi haklı çıkarmak için sosyo-matematiksel normlardan faydalanılmıştır. Böylece normların, bu çalışmada önemli bir rol oynadığı görülmüştür.

Bu çalışmanın bağlamı doğrultusunda elde edilen bulgular ve tartışmalar sonucunda, başka sorular da gündeme gelmiş ve gelecek çalışmalar için öneriler sunulmuştur. Örneğin, mevcut çalışma 2. sınıf öğrencilerinin çarpma becerilerinin toplu gelişimini ortaya koymaktadır. Öğrencilerin bireysel gelişimlerine odaklanılmamıştır. Bu nedenle, farklı öğrencilerin bireysel olarak nasıl geliştiğini, özellikle de belirli öğrencilerin bu kolektif gelişim sürecinden ne kazandıklarını ve sürece nasıl katkıda bulunduklarını araştırmak faydalı olabilir. Ek olarak, ön ve son testler yoluyla öğrenmelerini ölçmek, sınıf topluluğunda gerçekleşen öğrenme ve gelişim hakkında daha kapsamlı bir anlayışa sahip olmak için faydalı olabilir.

Bulgular incelendiğinde çarpma işleminin yanı sıra oran-orantı gibi çarpımsal düşünme gerektiren konulara dair de çıktılar gözlenmiştir. Çarpımsal düşünme gerektiren konuların çarpmanın yanı sıra nasıl geliştiğinin incelenmesinin geliştirilen öğrenme rotasının ve öğretim dizisinin önemini ortaya koymak adına değerli olacağı düşünülmektedir. Örneğin, mevcut tasarımın bölme öğretimi üzerindeki etkisini ortaya çıkarmak için öğrencilerin bölme işlemine ilişkin sınıf matematik uygulamaları belgelenebilir. Bu amaçla, çarpımsal düşünme gerektiren kavramların evriminin boylamsal bir çalışma ile gözlemlenmesi ve incelenmesi önerilmektedir.

Benzer şekilde, üçüncü sınıfta çarpma için başka bir öğrenme rotası ve ilgili öğretim dizisi geliştirmek için başka bir tasarım araştırması önerilmektedir. Mevcut çalışmada, öğrenciler üçüncü sınıfla ilgili kazanımları sezgisel olarak kavramışlardır. Bu nedenle ekip, öğrencilerin gelecek yıl öğrenmek için daha az hedefleri olduğu sonucuna varmıştır. Bu bağlamda, bu çalışmanın katılımcıları için bir sonraki yıl için hedeflerin ve derslerin tasarlanması önerilebilir.

Son olarak, öğretmenlerin konu bilgisini ve pedagojik alan bilgilerini geliştirmek için çeşitli konuları dikkate alarak GME temelli öğrenme rotaları oluşturma, normları belirleme ve sınıfta argümantasyonu sürdürme konusunda mesleki gelişim programları geliştirilebilir. Mevcut çalışma, müdahaleleri geliştirmek ve belirli bağlamlara daha uygun teoriler oluşturmak için metodolojik bir araç sunmaktadır. Sonuç olarak, bu çalışma, eğitmenlere öğretim araçlarına da ulaşabilecekleri, uygulanabilir yeni bir mesleki gelişim stratejisi sağlayabilir. Benzer şekilde mesleki gelişim programları hazırlanabilir ve uygulanabilir.

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EDUCATION

Degree	Institution	Year of Graduation
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BS	Anadolu University, Sociology	Expected 2024
BS	METU Elementary Math Education	2013
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WORK EXPERIENCE

Year	Place	Enrollment
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Feb. 2015-Oct. 2022	Middle East Technical University, Department of Elementary Education	Research Assistant
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Sept. 2013- Sept. 2014	Ministry of Education, Turkey	Mathematics Teacher

FOREIGN LANGUAGES

Upper Intermediate English, Intermediate German

PUBLICATIONS

Journal Papers in International and National Indexed Journals

Kandil, S., & Işıksal-Bostan, M. (2019). Effect of inquiry-based instruction enriched with origami activities on achievement, and self-efficacy in geometry. *International Journal of Mathematical Education in Science and Technology, 50*(4), 557-576. DOI: 10.1080/0020739X.2018.1527407

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Projects

Project Title: Ortaokul Matematik Öğretmen Adaylarının Çevre ve Alan Konusuna Yönelik Pedagojik Alan Bilgilerinin İncelenmesi
Project length: 12 Month (2018-2019)
Funding institution: Middle East Technical University, Turkey (GAP-501-2018-2975)
Role in the project: Researcher
Project coordinator: Prof. Dr. Mine Işıksal Bostan

Project Title: Developing a Learning Trajectory Based on Realistic Mathematics Education for 1st Grade Mathematics, Numbers and Operations Unit: A Design Research (First Step Project)
Project length: 24 Month (2017-2019)
Funding institution: The Scientific and Technological Research Council of Turkey (1001-116K078)
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SCHOLARSHIPS, GRANTS AND AWARDS

- Scholarship for Graduate Students, The Scientific and Technological Research Council of Turkey, 2017-2019.
- Performance Award (The most successful student in the Ph.D. program of the Department of Elementary Education) Middle East Technical University, Turkey, 2016-2018.
- High Honor Roll (M.S.) – Middle East Technical University, Turkey, 2016
- Honor Roll (B.S.) – Middle East Technical University, Turkey, 2013

I. THESIS PERMISSION FORM/TEZ İZİN FORMU

ENSTİTÜ / INSTITUTE

Fen Bilimleri Enstitüsü / Graduate School of Natural and Applied Sciences ☐

Sosyal Bilimler Enstitüsü / Graduate School of Social Sciences ☒

Uygulamalı Matematik Enstitüsü / Graduate School of Applied Mathematics ☐

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TEZİN ADI / TITLE OF THE THESIS (**İngilizce** / English): SUPPORTING THE DEVELOPMENT OF A SECOND-GRADE CLASSROOM'S CONCEPTIONS OF MULTIPLICATION THROUGH A HYPOTHETICAL LEARNING TRAJECTORY

TEZİN TÜRÜ / DEGREE: **Yüksek Lisans** / Master ☐ **Doktora** / PhD ☒

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