DECOMPOSITION OF A SPECIFIC CLASS OF (1,3) GROUPS

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ABSTRACT

DECOMPOSITION OF A SPECIFIC CLASS OF (1,3) GROUPS

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The classification (up to near isomorphism) of some class of almost completely decomposable groups with a regulating regulator is possible. Since almost completely decomposable groups can be written as a direct sum of indecomposable groups, for classification of almost completely decomposable groups it would be enough to find isomorphism classes of all indecomposable groups. The class of almost completely decomposable groups with a critical typeset in (1,3) configuration and a regulator quotient of exponent p^3 have 6 near isomorphism classes of indecomposable groups. We describe almost completely decomposable groups by coordinate matrices. If the coordinate matrix of an almost completely decomposable group is decomposable, then the group is decomposable. Hence the method used in this thesis to determine the decomposition of almost completely decomposable groups is turned to an equivalance problem of determining the decomposition of the corresponding coordinate matrices.

Keywords: Almost completely decomposable groups, Torsion free groups, Decom-

posability of almost completely decomposable groups.

ÖZ

(1,3) GRUPLARIN SPESİFİK BİR SINIFININ PARÇALANMASI

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Regulatörü regule olan hemen hemen ayrışan grupların bazı sınıflarının (yakın izomorfizma altında) sınıflandırılması mümkündür. Hemen hemen ayrışan gruplar ayrışamayan grupların direkt toplamı olarak yazılabildiği için, hemen hemen ayrışan grupların sınıflandırılmasında ayrışamayan grupların izomorfizma sınıflarını bulmak yeterli olacaktır. Kritik tip kümesi (1,3) düzeneğinde, bölüm regulatörün üssü p^3 olan hemen hemen ayrışan grupların 6 tane yakın izomorfizma sınıfı vardır.

Hemen hemen ayrışan grupları koordinat matrisleriyle betimleyebiliriz. Hemen hemen ayrışan bir grubun koordinat matrisi ayrışıyor ise, grubun kendisi de ayrışır. Bundan dolayı, hemen hemen ayrışan grupların ayrışmasını belirlemek için bu tezde kullandığımız metot, gruplara karşılık gelen koordinat matrislerin ayrışmasına denk gelen bir probleme dönmüştür.

Anahtar Kelimeler: Hemen hemen ayrışan gruplar, Torsiyonsuz gruplar, Hemen hemen ayrışan grupların parçalanması.

To my family

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CHAPTER 1

INTRODUCTION

An abelian group in which every element except identity has infinite order is called a torsion-free group.

A torsion-free abelian group of finite rank which contains a completely decomposable subgroup of finite index is called an almost completely decomposable group. Almost completely decomposable groups are very complicated to classify and only some special subclasses of almost completely decomposable groups can be classified under a modified isomorphism called near-isomorphism.

In this thesis we investigate the decomposition of a subclass of almost completely decomposable groups, namely (1,3) groups.

In [14] the near isomorphism classes of p-reduced, p-local (1,3) groups of exponent p^3 are determined by a long and detailed proof. In this thesis we first determine the near-isomorphism classes of indecomposable (1,3) groups with homocyclic regulator quotient of exponent p^3, then the near-isomorphism classes of indecomposable (1,3) groups with regulator quotient isomorphic to $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_p$ and finally we find the near-isomorphism classes of indecomposable (1,3) groups with regulator quotient isomorphic to $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_{p^2}$.

In this way we believe that the readers can follow the proof given in [14] easier.

A very good source for the definitions and notations are the books of A. Mader and L. Fuchs. For the detailed information about matrix represantations of almost completely decomposable groups you can read the papers published by Arnold,Mader,Mutzbauer and Solak.

In chapter 2, we define almost completely decomposable groups, pure subgroups, posets and types of almost completely decomposable groups.

In chapter 3, the definition and properties of two important isomorphism invariants of an almost completely decomposable group G, namely regulator and regulator quotient, are given.

Almost completely decomposable groups can be described by matrices, the so called coordinate matrices. This induces the problem of decomposability of almost completely decomposable groups to an equivalance problem of the decomposability of matrices.

In chapter 4, coordinate matrices and the relation between the decomposability of almost completely decomposable groups and the corresponding coordinate matrices are presented.

In chapter 5, the modified Gauss algorithm which we need to find the Smith Normal form of coordinate matrices of the given almost completely decomposable groups, is discussed

In chapter 6, we specialize on (1,3) groups of exponent p^3. The theory given in the previous chapter is formulated in this chapter particularly for (1,3) groups. We show how to construct the coordinate matrix of a (1,3) group and we list the allowed row and column operations to reduce it to its Smith Normal form. We also showed that it is possible to exclude some summands if the coordinate matrix has a 0-line or a cross.

In chapters 7,8 and 9 decomposition theorems of (1,3) groups of exponent p^3 are stated and proven.

CHAPTER 2

PRELIMINARIES

2.1 Almost Completely Decomposable Groups

A torsion-free abelian group G can be considered as an additive subgroup of the $\mathbb{Q}$-vector space $\mathbb{Q}\,G$, the divisible hull of G. The dimension of $\mathbb{Q}G$ is called the **rank of** G.

Let G be a torsion-free abelian group of finite rank. G is called **completely decomposable** if G is the direct sum of rank 1 groups. If G has only trivial direct summands then G is called **indecomposable**.

Definition 1 *A torsion-free abelian group G is said to be* ***almost completely decomposable*** *if G contains a completely decomposable subgroup H such that G/H is finite.*

Definition 2 *(Pure Subgroup) Let G be an abelian group and let H be a subgroup of G. H is called a pure subgroup of G if $mG \cap H = mH$ for all $m \in \mathbb{Z}$. In particular, if G is torsion free then from the definition of pure subgroup it follows that for all $n \in \mathbb{Z}$ the equation $a = nb$ where $b \in G$ and $a \in G$ implies $b \in H$.*

The group $H_*^G := \{x \in G \mid \exists_{n\in\mathbb{N}}\; nx \in H\} = G \cap \mathbb{Q}\,H$ is called the ***purification*** *of H in G. The purification H_*^G is the unique smallest pure subgroup of G containing H.*

2.2 Posets

Definition 3 *A partially ordered set or poset is a set T with a binary relation $\leq$ satisfying the followings:*

1. **the reflexive law:** $t \leq t$, for any $t \in T$
2. **the anti–reflexive law:** $t_1 \leq t_2$ and $t_2 \leq t_1$ imply $t_1 = t_2$, where $t_1, t_2 \in T$
3. **the transitive law:** $t_1 \leq t_2$ and $t_2 \leq t_3$ implies $t_1 \leq t_3$, where $t_1, t_2, t_3 \in T$

Note that, $t_1 \leq t_2$ means the same as $t_2 \geq t_1$ and $t_1 < t_2$ means that $t_1 \leq t_2$ and $t_1 \neq t_2$.

Some useful definitions about posets are given below:

1. Two elements $t_1, t_2 \in T$ are **comparable** if either $t_1 \leq t_2$ or $t_2 \leq t_1$. Otherwise the elements are **incomparable**.
2. A subset of T is called a **chain**, if any two elements of T are comparable. On the other hand, the **antichains** are subsets of T whose elements are pairwise incomparable.
3. A poset T is a **tree** if for each $t \in T$ the subset $\{x|x \leq t\}$ is a chain. Similarly, T is called an **inverted tree** if for each $t \in T$ the subset $\{x|x \geq t\}$ is a chain.
4. A **forest** is the disjoint union of trees and an **inverted forest** is a disjoint union of inverted trees. Inverted forests are also called **V–free**.

2.3 Types and Critical Typeset

Let G be a torsion-free abelian group and let p be a prime. The maximal integer m for which the equation $p^m x = y$ is solvable in G, is called p-height of y, denoted $h_p(y)$. If there is no such maximal integer exists then $h_p(y)$ is infinity. The sequence of p-heights,

$H(y) = (h_{p_1}(y), ..., h_{p_k}(y), ...)$ is said to be the **charachteristic** of y.

The equivalance classes of characteristics are called types. A type can be represented by any member in its equivalance class.

Types were invented to classify torsion-free groups of rank 1. A type $t(R)$ is an isomorphism class of a rank one group R. The set of types form a poset where $t(R_1) \leq t(R_2)$ if R_1 is isomorphic to a subgroup of R_2.

Let G be a torsion-free abelian group. The set of all types of elements of G is said to be a typeset of G. If G is an almost completely decomposable group then the finite set of types of direct summands of rank 1 is called the **critical typeset** of G.

The group G is called **homogenous** if every non-zero element of G is of the same type.

CHAPTER 3

REGULATOR AND REGULATOR QUOTIENT OF AN ALMOST COMPLETELY DECOMPOSABLE GROUP

Almost completely decomposable groups are torsion-free abelian groups of finite rank which contain completely decomposable subgroups of finite index.

Let G be an almost completely decomposable group. We call a completely decomposable subgroup of G of minimal index a **regulating subgroup**, namely, a completely decomposable subgroup R of G is called a **regulating subgroup** if $|G/R|$ is the least integer in the set $\{|G/H| : H$ is completely decomposable with G/H finite$\}$

The intersection of all regulating subgroups of G is again a completely decomposable group of finite index and is called **regulator** of G. This result is due to R. Burkhardt.

If R is the regulator of G, then G/R is called **the regulator quotient** of G.

The isomorphism types of the regulator and the regulator quotient are isomorphism invariants of an almost completely decomposable group.

If G contains exactly one regulating subgroup then we say G has a **regulating regulator**.

Remark: Let G be an almost completely decomposable group and let R be its regulator. Then R is completely decomposable. Moreover, if n is the exponent of G/R, then $R \subseteq G \subseteq n^{-1}R$.

The following result plays an important role in the decomposition problem of almost completely decomposable groups with a V-free critical typeset.

Theorem 3.0.1 *(Mutzbauer [7]) Let G be an almost completely decomposable group* *and let T be the critical typeset of G. If T is V-free then G has a regulating regulator.*

Definition 4 *Let G be an almost completely decomposable group with regulator R and regulator quotient G/R. Write $R = C_1x_1 \bigoplus ... \bigoplus C_nx_n$ where $x_i \in R$ and $C_i = \{c \in \mathbb{Q} \,|cx_i \in R\}$. If $p^{-1} \notin C_i$, then the set $\{x_1, ..., x_n\}$ is called a* ***p-basis*** *of R.*

Definition 5 *An almost completely decomposable group G is said to be* ***p-local*** *for a prime p if the regulator quotient G/R is a primary group, i.e. , G/R is a group of exponent dividing p.*

Definition 6 *An almost completely decomposable group G is called* ***p-divisible*** *for a prime p if $pG = G$ and G is called* ***p-reduced*** *if it contains no p-divisible subgroup other than $\{0\}$.*

Definition 7 *Let G be an almost completely decomposable group. Then G is called* ***clipped*** *if it does not have any rational direct summand.*

CHAPTER 4

COORDINATE MATRICES

Let $M = \begin{bmatrix} m_{ij} \end{bmatrix}$ be an integer matrix of size $r \times n$. The matrix M is called **decomposable** if there exist permutation matrices P and Q such that PMQ is of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. It is allowed that either A or B has no rows or no columns.

Definition 8 *Let G be a p-local, p-reduced almost completely decomposable group with a regulator R and let $M = \begin{bmatrix} m_{ij} \end{bmatrix}$ be an $r \times n$ integer matrix. M is called a* ***coordinate matrix*** *of the group G if $\{x_1, ..., x_n\}$ is a p-basis of R and $\{\gamma_1, ..., \gamma_r\}$ is a basis of the regulator quotient G/R such that*

$$g_i = p^{-k_i} \left(\sum_{j=1}^{n} m_{ij} x_j \right)$$

with $\langle \gamma_i \rangle \cong Z_{p^{k_i}}$.

Suppose that $k = k_1 \geq k_2 \geq ... \geq k_r$. If we group the generators g_i of the regulator quotient of equal orders, then the coordinate matrix M has the form

$$M = \left[\begin{array}{ccc|c} m_{11} & ... & m_{1n} & p^{-k_1} \\ \hline m_{21} & ... & m_{2n} & p^{-k_2} \\ \hline & & & \vdots \\ \hline m_{r1} & ... & m_{rn} & p^{-k_r} \end{array} \right]$$

where p^{k_i} =ord$(g_i + R)$ for $i = 1, ..., r$.

Almost completely decomposable groups can be represented by coordinate matrices. The following theorem shows us that the decomposability of an almost completely decomposable group G can be reduced to the decomposability of the corresponding coordinate matrix M of G. This theorem has a very important role in the decomposition theory of almost completely decomposable groups.

Theorem 4.0.1 *Let G be an almost completely decomposable group and let M be its coordinate matrix. G is decomposable if and only if M is decomposable. In particular, if G has a decomposable matrix M, i.e., $PMQ = M_1 \bigoplus M_2$, where P and Q are permutation matrices, then $G = G_1 \bigoplus G_2$ where G_i has the coordinate matrix M_i.*

Proof 1 *First suppose that M is decomposable. Then there are permutation matrices P and Q such that $PMQ = M_1 \bigoplus M_2$. The coordinate matrix is obtained by means of a p-basis B of the regulator R. Each column of B corresponds to a basis element and the columns of the M_i determine a partition $B = B_1 \cup B_2$ of the p-basis.*

According to this partition, we can write $R = R_1 \bigoplus R_2$. But then we can write **G** *as $G = G_1 \bigoplus G_2$ where $G_i = \langle R_i \rangle_*$ the purification of R_i in G, i.e., G is decomposable.*

Next assume that G is decomposable and let M be its coordinate matrix. By definition the group G can be written as $G_1 \bigoplus G_2$. We may write the regulator R of G as $R = R_1 \bigoplus R_2$ where R_i is the regulator of G_i and this leads us to be able to write M as a direct sum of the coordinate matrices of G_1 and G_2 because the columns of M are determined by the p-basis of R_1 and R_2.

4.1 Smith Normal Form

Theorem 4.1.1 *Let H be a non-singular, $k \times k$ integer matrix. Then there exist invertible integer matrices U, Y of size $k \times k$ such that*

$$UHY = \begin{bmatrix} d_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & 0 & \dots & 0 \\ \vdots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & d_{k-1} & 0 \\ 0 & \dots & 0 & 0 & 0 & d_k \end{bmatrix}$$

where d_i are positive integers and d_i divides d_{i+1} for $1 \leq i \leq m$. The numbers d_i are uniquely determined by H. UHY is called the Smith Normal Form of a matrix H.

We need to establish standard form of coordinate matrices of almost completely decomposable groups.

Smith Normal Forms of coordinate matrices are helpful to determine the decomposability of matrices. By Theorem 4.1.1, every integer matrix H is equivalent to a matrix H' in Smith Normal form, i.e., $H' = UHY$ where U, Y are invertible matrices.

In this thesis, we deal with matrices in Z_{p^k}. By Theorem 4.1.1, we get the Smith Normal form of H as

$$\begin{bmatrix} I & & & & \\ & pI & & & \\ & & \ddots & & \\ & & & p^{k-1}I & \\ & & & & 0 \end{bmatrix}$$

where I denotes the identity matrix and the empty spaces are 0-blocks.

Let G be an almost completely decomposable group and let $H = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be its coordinate matrix, where A, B, C and D are block matrices. We will use allowed row and column transformations to get the Smith Normal form of the subblocks of H. While applying this procedure, the submatrices 0 and the submatrices of the form $p^k I$ will be affected. However it is possible to reestablish them.

CHAPTER 5

MODIFIED GAUSS ELIMINATIONS

Almost completely decomposable groups can be written as a direct sum of indecomposables. This decomposition is not unique and therefore it is hopeless to determine the isomorphism classes of almost completely decomposable groups. But some classes of almost completely decomposable groups can be classified under a weaker form of isomorphism called *near isomorphism*.

Definition 9 *Let G_1 and G_2 be two torsion free groups of finite rank. If for every integer $n \in \mathbb{Z}^+$, there is a monomorphism $\alpha_n : G_1 \to G_2$ such that $[G_2 : \alpha_n(G_1)]$ is finite, n and $[G_2 : \alpha_n(G_1)]$ are relatively prime, then G_1 and G_2 are called **nearly isomorphic**, denoted $G_1 \cong_{nr} G_2$.*

Note that near isomorphism is a weaker form of isomorphism and isomorphic groups are also nearly isomorphic.

The following theorem includes some characterizations of near-isomorphism.

Theorem 5.0.1 *[4] Let G_1 and G_2 be nearly isomorphic almost completely decomposable groups. Then the following statements hold.*

1. *Let R_1 and R_2 be the regulators of G_1 and G_2 respectively. Then R_1 and R_2 are isomorphic.*
2. *$Rank(G_1) = Rank(G_2)$ and $T_{cr}(G_1) = T_{cr}(G_2)$ where T_{cr} denotes the critical typeset.*
3. *$G_1 \bigoplus R_2 \cong G_2 \bigoplus R_1$*

Almost completely decomposable groups can be represented by coordinate matrices. A row or a column transformation of the coordinate matrix M of an almost completely decomposable group G is equivalent to multiplying M with the corresponding elemantary matrices from left or right, respectively.

Let G and H be torsion free groups of finite rank. Then G and H are called nearly isomorphic if for every positive integer n, there are relatively prime integers m and n and homomorphisms $f : G \to H$ and $g : H \to G$ such that $fg = m1_H$ and $gf = n1_G$.

Let G be an almost competely decomposable group and M be its coordinate matrix. Row and column transformations of M are called **allowed** if the transformed coordinate matrix M' is the coordinate matrix of G or another group H, that is nearly isomorphic to G. Matrices are simplified by making entries equal to 0.

By a theorem of Faticoni-Schultz (see [5]) p-local almost completely decomposable groups can be classified up to near-isomorphism if the near isomorphism classes of indecomposable groups are known. Hence our motivation is to obtain a complete list of indecomposable groups.

Let G be an almost completely decomposable group and let M be the corresponding coordinate matrix. If we want to classify the near-isomorphism classes of indecomposable groups then we first simplify M to M' by allowed row and column transformations. The matrix M' that we obtained is the coordinate matrix of a nearly isomorphic group G'. If M' is decomposable then the group G' is decomposable and by a well-known theorem of Arnold, two nearly isomorphic torsion-free abelian groups have the same decomposition properties, i.e., if G' is decomposable then G is decomposable.

Lemma 5.0.2 *([10]) Let $G = A_1u_1 + A_2u_2 + A_3u_3 + Z_{p^{-k}}(a_1u_1 + a_2u_2 + a_3u_3)$ be an almost completely decomposable group where $\mathbb{Z} \subseteq A_i \subseteq \mathbb{Q}_p$ and a_i's are integers where $i \in \{1, 2, 3\}$*

Suppose that $a_3 \in p^m \mathbb{Q}_p / p^{m+1} \mathbb{Q}_p$ where $m \in \mathbb{N}$. Then $\langle u_1, u_2 \rangle_*^G = A_1u_1 \bigoplus A_2u_2$.

Proof 2 *It is clear that $\langle A_1u_1 + A_2u_2, p^{-m}(a_1u_1 + a_2u_2) \rangle$ is a subset of $\langle u_1, u_2 \rangle_*^G$*

By definition we can write $\langle u_1, u_2\rangle_*^G = (Qu_1 \bigoplus Qu_2) \cap G$.

Take an arbitrary element $b \in \langle u_1, u_2\rangle_*^G$ *then*

$$b = c_1u_1 + c_2u_2 \in (Qu_1 \bigoplus Qu_2) \cap G \tag{51}$$

where $c_1, c_2 \in \mathbb{Q}$.

and since $b \in G$ *we can also write*

$$b = d_1u_1 + d_2u_2 + d_3u_3 + \varepsilon p^{-k}(a_1u_1 + a_2u_2 + a_3u_3) \tag{52}$$

where $d_1 \in A_1, d_2 \in A_2, d_3 \in A_3$ *and* $\varepsilon \in \mathbb{Z}$.

Put $a_3 = p^m a_3'$ *where* a_3' *is a unit and* $m \in \mathbb{N}$. *Then equating the coefficients of the equations* (51) *and* (52) *we obtain*

$$d_3 + \varepsilon p^{-k}a_3 = d_3 + \varepsilon p^{-k}p^m a_3' = 0$$

Set $\varepsilon = p^{k-m}\varepsilon'$ *where* $k > m$ *and* $\varepsilon' \in \mathbb{Z}$. *We get*

$$\begin{aligned} c_1u_1 + c_2u_2 &= d_1u_1 + d_2u_2 + d_3u_3 + \varepsilon p^{-k}(a_1u_1 + a_2u_2 + p^m a_3'u_3) \\ &= d_1u_1 + d_2u_2 + p^{k-m}\varepsilon' p^{-k}(a_1u_1 + a_2u_2) + (\varepsilon p^{-k+m}a_3' + d_3)u_3 \end{aligned}$$

This implies $b \in \langle A_1u_1 + A_2u_2, p^{-m}(a_1u_1 + a_2u_2)\rangle$ *because* $\varepsilon p^{-k+m}a_3' + d_3 = 0$ *hence we proved that* $\langle u_1, u_2\rangle_*^G = A_1u_1 + A_2u_2 + \mathbb{Z}_{p^{-m}}(a_1u_1 + a_2u_2)$.

CHAPTER 6

(1,3) GROUPS

A (1,3) group is a p-local, p-reduced almost completely decomposable group with a critical typeset of (1,3) form.

Let G be a (1,3) group and let $R = R_1 \bigoplus R_2 \bigoplus R_3 \bigoplus R_4$ be the regulator of G. Direct summands R_i's are all homogoneous completely decomposable groups of rank $r_i \geq 1$ and type t_i. The rank of G is $n = r_1 + r_2 + r_3 + r_4$.

Let $M = [M_{ij}]$ be the coordinate matrix of G and let $(x_1, x_2, ..., x_n)$ be a p-basis of R. Assume $(x_1, ..., x_{r_1})$ is a p-basis of R_1, $(x_{r_1+1}, ..., x_{r_1+r_2})$ is a p-basis of R_2, $(x_{r_1+r_2+1}, ..., x_{r_1+r_2+r_3})$ is a p-basis of R_3 and $(x_{r_1+r_2+r_3+1}, ..., x_{r_1+r_2+r_3+r_4})$ is a p-basis of R_4, so the coordinate matrix M is divided in 4 blocks, say $\alpha, \beta_1, \beta_2, \beta_3$ of size $r \times r_i$, where $i \in \{1, 2, 3, 4\}$ and we get $M = [\alpha||\beta_1|\beta_2|\beta_3]$. The matrix $[\beta_1|\beta_2|\beta_3]$ is called the β-part of M. The coordinate matrix M is obtained by means of the bases of R and G/R.

If $(x_1, x_2, ..., x_n)$ is a p-basis of R and if $(r_1, r_2, ..., r_r)$ is a basis of G/R then the coordinate matrix M is of size $r \times n$ and has rank r. Each column of M corresponds to a type. For example, according to the division of the p-basis of R, the columns of α are of type t_1 and hence called t_1-columns of M.

Let $G = \langle R, g_1, ..., g_r \rangle$ be a (1,3) group with regulator R. The g_i's are the representatives of the basis of the regulator quotient $G/R = \bigoplus_{s=1}^{r} \mathbb{Z}_{p^{k_s}}$

If we group the generators g_i, then we may write the coordinate matrix M in the form

$$M = \left[\begin{array}{ccc|c} m_{11} & \dots & m_{1n} & p^{k_1} \\ \hline & & & \vdots \\ \hline m_{r1} & \dots & m_{rn} & p^{k_r} \end{array}\right]$$

where p^{k_i} =ord$(g_i + R)$ and $k_1 \geq k_2 \geq \dots \geq k_r$.

6.1 Allowed Column and Row Operations

Let G be an almost completely decomposable group and let M be the coordinate matrix of G. Row and column transformations of M are called *allowed* if the transfromed coordinate matrix M' is the coordinate matrix of G' where G' is near isomorphic to G.

By[14] the following row and column operations are allowed for (1,3) groups:

1. We can add any multiple of a row to any row below it.
2. We can add any multiple of $p^{k_i - k_j}$ times of row j to a row i where $i < j$
3. We can multiply any row by an integer y where y is relatively prime to p , i.e., any row can be multiplied by a p-unit.
4. Two columns of α or two columns of β_i for $i \in \{1, 2, 3\}$ can be interchanged.
5. Any column can be multiplied by a p-unit.
6. We can add any multiple of a column of β_i to a column of β_j for $j \geq i$

Allowed row operations are done by multiplying the coordinate matrix M from left by a lower triangular matrix X and allowed column operations are performed by multiplying M from right by an upper triangular matrix $Y = [Y_{ij}]$ which is of the form

$$Y = \begin{bmatrix} Y_{1,1} & 0 & 0 & 0 \\ 0 & Y_{2,2} & Y_{2,3} & Y_{2,4} \\ 0 & 0 & Y_{3,3} & Y_{3,4} \\ 0 & 0 & 0 & Y_{4,4} \end{bmatrix}$$

where $Y_{i,j}$ are $r_i \times r_j$ integer matrices and the diagonal blocks are p-invertible block matrices.

Lemma 6.1.1 ***(Regulator Criterion)*** *Let G be a $(1,3)$-group. Let $R = R_1 \bigoplus R_2 \bigoplus R_3 \bigoplus R_4$ be a completely decomposable subgroup of G of finite index. Then R is the regulator of G if and only if R_1 and $R_2 \bigoplus R_3 \bigoplus R_4$ are pure subgroups in G.*

Proof 3 *Let $\tau_1, \tau_2, \tau_3, \tau_4$ be the types of R_1, R_2, R_3 and R_4 respectively. Let $t(x)$ denote the type of x where $x \in G$.*

Define $R(\tau_i) = \{x \in G | t(x) \geq \tau_i\}$ Since G is a (1,3) group $R(\tau_1) = R_1$ and $R(\tau_2) = \{x \in G | t(x) \geq \tau_2\} = R_2 \bigoplus R_3 \bigoplus R_4$. By lemma 4.0.2 in [11] the subgroups R_1 and R_2, R_3, R_4 are pure subgroups of G.

Definition 10 *Let G be an almost completely decomposable group. G is called* ***clipped*** *if G has no direct summand of rank 1.*

We mean by a **line** of a matrix a row or a column. While an entry of a coordinate matrix is annihilated, other entries that were zero may become non-zero and such entries are called **fill-ins**. An integer that is relatively prime to p is called a **p-unit** or briefly a **unit**.

A matrix $M = [mij]$ has a **cross** located at (i_1, j_1) if $m_{i_1,j_1} \neq 0$ and $m_{i_1,j} = 0$, $m_{i,j_1} = 0$ for all $i \neq i_1$ and $j \neq j_1$.

If an entry m_{ij} is used to annihilate in its row and column by using allowed row and cloumn transformation to get a cross located at m_{ij}, then we write "m_{ij} leads to a cross".

If the fill-ins in a row or in a column caused by allowed row and column transformations can be removed by inverses of the transformations and the 0-blocks can be reestablished, then we write "we can annihilate in ...".

Lemma 6.1.2 *Let $M = [\alpha||\beta_1|\beta_2|\beta_3]$ be a coordinate matrix of size $r \times n$ of a clipped (1,3) group G. Then*

1. *α and $\beta = [\beta_1|\beta_2|\beta_3]$ both are of rank r.*
2. *The block matrix α can be transformed to the identity matrix by allowed column transformations.*
3. *The first l_1 rows of β_1 can be transformed to a p-diagonal matrix by allowed row and column transformations.*
4. *G is uniquely determined by $[\beta_1|\beta_2]$ up to near isomorphism.*
5. *If $k_1 = k_2 + 1$ then the first $l_1 + l_2$ rows of β_1 can be transformed to the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ where A, B are p-diagonal matrices.*

Proof 4

1. *This result is due to regulator criterion.*
2. *By regulator criterion R_1 and $R_2 \bigoplus R_3 \bigoplus R_4$ are pure subgroups in G and this is true if α and β part both have p-rank r. This implies that there is a unit in each row of α and in each row of β part of the coordinate matrix. By column transformations the unit in the first row of α can be moved to (1,1)'th position of α. By allowed row and column transformations the entries in the first column below this unit and the entries in the first row of α right to this unit can be made zero. Similarly, the unit in the second row can be moved to the (2,2)'th position of α and as described above, the second column and the second row of α can be annihilated up to this unit.*

   Continuing like this, α can be transformed to the form $[I|0]$. Since G is clipped, there is no 0-column in α. Hence α can be transformed to the identity matrix I and the proof finishes.

3. *We may transform the first l_1 rows of β_1 into a p-diagonal matrix because we can apply any row and column transformations on the first l_1 rows of β_1.*

4. *By part (2) we know that α can be transformed to the identity matrix. The column and row operations applied on $[\alpha|\beta]$ cause fill-ins in α but it is possible to reestablish α in the original form by column transformations in α only.*

5. *See lemma 15 in [14].*

If there are crosses, zero row or columns or units in the first block of β-part of the coordinate matrix $M = [\alpha||\beta_1|\beta_2|\beta_3]$ of a (1,3) group G, then we can read off summands of rank 1, 2 or 3.

The following lemma is due to Corollary 26 in [14].

Lemma 6.1.3 *Let G be a (1,3) group and let $M = [\alpha||\beta_1|\beta_2|\beta_3]$ be a coordinate matrix of G. Then*

1. *If there is a zero column in $[\beta_1|\beta_2]$ then G has a direct summand of rank 1.*

2. *If there is a zero row in $[\beta_1|\beta_2]$ then G has a direct summand of rank 2.*

3. *If there is a cross in $[\beta_1|\beta_2]$ then G has a summand either of rank 2 or rank 3.*

4. *If there is a double cross in $[\beta_1|\beta_2]$ then G has a summand either of rank 3 or rank 4.*

5. *If there is a unit in the first l_1 rows of β_1 then G has a direct summand of rank 2.*

6. *If $k_1 = k_2 + 1$ and β_1 has a unit in the first $l_1 + l_2$ rows, then G has a summand of rank 2.*

Proof 5 *3) If there is a cross in $[\beta_1|\beta_2]$ with a pivot a unit located at row i, then with this unit we can annihilate the i'th row in β_3 and this would display a direct summand of rank 2.*

If there is a cross in $[\beta_1|\beta_2]$ with a pivot in $p\mathbb{Z}$ located at row i then by regulator criterion there must be a unit in the i'th row of β_3. This would lead to a direct summand of rank 3.

5) If there is a unit in the first l_1 rows of β_1 then by row and column permutations we can bring this unit at position (1,1) in β_1. Then by allowed row and column transformations all the entries in the first column of β_1 and the first row in $[\beta_1|\beta_2|\beta_3]$ can be annihilated up to this unit. This would display a direct summand of rank 2.

6) If $k_1 = k_2 + 1$ and there is a unit in a row j where $l_1 \leq j \leq l_1 + l_2$ then there is a cross in the first $l_1 + l_2$ rows of β_1. By allowed row and column transformations using the unit as a pivot, the whole columns and rows in β-block up to this unit can be annihilated. This leads to a summand of rank 2.

CHAPTER 7

HOMOCYCLIC (1,3) GROUPS

Almost completely decomposable groups are direct sums of indecomposables. However, this decomposition is not unique and very complicated to deal with. Some classes of almost completely decomposable groups can be classified under near-isomorphism. Let G be a p-local almost completely decomposable group. Then G is, up to near isomorphism, uniquely a direct sum of indecomposable groups by a theorem of Faticoni-Schultz, see [5]. Hence to classify all p-reduced almost completely decomposable groups, it is sufficient to classify all indecomposable p-reduced almost completely indecomposable groups.

By Arnold's Theorem if G and H are two near isomorphic almost completely decomposable groups of finite rank then they have the same decomposition properties. By theorem 4.0.1 if the coordinate matrix of an almost completely decomposable group is decomposable then the group is decomposable. Hence our method consists in turning the decomposition question into equivalance problem for integer matrices.

Let G be a p-local, p-reduced (1,3) group with regulator R and regulator quotient G/R. Let $\delta = \begin{bmatrix}\alpha & || & \beta_1 & | & \beta_2 & | & \beta_3\end{bmatrix}$ be the coordinate matrix of G. By Proposition 27 in [14] if $[\beta_1|\beta_2]$-part of δ is decomposable, then G is decomposable.

In this chapter we will find all indecomposable (1,3)-groups with homocyclic regulator quotient of exponent p^3.

PROPOSITION 7.0.1 *The following two (1, 3)-groups G with homocyclic regulator quotient of exponent p^3 given by the isomorphism types of their regulator with fixed types, their regulator quotient and their coordinate matrix $\delta = \begin{bmatrix}\alpha & || & \beta_1 & | & \beta_2 & | & \beta_3\end{bmatrix}$ are indecomposable and pairwise not near-isomorphic.*

(i) $\delta = \begin{bmatrix} 1 & | & p^2 & | & p & | & 1 \end{bmatrix}$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3}$ *and* $rankG = 4$.

(ii) $\delta = \begin{bmatrix} 1 & 0 & | & p & | & 1 & | & 0 \\ 0 & 1 & | & 0 & | & p & | & 1 \end{bmatrix}$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_{p^3}$ *and* $rankG = 5$.

Proof 6 *(i) is obvious.*

(ii) It is enough to show that the following matrix has no 0-line modulo p^3

$$\begin{bmatrix} 1 & c \\ ap & 1 \end{bmatrix} \begin{bmatrix} p & 1 \\ 0 & p \end{bmatrix} = \begin{bmatrix} p & 1+cp \\ ap^2 & ap+p \end{bmatrix}$$

We need to show that by using allowed row operations it is not possibble to decompose $\left[\begin{smallmatrix} p & 1 \\ 0 & p \end{smallmatrix}\right]$. *The first row is never* $0 \mod p^3$. *The only possibility for the decomposition is* $a \equiv p \mod p^3$. *But then the entry at position (2,2) is not* 0 *modulo* p^3.

Denote (1,3)-groups with regulator quotient of exponent p^3 as $((1,3), p^3)$-groups.

Theorem 7.0.2 *There are precisely two near-isomorphism classes of indecomposable homocyclic* $((1,3), p^3)$*-groups as in Proposition 7.0.1.*

Proof 7 *Assume that G is a homocyclic* $((1,3), p^3)$ *group with regulator* R *and a coordinate matrix* $\delta = [\alpha||\beta_1|\beta_2|\beta_3]$. *Our aim is to find all indecomposable homocyclic* $(1,3)$*-groups that are direct summands of G. By Proposition 27 in [14] if* $[\beta_1|\beta_2]$*-part of* δ *is decomposable, then the group G is decomposable. Hence for the decomposability of a* $(1,3)$*-group it is enough to check the* $[\beta_1|\beta_2]$*-part of* δ. *Our method is to form successively the Smith Normal form's of sub-blocks of* β_1 *and* β_2. *If we find a summand then its class is either on the list given in Proposition 7.0.1 or it leads to a contradiction. By Proposition 7.0.1* $[\beta_1|\beta_2]$ *contains no* 0*-rows and there is no* 0*-column in* δ. *Furthermore, there can not be a cross or a double cross in* $[\beta_1|\beta_2]$. *Since a unit in* β_1 *leads to a cross, the Smith Normal form of* β_1 *is* $\begin{bmatrix} p^2I & 0 \\ 0 & pI \\ 0 & 0 \end{bmatrix}$. *Therefore* $[\beta_1|\beta_2]$ *is of the form*

$$[\beta_1|\beta_2] = \begin{bmatrix} p^2 I & 0 & | & A_1 \\ 0 & pI & | & A_2 \\ 0 & 0 & | & A_3 \end{bmatrix}$$

With a unit in A_3 we can annihilate in its column and in its row. But then we obtain a cross located at this unit. Hence we may write pA_3. Then if there is a unit in A_1, the entries in its row, in its column and the entries below this unit can be annihilated. This eliminations cause fill-ins in the sub-blocks of β_1 but they can removed by pI or are in $p^3\,\mathbb{Z}$ and can be neglected. Hence we may write pA_1. Thus we get

$$[\beta_1|\beta_2] = \begin{bmatrix} p^2 I & 0 & | & pA_1 \\ 0 & pI & | & A_2 \\ 0 & 0 & | & pA_3 \end{bmatrix}$$

The entries of A_2 are either units or zeros due to pI on the left. A 0-row in A_2 leads to a cross. Hence the Smith Normal form of A_2 is $\begin{bmatrix} I & 0 \end{bmatrix}$. This expands the block matrices A_2 and pA_3 and then $[\beta_1|\beta_2]$ transforms to

$$[\beta_1|\beta_2] = \begin{bmatrix} p^2 I & 0 & | & pA_1 & pA_1' \\ 0 & pI & | & I & 0 \\ 0 & 0 & | & pA_3 & pA_3' \end{bmatrix}$$

We annihilate pA_1 by I below. The resulting fill-ins in the first block of β_1 are in $p^2\,\mathbb{Z}$ and can be removed by $p^2 I$ on the left. The resulting coordinate matrix is

$$[\beta_1|\beta_2] = \begin{bmatrix} p^2 I & 0 & | & 0 & pA_1' \\ 0 & pI & | & I & 0 \\ 0 & 0 & | & pA_3 & pA_3' \end{bmatrix}$$

An entry $p \in pA_3'$ allows to annihilate in pA_3 and in pA_1'. This causes a cross in $[\beta_1|\beta_2]$. Hence we write $p^2 A_3$. The new coordinate matrix is

$$[\beta_1|\beta_2] = \begin{bmatrix} p^2I & 0 & | & 0 & pA_1' \\ 0 & pI & | & I & 0 \\ 0 & 0 & | & pA_3 & p^2A_3' \end{bmatrix}$$

The entries of pA_1' that are in $p^2\,\mathbb{Z}$ can be annihilated by p^2I on the left. There is no 0*-row in pA_1' to avoid a cross. Hence the Smith Normal form of pA_1' is $\begin{bmatrix} pI & 0 \end{bmatrix}$ and we get*

$$[\beta_1|\beta_2] = \begin{bmatrix} p^2I & 0 & | & 0 & pI & 0 \\ 0 & pI & | & I & 0 & 0 \\ 0 & 0 & | & pA_3 & p^2A_3' & p^2A_3'' \end{bmatrix}$$

The block matrix p^2A_3' can be annihilated by pI above it. The fill-ins in the last block row of β_1 are in $p^3\,\mathbb{Z}$ and can be disregarded. The resulting coordinate matrix is

$$[\beta_1|\beta_2] = \begin{bmatrix} p^2I & 0 & | & 0 & pI & 0 \\ 0 & pI & | & I & 0 & 0 \\ 0 & 0 & | & pA_3 & 0 & p^2A_3'' \end{bmatrix}$$

The first row and the columns (1) with (4) display a summand which is (i) on the list in Proposition 7.0.1. Omitting this summand we get

$$[\beta_1|\beta_2] = \begin{bmatrix} pI & | & I & 0 \\ 0 & | & pA_3 & p^2A_3'' \end{bmatrix}$$

An entry $p \in pA_3$ leads to a summand with regulator quotient exponent p^3 which is (ii) on the list in Proposition 7.0.1. Omitting this summand we may assume that the entries of A_3 are in $p^2\,\mathbb{Z}$ and we may write p^2A_3. But then p^2A_3 can be annihilated by I above it. The corresponding fill-ins in the second block row of β_1 are in $p^3\,\mathbb{Z}$ and can be neglected. The resulting coordinate matrix is

$$[\beta_1|\beta_2] = \begin{bmatrix} pI & | & I & 0 \\ 0 & | & 0 & p^2A_3'' \end{bmatrix}$$

A p^2 *in* p^2A_3'' *leads to a cross in* $[\beta_1|\beta_2]$*, a contradiction. Hence* p^2A_3'' *can be considered as* 0*-matrix. Thus, the last block row and the last block column of* $[\beta_1|\beta_2]$ *can not be present and this leads to a summand of rank* ≤ 2*, a contradiction.*

This finishes the proof.

CHAPTER 8

(1,3)-GROUPS WITH REGULATOR QUOTIENT ISOMORPHIC TO $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_p$

In this chapter we discuss the decomposability of (1,3)-groups with regulator quotient isomorphic to $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_p$.

In the following proposition, the list of indecomposable (1,3) groups G with regulator quotient $G/R \cong \mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_p$ is given.

PROPOSITION 8.0.1 *The following two (1, 3)-groups G_1 and G_2 with regulator qutioent isomorphic to $Z_{p^3} \bigoplus Z_p$ given by the isomorphism types of their regulator with fixed types, their regulator quotient and their coordinate matrix* $\delta = \begin{bmatrix} \alpha & || & \beta_1 & | & \beta_2 & | & \beta_3 \end{bmatrix}$ *are indecomposable and pairwise not near-isomorphic.*

(i)$\delta = \left[\begin{array}{cc|c|c|c} 1 & 0 & p^2 & p & 1 \\ \hline 0 & 1 & 0 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_p$ *and* $rank G_1 = 5$.

(ii)$\delta = \left[\begin{array}{cc|c|c|c} 1 & 0 & p & p & 1 \\ \hline 0 & 1 & 1 & 0 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_p$ *and* $rank G_2 = 5$.

Proof 8 *(i) Consider the following matrix:*

$$\begin{bmatrix} 1 & ap^2 \\ c & 1 \end{bmatrix} \begin{bmatrix} p^2 & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p^2 & p + ap^2 \\ cp^2 & cp + 1 \end{bmatrix} = \begin{bmatrix} p^2 & p + ap^2 \\ 0 & 1 \end{bmatrix}$$

Since the entry at position (1,1) is not 0, the first row is not 0. Similarly, since the entry at position (2,2) is not 0, the entire second column is not 0, which shows the $\left[\begin{smallmatrix} p^2 & p \\ 0 & 1 \end{smallmatrix}\right]$ *can not be decomposed.*

(ii) We take the $\begin{bmatrix}\beta_1|\beta_2\end{bmatrix}$*-part of* δ *and multiply it from the left and from the right by the matrices of the form given in Theorem 12 in [14] and get*

$$\begin{bmatrix}1 & ap^2\\ c & 1\end{bmatrix}\begin{bmatrix}p & p\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = \begin{bmatrix}p+ap^2 & b(p+ap^2)+p\\ 1 & b\end{bmatrix}$$

The entry at position (1,1) is not $0 \mod p^3$ *and the entry at position (1,2) is not* $0 \mod p$*, i.e., the first column is not 0.*

For a possible decomposition, let $b \equiv 0 \mod p^2$*. However, in this case the entry at position (1,2) is not* 0 *modulo* p^3*.*

Theorem 8.0.2 *There are two near-isomorphism classes of indecomposable* $((1,3),p^3)$ *groups with regulator qutioent isomorphic to* $Z_{p^3} \bigoplus Z_p$ *as in Proposition 8.0.1.*

Proof 9 *Assume that the group* G *is an indecomposable* $((1,3),p^3)$*-group with regulator* R *and regulator quotient* G/R *is isomorphic to* $\mathbb{Z}_{p^3} \bigoplus Z_p$*.*

Let $\delta = \begin{bmatrix}\alpha & || & \beta_1 & | & \beta_2 & | & \beta_3\end{bmatrix}$ *be the coordinate matrix of* G*. Our method consists of forming the Smith Normal forms of subblocks of* $[\beta_1|\beta_2]$ *since by Proposition 27 in [14] we know that if* $[\beta_1|\beta_2]$ *is decomposable then* G *is decomposable. If a summand is displayed, then it leads to a contradiction or we check its class in the list given in Proposition 8.0.1.*

In this way we will find all indecomposable $((1,3),p^3)$ *groups with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \bigoplus Z_p$*. Since we supposed that* G *is indecomposable,* $[\beta_1|\beta_2]$ *can not contain 0-rows, there can not be any 0-column in* δ*, and there can not be a cross or a double cross in* $[\beta_1|\beta_2]$*.*

We successively form Smith Normal form's of sub-blocks to split out the parts p^2I *and* pI*'s. Since a unit in the* p^3*-block of* β_1 *leads to a cross, the Smith Normal form of* β_1 *is* $\begin{bmatrix}p^2I & 0 & 0\\ 0 & pI & 0\\ 0 & 0 & 0\\ \hline B_1 & B_2 & B_3\end{bmatrix}$ *and* $[\beta_1|\beta_2]$ *is of the form*

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|c} p^2I & 0 & 0 & A_1 \\ 0 & pI & 0 & A_2 \\ 0 & 0 & 0 & A_3 \\ \hline B_1 & B_2 & B_3 & B_4 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p \end{array}$$

With a unit in B_3 we can annihilate in its row and in its column without any fill-ins. But then we obtain a cross located at this unit. Hence entries of B_3 are in $p\mathbb{Z}$ which can be regarded as zeros. However, zero entries of B_3 leads to a zero column and so we get

$$[\beta_1|\beta_2] = \left[\begin{array}{cc|c} p^2I & 0 & A_1 \\ 0 & pI & A_2 \\ 0 & 0 & A_3 \\ \hline B_1 & B_2 & B_4 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p \end{array}$$

If there is a unit in B_1, then we can annihilate in its column and in its row causing to a cross located at this unit. Hence the entries of B_1 is in $p\mathbb{Z}$, and are zero $\mod p$. Thus we have

$$[\beta_1|\beta_2] = \left[\begin{array}{cc|c} p^2I & 0 & A_1 \\ 0 & pI & A_2 \\ 0 & 0 & A_3 \\ \hline 0 & B_2 & B_4 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p \end{array}$$

Since by our assumption G is indecomposable, there is no unit in A_3 because this gives a summand. Hence we write pA_3. Similarly with a unit in A_1 we can annihilate in its column and in its row except for p^2I which leads to a vertical double cross located at that unit. Thus we can write pA_1. Therefore $\begin{bmatrix}\beta_1 & \beta_2\end{bmatrix}$ is of the form

$$[\beta_1|\beta_2] = \left[\begin{array}{cc|c} p^2I & 0 & pA_1 \\ 0 & pI & A_2 \\ 0 & 0 & pA_3 \\ \hline 0 & B_2 & B_4 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p \end{array}$$

The entries of B_2 are either units or zeros since it is located in p$\mathbb{Z}$ block. Hence the Smith Normal Form of B_2 is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. This expands the block matrices B_4 and A_2 then $[\beta_1|\beta_2]$ transforms to

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|c} p^2I & 0 & 0 & pA_1 \\ 0 & pI & 0 & A_2 \\ 0 & 0 & pI & A_2' \\ 0 & 0 & 0 & pA_3 \\ \hline 0 & I & 0 & B_4 \\ 0 & 0 & 0 & B_4' \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}$$

We annihilate B_4 by I left to it. The resulting fill-ins are in A_2 so they can be disregarded. The entries of B_4' are either units or zeros since it is located in p-block. Furthermore a zero row in B_4' leads to a zero row in $[\beta_1|\beta_2]$. Hence the Smith Normal Form of B_4' is $\begin{bmatrix} I & 0 \end{bmatrix}$. This expands the block matrices pA_1, A_2, A_2' and pA_3. So $[\beta_1|\beta_2]$ is transformed to

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|cc} p^2I & 0 & 0 & pA_1 & pA_1' \\ 0 & pI & 0 & A_2 & A_2'' \\ 0 & 0 & pI & A_2' & A_2''' \\ 0 & 0 & 0 & pA_3 & pA_3' \\ \hline 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}$$

With a unit in A_2 we can annihilate in its column and in its row except for pI. This annihilation leads to a direct summand. Hence we may assume that the entries of A_2 are in $p\mathbb{Z}$. Similarly we may write pA_2'' instead of A_2''. Moreover the entries of pA_2 which are in $p\mathbb{Z}$ can be annihilated by pI in β_1 in the same block row. This cause fill-ins in the 5'th block row of $[\beta_1|\beta_2]$ but they are annihilated by I below it. Hence we get

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|cc} p^2I & 0 & 0 & pA_1 & pA_1' \\ 0 & pI & 0 & 0 & pA_2'' \\ 0 & 0 & pI & A_2' & A_2''' \\ 0 & 0 & 0 & pA_3 & pA_3' \\ \hline 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}$$

Due to the presence of pI in the third block row of β_1 the entries of A_2''' are either units or zeros. Hence the Smith Normal Form of A_2''' is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. This expands the block matrices A_2', pA_1', pA_2'' and pA_3'. The matrix $[\beta_1|\beta_2]$ changes to

$$[\beta_1|\beta_2] = \left[\begin{array}{cccc|ccc} p^2I & 0 & 0 & 0 & pA_1 & pA_1' & pA_1'' \\ 0 & pI & 0 & 0 & 0 & pA_2'' & pA_2''' \\ 0 & 0 & pI & 0 & A_2 & I & 0 \\ 0 & 0 & 0 & pI & A_2' & 0 & 0 \\ 0 & 0 & 0 & 0 & pA_3 & pA_3' & pA_3'' \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}$$

Similarly the entries of A_2' are either units or zeros due to the pI block left to it. Note that a zero row in A_2' leads to a cross. Therefore the Smith Normal Form of A_2' is $\begin{bmatrix} I & 0 \end{bmatrix}$. This enlarges the block matrices pA_1, A_2 and pA_3 changing $[\beta_1|\beta_2]$ to

$$[\beta_1|\beta_2] = \left[\begin{array}{cccc|cccc} p^2I & 0 & 0 & 0 & pA_1 & pA_1''' & pA_1' & pA_1'' \\ 0 & pI & 0 & 0 & 0 & 0 & pA_2'' & pA_2''' \\ 0 & 0 & pI & 0 & A_2 & A_2' & I & 0 \\ 0 & 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pA_3 & pA_3''' & pA_3' & pA_3'' \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \\ p \end{array}$$

We may annihilate the submatrix A_2 by I below it and the resulting fill-ins in the last column of β_1-block can be annihilated by pI next to it. Moreover, A_2' can be annihilated in the same block row by I right to it. The resulting fill-ins can be either disregarded or they can be annihilated by using identity matrix in β_2 in p-block. The identity matrix in the 7'th row of β_2 can be annihilated by I above it. The resulting fill-ins in p-block are in $p\,\mathbb{Z}$ and hence can be ignored. This causes a 0-row in $[\beta_1|\beta_2]$ and will be neglected. Thus, we get

$$[\beta_1|\beta_2] = \left[\begin{array}{cccc|cccc} p^2I & 0 & 0 & 0 & pA_1 & pA_1' & pA_1'' & pA_1''' \\ 0 & pI & 0 & 0 & 0 & 0 & pA_2 & pA_2' \\ 0 & 0 & pI & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pA_3 & pA_3' & pA_3'' & pA_3''' \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}$$

With an entry in $p\,\mathbb{Z}\backslash p^2\,\mathbb{Z}$ in pA_3''', we annihilate in its column and in its row. This would lead to a cross at this entry, so we can write p^2A_3'''. An entry in pA_1''' that is in $p\,\mathbb{Z}\backslash p^2\,\mathbb{Z}$ leads to annihilation in its column and in its row except for p^2I. This causes a vertical double cross. So, we can write p^2A_1'''. The resulting $[\beta_1|\beta_2]$ is of the form

$$[\beta_1|\beta_2] = \left[\begin{array}{cccc|cccc} p^2I & 0 & 0 & 0 & pA_1 & pA_1' & pA_1'' & p^2A_1''' \\ 0 & pI & 0 & 0 & 0 & 0 & pA_2 & pA_2' \\ 0 & 0 & pI & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pA_3 & pA_3' & pA_3'' & p^2A_3''' \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}$$

The block matrix p^2A_1''' can be annilated by p^2I in the first block column in β_1. Moreover, the block matrix pA_1 can be annihilated by I in the 4'th row of β_2 which cause fill-ins in the first row fourth column of β_1, but they can be annihilated by p^2I in β_1.

Thus, the resulting matrix is,

$$
[\beta_1|\beta_2] = \left[\begin{array}{cccc|cccc}
p^2I & 0 & 0 & 0 & 0 & pA_1' & pA_1'' & 0 \\
0 & pI & 0 & 0 & 0 & 0 & pA_2 & pA_2' \\
0 & 0 & pI & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & pI & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & pA_3 & pA_3' & pA_3'' & p^2A_3''' \\
\hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0
\end{array}\right]
\begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}
$$

An entry in pA_2' that is in $p\,\mathbb{Z}\setminus p^2\,\mathbb{Z}$ leads to an annihilation in its block column and in its block row except for the pI in the second column of β_1. This leads to a summand (ii) listed in Proposition 8.0.1. Omitting this summand we may write p^2A_2'. Thus we have

$$
[\beta_1|\beta_2] = \left[\begin{array}{cccc|cccc}
p^2I & 0 & 0 & 0 & 0 & pA_1' & pA_1'' & 0 \\
0 & pI & 0 & 0 & 0 & 0 & pA_2 & p^2A_2' \\
0 & 0 & pI & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & pI & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & pA_3 & pA_3' & pA_3'' & p^2A_3''' \\
\hline
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0
\end{array}\right]
\begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}
$$

The block matrix p^2A_2' can be annihilated by pI in the second column of $[\beta_1|\beta_2]$. The resulting fill-ins are in $p\,\mathbb{Z}$ and can be disregarded since they are located in p-block. Moreover, the block matrix pA_1'' can be annihilated by I in the same block column. The resulting fill-ins are in $p^2\,\mathbb{Z}$ and can be annihilated by p^2I in the first column. Hence we get $[\beta_1|\beta_2]$ as

$$[\beta_1|\beta_2] = \left[\begin{array}{cccc|cccc} p^2I & 0 & 0 & 0 & 0 & pA_1' & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & 0 & pA_2 & 0 \\ 0 & 0 & pI & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pA_3 & pA_3' & pA_3'' & p^2A_3''' \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}$$

It is possible to annihilate pA_2 by pI in the second column of β_1. This will cause fill-ins which can be annihilated by I in the third block row. This operation leads another fill-ins in 6'th row but they can be disregarded since they are in p-block.

$$[\beta_1|\beta_2] = \left[\begin{array}{cccc|cccc} p^2I & 0 & 0 & 0 & 0 & pA_1' & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pA_3 & pA_3' & pA_3'' & p^2A_3''' \\ \hline 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \\ p \end{array}$$

The second and sixth row together with second column lead to a horizantal double cross. Hence by deleting these block rows and the second block column we obtain

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|cccc} p^2I & 0 & 0 & 0 & pA_1' & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & I & 0 \\ 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & pA_3 & pA_3' & pA_3'' & p^2A_3''' \\ \hline 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \end{array}$$

If there is an entry of pA_3' in $p\,\mathbb{Z}\setminus p^2\,\mathbb{Z}$, then it may used to annihilate the entries in the same row of this entry causing fill-ins in the 4'th, 6'th and 7'th columns. The fill-ins in the 4'th column are annihilated by I above it which cause fill-ins in the last block row but can be disregarded since they are in p-block. The fill-ins in the 6'th *column are annihilated by I above it causing fill-ins in the last block row. However, these fill-ins can be neglected. The fill-ins in 7'th column are already in $p\,\mathbb{Z}$ and can be disregarded. This entry together with I in the last block row of $[\beta_1|\beta_2]$ leads to a horizontal double cross. Hence the entires of pA_3' are in $p^2\,\mathbb{Z}$ but then pA_3' can be annihilated by I in the last block row of $[\beta_1|\beta_2]$. Hence we assume that pA_3' is zero and $[\beta_1|\beta_2]$ changed to*

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|cccc} p^2I & 0 & 0 & 0 & pA_1' & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & I & 0 \\ 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & pA_3 & 0 & pA_3'' & p^2A_3''' \\ \hline 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \end{array}$$

An entry $p \in pA_1'$ leads to a summand (i) listed in Proposition 8.0.1. Omitting this summand we may assume that the entries of A_1' are in $p^2\,\mathbb{Z}$ and we may write p^2A_1'.

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|cccc} p^2I & 0 & 0 & 0 & p^2A_1' & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & I & 0 \\ 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & pA_3 & 0 & pA_3'' & p^2A_3''' \\ \hline 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \end{array}$$

The block matrix p^2A_1' can be annihilated by I in the last block row. Hence we get

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|cccc} p^2I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & pI & 0 & 0 & 0 & I & 0 \\ 0 & 0 & pI & I & 0 & 0 & 0 \\ 0 & 0 & 0 & pA_3 & 0 & pA_3'' & p^2A_3''' \\ \hline 0 & 0 & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p \end{array}$$

The fifth block column of $[\beta_1|\beta_2]$ with the last block row leads to a cross. Deleting these columns and rows, we end up with the homocyclic case which is not possible.

This finishes the proof.

CHAPTER 9

(1, 3)-GROUPS WITH REGULATOR QUTIOENT ISOMORPHIC TO $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_{p^2}$

In this chapter, we first list the indecomposable (1,3) groups with regulator quotient isomorphic to $\mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_{p^2}$ and then we prove that there is no other indecomposable (1,3) group G with $G/R \cong \mathbb{Z}_{p^3} \bigoplus \mathbb{Z}_{p^2}$ except the ones given on the list.

PROPOSITION 9.0.1 *The following two (1, 3)-groups G_1 and G_2 with regulator qutioent isomorphic to $Z_{p^3} \bigoplus Z_{p^2}$ given by the isomorphism types of their regulator with fixed types, their regulator quotient and their coordinate matrix $\delta = \begin{bmatrix} \alpha & \| & \beta_1 & | & \beta_2 & | & \beta_3 \end{bmatrix}$ are indecomposable and pairwise not near-isomorphic.*

(i) $\delta = \left[\begin{array}{cc|c|c|c} 1 & 0 & p & 1 & 0 \\ \hline 0 & 1 & p & 0 & 1 \end{array}\right]$ *with regulator quotient isomorphic to $Z_{p^3} \bigoplus Z_{p^2}$ and rank$G_1 = 5$.*

(ii) $\delta = \left[\begin{array}{cc|c|c|c} 1 & 0 & 0 & p & 1 \\ \hline 0 & 1 & p & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to $Z_{p^3} \bigoplus Z_{p^2}$ and rank$G_2 = 5$.*

Proof 10 *(i) It is enough to state the matrix* $\begin{bmatrix} p & 1 \\ p & 0 \end{bmatrix}$ *has no 0-line modulo diag(p^3, p^2) and is not decomposable.*

Consider the following matrix $\begin{bmatrix} 1 & c \\ ap & 1 \end{bmatrix} \begin{bmatrix} p & 1 \\ p & 0 \end{bmatrix} = \begin{bmatrix} p+cp & 1 \\ ap^2+p & ap \end{bmatrix} = \begin{bmatrix} p+cp & 1 \\ p & ap \end{bmatrix}$
Since the entry at position (1,2) is not 0, the first row and the second column can not be completely zero. The entry at position (1,1) is not 0 for any value of c. If we set $a = p$ then the entry at position (2,2) is 0, but again we neither get a zero line nor a

cross.

(ii) We now multiply the matrix $\begin{bmatrix} 0 & p \\ p & 1 \end{bmatrix}$ *from left by* $\begin{bmatrix} 1 & ap \\ c & 1 \end{bmatrix}$ *and check whether the resulting matrix is decomposable modulo diag*(p^3, p^2). *Consider* $\begin{bmatrix} 1 & ap \\ c & 1 \end{bmatrix} \begin{bmatrix} 0 & p \\ p & 1 \end{bmatrix} =$ $\begin{bmatrix} ap^2 & p+ap \\ p & cp+1 \end{bmatrix}$. *If we set* $a = p$*, then the entry at position (1,1) is 0 but this does not make the entry 0 at position (1,2). The entry at position (2,2) is never 0 for any value of c. This shows that there will be no 0-row, 0-column or cross in* $[\beta_1|\beta_2]$.

Theorem 9.0.2 *There are precisely two near-isomorphism classes of indecomposable* $((1,3), p^3)$*- groups with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$ *as in Proposition 9.0.1.*

Proof 11 *Let* G *be a* $((1,3), p^3)$*-group with regulator* R *and* $G/R \cong \mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$. *Let* $\delta = [\alpha||\beta_1|\beta_2|\beta_3]$ *be the coordinate matrix of* G. *Our method consists of finding all indecomposable* $(1,3)$*-groups with regulator quotient isomorphic to* $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$ *that are direct summands of* G *by forming the Smith Normal form of the subblocks of* δ. *By Theorem 4.0.1 if* δ *is decomposable, then* G *is decomposable. Moreover, by Proposition 27 in [14] it is enough to check* $[\beta_1|\beta_2]$ *to determine the decomposability of* δ. *By Lemma 6.1.3,* $[\beta_1|\beta_2]$ *contains no 0-rows and there is no 0-column in* δ.

Note that a cross or a double cross leads to a direct summand. Hence we assume that $[\beta_1|\beta_2]$ *has no cross and no double cross. Write* $[\beta_1|\beta_2] = \begin{bmatrix} X & | & Y \\ \hline Z & | & T \end{bmatrix} \begin{matrix} p^3 \\ p^2 \end{matrix}$

where X, Y, Z, T *are block matrices.*

Since a unit in X *leads to a cross, the Smith Normal form of* X *is* $\begin{bmatrix} p^2 I & 0 & 0 \\ 0 & pI & 0 \\ 0 & 0 & 0 \end{bmatrix}$. *Therefore* $[\beta_1|\beta_2]$ *is of the form*

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|c} p^2I & 0 & 0 & A_1 \\ 0 & pI & 0 & A_2 \\ 0 & 0 & 0 & A_3 \\ \hline B_1 & B_2 & B_3 & B_4 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^2 \end{array}$$

With a unit in A_3 we can annihilate in its column and in its row. But then we obtain a cross located at this unit. Hence we may write pA_3. Due to pI in the second block row of X, the entries of A_2 are either units or zeros. Hence the Smith Normal form of A_2 is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$. Thus we get

$$[\beta_1|\beta_2] = \left[\begin{array}{cccc|cc} p^2I & 0 & 0 & 0 & A_{11} & A_{12} \\ 0 & pI & 0 & 0 & I & 0 \\ 0 & 0 & pI & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pA_{31} & pA_{32} \\ \hline pB_1 & pB_{21} & pB_{22} & pB_3 & B_{41} & B_{42} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \end{array}$$

The entries of A_{11} are in pZ due to I in the first block column of Y in p^3-block. The matrix B_{41} can be annihilated by the identity matrix I above it and pB_{22} can be annihilated by pI above it. This would lead to a cross located at pI in the X-block. Omitting this cross, we get the new coordinate matrix

$$[\beta_1|\beta_2] = \left[\begin{array}{ccc|cc} p^2I & 0 & 0 & pA_{11} & A_{12} \\ 0 & pI & 0 & I & 0 \\ 0 & 0 & 0 & pA_{31} & pA_{32} \\ \hline pB_1 & pB_{21} & pB_3 & 0 & B_{41} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^2 \end{array}$$

There is no 0-column in B_3 and so the Smith Normal form of B_3 is $\begin{bmatrix} pI \\ 0 \end{bmatrix}$ and $[\beta_1|\beta_2]$ transforms to

$$[\beta_1\beta_2] = \left[\begin{array}{ccc|cc} p^2I & 0 & 0 & pA_{11} & A_{12} \\ 0 & pI & 0 & I & 0 \\ 0 & 0 & 0 & pA_{31} & pA_{32} \\ \hline pB_1' & pB_{21'} & pI & 0 & B_{41} \\ pB_1 & pB_{21} & 0 & 0 & B_{42} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \end{array}$$

The matrices pB_1' and $pB_{21'}$ are annihilated by pI in the Z-block and we get

$$[\beta_1\beta_2] = \left[\begin{array}{ccc|cc} p^2I & 0 & 0 & pA_{11} & A_{12} \\ 0 & pI & 0 & I & 0 \\ 0 & 0 & 0 & pA_{31} & pA_{32} \\ \hline 0 & 0 & pI & 0 & B_{41} \\ pB_1 & pB_{21} & 0 & 0 & B_{42} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \end{array}$$

We form the Smith Normal Form of the submatrix pB_1 to get $\begin{bmatrix} pI & 0 \\ 0 & 0 \end{bmatrix}$. This expands the block matrices in its column and row and then $[\beta_1|\beta_2]$ transforms to

$$[\beta_1|\beta_2] = \left[\begin{array}{cccc|cc} p^2I & 0 & 0 & 0 & pA_{11} & A_{12} \\ 0 & p^2I & 0 & 0 & pA_{21} & A_{22} \\ 0 & 0 & pI & 0 & I & 0 \\ 0 & 0 & 0 & 0 & pA_{31} & pA_{32} \\ \hline 0 & 0 & 0 & pI & 0 & B_{41} \\ pI & 0 & pB_{21} & 0 & 0 & B_{42} \\ 0 & 0 & pB_{22} & 0 & 0 & B_{43} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

First we annihilate pB_{21} and then the block matrix p^2I in the first column of X-block by $pI \subset pB_1$ below. This causes a 0-row in the first block row in X. But then to avoid a cross we may assume that pA_{12}. The Smith Normal form of pB_{22} is $\begin{bmatrix} pI & 0 \\ 0 & 0 \end{bmatrix}$ and the resulting coordinate matrix is

$$[\beta_1|\beta_2] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & pA_{11} & pA_{12} & pA_{13} \\ 0 & p^2 I & 0 & 0 & 0 & A_{21} & A_{22} & A_{23} \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & pA_{31} & pA_{32} & pA_{33} \\ \hline 0 & 0 & 0 & 0 & pI & 0 & 0 & B_{41} \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & B_{42} \\ 0 & 0 & pI & 0 & 0 & 0 & 0 & B_{43} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_{44} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

If there is a unit in A_{22}, then we get a double cross. Hence the entries of $A_{22} \in p\,\mathbb{Z}$. But then pA_{22} can be annihilated by I below. The corresponding fill-ins can removed by $p^2 I$ in the second row of the X-block. To avoid a double cross, the entries of A_{23} are in $p\,\mathbb{Z}$. Moreover, it is obvious that there is no unit in B_{44} otherwise we get a cross. The new coordinate matrix is

$$[\beta_1|\beta_2] = \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & pA_{11} & pA_{12} & pA_{13} \\ 0 & p^2 I & 0 & 0 & 0 & A_{21} & 0 & pA_{23} \\ 0 & 0 & pI & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & pA_{31} & pA_{32} & pA_{33} \\ \hline 0 & 0 & 0 & 0 & pI & 0 & 0 & B_{41} \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & B_{42} \\ 0 & 0 & pI & 0 & 0 & 0 & 0 & B_{43} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pB_{44} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

Note that from here on we will write β_1 in a closed form to save place. The entries of the submatrices B_{41},B_{42}, B_{43} are either units or zero due to pI's in $\mathbb{Z}$-block. The

Smith Normal form of B_{43} is $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $[\beta_1|\beta_2]$ transforms to

$$[\beta_1|\beta_2] = \left[\begin{array}{c|ccccc} 0 & pA_{11} & pA_{11'} & pA_{12} & pA_{13} & pA_{13'} \\ p^2I & A_{21} & A_{21'} & 0 & pA_{23} & pA_{23'} \\ pI & I & 0 & 0 & 0 & 0 \\ pI & 0 & I & 0 & 0 & 0 \\ pI & 0 & 0 & I & 0 & 0 \\ 0 & pA_{31} & pA_{32} & pA_{33} & pA_{34} & pA_{35} \\ \hline pI & 0 & 0 & 0 & B_{41} & B_{41'} \\ pI & 0 & 0 & 0 & B_{42} & B_{42'} \\ pI & 0 & 0 & 0 & I & 0 \\ pI & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & pB_{44} & pB_{45} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

We can annihilate the matrices $pA_{13}, pA_{23}, B_{41}, B_{42}$ and pB_{44} by $pI \subset pB_{43}$. The fill-ins in the X and the Z-blocks can be annihilated by pI's in these blocks. Moreover, the block matrices pA_{31} and $pA_{11'}, pA_{32}$ can be annihilated by I's in the third and the fourth column of Y-block. The new coordinate matrix is

$$[\beta_1|\beta_2] = \left[\begin{array}{c|ccccc} 0 & pA_{11} & 0 & pA_{12} & 0 & pA_{13'} \\ p^2I & A_{21} & A_{21'} & 0 & 0 & pA_{23'} \\ pI & I & 0 & 0 & 0 & 0 \\ pI & 0 & I & 0 & 0 & 0 \\ pI & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & pA_{33} & pA_{34} & pA_{35} \\ \hline pI & 0 & 0 & 0 & 0 & B_{41'} \\ pI & 0 & 0 & 0 & 0 & B_{42'} \\ pI & 0 & 0 & 0 & I & 0 \\ pI & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & pB_{45} \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

We can form the Smith Normal form of the matrices $B_{41'}$ and $B_{42'}$ to get $\begin{bmatrix} I & 0 \end{bmatrix}$. This will expand the the corresponding blocks. Hence $[\beta_1|\beta_2]$ transforms to

$$[\beta_1|\beta_2] = \left[\begin{array}{c|ccccccc}
0 & pA_{11} & 0 & pA_{12} & 0 & pA_{13'} & pA_{13''} & pA_{13'''} \\
p^2 I & A_{21} & A_{21'} & 0 & 0 & 0 & pA_{23'} & pA_{23''} \\
pI & I & 0 & 0 & 0 & 0 & 0 & 0 \\
pI & 0 & I & 0 & 0 & 0 & 0 & 0 \\
pI & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & pA_{33} & pA_{34} & pA_{35} & pA_{36} & pA_{37} \\
\hline
pI & 0 & 0 & 0 & 0 & I & 0 & 0 \\
pI & 0 & 0 & 0 & 0 & 0 & I & 0 \\
pI & 0 & 0 & 0 & I & 0 & 0 & 0 \\
pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & pB_{45} & pB_{46}
\end{array}\right]
\begin{array}{c}
p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \\ p^2
\end{array}$$

By I in the second row of T block, the submatrices pB_{45} and $pA_{23'}$ can be annihilated. Since there is no cross in $[\beta_1|\beta_2]$ the entries of pA_{37} and pA_{13}''' are in $p^2\,\mathbb{Z}$. Then we can form the Smith normal form of pB_{46} to get $\begin{bmatrix} pI & 0 \end{bmatrix}$. The resulting matrix is

$$[\beta_1|\beta_2] = \left[\begin{array}{c|cccccccc}
0 & pA_{11} & 0 & pA_{12} & 0 & pA_{13'} & pA_{13''} & p^2A_{13'''} & p^2A_{14} \\
p^2 I & A_{21} & A_{21'} & 0 & 0 & 0 & 0 & pA_{23'} & pA_{23''} \\
pI & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
pI & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
pI & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & pA_{33} & pA_{34} & pA_{35} & pA_{36} & p^2A_{37} & p^2A_{37'} \\
\hline
pI & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
pI & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
pI & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0
\end{array}\right]
\begin{array}{c}
p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \\ p^2
\end{array}$$

The matrices $p^2A_{13'''}$ and p^2A_{37} can be annihilated by pI in the last block row of the T-block. If there is a $p \in pA_{13'}$, then we get a direct summand (ii) on the list in Proposition 9.0.1. with regulator quotient isomorphic to $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$. Omitting this summand we may assume that the entries of $pA_{13'}$ are all in $p^2\,\mathbb{Z}$. The same holds for

$pA_{13''}$. Thereafter the matrices $pA_{13'}$ and $pA_{13''}$ can be annihilated by the block unit matrices below them. Therefore, the new coordinate matrix is

$$[\beta_1|\beta_2] = \left[\begin{array}{c|cccccccc} 0 & pA_{11} & 0 & pA_{12} & 0 & 0 & 0 & 0 & p^2A_{14} \\ p^2I & A_{21} & A_{21'} & 0 & 0 & 0 & 0 & pA_{23'} & pA_{23''} \\ pI & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ pI & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ pI & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & pA_{33} & pA_{34} & pA_{35} & pA_{36} & 0 & p^2A_{37'} \\ \hline pI & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\ pI & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ pI & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ pI & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

A unit in $A_{21'}$ leads to a summand of rank ≤ 3 and also the entries of $A_{21'}$ which are in $p\,\mathbb{Z}$ can be annihilated by I below. Hence we may assume that $A_{21'} = 0$. But then we get a direct summand (i) on the list in Proposition 9.0.1. with regulator quotient isomorphic to $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$. Omitting this summand we get

$$[\beta_1|\beta_2] = \left[\begin{array}{c|ccccccc} 0 & pA_{11} & pA_{12} & 0 & 0 & 0 & 0 & p^2A_{14} \\ p^2I & A_{21} & 0 & 0 & 0 & 0 & pA_{23'} & pA_{23''} \\ pI & I & 0 & 0 & 0 & 0 & 0 & 0 \\ pI & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & pA_{33} & pA_{34} & pA_{35} & pA_{36} & 0 & p^2A_{37'} \\ \hline pI & 0 & 0 & 0 & I & 0 & 0 & 0 \\ pI & 0 & 0 & 0 & 0 & I & 0 & 0 \\ pI & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

A unit in A_{21} leads to a summand of rank ≤ 3 and hence the entries of A_{21} are in $p\,\mathbb{Z}$ and they can be annihilated by I below. So we assume that $A_{21} = 0$. A $p \in pA_{23''}$ leads to a direct summand of type (i) on the list in Proposition 7.0.1. But then the entries of $pA_{23''}$ are all in $p^2\,\mathbb{Z}$ and can be annihilated by p^2I on the left.

The same holds if there is a $p \in pA_{23'}$ and hence it can be annihilated by pI in the last column. Hence $pA_{23'} = 0$ and this would lead to across located at p^2I in the X-block. Omitting the row and column leading crosses, the new coordinate matrix becomes

$$[\beta_1|\beta_2] = \left[\begin{array}{c|ccccc} 0 & pA_{11} & pA_{12} & 0 & 0 & p^2A_{14} \\ pI & I & 0 & 0 & 0 & 0 \\ pI & 0 & I & 0 & 0 & 0 \\ 0 & 0 & pA_{33} & pA_{34} & pA_{35} & p^2A_{37'} \\ \hline pI & 0 & 0 & 0 & I & 0 \\ pI & 0 & 0 & I & 0 & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \\ p^2 \end{array}$$

A $p \in pA_{35}$ leads to a summand of type (ii) on the list in Proposition 9.0.1 with regulator quotient isomorphic to $\mathbb{Z}_{p^3} \oplus \mathbb{Z}_{p^2}$. Omitting this summand we may assume that $pA_{35} = 0$. Thereafter omitting the columns and rows leading to a horizontal double cross and permutating the first block row of $[\beta_1|\beta_2]$ to the fourth block row of $[\beta_1|\beta_2]$ the resulting matrix takes the form

$$[\beta_1|\beta_2] = \left[\begin{array}{cc|cccc} pI & 0 & I & 0 & 0 & 0 \\ 0 & pI & 0 & I & 0 & 0 \\ 0 & 0 & 0 & pA_{33} & pA_{34} & p^2A_{37'} \\ 0 & 0 & pA_{11} & pA_{12} & 0 & p^2A_{14} \\ \hline pI & 0 & 0 & 0 & I & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \end{array}$$

A zero column leads to a cross with a cross point located at pA_{34}. Hence the Smith normal form of pA_{34} is $\begin{bmatrix} pI \\ 0 \end{bmatrix}$ and the new coordinate matrix is

$$[\beta_1|\beta_2] = \left[\begin{array}{cc|cccc} pI & 0 & I & 0 & 0 & 0 \\ 0 & pI & 0 & I & 0 & 0 \\ 0 & 0 & 0 & pA_{33} & pI & p^2A_{37'} \\ 0 & 0 & 0 & pA_{33'} & 0 & p^2A_{37''} \\ 0 & 0 & pA_{11} & pA_{12} & 0 & p^2A_{14} \\ \hline pI & 0 & 0 & 0 & I & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \end{array}$$

The submatrices pA_{33} and $p^2A_{37'}$ can be annihilated by $pI \subset pA_{34}$. The resulting fill ins can be removed. A p in $pA_{33'}$ leads a summand with homocyclic regulator quotient. Omitting this summand and by the fact $pA_{33'}$ has no entries in $p^2\mathbb{Z}$ by I above it, we can conclude that $pA_{33'} = 0$. We can also conclude that $pA_{12} = 0$ by the same reasoning that we used to show that $pA_{33'} = 0$. But this causes to a direct summand ≤ 3. We omit this summand to get the new resulting matrix as

$$[\beta_1|\beta_2] = \left[\begin{array}{cc|cccc} pI & 0 & I & 0 & 0 & 0 \\ 0 & pI & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 \\ 0 & 0 & 0 & 0 & 0 & p^2A_{37''} \\ 0 & 0 & pA_{11} & pA_{12} & 0 & p^2A_{14} \\ \hline pI & 0 & 0 & 0 & I & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \end{array}$$

The block row $p^2A_{37''}$ does not exist to avoid a cross. The submatrix pA_{11} can be annihilated by I above it.The resulting fill ins can be removed. Hence we can assume that $pA_{11} = 0$. Then the coordinate matrix transforms to

$$[\beta_1|\beta_2] = \left[\begin{array}{cc|cccc} pI & 0 & I & 0 & 0 & 0 \\ 0 & pI & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & pI & 0 \\ 0 & 0 & 0 & pA_{12} & 0 & p^2A_{14} \\ \hline pI & 0 & 0 & 0 & I & 0 \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^3 \\ p^2 \end{array}$$

A $p \in pA_{12}$ leads to a homocyclic summand of type (ii) on the list in Proposition 7.0.1. Thereafter the block row p^2A_{14} is not present to avoid a cross. Therefore the new coordinate matrix is of the form

$$[\beta_1|\beta_2] = \left[\begin{array}{cc|ccc} pI & 0 & I & 0 & 0 \\ 0 & pI & 0 & I & 0 \\ 0 & 0 & 0 & 0 & pI \\ \hline pI & 0 & 0 & 0 & I \end{array}\right] \begin{array}{c} p^3 \\ p^3 \\ p^3 \\ p^2 \end{array}$$

The second column with the fourth column result to a direct summand of rank ≤ 3 *and hence can be omitted. But then the new coordinate matrix*

$$[\beta_1|\beta_2] = \left[\begin{array}{c|cc} pI & I & 0 \\ 0 & 0 & pI \\ \hline pI & 0 & I \end{array}\right] \quad \begin{array}{c} p^3 \\ p^3 \\ p^2 \end{array}$$

The block matrix pI *in the last block row can be annihilated by* pI *above it and this transforms* $[\beta_1|\beta_2]$ *to the form*

$$[\beta_1|\beta_2] = \left[\begin{array}{c|cc} pI & I & 0 \\ 0 & 0 & pI \\ \hline 0 & I & I \end{array}\right] \quad \begin{array}{c} p^3 \\ p^3 \\ p^2 \end{array}$$

The identity matrix in the third block column can be annihilated by I on its left and we get

$$[\beta_1|\beta_2] = \left[\begin{array}{c|cc} pI & I & I \\ 0 & 0 & pI \\ \hline 0 & I & 0 \end{array}\right] \quad \begin{array}{c} p^3 \\ p^3 \\ p^2 \end{array}$$

The identity matrix in the second block column in the p^3*-block can be annihilated by I right to it. The resulting fill-ins in the second block row can be annihilated by I in the* p^2*-block. Hence we get*

$$[\beta_1|\beta_2] = \left[\begin{array}{c|cc} pI & 0 & I \\ 0 & 0 & pI \\ \hline 0 & I & 0 \end{array}\right] \quad \begin{array}{c} p^3 \\ p^3 \\ p^2 \end{array}$$

The second block column of $[\beta_1|\beta_2]$ *with the last block row leads to a cross. Deleting these columns and rows, we end up with the homocyclic case which is not possible.*

This finishes the proof.

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