

DESCRIPTIVE COMPLEXITY OF SUBSETS OF THE SPACE OF FINITELY GENERATED GROUPS

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ABSTRACT. In this paper, we determine the descriptive complexity of subsets of the Polish space of marked groups defined by various group theoretic properties. In particular, using Grigorchuk groups, we establish that the sets of solvable groups, groups of exponential growth and groups with decidable word problem are Σ_2^0 -complete and that the sets of periodic groups and groups of intermediate growth are Π_2^0 -complete. We also provide bounds for the descriptive complexity of simplicity, amenability, residually finiteness, Hopficity and co-Hopficity. This paper is intended to serve as a compilation of results on this theme.

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1. INTRODUCTION

Under appropriate coding and identification, one can form a compact Polish space of n -generated marked groups. This space was introduced in [Gri84] with the aim of studying a Cantor set of groups with unusual properties related to growth and amenability and has been well-studied since then. (For example, see [Gri84], [Cha00], [Gri05], [CG05] and [dCGP07].) In addition to being an interesting mathematical object on its own, this space can also be used to show the existence of groups

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with certain properties, such as being amenable but not elementary amenable. (See [Ste96] and [WW17] for examples of such arguments.)

A theme that has been explored, though not studied in its full extent, is the analysis of the interplay between group theoretic properties and topological properties of the sets that they define in this space.

In this paper, we aim to collect results and open questions on this theme and initiate a systematic study for future research. We also provide some important corollaries of a classical construction of Grigorchuk and its modifications, regarding solvability, periodicity, decidability and growth properties, which, as far as we know, have remained unnoticed. In particular, we prove the following.

Theorem 1.1. *Let $n > 1$ be an integer and \mathcal{G}_n be the Polish space of n -generated marked groups. Then*

- a. *the set of solvable groups in \mathcal{G}_n is Σ_2^0 -complete.*
- b. *the set of periodic groups in \mathcal{G}_n is Π_2^0 -complete.*
- c. *the set of groups with exponential growth in \mathcal{G}_n is Σ_2^0 -complete.*
- d. *the set of groups with intermediate growth in \mathcal{G}_n is Π_2^0 -complete.*
- e. *the set of groups with decidable word problem in \mathcal{G}_n is Σ_2^0 -complete.*

This paper is organized as follows. In Section 2, we shall summarize the relevant definitions and results from descriptive set theory and recall the construction of the Polish space of n -generated marked groups, together with some convergence results in this space for expository purposes. In Section 3, we will first set up a framework to formalize the notion of “having a property P ” to prove some general facts regarding descriptive complexity, for example, to analyze the relationship between “being P ” and “being virtually P .” Then we will prove our main result and determine bounds for the descriptive complexity of various important group theoretic properties. In Section 4, we will briefly mention properties that potentially define non-Borel sets and conclude by listing some open questions that we think are of importance.

2. PRELIMINARIES

2.1. Borel hierarchy and complete sets. In this subsection, we will cover some classical results from descriptive set theory that are necessary for the remainder of this paper. We refer the reader to [Kec95] for a general background.

Recall that a *Polish space* is a completely metrizable separable topological space. For the rest of this subsection, let X be an uncountable Polish space. The Borel σ -algebra of X , i.e., the σ -algebra generated by the open subsets of X , can be stratified as follows. By transfinite recursion, for every countable ordinal $1 \leq \alpha < \omega_1$, define the following collections of subsets of X :

- $\Sigma_1^0 = \{U \subseteq X : U \text{ is open}\}$,
- $\Pi_\alpha^0 = \{X - S : S \in \Sigma_\alpha^0\}$ and $\Delta_\alpha^0 = \Sigma_\alpha^0 \cap \Pi_\alpha^0$ for every $1 \leq \alpha < \omega_1$, and
- $\Sigma_\alpha^0 = \{\bigcup A_n : A_n \in \Pi_{\gamma_n}^0, 1 \leq \gamma_n < \alpha, n \in \mathbb{N}\}$ for every $1 < \alpha < \omega_1$.

In the classical terminology, the sets in Δ_1^0 , Σ_1^0 , Π_1^0 , Σ_2^0 and Π_2^0 are the clopen, open, closed, F_σ and G_δ subsets of X respectively. It is easily checked that the Borel σ -algebra of X is exactly the union of these classes, which together constitute the *Borel hierarchy* of X . The Borel hierarchy of X can be pictured as follows, where

every class in the diagram is a subset of the classes on right of it.

$$\begin{array}{cccccc}
\Sigma_1^0 & \Sigma_2^0 & \Sigma_3^0 & \dots & \Sigma_\alpha^0 & \\
\Delta_1^0 & \Delta_2^0 & \Delta_3^0 & \dots & \Delta_\alpha^0 & \alpha < \omega_1 \\
\Pi_1^0 & \Pi_2^0 & \Pi_3^0 & \dots & \Pi_\alpha^0 &
\end{array}$$

It is well-known that, for an uncountable Polish space X , the Borel hierarchy does not collapse, i.e., the containments in the diagram above are all proper. This hierarchy provides a notion of complexity for Borel sets. Intuitively speaking, where a Borel set resides in this hierarchy measures how difficult it is to “define” this set.

Given a topological space Y and subsets $A \subseteq X$ and $B \subseteq Y$, we say that B is *Wadge reducible* to A (or, A *Wadge reduces* B), written $B \leq_W A$, if there exists a continuous map $f : Y \rightarrow X$ such that $f^{-1}[A] = B$.

Next will be introduced the notion of a complete set. Let Γ_α^0 be one of the classes Σ_α^0 and Π_α^0 . A set $A \subseteq X$ is said to be Γ_α^0 -complete if A is in Γ_α^0 and for every zero-dimensional Polish space Y and every Γ_α^0 -set $B \subseteq Y$ we have that $B \leq_W A$.

It is straightforward to check that if $A \subseteq X$ is Γ_α^0 -complete, then $X - A$ is $\overline{\Gamma_\alpha^0}$ -complete, where $\overline{\Gamma_\alpha^0}$ denotes the complementary class of Γ_α^0 . Moreover, any set Wadge reducing a Γ_α^0 -complete set is Γ_α^0 -complete. The following characterization of complete sets is well-known.

Theorem 2.1. [Kec95, 22.10] *Let X be a zero-dimensional Polish space and A be a subset of X . Then A is Σ_α^0 -complete (respectively, Π_α^0 -complete) if and only if A is in $\Sigma_\alpha^0 - \Pi_\alpha^0$ (respectively, in $\Pi_\alpha^0 - \Sigma_\alpha^0$).*

Using this, one can show that any countable dense subset A of a zero-dimensional perfect Polish space is Σ_2^0 -complete as follows. On the one hand, since singletons are closed and A is countable, A is in Σ_2^0 . On the other hand, it follows from the Baire category theorem that A is not in Π_2^0 and hence, A is Σ_2^0 -complete. For example, the set E of eventually constant sequences and the set C of recursive sequences in the Polish space $3^{\mathbb{N}} := \{0, 1, 2\}^{\mathbb{N}}$ are Σ_2^0 -complete, being both countable and dense. An example of a Π_2^0 -complete set that will be used in this paper is the set

$$I = \{(a_n) \in 3^{\mathbb{N}} : \forall \ell \exists i, j, k \geq \ell \ a_i = 0 \wedge a_j = 1 \wedge a_k = 2\}$$

that is, the set of sequences in $3^{\mathbb{N}}$ which contain infinitely many 0’s, 1’s and 2’s. Observe that I is in Π_2^0 and that the set of non-eventually constant sequences in the Cantor space $2^{\mathbb{N}}$, which is also Π_2^0 -complete, is Wadge reduced to I via the continuous map $(a_n) \mapsto (a_0, 2, a_1, 2, \dots)$. Thus, I is Π_2^0 -complete.

2.2. The Polish space of n -generated marked groups. In this subsection, following the constructions in [Gri84] and [CG05], we shall construct the Polish space of n -marked groups. In order to avoid trivialities, we shall assume for the rest of the paper that $n > 1$.

Recall that an n -marked group (G, S) consists of a group G together with a finite sequence of generators $S = (s_1, s_2, \dots, s_n)$. We allow the generating set to have repetitions and the identity element. Two n -marked groups $(G, (s_1, s_2, \dots, s_n))$ and $(H, (r_1, r_2, \dots, r_n))$ are said to be *isomorphic* if there exists a group isomorphism $f : G \rightarrow H$ such that $f(s_i) = r_i$ for all $1 \leq i \leq n$.

Let \mathbb{F}_n denote the free group with basis $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$. For every n -marked group $(G, (s_1, s_2, \dots, s_n))$, there is a canonical epimorphism $f : \mathbb{F}_n \rightarrow G$ such that $f(\gamma_i) = s_i$ for all $1 \leq i \leq n$. Therefore, every n -marked group induces a normal

subgroup \mathbb{F}_n , namely, the kernel of f . Conversely, every normal subgroup $N \trianglelefteq \mathbb{F}_n$ induces an n -marked group, namely, $(\mathbb{F}_n/N, (\gamma_1 N, \gamma_2 N, \dots, \gamma_n N))$. For the sake of readability, we shall denote the marked group $(\mathbb{F}_n/N, (\gamma_1 N, \gamma_2 N, \dots, \gamma_n N))$ by (G_N, S_N) .

It is readily verified that two n -marked groups are isomorphic if and only if they induce the same normal subgroup of \mathbb{F}_n . Consequently, for the purpose of coding these groups as the points of a Polish space, we may assume without loss of generality that the set of n -marked groups is the set

$$\mathcal{G}_n = \{N \in \mathcal{P}(\mathbb{F}_n) : N \trianglelefteq \mathbb{F}_n\}$$

One may endow $\mathcal{P}(\mathbb{F}_n)$ with a natural Polish topology induced by the (ultra)metric

$$d(N, K) = \begin{cases} 0 & \text{if } N = K \\ 2^{-\min\{i: \mathbf{g}_i \in N \Delta K\}} & \text{if } N \neq K \end{cases}$$

where $(\mathbf{g}_i)_{i \in \mathbb{N}}$ is a fixed enumeration of elements of \mathbb{F}_n . This metric is equivalent to the metric defined in [CG05, 2.2.a]. Moreover, if one also identifies $\mathcal{P}(\mathbb{F}_n)$ with the Cantor space $\{0, 1\}^{\mathbb{F}_n}$ consisting of binary sequences indexed by \mathbb{F}_n , this metric is compatible with the product topology on $\{0, 1\}^{\mathbb{F}_n}$ where each component has the discrete topology.

It is straightforward to check that the set \mathcal{G}_n is a closed subset of $\mathcal{P}(\mathbb{F}_n)$ and hence forms a compact zero-dimensional Polish space under the subspace topology. This topology is sometimes referred to as Grigorchuk topology or Chabauty topology. (See [Cha50].)

Next will be introduced two other metrics that induce the topology of \mathcal{G}_n . For every N in \mathcal{G}_n , let $W(N, k)$ denote the set of elements of N with length at most k . Consider the metric on \mathcal{G}_n given by

$$\nu(N, K) = \begin{cases} 0 & \text{if } N = K \\ 2^{-\max\{k: W(N, k) = W(K, k)\}} & \text{if } N \neq K \end{cases}$$

That is, the distance between the marked groups (G_N, S_N) and (G_K, S_K) is 2^{-k} if and only if k is the maximum integer such that the sets of words with symbol set $\{\gamma_1^{\pm 1}, \dots, \gamma_n^{\pm 1}\}$ of length at most k representing the identity in G_N and G_K respectively are the same.

Consider the Cayley graph (G_N, S_N) as a rooted directed labeled graph where the root is the identity element and the label set is $\{\gamma_1, \gamma_2, \dots, \gamma_n\}$. Let $\mathcal{B}[N, k]$ denote the closed ball with radius k centered at the root in this Cayley graph. Consider the metric on \mathcal{G}_n given by

$$\mu(N, K) = \begin{cases} 0 & \text{if } N = K \\ 2^{-\max\{k: \mathcal{B}[N, k] \cong \mathcal{B}[K, k]\}} & \text{if } N \neq K \end{cases}$$

where isomorphism is understood as the isomorphism of rooted directed labelled graphs. That is, the distance between two marked groups is 2^{-k} if and only if k is the maximum integer such that the closed balls with radius k centered at the identity element in the Cayley graphs of these marked groups are isomorphic.

A moment's thought reveals that $W(N, 2k+1) = W(K, 2k+1)$ if and only if $\mathcal{B}[N, k] \cong \mathcal{B}[K, k]$, that is, $\nu(N, K) \leq 2^{-(2k+1)}$ if and only if $\mu(N, K) \leq 2^{-k}$. It follows that μ and ν are equivalent metrics. Moreover, it is not difficult to check that μ and ν are compatible with the topology of \mathcal{G}_n . Since some of our arguments

are easier to understand if one adopts the suitable metric, we shall be working with any of μ , ν and d when needed.

Let us mention some important notational remarks. For the rest of the paper, while referring to points of the space \mathcal{G}_n , we may abuse the notation and simply write an abstract marked group (G, S) while we indeed refer to an element N of \mathcal{G}_n such that (G_N, S_N) is isomorphic to (G, S) . We will also write $B_\eta(N, k)$ to denote the open ball centered at N with radius k with respect to the metric η , where η may be any of ν , μ or d .

Note that, the Polish space \mathcal{G}_1 is homeomorphic to $\{\frac{1}{k} : k \in \mathbb{N}^+\} \cup \{0\}$ together with the Euclidean topology, where 0 corresponds to the infinite cyclic and $\frac{1}{k}$ correspond to the finite cyclic groups. Observe also that the Polish space \mathcal{G}_n embeds into \mathcal{G}_{n+1} via the map $(G, (s_1, \dots, s_n)) \mapsto (G, (s_1, \dots, s_n, 1_G))$.

Having introduced three compatible metrics with the topology of \mathcal{G}_n , we shall next recall some simple facts regarding limits of various sequences of marked groups in \mathcal{G}_n . Most of these appear either explicitly or implicitly in [Gri84]. For the self-containment of this paper as a survey, we shall provide the proofs of these facts, some of which appear to not have been explicitly written.

Let P be a property of groups. Recall that a group G is said to be *fully residually* P if for every positive integer m and any distinct elements $g_1, \dots, g_m \in G$, there exist a group H with the property P and a homomorphism $\varphi : G \rightarrow H$ such that $\varphi(g_1), \dots, \varphi(g_m) \in H$ are distinct. It is readily verified that if the property P is preserved under (finite) direct products, then the class of fully residually P groups coincides with the class of *residually* P groups, that is, those groups that satisfy the condition stated above for $m = 2$. It turns out that fully residually P groups are limits of groups with the property P in \mathcal{G}_n .

Proposition 1. [CG05] *Suppose that P is a property of groups which is inherited by subgroups. Let $N \in \mathcal{G}_n$ be such that G_N is fully residually P . Then (G_N, S_N) is a limit of marked groups with the property P .*

Proof. Let m be a positive integer. Since G_N is fully residually P , there exist a group H_m with the property P and a homomorphism $\varphi_m : G_N \rightarrow H_m$ such that the images of the elements in $\mathcal{B}[N, m]$ under φ_m are all distinct. Now set K_m to be the subgroup $\langle \varphi_m[S_N] \rangle \leq H_m$. Then $\mathcal{B}[N, m]$ embeds into the Cayley graph of $(K_m, \varphi_m[S_N])$. Moreover, by the inheritance of P by subgroups, K_m has the property P . It can be checked that

$$\mu((G_N, S_N), (K_m, \varphi_m[S_N])) \leq 2^{-m}$$

This shows that G_N is a limit of groups with the property P . □

We will now prove that any group in \mathcal{G}_n is a limit of finitely presented groups and that finitely presented groups have neighborhoods which entirely consists of their quotients.

Proposition 2. [Gri84] *For every $N \in \mathcal{G}_n$, there exists a sequence of finitely presented groups with limit (G_N, S_N) .*

Proof. Let N be \mathcal{G}_n and $(w_i)_{i \in \mathbb{N}}$ be an enumeration of the words in N . For every $m \in \mathbb{N}$, set N_m to be the normal closure of $\{w_i : 0 \leq i \leq m\}$ in \mathbb{F}_n , that is, $N_m \in \mathcal{G}_n$ is such that

$$G_{N_m} = \langle \gamma_1, \dots, \gamma_n \mid w_i, 0 \leq i \leq m \rangle$$

It is straightforward to check that, with respect to ν , the sequence $(N_m)_{m \in \mathbb{N}}$ converges to N . \square

Proposition 3. [CG05, Lemma 2.3] *Let $N \in \mathcal{G}_n$ be such that G_N is finitely presented. Then there exists $\epsilon > 0$ such that every element of $B_\mu(N, \epsilon)$ is a quotient of G_N .*

Proof. Let $\langle \gamma_1 \dots, \gamma_n \mid w_i, 0 \leq i \leq m \rangle$ be a finite presentation of G_N . Set k to be $\max\{|w_i|, 0 \leq i \leq m\}$ and choose $0 < \epsilon < 2^{-k}$. Then, by definition, for every $M \in B_\mu(N, \epsilon)$, the words w_i where $1 \leq i \leq m$ are relations of G_M and hence, G_M is a quotient of G_N . \square

Before concluding this section, we shall introduce a definition that is used to characterize groups which are limits of groups with a specific property. Let P be a property of groups. A group G is called *locally embeddable into groups with the property P* (in short, LEP) if for every finite subset $E \subseteq G$ there exist a group H with the property P and a *local embedding on E to H* , that is, a map $\phi : E \cup (E \cdot E) \rightarrow H$ such that ϕ is injective on E and $\phi(gh) = \phi(g)\phi(h)$ for every $g, h \in E$.

Proposition 4. [VG97] *Let P be a property of groups which is inherited by subgroups. Then, $G_N \in \mathcal{G}_n$ is a limit of marked groups in \mathcal{G}_n with the property P if and only if G_N is LEP.*

Proof. Suppose G_N is LEP. As G_N is countable, there exist finite subsets $E_1 \subseteq E_2 \subseteq \dots$ such that $G_N = \bigcup_{k \in \mathbb{N}^+} E_k$. Since G is LEP, for every $k \in \mathbb{N}^+$, we have a local embedding $\phi_k : E_k \rightarrow F_k$ to a group F_k with the property P . Let k be large enough so that $S_N \subseteq E_k$. Set $H_k = \langle \phi_k[S] \rangle \leq F_k$. Then the marked group $(H_k, \phi[S_k])$ belongs to \mathcal{G}_n and, by the inheritance of P by subgroups, H_k has the property P .

We claim that the sequence $(H_k, \phi_k[S])$ converges to (G_N, S_N) . Given $m > 0$, let k be large enough so that E_k contains $\mathcal{B}[N, m]$. Thus, $\mathcal{B}[N, m]$ embeds into the Cayley graph of $(H_k, \phi_k[S])$. Thus we have

$$\mu((G_N, S_N), (H_k, \phi_k[S_N])) \leq 2^{-m}$$

Conversely, suppose that there is a sequence $(H_k, S_k) = (G_{N_k}, S_{N_k})$ of groups with the property P in \mathcal{G}_n converging to (G_N, S_N) . Given any finite subset E of G , let m be large enough so that $E \cdot E \subseteq \mathcal{B}[N, m]$ and let k be large enough so that $\mathcal{B}[N, m] \cong \mathcal{B}[N_k, m]$. Using this isomorphism, one can easily define a local embedding $\phi : E \cup (E \cdot E) \rightarrow H_k$. \square

3. GROUP THEORETIC PROPERTIES AS TOPOLOGICAL PROPERTIES

In this section, we shall analyze the descriptive complexity of subsets of \mathcal{G}_n defined by group theoretic properties. In other words, we will determine at which level of the Borel hierarchy the set

$$\{N \in \mathcal{G}_n : G_N \text{ is } P\}$$

resides for various properties P .

3.1. Definability of properties. In order to formalize the notion of “a group being P ” for a property P , we are going to use the first-order infinitary logic $\mathcal{L}_{\omega_1\omega}$ in the language of group theory together with constant symbols for every element of \mathbb{F}_n . Recall that, one is allowed to have finite quantifications but countable conjunctions and disjunctions in $\mathcal{L}_{\omega_1\omega}$. Let φ be an $\mathcal{L}_{\omega_1\omega}$ -sentence. We shall say that a property P is defined by φ in the space \mathcal{G}_n if

$$\{N \in \mathcal{G}_n : G_N \models \varphi\} = \{N \in \mathcal{G}_n : G_N \text{ is } P\}$$

where the constant symbols for elements of \mathbb{F}_n are interpreted in the obvious way, i.e. g is interpreted as gN in the quotient group $G_N = \mathbb{F}_n/N$. A simple but important observation is that, since the underlying structures are countable and each element is named by a constant, any sentence is equivalent to a quantifier-free sentence in each G_N , as one can replace universal quantifiers by appropriate conjunctions and existential quantifiers by appropriate disjunctions, using the constant symbols.

Before we proceed, let us mention why restricting our attention to properties that are defined by some $\mathcal{L}_{\omega_1\omega}$ -sentence is sufficient. We are mostly interested in algebraic properties that define Borel sets in \mathcal{G}_n . We shall see later that each Borel set in \mathcal{G}_n can be defined by some quantifier-free $\mathcal{L}_{\omega_1\omega}$ -sentence in this augmented language.

3.2. Some group theoretic properties. In this subsection, we shall determine the exact descriptive complexity of being finite, finitely presented, abelian, nilpotent and torsion-free. The results in this section are all folklore.

3.2.1. Being finite. Let $N \in \mathcal{G}_n$ be such that G_N is finite. Set k to be the radius of the Cayley graph of (G_N, S_N) . Since the closed ball $\mathcal{B}[N, k]$ completely determines the group operation, we have that $B_\mu(N, 2^{-(k+1)}) = \{N\}$. It follows that being finite is a Σ_1^0 -property (and that finite groups are isolated.)

Moreover, it is easily seen that the limit of the sequence of the finite cyclic groups $(\mathbb{Z}/k\mathbb{Z}, (1))$ is $(\mathbb{Z}, (1))$ and hence, being finite is not a Π_1^0 -property. Therefore, the set of finite groups in \mathcal{G}_n is Σ_1^0 -complete.

3.2.2. Being finitely presented. Since there are countably many finitely presented marked groups, the set of finitely presented groups in \mathcal{G}_n is clearly in Σ_2^0 . On the other hand, it follows from Proposition 2 that the isolated points of \mathcal{G}_n are finitely presented and that the set of finitely presented groups is dense in the perfect part of \mathcal{G}_n . It then follows from a Baire category argument that this set cannot be in Π_2^0 . Thus the set of finitely presented groups in \mathcal{G}_n is in Σ_2^0 -complete.

3.2.3. Being abelian. It is clear that the property of being abelian is defined by the sentence

$$\bigwedge_{1 \leq i, j \leq n} [\gamma_i, \gamma_j] = e$$

Since each sentence of the form $[\gamma_i, \gamma_j] = e$ defines a Δ_1^0 -subset of \mathcal{G}_n , we have that the set of abelian groups in \mathcal{G}_n is Δ_1^0 .

This fact can also be observed as follows. For every $K, N \in \mathcal{G}_n$, if $\nu(K, N) < 2^{-3}$, then $W(N, 4) = W(K, 4)$, in which case the sentences of the form $[\gamma_i, \gamma_j] = e$ that are satisfied in G_N and G_K are the same. Thus, any ν -open ball with radius 2^{-3} of an abelian (respectively, non-abelian) group consists of abelian (respectively, non-abelian) groups.

3.2.4. *Being nilpotent.* Note that the property of being nilpotent is defined by the sentence

$$\bigvee_{k \in \mathbb{N}^+} \left(\bigwedge_{1 \leq i_1, \dots, i_{k+1} \leq n} [\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_{k+1}}] = e \right)$$

Each sentence of the form $[\gamma_{i_1}, \gamma_{i_2}, \dots, \gamma_{i_{k+1}}] = e$ defines a Δ_1^0 -subset of \mathcal{G}_n and hence, being nilpotent of class at most k is a Δ_1^0 -property and being nilpotent is a Σ_1^0 -property. On the other hand, since non-abelian free groups are fully residually finite- p (for any prime p) [Hal50], by Proposition 1, there exists a sequence of finite nilpotent groups converging to a non-abelian free group. This shows that being nilpotent is not a Π_1^0 -property and hence, the set of nilpotent groups in \mathcal{G}_n is Σ_1^0 -complete.

3.2.5. *Being torsion-free.* By definition, the set of torsion-free groups in \mathcal{G}_n is defined by the sentence

$$\bigwedge_{\delta \in \mathbb{F}_n} \left(e = \delta \vee \left(\bigwedge_{k \in \mathbb{N}^+} \delta^k \neq e \right) \right)$$

Since each sentence of the form $\delta^k \neq e$ defines a Δ_1^0 -subset of \mathcal{G}_n , the set of torsion-free groups in \mathcal{G}_n is in Π_1^0 . However, as before, since a sequence of non torsion-free (in particular, finite) groups can converge to a non-abelian free group in \mathcal{G}_n , this set is not in Σ_1^0 . Thus, the set of torsion-free groups in \mathcal{G}_n is Π_1^0 -complete.

3.3. Being P v. being virtually P. Having set up a formal framework for definability of properties using an infinitary logic, as an illustration of how this can be used to prove general facts regarding descriptive complexity of properties, we will now analyze the relationship between being P and being virtually P in this subsection. Recall that given a property P of groups, a group G is said to be *virtually P* if G has a finite index subgroup with the property P .

Before we prove the main result of this subsection, let us give a stratification of a special subclass of quantifier-free formulas of $\mathcal{L}_{\omega_1\omega}$. This stratification will allow us to prove that any Borel set in \mathcal{G}_n is defined by some $\mathcal{L}_{\omega_1\omega}$ sentence, a fact promised earlier. We would like to remind the reader that we have a constant symbol for each element of \mathbb{F}_n , which is necessary for this fact to hold.

Let $\Sigma_0^{\text{qf}} = \Pi_0^{\text{qf}}$ be the set of quantifier-free $\mathcal{L}_{\omega_1\omega}$ -sentences that are finite Boolean combinations of atomic sentences. For every $1 \leq \alpha < \omega_1$, we inductively define $\Sigma_\alpha^{\text{qf}}$ (respectively, Π_α^{qf}) to be the set of countable disjunctions (respectively, conjunctions) of sentences in $\bigcup_{0 \leq \beta < \alpha} \Pi_\beta^{\text{qf}}$ (respectively, $\bigcup_{0 \leq \beta < \alpha} \Sigma_\beta^{\text{qf}}$.) Observe that the sets of the form

$$\begin{aligned} \{N \in \mathcal{G}_n : g \in N\} &= \{N \in \mathcal{G}_n : G_N \models g = e\} \\ \{N \in \mathcal{G}_n : g \notin N\} &= \{N \in \mathcal{G}_n : G_N \models g \neq e\} \end{aligned}$$

where g ranges over \mathbb{F}_n , form a clopen subbase for the topology of \mathcal{G}_n . Thus the sets defined by finite Boolean combinations of atomic sentences form a clopen basis for the topology of \mathcal{G}_n . It follows that any set in Σ_1^0 is defined by some sentence in Σ_1^{qf} and any set in Π_1^0 is defined by some sentence in Π_1^{qf} . It can now be proven by induction that every set in Σ_α^0 (respectively, Π_α^0) is defined by some quantifier-free sentence in $\Sigma_\alpha^{\text{qf}}$ (respectively, Π_α^{qf} .) Conversely, another inductive argument implies that any sentence in $\Sigma_\alpha^{\text{qf}}$ (respectively, Π_α^{qf}) defines a set in Σ_α^0 (respectively, Π_α^0 .)

We would like to note that a better stratification of a broader class formulas of $\mathcal{L}_{\omega_1\omega}$ can be given. For example, see [Osi20, Section 3.2] for such a stratification and [Osi20, Lemma 5.3] for the counterpart of the fact we proved above. That said, our somewhat primitive stratification is going to suffice for our purposes.

We have the following proposition, whose proof will be partly used later to analyze the descriptive complexity of residual finiteness.

Lemma 3.1. *Suppose that “being P ” is in Σ_α^0 (respectively, Π_α^0) for all $n \in \mathbb{N}^+$. Then, for each integer $n > 1$, “being virtually P ” is in*

- Σ_3^0 whenever $\alpha < 3$.
- Σ_α^0 (respectively, $\Sigma_{\alpha+1}^0$) whenever $\alpha \geq 3$.

Proof. As mentioned before, for each $n \in \mathbb{N}^+$, we can choose some quantifier-free sentence φ_n in $\Sigma_\alpha^{\text{qf}}$ (respectively, Π_α^{qf}) which defines “being P ” in \mathcal{G}_n . Let $n > 1$ and fix an enumeration $(A_i)_{i \in \mathbb{N}}$ of non-empty finite subsets of \mathbb{F}_n . Fix $i \in \mathbb{N}$, set $p_i = |A_i|$ and let $A_i = \{a_1, \dots, a_{p_i}\}$.

Consider the sentence ϕ_{A_i} obtained from φ_{p_i} after

- replacing each γ_j by the word a_j for every $1 \leq j \leq p_i$, and
- replacing the other elements of \mathbb{F}_{p_i} by words in $\langle A_i \rangle$ accordingly.

Observe that an atomic sentence is turned into another atomic sentence $\mathcal{L}_{\omega_1\omega}$. Consequently, one can show by induction on the complexity of sentences that ϕ_{A_i} is in $\Sigma_\alpha^{\text{qf}}$ (respectively, Π_α^{qf}) and hence the set $\{N \in \mathcal{G}_n : G_N \models \phi_{A_i}\}$ is also in Σ_α^0 (respectively, Π_α^0 .)

For each $N \in \mathcal{G}_n$, let H_{N, A_i} denote the finitely generated subgroup

$$\langle A_i \rangle N / N = \langle a_j N : 1 \leq j \leq p_i \rangle \leq G_N$$

Let $N \in \mathcal{G}_n$. Then the marked group $(H_{N, A_i}, (a_1 N, \dots, a_{p_i} N))$ is isomorphic to the marked group

$$(G_{K_{N, A_i}}, S_{K_{N, A_i}}) = (\mathbb{F}_{p_i} / K_{N, A_i}, (\gamma_1 K_{N, A_i}, \gamma_2 K_{N, A_i}, \dots, \gamma_{p_i} K_{N, A_i}))$$

where K_{N, A_i} is the kernel of the canonical surjection from \mathbb{F}_{p_i} to H_{N, A_i} extending the map $\gamma_j \mapsto a_j N$. Moreover, for every word w with symbols from $\{\gamma_1^{\pm 1}, \dots, \gamma_{p_i}^{\pm 1}\}$, the word w is in \widehat{N} if and only if the word obtained from w after replacing each γ_j by a_j is in N . It follows that the set

$$\begin{aligned} \{N \in \mathcal{G}_n : H_{N, A_i} \text{ is } P\} &= \{N \in \mathcal{G}_n : G_{K_{N, A_i}} \text{ is } P\} \\ &= \{N \in \mathcal{G}_n : G_{K_{N, A_i}} \models \varphi_{p_i}\} \\ &= \{N \in \mathcal{G}_n : G_N \models \phi_{A_i}\} \end{aligned}$$

is in Σ_α^0 (respectively, Π_α^0 .) Now, we want to understand the complexity of the set of N 's for which H_{N, A_i} is of finite index in G_N . Note that, for every integer $j \geq 2$, we have $[G_N : H_{N, A_i}] < j$ if and only if $[\mathbb{F}_n : \langle A_i \rangle N] < j$. Thus

$$\begin{aligned} &\{N \in \mathcal{G}_n : [G_N : H_{N, A_i}] < j\} \\ &= \{N \in \mathcal{G}_n : [\mathbb{F}_n : \langle A_i \rangle N] < j\} \\ &= \{N \in \mathcal{G}_n : \forall g_1, \dots, g_j \in \mathbb{F}_n \exists k \neq \ell \in \{1, \dots, j\} \exists \delta_{k\ell} \in \langle A_i \rangle \delta_{k\ell}^{-1} g_k^{-1} g_\ell \in N\} \\ &= \bigcap_{g_1, \dots, g_j \in \mathbb{F}_n} \bigcup_{1 \leq k \neq \ell \leq j} \bigcup_{\delta_{k\ell} \in \langle A_i \rangle} \{N \in \mathcal{G}_n : \delta_{k\ell}^{-1} g_k^{-1} g_\ell \in N\} \end{aligned}$$

Since the sets $\{N \in \mathcal{G}_n : \delta_{k\ell}^{-1} g_k^{-1} g_\ell \in N\}$ are clopen for every k and ℓ , the set defined above is in $\mathbf{\Pi}_2^0$.

We claim that, for every $N \in \mathcal{G}_n$, the group G_N has a subgroup of finite index if and only if there exists $i \in \mathbb{N}$ such that $[G_N : H_{N,A_i}] \leq |A_i|$. Suppose that $L \leq G_N$ is of finite index. So $L = K/N$ for some $K \geq N$ with $[\mathbb{F}_n : K] = [G_N : L]$. Therefore $K = \langle A_i \rangle = \langle A_i \rangle N$, that is, $L = H_{N,A_i}$ for some $i \in \mathbb{N}$. By Schreier index formula

$$|A_i| \geq \text{rank}(K) = [\mathbb{F}_n : K](n-1) + 1 \geq [\mathbb{F}_n : K] = [G_N : L]$$

It follows that

$$\begin{aligned} \{N \in \mathcal{G}_n : G_N \text{ is virtually } P\} = \\ \bigcup_{i=0}^{\infty} \{N \in \mathcal{G}_n : H_{N,A_i} \text{ is } P \text{ and } [G_N : H_{N,A_i}] \leq |A_i|\} = \\ \bigcup_{i=0}^{\infty} \{N \in \mathcal{G}_n : H_{N,A_i} \text{ is } P\} \cap \{N \in \mathcal{G}_n : [G_N : H_{N,A_i}] \leq |A_i|\} \end{aligned}$$

which can easily be checked to be in Σ_3^0 whenever $\alpha < 3$, and in Σ_α^0 (respectively, $\Sigma_{\alpha+1}^0$) whenever $\alpha \geq 3$. \square

3.4. Solvability, periodicity and some growth properties. In this subsection, we shall determine the exact descriptive complexity of solvability, periodicity and having various growth properties. For a general background regarding the notion of the growth of a finitely generated group, we refer the reader to [Man12].

For later use, we shall first observe that the set of groups with polynomial growth in \mathcal{G}_n is in Σ_1^0 (and indeed is Σ_1^0 -complete.) By Gromov's theorem [Gro81], the class of groups with polynomial growth coincides with the class of finitely generated virtually nilpotent groups. On the other hand, finitely generated virtually nilpotent groups are finitely presented and hence, by Proposition 3, every virtually nilpotent group in \mathcal{G}_n has a neighborhood which entirely consists of its quotients, which are also virtually nilpotent. Thus, the set of virtually nilpotent (and hence, polynomial growth) groups in \mathcal{G}_n is in Σ_1^0 . On the other hand, this set is not in $\mathbf{\Pi}_1^0$ since, as before, there exists a sequence of polynomial growth (in fact, finite) groups whose limit is an exponential growth (in fact, non-abelian free) group. Thus, the set of groups with polynomial growth in \mathcal{G}_n is Σ_1^0 -complete.

We are now ready to state our main theorem.

Theorem 3.2. *Let $n > 1$ be an integer. Then*

- a. *the set of solvable groups in \mathcal{G}_n is Σ_2^0 -complete.*
- b. *the set of periodic groups in \mathcal{G}_n is $\mathbf{\Pi}_2^0$ -complete.*
- c. *the set of groups with exponential growth in \mathcal{G}_n is Σ_2^0 -complete.*
- d. *the set of groups with intermediate growth in \mathcal{G}_n is $\mathbf{\Pi}_2^0$ -complete.*
- e. *the set of groups with decidable word problem in \mathcal{G}_n is Σ_2^0 -complete.*

Proof. Observe that the property of being solvable is defined by the sentence

$$\bigvee_{k \in \mathbb{N}^+} \left(\bigwedge_{\delta \in \mathbb{F}_n^{(k)}} \delta = e \right)$$

Each sentence $\delta = e$ defines a Δ_1^0 -subset of \mathcal{G}_n and hence, being solvable of degree at most k is a Π_1^0 -property and being solvable is a Σ_2^0 -property.

By definition, the property of being periodic is defined by the sentence

$$\bigwedge_{\delta \in \mathbb{F}_n} \left(\bigvee_{k \in \mathbb{N}^+} \delta^k = e \right)$$

Since each sentence $\delta^k = e$ defines a Δ_1^0 -subset of \mathcal{G}_n , being periodic is a Π_2^0 -property.

A marked group (G_N, S_N) has exponential growth if there exist a real (equivalently, rational) number $a > 1$ with

$$\Gamma_N(x) \geq a^x$$

for every $x \in \mathbb{N}^+$, where the growth function $\Gamma_N(x)$ is the number of vertices in $\mathcal{B}[N, x]$. In other words, the set of groups in \mathcal{G}_n with exponential growth is

$$\bigcup_{a \in \mathbb{Q}_{>1}} \bigcap_{x \in \mathbb{N}^+} \{N \in \mathcal{G}_n : \Gamma_N(x) \geq a^x\}$$

Observe that, for every fixed rational $a > 1$ and fixed $x \in \mathbb{N}^+$, the inner-most set defines a Δ_1^0 -set because $\Gamma_N(x) = \Gamma_K(x)$ whenever $\mu(N, K) \leq 2^{-x}$. Hence, having exponential growth is a Σ_2^0 -property. (We would like to note that, the argument above indeed shows that, given two functions $f, g : \mathbb{N}^+ \rightarrow \mathbb{R}$, the set of groups in \mathcal{G}_n with growth function $\Gamma_N(x)$ satisfying $f(x) \leq \Gamma_N(x) \leq g(x)$ for all $x \in \mathbb{N}^+$ is in Σ_2^0 .)

By definition, the set of groups with intermediate growth in \mathcal{G}_n is the intersection of the complements of the sets of exponential growth groups and polynomial growth groups in \mathcal{G}_n , which are in Π_2^0 and Π_1^0 respectively. It follows that this set is in Π_2^0 . The set of groups with decidable word problem is countable and hence is in Σ_2^0 .

We shall next show that the sets of solvable groups, periodic groups, groups with exponential growth, groups with intermediate growth and groups with decidable word problem are complete in their respective point classes.

Let E and C denote the set of eventually constant sequences and the set of recursive sequences in $3^{\mathbb{N}}$ respectively. Let I denote the set of sequences in $3^{\mathbb{N}}$ which contain infinitely many 0's, 1's and 2's. We have observed in Section 2.1 that E and C are Σ_2^0 -complete and I is Π_2^0 -complete. It follows from Grigorchuk's construction [Gri84] and its slight modification in [BG14] that, for each $n \geq 2$, there is a continuous function $f : 3^{\mathbb{N}} \rightarrow \mathcal{G}_n$ such that

- $f[E]$ consists of solvable groups of exponential growth,
- $f[3^{\mathbb{N}} - E]$ consists of intermediate growth (and hence, non-solvable) groups,
- $f(\alpha)$ is periodic if and only if $\alpha \in I$,
- $f(\alpha)$ has decidable word problem if and only if $\alpha \in C$.

See Appendix A for the details of this construction. In other words, E is Wadge reducible to both the set of solvable groups and the set of groups with exponential growth, C is Wadge reducible to the set of groups with decidable word problem, $3^{\mathbb{N}} - E$ is Wadge reducible to the set of groups with intermediate growth and I is Wadge reducible to the set of periodic groups in \mathcal{G}_n . This completes the proof. \square

3.5. More group theoretic properties. In this subsection, we shall provide some upper and lower bounds for the descriptive complexities of the properties of being simple, amenable, residually finite, Hopfian and co-Hopfian.

3.5.1. *Being simple.* Recall that a group is simple if and only if the normal closure of any non-identity element is the whole group. In other words, the property of being simple is defined by the sentence

$$\bigwedge_{\gamma \in \mathbb{F}_n} \left(e = \gamma \vee \bigwedge_{\delta \in \mathbb{F}_n} \bigvee_{k \in \mathbb{N}^+} \left(\bigvee_{\delta_1, \dots, \delta_k \in \mathbb{F}_n} \bigvee_{\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}} \delta = (\gamma^{\delta_1})^{\epsilon_1} \dots (\gamma^{\delta_k})^{\epsilon_k} \right) \right)$$

Observe that each sentence $\delta = (\gamma^{\delta_1})^{\epsilon_1} \dots (\gamma^{\delta_k})^{\epsilon_k}$ defines a Δ_1^0 -subset of \mathcal{G}_n and hence, the set of simple groups in \mathcal{G}_n is in Π_2^0 . We conjecture that this set is indeed Π_2^0 -complete. While we were not able to prove this, one can easily show that this set is not in Σ_1^0 or Π_1^0 as follows.

Since the sequence of finite simple groups $(\mathbb{Z}_{p_k}, (1))$, where p_k is the k -th prime, converges to the group $(\mathbb{Z}, (1))$, the set of simple groups in \mathcal{G}_n is not in Π_1^0 . To show that this set is not in Σ_1^0 , let $G = \langle \gamma_1, \dots, \gamma_n \mid w_i, i \in \mathbb{N} \rangle$ be a simple group which is not finitely presented. (For example, in [Cam53], the author constructs continuum many 2-generated infinite simple groups, which implies via a counting argument that there exist 2-generated infinite simple groups that are not finitely presented.) For each $k \in \mathbb{N}$, set

$$G_k = \langle \gamma_1, \dots, \gamma_n \mid w_i, 0 \leq i \leq k \rangle$$

The relations of G_k are also relations of G and consequently, there exist surjective homomorphisms $\varphi_k : G_k \rightarrow G$. Since G is not finitely presented, the maps φ_k cannot be isomorphisms and hence, are not injections. It follows that G_k is not simple for any $k \in \mathbb{N}$. On the other hand, as in the proof of Proposition 2, the groups G_k converge to G in \mathcal{G}_n which implies that the set of simple groups in \mathcal{G}_n is not in Σ_1^0 .

3.5.2. *Being amenable.* Recall that, by Følner's condition, a countable group G is amenable if and only if for every finite subset $\mathbf{K} \subseteq G$ and every positive integer m , there exists a non-empty finite subset $\mathbf{F} \subseteq G$ such that

$$\frac{|g\mathbf{F}\Delta\mathbf{F}|}{|\mathbf{F}|} \leq \frac{1}{m}$$

for every $g \in \mathbf{K}$. Translating this condition to our setting, we have that the set of amenable groups in \mathcal{G}_n is

$$\bigcap_{K \in \mathcal{F}} \bigcap_{m \in \mathbb{N}^+} \bigcup_{F \in \mathcal{F}} \bigcap_{\gamma \in K} \left\{ N \in \mathcal{G}_n : \frac{|(\gamma N \cdot FN)\Delta FN|}{|FN|} \leq \frac{1}{m} \right\}$$

where $FN = \{\alpha N : \alpha \in F\}$ and \mathcal{F} denotes the set of finite subsets of \mathbb{F}_n . Observe that, for every fixed values of the parameters K, m, F and γ , the inner-most set is open. To see this, fix these parameters and choose $k \geq 1$ big enough so that

$$F^{-1}F, F^{-1}\gamma F \subseteq \{\mathbf{g}_i : 1 \leq i \leq k\}$$

If $N_1, N_2 \in \mathcal{G}_n$ and $d(N_1, N_2) < 2^{-k}$, then we have that $(F^{-1}F) \cap N_1 = (F^{-1}F) \cap N_2$ and $(F^{-1}\gamma F) \cap N_1 = (F^{-1}\gamma F) \cap N_2$.

The first equality implies that the map $\alpha N_1 \mapsto \alpha N_2$ from FN_1 to FN_2 and the map $\gamma \alpha N_1 \mapsto \gamma \alpha N_2$ from γFN_1 to γFN_2 are well-defined and bijective. The second equality implies that these maps agree on $FN_1 \cap \gamma FN_1$. It follows that $|FN_1| = |FN_2|$ and $|\gamma FN_1 \Delta FN_1| = |\gamma FN_2 \Delta FN_2|$ whenever $d(N_1, N_2) < 2^{-k}$. So,

for any sufficiently close $N_1, N_2 \in \mathcal{G}_n$, the corresponding inequalities for N_1 and N_2 will be both true or both false, which implies that the inner-most set is open.

It then follows that the set of amenable groups in \mathcal{G}_n is in $\mathbf{\Pi}_2^0$. We currently do not know whether or not this set is $\mathbf{\Pi}_2^0$ -complete. We strongly suspect that this is the case. However, it is clear that this set is not in $\mathbf{\Sigma}_1^0$ since finite groups can converge to a non-abelian free group. Also, it is not in $\mathbf{\Pi}_1^0$, since non-amenable groups can converge to a solvable group such as $\mathbb{Z}_2 \wr \mathbb{Z}$. More generally, it was proven in [BS80] that every infinitely presented metabelian group enjoys this property. (Also, see [BGdH13] for more examples of amenable groups as limits of non-amenable groups.)

3.5.3. Being residually finite. Recall that a finitely generated group G is residually finite if and only if for every non-identity $g \in G$ there exists a finite index (and hence, finitely generated) subgroup H of G such that $g \notin H$. Thus, retaining the notation in the proof of Lemma 3.1, one can write the set of residually finite groups in \mathcal{G}_n as

$$\bigcap_{\gamma \in \mathbb{F}_n} \left(\{N \in \mathcal{G}_n : \gamma \in N\} \cup \bigcup_{i=0}^{\infty} \{N \in \mathcal{G}_n : \gamma N \notin H_{N, A_i} \wedge [G_N : H_{N, A_i}] < \infty\} \right)$$

The condition that $\gamma N \notin H_{N, A_i}$ can be expressed as

$$\forall \delta \in \langle A_i \rangle \quad \gamma^{-1} \delta \notin N$$

Therefore, the set of residually finite groups in \mathcal{G}_n is

$$\bigcap_{\gamma \in \mathbb{F}_n} \left(\{N \in \mathcal{G}_n : \gamma \in N\} \cup \bigcup_{i=0}^{\infty} \bigcup_{j=2}^{\infty} \bigcap_{\delta \in \langle A_i \rangle} \{N \in \mathcal{G}_n : \gamma^{-1} \delta \notin N \wedge [G_N : H_{N, A_i}] < j\} \right)$$

It can be seen from the proof of Lemma 3.1 that, for every fixed value of the parameters, the innermost set on the right is in $\mathbf{\Pi}_2^0$ and hence the set of residually finite groups in \mathcal{G}_n is in $\mathbf{\Pi}_4^0$. Given the inelegancy of this result, the authors hope that this bound can be improved.

Let $\text{Sym}_f(\mathbb{Z})$ be the group of finitely supported permutations of \mathbb{Z} . Consider the group $G = \text{Sym}_f(\mathbb{Z}) \rtimes \mathbb{Z}$ where \mathbb{Z} acts on $\text{Sym}_f(\mathbb{Z})$ by the left shift. The group G is 2-generated and is not residually finite since it contains an infinite simple subgroup. Given any finite subset $E \subseteq G$, one can choose a large enough $k \geq 1$ so that there is a local embedding on $G \rightarrow S_k \rtimes \mathbb{Z}_k$ on E . It follows by Proposition 4 that G is a limit of finite groups. Thus, the set of residually finite groups in \mathcal{G}_n is not in $\mathbf{\Pi}_1^0$.

On the other hand, it is proven in [Sta06] that the Baumslag-Solitar groups $B(m, n)$ converges to \mathbb{F}_2 in \mathcal{G}_2 as $|m|, |n| \rightarrow \infty$. Since the groups $B(m, n)$ are not residually finite for distinct $m, n > 1$, the set of residually finite groups in \mathcal{G}_n is not in $\mathbf{\Sigma}_1^0$. We do not know whether or not this set is in $\mathbf{\Sigma}_2^0$ or $\mathbf{\Pi}_2^0$.

3.5.4. Being Hopfian and co-Hopfian. Recall that a group is Hopfian (respectively, co-Hopfian) if every epimorphism (respectively, monomorphism) from this group to itself is an isomorphism.

Let $F \subseteq \mathbb{F}_n$ be a subset of size n , say, $F = \{w_1, \dots, w_n\}$. For each $g \in \mathbb{F}_n$, let $g_* \in \mathbb{F}_n$ denote the image of the canonical homomorphism from \mathbb{F}_n to \mathbb{F}_n sending each γ_i to w_i . Then the map from G_N to G_N given by $gN \mapsto g_*N$ is well-defined whenever $h^{-1}g \in N$ implies $h_*^{-1}g_* \in N$. It is clear that $gN \mapsto g_*N$ is a homomorphism whenever it is well-defined. Conversely, for each homomorphism

$f : G_N \rightarrow G_N$, we can choose some finite subset $F = \{w_1, \dots, w_n\} \subseteq \mathbb{F}_n$ of size n (if necessary, by multiplying some w_i 's by elements from N) with the property that $f(\gamma_i N) = w_i N$ such that $h^{-1}g \in N$ implies $h_*^{-1}g_* \in N$.

Now fix a subset $F \subseteq \mathbb{F}_n$ of size n . Clearly the set of N 's in \mathcal{G}_n for which the map given by $gN \rightarrow g_*N$ is

- well-defined is defined by the sentence $\bigwedge_{g,h \in \mathbb{F}_n} (g = h \rightarrow g_* = h_*)$
- surjective is defined by the sentence $\bigwedge_{h \in \mathbb{F}_n} \bigvee_{g \in \mathbb{F}_n} g_* = h$
- injective is defined by the sentence $\bigwedge_{g,h \in \mathbb{F}_n} (g_* = h_* \rightarrow g = h)$

It follows that the set of Hopfian groups in \mathcal{G}_n is defined by the sentence

$$\bigwedge_{\substack{F \subseteq \mathbb{F}_n \\ F \text{ has size } n}} \left(\bigwedge_{g,h \in \mathbb{F}_n} (g = h \rightarrow g_* = h_*) \wedge \bigwedge_{h \in \mathbb{F}_n} \bigvee_{g \in \mathbb{F}_n} g_* = h \right) \rightarrow \bigwedge_{g,h \in \mathbb{F}_n} (g_* = h_* \rightarrow g = h)$$

Similarly, the set of co-Hopfian groups in \mathcal{G}_n is defined by the sentence

$$\bigwedge_{\substack{F \subseteq \mathbb{F}_n \\ F \text{ has size } n}} \left(\bigwedge_{g,h \in \mathbb{F}_n} (g = h \leftrightarrow g_* = h_*) \right) \rightarrow \left(\bigwedge_{h \in \mathbb{F}_n} \bigvee_{g \in \mathbb{F}_n} g_* = h \right)$$

Since each sentence of the form $g = h$ and $g_* = h_*$ defines a Δ_1^0 -set, it is easily seen that the sets of Hopfian and co-Hopfian groups in \mathcal{G}_n are in Π_3^0 and Π_2^0 respectively.

Since finite groups, which are co-Hopfian, can converge to a non-abelian free group, which is not co-Hopfian, the set of co-Hopfian groups in \mathcal{G}_n is not in Π_1^0 . As mentioned before, the Baumslag-Solitar groups $B(m, n)$, which are not Hopfian, can converge to \mathbb{F}_2 , which is Hopfian, in \mathcal{G}_2 and hence the set of Hopfian groups in \mathcal{G}_n is not in Σ_1^0 . We do not know whether or not the set of Hopfian groups (respectively, co-Hopfian groups) is in Π_1^0 (respectively, in Σ_1^0 .)

4. SOME REMARKS AND OPEN QUESTIONS

While we determined the descriptive complexity of some fundamental group theoretic properties, one can further this study by analyzing more technical properties. For example, a non-trivial result of [Sha00] shows that Kazhdan's property (T) is a Σ_1^0 -property. Since finite groups have the property (T) and free groups do not, this property is clearly not Π_1^0 and hence, the set of groups in \mathcal{G}_n with Kazhdan's property (T) is Σ_1^0 -complete.

Another direction in which this study can be taken is the analysis of how the complexity of being P changes if one considers being *just-non* P or residually P . Recall that, given a property P of groups, a group is said to be just-non P if it is not P and all its proper quotients are P .

Let us remind that a subset of a Polish space X is called analytic if it is the projection of some Borel subset of $X \times Y$ for some Polish space Y ; and called coanalytic if its complement is analytic. Given a property P which defines a Borel set in \mathcal{G}_n , a quick examination shows that, in general, the properties of being residually P and just-non P respectively define analytic and coanalytic sets. Though, these sets need not be non-Borel. For example, as we have observed earlier, being residually finite is a Borel property. We currently know of no Borel properties P for which being residually P or just-non P define non-Borel sets. However, such a result would seem to be important as it would show that there is no equivalent possible "Borel definition". (An interesting result that is along the same lines is the result

of Wesolek and Williams [WW17] which shows that, while amenability is a Borel property, the set of elementary amenable groups in \mathcal{G}_n is coanalytic non-Borel.) We would like to conclude this paper by listing some open questions that we think are of importance.

Question. What are the descriptive complexities of the sets of simple groups, amenable groups, residually finite groups, Hopfian groups and co-Hopfian groups in \mathcal{G}_n ? In particular, are the sets of simple groups, amenable groups and co-Hopfian groups Π_2^0 -complete?

Question. Are there group theoretic Borel properties P for which being just-non P or being residually P define non-Borel sets in \mathcal{G}_n ?

APPENDIX A. GRIGORCHUK GROUPS

In this section we will recall the construction of Grigorchuk [Gri84] and its slight modification from [BG14]. Although the original definition is in terms of measure preserving transformations of the unit interval, we will give here a definition in terms of automorphisms of the binary rooted tree.

Let T_2 denote the binary rooted tree, where we identify each vertex by a finite binary sequence in $\{0, 1\}^*$ (The root of T_2 is identified with the empty sequence). $Aut(T_2)$ denotes the automorphism group of T_2 . Given a vertex v of T_2 and an automorphism $f \in Aut(T_2)$, the *section* of f at v is the automorphism f_v defined uniquely by the equation:

$$f(vu) = f(u)f_v(u) \text{ for any vertex } u.$$

Also, the *root permutation* of f is the permutation $\sigma_f \in S_2$ defined by $\sigma_f(x) = y \iff f(x) = y$. One observes that the map $\Phi : Aut(T_2) \rightarrow Aut(T_2) \wr S_2$ defined by $\Phi(f) = (f_0, f_1; \sigma_f)$ is an isomorphism.

Let $\tau : 3^{\mathbb{N}} \rightarrow 3^{\mathbb{N}}$ be the shift i.e., $\tau(\alpha_0\alpha_1\alpha_2\dots) = \alpha_1\alpha_2\dots$. For each $\alpha = \alpha_0\alpha_1\dots \in 3^{\mathbb{N}}$ we will define a subgroup G_α of $Aut(T_2)$. Each group G_α is the subgroup generated by the four automorphisms denoted by $a, b_\alpha, c_\alpha, d_\alpha$ whose actions onto the tree is given (recursively) by the following:

$$a(0v) = 1v \text{ and } a(1v) = 0v \text{ for any } v \in \{0, 1\}^*$$

$$\begin{aligned} b_\alpha(0v) &= 0\beta(\alpha_0)(v) & c_\alpha(0v) &= 0\zeta(\alpha_0)(v) & d_\alpha(0v) &= 0\delta(\alpha_0)(v) \\ b_\alpha(1v) &= 1b_{\tau\alpha}(v) & c_\alpha(1v) &= 1c_{\tau\alpha}(v) & d_\alpha(1v) &= 1d_{\tau\alpha}(v), \end{aligned}$$

where

$$\begin{aligned} \beta(0) &= a & \beta(1) &= a & \beta(2) &= e \\ \zeta(0) &= a & \zeta(1) &= e & \zeta(2) &= a \\ \delta(0) &= e & \delta(1) &= a & \delta(2) &= a \end{aligned}$$

and e denotes the identity automorphism. In the language of sections, this can be written shortly as $(b_\alpha)_0 = \beta(\alpha_0)$ and $(b_\alpha)_1 = b_{\tau\alpha}$ etc. Note that the isomorphism Φ restricts to an embedding of G_α into $G_{\tau\alpha} \wr S_2$.

Denoting by $S_\alpha = (a, b_\alpha, c_\alpha, d_\alpha)$, we obtain a subset $\{(G_\alpha, S_\alpha) \mid \alpha \in 3^{\mathbb{N}}\} \subseteq \mathcal{G}_4$. Let $E \subseteq 3^{\mathbb{N}}$ denote the set of eventually constant sequences. In [Gri84], it was observed if two sequences $\alpha, \beta \in 3^{\mathbb{N}} - E$ have a common prefix of length $n \geq 1$, then the marked groups $(G_\alpha, S_\alpha), (G_\beta, S_\beta)$ are within 2^{n-1} in \mathcal{G}_4 . Thus, replacing the marked groups $(G_\alpha, S_\alpha), \alpha \in E$ by the appropriate limits (denoted again by (G_α, S_α)) in \mathcal{G}_4 , one obtains a closed subset $\mathcal{R} = \{(G_\alpha, S_\alpha) \mid \alpha \in 3^{\mathbb{N}}\} \subset \mathcal{G}_4$. Also,

this new family also has the property that G_α embeds into $G_{\tau\alpha} \wr S_2$ by a map analogous to Φ .

Let $C \subseteq 3^{\mathbb{N}}$ be the set of recursive sequences and $I \subseteq 3^{\mathbb{N}}$ denote the set of sequences which contain infinitely many 0's, 1's and 2's. Then we have the following.

Theorem A.1. [Gri84]

- 1) $\mathcal{R} \subseteq \mathcal{G}_4$ is homeomorphic to $3^{\mathbb{N}}$ via the map $\alpha \mapsto (G_\alpha, S_\alpha)$,
- 2) G_α is a periodic group if and only if $\alpha \in I$.
- 3) G_α has exponential growth for $\alpha \in E$ and intermediate growth for $\alpha \notin E$.
- 4) If $\alpha \in E$, then G_α is virtually metabelian.
- 5) G_α has decidable word problem if and only if $\alpha \in C$.

In [BG14, Theorem 2], it was proven that if $\alpha \in 3^{\mathbb{N}}$ is a constant sequence, then $G_\alpha \cong L \rtimes \mathbb{Z}_2$, where $L = \mathbb{Z}_2 \wr \mathbb{Z}$ is the Lamplighter group. In particular, G_α is solvable for constant α . Since G_α embeds into $G_{\tau\alpha} \wr S_2$, one deduces that for every $\alpha \in E$, G_α is solvable.

To obtain a family in \mathcal{G}_2 with similar properties, we follow [BG14]. For each $\alpha \in 3^{\mathbb{N}}$, let $T_\alpha = \{d_\alpha, ab_\alpha\} \subset G_\alpha$ and $L_\alpha = \langle T_\alpha \rangle \leq G_\alpha$. Also let $\mathcal{L} = \{(L_\alpha, T_\alpha) \mid \alpha \in 3^{\mathbb{N}}\} \subseteq \mathcal{G}_2$. We have the following.

Theorem A.2. [BG14]

- 1) $\mathcal{L} \subseteq \mathcal{G}_2$ is homeomorphic to $3^{\mathbb{N}}$ via the map $\alpha \mapsto (L_\alpha, T_\alpha)$,
- 2) L_α is a periodic group if and only if $\alpha \in I$.
- 3) L_α has exponential growth for $\alpha \in E$ and intermediate growth otherwise.
- 4) L_α is solvable if and only if $\alpha \in E$.
- 5) L_α has decidable word problem if and only if $\alpha \in C$.

REFERENCES

- [BG14] Mustafa Gökhan Benli and Rostislav Grigorchuk, *On the condensation property of the lamplighter groups and groups of intermediate growth*, Algebra Discrete Math. **17** (2014), no. 2, 222–231. MR 3287930
- [BGdlH13] Mustafa Gökhan Benli, Rostislav Grigorchuk, and Pierre de la Harpe, *Amenable groups without finitely presented amenable covers*, Bull. Math. Sci. **3** (2013), no. 1, 73–131. MR 3061134
- [BS80] Robert Bieri and Ralph Strebel, *Valuations and finitely presented metabelian groups*, Proc. London Math. Soc. (3) **41** (1980), no. 3, 439–464. MR 591649
- [Cam53] Ruth Camm, *Simple free products*, J. London Math. Soc. **28** (1953), 66–76. MR 52420
- [CG05] Christophe Champetier and Vincent Guirardel, *Limit groups as limits of free groups*, Israel J. Math. **146** (2005), 1–75. MR 2151593
- [Cha50] Claude Chabauty, *Limite d'ensembles et géométrie des nombres*, Bull. Soc. Math. France **78** (1950), 143–151. MR 0038983
- [Cha00] Christophe Champetier, *L'espace des groupes de type fini*, Topology **39** (2000), no. 4, 657–680. MR 1760424
- [dCGP07] Yves de Cornulier, Luc Guyot, and Wolfgang Pitsch, *On the isolated points in the space of groups*, J. Algebra **307** (2007), no. 1, 254–277. MR 2278053
- [Gri84] R. I. Grigorchuk, *Degrees of growth of finitely generated groups and the theory of invariant means*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (1984), no. 5, 939–985. MR 764305
- [Gri05] Rostislav Grigorchuk, *Solved and unsolved problems around one group*, Infinite groups: geometric, combinatorial and dynamical aspects, Progr. Math., vol. 248, Birkhäuser, Basel, 2005, pp. 117–218. MR 2195454
- [Gro81] Mikhael Gromov, *Groups of polynomial growth and expanding maps*, Inst. Hautes Études Sci. Publ. Math. (1981), no. 53, 53–73. MR 623534
- [Hal50] Marshall Hall, Jr., *A topology for free groups and related groups*, Ann. of Math. (2) **52** (1950), 127–139. MR 0036767

- [Kec95] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, New York, 1995.
- [Man12] Avinoam Mann, *How groups grow*, London Mathematical Society Lecture Note Series, vol. 395, Cambridge University Press, Cambridge, 2012. MR 2894945
- [Osi20] D Osin, *A topological zero-one law and elementary equivalence of finitely generated groups*, arXiv preprint arXiv:2004.07479 (2020).
- [Sha00] Yehuda Shalom, *Rigidity of commensurators and irreducible lattices*, Invent. Math. **141** (2000), no. 1, 1–54. MR 1767270
- [Sta06] Yves Stalder, *Convergence of Baumslag-Solitar groups*, Bull. Belg. Math. Soc. Simon Stevin **13** (2006), no. 2, 221–233. MR 2259902
- [Ste96] A. M. Stepin, *Approximation of groups and group actions, the Cayley topology*, Ergodic theory of \mathbf{Z}^d actions (Warwick, 1993–1994), London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996, pp. 475–484. MR 1411234
- [VG97] A. M. Vershik and E. I. Gordon, *Groups that are locally embeddable in the class of finite groups*, Algebra i Analiz **9** (1997), no. 1, 71–97. MR 1458419
- [WW17] Phillip Wesolek and Jay Williams, *Chain conditions, elementary amenable groups, and descriptive set theory*, Groups Geom. Dyn. **11** (2017), no. 2, 649–684. MR 3668055

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