TORSION GENERATORS OF THE TWIST SUBGROUP

TÜLİN ALTUNÖZ, MEHMETCİK PAMUK, AND OĞUZ YILDIZ

ABSTRACT. We showed that the twist subgroup of the mapping class group of a closed connected nonorientable surface of genus $g \geq 13$ can be generated by two involutions and an element of order g or $g-1$ depending on whether g is odd or even respectively.

1. Introduction

Let Σ_g denote a closed connected orientable surface of genus g. The mapping class group, $\mathrm{Mod}(\Sigma_g)$, is the group of the isotopy classes of orientation preserving diffeomorphisms of Σ_g. It is a classical result that $\mathrm{Mod}(\Sigma_g)$ is generated by finitely many Dehn twists about nonseparating simple closed curves [4, 6, 12]. The study of algebraic properties of mapping class group, finding small generating sets, generating sets with particular properties, is an active one leading to interesting developments. Wajnryb [23] showed that $\mathrm{Mod}(\Sigma_g)$ can be generated by two elements given as a product of Dehn twists. As the group is not abelian, this is the smallest possible. Later, Korkmaz [9] showed that one of these generators can be taken as a Dehn twist, he also proved that $\mathrm{Mod}(\Sigma_g)$ can be generated by two torsion elements. Recently, the third author showed that $\mathrm{Mod}(\Sigma_g)$ is generated by two torsions of small orders [24].

Generating $\mathrm{Mod}(\Sigma_g)$ by involutions was first considered by McCarthy and Papadopoulus [15]. They showed that the group can be generated by infinitely many conjugates of a single involution (element of order two) for $g \geq 3$. In terms of generating by finitely many involutions, Luo [14] showed that any Dehn twist about a nonseparating simple closed curve can be written as a product six involutions, which in turn implies that $\mathrm{Mod}(\Sigma_g)$ can be generated by $12g+6$ involutions. Brendle and Farb [2] obtained a generating set of six involutions for $g \geq 3$. Following their work, Kassabov [7] showed that $\mathrm{Mod}(\Sigma_g)$ can be generated by four involutions if $g \geq 7$. Recently, Korkmaz [10] showed that $\mathrm{Mod}(\Sigma_g)$ is generated by three involutions if $g \geq 8$ and four involutions if $g \geq 3$. Also, the third author improved his result showing that it is generated by three involutions if $g \geq 6$ [25].

The main aim of this paper is to find minimal generating sets of torsion elements for a particular subgroup, namely the twist subgroup, of the mapping class groups of nonorientable surfaces. Let N_g denote a closed connected nonorientable surface of genus g. The mapping class group,$\mathrm{Mod}(N_g)$, is defined to be the group of the isotopy classes of all diffeomorphisms of N_g. Compared to orientable surfaces less is known about $\mathrm{Mod}(N_g)$. Lickorish [11, 13] showed that it is generated by Dehn twists about two-sided simple closed curves and a so-called Y-homeomorphism (or

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a crosscap slide). Chillingworth [3] gave a finite generating set for $\mathrm{Mod}(N_g)$ that linearly depends on g. Szepietowski [21] proved that $\mathrm{Mod}(N_g)$ is generated by three elements and by four involutions.

The twist subgroup $\mathcal{T}_g$ of $\mathrm{Mod}(N_g)$ is the group generated by Dehn twists about two-sided simple closed curves. The group $\mathcal{T}_g$ is a subgroup of index 2 in $\mathrm{Mod}(N_g)$ [13]. Chillingworth [3] showed that $\mathcal{T}_g$ can be generated by finitely many Dehn twists. Stukow [19] obtained a finite presentation for $\mathcal{T}_g$ with $(g+2)$ Dehn twist generators. Later Omori [17] reduced the number of Dehn twist generators to $(g+1)$ for $g \geq 4$. If it is not required that all generators are Dehn twists, Du [5] obtained a generating set consisting of three elements, two involutions and an element of order $2g$ whenever $g \geq 5$ and odd.

In the present paper, we prove that $\mathcal{T}_g$ can be generated by two involutions and an element of order g or $g-1$ depending on the parity of g (see Theorems 2.4 and 2.5).

Main Theorem. *The twist subgroup $\mathcal{T}_g$ can be generated by two involutions and an element of order g or $g-1$ depending on whether g is odd or even respectively.*

Before we finish the introduction, let us point out that the twist subgroup $\mathcal{T}_g$ admits an epimorphism onto the automorphism group of $H_1(N_g;\mathbb{Z}_2)$ preserving the (mod 2) intersection pairing [16], which is isomorphic to (see [8] and [22])

$$\begin{cases} Sp(2h;\mathbb{Z}_2) & \text{if } g = 2h+1, \\ Sp(2h;\mathbb{Z}_2) \ltimes \mathbb{Z}_2^{2h+1} & \text{if } g = 2h+2. \end{cases}$$

Hence, the action of mapping classes on $H_1(N_g;\mathbb{Z}_2)$ induces an epimorphism from $\mathcal{T}_g$ to $Sp\big(2\lfloor\frac{g-1}{2}\rfloor;\mathbb{Z}_2\big)$, which immediately implies the following corollary:

Corollary 1.1. *The symplectic group $Sp\big(2\lfloor\frac{g-1}{2}\rfloor;\mathbb{Z}_2\big)$ can be generated by two involutions and an element of order g or $g-1$ depending on whether g is odd or even respectively.*

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2. Background and Results on Mapping Class Groups

Let N_g be a closed connected nonorientable surface of genus g. Note that the *genus* for a nonorientable surface is the number of projective planes in a connected sum decomposition. The *mapping class group* $\mathrm{Mod}(N_g)$ of the surface N_g is defined to be the group of the isotopy classes of diffeomorphisms $N_g \to N_g$. Throughout the paper we do not distinguish a diffeomorphism from its isotopy class. For the composition of two diffeomorphisms, we use the functional notation; if g and h are two diffeomorphisms, the composition gh means that h acts on N_g first.

A simple closed curve on a nonorientable surface N_g is said to be *one-sided* if a regular neighbourhood of it is homeomorphic to a Möbius band. It is called *two-sided* if a regular neighbourhood of it is homeomorphic to an annulus. If a is a two-sided simple closed curve on N_g, to define the Dehn twist t_a, we need to fix one of two possible orientations on a regular neighbourhood of a (as we did for the curve a_1 in Figure 1). Following [10] the right-handed Dehn twist t_a about a will be denoted by the corresponding capital letter A.

Now, let us recall the following basic properties of Dehn twists which we use frequently in the remaining of the paper. Let a and b be two-sided simple closed curves on N_g and $f \in \mathrm{Mod}(N_g)$.

- **Commutativity:** If a and b are disjoint, then $AB = BA$.
- **Conjugation:** If $f(a) = b$, then $fAf^{-1} = B^s$, where $s = \pm 1$ depending on whether f is orientation preserving or orientation reversing on a neighbourhood of a with respect to the chosen orientation.

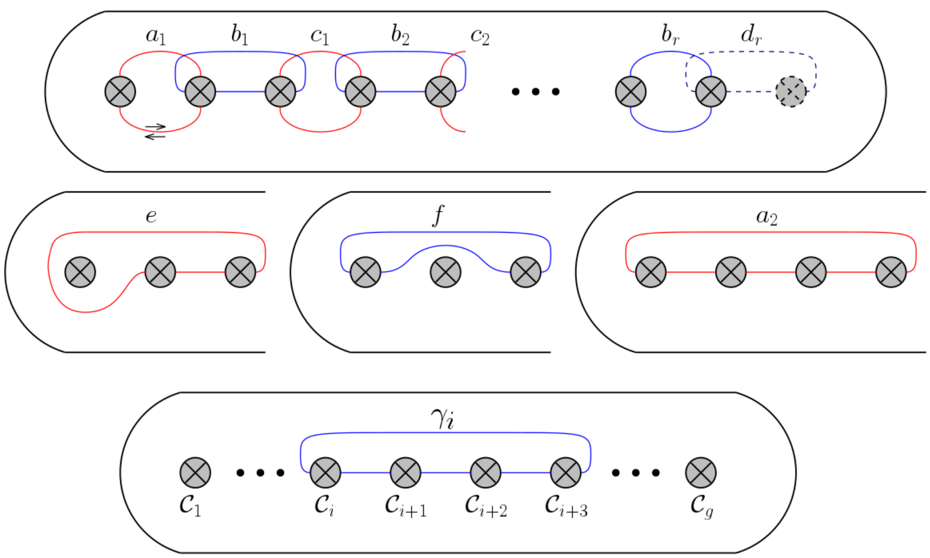

FIGURE 1. The curves a_1, a_2, b_i, c_i, e, f and γ_i on the surface N_g.

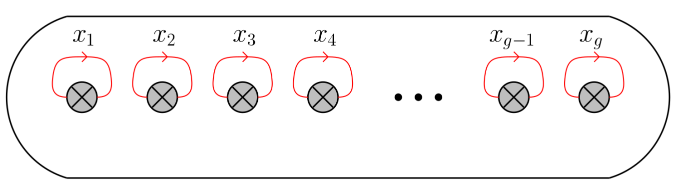

FIGURE 2. Generators of $H_1(N_g; \mathbb{R})$.

Consider the surface N_g shown in Figure 1. The Dehn twist generators of Omori can be given as follows (note that we do not have the curve d_r when g is odd).

Theorem 2.1. [17] *The twist subgroup $\mathcal{T}_g$ is generated by the following $(g+1)$ Dehn twists*

(1) $A_1, A_2, B_1, \ldots, B_r, C_1, \ldots, C_{r-1}$ *and* E *if* $g = 2r+1$ *and*
(2) $A_1, A_2, B_1, \ldots, B_r, C_1, \ldots, C_{r-1}, D_r$ *and* E *if* $g = 2r+2$.

Consider a basis $\{x_1, x_2 \ldots, x_{g-1}\}$ for $H_1(N_g;\mathbb{R})$ such that the curves x_i are one-sided and disjoint as in Figure 2. It is known that every diffeomorphism $f : N_g \to N_g$ induces a linear map $f_* : H_1(N_g;\mathbb{R}) \to H_1(N_g;\mathbb{R})$. Therefore, one can define a homomorphism $D : \mathrm{Mod}(N_g) \to \mathbb{Z}_2$ by $D(f) = \det(f_*)$. The following lemma from [11] tells when a mapping class falls into the twist subgroup $\mathcal{T}_g$.

Lemma 2.2. *Let $f \in \mathrm{Mod}(N_g)$. Then $D(f) = 1$ if $f \in \mathcal{T}_g$ and $D(f) = -1$ if $f \notin \mathcal{T}_g$.*

2.1. **A generating set for $\mathcal{T}_g$.** We start with presenting a generating set for $\mathcal{T}_g$. The diffeomorphism T is the rotation by $\frac{2\pi}{g}$ or $\frac{2\pi}{g-1}$ as shown on the right hand sides of Figures 3 and 4, respectively. Note that the rotation T satisfies $D(T) = 1$, which implies that T belongs to $\mathcal{T}_g$.

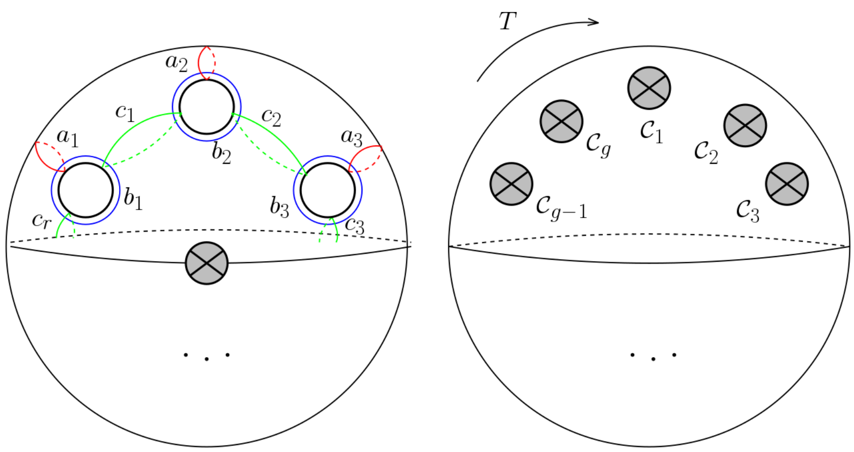

FIGURE 3. The models for N_g if $g = 2r + 1$.

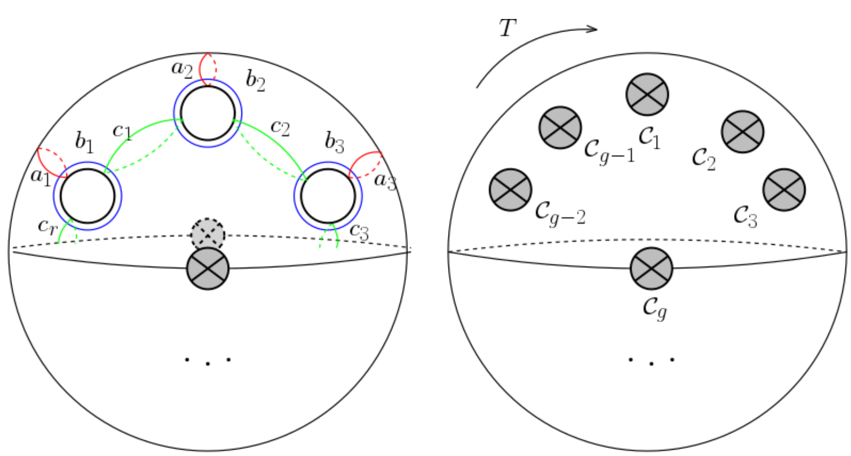

FIGURE 4. The models for N_g if $g = 2r + 2$.

Theorem 2.3. *The twist subgroup $\mathcal{T}_g$ is generated by the elements*

(1) $T, A_1A_2^{-1}, B_1B_2^{-1}$ *and* E *if* $g = 2r+1$ *and* $r \geq 3$,
(2) $T, A_1A_2^{-1}, B_1B_2^{-1}, D_r$ *and* E *if* $g = 2r+2$ *and* $r \geq 3$.

Proof. Let G be the subgroup of $\mathcal{T}_g$ generated by the following set

$$G = \begin{cases} \{T, A_1A_2^{-1}, B_1B_2^{-1}, E\} & \text{if } \quad g = 2r+1, \\ \{T, A_1A_2^{-1}, B_1B_2^{-1}, D_r, E\} & \text{if } \quad g = 2r+2, \end{cases}$$

where $r \geq 3$. It follows from Theorem 2.1 that we only need to prove that G contains the elements A_1, A_2, B_i and C_j shown in Figures 3 and 4 where $i = 1, \ldots, r$ and $j = 1, \ldots, r-1$. (We use the explicit homeomorphism constructed in [18, Section 3] to identify the models in these figures.)
Let $\mathcal{S}$ denote the set of isotopy classes of two-sided non-separating simple closed curves on N_g. Define a subset $\mathcal{G}$ of $\mathcal{S} \times \mathcal{S}$ as

$$\mathcal{G} = \{(a,b) : AB^{-1} \in G\}.$$

Using the arguments similar to the proof of [10, Theorem 5], the set $\mathcal{G}$ satisfies

- if $(a,b) \in \mathcal{G}$, then $(b,a) \in \mathcal{G}$ (symmetry),
- if (a,b) and $(b,c) \in \mathcal{G}$, then $(a,c) \in \mathcal{G}$ (transitivity) and
- if $(a,b) \in \mathcal{G}$ and $H \in G$ then $(H(a), H(b)) \in \mathcal{G}$ (G-invariance).

Hence, $\mathcal{G}$ defines an equivalence relation on $\mathcal{S}$.
We begin by showing that $B_iC_j^{-1}$ is contained in G for all i, j. It will follow from the definition of G and from the fact that $T(b_1, b_2) = (c_1, c_2)$, we have $C_1C_2^{-1}$ is in G (here, we use the notation $f(a,b)$ to denote $(f(a), f(b))$). Also, by conjugating $C_1C_2^{-1}$ with powers of T, one can show that the elements $B_iB_{i+1}^{-1}$ and $C_iC_{i+1}^{-1}$ are contained in G. Moreover, the subgroup G contains the elements $B_iB_j^{-1}$ and $C_iC_j^{-1}$ by the transitivity. To start with, since $B_2B_3^{-1} \in G$ and it is easy to check that

$$B_2B_3^{-1}A_2A_1^{-1}(b_2, b_3) = (a_2, b_3),$$

so that $A_2B_3^{-1}$ is contained in the subgroup G. We have

$$(A_1A_2^{-1})(A_2B_3^{-1})(B_3B_2^{-1}) = A_1B_2^{-1} \in G,$$

since each of the factors is contained in G. Hence, $T(a_1, b_2) = (b_1, c_2)$ implies that $B_1C_2^{-1}$ is also in G. Now, the subgroup G contains the element

$$B_1C_1^{-1} = (B_1C_2^{-1})(C_2C_1^{-1}).$$

Therefore, the elements $B_iC_i^{-1}$ is contained in G by conjugating with powers of T for all $i = 1, \ldots, r-1$. It follows from the transitivity that $B_iC_j^{-1}$ is in G. Note that, we have

- $(A_1B_2^{-1})(B_2C_1^{-1}) = A_1C_1^{-1}$,
- $(C_1A_1^{-1})(A_1A_2^{-1}) = C_1A_2^{-1}$, and
- $(C_2C_1^{-1})(C_1A_1^{-1}) = C_2A_1^{-1}$

from which it follows that the elements $A_1C_1^{-1}$, $C_1A_2^{-1}$ and $C_2A_1^{-1}$ belong to G. It can also be shown that

$$(B_2A_1^{-1})(C_1A_2^{-1})(C_2A_1^{-1})(b_2, a_1) = (d_1, a_1)$$

and

$$(A_1B_2^{-1})(A_1C_1^{-1})(A_1C_2^{-1})(A_1B_2^{-1})(a_2, a_1) = (d_2, a_1)$$

so that G contains $D_1A_1^{-1}$ and $D_2A_1^{-1}$ (here, the curves d_1 and d_2 are shown in [10, Figure 1]). Also, we have

$$(D_2A_1^{-1})(A_1C_1^{-1}) = D_2C_1^{-1} \in G.$$

By similar arguments as in the proof of [10, Theorem 5], the lantern relation implies the following identity

$$A_3 = (A_2C_2^{-1})(D_1A_1^{-1})(D_2C_1^{-1}).$$

Since the subgroup G contains each factor on the right hand side, the element A_3 belongs to G. It follows from

$$B_3 = A_3(B_3B_1^{-1})A_3(B_1B_3^{-1})A_3^{-1}$$

that B_3 is also contained in G. By conjugating B_3 with the powers of T, we get $A_1, B_1, C_1, \ldots B_{r-1}, C_{r-1}$ and B_r are all contained in G. Moreover,

$$A_2 = (A_2A_1^{-1})A_1 \in G.$$

Therefore, we conclude that $G = \mathcal{T}_g$. $\square$

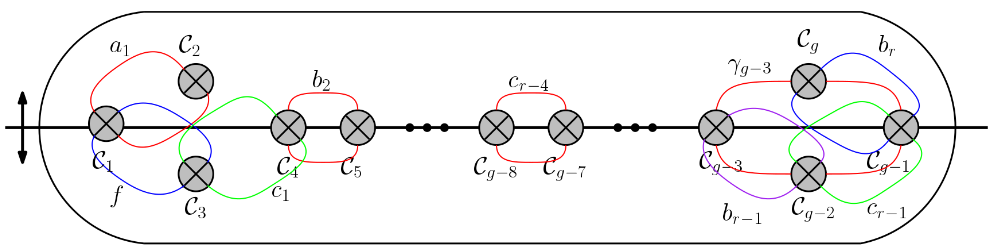

FIGURE 5. The involution σ if $g = 2r + 1$.

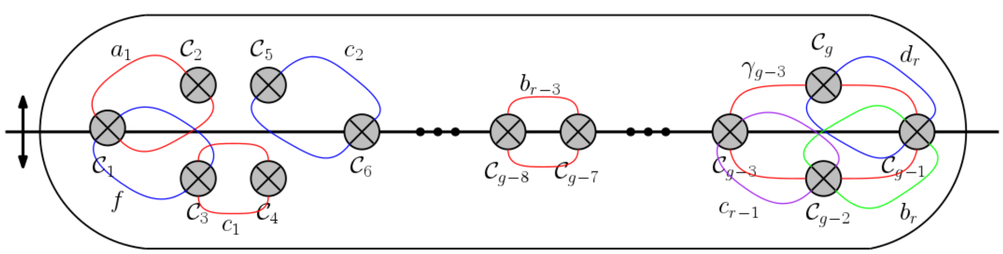

FIGURE 6. The involution σ if $g = 2r + 2$.

We consider the surface N_g where g-crosscaps are distributed on the sphere as in Figures 5 and 6. If $g = 2r + 1 \geq 13$, there is a reflection, σ, of the surface N_g in the xy-plane such that

- $\sigma(a_1) = f$, $\sigma(b_r) = c_{r-1}$,
- $\sigma(x_2) = x_3$, $\sigma(x_g) = x_{g-2}$ and
- $\sigma(x_i) = x_i$ if $i = 4, \ldots, g-3$ or $i = 1, g-1$.

with reverse orientation as in Figure 5. (Recall that x_i's are the generators of $H_1(N_g;\mathbb{R})$ as shown in Figure 2.) If $g = 2r+2 \geq 14$, there is a reflection, σ, of the surface N_g in the xy-plane such that

- $\sigma(a_1) = f$, $\sigma(b_r) = d_r$,
- $\sigma(x_2) = x_3$, $\sigma(x_3) = x_4$, $\sigma(x_g) = x_{g-2}$ and
- $\sigma(x_i) = x_i$ if $i = 6, \ldots, g-3$ or $i = 1, g-1$.

with reverse orientation as in Figure 6. Note that in both cases the reflection σ in in $\mathcal{T}_g$ since $D(\sigma) = 1$ for $g \geq 13$.

Now, for the remaining part of the paper, let Γ_i denote the right handed Dehn twist about the curve γ_i shown in Figure 1.

Theorem 2.4. *For odd $g \geq 13$, the twist subgroup $\mathcal{T}_g$ is generated by the three elements T, σ and $\sigma\Gamma_{g-3}C^{-1}_{\frac{g-9}{2}}$.*

Proof. Consider the surface N_g as in Figure 5. Since

$$\sigma(\gamma_{g-3}) = \gamma_{g-3} \text{ and } \sigma(c_{\frac{g-9}{2}}) = c_{\frac{g-9}{2}},$$

and σ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we have

$$\sigma\Gamma_{g-3}\sigma = \Gamma^{-1}_{g-3} \text{ and } \sigma C_{\frac{g-9}{2}}\sigma = C^{-1}_{\frac{g-9}{2}}.$$

Hence, it is easy to verify that $\sigma\Gamma_{g-3}C^{-1}_{\frac{g-9}{2}}$ is an involution. Let H be the subgroup of $\mathcal{T}_g$ generated by the following set

$$\{T, \sigma, \sigma\Gamma_{g-3}C^{-1}_{\frac{g-9}{2}}\},$$

where $g \geq 13$ and odd. It follows from Theorem 2.3 that we only need to prove that the elements $A_1A_2^{-1}$, $B_1B_2^{-1}$ and E are contained in the subgroup H.
Since

$$\Gamma_{g-3}C^{-1}_{\frac{g-9}{2}} = (\sigma)(\sigma\Gamma_{g-3}C^{-1}_{\frac{g-9}{2}}),$$

the element $\Gamma_{g-3}C^{-1}_{\frac{g-9}{2}}$ belongs to H.
It follows from

- $T^{13-g}(\gamma_{g-3}, c_{\frac{g-9}{2}}) = (\gamma_{10}, c_2)$,
- $T^{-4}(\gamma_{10}, c_2) = (\gamma_6, a_1)$ and
- $T^2(\gamma_6, a_1) = (\gamma_8, c_1)$

that the elements $\Gamma_{10}C_2^{-1}$, $\Gamma_6A_1^{-1}$ and $\Gamma_8C_1^{-1}$ are in H. Since

$$(\Gamma_6A_1^{-1})(C_2\Gamma_{10}^{-1})(\gamma_6, a_1) = (c_2, a_1),$$

the element $C_2A_1^{-1}$ is in H. Since

- $(\Gamma_6A_1^{-1})(A_1C_2^{-1}) = \Gamma_6C_2^{-1} \in H$ and
- $T^4(\gamma_6, c_2) = (\gamma_{10}, c_4)$,

$\Gamma_{10}C_4^{-1}$ is contained in H. Also, we have

$$(C_4\Gamma_{10}^{-1})(\Gamma_{10}C_2^{-1}) = C_4C_2^{-1} \in H.$$

Thus, $C_3C_1^{-1}$, $C_3C_5^{-1}$, $B_4B_2^{-1}$ and $C_2A_1^{-1}$ are contained in H by conjugating $C_4C_2^{-1}$ with some powers of T. Then, since

$$(C_4C_2^{-1})(B_4B_2^{-1})(c_2, a_1) = (b_2, a_1),$$

H contains $B_2A_1^{-1}$. Also, we get

$$(C_2A_1^{-1})(A_1B_2^{-1}) = C_2B_2^{-1} \in H.$$

Since

$$C_1C_3^{-1}B_2B_4^{-1}(c_3, c_5) = (b_4, c_5),$$

H contains $B_4C_5^{-1}$. Then,

$$T^{-4}(b_4, c_5) = (b_2, c_3)$$

implies that $B_2C_3^{-1} \in H$. It follows from the following equalities

- $C_2C_3^{-1} = (C_2B_2^{-1})(B_2C_3^{-1})$ and
- $T^{-2}(b_2, c_3) = (b_1, b_2)$

that $B_1B_2^{-1}$ belongs to H. On the other hand, since

- $(\Gamma_8C_1^{-1})(C_1C_3^{-1})(C_3C_5^{-1})(C_5B_4^{-1}) = \Gamma_8B_4^{-1}$ and
- $T^{-7}(\gamma_8, b_4) = (\gamma_1, a_1) = (a_2, a_1)$

that $A_1A_2^{-1}$ is contained in H.
Since the elements T, $A_1A_2^{-1}$ and $B_1B_2^{-1}$ are contained in the subgroup H, the generators $A_1, B_1, C_1, \ldots, B_{r-1}, C_{r-1}, B_r$ are in H by the proof of Theorem 2.1. Moreover,

$$A_1\sigma(a_1) = A_1(f) = e$$

leads to $E \in H$. This completes the proof. □

Theorem 2.5. *For even $g \geq 14$, the twist subgroup $\mathcal{T}_g$ is generated by the three elements T, σ and $\sigma\Gamma_{g-3}C^{-1}_{\frac{g-8}{2}}$.*

Proof. Consider the surface N_g as in Figure 6. Since

$$\sigma(\gamma_{g-3}) = \gamma_{g-3} \text{ and } \sigma(c_{\frac{g-8}{2}}) = c_{\frac{g-8}{2}},$$

and σ reverses the orientation of a neighbourhood of a two-sided simple closed curve, we have

$$\sigma\Gamma_{g-3}\sigma = \Gamma^{-1}_{g-3} \text{ and } \sigma C_{\frac{g-8}{2}}\sigma = C^{-1}_{\frac{g-8}{2}}.$$

Hence, it is easy to show that $\sigma\Gamma_{g-3}C^{-1}_{\frac{g-9}{2}}$ is an involution. Let K be the subgroup of $\mathcal{T}_g$ generated by the following set

$$\{T, \sigma, \sigma\Gamma_{g-3}C^{-1}_{\frac{g-8}{2}}\},$$

where $g \geq 14$ and even.
Since

$$\Gamma_{g-3}C^{-1}_{\frac{g-8}{2}} = (\sigma)(\sigma\Gamma_{g-3}C^{-1}_{\frac{g-8}{2}}),$$

the element $\Gamma_{g-3}C^{-1}_{\frac{g-8}{2}}$ is contained in K.
It follows from

$$T^{13-g}(\gamma_{g-3}, c_{\frac{g-8}{2}}) = (\gamma_{10}, c_2)$$

that the elements $\Gamma_{10}C_2^{-1}$ is in the subgroup K. Recall that this element also appears in the proof of Theorem 2.4. The remaining part of the proof follows as in the proof of the previous theorem. Also, note that the element D_r belongs to K since $\sigma(b_r) = d_r$. This finishes the proof. □

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Department of Mathematics, Middle East Technical University, Ankara, Turkey
E-mail address: atulin@metu.edu.tr
E-mail address: mpamuk@metu.edu.tr
E-mail address: oguzyildiz16@gmail.com