A solution of a problem about Erdős space

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Abstract

For Erdős space, $\mathfrak{E}$, let us define a topology, τ_{clopen}, which is generated by clopen subsets of $\mathfrak{E}$. A. V. Arhangel'skii and J. Van Mill asked whether the topology τ_{clopen} is compatible with the group structure on $\mathfrak{E}$. In this paper, we give a negative answer for this question.

Keywords: Erdős space, topological groups, Sequence spaces.

MSC: 22A99, 22A45, 46A45

1 Introduction and Terminology

We let $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{R}^+$ denote the sets of rational numbers, real numbers and positive real numbers, respectively. $\mathbb{N}^+$ denotes the set of positive natural numbers, i.e., $\mathbb{N}^+ = \{1, 2, 3, \ldots\}$. Let us consider the Banach space $\ell_2 \subseteq \mathbb{R}^{\mathbb{N}^+}$. This space consists of all sequences $x = (x_1, x_2, x_3, \ldots) \in \mathbb{R}^{\mathbb{N}^+}$ such that the series $\sum_{k=1}^{\infty} |x_k|^2$ is convergent. The topology on ℓ_2 is generated by the norm $||x|| = \sqrt{\sum_{k=1}^{\infty} |x_k|^2}$. The Erdős space $\mathfrak{E}$ is a subspace of ℓ_2 such that $\mathfrak{E}$ consists of the all sequences, the all components of which are rational, i.e., $\mathfrak{E} = \mathbb{Q}^{\mathbb{N}^+} \cap \ell_2$. The topology τ on $\mathfrak{E}$ is the subspace topology inherited from ℓ_2.

In this paper our main space is $(\mathfrak{E}, \tau)$. What we mean with an open ball $B(x, r)$ is the set $B(x, r) = \{y \in \mathfrak{E} : ||x - y|| < r\}$ where $r > 0$. If we say "O is an open set", we mean that $O \subseteq \mathfrak{E}$ and $O \in \tau$. Let O be a subset of $\mathfrak{E}$. We denote the *interior* of O by $int(O)$, i.e., $int(O) = \{x \in O : \exists V_x \in \tau(x \in V_x \subseteq O)\}$.

We need the following basic facts. These can be found in any proper book.

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Theorem 1.1. *([1] Theorem 1.3.12) Let G be a topological group and e the identity element of G. If U is an open subset of G and $e \in U$, there is an open subset V of G such that $e \in V$ and $V + V \subseteq U$.*

Theorem 1.2. *([3] p. 17, [2] p. 220) The only bounded clopen subset of $(\mathfrak{E}, \tau)$ is the emptyset.*

2 The solution of the problem

Let $(\mathfrak{E}, \tau)$ be the topological space as in the above. Let $\mathcal{B}$ be the set of all clopen subsets of $\mathfrak{E}$, i.e., $\mathcal{B} = \{U \in \tau : \mathfrak{E} - U \in \tau\}$. Take $\mathcal{B}$ as the base for a new topology τ_{clopen} on $\mathfrak{E}$. Is the topology τ_{clopen} compatible with the group structure on $\mathfrak{E}$? In [2], Question 8.9, A. V. Arhangel'skii and J. Van Mill asked this question. The following theorem states that the answer of the question is negative.

Theorem 2.1. *The topology τ_{clopen} is not compatible with the group structure on $\mathfrak{E}$.*

Proof. (Outline of this proof: First we define a clopen subset $A_{\alpha,\beta}$ and using this set we define a clopen subset O of $\mathfrak{E}$ with $0 \in O$. After that, we show that $V + V \nsubseteq O$ for any clopen subset V of $\mathfrak{E}$ with $0 \in V$.)

Fix any $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$ with the condition that $\alpha^2 \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}$.

Take any $x = (x_1, x_2, x_3, \dots) \in \mathfrak{E}$. If $\{m \in \mathbb{N}^+ : \sqrt{\sum_{k=1}^{m} |x_k|^2} > \alpha\} \neq \emptyset$, then let us say that $m_{x,\alpha}$ *exists* and define $m_{x,\alpha} = \min\{m \in \mathbb{N}^+ : \sqrt{\sum_{k=1}^{m} |x_k|^2} > \alpha\}$. If $\{m \in \mathbb{N}^+ : \sqrt{\sum_{k=1}^{m} |x_k|^2} > \alpha\} = \emptyset$, then let us say that $m_{x,\alpha}$ *does not exist.*

Now, define

$$\mathfrak{E}_\alpha = \{x \in \mathfrak{E} : m_{x,\alpha} \text{ does not exist}\}$$

and define

$$A_{\alpha,\beta} = \{x = (x_1, x_2, \dots) \in \mathfrak{E} : m_{x,\alpha} \text{ exists and } |x_l| < \beta \text{ for all } l > m_{x,\alpha}\} \cup \mathfrak{E}_\alpha.$$

Claim 1. The open ball $B(0, \alpha)$ is a subset of $A_{\alpha,\beta}$. (Here, $0 = (0, 0, 0, \dots)$ is the identity element of the topological group $(\mathfrak{E}, \tau)$).

Proof of Claim 1. Take any $x = (x_1, x_2, x_3, \dots) \in B(0, \alpha)$. Then, for all $m \in \mathbb{N}^+$, $\sqrt{\sum_{k=1}^{m} |x_k|^2} \leq ||x|| < \alpha$. Thus, $m_{x,\alpha}$ does not exist. So, $x \in \mathfrak{E}_\alpha \subseteq A_{\alpha,\beta}$. □

Claim 2. $A_{\alpha,\beta}$ is a closed subset of $\mathfrak{E}$, i.e., $\mathfrak{E} - A_{\alpha,\beta} \in \tau$.

Proof of Claim 2.

To see $\mathfrak{E} - A_{\alpha,\beta}$ is open, take and fix any $z = (z_1, z_2, z_3, \dots) \in \mathfrak{E} - O$. So, $m_{z,\alpha}$ exists, $\alpha < \sqrt{\sum_{k=1}^{m_{z,\alpha}} |z_k|^2}$ and there exists an $l_0 > m_{z,\alpha}$ with $|z_{l_0}| \geq \beta$

because $z \notin A_{\alpha,\beta}$. Then, $|z_{l_0}| > \beta$, because $z_{l_0} \in \mathbb{Q}$ and $\beta \notin \mathbb{Q}$. Thus, we can define $r_0 = \min\{|z_{l_0}| - \beta, \sqrt{\sum_{k=1}^{m_{z,\alpha}} |z_k|^2} - \alpha\}$ and take the open ball $B(z, r_0)$. Take any $y = (y_1, y_2, y_3, \ldots) \in B(z, r_0)$.

$\sqrt{\sum_{k=1}^{m_{z,\alpha}} |z_k|^2} - \sqrt{\sum_{k=1}^{m_{z,\alpha}} |y_k|^2} \leq \sqrt{\sum_{k=1}^{m_{z,\alpha}} |z_k - y_k|^2} \leq \sqrt{\sum_{k=1}^{\infty} |z_k - y_k|^2} = ||z - y|| < r \leq \sqrt{\sum_{k=1}^{m_{z,\alpha}} |z_k|^2} - \alpha$. So, $\sqrt{\sum_{k=1}^{m_{z,\alpha}} |z_k|^2} - \sqrt{\sum_{k=1}^{m_{z,\alpha}} |y_k|^2} < \sqrt{\sum_{k=1}^{m_{z,\alpha}} |z_k|^2} - \alpha$. Thus, $\alpha < \sum_{k=1}^{m_{z,\alpha}} |y_k|^2$.

So, $m_{y,\alpha}$ exists and because of the definition of $m_{y,\alpha}$, $m_{y,\alpha} \leq m_{z,\alpha}$.

Thus, $l_0 \geq m_{z,\alpha} \geq m_{y,\alpha}$ and $|z_{l_0}| - |y_{l_0}| \leq |z_{l_0} - y_{l_0}| = \sqrt{|z_{l_0} - y_{l_0}|^2} \leq \sqrt{\sum_{k=1}^{\infty} |z_k - y_k|^2} = ||z - y|| < r \leq |z_{l_0}| - \beta$. So, $|z_{l_0}| - |y_{l_0}| < |z_{l_0}| - \beta$. Thus, $\beta < |y_{l_0}|$.

Because $m_{y,\alpha}$ exists and there exists $l_0 > m_{y,\alpha}$ such that $\beta < |y_{l_0}|$, $y \notin A_{\alpha,\beta}$. Therefore $z \in B(z, r_0) \subseteq \mathfrak{E} - A_{\alpha,\beta}$. Hence, $A_{\alpha,\beta}$ is a closed subset of $\mathfrak{E}$. $\square$

Claim 3. $A_{\alpha,\beta}$ is an open subset of $\mathfrak{E}$, i.e., $A_{\alpha,\beta} \in \tau$.

Proof of Claim 3. Take and fix any $x = (x_1, x_2, x_3, \ldots) \in A_{\alpha,\beta}$. There are two cases: $m_{x,\alpha}$ does not exist or $m_{x,\alpha}$ exists.

Case 1: $m_{x,\alpha}$ does not exist.

Then, $\{m \in \mathbb{N}^+ : \sqrt{\sum_{k=1}^{m} |x_k|^2} > \alpha\} = \emptyset$. So, $\sqrt{\sum_{k=1}^{m} |x_k|^2} \leq \alpha$ for all $m \in \mathbb{N}^+$. For this case either $||x|| = \sqrt{\sum_{k=1}^{\infty} |x_k|^2} < \alpha$ or $||x|| = \alpha$.

If $||x|| < \alpha$, then from Claim 1, $x \in B(0, \alpha) \subseteq A_{\alpha,\beta}$.

Now, suppose $||x|| = \alpha$. Say $r_1 = \sqrt{\alpha^2 + \beta^2} - \alpha$. Then, to see the open ball $B(x, r_1)$ is a subset of $A_{\alpha,\beta}$, take any $y = (y_1, y_2, y_3, \ldots) \in B(x, r_1)$. If $m_{y,\alpha}$ does not exist, then $y \in \mathfrak{E}_\alpha \subseteq A_{\alpha,\beta}$. If $m_{y,\alpha}$ exists, then aiming for a contradiction, suppose there exists an $l > m_{y,\alpha}$ and $|y_l| \geq \beta$. Then, $||y|| \leq ||y - x|| + ||x|| < r_1 + ||x|| < \sqrt{\alpha^2 + \beta^2} - \alpha + \alpha = \sqrt{\alpha^2 + \beta^2}$. Therefore, $||y|| < \sqrt{\alpha^2 + \beta^2}$. So, $\alpha^2 + \beta^2 > ||y||^2 = \sum_{k=1}^{\infty} |y_k|^2 \geq |y_l|^2 + \sum_{k=1}^{m_{y,\alpha}} |y_k|^2 > \beta^2 + \alpha^2$. Thus, we get the contradiction $\alpha^2 + \beta^2 > \beta^2 + \alpha^2$. From the contradiction, we get $|y_l| \leq \beta$. Becuse $|y_l| \in \mathbb{Q}$, $\beta \notin \mathbb{Q}$ and $l > m_{y,\alpha}$ is arbitrary, $|y_l| < \beta$ for all $l > m_{y,\alpha}$. Hence, $y \in A_{\alpha,\beta}$. Thus, $x \in int(A_{\alpha,\beta})$.

Case 2: $m_{x,\alpha}$ exists.

Because $||x|| = \sqrt{\sum_{k=1}^{\infty} |x_k|^2} < \infty$, for $\varepsilon = \frac{\beta}{2}$ there exists an $l_0 \in \mathbb{N}^+$ such that $\sqrt{\sum_{k=l_0}^{\infty} |x_k|^2} < \varepsilon = \frac{\beta}{2}$. Thus,

$$\sqrt{\sum_{k=l_0}^{\infty} |x_k|^2} < \frac{\beta}{2}. \tag{1}$$

Subclaim 1. For any $y = (y_1, y_2, \ldots) \in \mathfrak{E}$ if $||y - x|| < \frac{\beta}{2}$, then $|y_l| < \beta$ for all $l \geq l_0$.

Proof of Subclaim 1.

$-\sqrt{\sum_{k=l_0}^{\infty}|x_k|^2}+\sqrt{\sum_{k=l_0}^{\infty}|y_k|^2} \leq \sqrt{\sum_{k=l_0}^{\infty}|x_k-y_k|^2} \leq \sqrt{\sum_{k=1}^{\infty}|x_k-y_k|^2}$ $= ||x-y|| < \frac{\beta}{2}$. So, $-\sqrt{\sum_{k=l_0}^{\infty}|x_k|^2}+\sqrt{\sum_{k=l_0}^{\infty}|y_k|^2} < \frac{\beta}{2}$. Thus, from (1), $\sqrt{\sum_{k=l_0}^{\infty}|y_k|^2} < \frac{\beta}{2}+\sqrt{\sum_{k=l_0}^{\infty}|x_k|^2} = \beta$. Therefore, $|y_l| \leq \sqrt{\sum_{k=l_0}^{\infty}|y_k|^2} < \beta$ where $l \geq l_0$. So, $|y_l| < \beta$ for all $l \geq l_0$. The proof of Subclaim 1 is completed.

Now, say

$$h_x = \min\{\big|\beta - |x_i|\big| : i \leq l_0\}$$

and

$$a_x = \begin{cases} \alpha - \sqrt{\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2} & : m_{x,\alpha} > 1 \\ 1 & : m_{x,\alpha} = 1 \end{cases}$$

(We don't need to case $a_x = 1$. To get a well defined r, we write it. Note that if $m_{x,\alpha} > 1$, then $\alpha - \sqrt{\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2} > 0$, because $\sqrt{\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2} \leq \alpha$, so, $\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2 \leq \alpha^2$. Because $\alpha^2 \notin \mathbb{Q}$ and $\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2 \notin \mathbb{Q}$, $\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2 < \alpha^2$. Thus, $\sqrt{\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2} < \alpha$.)

Now, define

$$r = \min\{\sqrt{\sum_{k=1}^{m_{x,\alpha}}|x_k|^2} - \alpha, \frac{\beta}{2}, h_x, a_x\}.$$

To see the open ball $B(x,r)$ is a subset of $A_{\alpha,\beta}$, take any $y = (y_1, y_2, y_3, \ldots) \in B(x,r)$.

$\sqrt{\sum_{k=1}^{m_{x,\alpha}}|x_k|^2} - \sqrt{\sum_{k=1}^{m_{x,\alpha}}|y_k|^2} \leq \sqrt{\sum_{k=1}^{m_{x,\alpha}}|x_k-y_k|^2} \leq \sqrt{\sum_{k=1}^{\infty}|x_k-y_k|^2}$ $= ||x-y|| < r \leq \sqrt{\sum_{k=1}^{m_{x,\alpha}}|x_k|^2} - \alpha$. Thus, $\sqrt{\sum_{k=1}^{m_{x,\alpha}}|x_k|^2} - \sqrt{\sum_{k=1}^{m_{x,\alpha}}|y_k|^2} < \sqrt{\sum_{k=1}^{m_{x,\alpha}}|x_k|^2} - \alpha$ and so, $\sqrt{\sum_{k=1}^{m_{x,\alpha}}|y_k|^2} > \alpha$. Therefore, $m_{y,\alpha}$ exists and $m_{y,\alpha} \leq m_{x,\alpha}$.

Now, we will see that $m_{y,\alpha} = m_{x,\alpha}$. If $m_{x,\alpha} = 1$, then $m_{y,\alpha} = m_{x,\alpha}$ because $m_{y,\alpha} \leq m_{x,\alpha}$. If $m_{x,\alpha} > 1$, take any $n < m_{x,\alpha}$. Then, $-\sqrt{\sum_{k=1}^{n}|x_k|^2} + \sqrt{\sum_{k=1}^{n}|y_k|^2} \leq \sqrt{\sum_{k=1}^{n}|x_k-y_k|^2} \leq \sqrt{\sum_{k=1}^{\infty}|x_k-y_k|^2} = ||x-y|| < r \leq a_x \leq \alpha - \sqrt{\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2}$.

So, $-\sqrt{\sum_{k=1}^{n}|x_k|^2} + \sqrt{\sum_{k=1}^{n}|y_k|^2} < \alpha - \sqrt{\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2}$. Therefore $\sqrt{\sum_{k=1}^{n}|y_k|^2} < \alpha + \sqrt{\sum_{k=1}^{n}|x_k|^2} - \sqrt{\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2}$. Because $n \leq m_{x,\alpha} - 1$, $\sqrt{\sum_{k=1}^{n}|x_k|^2} - \sqrt{\sum_{k=1}^{m_{x,\alpha}-1}|x_k|^2} \leq 0$. Thus, $\sqrt{\sum_{k=1}^{n}|y_k|^2} < \alpha < \sqrt{\sum_{k=1}^{m_{y,\alpha}}|y_k|^2}$. So, $n < m_{y,\alpha}$. Hence, $n < m_{y,\alpha}$ for all $n < m_{x,\alpha}$. Therefore, $m_{y,\alpha} \geq m_{x,\alpha}$. So, $m_{y,\alpha} = m_{x,\alpha}$.

Fix any arbitrary $l > m_{y,\alpha} = m_{x,\alpha}$. We will see $|y_l| < \beta$. If $l \geq l_0$, then because $||x-y|| < r \leq \frac{\beta}{2}$, from Subclaim 1, $|y_l| < \beta$. If $l_0 \geq l > m_{y,\alpha} = m_{x,\alpha}$ where $l_0 > m_{y,\alpha}$, then $-|x_l| + |y_l| \leq |x_l - y_l| \leq ||x-y|| < r \leq \big|\beta - |x_l|\big|$. On the other hand, $|x_l| < \beta$ because $x \in A_{\alpha,\beta}$, $m_{x,\alpha}$ exists and $l > m_{x,\alpha}$. Thus, $-|x_l| + |y_l| < \big|\beta - |x_l|\big| = \beta - |x_l|$. Hence, $|y_l| < \beta$.

For Case 2, because $m_{y,\alpha}$ exists and $|y_l| < \beta$ for all $l > m_{y,\alpha}$, $y \in A_{\alpha,\beta}$. Therefore, $B(x, r) \subseteq A_{\alpha,\beta}$ and so, $x \in int(A_{\alpha,\beta})$.

Therefore, $A_{\alpha,\beta}$ is an open subset of $\mathfrak{E}$. $\square$

Let (α_n) be a strictly increasing sequence of positive real numbers ($\alpha_1 < \alpha_2 < \alpha_3 < \ldots$) such that $(\alpha_n) \to \infty$ and $\alpha_n^2 \notin \mathbb{Q}$ for all $n \in \mathbb{N}^+$. And let (β_n) be a strictly decreasing sequence of positive real numbers such that $(\beta_n) \to 0$ and $\beta_n \notin \mathbb{Q}$ for all $n \in \mathbb{N}^+$. Then, we can give the following claim.

Claim 4. Let $O = \bigcap_{n\in\mathbb{N}^+} A_{\alpha_n,\beta_n}$. Then, the identity element $0 \in O$ and O is a clopen subset of $\mathfrak{E}$, i.e., $O \in \tau$ and $\mathfrak{E} - O \in \tau$.

Proof of Claim 4. From Claim 1, $0 \in A_{\alpha_n,\beta_n}$ for all $n \in \mathbb{N}^+$. Thus, $0 \in O$.

From Claim 2, each A_{α_n,β_n} is closed. Thus, $O = \bigcap_{n\in\mathbb{N}^+} A_{\alpha_n,\beta_n}$ is a closed in $(\mathfrak{E}, \tau)$.

To see O is open in $(\mathfrak{E}, \tau)$, take and fix any $x \in O$. Because $(\alpha_n) \to \infty$, there is an $n_0 \in \mathbb{N}^+$ such that $||x|| < \alpha_{n_0}$. Thus, $x \in B(0, \alpha_{n_0})$. Define $W = (\bigcap_{n\leq n_0} A_{\alpha_n,\beta_n}) \cap (B(0, \alpha_{n_0}))$. So, clearly W is open, and $x \in W$ because $x \in B(0, \alpha_{n_0})$ and $x \in \bigcap_{n\in\mathbb{N}^+} A_{\alpha_n,\beta_n}$. Now, fix any $m \in \mathbb{N}^+$. We will see that $W \subseteq A_{\alpha_m,\beta_m}$. If $m \leq n_0$, then $W \subseteq \bigcap_{n\leq n_0} A_{\alpha_n,\beta_n} \subseteq A_{\alpha_m,\beta_m}$. Suppose $m \geq n_0$. Because (α_n) is strictly increasing sequence, $\alpha_m \geq \alpha_{n_0}$. Then, $W \subseteq B(0, \alpha_{n_0}) \subseteq B(0, \alpha_m)$. Thus, $W \subseteq B(0, \alpha_m) \subseteq A_{\alpha_m,\beta_m}$ because we know that $B(0, \alpha_m) \subseteq A_{\alpha_m,\beta_m}$ from Claim 1. Thus, $W \subseteq O = \bigcap_{n\in\mathbb{N}^+} A_{\alpha_n,\beta_n}$ because m is an arbitrary element of $\mathbb{N}^+$. Hence, $x \in W \subseteq O$ and W is open. Therefore, O is an open subset of $\mathfrak{E}$. $\square$

Claim 5. If V is any open unbounded subset of $(\mathfrak{E}, \tau)$ such that $0 = (0, 0, 0, \ldots) \in V$, then $V + V \nsubseteq O$.

Proof of Claim 5. Fix any open unbounded subset V of $\mathfrak{E}$ such that the identity element $0 \in V$. Then, there exists an $r^* > 0$ such that the open ball $B(0, r^*)$ is a subset of V. We can find an $n^* \in \mathbb{N}^+$ such that $0 < \frac{1}{n^*} < r^*$. Because $(\alpha_n) \to \infty$ and $(\beta_n) \to 0$, there exist $m_1, m_2 \in \mathbb{N}^+$ such that $0 < \beta_{m_1} < \frac{1}{n^*}$ and $n^* < \alpha_{m_2}$. Say $m^* = \max\{m_1, m_2\}$. Then, $\beta_{m^*} < \frac{1}{n^*}$ and $n^* < \alpha_{m^*}$. Because V is unbounded, there exists an $x \in V$ such that $\alpha_{m^*} < ||x||$. Thus, there exists an $m \in \mathbb{N}^+$ such that $\alpha_{m^*} < \sqrt{\sum_{k=1}^{m} |x_k|^2}$. Therefore, $m_{x,\alpha_{m^*}}$ exists. Now, fix any $l^* > m_{x,\alpha_{m^*}}$ and a rational number q such that $\frac{1}{\beta_{m^*}} < q < r^*$. Let $e^{l^*} = (e_1, e_2, \ldots) \in \mathfrak{E}$ such that $e_{l^*} = 1$ and $e_k = 0$ where $k \neq l^*$.

Case 1: $x_{l^*} \geq 0$.

Say $y = q.e^{l^*}$. So, $y_{l^*} = q$ and $y_k = 0$ for all $k \neq l^*$ where $y = (y_1, y_2, y_3, \ldots)$. Then, $y \in B(0, r^*)$ because $||y - 0|| = ||y|| = q < r^*$. So, $y \in V$. Now, define $z = x + y$. Thus, $z \in V + V$. Also $\sqrt{\sum_{k=1}^{m_{x,\alpha_{m^*}}} |z_k|^2} = \sqrt{\sum_{k=1}^{m_{x,\alpha_{m^*}}} |x_k|^2} > \alpha_{m^*}$. So, $m_{z,\alpha_{m^*}}$ exists and from definition of $m_{z,\alpha_{m^*}}$, $m_{z,\alpha_{m^*}} \leq m_{x,\alpha_{m^*}} \leq l^*$. Thus, $|z_{l^*}| = |x_{l^*} + q| = x_{l^*} + q > \frac{1}{\beta_{m^*}}$. So, $z \notin A_{\alpha_{m^*},\beta_{m^*}}$. Therefore, $z \notin O$. Hence $z \in V + V$ and $z \notin O$.

Case 2: $x_{l^*} < 0$.

Say $y = -q.e^{l^*}$. Define $z = x+y$, in a similar manner to Case 1, $z \in V+V$ and $z \notin O$.

Therefore $V + V \nsubseteq O$. $\square$

From Claim 4 and Claim 5, there is a clopen subset O of $(\mathfrak{E}, \tau)$ with $0 \in O$ such that $V+V \nsubseteq O$ if V is any open unbounded subset of $(\mathfrak{E}, \tau)$ with $0 \in V$. From Theorem 1.2, if V is any clopen subset of $(\mathfrak{E}, \tau)$ with $0 \in V$, then V is unbounded. Therefore, $O \in \tau_{clopen}$ with $0 \in O$ such that $V + V \nsubseteq O$ for any $V \in \tau_{clopen}$ with $0 \in V$. Hence, from Theorem 1.1, the topology τ_{clopen} is not compatible with the group structure on $\mathfrak{E}$.

$\square$

In Claim 5 which is in the proof above, for the clopen set O, actually we showed that if K is unbounded subset of $\mathfrak{E}$, then $K + B(0, \varepsilon) \nsubseteq O$ for any $\varepsilon > 0$.

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