



Continuity problem for singular BSDE with random terminal time

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Abstract. We study a class of non-linear Backward stochastic differential equations (BSDE) with a superlinear driver process f adapted to a filtration \mathbb{F} and over a random time interval $\llbracket 0, S \rrbracket$ where S is a stopping time of \mathbb{F} . The terminal condition ξ is allowed to take the value $+\infty$, i.e., singular. We call a stopping time S solvable with respect to a given BSDE and filtration if the BSDE has a minimal supersolution with terminal value ∞ at terminal time S . Our goal is to show existence of solutions to the BSDE for a range of singular terminal values under the assumption that S is solvable. We will do so by proving that the minimal supersolution to the BSDE is a solution, i.e., it is continuous at time S and attains the terminal value with probability 1. We consider three types of terminal values: 1) Markovian: i.e., ξ is of the form $\xi = g(\Xi_S)$ where Ξ is a continuous Markovian diffusion process, S is a hitting time of Ξ and g is a deterministic function 2) terminal conditions of the form $\xi_1 = \infty \cdot \mathbf{1}_{\{\tau \leq S\}}$ and 3) $\xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}$ where τ is another stopping time. For general ξ we prove that minimal supersolution has a limit at time S provided that \mathbb{F} is left continuous at time S . Finally, we discuss the implications of our results about Markovian terminal conditions to the solution of non-linear elliptic PDE with singular boundary conditions.

1. Introduction and definitions

A backward stochastic differential equation (BSDE) is a stochastic differential equation (SDE) with a prescribed terminal condition. They have been intensively studied since the seminal papers [Bismut \(1973\)](#); [Pardoux and Peng \(1990\)](#); they arise naturally in stochastic optimal control problems (see among others [Yong and Zhou \(1999\)](#)), they provide a probabilistic representation of semi-linear partial differential equations (PDE) extending the Feynman-Kac formula ([Pardoux](#)

Received by the editors May 3rd, 2021; accepted July 15th, 2022.

2010 *Mathematics Subject Classification.* 35J75, 60G40, 60G99, 60H30, 60H99.

Key words and phrases. Backward stochastic differential equation, stopping time, singularity, continuity problem.

This work is supported by TUBITAK (The Scientific and Technological Research Council of Turkey) through project number 118F163. We are grateful for this support.

and Răşcanu (2014)) and they have found numerous applications in finance and insurance (Delong (2013); El Karoui et al. (1997)).

If the driver term of the BSDE has superlinear growth the solution of the BSDE can blow up in finite time, this allows one to specify ∞ as a possible terminal value for such BSDE; when the terminal value is allowed to take ∞ it is called “singular”¹. In Ahmadi et al. (2021); Kruse and Popier (2016b); Popier (2006); Sezer et al. (2019), we study non-linear BSDE with singular terminal condition at a deterministic terminal time T . Such BSDE generalize parabolic diffusion-reaction PDE with singular final trace (Graewe et al. (2018); Marcus and Véron (1999); Popier (2017)) and they can be used to represent value functions of a class of stochastic optimal control problems with terminal constraints (Ankirchner et al. (2014); Graewe et al. (2018); Kruse and Popier (2016b) and the references therein).

In this paper, we focus on BSDE with singular terminal conditions over a random time horizon. We adopt the general framework for BSDE with terminal singular values established in Kruse and Popier (2016a,b, 2017) and consider BSDE of the following form

$$dY_t = -f(t, Y_t, Z_t, U_t)dt + Z_t dW_t + \int_{\mathcal{E}} U_t(e) \tilde{\pi}(de, dt) + dM_t, \quad Y_S = \xi, \quad (1.1)$$

where

- W is a d -dimensional Brownian motion;
- π is a Poisson random measure with intensity $\mu(de)dt$ on the space $\mathcal{E} \subset \mathbb{R}^m \setminus \{0\}$. The measure μ is σ -finite on \mathcal{E} such that

$$\int_{\mathcal{E}} (1 \wedge |e|^2) \mu(de) < +\infty.$$

The compensated Poisson random measure $\tilde{\pi}$ is defined by: $\tilde{\pi}(de, dt) = \pi(de, dt) - \mu(de)dt$. W and $\tilde{\pi}$ are supported by the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$. Filtration \mathbb{F} is supposed to be complete and right continuous. W and $\tilde{\pi}$ are martingales w.r.t. the filtration \mathbb{F} .

The unknown that is sought is the quadruple (Y, Z, U, M) ; the solution component M is required to be a local martingale orthogonal to $\tilde{\pi}$. The function $f : \Omega \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{B}_{\mu}^2 \rightarrow \mathbb{R}$ is called the *generator* (or *driver*) of the BSDE. Finally S is a stopping time of the filtration \mathbb{F} and ξ is an \mathcal{F}_S measurable random variable, which is singular, i.e., $\mathbb{P}(\{\xi = \infty\}) > 0$. Precise conditions on all of these terms are spelled out in subsections 1.2 and 1.3 below. A quadruple (Y, Z, U, M) is said to be a supersolution of (1.1) if it satisfies the first equation in (1.1) and

$$\liminf_{t \rightarrow +\infty} Y_{t \wedge S} \geq \xi, \quad \text{almost surely,} \quad (1.2)$$

holds.

In what follows we will indicate the terminal condition ξ (and sometimes the terminal time S) as a superscript of the (super)solution, e.g., $Y^{S,k}$ will denote the Y component of the solution with terminal condition $Y_S = k$; we omit the terminal time from the superscript when an emphasis on the terminal time is not necessary (see Section 8 for a list of the symbols and notation used in the paper).

A supersolution $(Y^\xi, Z^\xi, U^\xi, M^\xi)$ is called minimal if $Y^\xi \leq Y$ for any other supersolution (Y, Z, U, M) . We say $(Y^\xi, Z^\xi, U^\xi, M^\xi)$ solves the BSDE with singular terminal condition ξ if it satisfies the first equation in (1.1) and

$$\lim_{t \rightarrow +\infty} Y_{t \wedge S}^\xi = \xi; \quad (1.3)$$

¹Note that singular could also mean that the terminal value does not belong to any integrability space (see Condition (1.6)).

i.e., to go from a supersolution to a solution we need to replace the \liminf in (1.2) with \lim and \geq with $=$. In the rest of this paper whenever we refer to the “solution” of a BSDE with a singular terminal value, it will be in the sense of (1.3). The condition (1.3) means that the process Y is continuous at time S ; for this reason we refer to the problem of establishing that a candidate solution satisfies (1.3) as the “*continuity problem*.” The present paper is devoted to the study of this problem for a range of terminal values; we introduce the terminal values we focus on in the next subsection.

Just as BSDE over deterministic time intervals generalize parabolic PDE, BSDE over random time intervals are generalizations of elliptic PDE; we provide further comments on this connection, on the motivation for the study of BSDE over a random time horizon with singular terminal values and on the implications of continuity results for BSDE theory and for constrained stochastic optimal control at the end of the next subsection.

We call a terminal condition “Markovian” if it is of the form $\xi = g(\Xi_S)$ where, $g : \mathbb{R}^d \mapsto \mathbb{R} \cup \{\infty\}$, Ξ is a Markov diffusion process and S is the first time Ξ hits a smooth boundary ∂D of a domain $D \subset \mathbb{R}^d$. For such exit times, existence of minimal supersolutions for (1.1) is proved in Kruse and Popier (2016b) for a general singular terminal condition (see subsection 1.3 below). The work Popier (2007) proves that these minimal supersolutions are in fact solutions for the case $\mathbb{F} = \mathbb{F}^W$ and for the specific generator $f(y) = -y|y|^{q-1}$ and for Markovian terminal conditions. The works Kruse and Popier (2016a, 2017) develop solutions to (1.1) when ξ belongs to some integrability space. The goal of the present work is to prove that the minimal supersolution of (1.1) satisfies (1.3) (and therefore is a solution) for several classes of singular terminal conditions and several assumptions on S . We outline these classes and assumptions in the next subsection.

1.1. *Outline of results.* In two previous works Sezer et al. (2019) and Ahmadi et al. (2021) that prove continuity results for deterministic terminal times, two of the main ingredients are the minimal supersolution Y^∞ with terminal condition ∞ at terminal time and the a priori upperbounds on supersolutions; both of these, are readily available in the prior literature for deterministic terminal times (for the one dimensional Brownian case treated in Sezer et al. (2019), Y^∞ is deterministic and has an explicit formula). For random terminal times the existence of Y^∞ and a priori upperbounds are known only for exit times of Markov diffusions from smooth domains. One of the main ideas of the present work is to impose the existence of Y^∞ as an assumption on the stopping time S and base most of our arguments on this assumption. We call the terminal stopping time S *solvable* with respect to the BSDE (1.1) if there exists a supersolution to the BSDE with terminal value ∞ at terminal time S (see Definition 1.8). Deterministic times and exit times of Markov diffusion processes are solvable for a wide range of BSDE; times that have a strictly positive density around 0 are not solvable (Kruse and Popier (2016b)). Many of our arguments are based on this solvability concept; some basic consequences of solvability are given in Section 2. In particular, if S is solvable, the BSDE (1.1) has a minimal supersolution for any singular terminal condition $\xi \geq 0$ (Lemma 2.1). In addition to S being solvable, in many arguments we assume \mathbb{F} to be left continuous at S for the following reason. Because the filtration \mathbb{F} is assumed to be general (a priori only completeness and right-continuity is assumed) there is no way to control the jumps of the additional local martingale component M of the solution at the terminal time. To avoid such jumps, we suppose that \mathbb{F} is left-continuous at time S .

We now indicate the main results of the present work. In Section 3 we assume S to be solvable and consider the problem of proving the existence of $\lim_{t \rightarrow \infty} Y_t^\xi$ for an arbitrary singular terminal condition $\xi \geq 0$. For S deterministic, the work Popier (2016) establishes the existence of this limit under some additional conditions on the generator f . Here we show that these assumptions are also sufficient for a random terminal time (Section 3) provided that it is solvable.

The next three sections focus on the continuity problem. Section 4 treats Markovian terminal conditions. To the best of our knowledge, Popier (2007) is the only paper that proves continuity

results for a singular terminal condition at a random time S ; Popier (2007) assumes $f(y) = -y|y|^{q-1}$, ξ to be Markovian and \mathbb{F} Brownian. The results in Section 4 generalize the results in Popier (2007) to a general filtration \mathbb{F} and driver f keeping the terminal condition Markovian. An important step is a bound on the expected value of an integral over the solution processes and $\text{dist}(\Xi)$, where $\text{dist}(x) = \inf_{y \in \partial D} |x - y|$ (see Lemma 4.3). One of the main ingredients in the proof is the a priori upperbound on Y derived in Kruse and Popier (2016b). When \mathbb{F} is Brownian and f is deterministic, the solution of the BSDE with a Markovian terminal condition can be used to construct a viscosity solution of an associated elliptic PDE. This is discussed in subsection 4.1.

Sections 5 and 6 focus on the continuity problem for non-Markovian terminal conditions² of the form $\xi_1 = \infty \cdot \mathbf{1}_{\{\tau \leq S\}}$ (Section 5) and $\xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}$ (Section 6) where τ is another stopping time of \mathbb{F} . The results in these sections generalize results from Sezer et al. (2019) (the one dimensional Brownian case) and Ahmadi et al. (2021) (the general filtration, driver case) treating the same type of terminal conditions where S is assumed to be deterministic. Events of the form $\{\tau \leq S\}$ naturally arise when one modifies constraints on stochastic optimal control problems based on the values the state process of the problem takes. We refer to Sezer et al. (2019); Ahmadi et al. (2021) for more comments on why we pay particular attention to these type of non-Markovian terminal conditions. Solution of the continuity problem for general terminal conditions of the form $\infty \cdot \mathbf{1}_A$ for arbitrary $A \in \mathcal{F}_S$ is an open problem even for the one dimensional Brownian case and S deterministic.

Section 5 provides two arguments to prove

$$\lim_{t \rightarrow +\infty} Y_{t \wedge S}^{\xi_1} = \xi_1. \quad (1.4)$$

The first one is an adaptation of the argument given for the same type of terminal condition in Ahmadi et al. (2021). It involves the construction of an auxiliary linear process that dominates Y^{ξ_1} and that is known to have the desired limit property at the terminal time S . The main assumption on τ for the construction of the upperbound in Ahmadi et al. (2021) is that τ has bounded density at the terminal time; in the current setting this is replaced with the assumption that the random variable $\mathbf{1}_{\{\tau \leq S\}} Y_\tau^\infty$ has a bounded ϱ -moment for some $\varrho > 1$ (see (5.2)). The other main ingredient in the construction of the upperbound process in Ahmadi et al. (2021) is the a priori upperbounds on the supersolution of BSDE; in the current context this is replaced by the solvability assumption on S . Subsection 5.2 presents a new argument for the terminal value ξ_1 that is completely based on the original BSDE (i.e., it doesn't involve the solution of an auxiliary linear BSDE). To simplify arguments this subsection assumes \mathbb{F} to be generated only by the Brownian motion W . The only assumption on τ is that it be solvable. Let $Y^{\tau, \infty}$ be the supersolution of the BSDE with terminal condition ∞ at terminal time τ . The main idea of this argument is the use of the process $Y^{\tau, \infty}$ as an upperbound to prove (1.4). Working directly with the original BSDE in constructing upperbounds can lead to less stringent conditions on model parameters. As an example, we consider in subsection 5.3 the case $S = T$ and $\tau = \inf\{t : |W_t| = L\}$ which was originally studied in Sezer et al. (2019) using essentially a special case of the argument based on the linear auxiliary process which requires the q parameter in assumption (B1) to satisfy $q > 2$. The new proof given in subsection 5.3 based on the new argument based on solvable stopping times establishes (1.4) for the minimal supersolution assuming only $q > 1$.

The argument in Section 6 that proves that the minimal supersolution corresponding to ξ_2 is in fact a solution follows closely the argument given for the same type of terminal condition in Ahmadi et al. (2021) for the case $S = T$ deterministic. The assumptions in this section are: S is solvable and $\mathbb{P}(S = \tau) = 0$; no solvability is required for τ . To simplify arguments \mathbb{F} is assumed to be generated by the Brownian motion only.

²We define $\infty \cdot 0 = 0$.

Comments on implications for stochastic optimal control, PDE and BSDE theory. BSDE with random terminal times are a generalization of elliptic semi-linear PDE (extension of the Feynman-Kac formula, see Darling and Pardoux (1997); Pardoux (1999); Pardoux and Răşcanu (2014)). The works Dynkin and Kuznetsov (1998); Le Gall (1997); Marcus and Véron (1998); Marcus and Veron (1998) show that the solution of some of these PDE can exhibit a singularity of the following form on the boundary of the domain D

$$\lim_{x \rightarrow \partial D} u(x) = +\infty.$$

This boundary behavior generalizes to

$$\lim_{t \rightarrow +\infty} Y_{t \wedge S} = +\infty$$

for BSDE of the form (1.1) (the clearest connection between the last two condition arises when S is a first hitting time of a Markov diffusion process, see subsection 4.1). Minimal supersolutions of BSDE of the type (1.1) with ∞ -valued terminal values at random terminal times can also be used to express the value function of a class of stochastic optimal control problems over a random time horizon $\llbracket 0, S \rrbracket$ with terminal constraints of the form $\mathbf{1}_A \cdot q_S = 0$, for some $A \in \mathcal{F}_S$, where q is the controlled process (see Kruse and Popier (2016b)).

Strengthening (1.11) to (1.3) (i.e., going from a supersolution to a solution) has implications both for BSDE theory and for stochastic optimal control applications. Consider two distinct terminal values ξ^1 and ξ^2 ; with (1.11) it is impossible to tell whether the corresponding minimal supersolutions are distinct. Whereas (1.3) guarantees that distinct solutions Y^1 and Y^2 correspond to distinct terminal values ξ^1 and ξ^2 . In stochastic optimal control / finance applications a non-tight optimal control (corresponding to strict inequality in (1.11)) can be interpreted as a strictly super-hedging trading strategy. Continuity results overrule such strategies. For more comments on these points we refer the reader to Ahmadi et al. (2021).

The next two subsections give the definitions, assumptions and results we employ from previous works (subsection 1.2 concerns integrable terminal conditions and subsection 1.3 concerns singular terminal values). The only novelty is Definition 1.8, the definition of a solvable stopping time. We comment on possible future work in the Conclusion (Section 7).

1.2. *Integrable data.* Let us start with the definition of solution for BSDE (1.1).

Definition 1.1 (Classical solution). A process $(Y, Z, U, M) = (Y_t, Z_t, U_t, M_t)_{t \geq 0}$, such that

- Y is progressively measurable and càdlàg ,
- Z is a predictable process with values in \mathbb{R}^d ,
- M is a local martingale orthogonal to W and $\tilde{\pi}$,
- U is also predictable and such that for any $t \geq 0$

$$\int_0^t \int_{\mathcal{E}} (|U_s(e)|^2 \wedge |U_s(e)|) \mu(de) < +\infty,$$

is a solution to the BSDE (1.1) with random terminal time S with data $(\xi; f)$ if on the set $\{t \geq S\}$ $Y_t = \xi$ and $Z_t = U_t = M_t = 0$, \mathbb{P} -a.s., $t \mapsto f(t, Y_t, Z_t, U_t) \mathbf{1}_{t \leq T}$ belongs to $L^1_{loc}(0, \infty)$ for any $T \geq 0$, the stochastic integrals with respect to W and $\tilde{\pi}$ are well-defined and, \mathbb{P} -a.s., for all $0 \leq t \leq T$,

$$\begin{aligned} Y_{t \wedge S} &= Y_{T \wedge S} + \int_{t \wedge S}^{T \wedge S} f(u, Y_u, Z_u, U_u) du - \int_{t \wedge S}^{T \wedge S} Z_u dW_u \\ &\quad - \int_{t \wedge S}^{T \wedge S} \int_{\mathcal{E}} U_u(e) \tilde{\pi}(de, du) - \int_{t \wedge S}^{T \wedge S} dM_u. \end{aligned} \tag{1.5}$$

For precise definitions of the stochastic integral with respect to $\tilde{\pi}$ and orthogonality, we refer to Jacod and Shiryaev (2003).

Theorem 3 of [Kruse and Popier \(2016a, 2017\)](#), ensures the existence and uniqueness of a classical solution, under some conditions on the terminal value ξ and on the generator f . Let us state these conditions and following them the theorem (Theorem 1.4 below).

Firstly the following integrability condition is assumed: for some $r > 1$

$$\mathbb{E} \left[e^{r\rho S} |\xi|^r + \int_0^S e^{r\rho t} |f(t, 0, 0, \mathbf{0})|^r dt \right] < +\infty. \tag{1.6}$$

The constant ρ depends on r and on the generator f (see Remark 1.3). We suppose that $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m \times \mathfrak{B}_\mu^2 \rightarrow \mathbb{R}$ is a random measurable function, such that for any $(y, z, \psi) \in \mathbb{R} \times \mathbb{R}^m \times \mathfrak{B}_\mu^2$, the process $f(t, y, z, \psi)$ is progressively measurable. For notational convenience we write $f_t^0 = f(t, 0, 0, \mathbf{0})$, where $\mathbf{0}$ denotes the function mapping \mathcal{E} to $0 \in \mathbb{R}$. The space \mathfrak{B}_μ^2 is defined³ as follows:

$$\mathfrak{B}_\mu^2 = \begin{cases} \mathbb{L}_\mu^2 & \text{if } r \geq 2, \\ \mathbb{L}_\mu^1 + \mathbb{L}_\mu^2 & \text{if } r < 2, \end{cases}$$

where $\mathbb{L}_\mu^p = \mathbb{L}^p(\mathcal{E}, \mu; \mathbb{R})$ is the set of measurable functions $\psi : \mathcal{E} \rightarrow \mathbb{R}$ such that

$$\|\psi\|_{\mathbb{L}_\mu^p}^p = \int_{\mathcal{E}} |\psi(e)|^p \mu(de) < +\infty.$$

The next conditions are adapted from [Kruse and Popier \(2016b\)](#):

- (A1) The function $y \mapsto f(t, y, z, \psi)$ is continuous and monotone: there exists $\chi \in \mathbb{R}$ such that a.s. and for any $t \geq 0$ and $z \in \mathbb{R}^m$ and $\psi \in \mathfrak{B}_\mu^2$

$$(f(t, y, z, \psi) - f(t, y', z, \psi))(y - y') \leq \chi(y - y')^2.$$

- (A2) For every $j > 0$ and $n \geq 0$, the process

$$\Upsilon_t(j) = \sup_{|y| \leq j} |f(t, y, 0, \mathbf{0}) - f_t^0|$$

is in $L^1((0, n) \times \Omega)$.

- (A3) There exists a progressively measurable process $\kappa : \Omega \times [0, T] \times \mathcal{E} \times \mathbb{R} \times \mathbb{R}^m \times (\mathfrak{B}_\mu^2)^2 \rightarrow \mathbb{R}$ such that for any (y, z, ψ, ϕ) , with $\kappa(\cdot, \cdot, y, z, \phi, \psi) = \kappa^{y, z, \psi, \phi}(\cdot)$,

$$f(t, y, z, \psi) - f(t, y, z, \phi) \leq \int_{\mathcal{E}} (\psi(e) - \phi(e)) \kappa_t^{y, z, \psi, \phi}(e) \mu(de),$$

with $\mathbb{P} \otimes \text{Leb} \otimes \mu$ -a.e. for any (y, z, ψ, ϕ) , $-1 \leq \kappa_t^{y, z, \psi, \phi}(e)$ and $|\kappa_t^{y, \psi, \phi}(e)| \leq \vartheta(e)$ where ϑ belongs to the dual space of \mathfrak{B}_μ^2 , that is \mathbb{L}_μ^2 or $\mathbb{L}_\mu^\infty \cap \mathbb{L}_\mu^2$.

- (A4) There exists a constant L_z such that a.s.

$$|f(t, y, z, \psi) - f(t, y, z', \psi)| \leq L_z |z - z'|$$

for any (t, y, z, z', ψ) .

Remark 1.2. We can replace (A3) by the Lipschitz condition: there exists a constant L_ϑ such that

$$|f(t, y, z, \psi) - f(t, y, z, \phi)| \leq L_\vartheta \|\psi - \phi\|_{\mathfrak{B}_\mu^2}.$$

As explained at the beginning of [Kruse and Popier \(2016a, Section 5\)](#), (A3) implies Lipschitz regularity of f with respect to ψ , with L_ϑ equal to the norm $\|\vartheta\|_{(\mathfrak{B}_\mu^2)^*}$ of ϑ in the dual space of \mathfrak{B}_μ^2 . However (A3) is sufficient to ensure comparison principle for the solution of BSDEs (see [Pardoux and Răşcanu \(2014, Proposition 5.34\)](#), [Delong \(2013, Theorem 3.2.1\)](#) or [Kruse and Popier \(2016a, Remark 4\)](#)).

³For the definition of the sum of two Banach spaces, see for example [Kreĭn et al. \(1982\)](#). The introduction of \mathfrak{B}_μ^2 is motivated in [Kruse and Popier \(2017\)](#).

We denote

$$K^2 = \frac{1}{2}(L_z^2 + L_\vartheta^2). \tag{1.7}$$

Remark 1.3. Recall the constants r and ρ appearing in (1.6). The constant ρ satisfies

$$\rho > \nu = \nu(r) := \begin{cases} \chi + K^2 & \text{if } r \geq 2, \\ \chi + \frac{K^2}{r-1} + \frac{L_\vartheta^2}{\varepsilon(L_\vartheta, r)} & \text{if } r < 2 \end{cases} \tag{1.8}$$

where the constant $0 < \varepsilon(L_\vartheta, r) < r - 1$ depends only on L_ϑ and r (see Kruse and Popier (2017), Section 4). The additional term in ν disappears if the generator does not depend on the jump part ψ (that is, if $L_\vartheta = 0$). We have the following bounds on $\varepsilon = \varepsilon(L_\vartheta, r)$:

$$0 < \varepsilon \leq (r - 1) \left(2(\alpha(L_\vartheta, r) + 1)^2 - 1 \right)^{-\frac{2-r}{2}},$$

and $\alpha(L_\vartheta, r)$ has to be chosen such that for any $x \geq \alpha(L_\vartheta, r)$,

$$\frac{1}{2^{r/2}}x^r - 2^{r/2} - 1 - r(2L_\vartheta + 1)x \geq 0.$$

The right side is an increasing function with respect to $r \in (1, 2)$ and decreasing with respect to $L_\vartheta \geq 0$. Hence when r is close to one and L_ϑ is large, ε is very small and thus ρ becomes large.

In Kruse and Popier (2016a, 2017), a second integrability condition is supposed:

$$\mathbb{E} \left[\int_0^S e^{r\rho t} |f(t, e^{-\nu t}\xi_t, e^{-\nu t}\eta_t, e^{-\nu t}\gamma_t)|^r dt \right] < +\infty, \tag{1.9}$$

where $\xi_t = \mathbb{E}(e^{\nu S}\xi | \mathcal{F}_t)$ and (η, γ, N) are given by the martingale representation:

$$e^{\nu S}\xi = \mathbb{E}(e^{\nu S}\xi) + \int_0^\infty \eta_s dW_s + \int_0^\infty \int_{\mathcal{E}} \gamma_s(e) \tilde{\pi}(de, ds) + N_S$$

with

$$\mathbb{E} \left[\left(\int_0^\infty |\eta_s|^2 ds + \int_0^\infty \int_{\mathcal{E}} |\gamma_s(e)|^2 \pi(de, ds) + [N]_S \right)^{r/2} \right] < +\infty.$$

We now state Kruse and Popier (2016a, 2017, Theorem 3):

Theorem 1.4. *If (A1) to (A4) hold and ξ and f^0 satisfy assumptions (1.6) and (1.9), then the BSDE (1.1) has a unique solution (Y, Z, U, M) in the sense of Definition 1.1 such that for any $0 \leq t \leq T$*

$$\begin{aligned} & \mathbb{E} \left[e^{r\rho(t \wedge S)} |Y_{t \wedge S}|^r + \int_0^{T \wedge S} e^{\rho s} |Y_s|^r ds + \int_0^{T \wedge S} e^{r\rho s} |Y_s|^{r-2} |Z_s|^2 \mathbf{1}_{Y_s \neq 0} ds \right] \\ & + \mathbb{E} \left[\int_0^{T \wedge S} e^{r\rho s} |Y_{s-}|^{r-2} \mathbf{1}_{Y_{s-} \neq 0} d[M]_s^c \right] \\ & + \mathbb{E} \left[\int_{t \wedge S}^{T \wedge S} \int_{\mathcal{E}} e^{r\rho s} (|Y_{s-}|^2 \vee |Y_{s-} + U_s(e)|^2)^{r/2-1} \mathbf{1}_{|Y_{s-}| \vee |Y_{s-} + U_s(e)| \neq 0} |U_s(e)|^2 \pi(de, ds) \right] \\ & + \mathbb{E} \left[\sum_{0 < s \leq T \wedge S} e^{r\rho s} |\Delta M_s|^2 (|Y_{s-}|^2 \vee |Y_{s-} + \Delta M_s|^2)^{r/2-1} \mathbf{1}_{|Y_{s-}| \vee |Y_{s-} + \Delta M_s| \neq 0} \right] < +\infty. \end{aligned}$$

And

$$\mathbb{E} \left[\left(\int_0^S e^{2\rho s} |Z_s|^2 ds \right)^{r/2} + \left(\int_0^S e^{2\rho s} \int_{\mathcal{E}} |U_s(e)|^2 \pi(de, ds) \right)^{r/2} + \left(\int_0^S e^{2\rho s} d[M]_s \right)^{r/2} \right] \leq C \mathbb{E} \left[e^{r\rho S} |\xi|^p + \int_0^S e^{r\rho s} |f(s, 0, 0, \mathbf{0})|^r ds \right].$$

The constant C depends only on r, K and χ .

In general (1.9) is not easy to check. Nonetheless if ξ is bounded, we can take $\nu = 0$ in (1.9) and assume that:

$$\mathbb{E} \left[\int_0^S e^{r\rho t} |f(t, \xi_t, \eta_t, \gamma_t)|^r dt \right] < +\infty,$$

where $\xi_t = \mathbb{E}(\xi | \mathcal{F}_t)$ and

$$\xi = \mathbb{E}(\xi) + \int_0^\infty \eta_s dW_s + \int_0^\infty \int_{\mathcal{E}} \gamma_s(e) \tilde{\pi}(de, ds) + N_S.$$

1.3. *Supersolutions for singular terminal conditions.* Theorem 1.4 gives sufficient conditions to ensure the existence and uniqueness of the classical solution (Y, Z, U, M) . When the terminal condition is singular, that is, if ξ does not belong to any $\mathbb{L}^p(\Omega)$ for some $p > 1$, we adopt the following definition:

Definition 1.5 (Supersolution for singular terminal condition). We say that a quadruple of processes (Y, Z, U, M) is a supersolution to the BSDE (1.1) with singular terminal condition $Y_S = \xi \geq 0$ if it satisfies:

- (1) There exists some $\ell > 1$ and an increasing sequence of stopping times S_n converging to S such that for all $n > 0$ and all $t \geq 0$

$$\mathbb{E} \left[\sup_{r \in [0, t]} |Y_{r \wedge S_n}|^\ell + \left(\int_0^{t \wedge S_n} |Z_r|^2 dr \right)^{\ell/2} + \left(\int_0^{t \wedge S_n} \int_{\mathcal{E}} |U_r(e)|^2 \pi(de, dr) \right)^{\ell/2} + [M]_{t \wedge S_n}^{\ell/2} \right] < +\infty;$$

- (2) Y is non-negative;
- (3) for all $0 \leq t \leq T$ and $n > 0$:

$$Y_{t \wedge S_n} = Y_{T \wedge S_n} + \int_{t \wedge S_n}^{T \wedge S_n} f(u, Y_u, Z_u, U_u) du - \int_{t \wedge S_n}^{T \wedge S_n} Z_u dW_u - \int_{t \wedge S_n}^{T \wedge S_n} \int_{\mathcal{E}} U_u(e) \tilde{\pi}(de, du) - \int_{t \wedge S_n}^{T \wedge S_n} dM_u. \tag{1.10}$$

- (4) On the set $\{t \geq S\}$: $Y_t = \xi, Z = U = M = 0$ a.s. and (1.11) holds:

$$\liminf_{t \rightarrow +\infty} Y_{t \wedge S} \geq \xi, \quad \text{a.s.} \tag{1.11}$$

We say that (Y, Z, U, M) is a minimal supersolution to the BSDE (1.1) if for any other supersolution (Y', Z', U', M') we have $Y_t \leq Y'_t$ a.s. for any $t > 0$.

To lighten the presentation, in the present paper we assume ξ and f^0 to be non-negative. Under the conditions of Theorem 1.4 and this sign assumption on ξ and f^0 , the comparison principle (Kruse and Popier (2016a, Proposition 4)) implies $Y_t \geq 0$ for $t \geq 0$.

Remark 1.6. The non-negativity condition can be replaced in general by: Y is bounded from below by a process \bar{Y} such that $\mathbb{E} \sup_{t \geq 0} |\bar{Y}_{t \wedge S}|^\ell < +\infty$ (see [Kruse and Popier \(2016b\)](#), Definition 1).

The most basic and common way of constructing a candidate supersolution for a given singular terminal condition $Y_S = \xi \geq 0$ is by approximation from below. Let $Y^{\xi \wedge k}$ be the classical solution of the BSDE for the bounded terminal condition $Y_S = \xi \wedge k$; then a candidate supersolution⁴ for the terminal condition is $Y^\xi = \lim_{k \nearrow \infty} Y^{\xi \wedge k}$ (cf. (1.17)) where the existence of the limit follows from the comparison principle; as noted in [Popier \(2006\)](#), this technique always gives a process that satisfies (1.11):

Lemma 1.7. *Let $Y^{\xi \wedge k}$ and Y^ξ be as above. If the filtration \mathbb{F} is left-continuous at time S , then Y^ξ satisfies (1.11). In particular,*

$$\lim_{t \rightarrow \infty} Y_{t \wedge S}^\xi = \xi$$

over the event $\{\xi = \infty\}$.

Proof: The left-continuity of the filtration \mathbb{F} at time S implies that the martingale part of the BSDE, M and the stochastic integral w.r.t. $\tilde{\pi}$, has no jump at time S . Thus

$$\liminf_{t \rightarrow \infty} Y_{t \wedge S}^\xi \geq \liminf_{t \rightarrow \infty} Y_{t \wedge S}^{\xi \wedge k} = \xi \wedge k \tag{1.12}$$

for all k . Letting $k \nearrow \infty$ implies

$$\liminf_{t \rightarrow \infty} Y_{t \wedge S}^\xi \geq \xi. \tag{1.13}$$

Over the event $\{\xi = \infty\}$ we have:

$$\infty \geq \limsup_{t \rightarrow \infty} Y_{t \wedge S}^\xi \geq \liminf_{t \rightarrow \infty} Y_{t \wedge S}^\xi \geq \xi = \infty$$

which gives (1.13). □

We next introduce a concept that we think provides a general and natural framework for the study of BSDE (1.1) with singular terminal conditions when the terminal time is a stopping time.

Definition 1.8. A stopping time S will be called solvable with respect to the BSDE (1.1) if the filtration \mathbb{F} is left-continuous at time S and if the BSDE (1.1) has a supersolution on the time interval $\llbracket 0, S \rrbracket$ with terminal condition $Y_S = \infty$ that is defined as the limit of the solution of the same BSDE with terminal condition equal to the constant k , as k tends to ∞ .

Most of our arguments will be based on solvable stopping times. From [Kruse and Popier \(2016b\)](#), we know that every deterministic time S is solvable provided Conditions **(A)** (given above), and **(B1)**, **(B2)** (given below) hold. Exit times of diffusions from smooth domains provide another example of a solvable stopping time, see Theorem 1.10 below (a restatement of [Kruse and Popier \(2016b\)](#), Theorem 2) in terms of solvable times). [Kruse and Popier \(2016b\)](#), Example 1) shows that any stopping time that has a strictly positive density around 0 is non-solvable. Section 2 lists some immediate consequences of the definition above that will be useful in the rest of this article.

1.3.1. *Additional conditions on f .* For a singular non-negative terminal value ξ , the conditions (1.6) and (1.9) are false. Hence following [Kruse and Popier \(2016b\)](#), we add some hypotheses concerning the generator f and the terminal random time S .

(B1) There exists a constant $q > 1$ and a positive and bounded process η such that for any $y \geq 0$

$$f(t, y, z, \psi) \leq -\frac{y}{\eta_t} |y|^{q-1} + f(t, 0, z, \psi).$$

⁴More precisely, the Y component of a candidate supersolution; we will often refer to the Y component as a solution/supersolution to keep the discussion shorter.

- (B2) The process f^0 is non-negative and bounded⁵.
- (B3) There exists $\delta > \delta^*$ such that $\mathbb{E} [e^{\delta S}] < +\infty$. The threshold δ^* only depends on χ, L_z and L_ϑ (see Remark 1.9 below).
- (B4) There exists $m > m^*$ such that for any j

$$\mathbb{E} \int_0^S |\Upsilon_t(j)|^m dt < +\infty.$$

Process $\Upsilon_t(j)$ is defined in Assumption (A2). The value of m^* depends on χ, L_z, L_ϑ and δ^* , as well as on the choice of δ in (B3) (see Remark 1.9 below).

We further suppose that the generator $(t, y) \mapsto -y|y|^{q-1}/\eta_t$ satisfies the (A) and (B) assumptions, which means that η satisfies:

$$\mathbb{E} \int_0^T \frac{1}{\eta_t^m} dt < +\infty. \tag{1.14}$$

Remark 1.9. The values of δ^* and m^* are given in Kruse and Popier (2016b) in the case $L_\vartheta = 0$ and in Popier (2021, Section 9.1) for the general case. Since Conditions (B) have to hold also for the generator defined by the upper bound in (B1), χ should be replaced by $\max(\chi, 0)$ in the formulas for δ^* and m^* . In other words we provide formulas only for $\chi \geq 0$. When L_ϑ is positive, the thresholds are not explicit and have to be numerically computed. Some tables for these values are given in Popier (2021). Nonetheless if f does not depend on U (as in Section 4.1), then

$$\delta^* = \begin{cases} 2(\chi + K^2) & \text{if } \chi \leq K^2 \\ \chi \left(1 + \frac{K}{\sqrt{\chi}}\right)^2 & \text{if } \chi > K^2 \end{cases}$$

and

$$m^* = \begin{cases} \frac{2\delta}{\delta - 2(K^2 + \chi) + (\sqrt{\delta} - 2K)^2 \mathbf{1}_{\delta > 4K^2}} & \text{if } \chi \leq K^2, \\ \frac{\frac{\delta}{\sqrt{\delta} + \sqrt{\chi} - K} \times \frac{1}{\sqrt{\delta} - \sqrt{\delta^*}}}{\sqrt{\delta} - \sqrt{\delta^*}} & \text{if } \chi > K^2. \end{cases}$$

In particular if $y \mapsto f(t, y, z, \psi)$ is non-increasing, that is for $\chi = 0$, then we have:

$$\delta^* = 2K^2, \quad m^* = \frac{2\delta}{\delta - 2K^2}.$$

Furthermore if f depends only on y , i.e. $\chi = K^2 = 0, \delta^* = 0$ and $m^* = 1$.

1.3.2. *Known results for exit times.* Let $(Y^{\xi \wedge k}, Z^{\xi \wedge k}, U^{\xi \wedge k}, M^{\xi \wedge k})$ be the unique solution⁶ of the BSDE: for any $t < T$

$$\begin{aligned} Y_{t \wedge S}^{\xi \wedge k} &= Y_{T \wedge S}^{\xi \wedge k} + \int_{t \wedge S}^{T \wedge S} f(u, Y_u^{\xi \wedge k}, Z_u^{\xi \wedge k}, U_u^{\xi \wedge k}) du \\ &\quad - \int_{t \wedge S}^{T \wedge S} Z_u^{\xi \wedge k} dW_u - \int_{t \wedge S}^{T \wedge S} \int_{\mathcal{E}} U_u^{\xi \wedge k}(e) \tilde{\pi}(de, du) - \int_{t \wedge S}^{T \wedge S} dM_u^{\xi \wedge k}, \end{aligned} \tag{1.15}$$

with the truncated terminal condition:

$$\mathbb{P} - \text{a.s.}, \text{ on the set } \{t \geq S\}, \quad Y_t^{\xi \wedge k} = \xi \wedge k, \quad Z_t^{\xi \wedge k} = U_t^{\xi \wedge k} = M_t^{\xi \wedge k} = 0. \tag{1.16}$$

⁵In this paper ξ is non-negative; in general we should assume that ξ^- and $(f^0)^-$ are integrable (see Kruse and Popier (2016b)).

⁶In this part, S and ξ are fixed; we remove the superscripts S and ξ in the notations of the solutions.

From [Kruse and Popier \(2016b, Proposition 5\)](#), under **(A)**, **(B2)**, **(B3)** and **(B4)**, there exists a unique solution $(Y^{\xi \wedge k}, Z^{\xi \wedge k}, \psi^{\xi \wedge k}, M^{\xi \wedge k})$ to the BSDE (1.15) and (1.16). By the comparison principle for BSDEs, the sequence $Y^{\xi \wedge k}$ is non-decreasing and converges to some process

$$Y^\xi \doteq \lim_k Y^{\xi \wedge k} \tag{1.17}$$

As in the case of deterministic terminal times, the key step in [Kruse and Popier \(2016b\)](#) in establishing that Y^ξ is a minimal supersolution to the BSDE (1.1) is to obtain an a priori estimate on $Y^{\xi \wedge k}$, independent of the constant k . In terms of the concept of solvable stopping times introduced above in [Definition 1.8](#), the role of the apriori estimate is to ensure that the stopping time S is solvable. To have such an estimate, [Kruse and Popier \(2016b\)](#) restricts attention to the case where S is the first hitting time of a diffusion, namely

$$S = S_D = \inf\{t \geq 0, \Xi_t \notin D\}, \tag{1.18}$$

where the forward process Ξ in \mathbb{R}^d is the strong solution to the stochastic differential equation

$$d\Xi_t = b(\Xi_t)dt + \sigma(\Xi_t)dW_t \tag{1.19}$$

with some initial value $\Xi_0 \in \mathbb{R}^d$. The functions $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ satisfy a global Lipschitz condition: there exists some $C > 0$ such that

$$\forall x, y \in \mathbb{R}^d \quad \|\sigma(x) - \sigma(y)\| + \|b(x) - b(y)\| \leq C\|x - y\|. \tag{1.20}$$

The domain D is an open bounded subset of \mathbb{R}^d , whose boundary is at least of class C^2 (see for example [Gilbarg and Trudinger \(2001, Section 6.2\)](#) for the definition of a regular boundary); Ξ_0 is assumed to be fixed and in D .

Note that the condition **(B3)** imposes some implicit hypotheses between the generator f , the set D and the coefficients of the SDE (1.19). [Kruse and Popier \(2016b, Lemma 2\)](#) details some sufficient conditions on the coefficients b and σ .

For $B \subset \mathbb{R}^d$ let dist_B denote the signed distance function to B :

$$\text{dist}_B(x) = \begin{cases} \inf_{y \notin B} \|x - y\|, & x \in B \\ -\inf_{y \in B} \|x - y\|, & x \notin B. \end{cases} \tag{1.21}$$

For $D = B$ we simply write dist . [Kruse and Popier \(2016b, Proposition 6\)](#) is a Keller-Osserman type inequality (see [Keller \(1957\)](#); [Osserman \(1957\)](#)): there exists a constant C such that:

$$0 \leq Y_{t \wedge S}^{\xi \wedge k} \leq Y_{t \wedge S}^\xi \leq \frac{C}{\text{dist}(\Xi_{t \wedge S})^{2(p-1)}}. \tag{1.22}$$

Constant $p > 1$ is the Hölder conjugate of q .

For $n \geq 1$ define

$$S_n = \inf \left\{ t \geq 0, \text{dist}(\Xi_t) \leq \frac{1}{n} \right\}, \tag{1.23}$$

where $\text{dist}(\Xi_t)$ denotes the distance between the position of Ξ at time t and the boundary of D . The main result [Kruse and Popier \(2016b, Theorem 2\)](#) (expressed in terms of solvable stopping times) is:

Theorem 1.10. *If S is the exit time given by (1.18), and if \mathbb{F} is left-continuous at time S , under Assumptions **(A)** and **(B)**, S is a solvable stopping time ([Definition 1.8](#)). Moreover there exists a minimal supersolution $(Y^\xi, Z^\xi, U^\xi, M^\xi)$ to the BSDE (1.1) with singular terminal condition $Y_S = \xi$ ([Definition 1.5](#)).*

2. Solvable stopping times and minimal supersolutions

The next lemmas are useful consequences of the notion of solvable stopping times. First, note that the left-continuity assumption of \mathbb{F} at time S is true for example if S is predictable and if \mathbb{F} is a quasi-left continuous filtration (that is for any predictable stopping time τ , we have $\mathcal{F}_{\tau-} = \mathcal{F}_\tau$). This property of the filtration rules out the possibility that any of the involved processes has jumps at predictable, and a fortiori deterministic times. An important example is the filtration generated by the Brownian motion W and the orthogonal Poisson random measure π and S given by (1.18).

Lemma 2.1 (Minimality). *Assume that S is solvable and suppose that the generator f satisfies Conditions (A). Then the BSDE (1.1) has a minimal supersolution $(Y^\infty, Z^\infty, U^\infty, M^\infty)$ on the time interval $\llbracket 0, S \rrbracket$ with terminal condition $Y_S = \infty$.*

Proof: The arguments of minimality can be found in Kruse and Popier (2016b, Propositions 4 and 7). The adaptation is straightforward in our setting since the arguments are not based on a particular form of the stopping time S . Only left-continuity of the filtration is important. \square

Note that we haven't used Assumptions (B) in the above result, since solvability implies the existence of a supersolution.

Lemma 2.2. *Assume that S is solvable and suppose that generator f satisfies Conditions (A), (B2), (B3) and (B4). Then the BSDE (1.1) with a singular terminal value ξ at time S , has a minimal supersolution $(Y^\xi, Z^\xi, U^\xi, M^\xi)$ on the time interval $\llbracket 0, S \rrbracket$ with terminal condition $Y_S = \xi$.*

Proof: Let us denote by Y^k the first component of the solution of the BSDE (1.1) with terminal condition k . Since S is solvable, and with (A), Y^k is an increasing sequence converging to Y^∞ .

Again from Kruse and Popier (2016b, Proposition 5), under (A), (B2), (B3) and (B4), there exists a unique solution $(Y^{\xi \wedge k}, Z^{\xi \wedge k}, U^{\xi \wedge k}, M^{\xi \wedge k})$ to the BSDE (1.15) and (1.16). By the comparison principle, a.s for any $t \geq 0$

$$Y_t^{\xi \wedge k} \leq Y_t^k \leq Y_t^\infty.$$

Hence we obtain an upper estimate on $Y^{\xi \wedge k}$, independent of k , which replaces the upper bound (1.22). Arguing now as in Kruse and Popier (2016b), we obtain the desired minimal supersolution $(Y^\xi, Z^\xi, U^\xi, M^\xi)$. \square

This lemma allows us to give an alternative proof of Theorem 1.10. To wit, following the arguments of Kruse and Popier (2016b), under Assumptions (A) and (B), first prove that the exit time S defined by (1.18) is solvable. The arguments are simplified since we only have to work with the deterministic terminal conditions k at time S . Then from the previous lemma, solvability of S implies that for any terminal condition ξ , there exists a minimal supersolution $(Y^\xi, Z^\xi, U^\xi, M^\xi)$.

In Sections 5 and 6 it will be convenient to choose the stopping times S_n approximating S in Definition 1.5 in such a way that the minimal supersolution Y^ξ for a given singular terminal value $\xi \geq 0$ remains bounded up to time S_n . The next lemma asserts that S_n can always be chosen in this way:

Lemma 2.3. *Suppose a stopping time S is solvable. Let $(Y^\xi, Z^\xi, U^\xi, M^\xi)$ be the minimal supersolution of (1.1) with singular terminal condition $Y_T = \xi$ constructed as the limit of solutions with terminal condition $\xi \wedge k$. Then the sequence S_n in Definition 1.5 can be chosen so that*

$$Y_t \leq n \text{ for } t < S_n. \tag{2.1}$$

Proof: Let Y^∞ denote the first component of the (minimal) supersolution for terminal condition ∞ at time S and let S_n^∞ be the sequence of S_n in Definition 1.5 for the same terminal condition ∞ . It follows from (1.10) and (1.11) that Y^∞ has càdlàg sample paths on $\llbracket 0, S \rrbracket$ and $\lim_{t \rightarrow \infty} Y_{t \wedge S}^\infty = \infty$. This implies that the hitting times

$$\sigma_n^\infty \doteq \inf\{t : Y_{t \wedge S}^\infty \geq n\} \tag{2.2}$$

satisfy: $\sigma_n^\infty \leq S$ and it is a non-decreasing sequence. From the first property of a supersolution, this sequence converges almost surely to S . Now suppose that $\sigma_N^\infty = S$ for some N (and thus for any $n \geq N$). It would mean that Y^∞ has a jump at time S . In other words, the martingale parts have a jump at time S . But this is not possible since the filtration is assumed left continuous at S (Definition 1.8). Thus

$$\sigma_n^\infty \nearrow S \text{ as } n \nearrow \infty. \tag{2.3}$$

Then if we replace the stopping times S_n^∞ in Definition 1.5 with $S_n^\infty \doteq S_n^\infty \wedge \sigma_n^\infty$ all of the conditions of the definition remain valid; furthermore

$$Y_t^\infty \leq n \text{ for } t < S_n^\infty, \tag{2.4}$$

holds. This proves the lemma for the terminal condition ∞ .

Let Y^k denote the solution of (1.1) with terminal condition $Y_S = k$. Then by definition $Y_{t \wedge S}^k \nearrow Y_{t \wedge S}^\infty$. This and (2.4) imply

$$Y_t^k \leq Y_t^\infty \leq n \text{ for } t < S_n^\infty. \tag{2.5}$$

Let Y^ξ be the minimal supersolution of (1.1) with terminal condition $Y_S = \xi$ and let $Y^{\xi \wedge k}$ be the solution of (1.1) with terminal condition $Y_S = \xi \wedge k$. By the assumption of the lemma

$$Y_{t \wedge S}^{\xi \wedge k} \nearrow Y_{t \wedge S}^\xi \tag{2.6}$$

as $k \nearrow \infty$. By comparison principle for the solution of BSDE we have $Y_{t \wedge S}^{\xi \wedge k} \leq Y_{t \wedge S}^k$. This, (2.5), (2.6), the definition (2.2) of σ_n^∞ and letting $k \nearrow \infty$ imply

$$Y_t^\xi \leq Y_t^\infty \leq n \text{ for } t < S_n^\infty. \tag{2.7}$$

Let S_n^ξ be the sequence of stopping time appearing in Definition 1.5 of the supersolution Y^ξ . Define $\tau_n^\xi \doteq S_n^\xi \wedge S_n^\infty$. From (2.1) and from the assumption that $S_n^\xi \nearrow S$ we infer $\tau_n^\xi \nearrow S$. This implies that if we replace S_n^ξ with τ_n^ξ all of the conditions appearing in the definition of the supersolution Y^ξ continue to hold; by (2.7) this sequence of stopping times also satisfy

$$Y_t^\xi \leq Y_t^\infty \leq n \text{ for } t < \tau_n^\xi. \tag{2.8}$$

This proves the lemma for the terminal condition ξ . □

If we work with the filtration \mathbb{F}^W generated by the Brownian motion W , then BSDE (1.1) reduces to the following:

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t. \tag{2.9}$$

Corollary 2.4. *In the Brownian filtration \mathbb{F}^W , if S is solvable, then (2.1) becomes:*

$$Y_t \leq n \text{ for } t \leq S_n. \tag{2.10}$$

Proof: Since the trajectories of Y are now continuous, (2.1) can be strengthened to (2.10). □

Remark 2.5. The estimate (1.22) implies that Y_t^ξ of (1.17) satisfies $Y_t^\xi \leq Cn^{2(p-1)}$ almost surely if $t \leq S_n$ where S_n is as in (1.23). Therefore, the above lemma can be seen, up to a reparametrization of the upper bound (the above lemma uses n as an upper bound on the minimal supersolution up to time S_n , instead of $Cn^{2(p-1)}$), as a generalization of the way S_n is chosen in (1.23).

3. On the existence of a limit

We suppose that S is a solvable stopping time and that Conditions **(A)** and **(B)** hold. Hence, from Lemma 2.2, for any ξ , we can consider the minimal supersolution $(Y^\xi, Z^\xi, U^\xi, M^\xi)$ of the BSDE (1.1) with terminal condition ξ at time S , which is obtained as the increasing limit of the solution with terminal condition $\xi \wedge k$. Definition 1.5 of a supersolution requires

$$\liminf_{t \rightarrow +\infty} Y_{t \wedge S}^\xi \geq \xi \text{ almost surely}$$

(see (1.11)). If the terminal condition $\xi = +\infty$ almost surely then we immediately obtain that

$$\liminf_{t \rightarrow +\infty} Y_{t \wedge S}^\infty = \lim_{t \rightarrow +\infty} Y_{t \wedge S}^\infty = +\infty.$$

Therefore, for $\xi = \infty$, Y^∞ is continuous at time S and equals the terminal condition almost surely. In this section, we focus on the existence of the limit for an arbitrary terminal singular condition ξ ; that is, we will study whether

$$\liminf_{t \rightarrow +\infty} Y_{t \wedge S}^\xi = \lim_{t \rightarrow +\infty} Y_{t \wedge S}^\xi$$

holds almost surely for a singular terminal condition $\xi \geq 0$. This question was answered in the positive in Popier (2016) for deterministic terminal times; the goal of this section is to extend these results to solvable terminal times.

Roughly speaking, the main idea in Popier (2016) is the following: the limit of $Y_{\cdot \wedge S}^\xi$ exists provided we know the precise behavior of the generator f with respect to y . The full argument can be found in Popier (2016); in what follows we will provide an outline and emphasize the necessary changes to treat the random terminal time S . We break the generator f into four parts:

$$\begin{aligned} f(s, y, z, \psi) &= [f(s, y, z, \psi) - f(s, 0, z, \psi)] + [f(s, 0, z, \psi) - f(s, 0, 0, \psi)] \\ &\quad + [f(s, 0, 0, \psi) - f(s, 0, 0, \mathbf{0})] + f_s^0 \\ &= \phi(s, y, z, u) + \varpi(s, z, \psi) + \theta(s, \psi) + f_s^0. \end{aligned} \tag{3.1}$$

We adopt the following further assumptions on f from Popier (2016):

(C1) The generator f satisfies

$$\frac{1}{\varsigma_t} \mathfrak{f}(y) \leq f(t, y, z, \psi) - f(t, 0, z, \psi), \quad \forall y \geq 0, \forall (t, z, \psi),$$

where

- ς is positive and $\mathbb{E} \int_0^S \frac{1}{\varsigma_s} ds < +\infty$;
- \mathfrak{f} is a negative, decreasing and of class C^1 function and concave on \mathbb{R}_+ with $\mathfrak{f}(0) < 0$ and $\mathfrak{f}'(0) < 0$.

(C2) One of the next three cases holds:

- **Case 1.** f does not depend on ψ or $\theta(t, \psi) \geq 0$;
- **Case 2.** The value ϑ of **(A3)** belongs to $\mathbb{L}_\mu^1(\mathcal{E})$ and there exists a constant $\kappa_* > -1$ such that $\kappa_s^{0,0,\psi,0}(e) \geq \kappa_*$ a.e. for any (s, ψ, e) ;
- **Case 3.** μ is a finite measure on \mathcal{E} .

Since Conditions **(B)** should hold, in particular **(B1)**, we deduce that $\frac{1}{\varsigma_t} \mathfrak{f}(y) \leq -\frac{1}{\eta_t} y|y|^{q-1}$ for any $t \geq 0$ and y . Thus w.l.o.g. $\mathfrak{f}(y) \leq -y|y|^q$ and $\varsigma_t \leq -\mathfrak{f}(1)\eta_t = C\eta_t$ for some positive constant C . We can always add to \mathfrak{f} a linear function like $-y - 1$ such that $\mathfrak{f}(0) < 0$ and $\mathfrak{f}'(0) < 0$. Let us define the function Θ on $(0, +\infty)$ by

$$\Theta(x) = \int_x^{+\infty} \frac{-1}{\mathfrak{f}(y)} dy. \tag{3.2}$$

Recall that f is continuous and negative on \mathbb{R}_+ . Thus from the condition $f(y) \leq -y|y|^q$, the function $\Theta : [0, +\infty) \rightarrow (0, \Theta(0)]$ is well defined, decreasing, of class C^1 , and bijective. Let $\Theta^{-1} : (0, \Theta(0)] \rightarrow [0, +\infty)$ be the inverse of Θ .

The next theorem shows that process Y^ξ is càdlàg on \mathbb{R}_+ when filtration \mathbb{F} is complete and right-continuous. No additional assumption (left-continuity) on the filtration is needed here.

Theorem 3.1. *Assumptions (A), (B) and (C) hold. Then the minimal supersolution Y^ξ is equal to: a.s. for any $t \geq 0$*

$$Y_{t \wedge S}^\xi = \Theta^{-1} \left(\mathbb{E} \left[\Theta(\xi) - \Phi_{t \wedge S}^+ + \Phi_{t \wedge S}^- \middle| \mathcal{F}_t \right] \right).$$

The processes Φ^+ and Φ^- are two non-negative càdlàg supermartingales with a.s. $\lim_{t \rightarrow +\infty} \Phi_{t \wedge S}^- = 0$.

Now Φ^+ being a non-negative càdlàg supermartingale, we can deduce the existence of the following limit:

$$\lim_{t \rightarrow +\infty} \Phi_{t \wedge S}^+ := \Phi_{S-}^+.$$

Therefore, the limit of Y^ξ exists

$$\lim_{t \rightarrow +\infty} Y_{t \wedge S}^\xi = \Theta^{-1} (\Theta(\xi) - \Phi_{S-}^+) \geq \xi.$$

In other words, Y^ξ is a càdlàg process.

Proof: We follow the arguments developed in the proof of [Popier \(2016, Lemma 2.3\)](#). We only have to handle the stopping time S . Since $Y_t^{\xi \wedge k}$ is bounded from below by zero, we can apply Itô's formula: for $0 \leq t \leq T$

$$\begin{aligned} \Theta(Y_{t \wedge S}^{\xi \wedge k}) &= \Theta(Y_{T \wedge S}^{\xi \wedge k}) + \int_{t \wedge S}^{T \wedge S} \Theta'(Y_{s-}^{\xi \wedge k}) f(s, Y_s^{\xi \wedge k}, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) ds \\ &\quad - \int_{t \wedge S}^{T \wedge S} \Theta'(Y_{s-}^{\xi \wedge k}) Z_s^{\xi \wedge k} dW_s - \int_{t \wedge S}^{T \wedge S} \Theta'(Y_{s-}^{\xi \wedge k}) \int_{\mathcal{E}} U_s^{\xi \wedge k}(e) \tilde{\pi}(de, ds) \\ &\quad - \int_{t \wedge S}^{T \wedge S} \Theta'(Y_{s-}^{\xi \wedge k}) dM_s^{\xi \wedge k} - \frac{1}{2} \int_{t \wedge S}^{T \wedge S} \Theta''(Y_{s-}^{\xi \wedge k}) |Z_s^{\xi \wedge k}|^2 ds - \frac{1}{2} \int_{t \wedge S}^{T \wedge S} \Theta''(Y_{s-}^{\xi \wedge k}) d[M^{\xi \wedge k}]_s^c \\ &\quad - \int_{t \wedge S}^{T \wedge S} \int_{\mathcal{E}} \left[\Theta(Y_{s-}^{\xi \wedge k} + U_s^{\xi \wedge k}(e)) - \Theta(Y_{s-}^{\xi \wedge k}) - \Theta'(Y_{s-}^{\xi \wedge k}) U_s^{\xi \wedge k}(e) \right] \pi(ds, de) \\ &\quad - \sum_{t \wedge S < s \leq T \wedge S} \left[\Theta(Y_{s-}^{\xi \wedge k} + \Delta M_s^{\xi \wedge k}) - \Theta(Y_{s-}^{\xi \wedge k}) - \Theta'(Y_{s-}^{\xi \wedge k}) \Delta M_s^{\xi \wedge k} \right] \\ &= \mathbb{E}^{\mathcal{F}_t} \Theta(Y_{T \wedge S}^{\xi \wedge k}) - \Phi_{t \wedge S, T \wedge S}^{\xi \wedge k} \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \Phi_{t \wedge S, T \wedge S}^{\xi \wedge k} &= \\ &\quad - \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^{T \wedge S} \Theta'(Y_{s-}^{\xi \wedge k}) f(s, Y_s^{\xi \wedge k}, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) ds + \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^{T \wedge S} \Theta''(Y_{s-}^{\xi \wedge k}) |Z_s^{\xi \wedge k}|^2 ds \\ &\quad + \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^{T \wedge S} \Theta''(Y_{s-}^{\xi \wedge k}) d[M^{\xi \wedge k}]_s^c \\ &\quad + \mathbb{E}^{\mathcal{F}_t} \sum_{t \wedge S < s \leq T \wedge S} \left[\Theta(Y_{s-}^{\xi \wedge k} + \Delta M_s^{\xi \wedge k}) - \Theta(Y_{s-}^{\xi \wedge k}) - \Theta'(Y_{s-}^{\xi \wedge k}) \Delta M_s^{\xi \wedge k} \right] \\ &\quad + \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^{T \wedge S} \int_{\mathcal{E}} \left[\Theta(Y_{s-}^{\xi \wedge k} + U_s^{\xi \wedge k}(e)) - \Theta(Y_{s-}^{\xi \wedge k}) - \Theta'(Y_{s-}^{\xi \wedge k}) U_s^{\xi \wedge k}(e) \right] \pi(ds, de). \end{aligned}$$

We use the decomposition (3.1) of the generator f . Since Θ is non-increasing and convex, the next terms are non-negative:

$$\begin{aligned} & \mathbb{E}^{\mathcal{F}_t} \sum_{t \wedge S < s \leq T \wedge S} \left[\Theta(Y_{s-}^{\xi \wedge k} + \Delta M_s^{\xi \wedge k}) - \Theta(Y_{s-}^{\xi \wedge k}) - \Theta'(Y_{s-}^{\xi \wedge k}) \Delta M_s^{\xi \wedge k} \right] \\ & + \frac{1}{2} \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^{T \wedge S} \Theta''(Y_{s-}^{\xi \wedge k}) d[M^{\xi \wedge k}]_s^c \\ & - \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^{T \wedge S} \Theta'(Y_{s-}^{\xi \wedge k}) f_s^0 ds \end{aligned}$$

and we can use monotone convergence theorem to pass to the limit as T tends to $+\infty$.

Since $\varpi(s, z, \psi) \geq -L_z|z|$ (see (3.1) and (A4)) and using the concavity of \mathfrak{f} , we obtain that

$$-\Theta'(Y_{s-}^{\xi \wedge k}) \varpi(s, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) + \frac{1}{2} \Theta''(Y_{s-}^{\xi \wedge k}) |Z_s^{\xi \wedge k}|^2 \geq \frac{L_z^2}{2\mathfrak{f}'(Y_s^{\xi \wedge k})} \geq \frac{L_z^2}{2\mathfrak{f}'(0)}.$$

And

$$-\Theta'(Y_{s-}^{\xi \wedge k}) \phi(s, Y_s^{\xi \wedge k}, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) \geq -\frac{1}{\varsigma_s}.$$

Since $\mathbb{E} \int_0^S (\varsigma_s)^{-1} ds < +\infty$ and from (B3), we deduce that the negative part of

$$-\Theta'(Y_{s-}^{\xi \wedge k}) \left[f(s, Y_s^{\xi \wedge k}, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) - f(s, 0, 0, U_s^{\xi \wedge k}) \right] + \frac{1}{2} \Theta''(Y_{s-}^{\xi \wedge k}) |Z_s^{\xi \wedge k}|^2$$

is bounded in L^1 , uniformly with respect to (T, k) . The remaining term is

$$\begin{aligned} & - \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^{T \wedge S} \Theta'(Y_{s-}^{\xi \wedge k}) \left[f(s, 0, 0, U_s^{\xi \wedge k}) - f_s^0 \right] ds \\ & + \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^{T \wedge S} \int_{\mathcal{E}} \left[\Theta(Y_{s-}^{\xi \wedge k} + U_s^{\xi \wedge k}(e)) - \Theta(Y_{s-}^{\xi \wedge k}) - \Theta'(Y_{s-}^{\xi \wedge k}) U_s^{\xi \wedge k}(e) \right] \pi(ds, de). \end{aligned}$$

Assume that f does not depend on ψ or that $\theta(s, \psi) \geq 0$ (Case 1). Again from the convexity of Θ , this last term is non-negative. Our previous arguments show that we can pass to the limit when T goes to $+\infty$ in (3.3):

$$\Theta(Y_{t \wedge S}^{\xi \wedge k}) = \mathbb{E}^{\mathcal{F}_t} \Theta(\xi \wedge k) - \Phi_{t \wedge S, S}^{\xi \wedge k}.$$

Then by the monotone convergence theorem, we obtain the convergence (in L^1) of $\Phi_{t \wedge S, S}^{\xi \wedge k}$ to some process Φ_t and:

$$\Theta(Y_{t \wedge S}) = \mathbb{E}^{\mathcal{F}_t} [\Theta(\xi)] - \Phi_{t \wedge S}. \tag{3.4}$$

We can decompose the process Φ :

$$\Phi_{t \wedge S} = \Phi_{t \wedge S}^+ - \Phi_{t \wedge S}^-,$$

such that Φ^+ and Φ^- are non-negative càdlàg supermartingales with:

$$\Phi_{t \wedge S}^- \leq \mathbb{E}^{\mathcal{F}_t} \int_{t \wedge S}^S \left(\frac{1}{\varsigma_s} - \frac{L_z^2}{2\mathfrak{f}'(0)} \right) ds.$$

In particular a.s.

$$\lim_{t \rightarrow +\infty} \Phi_{t \wedge S}^- = 0.$$

For the Case 2 and the Case 3, we can exactly use the same arguments as in Popier (2016). We skip them here. This achieves the proof of the theorem. \square

Remark 3.2. A careful reading shows that (B1) is not needed in the above argument; we only need that the function Θ is well-defined.

In this section, we gave sufficient conditions on the generator f to ensure that Y^ξ is a càdlàg process, that is (1.11) becomes: a.s.

$$\lim_{t \rightarrow +\infty} Y_{t \wedge S}^\xi = \liminf_{t \rightarrow +\infty} Y_{t \wedge S}^\xi \geq \xi.$$

In the next three sections we study whether this relation can be strengthened further to

$$\lim_{t \rightarrow +\infty} Y_{t \wedge S}^\xi = \xi,$$

i.e., whether the minimal supersolution $(Y^\xi, Z^\xi, U^\xi, M^\xi)$ for the singular terminal condition ξ is in fact a solution, for three classes of terminal conditions.

4. Markovian terminal conditions

The aim of this section is to prove that equality holds in (1.11) when ξ is a deterministic function of the value at time S of a forward diffusion. We assume that Conditions **(A)** and **(B)** hold and that S is given by (1.18); in particular, S is a solvable stopping time (Theorem 1.10). The process Ξ is as in (1.19) with initial condition $\Xi_0 \in D^o$. We further suppose that

(D1) The terminal data ξ satisfies

$$\xi = g(\Xi_S),$$

where $g : \mathbb{R}^d \rightarrow \overline{\mathbb{R}}_+$ is a function such that $F_\infty = \{g = +\infty\} \cap \partial D$ is a closed set.

(D2) On $\mathbb{R}^d \setminus F_\infty$, g is locally bounded, that is, for all compact set $\mathcal{K} \subset \mathbb{R}^d \setminus F_\infty$,

$$g \mathbf{1}_{\mathcal{K}} \in L^\infty(\mathbb{R}^d).$$

(D3) The boundary ∂D belongs to C^3 .

To obtain the continuity, we start with a technical result. We already know that estimate (1.22) holds:

$$0 \leq Y_{t \wedge S}^{\xi \wedge k} \leq Y_{t \wedge S}^\xi \leq \frac{C}{\text{dist}(\Xi_{t \wedge S})^{2(p-1)}}.$$

The constant C depends on q , D and the bound on b and σ . Here we construct another estimate which depends also on the function g .

Lemma 4.1. *If \mathcal{O} is an open set such that $\overline{\mathcal{O}} \cap F_\infty = \emptyset$ and $\mathcal{O} \cap \partial D \neq \emptyset$, then there exists a constant $C = C(\mathcal{O}, g, q, b, \sigma, D)$ and an open set $D_{\mathcal{O}}$ such that $\mathcal{O} \cap \overline{D}$ and D are included in $D_{\mathcal{O}}$ and*

$$\mathbb{P} - \text{a.s. } \forall k \in \mathbb{N}, \forall t \geq 0, Y_t^{\xi \wedge k} \leq \frac{C}{(\text{dist}_{D_{\mathcal{O}}}(\Xi_{t \wedge S}))^{2(p-1)}}. \tag{4.1}$$

Recall that S is always the first exit time from \overline{D} .

The proof is a straightforward adaptation of Popier (2007, Proposition 7) and Kruse and Popier (2016b, Proposition 6). The second technical result concerns (Z^ξ, U^ξ) , and it is the extension of Popier (2007, Propositions 4 and 8) (a similar result was not proven in Kruse and Popier (2016b)). Before the statement of our result, let us recall the following property of the distance:

Lemma 4.2. *For any $\epsilon > 0$, if $D_\epsilon = \{x \in \mathbb{R}^d, |\text{dist}(x)| \leq \epsilon\}$, then there exists a positive constant ϵ_1 such that $\text{dist} \in C^2(D_{\epsilon_1})$.*

Proof: See Gilbarg and Trudinger (2001, Lemma 14.16). □

Lemma 4.3. *Under assumptions **(A)** and **(B)**, for any $\alpha > 4(p - 1) + 1$, there exists a constant C such that*

$$\mathbb{E} \int_0^S \left(\|Z_r^\xi\|^2 + \int_{\mathcal{E}} |U_r^\xi(e)|^2 \mu(de) \right) \text{dist}(\Xi_r)^\alpha dr \leq C.$$

*This inequality holds if we replace Z^ξ and U^ξ by $Z^{\xi \wedge k}$ and $U^{\xi \wedge k}$. If Condition **(D)** holds, then we can replace dist by $\text{dist}_{D_{\mathcal{O}}}$, with a modification of the value of the constant C .*

In the proof we will use the following definition:

$$D_\lambda = \{x \in \mathbb{R}^d, |\text{dist}(x)| \leq \lambda\} \tag{4.2}$$

for $\lambda > 0$.

Proof: The beginning of the proof is similar to Kruse and Popier (2016b, Proposition 6). Since D is bounded there exists a constant $R > 0$ such that $0 \leq \text{dist}(x) \leq R$ for all $x \in \bar{D}$. Using Lemma 4.2, let $\varphi \in C^\infty(\mathbb{R}^d, [0, 1])$ with $\varphi = 1$ on $\mathbb{R}^d \setminus D_{\epsilon_1}$ and $\varphi = 0$ on $D_{\epsilon_1/2}$. We define a function $\zeta \in C^2(\mathbb{R}^d, \mathbb{R}_+)$ such that $\zeta = (1 - \varphi)\text{dist} + R\varphi$ on \bar{D} . Since $\zeta \geq \text{dist} \geq 0$ on \bar{D} , $x \mapsto |\zeta(x)|^{4(p-1)+\varepsilon}$ is not in $C^2(\mathbb{R}^d)$, but this function belongs to $C^2(D \setminus D_{\epsilon_1})$ and we can define this function on the rest of $(\mathbb{R}^d \setminus D) \cup D_{\epsilon_1}$ in order to have the required regularity. For $n > 1/\epsilon_1$, define

$$S_n = \inf\{t \geq 0, \Xi_t \in D_{1/n}\}.$$

Take n sufficiently large such that $\Xi_0 \in D \setminus D_{1/n}$. In the following, ∇ denotes the gradient and ∇^2 the Hessian matrix. The Itô formula leads to:

$$\begin{aligned} \left(Y_{t \wedge S_n}^{\xi \wedge k}\right)^2 \zeta(\Xi_{t \wedge S_n})^\alpha &= \left(Y_0^{\xi \wedge k}\right)^2 \zeta(\Xi_0)^\alpha + \int_0^{t \wedge S_n} \|Z_r^{\xi \wedge k}\|^2 \zeta(\Xi_r)^\alpha dr \\ &+ 2 \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \zeta(\Xi_r)^\alpha dY_r^{\xi \wedge k} + \int_0^{t \wedge S_n} \zeta(\Xi_r)^\alpha \int_{\mathcal{E}} \left|U_r^{\xi \wedge k}(e)\right|^2 \pi(de, dr) \\ &+ \int_0^{t \wedge S_n} \zeta(\Xi_r)^\alpha d[M^{\xi \wedge k}]_r^c + \sum_{0 < s \leq t \wedge S_n} \zeta(\Xi_r)^\alpha (\Delta M_r^{\xi \wedge k})^2 \\ &+ \alpha \int_0^{t \wedge S_n} (Y_r^{\xi \wedge k})^2 \zeta(\Xi_r)^{\alpha-1} \nabla \zeta(\Xi_r) (b(\Xi_r)dr + \sigma(\Xi_r)dW_r) \\ &+ \frac{\alpha}{2} \int_0^{t \wedge S_n} (Y_r^{\xi \wedge k})^2 [(\alpha - 1)\zeta(\Xi_r)^{\alpha-2} \|\sigma(\Xi_r)\nabla \zeta(\Xi_r)\|^2 \\ &\quad + \zeta(\Xi_r)^{\alpha-1} \text{tr}(\sigma\sigma^*(\Xi_r)\nabla^2 \zeta(\Xi_r))] dr \\ &+ 2\alpha \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \zeta(\Xi_r)^{\alpha-1} Z_r^{\xi \wedge k} \nabla \zeta(\Xi_r) \sigma(\Xi_r) dr. \end{aligned} \tag{4.3}$$

Taking the expectation removes all martingale terms. From (1.22), since $\alpha > 4(p - 1)$, we know that there exists a constant such that for any k and all $t \geq 0$,

$$\left(Y_{t \wedge S_n}^{\xi \wedge k}\right)^2 \zeta(\Xi_{t \wedge S_n})^{\alpha-1} \leq C.$$

Thereby the terms

$$\begin{aligned} &\alpha \int_0^{t \wedge S_n} (Y_r^{\xi \wedge k})^2 \zeta(\Xi_r)^{\alpha-1} \nabla \zeta(\Xi_r) b(\Xi_r) dr \\ &+ \frac{\alpha}{2} \int_0^{t \wedge S_n} (Y_r^{\xi \wedge k})^2 [(\alpha - 1)\zeta(\Xi_r)^{\alpha-2} \|\sigma(\Xi_r)\nabla \zeta(\Xi_r)\|^2 \\ &\quad + \zeta(\Xi_r)^{\alpha-1} \text{tr}(\sigma\sigma^*(\Xi_r)\nabla^2 \zeta(\Xi_r))] dr \end{aligned}$$

are bounded by

$$C \left(\mathbb{E} \int_0^S \zeta^{\alpha-4(p-1)-1}(\Xi_r) dr + \mathbb{E} \int_0^S \zeta^{\alpha-4(p-1)-2}(\Xi_r) dr \right).$$

Since $\alpha - 4(p - 1) > 1$, the arguments developed in the proof of [Popier \(2007, Proposition 4\)](#) show that these integrals are finite. The Cauchy–Schwarz inequality leads to

$$\begin{aligned} & \left| \mathbb{E} \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \zeta(\Xi_r)^{\alpha-1} Z_r^{\xi \wedge k} \nabla \zeta(\Xi_r) \sigma(\Xi_r) dr \right| \\ & \leq \left(\mathbb{E} \int_0^{t \wedge S_n} \|Z_r^{\xi \wedge k}\|^2 \zeta(\Xi_r)^\alpha dr \right)^{1/2} \\ & \quad \times \left(\mathbb{E} \int_0^{t \wedge S_n} (Y_r^{\xi \wedge k})^2 \zeta(\Xi_r)^{\alpha-2} \|\nabla \zeta(\Xi_r) \sigma(\Xi_r)\|^2 dr \right)^{1/2}. \end{aligned}$$

But since $\nabla \zeta$ and σ are bounded,

$$\mathbb{E} \int_0^{t \wedge S_n} (Y_r^{\xi \wedge k})^2 \zeta(\Xi_r)^{\alpha-2} \|\nabla \zeta(\Xi_r) \sigma(\Xi_r)\|^2 dr \leq C \mathbb{E} \int_0^S \zeta^{\alpha-4(p-1)-2}(\Xi_r) dr < +\infty.$$

Compared to [Popier \(2007\)](#), the novelties are the generator f and the terms $U^{\xi \wedge k}$ and $M^{\xi \wedge k}$. First using [\(3.1\)](#):

$$\begin{aligned} & -2 \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} f(r, Y_r^{\xi \wedge k}, Z_r^{\xi \wedge k}, U_r^{\xi \wedge k}) \zeta(\Xi_r)^\alpha dr \\ & = -2 \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} f_r^0 \zeta(\Xi_r)^\alpha dr - 2 \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \theta(r, U_r^{\xi \wedge k}) \zeta(\Xi_r)^\alpha dr \\ & - 2 \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \varpi(r, Z_r^{\xi \wedge k}, U_r^{\xi \wedge k}) \zeta(\Xi_r)^\alpha dr - 2 \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \phi(r, Z_r^{\xi \wedge k}, U_r^{\xi \wedge k}) \zeta(\Xi_r)^\alpha dr. \end{aligned}$$

We know that $|Y_r^{\xi \wedge k} f_r^0 \zeta(\Xi_r)^\alpha| \leq C$. From [\(A4\)](#)

$$\varpi(r, Z_r^{\xi \wedge k}, U_r^{\xi \wedge k}) = \varpi_r^{\xi \wedge k} Z_r^{\xi \wedge k}$$

with $|\varpi_r^{\xi \wedge k}| \leq L_z$. Again by the Cauchy-Schwarz inequality and the previous arguments:

$$\begin{aligned} & \left| \mathbb{E} \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \varpi(r, Z_r^{\xi \wedge k}, U_r^{\xi \wedge k}) \zeta(\Xi_r)^\alpha dr \right| \\ & \leq C \left(\mathbb{E} \int_0^{t \wedge S_n} \|Z_r^{\xi \wedge k}\|^2 \zeta(\Xi_r)^\alpha dr \right)^{1/2}. \end{aligned}$$

From [\(A3\)](#) and similar arguments, we also have:

$$\begin{aligned} & \left| \mathbb{E} \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \theta(r, U_r^{\xi \wedge k}) \zeta(\Xi_r)^\alpha dr \right| \\ & \leq C \left(\mathbb{E} \int_0^{t \wedge S_n} \int_{\mathcal{E}} (U_r^{\xi \wedge k}(e))^2 \mu(de) \zeta(\Xi_r)^\alpha dr \right)^{1/2}. \end{aligned}$$

Note that with [\(B1\)](#)

$$2 \mathbb{E} \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} \phi(r, Y_r^{\xi \wedge k}, Z_r^{\xi \wedge k}, U_r^{\xi \wedge k}) dr \leq -2 \mathbb{E} \int_0^{t \wedge S_n} \frac{1}{\eta_r} \left| Y_r^{\xi \wedge k} \right|^q dr \leq 0.$$

Up to some localization procedure we have

$$\begin{aligned} & \mathbb{E} \int_0^{t \wedge S_n} \zeta(\Xi_r)^\alpha \int_{\mathcal{E}} \left| U_r^{\xi \wedge k}(e) \right|^2 \pi(de, dr) \\ & = \mathbb{E} \int_0^{t \wedge S_n} \zeta(\Xi_r)^\alpha \int_{\mathcal{E}} \left| U_r^{\xi \wedge k}(e) \right|^2 \mu(de) dr. \end{aligned}$$

Coming back to (4.3) and taking the expectation, we obtain:

$$\begin{aligned} & \mathbb{E} \left(Y_{t \wedge S_n}^{\xi \wedge k} \right)^2 \zeta(\Xi_{t \wedge S_n})^\alpha - \mathbb{E} \left(Y_0^{\xi \wedge k} \right)^2 \zeta(\Xi_0)^\alpha \\ & - \alpha \mathbb{E} \int_0^{t \wedge S_n} (Y_r^{\xi \wedge k})^2 \zeta(\Xi_r)^{\alpha-1} \nabla \zeta(\Xi_r) b(\Xi_r) dr \\ & - \frac{\alpha}{2} \mathbb{E} \int_0^{t \wedge S_n} (Y_r^{\xi \wedge k})^2 [(\alpha - 1) \zeta(\Xi_r)^{\alpha-2} \|\sigma(\Xi_r) \nabla \zeta(\Xi_r)\|^2 \\ & \quad + \zeta(\Xi_r)^{\alpha-1} \text{tr}(\sigma \sigma^*(\Xi_r) \nabla^2 \zeta(\Xi_r))] dr \\ & + 2 \mathbb{E} \int_0^{t \wedge S_n} Y_r^{\xi \wedge k} f_r^0 \zeta(\Xi_r)^\alpha dr \\ & \geq \mathbb{E} \int_0^{t \wedge S_n} \|Z_r^{\xi \wedge k}\|^2 \zeta(\Xi_r)^\alpha dr - C \left(\mathbb{E} \int_0^{t \wedge S_n} \|Z_r^{\xi \wedge k}\|^2 \zeta(\Xi_r)^\alpha dr \right)^{1/2} \\ & - C \left(\mathbb{E} \int_0^{t \wedge S_n} \int_{\mathcal{E}} (U_r^{\xi \wedge k}(e))^2 \mu(de) \zeta(\Xi_r)^\alpha dr \right)^{1/2} \\ & + \mathbb{E} \int_0^{t \wedge S_n} \zeta(\Xi_r)^\alpha \int_{\mathcal{E}} |U_r^{\xi \wedge k}(e)|^2 \mu(de) dr. \end{aligned}$$

The left-hand side of the inequality is bounded, uniformly with respect to k, t and n . Hence for all k, t and n ,

$$\mathbb{E} \int_0^{t \wedge S_n} \left(\|Z_r^{\xi \wedge k}\|^2 + \int_{\mathcal{E}} |U_r^{\xi \wedge k}(e)|^2 \mu(de) \right) \zeta(\Xi_r)^\alpha dr \leq C.$$

By Fatou’s lemma,

$$\mathbb{E} \int_0^S \left(\|Z_r^\xi\|^2 + \int_{\mathcal{E}} |U_r^\xi(e)|^2 \mu(de) \right) \zeta(\Xi_r)^\alpha dr \leq C.$$

Since $\zeta \geq \text{dist}$ on \overline{D} , we obtain the announced result. If (D) holds, we modify the above arguments so that (4.1) is used instead of (1.22). □

Theorem 4.4. *Assume that (A), (B) and (D) hold. Then a.s.*

$$\liminf_{t \rightarrow +\infty} Y_{t \wedge S}^\xi = \xi.$$

Proof: The proof is based on the arguments developed in Popier (2007, Theorem 2) and Popier (2016, Theorem 3.5). Below we only provide an outline and emphasize the modifications needed to handle the solvable time S .

Recall that $F_\infty = \{g = +\infty\} \cap \partial D$ is a closed set, that \mathcal{O} is an bounded open set such that $\overline{\mathcal{O}} \cap F_\infty = \emptyset$ and $\mathcal{O} \cap \partial D \neq \emptyset$. Now we take a function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}_+$ of class C^2 and with a compact support included in \mathcal{O} . For $\beta > 0$ we apply the Itô formula to the process $e^{-\beta t} Y_t^{\xi \wedge k} \varphi(\Xi_t)$:

$$\begin{aligned} & \mathbb{E} \left[e^{-\beta S} (g \wedge k)(\Xi_S) \varphi(\Xi_S) \right] = \mathbb{E} \left[e^{-\beta(t \wedge S)} Y_{t \wedge S}^{\xi \wedge k} \varphi(\Xi_{t \wedge S}) \right] \\ & - \beta \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} Y_s^{\xi \wedge k} \varphi(\Xi_s) ds - \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) f(s, Y_s^{\xi \wedge k}, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) ds \\ & + \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} Y_s^{\xi \wedge k} \mathcal{L} \varphi(\Xi_s) ds + \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \nabla \varphi(\Xi_s) \sigma(\Xi_s) Z_s^{\xi \wedge k} ds. \end{aligned} \tag{4.4}$$

$\beta > 0$ is here only to avoid time integrability trouble. Again we decompose f using (3.1).

$$\begin{aligned} & \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) f(s, Y_s^{\xi \wedge k}, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) ds \\ &= \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) f_s^0 ds \\ &+ \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) \left[f(s, Y_s^{\xi \wedge k}, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) - f(s, 0, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) \right] ds \\ &+ \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) \varpi_s^{\xi \wedge k} Z_s^{\xi \wedge k} ds + \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) \theta(s, U_s^{\xi \wedge k}) ds. \end{aligned}$$

Using the previous lemma and the Cauchy-Schwarz inequality, arguing as in Popier (2007), we deduce the existence of some constant C , independent of k and t , such that

$$\mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \left| \varphi(\Xi_s) \left(\varpi_s^{\xi \wedge k} Z_s^{\xi \wedge k} + \theta(s, U_s^{\xi \wedge k}) \right) + \nabla \varphi(\Xi_s) \sigma(\Xi_s) Z_s^{\xi \wedge k} \right| ds \leq C.$$

The key argument is the choice of $D_{\mathcal{O}}$ such that on this set $\text{dist}_{D_{\mathcal{O}}}$ remains bounded away from zero by some positive constant. From Lemma 4.1,

$$\mathbb{E} \int_{t \wedge S}^S e^{-\beta s} Y_s^{\xi \wedge k} |\varphi(\Xi_s) + \mathcal{L}\varphi(\Xi_s)| ds \leq C.$$

Hence all terms in (4.4), except maybe

$$-\mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) \left[f(s, Y_s^{\xi \wedge k}, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) - f(s, 0, Z_s^{\xi \wedge k}, U_s^{\xi \wedge k}) \right] ds,$$

are uniformly bounded. Thus this remaining term is also bounded and, thanks to (B1), is greater than

$$\mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) (Y_s^{\xi \wedge k})^q ds.$$

The dominated convergence theorem and again Lemma 4.3 imply that, up to a suitable subsequence, we can pass to the limit on k in (4.4) to obtain for any $t \geq 0$:

$$\begin{aligned} & \mathbb{E} \left[e^{-\beta S} g(\Xi_S) \varphi(\Xi_S) \right] = \mathbb{E} \left[e^{-\beta(t \wedge S)} Y_{t \wedge S}^{\xi} \varphi(\Xi_{t \wedge S}) \right] \\ & - \beta \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} Y_s^{\xi} \varphi(\Xi_s) ds - \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \varphi(\Xi_s) f(s, Y_s^{\xi}, Z_s^{\xi}, U_s^{\xi}) ds \\ & + \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} Y_s^{\xi} \mathcal{L}\varphi(\Xi_s) ds + \mathbb{E} \int_{t \wedge S}^S e^{-\beta s} \nabla \varphi(\Xi_s) \sigma(\Xi_s) Z_s^{\xi} ds. \end{aligned}$$

Using Fatou’s lemma and letting t go to $+\infty$, we deduce that

$$\mathbb{E} \left[e^{-\beta S} g(\Xi_S) \varphi(\Xi_S) \right] \geq \mathbb{E} \left[e^{-\beta S} \varphi(\Xi_S) \liminf_{t \rightarrow +\infty} Y_{t \wedge S}^{\xi} \right].$$

The conclusion follows since a.s.

$$\liminf_{t \rightarrow +\infty} Y_{t \wedge S}^{\xi} \geq g(\Xi_S).$$

We emphasize again that the technical details are in Popier (2007, 2016) and are skipped above. Since Ξ is continuous, several technical issues of Popier (2016) are avoided here. \square

Combining the last result with Theorem 3.1 we get our continuity result for Markovian terminal conditions:

Corollary 4.5. *Assume that (A), (B), (C), (D) hold. Then a.s.*

$$\lim_{t \rightarrow +\infty} Y_{t \wedge S}^\xi = \xi,$$

i.e., the minimal supersolution Y^ξ of the BSDE (1.1) with terminal condition $\xi = g(\Xi_S)$ is a solution.

4.1. *Related elliptic PDE.* Since Darling and Pardoux (1997); Pardoux (1999), it is well known that BSDEs with random terminal time and elliptic PDE are strongly related. Inspired by Le Gall (1997); Marcus and Véron (1998); Marcus and Veron (1998), Popier (2007) extended such result to singular boundary value for the elliptic PDE, when the generator f is of the form $-y|y|^{q-1}$, $q > 1$. Let us now assume that S is given by (1.18), that f is a deterministic function⁷, and that the terminal condition is given by (D1), namely $\xi = g(\Xi_S)$. In the rest of this section, to emphasize the role of the initial position x , we indicate it as a subscript of the minimal supersolution Y^ξ . We consider the system: for any $x \in D$ and $t \geq 0$

$$\Xi_{x,t} = x + \int_0^t b(\Xi_{x,r})dr + \int_0^t \sigma(\Xi_{x,r})dW_r, \tag{4.5}$$

$$Y_{x,t}^\xi = g(\Xi_{x,S}) + \int_t^S f(\Xi_{x,r}, Y_{x,r}^\xi, Z_{x,r}^\xi)dr - \int_t^S Z_{x,r}^\xi dW_r. \tag{4.6}$$

Of course, Equation (4.6) of this forward-backward SDE has to be understood in the sense of Definition 1.5.

We consider the elliptic PDE

$$\begin{cases} -\mathcal{L}v - f(x, v, \nabla v \sigma^*) = 0 & \text{on } D; \\ v = g & \text{on } \partial D, \end{cases} \tag{4.7}$$

where the operator \mathcal{L} is the infinitesimal generator of Ξ .

The following definition can be found in Barles (1993), Barles (1994) (or Pardoux (1999), Crandall et al. (1992) for v continuous). If v is a function defined on \bar{D} , we denote by v^* (respectively v_*) the upper- (respectively lower-) semicontinuous envelope of v : for all $x \in \bar{D}$

$$v^*(x) = \limsup_{x' \rightarrow x, x' \in \bar{D}} v(x') \quad \text{and} \quad v_*(x) = \liminf_{x' \rightarrow x, x' \in \bar{D}} v(x').$$

The next definition holds for bounded boundary condition g .

Definition 4.6 (Viscosity solution).

- $v : \bar{D} \rightarrow \mathbb{R}$ is called a **viscosity subsolution** of (4.7) if $v^* < +\infty$ on \bar{D} and if for all $\phi \in C^2(\mathbb{R}^d)$, whenever $x \in \bar{D}$ is a point of local maximum of $v^* - \phi$,

$$\begin{aligned} -\mathcal{L}\phi(x) - f(x, v^*(x), \nabla\phi(x)\sigma^*(x)) &\leq 0 \quad \text{if } x \in D; \\ \min(-\mathcal{L}\phi(x) - f(x, v^*(x), \nabla\phi(x)\sigma^*(x)), v^*(x) - g(x)) &\leq 0 \quad \text{if } x \in \partial D. \end{aligned}$$

- $v : \bar{D} \rightarrow \mathbb{R}$ is called a **viscosity supersolution** of (4.7) if $v_* > -\infty$ on \bar{D} and if for all $\phi \in C^2(\mathbb{R}^d)$, whenever $x \in \bar{D}$ is a point of local minimum of $v_* - \phi$,

$$\begin{aligned} -\mathcal{L}\phi(x) - f(x, v^*(x), \nabla\phi(x)\sigma^*(x)) &\geq 0 \quad \text{if } x \in D; \\ \max(-\mathcal{L}\phi(x) - f(x, v^*(x), \nabla\phi(x)\sigma^*(x)), v(x) - g(x)) &\geq 0 \quad \text{if } x \in \partial D. \end{aligned}$$

- $v : \bar{D} \rightarrow \mathbb{R}$ is called a **viscosity solution** of (4.7) if it is both a viscosity sub- and supersolution.

If the boundary condition is singular, we adapt the preceding definition.

⁷If the terminal time and the terminal values are deterministic functions of Ξ_S , then the solution of the BSDE (1.1) satisfies $U = M = 0$. Hence we can assume w.l.o.g. that f does not depend on U here.

Definition 4.7 (Unbounded viscosity solution). We say that v is a viscosity solution of the PDE (4.7) with unbounded terminal data g if v is a viscosity solution on D in the sense of Definition 4.6 and if

$$g(x) \leq \lim_{\substack{x' \rightarrow x \\ x' \in D, x \in \partial D}} v_*(x') \leq \lim_{\substack{x' \rightarrow x \\ x' \in D, x \in \partial D}} v^*(x') \leq g(x).$$

Remark that this definition implies that $v^* < +\infty$ and $v_* > -\infty$ on D . Define

$$u^{(k)}(x) = Y_{x,0}^{g \wedge k}.$$

Let us further assume:

- $g : \partial D \rightarrow \overline{\mathbb{R}}_+$ is continuous,
- f is continuous on $\overline{D} \times \mathbb{R} \times \mathbb{R}^d$.

Then under Conditions (A), (B) and (D) Pardoux and Răşcanu (2014, Theorem 5.74), implies that $u^{(k)}$ is continuous on \overline{D} and is a viscosity solution of the elliptic PDE (4.7) with boundary data $g \wedge k$. Recall that the sequence $Y_x^{\xi \wedge k}$ converges to Y_x^ξ . If we define

$$u(x) = Y_{x,0}^\xi,$$

then u is the supremum of the continuous functions $u^{(k)}$, is non-negative and lower-semicontinuous on \overline{D} and satisfies:

$$\forall x \in \overline{D}, \quad u(x) \leq \frac{C}{\text{dist}^{2(p-1)}(x)}.$$

Following the arguments of Popier (2007) with some modifications, we have:

Proposition 4.8. *If Conditions (A), (B) and (D) hold, and if f and g are continuous functions, then the function u defined by $u(x) = Y_{x,0}^\xi$ is a viscosity solution of the elliptic PDE in the sense of Definition 4.7.*

Moreover suppose that the matrix $\sigma\sigma^$ is uniformly elliptic: there exists a constant $\alpha > 0$ such that*

$$\forall x \in \mathbb{R}^d, \quad \sigma\sigma^*(x) \geq \alpha Id. \tag{4.8}$$

If the map $(x, y, z) \mapsto (b(x), \sigma(x), f(x, y, z))$ is of class C^1 , then u belongs to $C^0(\overline{D}, [0, +\infty)) \cap C^2(D, [0, +\infty))$.

5. Terminal condition ξ_1

In this section we study terminal conditions of the form

$$\xi_1 = \infty \cdot \mathbf{1}_{\{\tau \leq S\}} \tag{5.1}$$

where τ is another stopping time. We know from Ahmadi et al. (2021, Section 2) that when $S = T$ is deterministic and τ has a bounded density around the terminal time T , the minimal supersolution Y^{ξ_1} of the BSDE (1.1) with terminal condition ξ_1 satisfies

$$\lim_{t \rightarrow T} Y_T^{\xi_1} = \xi_1.$$

Our goal is to prove similar continuity results when S is a stopping time. For this we will consider two approaches: the first is an extension of the approach taken in Ahmadi et al. (2021, Section 2), the first subsection focuses on this. We present a new approach in the second subsection.

5.1. *First approach.* The approach of [Ahmadi et al. \(2021, Section 2\)](#), treating the case $S = T$, where $T > 0$ is a fixed deterministic time, can be summarized as follows:

- (1) Assume that τ has a bounded density around the terminal time T .
- (2) Let $Y^{T,\infty}$ be the minimal supersolution of (1.1) on the interval $[0, T]$ with terminal condition $Y_T = \infty$; define the auxiliary terminal condition

$$\xi_1^{(\tau)} = \mathbf{1}_{\{\tau \leq T\}} Y_\tau^{T,\infty}.$$

- (3) Use the bounded density assumption and a priori upperbounds on $Y^{T,\infty}$ to prove

$$\mathbb{E} \left[\left(\xi_1^{(\tau)} \right)^\varrho \right] < \infty \tag{5.2}$$

for some $\varrho > 1$; in particular, $\xi_1^{(\tau)}$ is not a singular terminal condition.

- (4) Let \widehat{Y} be the solution of a linear BSDE with terminal condition $\xi_1^{(\tau)}$ whose driver term is chosen to guarantee $Y^{\xi_1} < \widehat{Y}$.
- (5) Derive the continuity of Y^{ξ_1} at time T from that of \widehat{Y} .

This argument requires a modification when the terminal time S is random because 1) a priori upperbounds on supersolutions with explicit expressions are not in general available and 2) even if such bounds were available, assumptions only on the distribution of τ (such as the bounded density assumption in the first item of the list above) would not be sufficient because, when S is random, the expectation in (5.2) depends on the joint distribution of τ and S . In the light of these observations, we take (5.2) as our starting point for the next theorem. Proposition 5.2 gives an example of a case where (5.2) is satisfied.

Note that (5.2) implies that $\mathbb{P}(\tau = S) = 0$. Indeed, if $\mathbb{P}(\tau = S) > 0$, then

$$\mathbb{E} \left[\left(\xi_1^{(\tau)} \right)^\varrho \right] \geq \mathbb{E} \left[\mathbf{1}_{\{\tau=S\}} (Y_S^\infty)^\varrho \right] = +\infty.$$

Theorem 5.1. *Assume that the stopping time S is solvable and Conditions (A) and (B) hold. Let τ be a stopping time such that there exists $\varrho > \varrho^*$ so that (5.2) holds (where the threshold ϱ^* depends on δ and δ^* in (B3)). Then Y^{ξ_1} satisfies*

$$\lim_{t \rightarrow +\infty} Y_{t \wedge S}^{\xi_1} = \xi_1, \tag{5.3}$$

i.e., the minimal supersolution Y^{ξ_1} is in fact a solution.

Proof: We adopt the argument in [Ahmadi et al. \(2021\)](#) given for deterministic terminal times (see the list above) to solvable terminal times as follows. Since S is solvable, there exists a minimal supersolution $(Y^\infty, Z^\infty, U^\infty, M^\infty)$ to the BSDE (1.1) with terminal condition $+\infty$ at time S .

Define the generator

$$g(t, y, z, \psi) = \chi y + f(t, 0, z, \psi);$$

g is linear in y and satisfies (A). We would like to solve the BSDE defined by g with terminal condition $Y_S = \xi_1^{(\tau)} = \mathbf{1}_{\{\tau \leq S\}} Y_\tau^\infty$; $\xi_1^{(\tau)}$ is $\mathcal{F}_{\tau \wedge S}$ -measurable and therefore \mathcal{F}_S -measurable. Let us check that (1.6) holds, namely for some $r > 1$ and $\rho > \nu(r)$

$$\mathbb{E} \left[e^{r\rho S} |\xi_1^{(\tau)}|^r + \int_0^S e^{r\rho t} |g(t, 0, 0, \mathbf{0})|^r dt \right] < +\infty.$$

Note that $g(t, 0, 0, \mathbf{0}) = f_t^0$ and (B2) holds. From the proof of [Kruse and Popier \(2016b, Proposition 5\)](#), using (B3), there exists $r > 1$ and $\rho > \nu(r)$ such that $r\nu(r) < \delta$. Hence we can find $\gamma > 1$ such that $\mathbb{E}(e^{r\rho\gamma S}) < +\infty$. Hölder's inequality gives:

$$\mathbb{E} \left[e^{r\rho S} |\xi_1^{(\tau)}|^r \right] \leq (\mathbb{E} e^{r\rho\gamma S})^{1/\gamma} \left(\mathbb{E} |\xi_1^{(\tau)}|^{r\widehat{\gamma}} \right)^{1/\widehat{\gamma}}.$$

If $\varrho \geq r\widehat{\gamma} = \varrho^* > 1$, then we deduce that $\mathbb{E}|\xi_1^{(\tau)}|^{r\widehat{\gamma}} < +\infty$ and (1.6) is satisfied.

Then we have to verify that (1.9) holds for $\xi_1^{(\tau)}$. This can be done by linearizing g and using the same arguments as for (1.6). Applying Theorem 1.4 leads to the existence and the uniqueness of the solution $(\widehat{Y}, \widehat{Z}, \widehat{U}, \widehat{M})$.

We next prove that \widehat{Y} does serve as an upper bound on $Y^{\xi_1 \wedge k}$, the solution of the BSDE (1.1) with terminal condition $\xi_1 \wedge k = k\mathbf{1}_{\{\tau \leq S\}}$ at time S : a.s. for any $t \geq 0$

$$Y_{t \wedge \tau \wedge S}^{\xi_1 \wedge k} \leq \widehat{Y}_{t \wedge \tau \wedge S}.$$

Indeed by the comparison principle, $Y^{\xi_1 \wedge k} \leq Y^\infty$. Hence a.s.

$$Y_{\tau \wedge S}^{\xi_1 \wedge k} = Y_\tau^{\xi_1 \wedge k} \mathbf{1}_{\tau \leq S} \leq Y_\tau^\infty \mathbf{1}_{\tau \leq S} = \xi_1^{(\tau)}.$$

Since $f(t, y, z, \psi) \leq g(t, y, z, \psi)$ by Condition (A1), we deduce the desired result.

We conclude using a linearization procedure (see Ahmadi et al. (2021, Lemma 3)) that a.s. on the \mathcal{F}_S -measurable set $\{\tau > S\}$, that $\lim_{t \rightarrow +\infty} \widehat{Y}_{t \wedge S} = 0$. Therefore, a.s. on the same set

$$0 \leq \lim_{t \rightarrow +\infty} Y_{t \wedge S}^{\xi_1} \leq \lim_{t \rightarrow +\infty} \widehat{Y}_{t \wedge S} = 0 = \xi_1. \tag{5.4}$$

That (5.3) holds over the event $\{\xi_1 = \infty\} = \{\tau \leq S\}$ follows from the fact that Y^{S, ξ_1} is constructed by approximation from below (see Lemma 1.7). This and (5.4) imply (5.3). \square

Comments on the choice of ϱ^* appearing in the above theorem: if f depends only on y and is non-increasing ($\chi = K^2 = 0$), we have $\delta^* = 0$ from Remark 1.9. Since δ can be chosen arbitrarily small we can set $\varrho^* = 1$. Then it is sufficient to have $q > 3$ to satisfy the constraints on ϱ in Theorem 5.1 and in Proposition 5.2 below.

One of the key assumptions of the previous theorem is the bound (5.2); let us develop an example for which this assumption holds. We assume that S is the first exit time of Ξ given by (1.18), $S = S_D = \inf\{t \geq 0, \Xi_t \notin D\}$; we assume enough regularity on D so that (1.22) holds:

$$0 \leq Y_{t \wedge S}^\infty \leq \frac{C}{\text{dist}(\Xi_{t \wedge S})^{2(p-1)}},$$

for some constant $C > 0$. We also suppose that σ is uniformly elliptic (Equation (4.8)), so that by Friedman (1964), for $\Xi_0 = x \in D$, Ξ_t has a density $\phi(t, x, \cdot)$. Under these assumptions, to prove (5.2) it suffices to prove

$$\mathbb{E} \left[\mathbf{1}_{\{\tau \leq S\}} \frac{1}{\text{dist}(\Xi_\tau)^{\varrho 2(p-1)}} \right] < \infty, \tag{5.5}$$

for some $\varrho > 1$. Theorem 5.1 above gives:

$$\lim_{t \rightarrow \infty} Y_{t \wedge S}^{\xi_1} = \xi_1,$$

assuming (5.5).

The expectation in (5.5) depends on the joint distribution of (τ, S, Ξ_S) . We are not aware of results available in the current literature that would imply (5.5) under broad and general assumptions on these variables. A basic case that can be treated with techniques that we know of is when τ is independent of Ξ (and therefore of S). The next proposition proves (5.5) in this setting.

Proposition 5.2. *Suppose that S is the first exit time of Ξ given by (1.18), that σ is uniformly elliptic, and that τ is independent of Ξ . If $q > 1 + 2\varrho$, then*

$$\mathbb{E} \left[\mathbf{1}_{\{\tau \leq S\}} \frac{1}{\text{dist}(\Xi_\tau)^{\varrho 2(p-1)}} \right] < \infty, \tag{5.6}$$

Proof: The equality $1/p+1/q = 1$ and $q > 1+2\varrho$ imply $2(p-1)\varrho < 1$. Let us denote the distribution of τ by F_τ . The expectation (5.5) can then be written as

$$\mathbb{E} \left[\mathbf{1}_{\{\tau \leq S\}} \frac{1}{\text{dist}(\Xi_\tau)^{\varrho 2(p-1)}} \right] = \int_0^\infty \mathbb{E} \left[\mathbf{1}_{\{t \leq S\}} \frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}} \right] dF_\tau(t).$$

Since S is the exit time of Ξ from a smooth domain with uniformly elliptic diffusion matrix, we have:

$$\mathbb{E} \left[\mathbf{1}_{\{\tau \leq S\}} \frac{1}{\text{dist}(\Xi_\tau)^{\varrho 2(p-1)}} \right] = \int_0^\infty \mathbb{E} \left[\mathbf{1}_{\{t < S\}} \frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}} \right] dF_\tau(t)$$

that $\{\Xi_t \in D\} \supset \{t < S\}$ implies

$$\leq \int_0^\infty \mathbb{E} \left[\mathbf{1}_{\{\Xi_t \in D\}} \frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}} \right] dF_\tau(t). \quad (5.7)$$

We next bound

$$\mathbb{E} \left[\mathbf{1}_{\{\Xi_t \in D\}} \frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}} \right].$$

For $\Xi_0 = x \in D$, let $\phi(t, x, \cdot)$ be the density of Ξ_t . The expectation above then can be written as

$$\mathbb{E} \left[\mathbf{1}_{\{\Xi_t \in D\}} \frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}} \right] = \int_D \phi(t, x, y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy. \quad (5.8)$$

Let D_ε and ε_1 be as in Lemma 4.2, where we choose ε_1 small enough so that $x \notin D_{\varepsilon_1}$. The continuity of dist implies that D_ε is closed; D_ε is therefore compact since $D_\varepsilon \subset D$ and D is bounded. This, the continuity of dist and $x \notin D_{\varepsilon_1}$ imply

$$C_1 \doteq \inf_{y \in D_{\varepsilon_1}} |x - y| > 0. \quad (5.9)$$

Since b and σ are Lipschitz continuous and since σ is uniformly elliptic, from Friedman (1964, page 16) we have the following Aronson's estimate on $\phi(t, x, y)$:

$$\phi(t, x, y) \leq \frac{C_2}{t^{d/2}} e^{-\frac{\lambda_0 |y-x|^2}{4t}}.$$

This and (5.9) imply

$$\phi(t, x, y) \leq \frac{C_2}{t^{d/2}} e^{-\frac{\lambda_0 C_1^2}{4t}},$$

for $y \in D_{\varepsilon_1}$. The right side of this inequality is continuous and bounded for $t \in [0, \infty]$. Therefore

$$C_3 \doteq \sup_{t \in [0, \infty], y \in D_{\varepsilon_1}} \phi(t, x, y) \leq \sup_{t \in [0, \infty], y \in D_{\varepsilon_1}} \frac{C_2}{t^{d/2}} e^{-\frac{\lambda_0 C_1^2}{4t}} < \infty. \quad (5.10)$$

We now decompose (5.8) into two integrals over D_{ε_1} and $D \setminus D_{\varepsilon_1}$:

$$\begin{aligned} \mathbb{E} \left[\mathbf{1}_{\{\Xi_t \in D\}} \frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}} \right] &= \int_D \phi(t, x, y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy \\ &= \int_{D \setminus D_{\varepsilon_1}} \phi(t, x, y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy + \int_{D_{\varepsilon_1}} \phi(t, x, y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy \\ &\leq \frac{1}{\varepsilon_1^{2\varrho(p-1)}} + \int_{D_{\varepsilon_1}} \phi(t, x, y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy. \end{aligned} \quad (5.11)$$

the last inequality coming from: $\text{dist}(y) > \varepsilon_1$ for $y \in D \setminus D_{\varepsilon_1}$.

It remains to bound the last integral. For this, recall that dist is C^2 over D_{ε_1} (Lemma 4.2). Furthermore, ∂D is the 0-level curve of dist , in particular, for $y \in \partial D$, the gradient $\nabla \text{dist}(y)$ is

normal to ∂D . ∂D is a C^1 surface, with non-vanishing normal at every point. It follows from these and the definition of dist that ∇dist satisfies $|\nabla \text{dist}(y)| = 1$ for $y \in \partial D$. Now define

$$E_\epsilon = \{y \in D : \text{dist}(y) > \epsilon\} = D \setminus D_\epsilon.$$

That dist is $C^2(D_{\epsilon_1})$ implies that ∂D_{ϵ_1} is a C^2 bounded surface and that the function

$$A(\epsilon) = \text{Area}(\partial E_\epsilon)$$

is C^1 over the interval $[0, \epsilon_1]$. In particular, it is continuous and satisfies

$$C_4 \doteq \sup_{\epsilon \in [0, \epsilon_1]} A(\epsilon) < \infty. \tag{5.12}$$

This and the definition of dist imply $|\nabla \text{dist}(y)| = 1$ for $y \in \partial D_\epsilon$ for $\epsilon \leq \epsilon_1$. We are now in a setting where we can apply the co-area formula [Evans \(2010, Theorem 5, page 713\)](#), which gives

$$\int_{D_{\epsilon_1}} \phi(t, x, y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy = \int_0^{\epsilon_1} \left(\int_{\partial E_\epsilon} \phi(t, x, y) dS \right) \frac{1}{\epsilon^{\varrho 2(p-1)}} d\epsilon$$

$\partial E_\epsilon \subset D_{\epsilon_1}$ and [\(5.10\)](#) imply

$$\leq \int_0^{\epsilon_1} \left(\int_{\partial E_\epsilon} C_3 dS \right) \frac{1}{\epsilon^{\varrho 2(p-1)}} d\epsilon$$

This and [\(5.12\)](#) give

$$\leq C_3 C_4 \int_0^{\epsilon_1} \frac{1}{\epsilon^{\varrho 2(p-1)}} d\epsilon.$$

Recall that $\varrho 2(p-1) < 1$. This and the last line imply

$$\int_{D_{\epsilon_1}} \phi(t, x, y) \frac{1}{\text{dist}(y)^{\varrho 2(p-1)}} dy < C_5, \tag{5.13}$$

where

$$C_5 \doteq C_3 C_4 \int_0^{\epsilon_1} \frac{1}{\epsilon^{\varrho 2(p-1)}} d\epsilon < \infty.$$

The bound [\(5.13\)](#) we have just derived and [\(5.11\)](#) imply

$$\mathbb{E} \left[\mathbf{1}_{\{\Xi_t \in D\}} \frac{1}{\text{dist}(\Xi_t)^{\varrho 2(p-1)}} \right] \leq \frac{1}{\epsilon_1^{\varrho 2(p-1)}} + C_5.$$

This and [\(5.7\)](#) imply [\(5.6\)](#). □

5.2. *A new argument for ξ_1 .* In the rest of the paper, to clearly state the ideas and for a less technical presentation, we will restrict our attention to the Brownian framework, i.e., we assume that $\mathbb{F} = \mathbb{F}^W$ is the filtration generated by the d -dimensional Brownian motion W . Therefore [\(1.1\)](#) reduces to [\(2.9\)](#), that is:

$$dY_t = -f(t, Y_t, Z_t)dt + Z_t dW_t. \tag{5.14}$$

The continuity arguments in Section 5.1 above and in [Ahmadi et al. \(2021, Section 2\)](#) use the solution of a linear auxiliary BSDE as an upper bound to the minimal supersolution. In this section we would like to explore a new upper bound that is based directly on the original nonlinear BSDE. As will be seen, whenever applicable, this is more natural and leads to less strict conditions on the parameter q of Condition [\(B1\)](#). We assume τ and S to be solvable in the sense of Definition 1.8. Let $Y^{S, \infty}$ and $Y^{\tau, \infty}$ denote the ∞ -supersolutions⁸ corresponding to τ and S . The main idea of

⁸When we refer to Y as the solution, we mean the first component Y of a solution (Y, Z) .

the present section as compared to that of Section 5.1 and Ahmadi et al. (2021, Section 2) is the following: we replace the upper bound process \widehat{Y} of the proof of Theorem 5.1 with $Y^{\tau,\infty}$.

Theorem 5.3. *Suppose τ and S are solvable in the sense of Definition 1.8. Then the minimal supersolution Y^{S,ξ_1} of (2.9) with terminal condition $Y_S = \xi_1 = +\infty \cdot \mathbf{1}_{\{\tau \leq S\}}$ satisfies*

$$\lim_{t \rightarrow \infty} Y_t^{S,\xi_1} = \xi_1. \tag{5.15}$$

Proof: As before, that (5.15) holds over the event $\{\tau \leq S\} = \{\xi_1 = \infty\}$ follows from the fact that Y^{S,ξ_1} is constructed by approximation from below (Lemma 1.7). Therefore, it suffices to prove (5.15) over the event $\{S < \tau\}$.

For $L > 0$, $\xi_1 \wedge L$ is a bounded random variable; let $Y^{S,\xi_1 \wedge L}$ be the continuous classical solution of the BSDE (5.14) with terminal value $Y_S = \xi_1 \wedge L$. By definition $Y^{S,\xi_1} = \lim_{L \nearrow \infty} Y^{S,\xi_1 \wedge L}$. For ease of notation set

$$\widehat{\xi}_1 = Y_{\tau \wedge S}^{S,\xi_1 \wedge L},$$

which is a $\mathcal{F}_{\tau \wedge S}$ measurable random variable. By the definition of $Y^{S,\xi_1 \wedge L}$, we have $Y_S^{S,\xi_1 \wedge L} = \xi_1 \wedge L$ and in particular $Y_S^{S,\xi_1 \wedge L} = 0$ over the event $\{S < \tau\}$. This implies

$$\widehat{\xi}_1 = Y_{\tau \wedge S}^{S,\xi_1 \wedge L} = Y_{\tau}^{S,\xi_1 \wedge L} \mathbf{1}_{\{\tau \leq S\}}. \tag{5.16}$$

For $L_1 > 0$, let Y^{τ,L_1} be the classical solution of the BSDE (5.14) with terminal condition $Y_{\tau} = L_1$. Note that $Y_{\tau \wedge S}^{\tau,L_1} = L_1 \mathbf{1}_{\{\tau \leq S\}} + Y_S^{\tau,L_1} \mathbf{1}_{\{S < \tau\}}$. This, $Y^{\tau,L_1} \geq 0$ and (5.16) imply

$$\widehat{\xi}_1 \wedge L_1 = Y_{\tau \wedge S}^{S,\xi_1 \wedge L} \wedge L_1 \leq Y_{\tau \wedge S}^{\tau,L_1}. \tag{5.17}$$

Let $Y^{\tau \wedge S, \widehat{\xi}_1 \wedge L_1}$ be the classical solution of the BSDE (5.14) with terminal condition $Y_{\tau \wedge S} = \widehat{\xi}_1 \wedge L_1$. Y^{τ,L_1} is the classical solution of (5.14) with terminal condition $Y_{\tau} = L_1$; therefore, its restriction to the time interval $[0, \tau \wedge S]$ is again a classical solution of the same BSDE with terminal condition $Y_{\tau \wedge S} = Y_{\tau \wedge S}^{\tau,L_1}$. The inequality (5.17) and the comparison principle (applied to $Y^{\tau \wedge S, \widehat{\xi}_1 \wedge L_1}$ and the restriction of Y^{τ,L_1} to the time interval $[0, \tau \wedge S]$) gives

$$Y_t^{\tau \wedge S, \widehat{\xi}_1 \wedge L_1} \leq Y_t^{\tau,L_1}, \quad t \leq \tau \wedge S. \tag{5.18}$$

$Y^{S,\xi_1 \wedge L}$ is the classical solution of the BSDE (5.14) with terminal condition $Y_S = \xi_1 \wedge L$. Therefore, its restriction to the time interval $[0, \tau \wedge S]$ is again a classical solution of the same BSDE with terminal condition $\widehat{\xi}_1 = Y_{\tau \wedge S}^{S,\xi_1 \wedge L}$. By the comparison principle and the continuity of classical solutions of BSDE with respect to the terminal value (applied to the processes $Y^{\tau \wedge S, \widehat{\xi}_1 \wedge L_1}$ and the restriction of $Y^{S,\xi_1 \wedge L}$ to the interval $[0, \tau \wedge S]$) we have:

$$\lim_{L_1 \nearrow \infty} Y_t^{\tau \wedge S, \widehat{\xi}_1 \wedge L_1} = Y_t^{S,\xi_1 \wedge L}, \quad t \leq \tau \wedge S.$$

That τ is solvable means that $Y^{\tau,L_1} \nearrow Y^{\tau,\infty}$ as $L_1 \nearrow \infty$ where the last process is the Y component of the minimal supersolution of (5.14) with terminal condition $Y_{\tau} = \infty$. This, (5.18) and the last display imply

$$Y_t^{S,\xi_1 \wedge L} \leq Y_t^{\tau,\infty}, \quad t \leq \tau \wedge S.$$

Let $\tau_n \nearrow \tau$ be the sequence of increasing stopping times in Definition 1.5 associated with $Y^{\tau,\infty}$. By Corollary 2.4, $Y^{\tau,\infty}$ is bounded by n in the interval $[0, \tau_n]$. This and the last display imply

$$Y_t^{S,\xi_1 \wedge L} \leq n, \quad t \leq \tau_n \wedge S.$$

Letting $L \nearrow \infty$ we get

$$Y_t^{S,\xi_1} \leq n, \quad t \leq \tau_n \wedge S,$$

and in particular the restriction of Y^{S,ξ_1} to the time interval $\llbracket 0, \tau_n \wedge S \rrbracket$ is the continuous classical solution of the BSDE (5.14) with terminal value $Y_{\tau_n \wedge S}^{S,\xi_1} \leq n$; therefore

$$\lim_{t \rightarrow \infty} Y_{t \wedge S}^{S,\xi_1} = \lim_{t \rightarrow \infty} Y_{t \wedge \tau_n \wedge S}^{S,\xi_1} = Y_{\tau_n \wedge S}^{S,\xi_1} = \lim_{L \nearrow \infty} Y_{\tau_n \wedge S}^{S,\xi_1 \wedge L} = 0$$

over the event $\{S < \tau_n\}$. $\tau_n \nearrow \tau$ and $\mathbb{P}(\tau = S) = 0$ imply $\{S < \tau\} = \cup_{n=1}^\infty \{S < \tau_n\}$ almost surely. This and the last display imply (5.15) over the event $\{S < \tau\}$. This completes the proof of this theorem. \square

5.3. *An example in one space dimension.* In this subsection we go back to the setup studied in Sezer et al. (2019, Section 2): the driver is deterministic and only a function of y :

$$f(y) = -y|y|^{q-1},$$

the terminal time S is deterministic T and the terminal condition is

$$Y_T = \infty \cdot \mathbf{1}_{\{\tau \leq T\}} \tag{5.19}$$

where τ is the first exit time of W from the interval $(0, L)$. Since f is deterministic and the terminal condition depends only on W , the solution (Y, Z, U, M) of the BSDE (1.1) reduces to $(Y, Z, \mathbf{0}, 0)$ and the BSDE becomes:

$$Y_s = Y_t + \int_s^t f(Y_u) du + \int_s^t Z_u dW_u. \tag{5.20}$$

Theorem 2.1 of Sezer et al. (2019) states that for $q > 2$ the minimal supersolution of the BSDE (5.20) with terminal condition (5.19) is continuous at time T . Let y_t denote the solution of $\frac{dy}{dt} = -f(y)$ on the interval $[0, T]$ with terminal value $y_T = \infty$, i.e.,

$$y_t \doteq ((q - 1)(T - t))^{1-p}, \quad t < T, \quad 1/p + 1/q = 1. \tag{5.21}$$

The proof of Sezer et al. (2019, Theorem 2.1) is based on the following integrability result:

$$\mathbb{E}[y_\tau \mathbf{1}_{\{\tau \leq T\}}] = \mathbb{E}[y_\tau \mathbf{1}_{\{\tau < T\}}] < \infty. \tag{5.22}$$

As in the proof of Theorem 5.1, Sezer et al. (2019) constructs a linear process that is continuous at time T to find a continuous upperbound on the minimal supersolution (which implies the continuity of the minimal supersolution); the bound (5.22) ensures that the upper bound linear process is well defined. The bound (5.22) requires $q > 2$ and that is the reason why this was assumed in Sezer et al. (2019) in its treatment of the terminal condition (5.19). We will now derive the same continuity result under the assumption $q > 1$ using Theorem 5.3 above.

To apply Theorem 5.3 to the present setup we need T and τ to be solvable. This essentially means that the BSDE has supersolutions with terminal value ∞ at these terminal times. The supersolution for terminal time T is the deterministic process $t \mapsto y_t$. That τ is solvable can be derived from (1.22). Instead of invoking this general result, in the following lemma we will make use of the simple nature of f and W to explicitly construct the supersolution $Y^{\tau, \infty}$ with terminal condition $Y_\tau = \infty$. Following Polyanin and Zaitsev (2003, page 307) we will use

$$\mathbf{x}(v, v_l) \doteq v_l^{1-\frac{q+1}{2}} \left(\frac{q+1}{4}\right)^{1/2} \int_1^{v/v_l} (u^{q+1} - 1)^{-1/2} du. \tag{5.23}$$

to construct solutions to the ODE

$$\frac{1}{2} \frac{d^2 V}{dx^2} - V^q = 0. \tag{5.24}$$

The function \mathbf{x} is strictly increasing in v , furthermore, $q > 1$ implies $\mathbf{x}(\infty, v_l) < \infty$. Define

$$\mathbf{L}(v_l) = \mathbf{x}(\infty, v_l).$$

Let $\mathbf{x}^{-1}(\cdot, v_l)$ denote the inverse of $\mathbf{x}(\cdot, v_l)$. Now define

$$\mathbf{v}(x, v_l) \doteq \mathbf{x}^{-1}(|x - L/2|, v_l).$$

Lemma 5.4. *On the interval $[L/2 - \mathbf{L}(v_l), L/2 + \mathbf{L}(v_l)]$, $\mathbf{v}(\cdot, v_l)$ satisfies (5.24) with boundary conditions ∞ on both sides.*

Proof: Direct calculation using the definition (5.23) of \mathbf{x} . □

To construct a supersolution of (5.20), we want to solve (5.24) in the interval $[0, L]$ with ∞ boundary condition. Note that $\mathbf{L}(0) = \infty$ and $\mathbf{L}(\infty) = 0$ and \mathbf{L} is a decreasing smooth function. It follows that there is a unique v^* such that $\mathbf{L}(v^*) = L/2$. Then for $v_l = v^*$, $\mathbf{v}(x, v^*)$ solves (5.24) in the interval $[0, L]$ with ∞ on the boundary. For our argument we also need solutions to (5.24) in the time interval $[0, L]$ with boundary condition n on both sides. For this purpose, the next lemma constructs a sequence $0 < v_n \nearrow v^*$ such that $\mathbf{x}(n, v_n) = L/2$.

Lemma 5.5. *There exists a sequence $0 < v_n \nearrow v^*$ such that $\mathbf{x}(n, v_n) = L/2$.*

Proof: Recall that v^* is the unique solution of $\mathbf{x}(\infty, v^*) = L/2$, i.e.,

$$(v^*)^{1-\frac{q+1}{2}} \left(\frac{q+1}{4}\right)^{1/2} \int_1^\infty (u^{q+1} - 1)^{-1/2} du = L/2.$$

This implies in particular

$$\mathbf{x}(1, v^*) = (v^*)^{1-\frac{q+1}{2}} \left(\frac{q+1}{4}\right)^{1/2} \int_1^{1/v^*} (u^{q+1} - 1)^{-1/2} du < L/2.$$

Furthermore, the function $v_l \mapsto \mathbf{x}(1, v_l)$ is continuous on $(0, v^*]$ and increases to ∞ as $v_l \searrow 0$. This implies that there exists $v_1 < v^*$ satisfying $\mathbf{x}(1, v_1) = L/2$. Now note $\mathbf{x}(2, v_1) > L/2$ and $\mathbf{x}(2, v^*) < L/2$. Applying the same argument gives $v_2 \in (v_1, v^*)$ satisfying $\mathbf{x}(2, v_2) = L/2$. Repeating the same argument inductively gives us an increasing sequence v_n bounded by v^* solving $\mathbf{x}(n, v_n) = L/2$. The limit v^{**} of this sequence satisfies $\mathbf{x}(\infty, v^{**}) = L/2$. Recall that v^* is the unique solution of this equation. This yields $v_n \nearrow v^*$. □

We can now state and prove the generalization of Sezer et al. (2019, Theorem 4) to $q > 1$:

Theorem 5.6. *For $q > 1$ the minimal supersolution of (5.20) with terminal condition $Y_T = \infty \cdot \mathbf{1}_{\{\tau \leq T\}}$ is continuous at time T .*

Proof: By the previous lemma there exists $v_n \nearrow v^*$ that solves $\mathbf{x}(n, v_n) = L/2$. It follows from this and Lemma 5.4 that $\mathbf{v}(\cdot, v_n)$ solves (5.24) on $[0, L]$ with terminal condition n on both sides and that $\mathbf{v}(\cdot, v_n) \rightarrow \mathbf{v}(\cdot, v^*)$. The comparison principle for the equation (5.24) implies that in fact $\mathbf{v}(\cdot, v_n) \nearrow \mathbf{v}(\cdot, v^*)$. Now define the processes

$$Y_t^{\tau, n} = \mathbf{v}(W_t, v_n), Y_t^{\tau, \infty} = \mathbf{v}(W_t, v^*).$$

Itô's formula implies that $Y_t^{\tau, n}$ solves (5.20) with terminal condition $Y_\tau = n$. Define τ_n be the first time W hits $[1/n, L - 1/n]$. Itô's formula implies $Y^{\tau, \infty}$ satisfies (1.10) (with $\beta_n = \tau_n$) and the definition of $\mathbf{v}(\cdot, v^*)$ and the continuity of the sample paths of W imply (1.11) with $\xi = \infty$. Therefore, $Y^{\tau, \infty}$ is a supersolution of (5.20) with terminal condition $Y_\tau = \infty$. Furthermore, $\mathbf{v}(\cdot, v_n) \nearrow \mathbf{v}(\cdot, v^*)$ implies $Y_t^{\tau, n} \nearrow Y_t^{\tau, \infty}$. These imply that τ satisfies all of the conditions of being solvable. T is also solvable because it is deterministic. Theorem 5.3 now implies the statement of the present theorem. □

6. Terminal condition ξ_2

Let's assume S solvable and τ a stopping time for which $\mathbb{P}(S = \tau) = 0$. Define

$$\xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}. \tag{6.1}$$

Our goal now is to prove that the minimal supersolution corresponding to this type of terminal condition is in fact a solution, i.e., it satisfies $\lim_{t \rightarrow \infty} Y_{S \wedge t}^{\xi_2} = \xi_2$. Our proof involves pasting processes at stopping times to obtain candidate solutions; to simplify arguments that this procedure results in solutions we assume that \mathbb{F} is generated only by W . Here is the main result of this section:

Theorem 6.1. *Suppose S is solvable and τ is an arbitrary stopping time such that $\mathbb{P}(S = \tau) = 0$. Then the BSDE (2.9) has a minimal supersolution Y^{S, ξ_2} in the time interval $\llbracket 0, S \rrbracket$ with terminal condition $Y_S = \xi_2 = \infty \cdot \mathbf{1}_{\{\tau > S\}}$. Furthermore this supersolution is in fact a solution:*

$$\lim_{t \rightarrow \infty} Y_{S \wedge t}^{S, \xi_2} = \xi_2. \tag{6.2}$$

This generalizes Ahmadi et al. (2021, Theorem 2) which treats deterministic terminal times, to random terminal times. The main idea of the proof of Ahmadi et al. (2021, Theorem 2) generalized to the current setup is as follows: we construct a sequence of supersolutions to (2.9) with terminal conditions $Y_S = \infty \cdot \mathbf{1}_{\{\tau > S_n\}}$ where S_n is the sequence of stopping times approximating S . Note that these processes are all defined over the time interval $\llbracket 0, S \rrbracket$; $S_n < S$ allows one to prove that they are all continuous at time S . This, $\infty \cdot \mathbf{1}_{\{\tau > S_n\}} \geq \infty \cdot \mathbf{1}_{\{\tau > S\}}$ and the comparison principle for BSDE allow one to argue that Y^{S, ξ_2} converges to its terminal condition at time S , which is the result we seek.

Let us define several processes that will be useful in the proof of Theorem 6.1; they are all defined as solutions of the BSDE (2.9) over the time interval $\llbracket 0, S \rrbracket$ with different terminal conditions at time S :

- $Y^{S, L}$ corresponds to the terminal condition L ,
- $Y^{S, 0}$ to the terminal condition 0 ,
- $Y^{S, L \cdot \mathbf{1}_{\{\tau > S_n\}}}$ to the terminal condition $L \cdot \mathbf{1}_{\{\tau > S_n\}}$.

These terminal conditions are \mathcal{F}_S -measurable and bounded. Hence from Theorem 1.4 and the conditions (B), the corresponding solutions are well defined and unique (in the sense of Definition 1.1).

Let $Y^{S_n, \hat{\xi}_2}$ be the solution of (2.9) in the time interval $\llbracket 0, S_n \rrbracket$ with terminal condition

$$Y_{S_n} = \hat{\xi}_2 = Y_{S_n}^{S, L} \cdot \mathbf{1}_{\{\tau > S_n\}} + Y_{S_n}^{S, 0} \cdot \mathbf{1}_{\{\tau \leq S_n\}};$$

$\hat{\xi}_2$ depends on L and n . The estimates on $Y^{S, L}$ and $Y^{S, 0}$ in Theorem 1.4 imply the existence and uniqueness of $Y^{S_n, \hat{\xi}_2}$. We begin our argument with the following lemma.

Lemma 6.2. *The process $Y^{S, L \cdot \mathbf{1}_{\{\tau > S_n\}}}$ has the following structure:*

$$Y^{S, L \cdot \mathbf{1}_{\{\tau > S_n\}}} = \mathcal{Y}_t \doteq Y_t^{S_n, \hat{\xi}_2} \mathbf{1}_{\{t \leq S_n\}} + Y_t^{S, 0} \cdot \mathbf{1}_{\{t > S_n\}} \cdot \mathbf{1}_{\{\tau \leq S_n\}} + Y_t^{S, L} \cdot \mathbf{1}_{\{t > S_n\}} \cdot \mathbf{1}_{\{\tau > S_n\}}. \tag{6.3}$$

Proof: First, $S_n < S$ implies that \mathcal{Y} is an adapted and continuous process with bounded terminal value $L \cdot \mathbf{1}_{\{\tau > S_n\}}$; in particular, Y satisfies the terminal condition $Y_S = L \cdot \mathbf{1}_{\{\tau > S_n\}}$ that $Y^{S, L \cdot \mathbf{1}_{\{\tau > S_n\}}}$ satisfies. Let us show that \mathcal{Y} satisfies also (2.9). Parallel to the definition of \mathcal{Y} , define

$$Z_t = Z_t^{S_n, \hat{\xi}_2} \mathbf{1}_{\{t \leq S_n\}} + Z_t^{S, 0} \cdot \mathbf{1}_{\{t > S_n\}} \cdot \mathbf{1}_{\{\tau \leq S_n\}} + Z_t^{S, L} \cdot \mathbf{1}_{\{t > S_n\}} \cdot \mathbf{1}_{\{\tau > S_n\}}.$$

For any $0 \leq t \leq T$, there are three cases to consider.

Case 1: $0 \leq t \leq T \leq S_n < S$. Since $Y^{S_n, \hat{\xi}_2}$ solves (2.9) on $\llbracket 0, S_n \rrbracket$:

$$\begin{aligned} \mathcal{Y}_{t \wedge S} &= Y_t^{S_n, \hat{\xi}_2} = Y_T^{S_n, \hat{\xi}_2} + \int_t^T f(u, Y_u^{S_n, \hat{\xi}_2}, Z_u^{S_n, \hat{\xi}_2}) du - \int_t^T Z_u^{S_n, \hat{\xi}_2} dW_u \\ &= \mathcal{Y}_{T \wedge S} + \int_{t \wedge S}^{T \wedge S} f(u, \mathcal{Y}_u, \mathcal{Z}_u) du - \int_{t \wedge S}^{T \wedge S} \mathcal{Z}_u dW_u. \end{aligned}$$

Case 2: $S_n < t \leq T$:

$$\begin{aligned} \mathcal{Y}_{t \wedge S} &= Y_{t \wedge S}^{S, 0} \cdot \mathbf{1}_{\tau \leq S_n} + Y_{t \wedge S}^{S, L} \cdot \mathbf{1}_{\tau > S_n} \\ &= \mathcal{Y}_{T \wedge S} \\ &+ \int_{t \wedge S}^{T \wedge S} [f(u, Y_u^{S, 0}, Z_u^{S, 0}) \cdot \mathbf{1}_{\{\tau \leq S_n\}} + f(u, Y_u^{S, L}, Z_u^{S, L}) \cdot \mathbf{1}_{\{\tau > S_n\}}] du \\ &- \int_{t \wedge S}^{T \wedge S} [Z_u^{S, 0} \cdot \mathbf{1}_{\{\tau \leq S_n\}} + Z_u^{S, L} \cdot \mathbf{1}_{\{\tau > S_n\}}] dW_u \\ &= \mathcal{Y}_{T \wedge S} + \int_{t \wedge S}^{T \wedge S} f(u, \mathcal{Y}_u, Z_u^{S, 0} \cdot \mathbf{1}_{\{\tau \leq S_n\}} + Z_u^{S, L} \cdot \mathbf{1}_{\{\tau > S_n\}}) du \\ &- \int_{t \wedge S}^{T \wedge S} [Z_u^{S, 0} \cdot \mathbf{1}_{\{\tau \leq S_n\}} + Z_u^{S, L} \cdot \mathbf{1}_{\{\tau > S_n\}}] dW_u \\ &= \mathcal{Y}_{T \wedge S} + \int_{t \wedge S}^{T \wedge S} f(u, \mathcal{Y}_u, \mathcal{Z}_u) du - \int_{t \wedge S}^{T \wedge S} \mathcal{Z}_u dW_u \end{aligned}$$

since both sets $\{\tau \leq S_n\}$ and $\{\tau > S_n\}$ are \mathcal{F}_{S_n} -measurable.

Case 3: $0 \leq t \leq S_n < T$:

$$\begin{aligned} \mathcal{Y}_{t \wedge S} &= Y_t^{S_n, \hat{\xi}_2} = \mathcal{Y}_{S_n} + \int_{t \wedge S}^{S_n} f(u, \mathcal{Y}_u, Z_u^{S_n, \hat{\xi}_2}) du - \int_{t \wedge S}^{S_n} Z_u^{S_n, \hat{\xi}_2} dW_u \\ &= Y_{S_n}^{S, 0} \cdot \mathbf{1}_{\tau \leq S_n} + Y_{S_n}^{S, L} \cdot \mathbf{1}_{\tau > S_n} \\ &+ \int_{t \wedge S}^{S_n} f(u, \mathcal{Y}_u, Z_u^{S_n, \hat{\xi}_2}) du - \int_{t \wedge S}^{S_n} Z_u^{S_n, \hat{\xi}_2} dW_u \\ &= \mathcal{Y}_{T \wedge S} + \int_{S_n}^{T \wedge S} f(u, \mathcal{Y}_u, Z_u^{S, 0} \cdot \mathbf{1}_{\tau \leq S_n} + Z_u^{S, L} \cdot \mathbf{1}_{\tau > S_n}) du \\ &- \int_{S_n}^{T \wedge S} [Z_u^{S, 0} \cdot \mathbf{1}_{\tau \leq S_n} + Z_u^{S, L} \cdot \mathbf{1}_{\tau > S_n}] dW_u \\ &+ \int_{t \wedge S}^{S_n} f(u, \mathcal{Y}_u, Z_u^{S_n, \hat{\xi}_2}) du - \int_{t \wedge S}^{S_n} Z_u^{S_n, \hat{\xi}_2} dW_u \\ &= \mathcal{Y}_{T \wedge S} + \int_{t \wedge S}^{T \wedge S} f(u, \mathcal{Y}_u, \mathcal{Z}_u) du - \int_{t \wedge S}^{T \wedge S} \mathcal{Z}_u dW_u. \end{aligned}$$

Hence we have verified that $(\mathcal{Y}, \mathcal{Z})$ solves the BSDE (2.9). The statement of the lemma follows from the uniqueness of such a solution (Theorem 1.4). \square

We now give

Proof of Theorem 6.1: Let $Y^{S, \xi_2 \wedge L}$ be the solution of (2.9) with bounded terminal condition $Y_S = \xi_2 \wedge L = L \cdot \mathbf{1}_{\{\tau > S\}}$. As usual, we define Y^{S, ξ_2} via approximation from below: $Y^{S, \xi_2} = \lim_{L \nearrow \infty} Y^{S, \xi_2 \wedge L}$.

That S is solvable and Lemma 2.2 imply that Y^{S,ξ_2} is the minimal supersolution we seek. It remains to prove (6.2), i.e., $\lim_{t \rightarrow \infty} Y_{t \wedge S}^{S,\xi_2} = \xi_2$. Once again, this holds over the event $\{\tau > S\} = \{\xi_2 = \infty\}$ by construction (approximation from below, Lemma 1.7). Therefore, we only focus on the proof of (6.2) over the event $\{\tau \leq S\}$. Recall the process $Y^{S,L \cdot \mathbf{1}_{\{\tau > S_n\}}}$ of (6.3) that is the solution of (2.9) over the interval $\llbracket 0, S \rrbracket$ with terminal condition $Y_S = L \cdot \mathbf{1}_{\{\tau > S_n\}}$. That $S_n \leq S$ implies

$$L \cdot \mathbf{1}_{\{\tau > S\}} \leq L \cdot \mathbf{1}_{\{\tau > S_n\}}$$

This and the comparison principle imply

$$Y_t^{S,\xi_2 \wedge L} \leq Y_t^{S,L \cdot \mathbf{1}_{\{\tau > S_n\}}}, \quad \text{for } t \leq S.$$

Lemma 6.2 implies

$$Y_t^{S,\xi_2 \wedge L} \leq Y_t^{S,0}, \text{ for } t \in \llbracket S_n, S \rrbracket$$

over the event $\{\tau \leq S_n\}$. Combining the last two displays we get

$$Y_t^{S,\xi_2 \wedge L} \leq Y_t^{S,0}, \text{ for } t \in \llbracket S_n, S \rrbracket$$

over the event $\{\tau \leq S_n\}$. The right side of the last inequality doesn't depend on L . Letting $L \nearrow \infty$ on the left gives

$$Y_t^{S,\xi_2} \leq Y_t^{S,0}, \text{ for } t \in \llbracket S_n, S \rrbracket.$$

over the event $\{\tau \leq S_n\}$. The right side of the above inequality is a classical solution of the BSDE (2.9) with 0 terminal condition. Therefore, taking limits of both sides above give

$$\limsup_{t \rightarrow \infty} Y_{t \wedge S}^{S,\xi_2} \leq \lim_{t \rightarrow \infty} Y_{t \wedge S}^{S,0} = 0.$$

By its construction, $Y^{S,\xi_2} \geq 0$. This and the last display imply

$$\lim_{t \rightarrow \infty} Y_{t \wedge S}^{S,\xi_2} = 0.$$

over the event $\{\tau \leq S_n\}$. Finally, $S_n \nearrow S$ and $\mathbb{P}(\tau = S) = 0$ imply $\bigcup_{n=1}^\infty \{\tau \leq S_n\} = \{\tau \leq S\}$. This and the last display imply

$$\lim_{t \rightarrow \infty} Y_{t \wedge S}^{S,\xi_2} = 0 = \xi_2$$

over the event $\{\tau \leq S\}$. This completes the proof of the theorem. □

7. Conclusion

The present work develops solutions to the BSDE (1.1) with random terminal time S for a range of singular terminal values. We do this by proving that the minimal supersolution is continuous at S and attains the terminal value by constructing upperbound processes that force the desired continuity at S on the minimal supersolution. A key ingredient of our arguments is the concept of a solvable stopping time with respect to the given BSDE and filtration, introduced in the present work. Solvability means that the BSDE has a supersolution with value ∞ at the given stopping time. In our arguments we assume the terminal time S to be solvable. We note that a stopping time that has a positive density around 0 is not solvable. We also note that deterministic times as well as exit times of continuous diffusion processes from smooth domains are solvable. A natural direction for future work is to further understand the concept of solvability and identify other classes of solvable/non-solvable stopping times.

8. Symbols and notation

- (1) $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space on which all of the random variables are defined; \mathbb{F} is a complete, right continuous filtration of \mathcal{F} supporting a d -dimensional standard Brownian motion W and a Poisson random measure π on $\mathcal{E} \subset \mathbb{R}^m \setminus \{0\}$ with intensity μ (see 1.1).
- (2) (Y, Z, U, M) are the components of a (super)solution to the BSDE (1.1), which is defined by the driver f , the terminal condition ξ and terminal time S , a stopping time (Definitions 1.1 and 1.5).
- (3) Constants and functions appearing in assumptions (A1)-(A4): $\chi, \Upsilon_t, f_t^0, \kappa, \vartheta, L_z, L_\vartheta$.
- (4) Constants appearing in other assumptions of Theorem 1.4 (Kruse and Popier (2016a, 2017, Theorem 3)) guaranteeing the existence of a unique classical solution to the BSDE (1.1): $r > 1, \rho, K$ (see (1.7)), ν (see (1.8)).
- (5) Constants appearing in assumptions (B1)-(B4): $q > 1, \delta > \delta^*, m > m^*$.
- (6) Constants appearing in Definition 1.5 of a supersolution: $\ell > 1$.
- (7) $(Y^\infty, Z^\infty, U^\infty, M^\infty)$ is the minimal supersolution with terminal condition $+\infty$ a.s. at the solvable stopping time S .
- (8) More generally, superscripts to (Y, Z, U, M) are used to denote terminal times and conditions for (super)solutions; an example: $Y^{\tau \wedge S, \xi_1 \wedge k}$ denotes the Y component of the solution of the BSDE (1.1) with terminal condition $Y_{\tau \wedge S} = \xi_1 \wedge k$. We omit the terminal time from the superscript when an emphasis on the terminal time is not necessary.
- (9) Itô's diffusion Ξ , its drift b and diffusion coefficient σ (1.19).
- (10) \mathcal{L} : infinitesimal generator of Ξ (4.7).
- (11) D : open bounded subset of \mathbb{R}^d with C^2 boundary,
- (12) For a set $B \subset \mathbb{R}^d$, $\text{dist}_B : \mathbb{R}^d \rightarrow \mathbb{R}$, $\text{dist}_B(x) = \inf_{y \notin B} \|x - y\|$ for $y \in B$, $\text{dist}_B(x) = -\inf_{y \in B} \|x - y\|$, $x \notin B$; if $B = D$ we write dist .
- (13) $D_\lambda = \{x \in \mathbb{R}^d, |\text{dist}(x)| \leq \lambda\}$ (4.2).
- (14) For a set $U \subset \mathbb{R}^d$, \bar{U} denotes its closure.
- (15) Decomposition of f into four parts: $f = \phi + \varpi + \theta + f^0$ (3.1).
- (16) Constants and functions appearing in assumptions (C1)-(C2): $\varsigma_t > 0, \mathfrak{f}, \kappa_* > -1$; Θ of (3.2) is defined in terms of \mathfrak{f} .
- (17) C, C_1, C_2 , etc. are used throughout the text to denote a positive real constant that is independent of (t, x, ω) ; the dependence of a constant on model parameters is indicated wherever it is used.
- (18) Constants and functions appearing in assumptions (D1)-(D3): g, F_∞ .
- (19) $\xi_1 = \mathbf{1}_{\{\tau \leq S\}}$, τ is a stopping time of \mathbb{F} , (5.1).
- (20) $\xi_2 = \mathbf{1}_{\{\tau > S\}}$, τ is a stopping time of \mathbb{F} , (6.1).

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