

Compatibility of the Dimensional Reduction and Variation Procedures for a Quadratic Curvature Model with a Kaluza-Klein Ansatz

S. Başkal¹ and S. Çelik²

Physics Department, Middle East Technical University
06531, Ankara, Turkey

Abstract

The introduction of extra dimensions is an invaluable strategy for the unification of gravity with other physical fields. Nevertheless, the matter in hand is to be eventually reduced to the actual 4D spacetime. The Kaluza-Klein theory is no exception to this well-known scheme. There are two procedures to obtain the field equations from a higher dimensional action. One can either take the variation of the effective action in that higher dimension and then reduce the resulting equations or reduce the higher dimensional action to the actual 4D and henceforward take the variations with respect to the constituent fields of the theory. Here, for the case of a quadratic curvature model with a Kaluza-Klein ansatz the field equations are obtained from the reduced action and compatibility of these two procedures is discussed in detail.

1 Introduction

In 1919, Weyl introduced a scale invariant effective action with a quadratic curvature [1]

$$\int d^4x \sqrt{-g} R_{jkmn} R^{jkmn} \quad (1)$$

as an alternative model to Einstein's gravity. In 1974, Yang introduced the underlying gauge structure of gravity, where the Riemann tensor represents gravitational field [2]. In this work we shall refer (1) as the Weyl-Yang (WY) action. Since then and in between, numerous authors, to name a few: Stelle[3], Stephenson[4], Kilmister and Newman[5], Higgs[6], Fairchild and Edward [7] have favored this particular model in their articles. In addition to being a scale invariant gauge model, it includes the solutions of Einstein's equations as a first integral, while its non-desirable properties can mostly be circumvented [7, 8, 9].

The Kaluza-Klein (KK) theory unifies electromagnetism with gravitation. Unifying gravity with electromagnetism by some suitable trivial and/or non-trivial couplings with the generic F^n , $(DF)^n$ and RF^2 terms in the action is shown to be interesting [10, 11, 12]. They account for photon graviton oscillations [13] and also drew attention from an astrophysical point of view [14]. However, including terms involving the Riemann, Ricci and curvature scalar contracted with the electromagnetic field F , seems to be done (apart from the well-known basic principles) rather freely, while the explicit

¹electronic address: baskal@newton.physics.metu.edu.tr

²electronic address: sinancel@metu.edu.tr

form of the couplings is predetermined in the KK theory as they arise uniquely from the reduction mechanism.

Quadratic curvature Lagrangians are also considered in the framework of the KK theory [15, 16]. Later on, Baskal and Kuyrukcu gave the field equations and the energy-momentum tensor for the WY type of KK gravity [17]. In that article they took the variation of the effective action in five dimensions by using the Palatini approach which considers the 5D connection and the 5D metric as independent variables and then they reduced the resulting equations to the actual four dimensions.

This work can be considered as an extension to [17]. Here, we shall reverse the succession of mechanisms involved therein. We shall first reduce the 5D WY action to four dimensions and thereafter we shall take the variations with respect to the constituent fields of the theory, namely with respect to the 4D connection, metric, electromagnetic vector potential and the scalar dilaton field, as they become manifest after the reduction. The main purpose of this work is to examine the compatibility of these two procedures where the order of the implementations of variation and reduction mechanisms are reversed, within the context of the WY model with a KK ansatz.

2 Preliminaries

The Kaluza-Klein theory operates on a five dimensional manifold with a coordinate system

$$\hat{x}^A = (x^a, x^5) \quad (2)$$

where one of the spacelike dimensions is spontaneously compactified to a circle, whose radius is in the order of Planck's length. The indices with capital Latin letters take values $A = 0, 1, 2, 3, 5$, whereas all the indices with lowercase Latin letters take values $a = 0, 1, 2, 3$. The KK metric in five dimensions can conveniently be written as

$$\hat{G}(x^j, x^5) = G(x^j) + \varphi(x^j)[A(x^j) + dx^5] \otimes \varphi(x^j)[A(x^j) + dx^5] \quad (3)$$

where

$$G(x^j) = \eta_{ab} e^a \otimes e^b \quad (4)$$

The following choice of the orthogonal basis 1-forms

$$\hat{E}^a(x^j, x^5) = e^a(x^j) \quad \text{and} \quad \hat{E}^5 = \varphi(x^j)(A(x^j) + dx^5) \quad (5)$$

allows the KK metric to be expressed succinctly

$$\hat{G}(x^j, x^5) = \eta_{AB} \hat{E}^A \otimes \hat{E}^B \quad (6)$$

with $\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1)$ and $\hat{E}^A = (\hat{E}^a, \hat{E}^5)$, satisfying $\iota_{X_B}(\hat{E}^A) = \delta_B^A$. In terms of the coordinate basis, fünf-beins can be expressed as $\hat{E}^A = h^A_{\mu} dx^{\mu}$, where $\mu = 0, 1, 2, 3, 5$. All physical fields are independent of the fifth coordinate x^5 .

Taking the metric (6) into account, the components of the Riemann tensor had been found earlier in [15] and in [17]:

$$\begin{aligned} \hat{R}_{abmn} &= R_{abmn} - \frac{1}{4}\varphi^2(2F_{ab}F_{mn} + F_{am}F_{bn} - F_{an}F_{bm}) \\ \hat{R}_{ab5m} &= \frac{1}{2}\varphi D_m F_{ab} + \frac{1}{2}(2\varphi_m F_{ab} + \varphi_b F_{am} - \varphi_a F_{bm}) \\ \hat{R}_{5a5m} &= -\varphi^{-1} D_m \varphi_a - \frac{1}{4}\varphi^2 F_{aj} F^j_m \end{aligned} \quad (7)$$

3 Kaluza-Klein reduction for a quadratic curvature

One way to obtain the field equations in the standard KK theory is to vary the 5D action

$$\int \hat{R} \sqrt{-\hat{g}} d^5x \quad (8)$$

with respect to the 5D metric \hat{g}_{AB} which yields

$$\hat{G}_{AB} = 0 \quad (9)$$

where

$$\hat{G}_{AB} \equiv \hat{R}_{AB} - \frac{1}{2} \hat{g}_{AB} \hat{R} \quad (10)$$

is the Einstein tensor in 5D. Then (9) is reduced to 4D to obtain equations of gravity coupled with electromagnetism and a scalar field. One may well start from reducing (8) to 4D,

$$\int \varphi [R - \frac{1}{4} \varphi^2 F^{ab} F_{ab} - 2\varphi^{-1} D_a \varphi^a] \sqrt{-g} d^4x \quad (11)$$

and then take the variations with respect to the emerging fields, namely g_{ab} , A_k , and φ . The question is whether these two approaches produce the same results. Equations obtained from those two procedures are shown to be equivalent with some straightforward manipulations [18].

3.1 The 5D Palatini approach to the WY action

A similar line of thought begs the same type of question for the WY action in 5D. The five dimensional gravitational action

$$\mathcal{I} = \frac{1}{16\pi\hat{G}} \int d^5x \hat{\mathcal{L}} \quad (12)$$

explicitly reads as

$$\mathcal{I} = \frac{1}{16\pi\hat{G}} \int d^5x \sqrt{-\hat{g}} \hat{R}_{JKLMN} \hat{R}^{JKMN} \quad (13)$$

when the Lagrangian is formed from a 5D quadratic curvature.

Using the Palatini method where variation of (13) is taken directly with respect to the 5D connection Γ_{MN}^J and the 5D metric \hat{g}^{AB} renders the field equations :

$$D_K \hat{R}^K{}_{JMN} = 0 \quad (14)$$

and the energy-momentum tensor

$$\hat{T}_{AB} = \hat{R}_{AKMN} \hat{R}_B{}^{KMN} - \frac{1}{4} \hat{g}_{AB} R_{JKLMN} R^{JKMN} \quad (15)$$

respectively, where

$$\hat{T}_{AB} = \frac{1}{2\sqrt{-\hat{g}}} \frac{\delta \hat{\mathcal{L}}}{\delta \hat{g}^{AB}} \quad (16)$$

Then one proceeds to reduce (14) and (15) to obtain the field equations and the energy-momentum tensor in the actual 4D spacetime [17]. In the next section, we shall reverse the order of this mechanism. We shall first reduce (13) into 4D and then we shall take the variations with respect to the fields emerging after the reduction.

3.2 Reduced form of the WY action

We start by expressing the 5D invariant by a natural splitting the Riemann tensor in terms of its spacetime, fifth and the mixed components, also keeping in mind that $\sqrt{-\hat{g}} = \sqrt{-g}\varphi$. Then, the action becomes

$$\mathcal{I} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \varphi (\hat{R}_{jkmn} \hat{R}^{jkmn} + 4\hat{R}_{5kmn} \hat{R}^{5kmn} + 4\hat{R}_{j5m5} \hat{R}^{j5m5}) \quad (17)$$

We define $L = \varphi L_q$, where

$$L_q = \hat{R}_{jkmn} \hat{R}^{jkmn} + 4\hat{R}_{5kmn} \hat{R}^{5kmn} + 4\hat{R}_{j5m5} \hat{R}^{j5m5} \quad (18)$$

Taking (7) into account our action becomes

$$\begin{aligned} \mathcal{I} &= \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left\{ \varphi R_{jkmn} R^{jkmn} - \frac{3}{2} \varphi^3 R^j{}_{kmn} F_j{}^k F^{mn} \right. \\ &\quad + \frac{1}{8} \varphi^5 [3F_{jk} F^{jk} F_{mn} F^{mn} + 5(F_{jm} F^{mn} F_{ni} F^{ij})] + \varphi^3 D_k F_{mn} D^k F^{mn} \\ &\quad + 4\varphi^{-1} D_m \varphi_n D^m \varphi^n - 2\varphi^2 F^{mk} F^n{}_k D_m (D_n \varphi) \\ &\quad + 8\varphi^2 (D_k F_{mn} \varphi^m F^{kn} + D_k F_{mn} \varphi^k F^{mn}) \\ &\quad \left. + 6\varphi (\varphi_k \varphi^k F^{mn} F_{mn} + \varphi_m \varphi_n F^{mk} F^n{}_k) \right\} \quad (19) \end{aligned}$$

4 Field equations from the reduced action

In this section we shall treat each term in (19) one by one and collect them at the end. So we define

$$\mathcal{I}_i = \int d^4x \mathcal{L}_i = \int d^4x \sqrt{-g} L_i \quad (20)$$

where the subscript i is representing each term. This way we have the advantage of analysing their equivalent forms, in addition to referring to the literature for some physical content.

- \mathcal{I}_1

The action for the first term is

$$\mathcal{I}_1 = \int d^4x \sqrt{-g} \varphi R_{jkmn} R^{jkmn} \quad (21)$$

By treating the gravitational field R_{jkmn} as a gauge field [2], dynamics resulting from a quadratic curvature Lagrangian has been studied extensively [7, 8, 19] (and the references therein). An immediate generalization would be the construction of this formalism for the KK ansatz [15, 17]. In fact, it is these latter developments that paved the way for this current work.

Now, consider

$$\delta \mathcal{L}_1 = 2\varphi R_{jkmn} \delta R^{jkmn} \sqrt{-g} \quad (22)$$

to vary with respect to the connection. Let us take the Palatini equation into account [20]:

$$\delta R^j{}_{kmn} = D_m (\delta \Gamma^j{}_{kn}) - D_n (\delta \Gamma^j{}_{km}) \quad (23)$$

and

$$D_m(\sqrt{-g}\varphi R_j^{kmn}\delta\Gamma_{kn}^j) = D_m(\sqrt{-g}\varphi R_j^{kmn})\delta\Gamma_{kn}^j + \sqrt{-g}\varphi R_j^{kmn}D_m\delta\Gamma_{kn}^j \quad (24)$$

Substituting (23) and using (24) we have where the first and third terms are tensor densities of weight +1. Apart from a total divergence, we have

$$\delta\mathcal{I}_1 = 4 \int d^4x D_n(\sqrt{-g}\varphi R_j^{kmn})\delta\Gamma_{km}^j \quad (25)$$

The expression resulting from above is

$$\gamma_1 := -4\sqrt{-g}D_m(\varphi R_j^{kmn}) \quad (26)$$

Variation of \mathcal{I}_1 with respect to φ is

$$\delta\mathcal{I}_1 = \int d^4x \sqrt{-g}\delta\varphi R_{jkmn}R^{jkmn} \quad (27)$$

So, we get

$$\Phi_1 := \sqrt{-g}R_{jkmn}R^{jkmn} \quad (28)$$

In order to vary with respect to g^{ab} , we rewrite our Lagrangian in terms of the hidden metrics Using the symmetry properties of the Riemann tensor and simplifying we have

$$\delta\mathcal{L}_1 = \varphi(2R_{anjk}R_b^{nj} - \frac{1}{2}g_{ab}R_{mnjk}R^{mnjk})\sqrt{-g}\delta g^{ab} \quad (29)$$

We define the energy-momentum tensor T_{ab} as the coefficient of δg^{ab} such that

$$T_{ab}\delta g^{ab} = \frac{1}{2}\frac{1}{\sqrt{-g}}\delta\mathcal{L} \quad (30)$$

Therefore, we obtain

$$T_{ab} = \varphi(R_{anjk}R_b^{nj} - \frac{1}{4}g_{ab}R_{mnjk}R^{mnjk}) \quad (31)$$

• \mathcal{I}_2

For the second term we have

$$\mathcal{I}_2 = \int d^4x \sqrt{-g}\varphi^3 R^j{}_{kmn}F_j{}^k F^{mn} \quad (32)$$

Early works dealing with Lagrangians given in the above form are mostly formal mathematical studies in essence [21, 22, 23]. Later on the motivations for the inclusion of RF^2 became diverse[10]. An exhausted list of $R_x F^2$ non-minimal couplings are presented in [24], where R_x refers to an element of the set $R_x = \{R, R_{ij}, R_{ijnm}\}$, along with the condition that couplings of F^2 with respect to R_x are linear. These authors argue that such couplings depict drastic conditions where the gravitational and electromagnetic fields are intense and the speed of the gravitational-electromagnetic wave differs from that of light in vacuum. We see that one such particular coupling as in (32) singles out naturally through the reduction procedure of the model we have, and excludes all other forms that might be added arbitrarily by hand.

We first start by varying (32) with respect to A_i

$$\delta\mathcal{I}_2 = \int d^4x \sqrt{-g}\varphi^3 \{R^{jk}{}_{mn}(D_j\delta A_k - D_k\delta A_j)F^{mn} + R_{jk}{}^{mn}F^{jk}(D_m\delta A_n - D_n\delta A_m)\} \quad (33)$$

Apart from total divergences, we obtain

$$\delta\mathcal{I}_2 = -4 \int d^4x \sqrt{-g} \varphi^3 D_j R^{j i}_{mn} F^{mn} \delta A_i \quad (34)$$

which gives the expression

$$a_2 := -4 \sqrt{-g} D_k (\varphi^3 R^{kj}_{mn} F^{mn}) \quad (35)$$

Now, we vary with respect to the connection Γ^j_{nk}

$$\delta\mathcal{I}_2 = \int d^4x \sqrt{-g} \varphi^3 (\delta R^j_{kmn}) F_j^k F^{mn} \quad (36)$$

to obtain

$$\delta\mathcal{I}_2 = 2 \int d^4x D_m (\sqrt{-g} \varphi^3 F_j^k F^{mn}) \delta \Gamma^j_{nk} \quad (37)$$

So, we have

$$\gamma_2 := 2 D_m (\sqrt{-g} \varphi^3 F_j^k F^{mn}) \quad (38)$$

By varying with respect to φ

$$\delta\mathcal{I}_2 = 3 \int d^4x \sqrt{-g} R^j_{kmn} F_j^k F^{mn} \varphi^2 \delta \varphi \quad (39)$$

we end up with

$$\Phi_2 := 3 \sqrt{-g} \varphi^2 R^j_{kmn} F_j^k F^{mn} \quad (40)$$

Varying with respect to the metric g^{ab} yields

$$\begin{aligned} \delta\mathcal{I}_2 = & \int d^4x \varphi^3 (2R_{akmn} F_b^k F^{mn} \\ & + 2R_{akmn} F_b^k F^{mn} + \frac{1}{2} g_{ab} R_{jkmn} F^{jk} F^{mn}) \sqrt{-g} \delta g^{ab} \end{aligned} \quad (41)$$

Thus, we obtain

$$T_{ab} = \varphi^3 (R_{anj k} F_b^n + R_{bnj k} F_a^n - \frac{1}{4} g_{ab} R_{jkmn} F^{mn}) F^{jk} \quad (42)$$

• \mathcal{I}_3

The action for the third term is

$$\mathcal{I}_3 = \int d^4x \sqrt{-g} \varphi^5 F_{jk} F^{jk} F_{mn} F^{mn} \quad (43)$$

From a mathematical point of view F^n invariants were considered to be worthwhile in their own right, [25]. The most interesting results arising from F^4 terms has been found the context of QED in Minkowskian space, when photons interact with a strong magnetic field [26].

Specifically, one can evaluate following two quadratic invariants in terms of the electric E_i and magnetic B_i field vectors

$$\begin{aligned} F_{jk} F^{jk} F_{mn} F^{mn} &= 4(E^2 - B^2)^2 \\ F^{jk} F_{km} F^{mn} F_{nj} &= 2(E^2 - B^2)^2 + 4(E \cdot B)^2 \end{aligned} \quad (44)$$

to grasp the strength of the fields as compared to the two fundamental quadratic invariants $F_{jk} F^{jk} = 2(E^2 - B^2)$ and $F_{jk} * F^{jk} = (E \cdot B)$.

Now, we take the variation of \mathcal{I}_3 with respect to A_n . Apart from total divergences we obtain

$$\delta\mathcal{I}_3 = 8 \int d^4x \{ -D_m(\sqrt{-g}\varphi^5 F^{ij} F_{ij} F^{mn}) \delta A_n \} \quad (45)$$

which yields to the following expression

$$a_3 := -8D_k(\sqrt{-g}\varphi^5 F_{mn} F^{mn} F^{kj}) \quad (46)$$

Variation of \mathcal{I}_3 with respect to φ gives

$$\delta\mathcal{I}_3 = 5 \int d^4x \sqrt{-g} \varphi^4 F_{jk} F^{jk} F_{mn} F^{mn} \delta\varphi \quad (47)$$

From above, we have

$$\Phi_3 := 5\sqrt{-g}\varphi^4 F_{jk} F^{jk} F_{mn} F^{mn} \quad (48)$$

The variation of \mathcal{L}_3 with respect to g^{ab} becomes

$$\delta\mathcal{L}_3 = \varphi^5 F_{mn} F^{mn} (F_{aj} F_b{}^j + F^j{}_a F_{jb} + F_b{}^j F_{aj} + F^j{}_b F_{ja} - \frac{1}{2} g_{ab} F_{ij} F^{ij}) \sqrt{-g} \delta g^{ab} \quad (49)$$

After simplifications, the energy-momentum tensor takes the form

$$T_{ab} = \varphi^5 F_{mn} F^{mn} (2F_{ak} F_b{}^k - \frac{1}{4} g_{ab} F_{ij} F^{ij}) \quad (50)$$

• \mathcal{I}_4

Consider the variation of the fourth term as

$$\begin{aligned} \delta\mathcal{I}_4 = \int d^4x \sqrt{-g} \quad & \varphi^5 (\delta F_{jm} F^{mn} F_{ni} F^{ij} + F_{jm} \delta F^{mn} F_{ni} F^{ij} \\ & + F_{jm} F^{mn} \delta F_{ni} F^{ij} + F_{jm} F^{mn} F_{ni} \delta F^{ij}) \end{aligned} \quad (51)$$

To vary with respect to A_k , it can be observed that all of the four terms above can be treated similarly. Therefore it is sufficient to perform the variation only on one of them. For instance we may consider the third term, which becomes

$$\delta\mathcal{I}_{43} = -2 \int d^4x D_i(\sqrt{-g} \varphi^5 F^{im} F_{mj} F^{jk}) \delta A_k \quad (52)$$

Treating the remaining terms in a similar way and adding them up all, this expression simplifies as

$$a_4 := 8D_j(\sqrt{-g} \varphi^5 F^{jm} F_{mi} F^{ik}) \quad (53)$$

Variation with respect to φ yields

$$\Phi_4 := 5\varphi^4 F_{jm} F^{mn} F_{ni} F^{ij} \sqrt{-g} \quad (54)$$

We take the variation of \mathcal{L}_4 with respect to metric g^{ab} , to obtain the energy-momentum tensor

$$T_{ab} = \varphi^5 (2F_{am} F^i{}_b - \frac{1}{4} g_{ab} F_{jm} F^{ij}) F^{mn} F_{ni} \quad (55)$$

• \mathcal{I}_5

The action for the fifth term is

$$\mathcal{I}_5 = \int d^4x \sqrt{-g} \varphi^3 D_k F_{mn} D^k F^{mn} \quad (56)$$

Lagrangians in the form of $(DF)^2$ has been of interest since long, and is referred to Bopp-Podolsky (BP) electrodynamics [27, 28].

Now , we vary (56) with respect to A_n to have

$$\delta \mathcal{I}_5 = -4 \int d^4x D_m (\sqrt{-g} \varphi^3 D_k D^k F^{mn}) \delta A_n \quad (57)$$

Thus, we find the expression

$$a_5 := -4 D_m (\sqrt{-g} \varphi^3 D_k D^k F^{mn}) \quad (58)$$

To vary with respect to Γ_{bc}^a , we consider again

$$\delta \mathcal{I}_5 = \int d^4x \sqrt{-g} \varphi^3 [2\delta(D_k F_{mn}) D^k F^{mn}] \quad (59)$$

where

$$\delta(D_k F_{mn}) = -\delta \Gamma_{mk}^l F_{ln} - \delta \Gamma_{nk}^l F_{ml} \quad (60)$$

After carrying out simple calculations, we obtain

$$\gamma_5 := -4 \sqrt{-g} \varphi^3 F_{na} D^c F^{nb} \quad (61)$$

There can be three distinguishable forms for \mathcal{L} in (56), in regard to their contracted indices, that are non-trivially equivalent

$$\mathcal{L}^{(1)} = D_k F^{mk} D_n F_m{}^n, \quad \mathcal{L}^{(2)} = D_k F^{mn} D_n F_m{}^k, \quad \mathcal{L}^{(3)} = D_k F^{mn} D^k F_{mn} \quad (62)$$

We have $\mathcal{L}^{(3)} = 2\mathcal{L}^{(2)}$. The relation between $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ can be found by using total divergences, which becomes [29] :

$$\mathcal{L}^{(2)} - \mathcal{L}^{(1)} = D_n (F_m{}^k D_k F^{mn}) - D_k (F_m{}^k D_n F^{mn}) + F_m{}^k (D_k D_n - D_n D_k) F^{mn} \quad (63)$$

By using

$$(D_k D_n - D_n D_k) F^{mn} = R^m{}_{lkn} F^{ln} - R^{ln}{}_{kn} F^m{}_l \quad (64)$$

and arranging the indices we have an equivalent form for (56)

$$2 \int d^4x \sqrt{-g} \varphi^3 \{ R_{mn} F^{mj} F_j{}^n + R_{jmnk} F^{mk} F^{jn} + D_n F^{jn} D_m F_j{}^m \} \quad (65)$$

apart from total divergences, as usual.

To be on the safe side, we vary this form with respect to A_k and Γ_{bc}^a in regard to the question of whether an alternative form renders an expression comparable with the results of [17]. Since the results of our calculations are far from being affirmative, we do not write them here.

From the variation of \mathcal{I}_5 with respect to φ , we get

$$\Phi_5 := 3 \sqrt{-g} \varphi^2 D_k F_{mn} D^k F^{mn} \quad (66)$$

Varying with respect to the metric the energy-momentum tensor takes the form

$$T_{ab} = \frac{1}{2}\varphi^3(D_a F_{ij} D_b F^{ij} + 2D^k F_a{}^j D_k F_{bj} - \frac{1}{2}g_{ab}D^k F^{ij} D_k F_{ij}) \quad (67)$$

• \mathcal{I}_6

For the sixth term with respect to φ , we have

$$\delta\mathcal{I}_6 = \int d^4x \{ -\varphi^{-2}(D_m D_n \varphi D^m D^n \varphi)\delta\varphi + 2\varphi^{-1}(D^m D^n \varphi)D_m D_n \delta\varphi\} \sqrt{-g} \quad (68)$$

to obtain

$$\Phi_6 := \sqrt{-g}\varphi^{-2}[(D_m D_n \varphi)D^m D^n \varphi] + 2\sqrt{-g}D_n D_m (\varphi^{-1}D^m D^n \varphi) \quad (69)$$

Now varying with respect to Γ_{bc}^a

$$\delta\mathcal{I}_6 = -2 \int d^4x \sqrt{-g} \varphi^{-1}(\varphi_a D^c \varphi^b) \delta\Gamma_{bc}^a \quad (70)$$

we obtain

$$\gamma_6 := -2\sqrt{-g}\varphi^{-1}(\varphi_a D^c \varphi^b) \quad (71)$$

We vary with respect to g^{ab}

$$\delta\mathcal{I}_6 = \int d^4x \varphi^{-1} \{ D_a \varphi_n D_b \varphi^n + D_m \varphi_a D_m \varphi^b - \frac{1}{2}g_{ab} D_m \varphi_n D^m \varphi^n \} \sqrt{-g} \delta g^{ab} \quad (72)$$

and obtain

$$T_{ab} = \varphi^{-1}(D_a \varphi_n D_b \varphi^n - \frac{1}{4}g_{ab} D_m \varphi_n D^m \varphi^n) \quad (73)$$

• \mathcal{I}_7

For the seventh term variation with respect to A_k is

$$\begin{aligned} \delta\mathcal{I}_7 = \int d^4x \sqrt{-g} \varphi^2 \{ & g^{mi} g^{ks} g^{np} [(D_i \delta A_s - D_s \delta A_i) F_{pk} \\ & + (D_p \delta A_k - D_k \delta A_p) F_{is}] (D_m \varphi_n) \} \end{aligned} \quad (74)$$

yielding

$$a_7 := 2\sqrt{-g} D_i \{ \varphi^2 (F^{ki} D_j \varphi_k + F_j{}^k D_i \varphi_k) \} \quad (75)$$

From the variation with respect to Γ_{bc}^a , we have the expression

$$\gamma_7 := -\varphi^2 F^{ck} F^b{}_k (D_a \varphi) \sqrt{-g} \quad (76)$$

Variation with respect to φ , apart from the total divergences is

$$\delta\mathcal{I}_7 = \int d^4x \sqrt{-g} \{ D_n D_m (\varphi^2 F^{mk} F^n{}_k) + 2\varphi F^{mk} F^n{}_k D_m \varphi_n \} \delta\varphi \quad (77)$$

Then, we obtain

$$\Phi_7 := \sqrt{-g} D_n D_m (\varphi^2 F^{mk} F^n{}_k) + 2\sqrt{-g} \varphi F^{mk} F^n{}_k D_m \varphi_n \quad (78)$$

We consider δg^{ab} and proceed as we did earlier. So we have

$$T_{ab} = \frac{1}{2}\varphi^2 \{ F_a{}^k F^i{}_k D_i D_b \varphi + F_b{}^k F^i{}_k D_i D_a \varphi + F_a{}^i F_b{}^j D_i D_j \varphi - \frac{1}{2}g_{ab} F^{ik} F^j{}_k D_i D_j \varphi \} \quad (79)$$

• \mathcal{I}_8

To our knowledge we have not encountered a Lagrangian containing such a term, apart from this particular KK reduction. We start by varying with respect to the potentials A_k .

$$\begin{aligned} \delta\mathcal{I}_8 = \frac{1}{2} \int d^4x \sqrt{-g} \varphi^2 \{ & (D_k D_m \delta A_n - D_k D_n \delta A_m) \varphi^m F^{kn} + D^k F_m{}^n \varphi^m (D_k \delta A_n - D_n \delta A_k) \\ & + (D_k D_m \delta A_n - D_k D_n \delta A_m) \varphi^k F^{mn} + D_k F^{mn} \varphi^k (D_m \delta A_n - D_n \delta A_m) \} \end{aligned} \quad (80)$$

We consider

$$\begin{aligned} D_k(\varphi^m F^{kn} D_m \delta A_n) &= D_m [D_k(\varphi^m F^{kn}) \delta A_n] - D_m D_k(\varphi^m F^{kn}) \delta A_n \\ &+ \varphi^m F^{kn} (D_m D_k \delta A_n) \end{aligned} \quad (81)$$

Therefore, we obtain

$$\begin{aligned} a_8 := \frac{1}{2} \sqrt{-g} \varphi^2 \{ & D_m D_k [F^{kn} \varphi^m + F^{mk} \varphi^n + 2F^{mn} \varphi^k] \\ & + D_k [(D^k F^m{}_m) \varphi^m + (D^n F_m{}^k) \varphi^m] + 2D_m [(D_k F^{nm}) \varphi^k] \} \end{aligned} \quad (82)$$

In order to vary with respect to Γ_{bc}^a we rewrite $\delta\mathcal{I}_8$ as

$$\begin{aligned} \delta\mathcal{I}_8 &= \frac{1}{2} \int d^4x \sqrt{-g} \varphi^2 \{ (\partial_k F_{mn} - \delta\Gamma_{mk}^j F_{jn} - \delta\Gamma_{nk}^j F_{mj} [\varphi^m F^{kn}]) \\ &= \frac{1}{2} \int d^4x \sqrt{-g} \varphi^2 (-\delta_a^j \delta_m^b \delta_k^c F_{jn} - \delta_a^j \delta_n^b \delta_k^c F_{mj}) [\varphi^m F^{kn} + \varphi^k F^{mn}] \delta\Gamma_{bc}^a \end{aligned} \quad (83)$$

After simplifying, this becomes

$$\gamma_8 := \frac{1}{2} \sqrt{-g} \varphi^2 (\varphi^b F_{aj} F^{cj} - 2\varphi^c F_{aj} F^{bj}) \quad (84)$$

Variation with respect to φ is

$$\delta\mathcal{I}_8 = \frac{1}{2} \int d^4x \sqrt{-g} (2\varphi^{-1} \delta\varphi L_8 + \varphi^2 D_k F^m{}_n D_m \delta\varphi F^{kn} + \varphi^2 D^k F_{mn} D_k \delta\varphi F^{mn}) \quad (85)$$

Apart from total divergences, we have

$$\delta L_8 = \frac{1}{2} \{ (2D_k F_{mn} \varphi^{(m} F^{k)n}) - D_m [\varphi^2 (D_k F_{mn}) F^{kn} + \varphi^2 (D^m F_{kn}) F^{kn}] \} \delta\varphi \quad (86)$$

Then the expression becomes

$$\Phi_8 := \frac{1}{2} \sqrt{-g} \{ (2D_k F_{mn} \varphi^{(m} F^{k)n}) - D_m [\varphi^2 (D_k F_{mn}) F^{kn} + \varphi^2 (D^m F_{kn}) F^{kn}] \} \quad (87)$$

We take the variation with respect to g^{ab} to obtain

$$\begin{aligned} T_{ab} &= \frac{1}{4} \varphi^2 \{ (D_a F^{mn} \varphi_m F_{bn} + D_k F_{an} \varphi_b F^{kn} + D_k F_{am} \varphi^m F_b{}^k \\ &+ D_a F_{mn} \varphi_b F^{mn} + 2D_k F_{am} \varphi^k F_b{}^m) - \frac{1}{2} g_{ab} D_k F_{mn} \varphi^{(m} F^{k)n} \} \end{aligned} \quad (88)$$

- \mathcal{I}_9

We vary the ninth term with respect to A_n

$$\delta\mathcal{I}_9 = \int d^4x \sqrt{-g} \{2\varphi\varphi_k\varphi^k F^{mn}(D_m\delta A_n)\} \quad (89)$$

From above, we have the expression

$$a_9 := -2\sqrt{-g}D_m(\varphi\varphi_k\varphi^k F^{mn}) \quad (90)$$

Variation of \mathcal{I}_9 with respect to φ yields

$$\delta\mathcal{I}_9 = \int d^4x \{ -2D_k(\sqrt{-g}\varphi F^{mn}F_{mn}\varphi^k) + \varphi_k\varphi^k F^{mn}F_{mn}\sqrt{-g} \} \delta\varphi \quad (91)$$

So, we have

$$\Phi_9 := -2\sqrt{-g}D_k(\varphi F^{mn}F_{mn}\varphi^k) + \varphi_k\varphi^k F^{mn}F_{mn}\sqrt{-g} \quad (92)$$

The energy-momentum tensor

$$T_{ab} = \varphi(\varphi_k\varphi^k F_{ai}F^{bi} + \frac{1}{2}\varphi_a\varphi_b F_{mn}F^{mn} - \frac{1}{4}g_{ab}\varphi_k\varphi^k F_{mn}F^{mn}) \quad (93)$$

is obtained by varying with respect to the metric, as before.

- \mathcal{I}_{10}

Varying the last term \mathcal{I}_{10} with respect to A_k gives

$$\begin{aligned} \delta\mathcal{I}_{10} = \int d^4x \sqrt{-g} \{ & -D_m(\varphi\varphi^m\varphi^n F_n^k) + D_m(\varphi\varphi^k\varphi^n F_n^m) \\ & -D_n(\varphi\varphi^m\varphi^n F_m^k) + D_n(\varphi\varphi^m\varphi^k F_m^n) \} \sqrt{-g} \delta A_k \end{aligned} \quad (94)$$

Therefore, we have

$$a_{10} := 2D_m(\varphi\varphi^k\varphi^n F_n^m - \varphi\varphi^m\varphi^n F_n^k)\sqrt{-g} \quad (95)$$

Varying with respect to φ , we obtain the expression

$$\Phi_{10} := \sqrt{-g}\varphi_m\varphi_n F^{mk} F^n_k - 2D_m(\sqrt{-g}\varphi\varphi_n F^{mk} F^n_k) \quad (96)$$

Variation with respect to g^{ab} yields the energy-momentum tensor

$$T_{ab} = \varphi(\frac{1}{2}\varphi^i\varphi^j F_a^i F_b^j + \varphi_a\varphi_i F_{bk} F^{ik} + \varphi_b\varphi_i F_{ak} F^{ik} - \frac{1}{4}g_{ab}\varphi_i\varphi_j F^{ik} F^j_k) \quad (97)$$

5 The compatibility issue

We have seen that in the standard KK theory, there are three sets of equations, irrespective of the order of the successive procedures. They come from the variations with respect to three fields or from the splitting of \hat{G}_{AB} into its two pure and one mixed components. On the other hand, the usage of the Palatini method for the quadratic curvature model in [17], where we also consider the

independent variation of the 5D connection $\hat{\Gamma}_{BC}^A$, yield six equations. Of those three come from the splitting of $D_K \hat{R}^K_{JMN} = 0$, which are

$$D_K \hat{R}^K_{jmn} = 0, \quad D_K \hat{R}^K_{5mn} = 0, \quad D_K \hat{R}^K_{5m5} = 0 \quad (98)$$

since, from Bianchi identities we have

$$D_K \hat{R}^K_{5mn} = 2D_K \hat{R}^K_{[nm]5} \quad (99)$$

reducing the number from four to three. Another three come from (15). The component T_{55} is included in (19), apart from some numerical coupling constants. A similar situation is valid for G_{55} of (10), and that it is included in (11). Obviously we do not expect an expression for T_{a5} , coming from this current procedure.

In order to compare our results with those in reference [17], we shall simplify them by setting the scalar field φ to be a constant. This is the optimal amount of simplification, for comparative purposes. Even in this simplified case, the non-minimal couplings between the gravitational field and the electromagnetic (EM) field are considerably intricate. In addition to the pure gravitational part, it includes non-minimal couplings and non-linear electrodynamics. Besides, we have the advantage of ignoring the difference between the Einstein and Jordan frames.

By taking account of the calculations done for each term in the Lagrangian density (19), the field equation that comes from the variation of connection is

$$D_k R^k_{jmn} = \varphi^2 \left(\frac{3}{4} D_k (F^k_j F_{mn}) - F^k_{[m} D_n] F_{kj} \right) \quad (100)$$

This is the equation governing the gravitational field. In [9], Pavelle wrote: *"I will show that the KY equations require more restriction than simply the elimination of degenerate spacetimes. Indeed, it appears that pure spaces themselves allow the generation of unphysical solutions; and I suggest that one should not examine $D_k R^k_{jmn} = S_{jmn}$ unless the source S_{jmn} is nonvanishing."* Here, we see that the right hand side of (100) is the S_{jmn} term Pavelle is emphasising, whose existence and form appears naturally through the KK reduction procedure.

On the other hand from $D_K \hat{R}^K_{jmn} = 0$, one has

$$D_k R^k_{jmn} = -\varphi^2 D_{[m} (F_{n]k} F^k_j) - \frac{\varphi^2}{2} F_{j[m} D_k F^k_{|n]} + \frac{\varphi^2}{2} F_{mn} D_k F^k_j \quad (101)$$

Equations (100) and (101) do not seem to be equivalent, even if we consider equivalent expressions for the Lagrangian as in (65).

As for the equations governing the electromagnetic field, one may be tempted to take the variation directly with respect to F^{mn} [12]. As it is clear in that article, in order to get the proper equations in regard to the electromagnetic equations, one should take an additional derivative. If, such an approach had been adopted then we would find

$$D_k D^k F_{mn} = \frac{3}{2} R_{mnjk} F^{jk} - \frac{3}{4} \varphi^2 F_{mn} F^{jk} F_{jk} + \frac{5}{4} \varphi^2 F_{mj} F^{jk} F_{kn} \quad (102)$$

from the the variation with respect to F^{mn} . This equation has the same terms coming from the reduction of $D_K \hat{R}^K_{5mn} = 0$, which explicitly reads as:

$$D_k D^k F_{mn} = -R_{mnjk} F^{jk} + \frac{\varphi^2}{2} (F_{mn} F_{jk} F^{jk} - 2F_{mj} F^{jk} F_{kn}) \quad (103)$$

but with different constants. On the other hand, varying with respect to A_n renders

$$D^m D_k D^k F_{mn} = D^m \left\{ \frac{3}{2} R_{mnjk} F^{jk} - \frac{3}{4} \varphi^2 F_{mn} F^{jk} F_{jk} + \frac{5}{4} \varphi^2 F_{mj} F^{jk} F_{kn} \right\} \quad (104)$$

where, we observe that (102) just becomes its first integral.

Similarly, the overall energy-momentum tensor (again with φ not depending on x^A) becomes

$$\begin{aligned} T_{ab} &= \varphi \left\{ (R_{anjk} R_b{}^{nj}{}^k - \frac{1}{4} g_{ab} R_{mnjk} R^{mnjk}) \right. \\ &\quad - \frac{3}{2} \varphi^2 (R_{anjk} F_b{}^n + R_{bnjk} F_a{}^n - \frac{1}{4} g_{ab} R_{jkmn} F^{mn}) F^{jk} \\ &\quad + \frac{3}{8} \varphi^4 (2F_{ak} F_b{}^k - \frac{1}{4} g_{ab} F_{ij} F^{ij}) F_{mn} F^{mn} \\ &\quad + \frac{5}{8} \varphi^4 (2F_{am} F^i{}_b - \frac{1}{4} g_{ab} F_{jm} F^{ij}) F^{mn} F_{ni} \\ &\quad \left. + \frac{1}{2} \varphi^2 (D_a F_{ij} D_b F^{ij} + 2D^k F_a{}^j D_k F_{bj} - \frac{1}{2} g_{ab} D^k F^{ij} D_k F_{ij}) \right\} \end{aligned} \quad (105)$$

One can immediately expect that when the Lagrangian density is reduced, there will be an additional φ that multiplies all terms coming from the 4D form of the 5D $\sqrt{-\hat{g}}$. Therefore, when the variation is taken with respect to the 4D metric, there is an extra φ appearing in T_{ab} , as we see in (105) compared to \hat{T}_{ab} , the reduced form of \hat{T}_{AB} to the actual spacetime.

Now, let us consider the spacetime components of (15)

$$\hat{T}_{ab} = \hat{R}_{aKMN} \hat{R}_b{}^{KMN} - \frac{1}{4} \hat{g}_{ab} R_{JKMN} R^{JKMN}. \quad (106)$$

The T_{ab} component of this expression is reduced to become [17]:

$$\begin{aligned} {}^R T_{ab} &= \left\{ (R_{anjk} R_b{}^{nj}{}^k - \frac{1}{4} g_{ab} R_{mnjk} R^{mnjk}) \right. \\ &\quad - \frac{3}{2} \varphi^2 (\frac{1}{2} R_{anjk} F_b{}^n + \frac{1}{2} R_{bnjk} F_a{}^n - \frac{1}{4} g_{ab} R_{jkmn} F^{mn}) F^{jk} \\ &\quad + \frac{3}{8} \varphi^4 (F_{ak} F_b{}^k - \frac{1}{4} g_{ab} F_{ij} F^{ij}) F_{mn} F^{mn} + \frac{5}{8} \varphi^4 (F_{am} F^i{}_b - \frac{1}{4} g_{ab} F_{jm} F^{ij}) F^{mn} F_{ni} \\ &\quad \left. + \frac{1}{2} \varphi^2 (\frac{1}{2} D_a F_{ij} D_b F^{ij} + D^k F_a{}^j D_k F_{bj} - \frac{1}{2} g_{ab} D^k F^{ij} D_k F_{ij}) \right\} \end{aligned} \quad (107)$$

where again we considered the case with $\varphi = \text{constant}$.

We observe that, in our energy-momentum tensor, (105), apart from the terms coming from the variation of $\sqrt{-g}$, there is a factor of 2 (preceding some terms from lines two to last). Except for that, these two expressions are formally similar when compared with the results given in [17]. The factor 2 comes from the fact that, when the action is reduced first, otherwise hidden 4D metrics come into play to form invariants from the constituent fields.

6 Conclusion

In this work, we have obtained the field equations and the energy-momentum tensor from the dimensionally reduced 5D WY action in order to compare its results with those obtained in [17]. Our current approach is to reverse the order of variation and dimensional reduction mechanisms.

We see that both ways of ordering produce the same number of field equations, i.e., 3, however, now there is only one expression T_{ab} , while if varied first, one has additional components T_{a5}, T_{55} .

Although some striking similarities in the resulting field equations can be observed, it cannot be said that reversing the order of the applications yields the same equations. This is basically due to the fact that when first reduced otherwise hidden fields of the KK theory become emergent, as is seen in equation (19), and thus one has to take variations with respect to those hidden 4D metrics, connections and potentials. In addition to this, there are specific coupling constants for each term constituting the overall Lagrangian, that come into play and do not cancel out. Since, the WY is quadratic, but not linear as the Einstein-Hilbert action, it could be expected that some differences would occur. However, one has to carry out the calculations in order to locate and describe the sources of the differences and to obtain the exact forms of the equations.

This work and [17], use the Palatini method for variations. One may wish to examine other variational methods for both ordering procedures, such as considering an explicit dependence of the connection on the metric, or introducing a torsion in the system. Surely, the next natural step is to look for non-trivial exact solutions for some specific metrics.

References

- [1] Weyl, H.: Eine neue Erweiterung der relativitätstheorie. Ann. Phys. (Leipzig) IV. Folge 59, 103 (1919)
- [2] Yang, C.N.: Integral formalism for gauge fields. Phys. Rev. L 33, 445 (1974)
- [3] Stelle, K.S.: Renormalization of higher-derivative quantum gravity. Phys. Rev. D 16, 953 (1977)
- [4] Stephenson, G.: Quadratic Lagrangians and general relativity. II Nuovo Cimento Series 10 9, 263-269 (1958)
- [5] Kilmister, C.W., Newman, D.L.: Proc. Cambridge Phil. Soc. (Math. Phys. Sci.) 57, 851 (1961)
- [6] Higgs, P.W.: Nuovo Cimento 11, 816-820 (1959)
- [7] Fairchild Jr, Edward E.: Gauge theory of gravitation. Phys. Rev. D 14,384 (1976)
- [8] Pavelle, R.: Unphysical solutions of Yang's gravitational-field equations. Phys. Rev. L 34, 1114 (1975)
Pavelle R.: Yang's Gravitational Field Equations. Phys. Rev. L 33, 1461-1463 (1974)
- [9] Pavelle, R.: Unphysical characteristics of Yang's pure-space equations. Phys. Rev. L 37, 961 (1976)
- [10] Drummond, I.T., Hathrell,S.J.: QED vacuum polarization in a background gravitational field and its effect on the velocity of photons. Phys. Rev. D 22, 343 (1980)
- [11] Dereli T., Sert O.: Nonminimally coupled gravitational and electromagnetic fields: pp-wave solutions. Phys. Rev. D 83, 065005 (2011)
- [12] Dereli ,T., Şenlikoğlu, Y.: A nonminimally coupled, conformally extended Einstein-Maxwell theory of pp-waves. Turkish Journal of Physics, 44, 273-279 (2020)

- [13] Raffelt, G., Stodolsky, L.: Mixing of the photon with low-mass particles. *Phys. Rev. D* 37, 1237 (1988)
- [14] Lambiase, G., Prasanna, A.R.: Gauge invariant wave equations in curved space-times and primordial magnetic fields. *Phys. Rev. D* 70, 063502 (2004)
- [15] Dereli, T., Uçoluk, G.: Kaluza-Klein Reduction of Generalized Theories of Gravity and Non-minimal Gauge Couplings. *Classical and Quantum Gravity* 7, 1109 (1990)
- [16] Huang, W. H.: Kaluza-klein reduction of gauss-bonnet curvature. *Physics Letters B*, 203, 105-108 (1988)
- [17] Başkal, S., Kuyrukcu, H.: Kaluza-Klein reduction of a quadratic curvature model. *General Relativity and Gravitation* 45, 359-371 (2013)
- [18] Çelik S.: Compatibility of the Dimensional Reduction and Variation Procedures for a Quadratic Curvature Model with a Kaluza-Klein Ansatz Master of Science, Middle east Technical University, 92 (2021)
- [19] Başkal, S.: Radiation in Yang-Mills formulation of gravity and a generalized pp-wave metric. *Progress of Theoretical Physics* 102, 803-807 (1999)
- [20] d’Inverno, R.A.: *Introducing Einstein’s relativity*. Introducing Einstein’s relativity by RA D’Inverno. New York: Oxford University Press (1992)
- [21] Prasanna, A.R.: Electromagnetism and gravitation. *Lett. Nuovo Cim* 6, 420-423 (1973)
- [22] Horndeski, G.W.: Conservation of charge and the Einstein-Maxwell field equations. *Journal of Mathematical Physics* 17, 1980-1987 (1976)
- [23] Buchdahl, H.A.: On a Lagrangian for non-minimally coupled gravitational and electromagnetic fields. *Journal of Physics A: Mathematical and General* 12, 1037 (1979)
- [24] Balakin, A.B, Lemos, J.P.: Non-minimal coupling for the gravitational and electromagnetic fields: a general system of equations. *Classical and Quantum Gravity* 22, 1867 (2005)
- [25] Escobar, C.A., Urrutia, L.F.: Invariants of the electromagnetic field. *Journal of Mathematical Physics* 55, 032902 (2014)
- [26] Adler, S.L.: Photon splitting and photon dispersion in a strong magnetic field. *Annals of Physics* 67, 599-647 (1971)
- [27] Bopp, F.: Eine lineare theorie des elektrons. *Annalen der Physik* 430, 345-384 (1940)
- [28] Podolsky, B.: A generalized electrodynamics part I non-quantum.” *Phys. Rev.* 62, 68 (1942)
- [29] Cuzinatto, R. R., et al.: Bopp-Podolsky black holes and the no-hair theorem. *The European Physical Journal C* 78, 1-9 (2018)