MIDDLE SCHOOL MATHEMATICS TEACHERS' KNOWLEDGE OF EIGHTH-GRADE STUDENTS' ALGEBRAIC THINKING

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ABSTRACT

MIDDLE SCHOOL MATHEMATICS TEACHERS' KNOWLEDGE OF EIGHTH-GRADE STUDENTS' ALGEBRAIC THINKING

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The first purpose of this study is to investigate middle school mathematics teachers' knowledge of students' algebraic thinking. This purpose also includes the prerequisite knowledge needed to learn algebra. The second purpose of the study is to examine middle school mathematics teachers' anticipations and interpretations regarding their students' performances in algebra. The last purpose of the study is to discover the causal attributions of middle school mathematics teachers regarding their students' difficulties in algebra. To that end, the data were collected from five middle school mathematics teachers in the spring semester of the 2018-2019 and the fall semester of the 2019-2020 academic years. The data sources are a teacher questionnaire, audio recordings of the semi-structured interviews, and field notes. The findings presented that middle school mathematics teachers could possess limited information regarding the prerequisite knowledge students should have prior to learning algebra. Middle school mathematics teachers could anticipate students' possible solutions and performances in simple translations from verbal statements to symbolic expressions and solving equations. However, their anticipations were not aligned with students' performances in the tasks that required a relational understanding of equivalence, conceptual understanding of the notion of variable, and functional thinking. Although teachers could identify students' difficulties in corresponding items, they could not precisely express the underlying reasons for their difficulties. Lastly, teachers mainly attributed students' difficulties to external, stable, and uncontrollable factors. They consider that students' difficulties are mainly related to the students themselves, such as their cognitive processes, effort, or motivation.

Keywords: Middle School Mathematics Teachers, Algebraic Thinking, Variable, Functional Thinking, Causal attributions

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMENLERİNİN SEKİZİNCİ SINIF ÖĞRENCİLERİNİN CEBİRSEL DÜŞÜNMELERİ İLE İLGİLİ BİLGİLERİ

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Bu çalışmanın ilk amacı, ortaokul matematik öğretmenlerinin öğrencilerinin cebirsel düşünmeleri hakkındaki bilgilerini incelemektir. Bu amaç aynı zamanda öğrencilerin cebir öğrenmeden önce sahip olması gereken önkoşul bilgilerini de içermektedir. Araştırmanın ikinci amacı, ortaokul matematik öğretmenlerinin öğrencilerinin performanslarına ilişkin beklenti ve yorumlarını eşitlik, cebirsel ifadeler ve denklem, genelleştirilmiş aritmetik, değişken ve fonksiyonel düşünme bağlamında incelemektir. Çalışmanın son amacı, ortaokul matematik öğretmenlerinin öğrencilerinin, cebirdeki güçlüklerine ilişkin nedensel yüklemelerini keşfetmektir. Bu amaçla 2018-2019 bahar yarıyılı ve 2019-2020 güz yarıyılında beş ortaokul matematik öğretmeninden veri toplanmıştır. Veri kaynakları anket, yarı yapılandırılmış görüşmelerin ses kayıtları ve araştırmacı gözlem notlarıdır. Bulgular, ortaokul matematik öğretmenlerinin, öğrencilerin cebir öğrenmeden önce sahip olması gereken önkoşul bilgilerle ilgili sınırlı bilgi sağlayabildiklerini ortaya koymuştur. Ortaokul matematik öğretmenleri, sözlü ifadelerden sembolik ifadelere basit dönüşümler ve denklem çözme konusunda öğrencilerin olası çözümlerini ve performanslarını tahmin edebilmişlerdir. Ortaokul matematik öğretmenlerinin beklentilerinin, eşitliğin kavramsal olarak anlaşılması,

değişken kavramının kavramsal olarak anlaşılması ve fonksiyonel düşünmeyi gerektiren durumlarda öğrencilerin performanslarıyla uyumlu olmadığı gözlemlenmiştir. Öğretmenler, ilgili maddelerde öğrencilerin zorluklarını belirleyebilmelerine rağmen, zorlukların altında yatan nedenleri tam olarak ifade edememişlerdir. Son olarak, öğretmenler, öğrencilerin zorluklarını çoğunlukla dışsal, değişmez ve kontrol edilemeyen faktörlere bağlamışlar. Öğretmenler, öğrencilerinin zorluklarının; öğrencilerin bilişsel süreçleri, çabaları veya motivasyonları gibi dış faktörlere bağlı olduğunu düşünmüşlerdir.

Anahtar Kelimeler: Ortaokul Matematik Öğretmenleri, Cebirsel Düşünme, Değişken, Fonksiyonel Düşünme, Nedensel Yükleme

To my family

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LIST OF ABBREVIATIONS

MSMT: Middle School Mathematics Teacher

ADT: Algebra Diagnostic Test

PCK: Pedagogical Content Knowledge

SMK: Subject Matter Knowledge

MoNE: Ministry of National Education

NCTM: National Council of Teachers of Mathematics

EEEI: Equivalence, expressions, equations, and inequalities

CHAPTER 1

INTRODUCTION

1.1.Background of the Study

Algebra is described as a gatekeeper in school mathematics as it provides attainment to the following levels of education (Donovan et al., 2022). Several studies have agreed on the significance of algebraic thinking in mathematics and mathematics education (Asquith et al., 2007; Cai & Moyer, 2008; Hodgen et al., 2018; Kieran, 2004). In response to researchers' enthusiasm about students' inadequate comprehension of algebra, the algebra curriculum and teaching of algebra have become focal points for accomplishing recent reforms in mathematics education (Bednarz et al., 1996; Lacampagne et al., 1995; NCTM, 2000). Kaput (1998) clarified that the "key to algebra reform is integrating algebraic reasoning across all grades and all topics—to 'algebrafy' school mathematics" (p. 1). As Stephens (2008) proposed, "algebra" in K-12 is "a term that should be used to describe a *way of thinking* as opposed to simply something we *do* (e.g., collect like terms, isolate the variable, change signs when we change sides)" (p. 35). Kaput (2008) stated that school algebra mainly focused on symbol manipulation worldwide. Based on a widely accepted view, the emphasis should not be on comprehending rules to manipulate symbols and use algebraic procedures excellently but on developing algebraic thinking. There were two main central themes at the core of algebraic thinking: "making generalizations" and "using symbols to represent mathematical ideas and to represent and solve problems" (Carpenter & Levi, 2000, p. 5).

Kaput (1998) specifically described the thinking practices of algebraic reasoning as making generalizations, formalizations of symbol systems, and reasoning with symbolic forms. He argued that these thinking practices could be observed across three content strands: "algebra as the study of structures and systems abstracted from computations and relations," "algebra as the study of functions, relations, and joint variation," and "algebra as a cluster of (a) modeling and (b) phenomena-controlling languages" (p. 3). Kaput added that the two thinking practices underlined all three content strands. Similarly, Carpenter and Levi (2000) also described two central aspects of algebraic reasoning: making generalizations and using symbols to show mathematical ideas and solve problems. Carpenter et al. (2003) and Jacobs et al. (2007) referred to these ideas as relational thinking. It is crucial to engage students in relational thinking to improve their computational fluency (Koehler, 2004) and to promote their perceptions of algebraic expressions and equations as objects (Stephens, 2008).

Researchers indicated that algebraic thinking should be constructed in corporation with arithmetic thinking beginning in early grades (Blanton & Kaput, 2011; Blanton et al., 2015; Carpenter et al., 2003; Ryan & Williams, 2007). It is noted that perceiving arithmetic and algebra as separate areas and concentrating on symbol manipulation while learning algebra might prevent students' algebraic thinking. In contrast to some researchers who proposed that arithmetic and algebra were separate areas and a transition existed between them, Carraher et al. (2006) perceived algebra as a generalized arithmetic in which the notion of function has a major role. They argued that arithmetic is a part of algebra and that the "algebraic character" of arithmetic should be emphasized in elementary mathematics instruction.

The difficulties students experienced while learning algebra resulted in students being isolated from mathematics and giving up learning mathematics early in high school (Kaput, 2002). Hence, U.S. educators and researchers call for nationwide movements, *algebra for all*, to make algebra attainable for all students (Chazan, 1996; Moses, 1995; Moses & Cobb, 2001). In response to the movement of 'algebra for all,' the National Council of Teachers of Mathematics (NCTM) suggested that instructional programs should enable students "to understand patterns, relations, and functions," "to represent and analyze mathematical situations and structures using algebraic symbols," "to use mathematical models to represent and understand quantitative relationships," and "to analyze the change in various contexts" (NCTM, 2000, p. 37).

Identifying students' conceptions, difficulties, and errors in algebra could be a good start to determining these standards.

1.2.Students’ Conceptions, Difficulties, and Misconceptions in Algebra

Algebra is one of the crucial branches of mathematics, and it provides a gateway from arithmetic reasoning in elementary school to advanced and deeper mathematics in higher grades (Blanton & Kaput, 2005; Knuth et al., 2005). Blanton et al. (2011) described algebra as a “mathematical language that combines operations, variables, and numbers to express mathematical structure and relationships in succinct forms” (Blanton et al., 2011, p. 67). Erbaş (2005) asserted that “the road to algebra is never as smooth as one may wish,” which is common everywhere (p. 26). Thus, algebraic reasoning should be expanded across the curricula of all grades to get the late, isolated, and superficial algebra courses to become more coherent, profound, and powerful (Kaput, 1998). Several studies have documented students' difficulties and misconceptions in algebra (Alibali et al., 2007; Carraher & Schliemann, 2007; Kieran, 1992; Kilpatrick et al., 2001; Knuth et al., 2005; Knuth et al., 2006; Sfard, 1991). For example, Carraher and Schliemann (2007) asserted that most difficulties students faced in algebra were related to the operational view of the equal sign (Kieran, 1981), an emphasis on particular quantities in answers instead of generalized statements (Booth, 1984), a lack of knowledge related to fundamental properties of number and operation in arithmetic (MacGregor, 1996), and a lack of understanding of variable notation that demonstrated the relationship between quantities (Bednarz, 2001).

As Booth (1989) emphasized, “we first understand the structural properties of mathematical operations and relations which distinguish allowable transformations from those that are not” to accurately conduct algebraic transformations, which were a crucial part of middle and secondary school grades (p. 72). Booth (1988) stated that the difficulties that students experience in algebra could be related to their inadequate understanding of arithmetic or problems in arithmetic that were not corrected in the past instead of algebra itself. Filloy and Rojano (1989) defined it as a “cut-point separating one kind of thought from the other” based on the evolution from concrete arithmetic processes to abstract algebraic thinking (p. 19). Similarly, Herscovics and

Linchevski (1994) described the presence of a cognitive gap between arithmetic and algebra by mentioning “the students’ inability to operate spontaneously with or on the unknown” (p. 59).

Understanding the notion of variable is another core issue while doing algebra, as it is at the center of the connection between arithmetic and algebra (Blanton et al., 2015; Stephens, 2005; Usiskin, 1988). Many studies showed that students struggle to use variable notation to demonstrate quantities and the relationships of those quantities (Bednarz, 2001; McNeil et al., 2010; Stephens, 2005; Vergnaud, 1985). Jupri et al. (2020) highlighted that one of the indicators of students’ algebraic proficiency is having the symbol sense (Bokhove & Drijvers, 2010; Jupri et al., 2020; Van Stiphout et al., 2013), which indicates a relational understanding of symbols (Skemp, 1976). Arcavi (2005) described it as an analogy to number sense as the ability to capture the meaning and be aware of the essential structures of symbols and algebraic expressions. Elementary-grade students had typically introduced to the variable as a constant, unknown quantity (Blanton et al., 2011; Lloyd et al., 2011). Researchers proposed that additional roles of the variable should also be emphasized in a comprehensive approach, such as becoming a varying quantity, a generalized number, and a parameter (Blanton et al., 2015; Usiskin, 1988).

Many students were alienated from mathematics and did not continue to learn it before they came to high school because of the difficulties they had faced while traditionally learning algebra (Kaput, 2002). Based on the study of Kenney and Silver (1997), even twelfth-grade students had difficulty solving simple algebraic equations, transitioning from verbal to symbolic representations, and sharing and justifying their reasoning for their solutions. Sfard (2000) proposed that a student should manipulate a concept to understand it, but she asked how a student can use something without understanding it. She defined this dichotomy as a circularity that determines the process of learning. However, students usually focus on the procedures related to symbol manipulation instead of considering the underlying meaning in traditional algebra classrooms (Chazan, 2000). To illustrate, the students might spend too much time learning how to construct an equation of a line; however, they may not explain why the procedure worked while writing the equation and why they were constructing such an equation.

As Kieran (1992) argued, to overcome the lack of understanding, students mostly "resort to memorizing rules and procedures and...eventually come to believe that this activity represents the essence of algebra" (p. 390).

In Third International Mathematics and Science Study (TIMMS) examinations, real-world problems were asked students to get them to use algebraic models and explain the relationships, including algebraic procedures such as determining one of two quantities when one of them was given in a formula. Also, they were asked to solve problems, including linear equations and functions, to observe the change in the value of one variable when the value of the other variable changes (Mullis et al., 2020). Algebra items constitute 30% of the mathematics test in the TIMMS examination, including two dimensions; expressions, operations, and equations (20%) and relationships and functions (10%). Although Turkish eighth-grade students demonstrated a gradually increasing performance in algebra scores from year to year (MoNE, 2014; MoNE, 2016; MoNE, 2020), analysis of eight grade students' responses to algebra items in TIMMS 2019 showed that Turkish eighth-grade students' algebra scores were below the average mathematics score (MoNE, 2020). In addition to the research studies, the results of international examinations showed that Turkish eighth-grade students might be supported to improve their algebra performance. Knuth et al. (2005) defined middle school grades as the period in which students' arithmetic and early algebraic reasoning are linked to complex, abstract, algebraic reasoning. Thus, it is crucial to examine eighth-grade students' conceptions, difficulties, and errors to improve their performance in algebra.

Algebra reform should be considered a continuous strand for K-12 mathematics instead of fixing algebra to a "traditional ninth-grade course" (Asquith et al., 2007, p. 250). "By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more sophisticated work in algebra in the middle grades and high school" (NCTM, 2000, p. 37). To accomplish this purpose, the reconceptualization of algebra has been encountered more in elementary grades as algebraic ideas were integrated into early grades in recent studies (Blanton et al., 2015, 2019; Carraher et al., 2006; Kaput et al., 2007). This reconceptualization across K-12 implies that algebraic reasoning is more than doing manipulations with symbols (Asquith et al., 2007; Carpenter&Levi, 2000; Schifter, 1999). Blanton et al. (2015) pointed out that typical elementary mathematics curricula and traditional instruction might not provide a significant transition for students from the "concrete, arithmetic reasoning of elementary school" to the increasingly "complex, abstract algebraic reasoning required for middle school and beyond" (Blanton et al., 2015, p. 76). The requirement to improve students' algebra learning was expressed in policy documents, such as the report of the RAND Mathematics Study Panel (2003) and the National Mathematics Advisory Panel (NMAP, 2008).

Researchers indicated that students' frequent exposure to algebraic ideas from kindergarten to eighth grade got them to make the transition from arithmetic to algebra more smoothly as they have already understood such necessary notions, operation sense, equality, and generalization (Asquith et al., 2005; Carpenter et al., 2003; Kaput, 1998). For this reason, they have called for reforms in teaching and learning mathematics, requiring mathematics teachers to recognize the occasions to encourage students' algebraic thinking. Thus, it requires the development of an extended curriculum and the enrichment of teacher knowledge to strengthen the connection between arithmetic and algebraic reasoning, especially in middle grades, when the connection between those two forms of reasoning is presumably the most salient (Asquith et al., 2007). There is a strong connection between teacher knowledge and students' learning (Carpenter et al., 1988; Carpenter et al., 1989; Franke et al., 1998; Hill et al., 2005). In their seminal works, Carpenter and colleagues also found a strong relationship between students' achievement and teachers' knowledge of students' thinking (Carpenter et al., 1988; Carpenter et al., 1989; Franke et al., 1998). These experimental studies presented that the students perform better whose teachers participate in professional development programs focusing on research-based information of teachers' knowledge of students' thinking. Carpenter et al. (1989) argued that teachers must be more familiar with students' thinking. Their Cognitively Guided Instruction program showed that teachers' instructional practices might be changed by serving them well-organized information regarding children's actual thinking and using strategies in solving simple arithmetic story problems. Thus, teacher knowledge constitutes an essential characteristic of their classroom practices

(Borko & Putnam, 1996). As Shulman (1986) defined Pedagogical Content Knowledge (PCK), the knowledge "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (p. 9), teachers' knowledge of students' thinking is an essential component of PCK (Ball & Cohen, 1999; Kazemi & Franke, 2004). Hence, teachers' knowledge of students' algebraic thinking deserves much more attention in middle grades, the period where a significant transition occurs from concrete, arithmetic reasoning of elementary grades to the more complex, abstract algebraic reasoning necessary for high school mathematics and beyond (Asquith et al., 2007; MoNE, 2018).

1.3.Teachers' Knowledge related to Students' Conceptions, Difficulties, and Misconceptions in Algebra

Algebra has been described as a subject that was both hard to learn and hard to teach (Stacey et al., 2004; Watson, 2009). Stump and Bishop (2002) asserted, "one of the greatest challenges for mathematics teacher educators committed to reforming and improving mathematics education is to help preservice elementary and middle school teachers develop an appreciation for algebraic reasoning" (p. 1903). Researchers explained it as the cornerstone of mathematics reform, and teachers are one of the most crucial factors in developing students' algebraic reasoning (Blanton & Kaput, 2005; Kaput, 1998). Research studies demonstrated that the change in practice should be conducted in collaboration with teachers as they were professional partners in the algebra teaching process.

Nathan and Koedinger (2000b) expressed that mathematics teachers' interpretation and implementation of the curricula are mainly influenced by their knowledge and beliefs regarding instruction (Ball, 1988; Borko et al., 1992; Clark & Peterson, 1986; Raymond, 1997; Thompson, 1984), student learning (Ball, 1988; Carpenter et al., 1989; Fennema et al., 1992; Romberg & Carpenter, 1986), and mathematics (Cooney, 1985; Raymond, 1997). As Kaiser et al. (2017) noted, the cognitive perspective of teacher professionalism has been dominantly studied, especially with large-scale studies, concentrating on the knowledge facets of mathematics teachers in the last few years (Ball & Bass, 2000; Blömeke et al., 2014; Bruckmaier et al., 2016; Kunter et al., 2013). Shulman (1986) suggested two types of understanding of subject matter teachers need, knowing that and knowing why. These two types of knowledge are crucial while investigating teachers' knowledge regarding students' ways of thinking (Even & Tirosh, 1995). They described *knowing that* as a research-based or experienced-based knowledge of students' ways of thinking and common conceptions regarding a subject matter and *knowing why* as the common knowledge about the potential sources of underlying conceptions. Thus, both dimensions are crucial for teachers to recognize and interpret their thinking.

In recent years, researchers have also focused on teachers' noticing as the starting point while studying teacher professionalism (Kersting et al., 2012; Kersting et al., 2016; Santagata & Guarino, 2011; Santagata & Yeh, 2016). Kaiser et al. (2017) asserted that the relationship between these two perspectives, teachers' knowledge and noticing, related to teachers' competencies and professionalism has remained ambiguous. Researchers called for studies examining teachers' knowledge of students' algebraic thinking in elementary school mathematics (Asquith et al., 2007; Blanton & Kaput, 2003; Carpenter et al., 2003; Kaput et al., 2007; Stephens, 2006). While examining teachers' knowledge of students' algebraic thinking, teachers' perceptions of algebraic thinking also have a crucial role (Asquith et al., 2014). Based on Van Dooren et al. (2002), pre-service mathematics teachers tended to evaluate the students' solution strategies to arithmetic and algebra problems in parallel with their own solution preferences. Therefore, teachers' evaluations of students' solutions might be a precursor for their own arithmetic and algebra perceptions. Stephens (2008) asserted that teachers would avoid engaging students in activities requiring generalization and algebraic reasoning if they lacked the appropriate conceptions about mathematics. She stated that it is impossible for students to successfully comprehend algebraic ideas unless they perceive arithmetic as something that makes sense. Thus, it would be difficult for students if they learned mathematics as a set of rule-based or procedural operations instead of using sense-making abilities.

In response to the recent calls for algebra reform, researchers have moved toward the studies investigating primary teachers' knowledge of students' algebraic reasoning in elementary grades to integrate algebraic reasoning throughout the K-8 strand (Blanton et al., 2015, 2019; Blanton & Kaput, 2011; Carpenter et al., 2003; Carraher et al., 2006; Kaput et al., 2007). However, further studies are still needed to focus on the knowledge of MSMTs (MSMTs) on students' thinking regarding algebraic ideas in middle school grades, which refers to "a period that marks a significant transition from the concrete, arithmetic reasoning of elementary school mathematics to the increasingly complex, abstract algebraic reasoning required for high school mathematics and beyond" (Asquith et al., 2007, p. 251). Although there have been studies investigating MSMTs' knowledge of students' thinking in algebra (Asquith et al., 2007; Baş et al., 2011; Li, 2007; Tanışlı & Köse, 2013; Putnam et al., 1992; Stephens, 2006), this topic deserves much more attention as teachers' knowledge was a crucial determinant of their classroom practices (Asquith et al., 2007; Borko & Putnam, 1996).

Shulman (1986) suggested that the potential sources of underlying conceptions of students, *knowing why*, is as crucial as the knowledge of students' algebraic thinking, *knowing what*. Teachers were found to express the knowledge of students' difficulties and errors related to knowing what (e.g., Stump, 2001), but they failed to present particular sources of those difficulties and errors, which was related to knowing why aspect of teachers' professional knowledge (Erbaş, 2004). Thus, it would be beneficial to draw a picture of underlying reasons, *potential sources*, depicted by teachers regarding their students' performances in algebra.

1.4.Teachers' Causal Attributions for Students' Success and Difficulties

Some teachers have persistent myths regarding the learning of mathematics, which implies that "success in mathematics depends more on innate ability than on hard work" (National Research Council [NRC], 1991, p. 10). Weiner (1985, 2000, 2010) defined causal attribution as an individual's perception regarding the cause of success and failure, which subsequently affects his/her emotion, decision-making, and performance. Fritz Heider, the founder of attribution theory, established his work around the causality of individuals' behaviors internal to themselves and external to the environment (Stage et al., 1998). Attribution theory suggests that individuals attribute their failure and success to either internal or external factors (Dweck, 1986; Weiner, 1974). As Wang and Hall (2018) suggested, the attribution theory (Weiner, 1985, 2000, 2010) provides a comprehensive theoretical framework to examine how individuals perceive the causes behind their performances and others and the influences of these attributions on individuals' emotions, cognitions, and behaviors in an educational context. Integrating both interpersonal and intrapersonal perspectives, this theory also explains how teachers perceive students' difficulties and occupational stressors and how their attributions affect their teaching behaviors, interactions with students, and emotional well-being (Wang & Hall, 2018).

Some researchers attributed students' difficulties with algebra to developmental constraints or inadequate cognitive development of students (Collis, 1975; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Kuchemann, 1981; MacGregor, 2001). Filloy and Rojano (1989) argued that there exists a "cut-point separating one kind of thought from the other," which was called "a break in the development of operations on the unknown" (p. 19). They noted that concrete arithmetical thinking very slowly turned into more abstract algebraic thinking. Apart from the difficulties attributed to the intrapersonal factors (e.g., students' cognitive process, motivation, and math skills), interpersonal factors might also be attributed to students' success or difficulties, such as instructional quality, luck, or environmental circumstances. Wang and Hall (2018) reviewed seventy-nine attribution studies. They found that teachers generally attribute students' failure to the factors related to students themselves, although there exist some studies suggesting that teachers explain students' performance based on external and uncontrollable factors, such as prior learning experiences and previous teachers (Rolison & Medway, 1985; Hall et al., 1989; Bertrand & Marsh, 2015). For example, according to their studies on the early algebra approach, Carraher and Schliemann (2007) noted that students' difficulties are attributed to the shortcomings related to how arithmetic, generally elementary mathematics, is introduced to students.

Researchers suggested that attributions have a crucial role in teachers' expectations regarding students' future performance (Clarkson & Leder, 1984; Peterson & Barger, 1985). Shores and Smith (2010) reviewed the attribution studies from 1974 to 2008 in mathematics education and highlighted the continuing studies on attribution. They emphasized the need for future studies to examine attributions from teachers' perspectives and, subsequently impacts of teachers' attributions on students' mathematics learning. Based on the attribution theory of Weiner (2000, 2010), the way teachers anticipate the causes of their students' performance can influence their emotions, which in turn predict their behaviors in teaching. Therefore, it might be helpful to investigate MSMTs' causal attributions for students' performance to have information about their thinking on students' performance and predict their classroom behaviors. Moreover, Baştürk (2016) asserted that there was still a need to provide practical information for mathematics teachers. Such studies might yield crucial results for pre-service and in-service MSMTs to recognize their attributions and observe how these attributions affect their instruction (Shores & Smith, 2010) and which dimension of causal attributions (e.g., causality, controllability, and stability) outweighs which might provide information about teachers' decision-making processes while teaching algebra (Weiner, 2010). Examining MSMTs' causal attributions for students' difficulties might provide a picture of what MSMTs think about the sources of students' performance and what they know about students' algebraic thinking.

1.5.Purpose of the Study and Research Questions

There were multifaceted purposes of the current study. The first purpose of this study was to analyze eighth-grade students' conceptions, difficulties, and errors in algebra tasks related to in four big ideas in algebra, equivalence, expressions, equations, and inequalities, generalized arithmetic, variable, and functional thinking. The second purpose of the study is to examine MSMTs' knowledge related to students' conceptions, difficulties, and errors in learning algebra. This purpose also includes the prerequisite knowledge students should have to learn algebraic concepts. The third purpose of the study is to investigate MSMTs' anticipations and interpretations related to their students' performances in four big ideas in algebra, equivalence, expressions, equations, and inequalities, generalized arithmetic, variable, and functional thinking. The last purpose of the study is to depict the causes of students' difficulties and errors that MSMTs express in the tasks related to in four big ideas in algebra, equivalence, expressions, equations, and inequalities, generalized arithmetic, variable, and functional thinking. The research questions are constructed to address these purposes.

All in all, considering the purposes mentioned above, the following research questions will be addressed in the current study:

1. What is the nature of MSMTs' pedagogical content knowledge about students' understanding related to four big ideas?
   1.1. What is the prerequisite knowledge that MSMTs consider necessary to begin learning algebra?
   1.2. What do in-service MSMTs know about common conceptions and difficulties held by eighth-grade students related to four big ideas?
   1.3. What do in-service MSMTs know about the possible sources of difficulties and errors held by eighth-grade students related to four big ideas?
   1.4. What strategies do in-service MSMTs consider overcoming the difficulties held by eighth-grade students related to four big ideas?
2. To what extent MSMTs' knowledge aligned with the conceptions and difficulties of eighth-grade students in the algebra diagnostic test (ADT)?
   2.1. What are MSMTs' predictions related to the conceptions and difficulties of eighth-grade students in ADT?
   2.2. How do MSMTs' predictions compare to students' performance on algebraic thinking tasks in ADT?
   2.3. How does MSMTs' knowledge of students' learning influence their interpretations of common conceptions and difficulties of eighth-grade students in ADT?
3. How do teachers attribute the factors that impact students' performance in algebra?

1.6. Significance of the Study

Students' inadequate understanding and difficulties in algebra and the role of algebra as a gatekeeper in future opportunities in education and employment (Asquith et al., 2007; Ladson-Billings, 1998; Moses & Cobb, 2001; National Research Council [NRC], 1998) directed mathematics education community to call for an algebra reform (Kaput, 1995, 1998; Olive et al., 2002; Stacey & Mac Gregor, 2001). There is an agreement among researchers that the reconceptualization of school algebra is needed to make algebra a continuous subject in the K-12 strand (Asquith et al., 2007). With the help of the inclusion of algebraic reasoning in elementary grades, algebra has been perceived to be a subject accessible to students across all grades more than mastering symbolic manipulations (Asquith et al., 2007; Carpenter&Levi, 2000; Schifter, 1999). The expansion of algebraic ideas from the earlier grades might conclude with the need for some revisions, especially at the middle school level, where the transition between arithmetic and algebra is more apparent. Hence, devising an appropriate curriculum and supporting and extending teacher knowledge and practice to strengthen the connections between arithmetic and algebraic reasoning are needed (Asquith et al., 2007).

Also, Ball et al. (2001) suggested “sizing up students’ ideas and responding” (p. 453) and emphasized the importance of using particular knowledge related to students’ understanding as an effective tool to cope with sophisticated classroom settings. As Asquith et al. (2007) expressed, there have been limited studies on MSMTs’ knowledge of students’ algebraic thinking. Various stakeholders have emphasized the importance of what teachers are required to know, such as mathematicians (Askey, 1999; Milgram, 2005; Wu, 1999), mathematics education researchers (Carrillo-Yañez et al., 2018; Hill et al., 2007; Shulman, 1987), and organizations (Conference Board of the Mathematical Sciences [CBMS], 2012), NCTM, 2000; National Mathematics Advisory Panel [NMAP], 2008). Thus, unpacking teachers’ knowledge related to students’ algebraic thinking might also provide clues regarding their own algebraic knowledge for teaching middle graders (Ball et al., 2008). Hence, to improve students’ performance in algebra and extend their algebraic reasoning, the investigation of MSMTs’ knowledge of students’ algebraic thinking might be a logical next step. Hence, this study could contribute to the teacher knowledge literature based on MSMTs’knowledge of students’ algebraic thinking and students’ difficulties and errors in algebra.

The inadequacy of research on teachers’ knowledge and practice was one of the significant obstacles to enhancing the teaching of algebra (Doerr, 2004; Stein et al., 2011). As Doerr (2004) stated, there was a severe need for theory-building to explain what teachers need to know to teach algebra. Stephens (2008) pointed out that a few studies investigated the knowledge and beliefs of mathematics teachers in algebra. Teachers are expected to be able to analyze students' thinking and solution strategies in particular tasks and students' understanding in the lectures and subsequently to constitute mathematics instruction regarding their inferences (MoNE, 2017). Therefore, teachers should concentrate on each student's mathematical understanding in the classroom (Jacobs et al., 2010). That is, teachers are required to have adequate knowledge to predict, analyze, and interpret students' thinking and solutions for particular tasks and improve their understanding (Asquith et al., 2007; Ball et al., 2008; Carrillo-Yañez et al., 2018; Doerr, 2004; Stephens, 2006), which is a crucial competency for teachers (Bromme, 1997; Kaiser et al., 2017; MoNE, 2017; Weinert, 2001).

"Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM, 2000, p. 16). Researchers emphasized that even MSMTs "have little experience with the rich and connected aspects of algebraic reasoning" (Blanton & Kaput, 2005, p. 414). Studies showed that pre-service and in-service MSMTs have deficiencies in identifying students' conceptions and anticipating the underlying reasons for students' difficulties and misconceptions in algebra (Alapala, 2018; Asquith et al., 2007; Dede & Peker, 2007; Didiş-Kabar & Amaç, 2017; Gökkurt et al., 2016; Li, 2007; Stephens, 2004, 2006; Li, 2007; Şen-Zeytun et al., 2010; Tanışlı & Köse, 2013; Tirosh et al., 1998). Literature review showed that there had been limited research concentrated on the knowledge of in-service MSMTs' knowledge of students' algebraic reasoning and the underlying reasons for their difficulties and misconceptions (Asquith et al., 2007; Şen-Zeytun et al., 2010; Tirosh et al., 1998). As Blanton et al. (2011) asserted, teaching algebraic thinking requires a special knowledge that goes "beyond what most teachers experience in standard preservice mathematics courses" similar to other core topics in mathematics (NCTM, 2000, p. 17). The present study could yield findings related to how MSMTs anticipate and interpret students' algebraic thinking, difficulties, and errors, in what points they struggle to anticipate and interpret students' thinking, and what inferences they attained based on their students' performances in algebraic thinking tasks. Based on the findings of this study, crucial information and implications might be proposed to MSMTs, teacher educators, and mathematics education researchers regarding MSMTs’ knowledge of students’ algebraic thinking. Thus, the findings of the current study could provide insights into future professional development programs for MSMTs to extend their professional knowledge.

Students' written work is a genuine activity to interpret and intervene in students’ conceptions and difficulties in mathematics (Grosman et al., 2009; Jacobs & Philipp, 2004). Erbaş (1999) noted that students’ way of thinking in dealing with equations and solving problems needed to be investigated by gathering information from both students and teachers. Moreover, Doerr (2004) highlighted that how teachers learned to teach algebra and understood their own practice were the issues that should be further investigated in their own cultural contexts. However, there are limited studies in which the data were collected from teachers via their own students’ written work (Asquith et al., 2007; Stephens, 2004, 2006; Tirosh et al., 1998). Therefore, collecting data through students’ own responses to the variable, equation, and functional thinking tasks would be significant. Thus, instead of using already prepared samples for students’ solutions to particular tasks, the Algebra Diagnostic Test (ADT) was developed to investigate eighth-grade students' algebraic thinking and, subsequently, MSMTs’ knowledge regarding students’ algebraic thinking in their own cultural context. ADT included algebra tasks that required algebraic reasoning, as Kaput (2008) proposed. This study allows MSMTs to get feedback based on their eighth-grade students via algebraic reasoning tasks. Therefore, they could compare their predictions with their students' actual ADT performances. Thus, they might have the chance to criticize themselves about how they should anticipate and interpret students’ solutions and at which points students’ algebraic thinking and difficulties differed from the predictions of MSMTs. Moreover, they might have an opportunity to recognize their students’ conceptions, difficulties, and errors that they had not noticed before. They could consider possible reasons for students’ conceptions, difficulties, and errors in the tasks. After observing their students' ADT performance, they might recognize the issues that need more attention in learning algebra, such as consideration of the big ideas while preparing lesson plans and the emphasis on some concepts (e.g., the notion of variable, the meaning of the equality sign, and covariation) in algebra classes. Hence, they might perceive the findings of the study as feedback to be aware of their students' conceptions, possible difficulties, and errors in the big ideas of variable, equivalence and equations, generalized arithmetic, and functional thinking in their future algebra classes.

As we have learned more about middle school students' algebraic thinking (Blanton et al., 2015; Herscovics & Linchevski, 1994; Jupri & Drijvers, 2014; Kieran, 1992; MacGregor & Stacey, 1997, Warren, 2003), unpacking the knowledge of MSMTs' knowledge of students' algebraic thinking could be a logical enterprise for the next step (Asquith et al., 2007). Considering all these perspectives, this study aimed to investigate MSMTs' knowledge of eighth-grade students' conceptions, difficulties, and errors in the big ideas of variable, equivalence and equations, generalized arithmetic, and functional thinking.

1.7.Definition of Important Terms

In this section, the operational definitions of the key terms presented in this study were given to provide a clear portrait of the study and increase the intelligibility of the following chapters.

Algebra

In this study, algebra refers to "mathematical language that combines operations, variables, and numbers to express mathematical structure and relationships in succinct forms" (Blanton et al., 2011, p. 67). In addition, it represents relations between quantities and mathematical structures and includes the procedures to operate with those structures (Kieran, 1992; Usiskin, 1988, 1997). Also, it was considered a generalized arithmetic which includes using symbols to replace unknown quantities and to generalize arithmetic operations (Blanton et al., 2015; Kieran, 1992; Usiskin, 1988, 1997).

Algebraic thinking

Kriegler (2004) described that algebraic thinking integrates two crucial components, algebraic ideas (e.g., patterns, variables, and functions) and mathematical thinking tools (e.g., representation, problem-solving, and reasoning). In other words, algebraic thinking takes place in any activity which combines one of the big ideas with mathematical thinking tools, such as representing mathematical statements using variables and analyzing the change. Thus, in this study, algebraic thinking refers to the integration of two components, algebraic ideas and mathematical thinking tools.

Big ideas in algebra

Big ideas are the content strands of algebra developed in terms of Kaput's study (2008). This study investigates big ideas under five strands: equivalence, expressions, equations, and inequalities, generalized arithmetic, functional thinking, and variable (Blanton et al., 2015; Blanton et al., 2019).

Difficulty

In the present study, difficulty refers to the struggles or problems of students while doing mathematics. While investigating teachers’ causal attributions, difficulty also means becoming unsuccessful. Thus, “difficulty” was used instead of “failure” when students were unsuccessful in particular situations.

Middle school mathematics teachers

Middle school mathematics teachers are the individuals who teach middle-grade students from fifth to eighth grade. The teachers working as MSMTs in a public school in Zonguldak were selected using purposeful sampling.

Teachers’ knowledge of students’ thinking

Carrillo-Yañez et al. (2018) explained teachers’ knowledge of students’ thinking as the teachers' understanding of how students think about and learn mathematical concepts. It focuses on students’ reasoning, proceeding, difficulties, and misconceptions in mathematical content instead of the learner. In this study, teachers’ knowledge of students’ thinking is employed as the integration of “theories of mathematical learning, strengths and weaknesses in learning mathematics, ways pupils interact with mathematical content, and emotional aspects of learning mathematics" (Carrillo-Yañez et al., 2018, p. 247).

Causal attribution

Weiner (1985, 2000, 2010) defined causal attributions as individuals' perceptions of the causes of behaviors. In this study, causal attribution refers to underlying reasons depicted by MSMTs for their students' difficulty in algebraic thinking.

CHAPTER 2

LITERATURE REVIEW

This study aimed to investigate in-service MSMTs' knowledge regarding their student's learning in algebra. In this context, the study's first aim was to examine in-service MSMTs' knowledge of students' conceptions and which type of difficulties and errors they faced while learning algebra. Also, the researcher prepared an investigation of teachers' knowledge related to typical solutions students would give and difficulties and errors they might have in Algebra Diagnostic Test (ADT). The study's second goal was to analyze middle school students' performances in ADT. After ADT was conducted on students and the results were analyzed, the teachers were asked to give their interpretations concerning the results of the analyses based on students' performances in algebra. Therefore, the last goal of the study was to get an image of mathematics teachers' interpretations of their students' performances in ADT. This chapter reviewed the literature on students' learning of algebra and mathematics teachers' knowledge of students' learning of algebra. Regarding the research questions, the literature review has been categorized into four sections: middle school students' learning in algebra, mathematics teachers' competencies, mathematics teacher knowledge, and mathematics teachers' knowledge of students' learning in algebra.

2.1.Knowledge of Mathematics Teachers

Mathematics teachers have a significant role while preparing K-12 students for mathematics (An, Kulm, & Wu, 2004; Blömeke & Delaney, 2012). Therefore, the quality of teachers has become a critical concern for policymakers and educators for several years. Elbaz (1983) defined the knowledge of teachers as "the single factor which seems to have the greatest power to carry forward our understanding of the teacher's role" (p. 45). There is an interaction between teacher knowledge and student performance in mathematics (Baumert et al., 2009). Therefore, teacher knowledge has been controversial for researchers, teacher educators, and policymakers. The results of long-scale studies, such as TEDS-M and TIMMS, illustrated that the professional knowledge of mathematics teachers should be improved to enhance students' mathematics performance (Blömeke & Delaney, 2012; Schoenfeld, 2010). Shulman (1987) described teachers' professional knowledge by considering seven domains: general pedagogical knowledge, content knowledge, pedagogical content knowledge, curricular knowledge, knowledge of educational contexts, knowledge of learners and their characteristics, knowledge of values, purposes, educational ends, and their philosophical and historical grounds. Researchers investigated mathematics teachers' knowledge under two main categories, subject matter, and pedagogical content knowledge, in recent years (Ball et al., 2008; Carrillo-Yañez et al., 2018; Magnusson et al., 1999; Shulman, 1986). First, teachers' subject matter knowledge will be briefly described in the following part.

2.1.1.Subject Matter Knowledge for Teaching Mathematics

Researchers declared that teacher knowledge is a crucial element for effective teaching. 'What teachers should know' and 'what they are required to know' have been frequently discussed issues for many years in teacher education (Ball et al., 2008; Hill et al., 2007). Shulman (1986) proposed that teachers should conceptually understand and express *why* a particular proposition works. Furthermore, they should understand and express *that* the proposition does work. Teachers should be able to explain why a specific proposition worked, why it was worth knowing, and how it could be integrated with other disciplines instead of superficial knowledge of facts and concepts. As National Mathematics Advisory Panel suggested:

> …teachers must know in detail and from a more advanced perspective the mathematical content they are responsible for teaching . . . both prior to and beyond the level they are assigned to teach (National Mathematics Advisory Panel, 2008, p. 37).

Having solid subject matter knowledge gets a teacher to integrate the subject area with other areas, pose students challenging questions, and feel free to use any other sources outside the textbook (NCTM, 2000). Subject matter knowledge was defined by Shulman (1986) as the "amount or organization of knowledge per se in the mind of the teacher" (p. 9). Ball (1991) described subject matter knowledge as an integration of knowledge, feelings, and beliefs about mathematics, and she considered mathematical knowledge under two dimensions: knowledge *of* mathematics and knowledge *about* mathematics. As she identified, knowledge of mathematics referred to "understandings of particular topics (e.g., fractions and trigonometry), procedures (e.g., long division and factoring quadratic equations), and concepts (e.g., quadrilaterals and infinity), and the relationships among these topics, procedures, and concepts" (p. 6). They defined knowledge about mathematics as "understandings about the nature of knowledge in the discipline--where it comes from, how it changes, and how truth is established" (p. 6).

Ball, Thames, and Phelps (2008) proposed a model for teachers' mathematics knowledge (MKT) that demonstrated the unity of SMK and PCK as two halves of an elliptic model. The left side of the oval was SMK comprising three components; common content knowledge (CCK), and horizon content knowledge (HCK), specialized content knowledge (SCK) (See Figure 2.1).

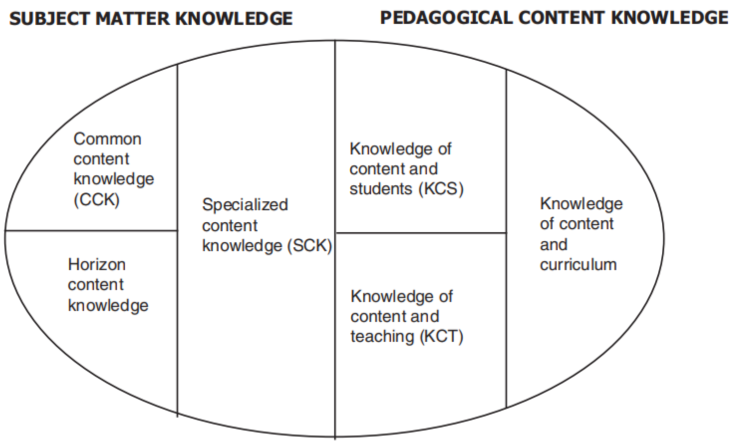


Figure 2. 1. Domains of Mathematical Knowledge for Teaching (Ball et al., 2008, p. 403)

Common content knowledge (CCK) refers to making simple calculations and solving problems in mathematics, the knowledge and skills required in other areas. The researchers highlighted that CCK did not refer to knowledge that everyone has. Instead, they pointed out that this dimension of knowledge was mathematical knowledge but not unique to teaching. CCK was similar to Shulman's SMK, but SCK is a new dimension (Hill et al., 2008). In contrast to CCK, SCK is mathematical knowledge required and 'necessary' for teaching. SCK is conceptual, including the knowledge of the effective use of mathematical language, appropriate use and construction of mathematical representations, and identification of the reasoning behind mathematical procedures. SCK is the conceptual knowledge of mathematical facts, such as correctly explaining the reasoning behind the procedure of inverting and multiplying while dividing fractions. Conversely, CCK is the mathematical knowledge to perform the invert and multiply process in procedural ways (Borko et al., 1992). As it is considered for all the dimensions of teacher knowledge, it is vague where CCK ends, and SCK begins since their boundaries are not sharply established (Baumert et al., 2009; Carrillo et al., 2013). The last dimension of SMK is horizon content knowledge (HCK), which refers to teachers' mathematical knowledge regarding the relationship of mathematical topics. With the help of this knowledge, teachers might get their students to connect the new knowledge with the already existing knowledge in mathematics.

Some Spanish researchers studied and revised the teacher knowledge model, called mathematics teachers' specialized knowledge (MTSK), comprising six facets of teacher knowledge and beliefs of teachers on mathematics and teaching and learning of mathematics at the core of the model (Aguilar-González et al., 2019, Carreño et al., 2013; Carrillo et al., 2013; Carrillo-Yañez et al., 2018). The model provides the differentiation suggested by Shulman (1986) and proposed by Ball et al. (2008) in two dimensions of teacher knowledge, namely mathematical knowledge (MK) and pedagogical content knowledge (PCK). The researchers identify sub-categories of MK as the knowledge of topics (KoT), the knowledge of the structure of mathematics (KSM), and the knowledge of practices of mathematics (KPM) for the dimension of subject matter knowledge (See Figure 2.2).

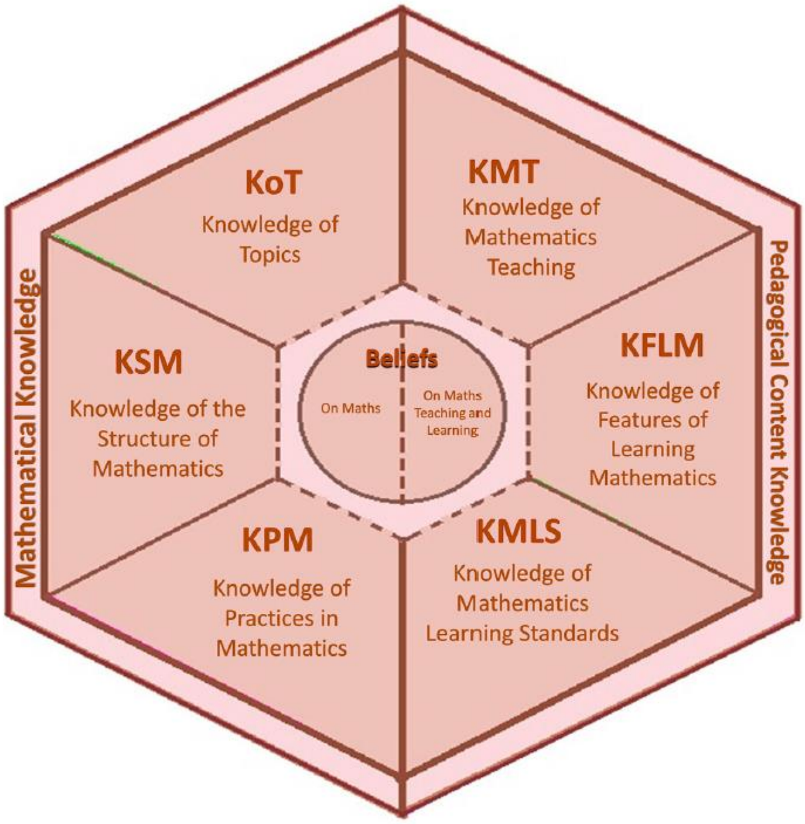


Figure 2. 2. A framework for mathematical knowledge for teaching (Carrillo-Yañez et al., 2018, p. 241)

KoT is a well-founded theoretical knowledge of mathematical calculation methods, procedures, and concepts. In other words, knowledge of the properties and their bases is attributable to mathematical content (Aguilar-González et al., 2019). To illustrate, the derivative concept can be illustrated as the gradient of a curve or maintained by the limit of finite increments. The researchers defined KSM as the knowledge of connections between the initial and subsequent mathematical concepts, which is a mathematical relation instead of being curricular (Montes et al., 2013). For example, although there is a mathematical relation between geometry and matrix algebra, they do not have to be consecutive in the curriculum. The last subdomain of SMK is The Knowledge of Practices in Mathematics (KPM), defined as “Knowing about demonstrating, justifying, defining, making deductions and inductions, giving examples, and understanding the role of counterexamples” (Carrillo-Yañez et al., 2018, p. 244).

This knowledge gets teachers to know a mathematical definition and its critical characteristics precisely and conceptualize how this knowledge is established in mathematics. KPM has some common points with syntactic knowledge of mathematics (Shulman, 1986), which means creation or exploration in mathematics (Carrillo et al., 2013). Researchers concluded that SMK not only refers to memorized facts and procedures in mathematics but also describes mathematics teachers' knowledge of whys and hows in mathematics (Ball et al., 2008; Shulman, 1986). Kahan et al. (2003) declared that having solid mathematics knowledge might provide "recognizing and seizing teachable moments" (p. 245), but it might not ensure students' conceptual learning of mathematics. Various studies proponent that SMK is a necessary prerequisite but not adequate for effective teaching (Krauss et al., 2008). As it has been included in various teacher knowledge models, PCK is also required to combine the knowledge of mathematics and the knowledge of teaching and learning (Aguilar-González et al., 2019; Ball et al., 2001; Ball et al., 2008; Shulman, 1986).

2.1.2.Pedagogical Content Knowledge of Mathematics Teachers

Shulman (1985) stated that teachers require extensive and highly organized knowledge to teach mathematics effectively. He described pedagogical content knowledge (PCK) as a "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (Shulman, 1987, p. 8). Shulman also defined PCK as the 'capacity' of teachers to express their content knowledge to students through pedagogically powerful and adaptive ways concerning their ages, abilities, and backgrounds. Magnusson et al. (1999) identified PCK as the understanding of teachers regarding students' understanding of a particular subject based on their interests and abilities. Moreover, Niess (2005) said that PCK is "the intersection of knowledge of the subject with knowledge of teaching and learning" (p. 510). Similar to the different definitions of PCK, various PCK models exist in the literature (Aguilar-González et al., 2019; Ball et al., 2008; Magnusson et al., 2009, Shulman, 1987). Kind (2009) studied several PCK models and concluded that representations and instructional strategies in subject matter and subject-specific learning difficulties of students were common components of different models of PCK. Shulman (1986, 1987) described PCK as the unity of two dimensions. He explained instructional strategies as “the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations” (p. 9). Furthermore, he identifies students’ subject-specific learning difficulties by considering the knowledge of misconceptions or naive ideas that came from previous learning and potential obstacles while learning the content.

Ball et al. (2008) constructed SMK and PCK as two main dimensions of teacher knowledge in their teacher knowledge model. As they suggested, PCK is comprised of three subdimensions, knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum (See Figure 2.1). In the model, KCS refers to teachers’ knowledge of students’ thinking and learning, such as considering which decimals are challenging for students. Hill et al. (2008) highlighted that KCS contributes a crucial foundation to PCK by focusing on the thinking and learning of students in mathematics. Secondly, KCT is related to teaching mathematics, such as knowing how to react to students’ difficulties in mathematical concepts. The last dimension of the model is knowledge of curriculum, which is structured on teachers’ knowledge about how the content should be shared with the students.

Similar to the teacher knowledge model of Ball et al. (2008), Carrillo-Yañez et al. (2018) specified three sub-domains of PCK regarding teaching and learning, which are called Knowledge of Mathematics Teaching (KMT) and Knowledge of Features of Learning Mathematics (KFLM), respectively (See Figure 2.2). Based on the third sub-dimension, Knowledge of Mathematics Learning Standards (KMLS), they had common points with Ball et al. (2008) by considering the requirement of teachers to be knowledgeable about the curriculum at any particular level. However, the researchers thought teachers’ knowledge should not be limited to curriculum knowledge only. This sub-domain should “enable the teacher to be critical and reflective in considering what the student should learn, and what focus should be taken, at any particular level, or period of development” (Carrillo-Yañez et al., 2018, p. 246). Rather than becoming an intersection of mathematics and pedagogical knowledge, they construed PCK as the pedagogical knowledge derived from mathematics.

The first sub-domain of PCK is knowledge of mathematics teaching (KMT), theoretical knowledge of teaching mathematics. KMT includes “awareness of the potential of activities, strategies, and techniques for teaching specific mathematical content, along with any potential limitations and obstacles which might arise” (Carrillo-Yañez et al., 2018, p. 247). Moreover, it involves the knowledge of teaching materials and resources such as manipulatives and technological devices. Beyond the use of those tools, as the researchers suggested, KMT provides a critical evaluation of how these materials enhance the teaching of mathematics and their limitations. For example, a balance scale can be used to teach addition and subtraction in equality. However, it may not be used when the terms include negative numbers. In addition, there are several types of representations for a specific mathematical concept, such as metaphors, explanations, and situations. KMT also includes this type of knowledge. The researchers summarized sub-categories of KMT as “theories of mathematics teaching, teaching resources (physical and digital), and strategies, techniques, tasks, and examples” (Carrillo-Yañez et al., 2018, p. 248).

Secondly, KFLM is related to the understanding of the teachers regarding how students think about and learn mathematical concepts. In this sub-domain, teacher knowledge focuses on mathematical content rather than the learner. It also focused on students’ reasoning, proceeding, difficulties, and misconceptions in mathematics (Carrillo-Yañez et al., 2018). KFLM dominantly refers to how students learn mathematics and the effects of teachers’ mathematics background on students’ learning of mathematics. It also comprises the knowledge of mathematics procedures, strategies students use, and different types of terminology students prefer to interact with the mathematical subject matter. KFLM also includes an emotional aspect regarding the learning of mathematics (Hannula, 2006). That is, awareness of math anxiety (Maloney et al., 2013) and factors affecting students’ motivation while learning mathematics. The researchers summarized the categories of KFLM as “theories of mathematical learning, strengths, and weaknesses in learning mathematics, ways pupils interact with mathematical content, and emotional aspects of learning mathematics” (Carrillo-Yañez et al., 2018, p. 247).

Lastly, KCMLS refers to “any instrument designed to measure students’ level of ability in understanding, constructing and using mathematics, and which can be applied at any specific stage of schooling” (Carrillo-Yañez et al., 2018, p. 248). Researchers declared that this knowledge could be constructed based on various sources such as teachers’ experiences with different sources, curriculum specifications, research literature, and documents outside the curriculum, such as NCTM (2000) or curriculum specifications of different countries. Also, this knowledge involves the sequence of topics. That is, a teacher should be knowledgeable on the required knowledge and skills of students both retrospectively, the knowledge of students’ previous learning about mathematical concepts, and prospectively, the knowledge which will be required in students’ learning of upcoming topics. As the researchers stated, multiplication is the number of times in grades 1 and 2, whereas it is taught as an abbreviated addition in grades 3 and 4 in Spain. Therefore, discrimination of such procedural and conceptual levels in multiplication or sequence of topics might be examples of the required knowledge of teachers for this category. The researchers summarized the sub-categories of KCMLS as “expected learning outcomes, the expected level of conceptual or procedural development, and sequencing of topics” (Carrillo-Yañez et al., 2018, p. 248).

Moreover, the Mathematics Teacher’s Specialised Knowledge (MTSK) model (Carrillo-Yañez et al., 2018) was constructed to specify the required and crucial knowledge categories for mathematics teachers, and it was a current teacher knowledge model which was built on well-known examples of teacher knowledge models in the literature (Ball et al., 2008; Shulman, 1987). The researchers gave special attention to and detailed explanations for teachers’ knowledge of how to cope with particular situations while teaching algebra, how students think and learn in particular algebra topics, and which aspects of curriculum and standards are needed by mathematics teachers to improve students’ algebraic thinking. The knowledge of in-service MSMTs was investigated considering the MTSK model of Carrillo-Yañez et al. (2018) in the current study. Since the KFLM dimension focused on the teachers’ understanding of how students think and learn mathematical concepts, mathematics teachers’ knowledge of students’ conceptions, difficulties, and errors were investigated regarding the KFLM dimension of the MTSK model.

In this study, teachers' interpretations of students' performances were also examined regarding the potential causes of students' difficulties and errors and teachers' inferences based on students' struggles. Causal attributions may affect individuals' expectations for future success, behaviors, and emotions (Graham & Williams, 2009; Weiner, 1992, 2000). Therefore, the causes of students' difficulties and errors expressed by teachers might be a precursor of teachers' perceived competencies and instructional decisions. Wang and Hall (2018) stated that causal attribution theory helps examine how teachers perceive challenges and occupational stressors of students and how teachers' attributions affect teacher-student interactions, instructional behaviors, and teachers' emotional well-being. Attribution theory examines how individuals interpret and explain the causes of individuals' behaviors or facts (Weiner, 1985, 1992, 2000, 2004). Therefore, the current study also investigated teachers' causal attributions for students' failures. In the following part, causal attribution theory will be described in detail.

2.2.Causal Attribution Theory

Attribution theory investigates how individuals make judgments and attempt to explain how they consider the causes of their own and others' behaviors (Weiner, 1985, 2010). Attributes might influence beliefs, emotions, and behaviors. As a result, attribution theory has significantly contributed to the research on motivation (Graham & Williams, 2009; Zimmerman & Schunk, 2008). Researchers stated that attributions significantly affect teachers' expectations regarding students' future academic performances (Clarkson & Leder, 1984; Peterson & Barger, 1985). Several studies have been conducted on attribution and mathematics (Bandura, 1981; Cobb et al., 1992; Fennema, 1980; Shores & Shannon, 2007; Weiner, 1974). Shores and Smith (2010) expressed that the most dangerous and persistent myth regarding mathematics education was that "success in mathematics depends more on innate ability than on hard work" (National Research Council [NRC], 1991, p. 10). Therefore, they concluded that the causes of students' difficulties in mathematics should be investigated carefully to improve students' success in mathematics, especially for low achievers. For this purpose, the initial phase might be investigating the attributions regarding the success or failure and teachers' consideration of real success (Baştürk, 2016). As Shores and Smith (2010) pointed out, it is crucial to get teachers to notice their own attributions and to understand how particular attributions affect their mathematics teaching.

As presented in Figure 2.3, researchers identified three main dimensions for the causal attributions: locus of causality, stability, and controllability (Weiner, 2010). The first dimension, the locus of causality, is related to the occurrence of outcomes of behaviors dependently or independently, in other words becoming internal or external. To illustrate, an individual's effort or ability refers to an internal locus of causality, whereas environmental circumstances or luck implies external factors. As Shores and Smith (2010) stated, if students were aware of their roles in their own success or failure, they would be more motivated in mathematics tasks and demonstrate more effort compared to those who did not accept the effects of their behaviors on the outcomes. The second dimension of causal attribution is stability, stable or unstable, which means an attribution may vary by time or not. For example, an individual's effort or luck may vary over time. However, task difficulty and low ability are stable issues as the characteristics of the task always remain the same. The third dimension is an attribution's controllability, whether controllable or uncontrollable. To illustrate, students' effort is a controllable attribution, whereas illness is an uncontrollable attribution. Students frequently use the ability, effort, task difficulty, and luck to explain their successes and failures in achievement settings. Although these are not the only attributions used by students, they are the most common. (Schunk, 2012; Weiner et al., 1971).

Shores and Smith (2010) reviewed attribution studies conducted between 1974 and 2008. The researchers concluded that it is crucial for teachers, especially in mathematics, to identify whether students attribute success or failure to effort, ability, luck, and task difficulty. So that teachers can understand students' behaviors when success and failure are attributed to an internal or external cause. The research on attribution and mathematics showed that when students attribute their success to external factors, they experience lower achievement than when they relate success to internal factors. Moreover, students typically linked success to effort and ability, whereas they attributed failure to luck and task difficulty.

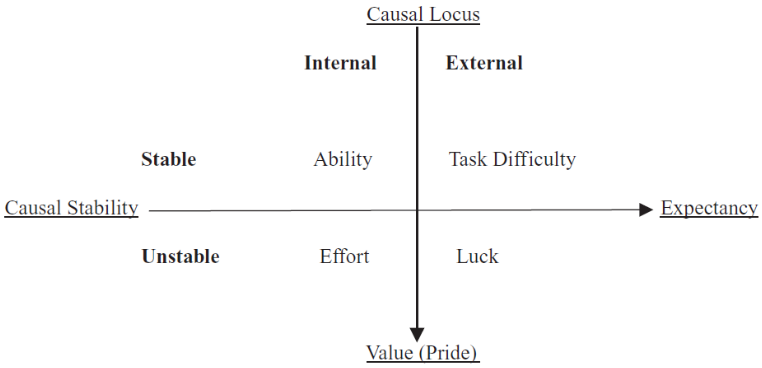


Figure 2. 3. Representation of the four main causes of behavior, their dimensional properties (locus and stability), and linkages to affect and expectancy (Weiner, 2010, p. 32)

Wang and Hall (2018) investigated teachers' causal attributions in seventy-nine published research studies since the 1970s based on the causal attributional theory of Weiner (2010) to observe how prevalent the particular attributional styles in teachers and the consequences of particular attributional styles on teachers and their students. The studies were analyzed based on causal attributions for students' performance, misbehavior, and occupational stress. The researchers examined how teachers' explanations for classroom stressors affect their instruction and student development. Based on the study, Medway (1979) observed that teachers attributed students' learning difficulties and behavioral challenges to ability-related factors, whereas they attributed behavioral problems to peer and family factors. Moreover, Georgiou (2008) suggested that experienced teachers tend to determine more controllable attributions related to students' failure and misbehavior, whereas novice teachers might have unrealistic thoughts about their ability to effectively improve students' academic performance (Pirrone, 2012). The review of the studies indicated that teachers' causal attributions might influence their emotions and instructional behaviors that significantly affect students' academic performance, behavior, and motivation. The researchers suggested that the literature lacks studies examining teachers' attributions' impact on students by examining actual student and teacher data.

Baştürk (2016) investigated pre-service mathematics teachers' causal attributions regarding success and failure in mathematics. Twenty-eight pre-service MSMTs participated in the study at a public university in Turkey. The researcher conducted a questionnaire that included an open-ended question related to the causes of students' success and failure in mathematics. The researcher found that pre-service teachers attributed students' failure in mathematics to four causes: "causes originating from students, causes originating from teaching and learning methods, causes originating due to the nature of mathematics itself, and physical causes" (Baştürk, 2010, p. 365). In this study, pre-service teachers most frequently mentioned the cause, the innate math talent, which is an internal, stable, and uncontrollable factor. Glasgow et al. (1997) showed that students who attributed their failures to a lack of ability, an uncontrollable factor, demonstrated lower performance in the classroom. Therefore, it might be inferred that if the teacher connects students' failure and their lack of innate talent, their students also probably think that way. Lastly, teachers who believe in innate math talent to be successful in mathematics may not perform much effort into the untalented students. He noted that math background, as an external, stable, and uncontrollable factor, is also one of the most frequently mentioned attributions for pre-service primary school teachers and pre-service MSMTs (Baştürk, 2012, 2016). Moreover, he found that teachers attributed the success or failure of students to loving and being interested in mathematics and noted that not loving math was seen as a barrier by pre-service teachers regarding students' learning of mathematics. In the study of Baştürk (2016), some pre-service teachers cited a lack of knowledge about how to study mathematics, the abstractness of math and little liaison with everyday life, and the abundance and sophistication of math topics as reasons for students' success or failure.

Bozkurt and Yetkin-Özdemir (2018) conducted research to describe reflection activities through a lesson study. The participants were three MSMTs with 8, 11, and 9 years of teaching experience. The researchers observed teachers' collaborative lesson study practices for five months and investigated teachers' reflection activities based on teacher self-regulation. Teachers'reflection activities were examined based on the themes, evaluation, causal attribution, and inference. Regarding causal attribution, the causes of successful and unsuccessful cases were coded based on controllability, becoming controllable or uncontrollable, in teachers' reflection activities. The results presented that the teachers tended to mention attributions for failure. The teachers mainly made controllable attributions, such as "determining excessive lesson content, inadequate planning on concrete materials, and not planning potential situations in detail" (p. 386). They considered that their failures were related to the factors under their control. Also, they provided uncontrollable attributions, such as "uncontrolled time losses-limited time, camera-observer effect, and lack of prior knowledge of students" (p. 386). Lastly, results indicated that the teachers attributed the failures to themselves more when they were engaged in lesson study activities since lesson study gets teachers to design their own instruction collaboratively and evaluate their instruction as a group.

Shores and Smith (2010) stated that attributions are crucial for teachers to understand why students are unsuccessful and what makes them fall behind academically. Therefore, new studies were needed to examine teachers' attributions for students' success and failure to observe how these factors impact the teachers' teaching and students' learning. To observe teachers' knowledge of students' algebraic thinking, difficulties, and misconceptions and teachers' attributions on students' success/failure, determining students' algebraic thinking, difficulties, and misconceptions might be helpful. The next part will review the studies on students' algebraic thinking, difficulties, and misconceptions.

2.3. Middle School Students' Algebraic Thinking, Difficulties, and Misconceptions

Algebra is a core topic that has been widely accepted as one of the most challenging topics in the mathematics curriculum, resulting in difficulties and misconceptions while students learn it (Blanton et al., 2015; Herscovics & Linchevski 1994; Jupri & Drijvers, 2014; Kieran 1992; Warren, 2003). Researchers suggested that students should be introduced to algebraic thinking in elementary grades beyond doing practice with manipulations in equations (Carpenter & Levi, 2000; Schifter, 1999).

Prerequisite Knowledge of Students Required for Learning Algebra. Research suggests that students should have some prerequisite knowledge before learning algebra. First, Miller and Smith (1994) proposed prerequisite *vocabularies* for teaching algebra, and they created 60 items list. Due to the researchers, students should know specific terms before learning algebra. Furthermore, some researchers focused on knowing *numbers* (Gallardo, 2002; Kieran, 1988; Watson, 1990; Wu, 2001). As Watson (1990) stated, a concrete understanding of numbers made students properly handle algebraic operations. Gallardo (2002) also noted that students' knowledge of negative numbers was essential to comprehending algebra. Therefore, students should have a solid understanding of integers while transitioning from arithmetic to algebra to solve equations and algebraic word problems correctly. Similarly, Kieran (1988) stressed the importance of the comprehension of integers by addressing students' difficulties with the division of integers by implying the lack of understanding of fractions. In addition, Wu (2001) highlighted the importance of teaching fractions for transforming from doing arithmetic calculations to comprehension of algebra. The prerequisite knowledge for students learning algebra is summarized in Table 2.1.

Table 2. 1. The prerequisite knowledge required to learn algebra in the literature

Prerequisite knowledge for learning algebra	**Research studies**
• Vocabularies (arithmetic or algebraic terms)	• (Miller & Smith, 1994)
• Numbers	• (Gallardo, 2002; Kieran, 1988; Watson, 1990; Wu, 2001)
• Proportionality	• (Blanton et al., 2015; Post et al., 1988)
• Computations	• (Booth, 1984)
• Equality	• (Falkner et al., 1999; Herscovics & Kieran, 1980; Kieran, 1981)
• Symbolism	• (Behr et al., 1976, 1980; Booth, 1986; Kieran, 1992; Küchemann, 1981; Macgregor & Stacey, 1997; Watson, 1990)
• Equation writing	• (Clement et al., 1981; Wollman, 1983)
• Representation of functions with graphics and symbolic expressions	• (Bottoms, 2003; Brenner et al., 1995; Markovits et al., 1988)

Post et al. (1988) draw attention to the extent of *proportionality* since it connects joint manipulations with numbers and patterns to algebra, a more abstract world. Since proportional reasoning requires a concrete understanding of rational numbers, the relationship of a unit and its parts, and comprehension of ratios, it contributes to students' algebraic reasoning development. In addition to the numbers, students should also understand the rules behind the *computations* among the numbers, such as commutativity, associativity, distributivity, inverse operations, and the order of operations (Booth, 1984). Researchers also studied students' understanding of the meaning of *equality* (Falkner et al., 1999; Herscovics & Kieran, 1980; Kieran, 1981). Carpenter, Levi, and Farnsworth (2000) stated that the correct interpretation of equal sign is one of the most important precursors for concrete algebraic reasoning since it allows students to comprehend the equality of both sides and use the equal sign properly to express generalizations.

Symbolism was also essential before learning algebra (Booth, 1986; Kieran, 1992; Küchemann, 1981). As researchers stated, using signs such as equality and plus sign might be interpreted by students in a typical way, as computations to be performed (Behr et al., 1976, 1980). Based on the literature, students might be confused about using some symbols in algebra. To illustrate, somebody could conjoin two and a half as $2\,\frac{1}{2}$ in arithmetic; however, it would not be correct to write 4a instead of 4+a in algebra (Booth, 1986; Kieran, 1988). In addition to using symbols, students confuse letters while transitioning from arithmetic to algebra (Macgregor & Stacey, 1997). As Watson (1990) stated, the introduction of variables is crucial at this point. Students should initially learn to find the pattern and then write it in words. After that, students should use variables to express the rule of a pattern. Studies show that there are common misconceptions that students have related to the meaning of letters in algebra. To illustrate, misinterpretation of letters as if they are representing objects or words, associating letters with their positions in the alphabet, and viewing letters as always representing a specific unknown are some examples of misconceptions that students had (Booth, 1986; Kieran, 1988; Macgregor & Stacey, 1997; Watson, 1990). *Equation writing* is one of the terms that should be investigated while considering the prerequisite knowledge for algebra. Transitioning algebraic expressions from verbal to algebraic form is one of the difficulties students might face while doing algebra (Clement, Narode, & Rosnick, 1981; Wollman, 1983). Also, students must understand the representation of functions graphically and algebraically before learning algebra (Bottoms, 2003; Markovits, Eylon, & Bruckheimer, 1988). Brenner et al. (1995) also stated that it is crucial to understand the functional relationship between variables and represent those relationships to succeed in algebraic reasoning.

The teachers should have adequate knowledge of students' thinking, difficulties, and misconceptions in algebra. There are several studies suggesting that students struggle with the concept of variable (Asquith et al., 2007; Blanton & Kaput, 2011; Blanton et al., 2017), equivalence, and equation (Blanton et al., 2015; Carpenter & Levi, 2000; Kieran 1992; McNeil & Alibali, 2005), and functional thinking (Blanton & Kaput, 2011; Carlson et al., 2002; Kaput, 1992, 1994; Rasmussen, 2001) in algebra. Therefore, the following section will present national and international studies conducted by researchers about students' algebraic thinking, difficulties, and misconceptions in algebra.

Middle School Students' Algebraic Thinking. Carpenter and Levi (2000) stated that the focus of algebra should not be manipulating the symbols. Instead, the goal should be developing algebraic thinking instead of using algebraic procedures. This part summarized studies showing students' algebraic thinking under equivalence, equations, and functional thinking.

Big ideas of algebra. Blanton et al. (2015) described five big ideas considering the content strands explained in the study of Kaput (2008) and the study of Shin et al. (2009). The researchers identified the five big ideas as "(a) equivalence, expressions, equations, and inequalities (EEEI); (b) generalized arithmetic; (c) functional thinking; (d) variable; and (e) proportional reasoning," which offers core algebraic thinking practices of "generalizing, representing, justifying, and reasoning with mathematical relationships" (Blanton et al., 2015, p. 43). The researchers' more recent study (Blanton et al., 2019) examined big ideas under three topics: EEEI; generalized arithmetic; and functional thinking in terms of algebraic thinking practices, generalize, represent, justify, and reason with. In their recent study, the researchers presented learning goals regarding algebraic thinking practices and three big ideas (Blanton et al., 2019).

Blanton et al. (2015) explained that the big idea of *equivalence, expressions, equations, and inequalities (EEEI)* consisted of "developing a relational understanding of the equal sign, representing and reasoning with expressions and equations in their symbolic form, and describing relationships between and among generalized quantities that may or may not be equivalent" (p. 43). The big idea of *variable* is described as the symbolic notation representing mathematical ideas and playing different roles in different mathematical contexts (Blanton et al., 2015). The researchers noted that the big idea of variable could be observed in other big ideas, such as a "varying, unknown quantity in the study of functional relationships and as a generalized number when examining the fundamental properties" (Blanton et al., 2015, p. 43). For this reason, the big idea of the variable was integrated throughout three big ideas, EEEI, generalized arithmetic, and functional thinking, in the revised Early Algebra Learning Progression (EALP) (Blanton et al., 2019). The researchers included recognizing and using the variable in problem situations and representations in EEEI in addition to the relational understanding of the equal sign and the properties of equality. Moreover, they described learning goals related to the representation and interpretation of algebraic expressions or equations in the big idea of EEEI. Next, the researchers emphasized the learning goals related to understanding different meanings of variables (i.e., a fixed quantity or a varying quantity), interpretations of equations with different formats, and reasoning in open-number sentences using structural relationships (Blanton et al., 2015; Blanton et al., 2019).

Knuth et al. (2005) emphasized that equivalence and variable were two of the most critical algebraic ideas to achieve algebraic reasoning. As researchers suggested, students should be able to perceive the equal sign as a precursor of equivalence, such as an interrelation between two quantities, instead of a command for the result of an arithmetic operation (Falkner et al., 1999; Kieran, 1981; Knuth et al., 2005; Rittle-Johnson & Alibali 1999). Knuth et al. (2005) stressed the importance of students' relational view of the equal sign referring to being "the same as" instead of the operational view, referring to "do something" (p. 69). Since students typically encounter the equations in the form of a + b = □ in elementary school, it might be problematic when confronted with the equations of ax + b = cx + d. At this point, the relational view of the equal sign became essential since students should understand that the equivalence relation was conserved throughout the transformations while solving the equation. Gallardo (2001) described this particular problem as the existence of a didactic cut that was anticipated in historical algebra textbooks between arithmetic and algebra related to “a resistance to operating on the unknown” (p. 127). As he noted, beginning algebra students’ obstacles in problem-solving approaches might be analogues with the pupils in history. This didactic cut was later determined on 12 and 13 years old students who experienced the transition from arithmetic to algebra (Filloy & Rojano, 1985, 1989; Herscovies & Linchevski, 1991; Rojano, 1985). Blanton et al. (2015) identified *generalized arithmetic* to explain the generalization of arithmetic relationships involving fundamental properties of numbers and operations and reasoning related to the structure of arithmetic expressions instead of their computational value. The researchers extended the description of generalized arithmetic by interpolating the use of words and/or variables to represent the generalization (i.e., examining the meaning of variable(s) in a representation or equation) (Blanton et al., 2019). Moreover, justifying the conjectures’ validity by exploring different arguments, identifying particular values to make the conjecture correct, and specifying the characteristics that make the conjecture valid for all values in a given domain were included in the big idea of generalized arithmetic. Lastly, identifying the generalization and using it to examine the validity of the new conjectures were accepted as learning goals for the big idea of generalized arithmetic.

Functional thinking refers to generalizing the relationships between covarying quantities and demonstrating and reasoning with those relationships using verbal representations, symbolic notations, tables, and graphs (Blanton et al., 2015). The researchers included identifying a recursive pattern, covariational relationship, and functional relationship in Blanton et al. (2019). They also emphasized the representation of variable quantities and different variables in verbal and symbolic notation, the representation of the recursive pattern, covariational relationship, and functional relationship, and the construction of coordinate graphs to represent a functional relationship of discrete data. Lastly, they included the justification of how the function rule demonstrated the problem situation and reasoning with how the function rule gives far function values, how we can interpret the behavior of the function, and how we determine the value of dependent or independent variables based on a function table or function rule. Carraher et al. (2006) stated that generalization was at the center of algebraic reasoning, and addressing the concept of function in the elementary mathematics curriculum facilitates the integration of algebra in the mathematics curriculum. Therefore, they described algebra as "a generalized arithmetic of numbers and quantities" in which function concept played a significant role (Carraher et al., 2006, p. 88). Various researchers emphasized the importance of functions in middle and high school curricula (Dubinsky & Harel, 1992; Yerushalmy & Schwartz, 1993). Tanışlı (2011) noted that functional thinking, understanding the relationship between varying quantities, was one of the essential components of algebraic thinking. Finally, the big idea of *proportional reasoning* was described as algebraic reasoning related to two generalized quantities whose ratios were invariant (Blanton et al., 2015).

Studies regarding students' understanding of equivalence and equations. A concrete understanding of the equal sign and its use in equations is critical for both having a deep conceptualization of arithmetic and learning algebra (Carpenter et al., 2003). However, many students lack a solid comprehension of the equal sign and frequently struggle with solving, transforming, and interpreting equations. Some of the studies related to students' understanding of equivalence and equations were presented in the following session.

Stephens et al. (2013) investigated the understanding of two hundred and ninety-third, fourth, and fifth-grade students regarding the meaning of the equal sign and equation structure before any algebraic instruction intervention was given. The researchers described three conceptions of the equal sign, operational, relational-computational, and relational-structural. Operational means that students perceive the equal sign as a command for doing something, such as computation, instead of a symbol that expresses a relationship. For example, based on the results, some students thought that the blank should be 8 in an open number sentence "5 + 3 = __ + 3" since "5 + 3 = 8" (p. 176). The relational-computational refers to the understanding that the equal sign demonstrates an equivalence relationship between two sides of the equation, and this equivalence is confirmed with computation. To illustrate, if a student expressed that six should be substituted in the blank in the open number sentence "7 + 3 = __ + 4" since the summation of each side would be ten separately (p. 176). The relational-structural view presents a more profound understanding that an equal sign is a symbol that expresses an equivalence relation between two expressions instead of two calculations. For example, in an open number sentence, “7 + 3 = __ + 4”, if a student considered that six should be put in the blank as four was one more than three, the student’s response was coded as structural (p. 176). Researchers observed that most students had an operational conception of the equal sign and experienced difficulty recognizing the underlying structure of equalities. Lastly, the researchers suggested that students should be challenged and directed to focus on relational thinking using true/false questions and open-number sentences.

Kızıltoprak and Köse (2017) investigated the development of students’ relational thinking skills with a teaching experiment method. The data was collected from six 5th-grade students using teaching episodes and pre and post-clinical interviews before and after the teaching sessions. Eight teaching sessions were prepared and conducted on students to observe how relational understanding of equal sign was established and developed in students’ minds. Results showed that students were successful at the end of the teaching sessions. Moreover, researchers observed a relationship between the development of basic arithmetic concepts and relational thinking such as minuend, subtrahend, and difference. Lastly, the students comprehended that the equal sign is not just for finding the result of an operation; instead, it can be used to present a relation between numbers, operations, and expressions.

Studies regarding students’ functional thinking. The following studies (Blanton and Kaput, 2004; Ng, 2018; Tanışlı, 2011) present students’ ways of functional thinking and how students’ functional thinking develops.

Blanton and Kaput (2004) studied Pre-K – 5th-grade elementary students to observe how they construct and express functions. The study was a 6-year teacher professional development project in an urban school district to get teachers to identify their instructional resources and teaching practices to develop students’ algebraic reasoning. Therefore, interviews were conducted with teachers. The data were investigated based on the forms of students’ representations, the progression of students’ mathematical language, the operations they used, and how they expressed the variation between quantities. The students were asked to respond to a task: “If there was one dog, how many eyes would there be? What if there were two dogs? Three dogs? 100 dogs? Do you see a relationship between the number of dogs and the total number of eyes? How would you describe this relationship? How do you know this works?” (p. 136). Moreover, they asked:

> Suppose you wanted to find out how many eyes and tails there were all together. How many eyes and tails are there for one dog? Two dogs? Three dogs? 100 dogs? How would you describe the relationship between the number of dogs and the total number of eyes and tails? How do you know this works? (p. 136)

Results illustrated that the pre-kindergarten students could use a t-chart to organize the data with their teachers’ guidance. Therefore, they could express that one dog had two eyes and one tail, or a total of three, and two dogs had four eyes and two tails, or six in total. The researchers pointed out that students could find far function values by counting visible objects rather than making predictions. They noted two important mathematical events for this group; they developed an understanding between numerals and objects and were introduced to a t-chart (function table) to organize covarying quantities. In kindergarten, children drew a dot for each eye and a long shape for each tail. They also focused on the pattern by drawing a t-chart. 1st-grade students could draw the t-chart without the teacher’s guidance and recognize the recursive pattern. Students noticed that they should count by 2s to find the number of eyes and by 3s to find the total number of eyes and tails. 2nd-grade students could determine the multiplicative relationship that the number of eyes equals double the number of dogs. Moreover, they predicted the number of eyes for 100 dogs by using this multiplicative relationship. 3rd-grade students could use t-charts fluently and express the rule multiplicatively both in words and symbols. Also, they could anticipate the number of eyes or the total number of eyes and tails for 100 dogs using the rule by writing “n *x* 2” or “2 *x* n” (p. 138). Based on the number of eyes, students could express, “It doesn’t matter how many dogs you have, you can just multiply it by 2” (p. 138). The 4^{th} and 5^{th} graders could also work similarly with 3rd graders. They could develop functions with fewer data in comparison to lower graders. Results indicated that very young learners were successful at functional thinking, and this study showed how their thinking might progress, to what extent they could study with patterns and relations, and how they could use the symbols to represent the relationships throughout Pre-K to 5^{th} grade.

Tanışlı (2011) studied with four 5th graders to examine students' functional thinking through linear function tables. She conducted task-based interviews to collect the data, including 16 items about linear function tasks. Since students do not use letters for the variables based on the 5th-grade in Turkish middle school mathematics curriculum, the researcher demonstrated the dependent and independent variables using "the number triangle" and "the number of square." She analyzed the results under two main themes: "realizing a pattern and ways of functional thinking" (p. 210). She observed that students looked for a recursive pattern either in the dependent or independent variable instead of observing the relationship of individual variables when they first introduced a function table. She concluded that students must first comprehend two things: "to focus on corresponding changes in the individual variables and to find the relation between corresponding pairs of variables" (p. 221). The study found that students could explore the correspondence relationship and think covariationally regardless of students' achievement level. Results also showed that the students could find and generalize the correspondence relationship using additive and multiplicative relationships. The researcher expressed that students successfully explained the correspondence relationship using the semi-symbolic rules, although 5th-grade students were unfamiliar with using algebraic symbols. The results suggested that 5th-grade students were successful while thinking covariationally, exploring correspondence relationships, and generalizing the relationship between variables. Lastly, the study illustrated that students do not use just one way of thinking while making a generalization and determining the relationships. Therefore, teachers and researchers should be conscious of the alternative thinking ways of students.

Ng (2018) investigated how students make generalizations in function-related tasks. Ten students participated in the study, and two different levels of interviews were done. The interview group consisted of first to third graders, and the other interview group included fourth to sixth graders. The interviews were conducted regarding function-machine tasks, which concentrated on input and output numbers, setting up a rule to find the output using the input, and finding a general rule. The interview tasks had an increasing structural complexity, beginning from a single operation and progressing to writing a functional rule with a letter. The researcher believed that students should recognize the relationship between input and output and be able to find the general rule in different tasks. The researcher observed that lower primary graders use semi-symbolic rules to express the rule of functions since they could not use algebraic letters at this level. The researcher found that the participant students could notice a relationship and make generalizations when a task with increasing difficulty was presented to them, even though they did not have an intervention.

Kieran (1992) found that students could not conceptualize structural aspects of algebra in general and noted that they "resort to memorizing rules and procedures and…eventually come to believe that this activity represents the essence of algebra" (p. 390). As Booth (1986) identified, the main goal of algebra is to learn how general relationships and procedures are represented so that various types of problems can be solved and new relationships can be constructed. However, students often consider algebra a collection of arbitrary manipulations. Therefore, as presented in the following part, they have difficulties learning algebra, as various research studies concluded with similar results.

Elementary and middle school students' difficulties in algebra. Algebra is considered hard to learn and teach (Stacey et al. 2004; Watson 2009). Students' difficulties in algebra were summarized based on the literature in Table 2.2. The following part explains students' difficulties while learning algebra.

*Students' difficulties with arithmetic. V*arious studies have presented that middle school students often struggle while adding or subtracting algebraic terms related to arithmetic operations (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996). In addition, as studies have shown, middle school students misuse commutative and associative properties, especially when performing subtraction or division. They fail to use the distributive property of multiplication over addition (Booth, 1988; Pillay et al., 1998). Jupri et al. (2014) interpreted that these difficulties resulted from students’ restricted mastery of priority rules in arithmetic operations such as addition, subtraction, division, multiplication, and properties related to numerical operations.

Table 2. 2. Preview of studies related to students’ difficulties, and misconceptions in algebra

Categories		**Areas where students have difficulties and misconceptions**	**Research studies**
Arithmetic		• Doing operations with algebraic expressions • Properties of arithmetic operations	• (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996) • (Booth, 1988; Pillay et al., 1998)
Variable		• Understanding the concept of variable	• (Asquith et al., 2007; Blanton & Kaput, 2011; Blanton et al., 2017)
Equal sign		• Understanding the meaning of the equal sign	• (Falkner et al., 1999; Kieran, 1989; Rittle-Johnson & Alibali, 1999)
Mathematization	**Horizontal**	• Interpreting symbolic expressions	• (Stephens, 2003)
		• Translating from verbal to symbolic representations	• (Clement, 1982; Jupri et al., 2014; Kenney & Silver, 1997; MacGregor & Stacey, 1997),
	Vertical	• Solving algebraic equations	• (Erbaş, 1999; Herscovics & Linchevski, 1994; Jupri et al., 2014)
Functional thinking		• Covariational relationship	• (Blanton & Kaput, 2011; Carlson et al., 2002; Kaput, 1992, 1994; Rasmussen, 2001)

Students’ difficulties with variable. Moreover, middle school students also have difficulties discriminating between different forms of literal symbols such as “placeholder, unknown, generalized number, and varying quantity” (Jupri et al., 2014, p. 686). A placeholder, a literal symbol, is viewed as an empty ‘container’ in which a numerical value can be preserved. An unknown means a literal symbol that might be used in problem-solving to solve an equation. A literal symbol serves as a pattern generalizer as a generalized number, representing equivalence such as 3x + 6x = 9x. A varying quantity implies a literal symbol used in a functional relationship as either an input expression or as the value of the output function. Therefore, the variable is considered a separate area since the different interpretations of the variable might cause difficulty in grasping the concept of the variable. Usiskin (1988) also clarified that the term variable has various meanings, such as a formula, an equation, an identity, a property, and a function. To illustrate, all of the following equations represent the product of two numbers; however, we usually call each of them differently:

1. A = LW
2. 40 = 5x
3. sin x = cos x . tan x
4. 1 = n . (lin)
5. y = kx

Figure 2. 4. Different forms of equations (Usiskin, 1988, p. 9)

Figure 2.4 shows that we generally refer to *a formula* in the first algebraic expression. The symbols A, L, and W refer to the quantities of area, length, and width, which have the sense of knowns. We usually call the second algebraic expression *an equation* in the second one and perceive x as an unknown. In the third equation, we commonly see *an identity* in which x is an argument of the function. In the fourth equation, becoming *a property* attracts attention, and n refers to a specific value of an arithmetic pattern. Lastly, we observe an equation of *a function* that shows a direct variation in the fifth equation, and ultimately we perceive the variability of x. Therefore, students might be confused when confronted with various types of a variable, as Usiskin (1988) highlighted. Küchemann (1978) also noted that students might perceive variables as labels and abbreviations instead of letters that refer to quantities, substitute values with letters regarding their position in the alphabet, and have difficulty doing operations with varying values as they are familiar with specific values.

Studies showed that students have struggled to interpret the variable (Asquith et al., 2007; Dede et al., 2002). As Usiskin (1988) described various meanings of a variable, students might have difficulty with these different meanings. As Küchemann (1978) noted, ignoring the letters is one type of student thinking based on variables. Dede et al. (2002) conducted a study with 8th-grade students to investigate whether they ignored the letter in an algebraic expression with variables. The results showed that 60% of the students responded incorrectly to the task "2 + 5x = ?" and some responded as "7", ignoring "x" in the algebraic expression. Moreover, some students tried to solve the algebraic expression by making it equal to a quantity related to the lack of closure obstacle (Tall & Thomas, 1991). MacGregor and Stacey (1997) found that students substitute quantities for the letters based on their order in the alphabet. For example, as the researchers observed, they typically accepted that a equals 1 and b equals 2 in an algebraic expression. Similarly, Ryan and Williams (2007) observed that students were prone to assign a particular value to the unknown such as a=1, b=2, and c=3. Also, they noted that students might read 5x as '5 times' since they confused the x in arithmetics and the multiplication symbol.

Soylu (2008) investigated 7th-grade students' interpretations related to the variable. The participants were fifty students and were asked to answer eight open-ended items regarding variables. Similar to the studies of MacGregor and Stacey (1997) and Ryan and Williams (2007), the researcher observed that students were prone to substitute a specific value instead of a variable in an algebraic expression. To illustrate, students typically found a numerical result by substituting a random number instead of using *n* in all the tasks. Moreover, the researcher observed that students ignored the variables, as Küchemann (1978) stated. For example, students considered that the result should be 5x or 5 in an algebraic expression of "2x+3=?". Moreover, the researcher found that students typically prefer to use x in their algebraic solutions, although they were given different symbolizations such as h, m, n, or y in the tasks.

Students' difficulties with the equal sign. The third category of students' algebraic difficulty is understanding the meaning of the equal sign. Researchers stated that equality was also one of the issues that students commonly had misunderstandings (Falkner et al., 1999; Kieran, 1989). Researchers suggested that a relational understanding of the equal sign is essential for performing the transformations to solve an equation and preserving the equivalence relation (Asquith et al., 2007; Kieran, 1992; Knuth et al., 2006). In arithmetic, students tend to see the equal sign as a procedure 'to do something' or 'give the answer' whereas, in algebra, it typically means ‘is algebraically equivalent to’ (Blanton et al., 2015; Herscovics & Linchevski, 1994; Jupri et al., 2014; Kieran, 1981; Pillay et al., 1998). In arithmetics, students may interpret 4+7 as adding 4 and 7 to get the answer 11 and may not consider 4+7 = 7+4; 5+6 = 6+5, or 11 = 4+7 as the alternative solutions for this task. However, the latter insight is required to understand equivalence, such as when rewriting x+2 = 3x+4 as x = 3x+2. However, they need to conceptually understand those equivalences while rewriting the expression x+5 = 4x+11 as x = 4x+6. Clement et al. (1981) found that even high school students had misconceptions regarding the meaning of the equal sign. Numerous studies on elementary and middle school students’ understanding of the equal sign have concluded that students could not have a relational understanding of the equal sign in general. Instead, they interpret it as the result or outcome of an arithmetic operation (Falkner et al., 1999; Kieran, 1981; Rittle-Johnson & Alibali, 1999).

Students’ difficulties with mathematization. Lastly, Jupri et al. (2014) described mathematization, a central idea in the realistic mathematics education (RME) approach (Freudenthal, 1991; Treffers, 1987), as one of the difficulties students might face in algebra. Mathematization includes translating a mathematical situation to the symbolic world of mathematics back and forth and moving with reorganizations and reconstructions in the symbolic world of mathematics (Treffers, 1987; Van den Heuvel-Panhuizen, 2003). As Jupri and Drijvers (2016) declared, mathematization has a cyclic structure that includes the separate phases of “understanding the problem, formulating a mathematical model from the problem, solving the problem expressed in the model, and interpreting the solution in terms of the original problem” (p. 2482). The cycle of mathematization is given by Jupri and Drijvers (2016), which was drawn based on the study of De Lange (2006).

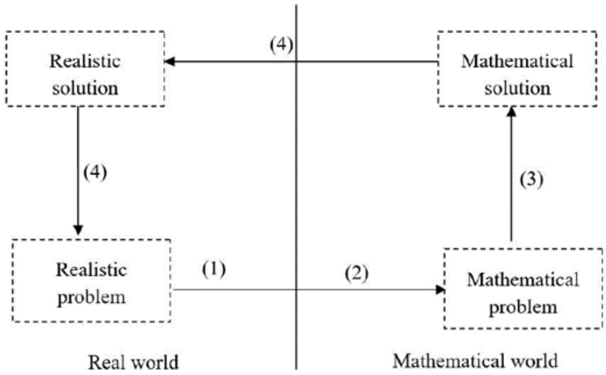


Figure 2. 5. The mathematization cycle (Jupri & Drijvers, 2016, p. 2485)

Based on Figure 2.5, a learner initially understands the problem and determines relevant mathematical information in the problem (1). Then, the real-world situation is converted into a mathematical problem by removing the irrelevant elements (2). After that, a mathematical model written based on the problem is solved by the learner (3). Lastly, the learner interprets the solution based on the realistic situation (4). Mathematization includes two aspects, *horizontal* and *vertical mathematization*. Horizontal mathematization is related to the transformation between the verbal conditions in mathematics to the symbolic world of algebra and vice versa, which is incredibly demanding for middle school students (MacGregor & Stacey, 1998; Treffers, 1987; Van den Heuvel-Panhuizen, 2003; Warren, 2003; Watson, 2009). Figure 2.6 presents the phases of mathematization, namely formulating a problem in a novel way, discovering relationships, and converting a real-world problem into a mathematical problem are examples of horizontal mathematization (De Lange, 1987).

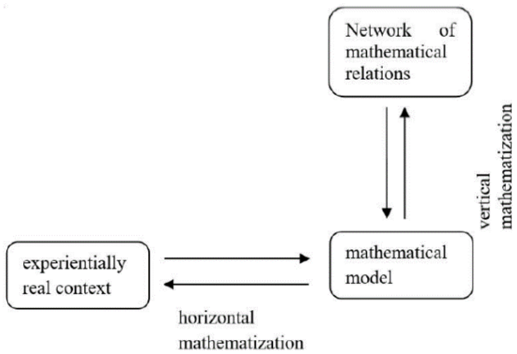


Figure 2. 6. Horizontal and vertical mathematization (Jupri & Drijvers, 2016, p. 2484)

Those activities might be corresponded with two steps of Polya's (1973) problem-solving heuristics, understanding the problem and devising a plan (See Figure 2.6). Vertical mathematization includes reorganizing the structure of mathematical objects (Treffers, 1987; Van den Heuvel-Panhuizen, 2003). Freudenthal (1991) declared that vertical mathematics was the process in which "symbols are shaped, reshaped, and manipulated mechanically, comprehendingly, reflectingly; this is vertical mathematization." (Freudenthal, 1991, p. 41-42). To illustrate, doing manipulations in algebraic expressions, combining different algebraic models while solving equations, and making generalizations and proofs are examples of activities in vertical mathematization (Treffers, 1987; Van den Heuvel-Panhuizen, 2000). Different obstacles related to vertical mathematization in algebra might be explained, such as the process-product duality (obstacle), parsing obstacle, the expected answer obstacle, and lack of closure obstacle. *The process-product duality (obstacle)* refers to the inability to discriminate between the view of the process and the product of this process, the object (Sfard, 1991; Tall & Thomas, 1991). For example, $x+5$ refers to both a process of addition and an algebraic object (Drijvers, 2003; Van Amerom, 2003). As Tall and Thomas (1991) highlighted, if a student perceives algebraic expressions as algebraic processes, they consider that $4(x + y)$ and $4x + 4y$ are quite different expressions as the first one implies the addition of x and y before the multiplication of $(x + y)$ with 4. In contrast, the second refers to the summation of the terms $4x$ and $4y$ after x and y are multiplied by four separately. The researchers suggested that teachers should get students to realize that the two expressions are the same since they always result in the same product. As the researchers declared, students should notice that $4a + 5$ always represents the same product of any process in which one takes a value, multiplies it by 4, and add 5 to the result. The researchers asserted that conceptualization of this procedure requires "the encapsulation of the process as an object so that one can talk about it without the need to carry out the process with particular values for the variable" (Tall & Thomas, 1991, p. 126). When *encapsulation* occurs, learners can regard two encapsulated objects as the 'same' if they always result in the same product.

Tall and Thomas (1991) also focused on the cognitive conflict between understanding natural language and algebra's symbolic world. They stated that natural language and algebra are written and read from left to right in most civilizations. However, this rule might be problematic in some situations in algebra. For example, students often read and process 2x+5 from left to right. However, 5+2x is read from left to right as ‘five plus two x’ but computed from right to left by calculating the multiplication ‘2x’ before the summation with 5. Tall and Thomas (1991) called this difficulty as *parsing obstacle*, which refers to the changeable sequence of algebraic processes in contrast to the sequence of natural language. The researchers also mentioned other aspects of the parsing obstacle, such as considering ab and a+b equal since they might read both as a and b by students. Moreover, students might read 4 + 5x as 4 + 5, resulting in 9, and conclude that the expression equals 9x. Tall and Thomas (1991) also explained an issue resulting from the similarity of the terms ‘and’ and ‘plus.’ In other words, students might confuse these terms as ab and a+b refer to the same meaning in students’ natural language.

Kieran (1981) argued that, before the introduction of algebra, children got used to mathematical situations where they could always obtain a numerical answer. Therefore, the same expectation continues for algebraic expressions. An arithmetic expression such as 4 + 5 can be considered a request to calculate the answer as 9. However, the outcome of the algebraic expression 4 + 5x could not be found if the value of x was not known. Tall and Thomas (1991) called this incorrect expectation, resulting in a numerical answer at the end of an algebraic expression, *the expected answer obstacle*. This obstacle also causes another difficulty called the *lack of closure obstacle*, by which students feel uncomfortable with an algebraic expression representing an algebraic process they cannot carry out. For example, students might consider that 5 + 7x = 12 since they thought the result should be a single term rather than an algebraic expression.

Based on the literature, students’ difficulties and misconceptions in algebra might be summarized under four topics: operations and rules in arithmetics, the concept of variable, understanding of the meaning of the equal sign, and difficulties related to mathematization and functional thinking (Erbaş, 1999; Falkner et al., 1999; Jupri et al., 2014; Kieran, 1989; Küchemann, 1978; Tall & Thomas, 1991; Treffers, 1987; Usiskin, 1988; Van den Heuvel-Panhuizen, 2003). Teachers’ recognition of students’ conceptions and misconceptions improves their abilities to make effective instructional decisions to contribute to students' thinking in algebra (Carpenter et al., 2003; Falkner et al., 1999; Stephens, 2006). Those issues regarding students' difficulties in algebra show that algebra teachers should be aware of students' learning goals and needs, make appropriate instructional decisions, and establish a classroom environment that improves students' mathematical thinking and sharing of their ideas. It depends on teachers' knowledge and beliefs about mathematics, their students, and teaching and learning to get teachers efficiently meet these demands (Stephens, 2004). The next part will investigate the studies regarding teachers' knowledge of students' algebraic thinking, difficulties, and misconceptions.

2.4.Teachers' Pedagogical Content Knowledge of Students' Algebraic Thinking, Difficulties, and Misconceptions

Shulman defined PCK as the knowledge "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (Shulman, 1986, p. 9). Teachers' knowledge of teaching mathematics is one of the essential components of effective teaching (Borko et al., 1996) as it provides significant implications for students' mathematics learning (Hill et al., 2005). Asquith et al. (2007) declared that there had been little research on MSMTs' knowledge of students' algebraic reasoning. Researchers concentrated on teachers' knowledge of students' understandings, conceptions, misconceptions, and errors as teacher knowledge models employed in various studies (Ball et al., 2008; Carrillo-Yañez et al., 2018; Kazemi & Franke, 2004). Introducing algebra to students in early grades might cause some requirements for teachers, especially in middle grades, as the transition between arithmetic and algebra is more explicit. Those requirements include developing students' algebraic reasoning and highly structured teacher knowledge and practice to construct the connections between arithmetic and algebra. The following part presents the national and international studies which addressed mathematics teachers' knowledge of students' algebraic thinking and difficulties.

Several studies have investigated the relationship between teachers' knowledge of students' thinking and achievement in algebra (Asquith et al., 2007; Baş et al., 2011;

Even & Tirosh, 1995; McCrory et al., 2012; Stephens, 2006; Şen-Zeytun et al., 2010; Tanışlı & Köse, 2013; Tirosh et al., 1998). Even and Tirosh (1995) proposed that teachers should not only express the knowledge of particular misconceptions that students might have but also why such misconceptions occur. With the help of this knowledge, they argued that teachers could anticipate an operational view of the equal sign in students' minds. Moreover, they might know why students have an operational view of the equal sign as their experiences conclude with an operational view.

Tirosh et al. (1998) studied the awareness of teachers on students' tendency to "finish" or "conjoin" open expressions. Four seventh-grade teachers participated in the study. Data were collected through lesson plans, lesson observations, and post-lesson interviews. Results showed that novice teachers were unaware of students' misconceptions, while experienced teachers were anticipating those students' tendencies. As the results suggested, novice teachers used the approach of 'collecting like terms' or applied rules and used 'fruit salad' analogies. In contrast, experienced teachers focused on identifying like terms before directly continuing to collect like terms. Also, they might use such strategies doing a substitution, considering the order of operations, and going back to get students' thoughts conflict and make students solve the task again. The researchers remarked on the requirement for unpacking teachers' knowledge of students' difficulties, errors, and misconceptions regarding their conceptions of different approaches and knowledge of the pros and cons of those approaches in future studies.

Stephens (2006) conducted a study on pre-service elementary teachers to examine their awareness of students' possible misconceptions and opportunities offered by the tasks related to equivalence and relational thinking. Thirty elementary pre-service teachers participated in the study at the beginning of the third semester in a five-semester elementary certification program. To observe the pre-service teachers' readiness, the study was implemented at the beginning of the course, which was related to teaching mathematics. Semi-structured interviews were conducted with the participants, including five tasks related to equivalence and relational thinking. The researcher clarified that teachers' knowledge of student thinking comprised teachers' awareness of several approaches students might employ while solving the tasks, from the approach of ‘the answer comes next’ to relational thinking of equal sign. Findings suggested that most pre-service teachers were aware of the purposes of tasks related to equivalence and relational thinking. Results showed that pre-service teachers tend to express computational strategies although they anticipated relational thinking solutions for some tasks. The researcher also found that pre-service teachers did not encourage strategies like relational thinking. She interpreted that the reason might be related to pre-service teachers’ viewing relational thinking as a “method” to solve a problem instead of a “way of thinking” in algebra to perceive expressions as objects and conceptualize arithmetic and algebraic expressions at an abstract level. To investigate pre-service teachers’ knowledge of students’ algebraic thinking, the researcher asked them about possible student solutions for particular tasks. Findings suggested that pre-service teachers successfully identified students’ strategies used in their solutions. The last research question was related to pre-service teachers’ knowledge of students’ misconceptions about the meaning of the equal sign. Based on the results, only 6 participants among 30 participants anticipated students’ misconceptions of operational thinking of equal sign, although it was one of the most typical misconceptions of students. Then, the researcher got pre-service teachers to analyze a student’s solution with an operational understanding of the equal sign.

Results showed that 26 participants could state that students could not understand the meaning of the equal sign, while the remaining participants associated it with students’ lack of attention. In another task, students were asked to answer, “16 + 15 = 31 is true. Is 16+15 – 9 = 31 – 9 true or false? Explain your answer.” A student responded, “False, if you minus 9 it won’t still equal 31.” (Stephens, 2006, p. 270). When teachers examine students’ answers, they cannot successfully explain the students’ misconceptions. The researcher found that only 7 participants attributed this error to students’ equal sign conception. 17 participants declared that the student ‘‘didn’t see’’ or ‘‘didn’t notice’’ the subtraction of 9 from both sides (p. 269). The researcher inferred that pre-service teachers accepted that students could use relational thinking, and they could notice such thinking in students’ work. However, she concluded that pre-service teachers were unfamiliar with the idea that students might have some misconceptions about the meaning of the equal sign. Moreover, the researcher noted that if pre-service teachers did not accept mathematical equivalence as a critical area that needs particular focus, they would not use problem-solving strategies which enhance such understanding.

Asquith et al. (2007) studied middle school students' performance on the tasks in equal sign and variable and also MSMTs' knowledge of students' understanding of these core concepts. The researchers interviewed 20 MSMTs to collect data about their predictions of students' performances in the tasks about the equal sign and variable. Moreover, they conducted algebra tasks on approximately 373 students regarding equal sign and variable. Therefore, they asked the teachers to identify how many of their students could correctly answer the tasks and to predict which strategies they might use while doing the tasks. They observed that teachers' predictions for students' responses to the items related to the variable substantially aligned with students' actual responses. However, teachers' predictions of students' performances for the tasks related to the equal sign did not correspond with students' actual performances. Researchers also asserted that teachers could rarely specify students' misconceptions regarding the equal sign and variable as an obstacle while solving problems. Although teachers were aware of students' misconceptions about the equal sign, they considered that their students did not hold such misconceptions as they had been exposed to it for many years.

Nathan and Koedinger (2000a, 2000b) investigated how teachers' beliefs about student thinking influence their instructional practices. Nathan and Koedinger (2000a) examined the views and beliefs of mathematics teachers who were teaching from 7th-grade to 12th-grade students and mathematics education researchers based on the difficulty levels of a set of algebra problems. In contrast to teachers' anticipations, students had less difficulty in solving algebra word and story problems than symbolic-equation problems. Nathan and Koedinger (2000a) found that high school teachers dominantly had a *symbol-precedence view* of student mathematical development. In other words, "arithmetic reasoning strictly precedes algebraic reasoning, and symbolic problem solving develops prior to verbal reasoning" (p. 209). According to Nathan and Koedinger, an alignment existed between teachers' views and the structure and content of algebra textbooks.

Nathan and Koedinger (2000b) conducted a similar study with the participation of elementary, middle, and high school teachers. The study concentrated on the accuracy of teachers' beliefs regarding students' performances while solving different algebra problems and possible influences that might affect teachers' instructional decisions and judgments. They concluded that teachers employ a symbol-precedence view related to students' mathematical development in which arithmetic reasoning occurs before algebraic reasoning and symbolic problem-solving precedes verbal reasoning. Peterson et al. (1989) attributed the inconsistency between teachers' beliefs and students' performance to inadequate PCK. Similarly, Nathan and Koedinger (2000b) took into account two dimensions: the professional and curricular standards for teaching mathematics and the content and structure of mathematics textbooks. Although most participant teachers held reform-based views regarding learning and teaching mathematics, they were not guided by those beliefs while making judgments on students' performance in arithmetic and algebra problems. Instead, the teachers possessed a symbol-precedence view of mathematical development far from their views on students' reasoning. Teachers aligned arithmetic problems as more complex than paired algebra problems. Nathan and Koedinger also observed that middle school teachers were most accurate in predicting students' performance in problem-solving in contrast to the view expressed by several high school teachers that symbolically presented problems were more straightforward to solve than verbally presented problems. The researchers concluded that high school teachers were least aware of their students' difficulties and were at least aligned with reform-based views.

Şen-Zeytun et al. (2010) examined the covariational reasoning abilities of mathematics teachers and their predictions regarding their students' performances. Researchers pointed out that students' covariational reasoning abilities were critical to construct and interpreting continuously changing events (Carlson et al., 2002; Kaput, 1994; Monk, 1992; Rasmussen, 2001). They investigated five mathematics teachers' covariational reasoning abilities through a model-eliciting activity, teachers' predictions about students' possible solutions for problems, and teachers' predictions of students' possible mistakes and misconceptions. Researchers used the covariation framework constructed by Carlson (1998) and Carlson et al. (2002) to describe students'covariational reasoning abilities. The results of in-service teachers' SMK showed that teachers performed poor covariational reasoning abilities and difficulties while representing and interpreting the graphs involving covariation. The results also presented that teachers viewed functions as correspondence rather than covariation. Consequently, teachers' predictions about students' reasoning abilities were also limited as their predictions were not beyond their own thoughts based on the problem. Researchers suggested that modeling activities, "though-revealing" in nature, would get teachers to develop PCK, especially for the knowledge of students' ways of thinking.

In another study, Baş et al. (2011) also investigated teachers' knowledge regarding their students' algebraic thinking and identified to what extent this knowledge reflects students' actual algebraic thinking. They conducted an activity for 49 ninth-grade students through a figural pattern generalization activity. Then, they unpacked the knowledge of three high school mathematics teachers by taking their predictions and expectations based on their students' algebraic thinking through interviews. Moreover, researchers observed how teachers' understanding of students' algebraic thinking changed after examining the students' worksheets. They concluded a considerable difference between teachers' expectations and students' algebraic thinking. As the results suggested, two teachers could predict various strategies students might use, whereas one could not foresee any strategy. Moreover, these two teachers thought their students were prone to use a variable as an unknown in a problem-solving context instead of a variable in a functional context. Conversely, the teacher who could not make accurate predictions based on students' preference for algebraic strategies considered that students could use the variable in a functional context. Researchers realized that teachers could understand students' ways of thinking after carefully examining students' worksheets. However, teachers could not notice some of the strategies students prefer while doing the task, such as the arithmetic sequence strategy that students most widely used in the study and the arithmetical thinking behind this strategy. Researchers inferred that these results might be related to teachers' lack of SMK and their tendency to interpret students' algebraic thinking from their aspects.

Similar to the study of Stephens (2006), Tanışlı and Köse (2013) investigated pre-service MSMTs' knowledge of students' algebraic thinking processes, difficulties, and misconceptions about the concepts of variable, equality, and equation. One hundred thirty fourth-year pre-service MSMTs from two different state universities participated in the study. Researchers preferred to choose the fourth-year students since they required the participants who had completed Mathematics Teaching I and II courses and got detailed instruction related to the PCK in elementary mathematics education. Researchers used a questionnaire with open-ended questions and clinical interviews to analyze pre-service teachers' knowledge of students' algebraic thinking processes, the ability to pose questions for detecting students' errors, and the ability of teachers to anticipate students' incorrect responses. In the questionnaire, one of the questions was, "Ayse is 4 cm. taller than Seda. If Seda is *n* cm. tall, how tall is Ayse?" (p. 5), and one of the 6th-grade students stated that "Ayse's height is 4n" (p. 5). When the researchers asked pre-service teachers what kind of questions they would ask this student, they concluded that a small number of participants' questions were competent enough, such as "What does 4 cm more mean? What does 4 times more mean? Does 4 more than Ayse's height equal to 4 times Ayse's height?" (p. 9). Rather, pre-service teachers mainly oriented leading questions such as "If Seda's height is n, and Ayse is 4 cm. taller than Seda, aren't we required to add 4 to Seda's height?" or concept teaching questions such as "Assume that Seda's height 101 cm. How tall is Ayse?" (p. 8). The results suggested that pre-service teachers often asked instructional questions referring to directive questions to teach the concept instead of guiding students to recognize their errors on their own. Moreover, another item asked, "In the expression 4n+7, what does the symbol n represent?" to investigate the pre-service teachers' knowledge of students' thinking processes. A student responded to the question as "n does not mean anything here because there is no symbol "=" in the expression. For example, in an expression such as 4n+7 = 11, n = 1." (p. 5). Results showed that although 62% of the pre-service teachers understood students' thinking processes, only 37% could state the reasons for students' erroneous thinking. Moreover, almost 40% of the pre-service teachers could not interpret students' thinking processes or could not explain them. Participants who could understand the student's thinking explained the student's response with statements such as "Without knowing the meaning of 'n,' he is focused on solving equations in the expressions of 4n+7 and 4n+7=11" and participants who could understand and explain the student's thinking gave statements such as "they couldn't completely understand the concept of variable" or "they didn't understand that n can represent more than one number" (p. 10). Moreover, the participants who could not understand students' thinking expressed statements such as "Since the expression 4n+7 equals nothing, there is no value of n." or "Ömer might be right. He might have considered n as a natural number."(p. 10). Researchers concluded that the pre-service teachers required an improvement in students' understanding of algebraic concepts, which presented the relationship between pre-service teachers' knowledge of students and their SMK and misconceptions. Furthermore, pre-service teachers need development in their SMK and misconceptions, preventing them from determining and explaining students' thinking processes and misconceptions.

2.5.Summary of the Literature Review

In the literature review part, theoretical frameworks were explained, and how they were used in the current study was briefly described. At first, different teacher knowledge models were reviewed based on their historical development process. The teacher knowledge model of Carrillo-Yañez et al. (2018) was employed in the current study, which was constructed based on the MKT framework of Ball et al. (2008). The focus of the study was the KFLM dimension of the mathematics teacher knowledge model of Carrillo-Yañez et al. (2018), as MSMTs' knowledge of students' algebraic thinking was investigated.

Next, the causal attribution theory of Weiner (1974, 1985) was reviewed since the attributions teachers made for students' difficulties in algebra were investigated. Also, the studies based on causal attributions and the studies investigating teachers' causal attributions for students' success and failure were reviewed. After that, the studies based on students' algebraic thinking and students' difficulties and misconceptions in algebra were included. The study by Wang and Hall (2018) presented that teachers' causal attributions may influence instructional behaviors that significantly affect students' academic performance, behavior, and motivation. Baştürk (2016) indicated that pre-service MSMTs frequently attributed students' success or failures to internal, stable, and uncontrollable factors such as innate math talent or motivation. Also, Bozkurt and Yetkin-Özdemir (2018) found that MSMTs tended to mention attributions for failure, and they mainly made controllable attributions based on the instructional decisions since they organize all the instructional processes in a lesson study.

Based on the algebraic thinking of students, firstly, the prerequisite knowledge required by students for learning algebra was examined. The prerequisite knowledge needed by students can be summarized as knowledge of arithmetic or algebraic terms (Miller & Smith, 1994), numbers (Gallardo, 2002; Kieran, 1988; Watson, 1990; Wu, 2001), proportionality (Blanton et al., 2015; Post, Behr, & Lesh, 1988), computations (Booth, 1984), equality (Falkner et al., 1999; Herscovics & Kieran, 1980; Kieran, 1981), symbolism (Behr et al., 1976, 1980; Booth, 1986; Kieran, 1992; Küchemann, 1981; Macgregor & Stacey, 1997; Watson, 1990), equation writing (Clement, Narode, & Rosnick, 1981; Wollman, 1983), representation of functions with graphics and symbolic expressions (Bottoms, 2003; Brenner et al., 1995; Markovits, Eylon, & Bruckheimer, 1988).

Elementary and middle school students' difficulties in algebra were investigated under the topics of arithmetic (Herscovics & Linchevski, 1994; Booth, 1988; Pillay et al., 1998), variable (Asquith et al., 2007; Blanton & Kaput, 2011; Blanton et al., 2017), the meaning of the equal sign (Falkner et al., 1999; Kieran, 1989; Rittle-Johnson & Alibali, 1999), mathematization (Clement, 1982; MacGregor & Stacey, 1997; Kieran, 1992; Sfard, 1991), and functional thinking (Blanton & Kaput, 2011; Carlson et al., 2002; Kaput, 1992, 1994; Rasmussen, 2001). Studies showed that most students had the conception of "finding the answer" or "doing the operation" when they were confronted with the equal sign (Blanton et al., 2011; McNeil & Alibali, 2005). Based on the concept of variable, students had several difficulties such as "letter ignored" (Küchemann, 1978, p. 25), lack of closure obstacle (Tall & Thomas, 1990), substitution (Ryan & Williams, 2007; MacGregor & Stacey, 1997), and using x as a multiplication symbol in arithmetic (Ryan & Williams, 2007).

Then, the studies examining teachers' and pre-service mathematics teachers' PCK, especially the knowledge of students' learning, difficulties, and misconceptions, were reviewed. A review of the studies showed that teachers' knowledge was limited in describing students' difficulties and misconceptions adequately in the concept of variable (Asquith et al., 2007; Tanışlı & Köse, 2013), equivalence (Asquith et al., 2007; Stephens, 2007; Tanışlı & Köse, 2013), relational thinking (Stephens, 2007), and covariational reasoning ability (Şen-Zeytun et al., 2010). Also, some studies suggested that pre-service and in-service teachers required an improvement in students' algebraic thinking, which might be related to teachers' knowledge of students' learning of algebra and their SMK (Baş et al., 2010; Tanışlı & Köse, 2013).

CHAPTER 3

METHODOLOGY

This study aimed to investigate the MSMTs' knowledge of eighth-grade students' algebraic thinking, difficulties, and errors. This chapter addresses an overview of the major design issues of the study, the characteristics of participants, research instruments, data collection and data analysis procedures, and reliability and validity issues. This chapter covers the research design and all details of the implementation of the study. In this perspective, it presents the details of the research questions, the research design of the study, the participants of the study, the context in which the study was conducted, and data collection tools and data analysis techniques used in the study. In addition, the issues of trustworthiness, the role of the researcher, and ethics issues were addressed at the end of the chapter.

3.1.Research Questions

The following research questions were investigated in this qualitative case study.

1. What is the nature of MSMTs' pedagogical content knowledge about students' understanding related to four big ideas?
    1.1. What is the prerequisite knowledge that MSMTs consider necessary to begin learning algebra?
    1.2. What do in-service MSMTs know about common conceptions and difficulties held by eighth-grade students related to four big ideas?
    1.3. What do in-service MSMTs know about the possible sources of difficulties and errors held by eighth-grade students related to four big ideas?
    1.4. What strategies do in-service MSMTs consider overcoming the difficulties held by eighth-grade students related to four big ideas?

2. To what extent MSMTs' knowledge aligned with the conceptions and difficulties of eighth-grade students in the algebra diagnostic test (ADT)?
    2.1. What are MSMTs' predictions related to the conceptions and difficulties of eighth-grade students in ADT?
    2.2. How do MSMTs' predictions compare to students' performance on algebraic thinking tasks in ADT?
    2.3. How does MSMTs' knowledge of students' learning influence their interpretations of common conceptions and difficulties of eighth-grade students in ADT?
3. How do teachers attribute the factors that impact students' performance in algebra?

3.2.Research Design

A qualitative research methodology was used in the current study to explore MSMTs' knowledge of students' understanding concerning the issues of four big ideas in algebra, EEEI, generalized arithmetic, variable, and functional thinking. "Qualitative researchers study things in their natural settings, attempting to make sense of, or interpret, phenomena in terms of the meanings people bring to them." (Denzin & Lincoln, 2005. p. 3). Brantlinger et al. (2005) expressed that "qualitative research is a systematic approach to understanding qualities, or the essential nature, of a phenomenon within a particular context" (p. 195). Therefore, it illustrated that there are multiple realities. Still, similar forms of reality shared across different groups of individuals, the qualitative research design was preferred for the current study to interpret the world using various ways (Guba & Lincoln, 1994).

In some qualitative studies, researchers may focus on some particular concepts (Corbin & Strauss, 2008) by asking "how" and "why" questions (Frankel & Wallen, 2006). At this point, interviews and observations are data collection tools that provide the researcher with vast data related to the phenomena. In this study, to explore the conceptions and difficulties of students and knowledge of teachers in terms of their student's performance based on four big ideas in algebra, the use of structured interviews, observations, and collecting and examining data through a diagnostic test would provide comprehensive and detailed information about the phenomena and the context (Yin, 2011). In this respect, a qualitative case study approach was employed to get extensive data in the natural setting of the research site. Therefore, students' conceptions and difficulties and teachers' knowledge based on their students' algebraic thinking was portrayed by employing different data collection tools. A brief explanation of the case study and its use in the current study was discussed under the following topic.

3.2.1.Case Study Research

Creswell (2007) stated that case study research is one of the types of qualitative research which aims to provide an in-depth description and analysis of *a bounded system (a case)* or *multiple bounded systems (cases)* over time. Moreover, an event, a program, an activity, or more than one individual may represent the unit of analysis by using multiple sources of information, such as observations, interviews, documents, and artifacts. Case study design provides answers to the challenging "how" and "why" questions, which may not be answered quickly by the use of empirical evidence (Stake, 1995; Yin, 2013). Also, case study design "confirm, challenge, or extend the theory" (Yin, 2009. p. 47) by stating particular cases in their natural settings and providing detailed descriptions of contexts. As Merriam (1998) identified, "The interest is in the process rather than outcomes, in the context rather than a specific variable, in discovery rather than confirmation." (p. 19). Therefore, it might be inferred that the process itself is as important as the results of a study in case study research. Also, Yin (1994) added that "A case study is an empirical inquiry that investigates a contemporary phenomenon within its real-life context, especially when the boundaries between phenomenon and context are not clearly evident." (p. 13). As shown in Figure 3.1, dotted lines between the phenomenon and the context indicate that the boundaries between the case and the context may not be sharply figured out. That is, the phenomenon cannot be considered apart from the context of the study. Influenced by the categorization of Stake (1995), Creswell (2007) identified the types of case studies in three forms: single instrumental case study, collective/multiple case study, and intrinsic case study. A researcher focuses on an issue or concern in a single instrumental case study and chooses a bounded system to illustrate the issue. An issue or problem is selected in a collective case study (or multiple case studies), but more than one single is used to demonstrate the issue. As Creswell stated, choosing the collective case study may get the inquirer to illustrate different perspectives on the issue by selecting more than one case. One of the differences between single case and multiple case studies is that multiple cases were employed to investigate the differences and similarities between cases (Stake, 1995). Moreover, the other difference is data analysis within and between different cases (Yin, 2003). Lastly, the researcher focuses on the issue in an intrinsic case study since it is unusual or unique. The intrinsic case study is preferred when the case is the focus of interest.

Merriam (1998) also categorized the case study under three topics: descriptive, interpretive, and evaluative. In a descriptive case study, the purpose is to present basic information about the phenomenon. In an interpretive case study, a substantial description is made to produce conceptual categories or strengthen the theoretical assumptions based on the phenomenon. Lastly, the evaluative case study provides a description, evaluation, and judgment of the phenomenon. The case study design was employed within the current study since the purpose was to investigate in-service MSMTs' knowledge of their 8th-grade students' algebraic thinking, difficulties, and errors in four big ideas in algebra. Since the current study aims to gain an in-depth understanding of the issue based on the selected case, it is characterized by the case study definitions of Creswell (2007) and Merriam (1998). Based on the categorization of Creswell (2007) and Merriam (1998), this study can be identified as a collective and interpretive case study since it includes two cases, 8th-grade students and their teachers in a public school, and since the primary purpose of the study is producing theoretical assumptions based on the judgments on the findings of the current study.

Yin (2003) also categorized case studies under four topics: single-case holistic, multiple-case holistic designs, single-case embedded, and multiple-case embedded designs. Whether a study is a single-case or multiple-case study is indicated by the number of cases in the research, and whether it is an embedded or holistic study is identified by the number of units of analysis. The model for the single-case embedded design is presented in Figure 3.1.

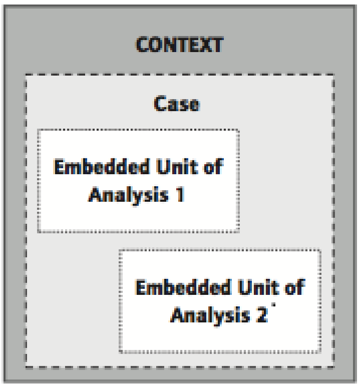


Figure 3. 1. Single-case embedded (multiple units of analysis) design (Yin, 2003, p. 63)

In the current study, a single-case embedded design was employed (Yin, 2003). The units of analysis were in-service MSMTs and 8th-grade students in a public school. The unit of analysis was MSMTs' knowledge based on 8th-grade students' algebraic thinking, difficulties, and errors (See Figure 3.2).

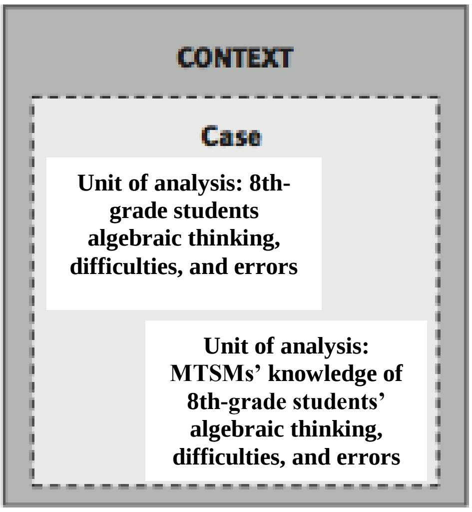


Figure 3. 2. Single-case embedded design of the study

3.2.2.Context of the Study

Baxter and Jack (2008) highlighted the importance of defining the context of the study in a case study. As the main focus of the current research is MSMTs, providing brief information about the school where the teachers work will help capture the setting of the study. The study was conducted in a public middle school known as the most successful middle school in the district, which was from the Western Black Sea Region in Turkey. There are approximately two thousand and five hundred students in the middle school, and six hundred and twenty of them are 8th-grade students. There are sixteen sections for 8th graders, and each classroom includes 40 or 41 students. That is, classrooms were very crowded in comparison to the classrooms of other schools in that district. Since the school is known as the most successful school in the city, parents are willing to get their students to be graduated from that school. However, this school's mathematics teachers said its success might be related to its crowdedness. Since there are more students, there are more successful students compared to the other schools. Moreover, the student's success in national examinations was very crucial for the school administrators. For this reason, several school-wide tests were conducted on 8th-grade students to rank and make them see how successful they are. In addition, there was a measurement and assessment center in the school, and the school's teachers prepared the school-wide examinations. The school administrators were helpful and cooperative with the educational research studies in that school.

3.2.3.Selection of the Participants

In the selection process of participants, the purpose of the study has a crucial role. If the goal is to generalize based on the gathered data, the appropriate sampling method would be probability sampling. Conversely, if a researcher's purpose is not to generalize the data, non-probabilistic sampling would be an appropriate sampling method (Merriam, 1998). Since the purpose of the current study is the investigation of 8th-grade students' algebraic thinking, difficulties, and errors and MSMTs' knowledge regarding students' conceptions and difficulties related to the big ideas in algebra, instead of making generalizations based on the results, a non-probabilistic sampling method will be conducted. Merriam (1998) expressed that criteria are essential when choosing study participants. Since the variable concept and functions were initially taught in the middle grades based on the middle school mathematics curriculum (MoNE, 2018), middle grades constituted a transition from arithmetic reasoning in elementary grades to complex algebraic reasoning in high school. Thus, middle-grade students were preferred for the study. 8th-grade students were selected for the study to have an image of the students' algebraic thinking and difficulties who are about to complete the middle school algebra instruction and continue with the high school algebra.

Since the researcher tried to get the data richer and wanted easy access to the school, the purposive sampling method was employed for the study. There were many students and teachers in the selected school compared to other public middle schools in the city. Therefore, it was chosen to get a considerable amount of data to discuss the problem in a broader context. Since there were several classroom observations and interviews throughout the study's data collection process, a central public middle school close to the city center was preferred for the study. Moreover, criterion sampling was preferred within purposive sampling to select participant MSMTs in the public school. As the study would investigate MSMTs' knowledge of 8th-grade students' algebraic thinking, only the MSMTs who teach 8th graders were invited to the study. After MSMTs had agreed to participate in the study, all 8th-grade students of participant MSMTs were invited to the study. Therefore, the study was conducted in a large, crowded public middle school in Western Black Sea Region in Turkey.

Table 3. 1. Summary of participant MSMTs' key qualifications

	Mr. Gürsoy	**Ms. Burcu**	**Mr. Yüce**	**Mr. Öner**	**Mr. Ferhan**
Level of education	Bachelor's degree	Bachelor's degree	Bachelor's degree	Bachelor's degree	Bachelor's degree
Graduated program	Elementary mathematics education	Mathematics	Mathematics	Elementary mathematics education	Elementary mathematics education
Teacher certification program	---	✓	✓	---	---
Professional teaching experience	7 years	19 years	25 years	18 years	11 years

As a result, five MSMTs and 620 eighth-grade students participated in the study in a public middle school in the Western Black Sea Region in Turkey. Lastly, pilot studies were conducted in two of the other public schools in the district. To overcome threats to validity, the school where the main research was conducted was different from the schools in which pilot studies were conducted. The participant eighth-grade students and MSMTs were introduced in the next part (See Table 3.1).

3.2.3.1. Mr. Gürsoy

Mr. Gürsoy has been teaching as an MSMT for seven years. He has a bachelor's degree in the department of elementary mathematics education from a faculty of education. He has been teaching eighth graders for several years. He is interested in the higher-level thinking skills of high-achiever students. In semi-structured interviews, he noted that he usually performed his teaching based on developing higher-level thinking skills of high-achiever students. Informal classroom observations conducted by the researcher showed that he often uses questions and problems requiring reasoning and analysis skills in his algebra classes. For this reason, I thought his courses addressed high-achiever students rather than students from all performance levels. In addition, he has strong acting and rhetoric skills in the classroom while teaching mathematics and uses his voice professionally. He defined algebra as mathematical operations including at least one unknown. As he stated, he first emphasized the meaning of the unknown to his students repeatedly; then explained what algebra is in his algebra classes.

3.2.3.2. Ms. Burcu

Ms. Burcu has taught as a middle school mathematics teacher for 19 years. She has a bachelor's degree in mathematics from the faculty of arts and sciences. She has been teaching eighth graders for several years. In semi-structured interviews, she said she loves struggling with advanced-level mathematical questions from high school or beyond. However, she had taught middle school students since she was appointed to middle schools when she began her career. During the informal observations, she highlighted the importance of conceptual learning of the concepts rather than rote learning of the rules in mathematics. She focused on doing more practice in her algebra classes. She defined algebra as identifying information using unknown and identifying a statement using unknown, addition, subtraction, multiplication, and division. She also described algebra as ‘expressions with unknowns.’

3.2.3.3. Mr. Yüce

Mr. Yüce has been teaching as a middle school mathematics teacher for 25 years. He has a bachelor’s degree in mathematics from a faculty of arts and sciences. He also took some educational courses for teacher certification during his university years. He has been teaching eighth graders for several years. He thinks daily life examples should be included in mathematics education to make concepts more concrete and meaningful. He also highlights the importance of using manipulatives while teaching mathematics. Moreover, he likes to use his gestures and mimics while teaching particular mathematical concepts, which gets all students to participate actively in the classroom. During the observation process, he prepared his algebra classes based on the content and examples in the mathematics curriculum. He defined algebra by using the definition of the mathematics textbooks: algebraic expressions including a letter and operations (addition, subtraction, multiplication, and division). Also, he emphasized that he tries to make algebra more concrete for students (e.g., explaining the meaning of x + 2 by saying that x + 2 is a box always giving two more than x).

3.2.3.4. Mr. Öner

Mr. Öner has been a middle school mathematics teacher for 18 years. He has a bachelor’s degree in the department of mathematics education from a faculty of education. He is a vice-principle in the school; therefore, he is busy with administrative tasks. He has taught eighth-graders for several years. For this reason, he teaches mathematics in just one class in the school. Based on informal classroom observations, it might be inferred that he prepared his algebra classes based on the content and examples in the mathematics curriculum. He first explains the topic to the students and practices on the smartboard by using the textbook used in that school. He defined algebra as the transition from concrete to abstract *(concepts)* and interpreting tangible things by our comprehension. He underlined beginning to teach algebra by using concrete objects (e.g., apple, orange) and making a transition among a concrete object, a box, and a letter, respectively, to represent a quantity.

3.2.3.5. Ms. Ferhan

Ms. Ferhan has taught as a middle school mathematics teacher for 11 years. She has a bachelor’s degree in the department of elementary mathematics education from a faculty of education, and she plans to have a master’s degree in mathematics education in the future. She has taught seventh and eighth graders for several years. In the interview, she stated that she was interested in students' peer learning and conducting mathematical activities in the classroom. Although she preferred mainly doing mathematical activities while teaching topics, she talked about not being able to do it anymore as there was an examination system and various issues they should have completed. She also attached great importance to making definitions in her classes. She defined algebra as “understanding the relationship between numbers, making the reasoning, and solving problems.” She expressed that all sciences need algebra. She emphasized teaching algebra with the help of mathematics in nature (e.g., fractals).

3.3.Data collection

The study aimed to investigate the knowledge of in-service MSMTs based on 8th-grade students’ algebraic thinking and difficulties related to four big ideas in algebra. The first part of the study was searching for the performance of 8th-grade students on particular algebra tasks. The second part investigated MSMTs’ knowledge based on students’ performances and difficulties with four big ideas in algebra. Therefore, the data collection procedure was divided into two phases: investigating students’ algebraic thinking and difficulties and investigating MSMTs' knowledge based on students’ algebraic thinking and difficulties with four big ideas in algebra. The following data collection tools were used to achieve the study's purpose: classroom observation, semi-structured interview, algebra diagnostic test (ADT), and a teacher questionnaire based on ADT.

3.3.1.Informal Classroom Observations

Merriam (1998) stated, “observational data represents a firsthand encounter with the phenomenon of interest rather than a secondhand account of the world obtained in the interview” (p. 94). That is, observations allow researchers to observe the phenomena in their natural settings. Therefore, the information gathered from the observations makes the picture more explicit and complete. For this reason, observations are one of the most critical data sources of qualitative studies. At the beginning of the study, the researcher made observations in algebra classes of three MSMTs to observe students’ algebraic thinking and difficulties throughout the algebra topics in three 7th-grade classrooms. After the test was developed, the researcher observed each MSMTs’ algebra classes at the beginning of the main study to have general information about how MSMTs taught algebraic concepts. Thus, the observations before the ADT were conducted to identify the main characteristics of participant MSMTs. Since there were several informal observations in the data collection procedure, it was impossible to follow the course schedules of several classrooms. Therefore, three classes were selected since each class had seven mathematics courses weekly, and there would be overlaps in the number of courses for more than three classrooms. Therefore, three classrooms were selected based on convenient sampling for informal classroom observations. A different teacher taught each classroom; therefore, informal observations of the sessions of the three teachers were done. Also, the classes of the remaining two teachers were observed outside of these three classes’ schedules.

3.3.2.Teacher Questionnaire based on ADT

A questionnaire was prepared based on Algebra Diagnostic Test (ADT) to investigate the conceptions of MSMTs. There were two or three sub-questions for each item in Algebra diagnostic test for teachers. The questionnaire aimed to examine MSMTs' knowledge of typical correct answers to students’ algebraic thinking and probable difficulties and errors. Moreover, teachers' predictions based on students' actual performances were also examined in the questionnaire. One of the aims of collecting data through the questionnaire was to give the participant teachers more time to think about the items of ADT and the probable answers of students. With the help of the questionnaire, it was considered that the interviews would provide more effective and generous data. The Turkish version of the questionnaire can be seen in Appendix D.

3.3.3.Semi-structured Interviews with Teachers

Interviews are one of the most critical data sources in case study research (Yin, 2003). Questionnaires are valuable tools for collecting data; however, interviews provide more detailed and deeper data for a researcher. Moreover, although observations provide generous information, interviews might give information that may not be observed, such as thoughts, intentions, and feelings (Merriam, 1998). Interviews are categorized by Merriam (1998) under three topics, namely highly structured, semi-structured, and unstructured. In highly-structured interviews, predetermined items are posed in a structured order. In semi-structured interviews, there are predetermined items, including "how" and "why" questions. Moreover, those questions can be completed using follow-up questions in an unstructured order. In an unstructured interview, the researcher uses questions to collect data about an issue without making a plan and may use the data for subsequent interviews. The interview questions before the ADT was conducted were presented in Table 3.2.

Table 3. 2. Interview questions before the ADT was conducted

1. Please identify the prerequisite knowledge that students should have prior to learning algebra.
    - Do you think your students have adequate knowledge of these subjects before moving on to algebra?
2. What resources do you use when explaining the subject of algebra? How do you use these resources?
3. At which points do students have difficulty in algebra based on your experiences?
    - How do you identify the points where students have difficulty?
    - Is there any method you use to overcome these difficulties in lecturing and problem-solving?

Table 3.2 (continued)

4. Based on your experiences, what errors do students usually make in algebra?

Is there a method you use to identify the misconceptions that students may have and the errors that students might make?

5. What would be a typical example given by students for each question in the test?
6. What questions do you think your students will answer correctly or unusually? (Please indicate the percentage of students who can answer correctly for each question.)
7. Are there any questions in this test that your students might find difficult or give incorrect answers to?
   - If your answer is yes, which questions in the test and which parts of these questions might be difficult for your students?
   - Why do you think that your students might have difficulty?
   - At which points can your students make errors in the items? (Please indicate the percentage of students who can answer the questions incorrectly for each question.)
8. What might be the causes of these errors? Please explain each item.

**The Turkish version of the interview form is given in Appendix E.*

The semi-structured interview was preferred in the current study since it allows a researcher to collect the data by asking follow-up questions if required. The questions in the semi-structured interview were prepared based on the literature (Ball et al., 2008; Carrillo-Yañez et al., 2018; Magnusson et al., 1999) to examine the knowledge of five MSMTs on 8th-grade students' algebraic thinking and difficulties based on four big ideas. The interviews were conducted with five MSMTs before and after the ADT was conducted. Table 3.3. presented the questions included in the interview, conducted after the main ADT was conducted on students.

Table 3. 3. Interview questions after the ADT was conducted

1. The results of the content analysis of the responses given by the students to the items in ADT are as follows (analysis results are shown to the MSMTs). • What do you think about these results?

Table 3.3 (continued)

• What might be the reasons for these errors presented in the analysis results of ADT? (Specific questions can also be asked, such as showing the results of the 3rd question. About … percent of the students correctly responded to Item 3; what might be the reason for their struggle?) • What might be the origin of these errors? Please explain 2. Is there a question you think most students will answer correctly but that results in the opposite? If yes, contrary to your opinion, what might be the reason for the majority of incorrect answers given by the students to this item? 3. Do you have any suggestions to eliminate these difficulties and errors observed in the students' responses? If so, what are they? Please explain each item. • Do you observe these difficulties and errors in your algebra classes? 4. If yes, do you have any suggestions to overcome these difficulties and errors? 5. Can you apply the suggestions you have stated in your class when you need them? • If yes, please describe how you implemented these recommendations. If not, please explain your reasons for not being able to apply.

**The Turkish version of the interview form is given in Appendix F.*

The interviews' total duration for each participant was between 40 minutes to 60 minutes. The Turkish version of the semi-structured interview forms can be found in Appendix E and F.

3.3.4.Algebra Diagnostic Test (ADT)

The researcher prepared an Algebra Diagnostic Test (ADT) to explore 8th-grade students' algebraic thinking, difficulties, and errors in EEEI, generalized arithmetic, variable, and functional thinking tasks (See Appendix C). The the preparation process for the test is presented in the following section.

3.3.4.1. Informal Classroom Observations for Preparation of ADT

At the beginning of the first pilot study, the researcher did informal classroom observations during the algebra classes. While making observations, the researcher took notes concerning the course content, the questions posed by students and by teachers to students, and the teaching of mathematics teachers. Informal observations lasted for a month throughout the algebra topics in three classrooms in a public middle school throughout December 2017. These observations got the researcher to investigate students' algebraic thinking, what type of difficulties students had, and which methods teachers used to teach algebra. Although informal observations could not answer those questions totally, the literature review and informal observation of an actual classroom environment gave the researcher some clues about the points that should be focused on in this study, especially for the development of the ADT.

3.3.4.2. Construction of ADT

A diagnostic test was developed to see what students knew and which type of difficulties they had based on the algebraic topics. The initial step was the identification of the test specifications while preparing ADT.

3.3.4.3. Preparation of Test Specifications

To draw the boundaries of the test, specific criteria were investigated. First, the researcher researched broad literature to decide which topics should be studied in middle school algebra. As the studies in the literature suggested, there were highly discrete topics for learners, namely the notions of unknown, variable, equality, and functional thinking. Moreover, classroom observations were done throughout the topic, and unstructured interviews with in-service MSMTs were done to identify the content of the test. As the classroom observations and unstructured interviews suggested, students struggled to solve problems by using equations, drawing the graphics of linear equations, transforming equations and graphs back and forth, and constructing the algebraic expression of a linear relationship. The observations and interviews suggested that middle school students struggled with specific topics in algebra. Therefore, the focus of the study was chosen to investigate students' algebraic thinking and difficulties with the notions of four big ideas in algebra. Moreover, the meaning of the equal sign and the notions of equivalence and equation were also considered since a concrete understanding of those concepts constitutes the structure of algebraic thinking (English & Sharry, 1996; Sfard & Linchevski, 1994. Stacey & Macgregor, 2000). Therefore, the study has been shaped around four big ideas in algebra.

3.3.4.4. Development of ADT

One of the aims of this study was to investigate 8^{th}-grade students' algebraic thinking and difficulties with particular concepts in algebra. For this reason, a diagnostic test was required to investigate students' difficulties in detail. During the test construction procedure, the studies of Blanton et al. (2015) and Kaput (2008) were considered. In the study of Blanton et al. (2015), *five big ideas* were identified based on the content strands of Kaput (2008) and literature on early algebra (Blanton et al., 2011; Carraher & Schliemann, 2007). Those five big ideas are EEEI; generalized arithmetic; functional thinking; and variable; and proportional reasoning (See Table 3.4). This study focused on the first four big ideas, EEEI, generalized arithmetic, functional thinking, and variable. Since proportional thinking has enormous scope in mathematics education, this study mainly focused on the first four big ideas. While preparing the ADT, items from each big idea were tried to be included in the test. Therefore, the researcher prepared the test considering the four big ideas: EEEI; generalized arithmetic; functional thinking; and variable.

The primary purpose of ADT was to diagnose students' algebraic thinking and difficulties with particular concepts instead of measuring students' knowledge of algebra throughout the middle school algebra curriculum. Therefore, the items did not include all the algebra objectives in the middle school mathematics curriculum. A summary of algebra objectives addressed for 6th, 7th, and 8th grade in the middle school mathematics curriculum is given in Table 3.5 (MoNE, 2017, p. 65-77).

Table 3. 4. Brief explanations of big ideas integrated in ADT (Blanton et al., 2015, p. 43)

Big Idea	Description of the big idea
• Equivalence, expressions, equations, and inequalities (EEEI)	Includes developing a relational understanding of the equal sign, representing and reasoning with expressions and equations in their symbolic form, and describing relationships between and among generalized quantities that may or may not be equivalent.
• Generalized arithmetic	Involves generalizing arithmetic relationships, including fundamental properties of number and operation (e.g., the commutative property of addition), and reasoning about the structure of arithmetic expressions rather than their computational value.
• Functional thinking	Involves generalizing relationships between covarying quantities and representing and reasoning with those relationships through natural language, algebraic (symbolic) notation, tables, and graphs
• Variable	Refers to symbolic notation as a linguistic tool for representing mathematical ideas in succinct ways and includes the different roles variable plays in different mathematical contexts.

Some topics were not included in the test, such as inequality, identities, and slope. Since these topics were beyond the focus of the study, they were not included in the test. As inequality was not included in the study, this study investigated the first big idea, equivalence, expressions, and equations. There were no algebra objectives in the 5th-grade mathematics curriculum. Thus, the algebra objectives from the 6th, 7th, and 8th-grade mathematics curriculum are illustrated in Table 3.5. After deciding on the test's specifications, the ADT was developed in four steps. The first step was preparing an item pool for the test. Then, the second step was preparing the draft version of the pilot ADT and conducting it on students to do possible revisions. The third step was making required revisions on the pilot ADT after analyses of the data and conducting it on students again. The final phase of the test development was conducting analyses of the data, making required revisions to the pilot ADT, and acquiring the final version of the ADT.

Therefore, the steps can be summarized as follows:

1) Item development and preparation of an item pool

2) Development of pilot ADT

3) Conducting the pilot ADT on students

4) Analyses of data collected through pilot ADT

Table 3. 5. A summary of algebra objectives addressed for 6th, 7th, and 8th grade in the middle school mathematics curriculum (MoNE, 2017, p. 65-77)

6th grade	**M.6.2.1.Algebraic expressions** M. 6.2.1.1. Students will be able to write the algebraic expression for given verbal statements, and students will be able to write the verbal statements for given algebraic expressions. M. 6.2.1.3. Students will be able to explain the meaning of simple algebraic expressions.
7th grade	**M.7.2.1.Algebraic expressions** M. 7.2.1.1. Students will be able to do addition and subtraction with algebraic expressions. M. 7.2.1.2. Students will be able to multiply a natural number with an algebraic expression. M. 7.2.1.3. Students will be able to identify the rule of number patterns with symbols; they will be able to identify the term of the pattern whose rule is given with a symbol. **M.7.2.2.Equality and equation** M.7.2.2.1. Students will be able to understand the conservation of equality in equations. M.7.2.2.2. Students will be able to identify the first-degree equations and write first-degree equations for real-world situations. M.7.2.2.3. Students will be able to solve first-degree equations with one unknown. M.7.2.2.4. Students will be able to solve the problems which require writing first-degree equations with one unknown.

Table 3.5. (continued)

M.8.2.2. Linear Equations
M.8.2.2.1. Students will be able to solve first-degree equations with one unknown.
M.8.2.2.2. Students will be able to recognize the characteristics of the coordinate axis and show ordered pairs.
M.8.2.2.3. Students will be able to identify how two variables with a linear relationship covariate by table, graphic, and equation.
M.8.2.2.4. Students will be able to construct the graphics of linear equations.
M.8.2.2.5. Students will be able to construct equations, tables, and graphics for real-world situations, including linear relationships.

In the item pool preparation process, a broad research literature was done to develop appropriate items for the test (Açıl, 2015; Asquith et al., 2007; Benneth, 2015; Blanton et al., 2015; Brown et al., 1984; Clement, 1982; Falkner et al., 1999; Fujii & Stephens, 2008; Kendal& Stacey, 2004; Mullis et al., 2004; Stacey & MacGregor, 2000; Wagner & Parker, 1993; Yaman, 2004). With the help of the literature research, informal classroom observations, and unstructured interviews with the teachers, an item pool including 30 items was prepared. The items in the pool were in the form of open-ended, multiple-choice, and true-false items. Since the study focused on investigating students' conceptions and difficulties in certain algebraic concepts, open-ended questions were preferred for the study to get detailed information from students. Therefore, the number of items should not be so much because of the consideration of the time issue. Thus, the researcher and three mathematics education professionals conducted an item selection process. In this process, the purpose, the content, and the structure of the items in the pool were considered. At the end of this process, ten open-ended items were included in the draft version of the ADT (See Appendix E).

The development of the pilot ADT was comprised of four phases. Firstly, each item in the pool was investigated based on objectives within the middle school mathematics curriculum (MoNE, 2017, p. 65-77) and big ideas (Blanton et al., 2015), namely EEEI, generalized arithmetic, functional thinking, and variable (Blanton et al., 2015). Therefore, the aim was to balance the distribution of items concerning these two perspectives, namely big ideas and curriculum objectives (See Table 3.6).

Table 3. 6. Categorization of an item in the item pool based on objectives and big ideas [The item was adapted from Açıl (2015)]

Objective	M. 6.2.1.1 Students will be able to write the algebraic expression for given verbal statements, and students will be able to write the verbal statements for given algebraic expressions. M. 6.2.1.3 Students will be able to explain the meaning of simple algebraic expressions.
Big Idea	EEEI
What is the algebraic expression of the statement "The time remaining after x minutes in a 50 minutes examination"?	

Some questions were removed from the test since similar items measured the same objectives and big ideas. One of the items excluded from the test was the following item related to the big idea of functional thinking (See Table 3.7).

Table 3. 7. An item removed from the item pool (Stacey & MacGregor, 2000, p. 143)

Objective	M. 7.2.1.3 Students will be able to identify the rule of number patterns with symbols; they will be able to identify the term of the pattern whose rule is given with a symbol. M.8.2.2.3. Students will be able to identify how two variables with a linear relationship are changing concerning each other by table, graphic, and equation.
Big Idea	Functional thinking, proportional reasoning

a) The results of an experiment that measured two quantities, L and Q, were:

L	**Q**
3	9
5	15
9	27
21	63

b) What would you expect Q to be when L is 30?

c) What would L be when Q is 99?

d) Describe how you would find Q if you were told what L is.

e) Use algebra to write a rule connecting L and Q.

Therefore, there were ten items in the draft version of Pilot Test 1 after the items were removed. Then, a mathematician, three mathematics education professionals, and four MSMTs evaluated the draft version of the pilot test. Lastly, the final version of Pilot Test 1 was constructed by making some revisions to the expression of the items and the use of terms and symbols in some questions.

3.4.Implementation of the Study

There were six main phases of this study. The summary of the test administration processes is given in Table 3.8. The first phase was the development of the algebra diagnostic test. Then, the second phase was pilot testing I and II.

Table 3. 8. The summary of the study's administration process

Round	**Time Interval**	**Participants**	**Purpose**	**Instruments**
Pilot Test I	March 2018	140 seventh-grade students	Pilot testing (Revision)	Ten item-version of the pilot test
Pilot Test II	May 2018	136 seventh-grade students	Pilot testing (Revision)	Eleven item-version of the revised pilot test
Questionnaire	February 2019	Five mathematics teachers	Surveying on ADT	A survey based on ADT
Interviews before ADT	February 2019	Five mathematics teachers	Doing pre-interview	Semistructured interview before ADT
Main Field Testing (ADT)	March 2019	620 eight-grade students	Final testing	Seventeen item-version of ADT
Interviews after ADT	September 2019-March 2020	Five mathematics teachers	Doing post-interview	Semistructured interview after ADT

the third phase was conducting interviews with teachers before the main field testing, the fourth one was the administration of the main ADT, the last phase was conducting interviews with teachers after the field testing.

3.4.1.Administration of the Pilot Test I

Pilot Test I was conducted on 140 seventh-grade students in a public middle school in the Western Black Sea Region in the spring semester of the 2017-2018 academic year. Students from five classrooms took the test and had 40 minutes (a class hour) to answer the items. They were given additional time (10 minutes) at the end of 40 minutes if they required more time. The test could not be conducted on all seventh graders simultaneously because half of the seventh graders went to school in the morning and the other in the afternoon. Therefore, two sections were taken on the test before, and the remaining three were taken in sequential two days on March 22-23, 2018. During the test, the researcher walked around the students and asked whether the items were understood. If they asked the researcher to explain the items, the researcher tried to explain what the question was asking.

> Could you illustrate the truthiness of the following equality without calculating the results of the multiplications 7×22 ve 14×11? Please give a brief explanation by stating the reasons.
>
> $$7 \times 22 = 14 \times 11$$

Figure 3. 3. Item 1 in Pilot Test I [Adapted from Benneth (2015)]

Moreover, the researcher tried to make the same explanations in each classroom to prevent potential differences that might have occurred. I realized that the first question was not understood by most of the students (See Figure 3.3). For this reason, I explain what is required for the solution of this item in each classroom. After the test was conducted on students, the researcher made some revisions, such as removing some of the items that did not work efficiently and revising the statements of the items to improve students' understanding. The revision process of Pilot Test I is described in the following part.

3.4.2.Revision of Pilot Test I

Content analysis was used to analyze students' answers to ADT to investigate which type of solutions students gave and explore the codes created from students' responses. Therefore, students' responses were analyzed, and codes for each question were listed with frequency and ratio information for each code, as given in Table 3.9 for the first question.

Table 3. 9. Content analysis of Item 1 in Pilot Test I

Response		**F***	**R(%)***
We can show the equivalence	"They are multiple of each other" or "we can do factorization" responses	15	10.71
	"One of them becomes double while the other becomes half."	14	10
	"22 is double of 11; 14 is double of 7"	49	35
	Inadequate explanations	10	7.14
Total		**88**	**62.85**
We cannot show the equivalence	"We should do the calculation."	7	5
	Just the response "no"	10	7.14
Total		**17**	**13.14**
Unspecified responses	Missing answer	21	15
	Incomprehensible and erroneous answers	8	10
Total		**35**	**25**
General Total		**140**	**100**

**F: Frequency, R: Ratio*

The analysis results related to Item 10 in Pilot Test 1 showed that the number of correct responses given by students was low. Thus, Item 10 was revised by decreasing the number of sub-questions within the item. Since there were six sub-categories in Item 10, it was long-chained. It was realized that most students did not answer the last sub-questions in the item, which might be caused by its taking too much time. Therefore, three sub-categories (c, e, and f) of Item 10 were eliminated since most students did not answer. The results of the analysis of Item 10a in Pilot Test I can be seen in Table 3.10.

Table 3. 10. Content analysis of Item 10a in Pilot Test I

10) A group of friends decided to join an organized trip from Zonguldak to Muğla, and they drove the 800 km road at a constant speed of 100 km per hour.

a) Fill in the below table showing the distance traveled at the end of each hour during the journey.

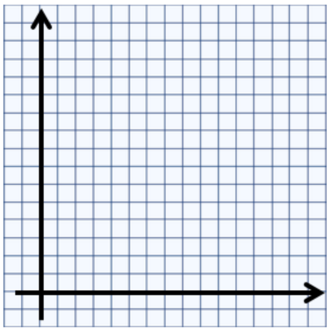
Time elapsed (hour)	Distance (km)

Code	**Frequency**	**Ratio (%)**
Correct	85	60.71
Partially correct (incomplete correct responses)	18	12.86
Incorrect	18	12.86
Missing	19	13.57
Total	**140**	**100**

In Item 10b, students were asked to draw a graphic based on the information in the table given in Item 10a (See Table 3.11).

Table 3. 11. Content analysis of Item 10b in Pilot Test I

b) According to the information in the table above, draw the graph on the coordinate plane below, with the x-axis showing the elapsed time (hours) and the y-axis showing the distance traveled (km).



Code	Frequency	Ratio (%)
Correct responses	52	37.14
Partially correct responses	33	23.57
Showing with bar graph (incorrect response)	8	5.71
Incorrect responses	9	6.42
Missing responses	38	27.14
Total	**140**	**100**

Item 10c was related to making students reasoning for drawing the graphic, as presented in Table 3.12.

Table 3. 12. Content analysis of Item 10c in Pilot Test I

c) Is it correct to connect the points on the graph on the same line? Why?

Code	Frequency	Ratio (%)
Correct responses	30	21.43
Partially correct responses	38	27.14
Incorrect responses	17	12.14
Just “yes” responses	25	17.86
Unspecified responses	2	1.43
Missing responses	28	20
Total	**140**	**100**

As illustrated in Table 3.13, Item 10d was related to writing the general rule based on the given problem in which x represented the distance and t represented the time.

Table 3. 13. Content analysis of Item 10d in Pilot Test I

d) Write the general rule allowing you to find the x values based on the t values.		
Code	**Frequency**	**Ratio (%)**
Correct responses	17	12.14
Partially correct responses	13	9.29
Incorrect responses	44	31.42
Unspecified responses	7	5
Missing responses	59	42.14
Total	**140**	**100**

As presented in Table 3.14, Item 10e was related to constructing the general rule based on the given problem by changing the symbols so that t represented the distance and x represented the time.

Table 3. 14. Content analysis of Item 10e in Pilot Test I

e) Write the general rule allowing you to find the t values based on the x values.		
Code	**Frequency**	**Ratio (%)**
Correct responses	13	9.29
Partially correct responses	3	2.14
Incorrect responses	50	35.71
x=100t instead of t=100x	8	5.71
Missing responses	66	47.14
Total	**140**	**100**

In Item 10f, the distance traveled for a particular instant was asked, as shown in Table 3.15. As the frequency of the missing responses dramatically increased in Item 10d, 10e, and 10f, those three sub-questions were removed from Item 10. The remaining items were also revised with minor changes in their statements to make them more explicit and comprehensive. Analysis of students' responses in Pilot Test I was presented in Appendix A. The preparation procedure for Pilot Test II was explained briefly in the following section.

Table 3. 15. Content analysis of Item 10f in Pilot Test I

f) Find the distance traveled in $\frac{3}{5}$ hours using the equation you wrote.

Code	Frequency	Ratio (%)
Correct responses	24	17.14
Correct but incomprehensible responses	3	2.13
Partially correct responses	2	1.43
Incorrect responses	027	19.29
Missing responses	84	60
Total	**140**	**100**

3.4.3.Administration of the Pilot Test II

One hundred thirty-six seventh-grade students participated in the administration process of Pilot Test II in another public middle school in the same district in the spring semester of the 2017-2018 academic year. Students from 5 classrooms took the test and had 40 minutes (a class hour) to answer the items. They were given additional time (10 minutes) at the end of 40 minutes if they required more time. The test was conducted on the students simultaneously in contrast to the application of Pilot Test I. The students took the test on May 24, 2018. During the test, the researcher walked around the students and asked whether the items were understood. The students posed fewer questions compared to the administration process of Pilot Test I. The researcher tried to make the same explanations in each classroom to prevent potential differences that might have occurred because of the explanations. I realized that the first question was still not understood by some of the students (See Figure 3.4). Thus, I again explained what is required to solve this item in each classroom. I thought Item 1 was a non-routine question for them; therefore, they might have struggled to understand it. Pilot Test II was given in Appendix B. The analysis of students' responses to some of the items in Pilot Test II is described in the next part in detail.

3.4.4.Item Analysis on Pilot Test II

The students' responses in Pilot Test II were analyzed, and scores for each student were produced to analyze the items in Pilot Test II. Therefore, students’ responses were labeled as 1 if correct and 0 if incorrect to get a total score for each student. As a result, the results of item analysis for 136 students were given in Table 3.16 *(M = 5.05. SD = 4.68)*. Based on the analysis, it could be concluded that Item 3, Item 4b, and Item 5 were the most difficult items in the test. Moreover, since the item discrimination values of all items were above 0.25, and most of them were above 0.40, it can be concluded that all items provided the criterion for item discrimination. Using the scores of students in algebra examinations in school showed a statistically significant correlation between their scores in algebra tests in the school and the Pilot Test II *(r(134) = .68, p = 0.05)*. Since the coefficient value was between 0.6 and 0.8, it could be inferred that there is a positive and high correlation between those two scores. Also, it might be assumed that the reliability of Pilot Test II was also high.

Table 3. 16. The results of item analysis in Pilot Test II

Item No	**Maximum points for the items**	**Item Difficulty**	**Item Discrimination**
Item 1	1	0.39	0.51
Item 2	1	0.62	0.56
Item 3	1	**0.09**	0.45
Item 4 (sub-category a)	1	0.39	0.66
Item 4 (sub-category b)	1	**0.18**	0.67
Item 5	1	**0.18**	0.46
Item 6	1	0.35	0.53
Item 7 (sub-category a)	1	0.47	0.64
Item 7 (sub-category b)	1	0.42	0.66
Item 8 (sub-category a)	1	0.22	0.56
Item 8 (sub-category b)	1	0.67	0.45
Item 9 (sub-category a)	1	0.34	0.68
Item 9 (sub-category b)	1	0.40	0.73

Table 3.16 (continued)

Item 10 (sub-category a)	1	0.55	0.27
Item 10 (sub-category b)	1	0.41	0.34
Item 10 (sub-category c)	1	0.21	0.63
Item 11 (sub-category a)	1	0.41	0.47
Item 11 (sub-category b)	1	0.30	0.55
Item 11 (sub-category c)	1	0.32	0.66
Maximum Total Score	19		

3.4.5.Revision of Pilot Test II

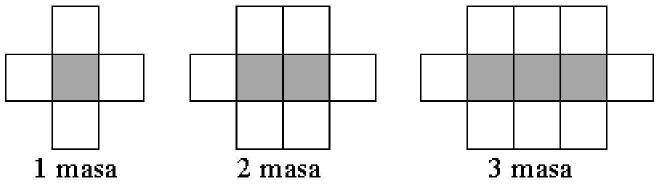
Students' responses to Pilot Test II were analyzed using content analysis to explore which type of solutions students gave and to create the codes. The students' responses were analyzed, and codes for each question were listed with frequency and percent information, as shown in Table 3.17 for Item 7. Since most students gave incorrect responses or left Item 7 blank, an alternative item similar to Item 7 was included in the revised version of Pilot Test II to observe whether students were making the same error in similar tasks. The first step referred to adding –3 on both sides of the equation, and the second step referred to dividing both sides by –2 in the codes.

Table 3. 17. Content analysis of Item 7 in Pilot Test II

7) Solve $-3 - 2x = -9$ by showing the solution steps.		**Frequency**	**Ratio (%)**
Correct	Correct response with the accurate first and second step	43	31.62
	Finding x by doing substitution	2	1.47
Incorrect	The erroneous first step and an accurate second step	7	5.15
	Erroneous second step	17	12.5
	Both steps are erroneous	17	12.5
	Unspecified (e.g., 2+3x=5x)	18	13.24
Missing		32	23.53
Total		**136**	**100**

Similarly, most students responded incorrectly to Item 11, which requires functional thinking (see Figure 3.4).

11) In the figure below, there are square tables (gray colored) and chairs (white colored) next to each other. The figure below shows the distribution of the number of chairs on the attached squared tables.



a) Find the total number of chairs placed on the tables when 10 tables are brought together.
b) Write the rule (equation) of the pattern formed between the variables x and y, including x as the number of tables brought together and y as the number of chairs placed on the tables.
c) Find the number of tables combined if the number of chairs placed on the tables is 152 when some tables are brought side by side.

Figure 3. 4. Item 11 in Pilot Test II [Adapted from Stephens et al. (2017)]

Moreover, the figures representing the tables and chairs were revised to eliminate potential misunderstandings among students. Since students might have given no answers to the items with sub-categories, most of the sub-categories were changed to separate items in the final ADT. Therefore, students would perceive those items as separate rather than sequential sub-categories of the same question. As a result, Items 8 and 9, Items 10 and 11, Items 12, 13, and 14, and Items 15, 16, and 17 were converted into separate items from Item 9, Item 8, Item 11, and Item 10, respectively. The analyses of the responses provided by students for Item 11 were given in Table 3.18, Table 3.19, and Table 3.20. Since most students answered the sub-categories of the item incorrectly, some minor revisions were also made.

Table 3. 18. Content Analysis of Item 11a in Pilot Test II

Response	Frequency	Ratio (%)
Correct responses	29	21.32
Just 22 response	24	17.65
Incorrect/Unspecified responses	59	43.38
Missing response	24	17.65
Total	**136**	**100**

Based on the results in Item 11a, most students could not express the correct answer or show their work (See Table 3.18). Approximately half of the students gave missing responses.

Table 3. 19. Content Analysis of Item 11b in Pilot Test II

Response	Frequency	Ratio (%)
Correct responses (e.g., 2n + 2 = y)	18	13.24
Partially correct (e.g., 2n + 2)	6	4.41
Incorrect/Unspecified	56	41.18
Missing response	56	41.18
Total	**136**	**100**

Table 3.19 presented that a few students could give the algebraic equation $2n + 2 = y$; some wrote the algebraic expression $2n + 2$ instead of an equation. As students were asked to write the relationship symbolically, the number of correct responses decreased in Item 11b compared to Item 11a and Item 11b.

Table 3. 20. Content Analysis of Item 11c in Pilot Test II

Response	Frequency	Ratio (%)
Correct responses (75)	28	20.59
Just 75 response	2	1.47
Incorrect/Unspecified responses	62	45.59
Missing response	44	32.35
Total	**136**	**100**

Although there was a decrease in Item 11b, the number of students' correct responses increased in Item 11c, as presented in Table 3.20. The analysis results of each item in Pilot Test II can be reviewed in Appendix B. The final test (ADT) data collection process was explained in detail in the next part.

3.4.6.Data Collection Process of the Final ADT

Six hundred twenty students participated in the final administration process of ADT in a public middle school. Some of the objectives in the 7th-grade middle school mathematics curriculum were transferred into the 8th-grade middle school mathematics curriculum at the beginning of the 2018-2019 academic year because of the revision of the middle school mathematics curriculum. Thus, the participants of the current study were changed to 8th-grade students since some of the items in ADT included the objectives of the 8th-grade mathematics curriculum. The participants were 8th-grade students in a public middle school in the Western Black Sea Region in Turkey. Six hundred twenty students who graduated from different primary schools participated in the test. At the beginning of the semester, the students are distributed to sections without any ability grouping. Therefore, there are students from all achievement levels in each classroom in the school. Students had 50 minutes to solve the questions in the final ADT (See Appendix C). The test was simultaneously conducted in sixteen classrooms on April 2, 2019, each one including approximately 35 students. The middle school teachers helped to conduct the test, and the researcher tried to answer the students' questions in each classroom. Since there were several sections, two researchers also helped the researcher to administer the test and make the necessary explanations. Therefore, the test was conducted on 620 eighth-grade students with the help of three educational researchers and 16 teachers.

3.5.Data Analysis of the Final Test (ADT)

After Pilot Test I and II, the final version of ADT was generated. The distribution of the big ideas (Blanton et al., 2015) in ADT is summarized in Table 3.21. The ADT analyses were based on already constructed codes in Pilot Test I and II. When the codes obtained by the researcher reached saturation and started to repeat, the remaining tests were excluded from the data analysis process to save time. Thus, the tests of 267 students were analyzed by choosing two classes of each MSMT. Moreover, additional codes were included in the analysis of the ADT results.

Table 3. 21. The distribution of four big ideas in ADT

Item no	**Big ideas**
Item 1	Equivalence & generalized arithmetic
Item 2	Expressions
Item 3	Variable
Item 4a	Equations
Item 4b	Variable
Item 5	Equations
Item 6	Functional thinking
Item 7a	Equations
Item 7b	Equations & generalized arithmetic
Item 8	Functional thinking
Item 9	Functional thinking
Item 10	Functional thinking
Item 11	Functional thinking
Item 12	Functional thinking
Item 13	Functional thinking
Item 14	Functional thinking
Item 15	Functional thinking
Item 16	Functional thinking
Item 17	Functional thinking

The students' responses were coded as ‘relational-structural’ if they provided the process of operations demonstrating equality instead of multiplying the values on both sides. Students’ responses were coded as ‘relational-computational’ if they considered that the equivalence could not be presented without multiplying the values on both sides. Since the category of ‘operational’ was not applicable to Item 1, it was not included in the coding process. Item 1 was an equality task that asked students to show the equality of the results of two different multiplications. Unspecified and missed responses were evaluated outside these two categories (See Table 3.22).

Table 3. 22. Analysis of students' responses to Item 1 [Adapted from Benneth (2015)]

Item 1: Equality task

Big ideas: Equivalence, generalized arithmetic

Could you illustrate the truthiness of the following equality without calculating the results of the multiplications 7 × 22 ve 14 × 11? Please give a brief explanation by stating the reasons.

$$7 \times 22 = 14 \times 11$$

	Codes	Frequency	Ratio (%)
Relational-structural	22 equals the multiplication of 11 and 2; 14 equals the multiplication of 7 and 2. representing it using different values 6 · 4 = 3 · 8.	2	0.75
	Showing with "7 · 22 = 7 · 2 · 11 or 7 · 2 · 11 = 14 · 11"	9	3.37
	Showing equality by saying they are multiple of each other/ We can divide both sides by the common divisor.	35	13.11
	One is multiplied by 2 when the other is divided by 2.	16	5.99
	Just the response of "we can illustrate it."	6	2.25
	Both numbers are doubled.	5	1.87
	One is twice the other.	3	1.12
	They are equal when prime factorization is done on both sides.	3	1.12
	Algebraic demonstration (Let 7 is x and 14 is 2x; 11 is y and 22 is 2y. Then, x · 2y = 2x · y)	2	0.75
	We can illustrate it by constructing an equation.	2	0.75
	22 = 11 · 14 : 7 ; 22 = 11 · 2; 22 = 22 or 7 (20+2) = (10 + 4) · 11 140+14 = 110 + 44	1	0.37
	Total	**84**	**31.45**
Relational-computational	22 equals the multiplication of 11 and 2; 14 equals the multiplication of 7 and 2.	67	25,09
	We cannot illustrate it.	40	14,98
	We should do multiplication.	18	7,74
	We cannot illustrate it since there is no unknown.	1	0,37
	I would multiply 22 and 11. and also 7 and 14.	1	0,37
	Total	**127**	**48.55**

Table 3.22 (continued)

Unspecified responses	26	9.74
Missed responses	30	11.24
Total	**56**	**20.98**
General total	**267**	**100**

The second item in ADT was related to the translation from the real-world context to the symbolic world of algebra. The coding scheme, frequencies, and percentages of students' responses are given in Table 3.23. Most students gave correct responses for Item 2, which asked about the symbolic expression of a mathematical problem. Based on the results, approximately 68% of the students could answer 50 – x. The remaining students (32%) gave incorrect responses (e.g., x – 50, x + 50, 50 - x = x, etc.) or left the item blank. Students' incorrect answers showed that some struggled to use algebraic symbols to express a verbal statement in symbolic form.

Table 3. 23. Analysis of students' responses to Item 2

Item 2: Translating the verbal statement to symbolic expression

Big idea: Expressions

Write the algebraic expression of the statement "the remaining time after x minutes during a 50 minutes long examination.

	Codes	Frequency	Ratio (%)
Correct responses	50 – x	106	39.70
	50 – x = y	75	28.09
	Total	**181**	**67.79**
Incorrect responses	x – 50	8	3.00
	x + 50	6	2.25
	50 · x	6	2.25
	50 = x	4	1.50
	Unspecified responses (e.g. 50-x=x, 50x-x=y, x – 50y, 50x = x)	34	12.73
	Total	**58**	**21.72**
	Missing responses	28	10.49
General total		**267**	**100.00**

The coding scheme of the responses given to Item 3 is shown in Table 3.24. Students' responses were categorized based on the correctness of the responses and the reasoning behind the answer, such as considering a variable or substituting one or more values. The students' answers were coded as ‘variable’ if they explained their reasoning based on the *changeability* of n. They stated they could not determine which was larger as the variable could take multiple values. If students justified their answers by substituting a value for n and drew a conclusion, their responses would be coded as single-value explanations.

Moreover, if they provided their answers substituting different values for n, their responses were coded as multiple-value explanations. 13.84% of students demonstrated their reasoning by substituting one or more value(s) for n. Moreover, the answers to which students gave operational but wrong reasons were coded as operational explanations. Approximately 25% of students correctly answered Item 3, explaining why we could not express which algebraic term was greater, as it changes based on the value of n. Also, 17.23% of the students provided partially correct answers, such as describing the situations of $3n > n + 6$ or $n + 6 > 3n$ without mentioning $3n = n + 6$ and substituting one or more values to n to decide which one was greater, namely single-value or multiple-value explanations. 40% of the students gave incorrect or unspecified responses to Item 3, and 16.85% provided incorrect operational answers by using wrong reasoning related to the structure of algebraic expressions.

It is confounding that 11.24% of the students expressed that ‘multiplication always gives greater results than addition.’ This information might show that students had erroneous thoughts about basic arithmetic operations, multiplication, and addition, considering that one of them was always greater, instead of thinking about the changeability of the results of those operations. Moreover, students had wrong ideas based on the meaning of algebraic expressions such as 3n, n + 6, and 6n. Some students thought 6n and n + 6 were equal and compared the expressions 3n and n + 6 based on such incorrect arguments. The results of the analysis of Item 3 are summarized in Table 3.24.

Table 3. 24. Analysis of students' responses for Item 3 (Asquith et al., 2007, p. 255)

Item 3: Which is larger task

Big idea: Variable

Let n be an integer. Could you tell which is the larger, 3n or n+6? Please explain your answer.

		Codes	Frequency	Ratio (%)
	Correct Variable	It varies based on the value of n.	32	11.98
		Explanation of different cases of n<3. n=3. and n>3 by expressing that we cannot say.	12	4.49
		3n>n+6; 2n>6; n=4.5.6.… 3n=n+6;2n=6;n=3 3n<n+6; 2n<6; n=2.1	23	8.61
		Total	**67**	**25.09**
Partially correct	Variable	n + 6 is greater when n<3; 3n is larger when n>3.	6	2.25
		n + 6 is greater up to a particular value, and 3n is greater after this specific value.	2	0.75
		We cannot determine.	11	4.12
	Single-value explanations	n + 6 is greater since 3n is smaller for all negative numbers.	2	0.75
		Justifying that 3n > n+6. 3n = n + 6 or n + 6 > 3n by substituting a single value for n. For example; let n=1; therefore, n + 6 > 3n.	16	5.98
	Multiple-values explanations	Justifying that 3n > n + 6 or n + 6 > 3n by substituting multiple values for n. For example; let n=2 therefore n + 6 > 3n and let n=5 therefore 3n > n+6.	21	7.86
		Total	**46**	**17.23**

Table 3.24 (continued)

Incorrect/Unspecified	**Incorrect operational explanations**	3n since multiplication always gives greater results than addition.	10	30	11.24
		n + 6 because 6 is greater than 3.	6	6	2.25
		n + 6 because there is an addition of a number and n.	17	5	1.87
		n + 6 because 6n > 3n (Claiming that n + 6 and 6n are equal terms.)	2	3	1.12
		3n since the quotient of n is greater.	23	1	0.37
	Unspecified explanations	Unspecified responses	5	26	9.74
		Erroneous responses by doing substitution	14	6	2.25
		Just saying 3n	3	25	9.36
		Just saying n + 6	12	5	1.87
		Total		**107**	**40.07**
Missing Answer			M	47	17.60
General Total				**267**	**100.00**

The analysis of students' responses for Item 4a is given in Table 3.25. Results showed that 37.83% of the students could correctly construct the equation based on the given problem. It was observed that some students tend to write 3x = 84 without specifying the extended version of the equation as x – 1 + x + x + 1 = 84. The responses are written correctly but not constructed as an equation (x + x + 1 + x + 2) and were coded as partially correct. Approximately half of the students (50.94%) responded to the item incorrectly constructing incorrect equations. Several students (17.60%) translated the problem into symbolic form as x + y + z = 84. It was not wrong mathematically but not in the form we expected as we asked them to write the equation of adding three consecutive numbers. As this equation did not demonstrate the relationship among the expressions of consecutive numbers, it was accepted as an incorrect response. Moreover, there were other incorrect answers, such as x + x + x = 84, x + x + 2 + x + 4 = 84, and x + 2x + 3x = 84. Those answers might show that students struggled to write different entities based on the same variable in a mathematical problem. These examples might also show that students had difficulty relating two or more algebraic expressions using a particular variable.

Table 3. 25. Analysis of students' responses for Item 4a [Adapted from Mullis et al. (2004)]

Item 4a: Writing the algebraic expression of a verbal statement and identifying the unknown				
Big ideas: Equations, expressions, generalized arithmetic				
"The summation of 3 consecutive natural numbers is 84."				
4a) Please write the equation of the statement above.				
	Codes	**Code no**	**Frequency**	**Ratio (%)**
Correct	Writing the equation (x+x+1+x+2=84 or x-1+x+x+1=84)	1	91	34.08
	Writing just 3x=84 (without specifying x-1+x+x+1=84)	9	10	3.75
	Total		**101**	**37.83**
Partially correct	x+x+1+x+2	8	2	0.75
	Total		**2**	**0.75**
Incorrect	x+y+z=84	4	47	17.60
	x+x+x=84	2	14	5.24
	x+x+2+x+4=84	7	9	3.37
	x+2x+3x=84	6	8	3.00
	3x+84	5	8	3.00
	Unspecified responses	3	40	14.98
Total			**136**	**50.94**
General Total			**267**	**100.00**

In the second part of Item 4, students were asked to identify the meaning of the unknown they used to construct the equation in Item 4b. The analysis of students' responses showed that only 19.48% of the students could correctly identify the meaning of the unknown as *the smallest of the consecutive numbers* or *the first number* in Item 4b. Moreover, some students (5.99%) just wrote the numerical value of the unknown, such as x = 27, by solving the equation in Item 4a. More than half of the students could not express the meaning of the unknown. 37.83% of the students could write the equation in Item 4a (see Table 3.25), and 19.48% of the students could explain the meaning of the unknown (See Table 3.26). In other words, some students could not identify the meaning of x, although they could write the equation based on the problem.

Table 3. 26. Analysis of students' responses for Item 4b

	Codes	**Frequency**	**Ratio (%)**
The verbal meaning of the unknown	The smallest of the consecutive number or the first number	52	19.48
Total		**52**	**19.48**
Specific number	Writing x=27, solving the equation	15	5.62
	Just writing x=27	1	0.37
Total		**16**	**5.99**
Incorrect answers	Not being able to write what the unknown refers to	141	52.81
Total		**141**	**52.81**
Missing responses		58	21.72
General total		**267**	**100**

The analysis of students' responses to Item 5 was summarized in Table 3.27. The purpose of Item 5 was to investigate students' understanding of negative and rational numbers. Results showed that half of the students could express that the argument of that student was incorrect. 17.60% of the students clarified that the result could be two if eight is added with a negative number. Moreover, some responses declared that 'the unknown might be a negative rational number' (7.12%), 'the unknown might be a real or rational number' (1.49%), or 'the number may not be positive and an integer.' (1.12%). Also, some students explained their reasoning by stating that 'they can find the value of c by solving the equation' (11.61%). Some students (31.09%) argued that the consideration of the student was correct.

Table 3. 27. Analysis of students' responses to Item 5

Item 5: Awareness of negative numbers and rational numbers

Big idea: Equation

The teacher writes the equation 8 + 9c = 2 on the blackboard and wants students to solve it in an algebra class. Then, a student states that the equation is incorrect since the summation of 8 and a number never equals 2. Do you think that the argument of that student is correct? Please give a brief explanation.

	Codes	**Frequency**	**Ratio (%)**
The argument is incorrect	If 8 is added with a negative number, the result can be 2.	49	18.35
	I can find the value of c by solving the equation.	31	11.61
	The unknown might be a negative rational number.	19	7.12
	Just 'it is incorrect' response	12	4.49
	The unknown might be a real or rational number.	4	1.49
	The number may not be positive and an integer.	3	1.12
	Unspecified responses	18	6.74
Total		**137**	**51.31**
The argument is correct	The summation of 8 and a number cannot equal 2.	23	8.61
	The summation of 8 and any multiple of 9 (even if negative) cannot equal 2.	11	4.12
	c = $\frac{-9}{6}$ is not equal to 2.	9	3.37
	The equation of $\frac{6}{6} = \frac{-9c}{6}$ cannot be solved.	7	2.62
	Finding the equation and just saying that c = $\frac{-2}{3}$	6	2.25
	The answer is 6 when we transfer 8 to the other side of the equation.	1	0.37
	Unspecified responses	25	9.36
Total		**83**	**31.09**
Missing responses		47	17.60
General total		**267**	**100**

As it was predicted, they did not take into consideration the negative numbers or rational numbers. For example, some students thought that 'summation of 8 and a number could not equal 2' (8.61%) or 'summation of 8 and any multiple of 9. even if it is a negative number, cannot be equal to 2' (4.12%). Moreover, some students solved the equation erroneously and concluded that c = $\frac{-9}{6}$ is not equal to 2 (3.37%). Or some of them thought that the equation of $\frac{6}{6} = \frac{-9c}{6}$ could not be solved as they might have difficulty with rational numbers and could not continue the procedure (2.62%). Moreover, unspecified responses (9.36%) were invalid and not within the scope of possible answers to Item 5. The detailed analysis of students' responses to Item 5 is presented in Table 3.27.

Based on the analysis of Item 6, 30.71% of the students correctly answered the item. Just 19.85% of the students responded by considering algebraic manipulations done on a = 3b + 4 as being aware that the variable can take multiple values. The remaining 10.86% of the students substituted single or multiple values to b to observe the change in the value of a. Therefore, they substituted a value or more to b and found the difference between the last and first values of a when b is increased by 2. Also, there were partially correct responses (35.21%) that stated 'a increases the same as the increase of b' and just the answer of 'it increases by 6.' 24.69% of the students gave incorrect responses to Item 6. such as just the answer of 'It increases by 2 as a changes similar to the change of b', 'the equation becomes a = 5b + 4', and 'the value of a doubles.' Moreover, students gave unspecified responses to Item 6, as summarized in Table 3.28.

Table 3. 28. Analysis of students' responses to Item 6

Item 6: The relationship between two variables

Big idea: Functional thinking

Let a = 3b + 4. How does the value of a changes as b is increased by 2? Please give a brief explanation.

	Codes	**Frequency**	**Ratio (%)**
Variable	a increases by 6 since b is multiplied by 3, is 3·2 = 6	18	6.74
	a = 3(b + 2) + 4 = 3b + 6 + 4; therefore, a increases by 6	8	3.00
	Total	**53**	**19.85**

Table 3.28 (continued)

Single or multiple value(s)	Substituting a single value for b in equation a = 3b + 4.	13	4.87
	Substituting more than one value for b in equation a = 3b + 4.	14	5.24
	It increases by 6, and it might be found by substitution.	2	0.75
	Total	**29**	**10.86**
Partially correct	Increases as b increases.	52	19.48
	It increases by 6.	42	15.73
	Total	**94**	**35.21**
Incorrect responses	It increases by 2 as a changes similar to the change of b.	27	10.11
	The equation becomes a = 5b + 4	14	5.24
	The value of ‘a’ doubles	10	3.75
	Decreases	5	1.87
	Unspecified responses	10	3.75
	Total	**66**	**24.69**
Missing responses		52	19.48
General total		**267**	**100**

The analysis of Item 7 showed that approximately half of the students correctly responded to two parts of the item (See Table 3.28). In Item 7a, adding both sides 3 was coded as the first step, and dividing both sides by -2 was coded as the second step to analyze students’ responses. In Item 7a, students frequently made an error in dividing both sides with -2. Students generally divided both sides by 2 instead of -2, showing they had difficulty operating with negative numbers (See Table 3.29).

Table 3. 29. Analysis of students’ responses for Item 7a

Item 7a: Solving equations

Big idea: Equation

Please solve the equations by showing your work.

$$-3 - 2x = -9$$

Table 3.29 (continued)

	Codes	Frequency	Ratio (%)
Correct	x = 3. the first and second steps are correct.	143	53.56
	Rather than writing x = 3. just 3.	10	3.75
	Total	**153**	**57.30**
Incorrect	The first step is erroneous.	13	4.87
	The second step is erroneous.	35	13.11
	The first and second steps are erroneous.	11	4.12
	Unspecified	11	4.12
	Total	**70**	**26.22**
Missing responses		44	16.48
General Total		**267**	**100**

In the second part of Item 7, half of the students could solve the equation. The incorrect responses showed that the distribution of the quotient into the parenthesis and adding both sides to the same value were the most frequently observed errors while solving the equation. The results also suggested that students had more difficulty in Item 7b compared to Item 7a. There were two reasons for students' struggle. Firstly, there was a parenthesis in Item 7b, and students were required to use the distributive property in arithmetics. Secondly, including unknowns on both sides might confuse students, compared to the equations with an unknown on one side of the equality. The analysis of Item 7b was summarized in Table 3.30.

Table 3. 30. Analysis of students' responses for Item 7b

Item 7b: Solving equations

Big idea: Equation, generalized arithmetic

Please solve the equations by showing your work.

$$3x + 2 = -7(x - 6)$$

Table 3.30 (continued)

	Codes	Frequency	Ratio (%)
Correct	x = 4. the first and second steps are correct.	133	49.81
Total		**133**	**49.81**
	The first step is erroneous	14	5.24
	The second step is erroneous	11	4.12
	The third step is erroneous	6	2.25
Incorrect	The second and third steps are erroneous	2	0.75
	All steps are erroneous	32	11.99
	Unspecified responses	11	4.12
Total		**76**	**28.46**
Missing responses		58	21.72
General Total		**267**	**100**

In Item 8, students were asked to write the algebraic equation based on a real-world problem and then find the information based on a particular instant in the second part of the item (See Table 3.31).

Table 3. 31. Analysis of students' responses to Item 8 [Adapted from Açıl (2015]

Please answer Items 8 and 9 based on the graphics below.
Item 8: Finding the rule of function
Big idea: Functional thinking, equation

The graphic below shows the amount of elongation by months of a sapling growing equally every month.

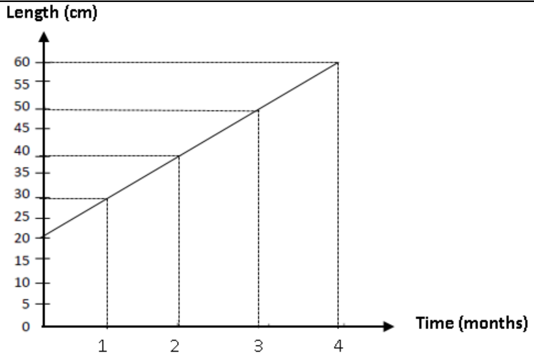


Table 3.31 (continued)

The x-axis in the graph represents the elapsed time (months), and the y-axis represents the sapling's height (cm). Write an equation showing the relationship between this sapling's length and time elapsed.			
	Codes	**Frequency**	**Ratio (%)**
Correct	y=20+10x	102	38.20
	The length of sapling = 20+10 · time elapsed	1	0.37
	Total	**103**	**38.57**
Partially correct	20+10x	9	3.37
Incorrect	Unspecified	100	37.45
	y=x+20 / y=10x	7	2.62
	x=20+10y	2	0.75
	Total	**109**	**40.82**
Missing responses		46	17.23
General total		**267**	**100**

In Item 8, 38.57% of the students could correctly write the equation based on the given situation, and 3.37% wrote the algebraic expression 20+10x instead of constructing the equation using equality. Students' responses showed several invalid equations (40.82%), such as y = x + 20 and x = 20 + 10y. The ratio of incorrect responses might present that students struggled to construct equations based on real-life situations. Students performed better in Item 9 than in Item 8, as they were asked to find the length of the sapling at a specific time instead of constructing an equation.

Table 3. 32. Analysis of students' responses to Item 9

Item 9: Finding the value of the function for a specific instant **Big idea: Functional thinking**
According to the graphic, what will be the length of this sapling eight months after planting? Please show your work, including the equation-solving steps.

Table 3.32 (continued)

	Codes	Frequency	Ratio (%)
Correct	y=20+8x 20+10·8=100	75	28.09
	By counting	46	17.23
	20+10.8=100 (by arithmetic)	23	8.61
	Counting on the graphics	3	1.12
	Total	**148**	**55.43**
PC*	Just 100	15	5.62
	Unspecified	57	21.35
Incorrect	Calculation of the direct proportion without considering the sapling's initial height	15	5.62
	Incorrect result because of an erroneous equation	8	3.00
	Incorrect results by counting	5	1.87
	Total	**84**	**31.46**
	Missing responses	20	7.49
General Total		**267**	**100.00**

**PC: Partially correct*

55.43% of the students could correctly answer the item using different methods, such as substituting 8 to x (28.09%), counting (18.35%), or calculating the length using arithmetics (8.61%). The results presented that some students did not prefer to use the equation, although they could write it correctly in Item 8. Students' errors generally were related to not considering the sapling's initial length, incorrect equations based on the problem, and unspecified responses (See Table 3.31 & Table 3.32). The analysis of students' responses for Item 10 showed that students struggled to write the equation of the real-life situation in the problem (See Table 3.33). As results suggested, fewer students (25.47%) could write the equation correctly compared to Item 8. which showed that they struggled to write the equation in this item more. Students' difficulty might be related to writing (x-1) in the equation as they struggled at the same point in Item 4 while writing x, x+1. and x+2 for the consecutive three numbers. It might be inferred that students had difficulty writing interconnected algebraic expressions. Approximately half of the students gave incorrect equations for the algebraic problem, some of which were closer to the correct answer but erroneous (52.43%).

Table 3. 33. Analysis of students' responses to Item 10

Please answer Items 10 and 11 based on the following problem.

Item 10: Finding the rule of function
Big idea: Functional thinking

In a cafe, the fee for breakfast is 20 TL. Customers who order breakfast at this cafe are not charged for the first tea they drink. For each subsequent tea order, an additional fee of 3 TL is charged.

10)Write the equation demonstrating the relationship between the variables x and y, where the number of teas consumed by a person who orders breakfast is x cups, and the total price paid is y TL.

	Codes	Frequency	Ratio (%)
Correct	y = 20 + 3(x-1)	52	24.34
	y = 17 + 3x	3	1.12
	Total	**55**	**25.47**
Incorrect	20x+3y / 1x+3x=20 / 20=x+3 / 20=3x-3 /20+3(x1)=y / 3x+20x / y=3x / x+20=y /20+2x=2y /x+20=y-3	66	24.72
	3x+20 / x-1·3+20 / y=3x-3 / y= (20+x)-3 / y=3x / x+20=y / y=3x-1+20 / 20+(x-1)=y / y=3x-x	51	19.10
	Unspecified responses	23	8.61
	Total	**140**	**52.43**
Missing responses		59	22.10
General total		**267**	**100**

The results of Item 11 illustrated that 65.54% of the students correctly calculated the total payment the customer should pay. Similar to the responses to Item 10, students preferred calculating the price using arithmetic instead of the equation. Although 25.47% of the students could write the equation, 15.73% used the equation to find the price the customer should pay. 9.74% of the students did not consider the first tea, which was free, and calculated that 5 · 3=15 and 20+15=35 TL. The responses of students for Item 11 are represented in Table 3.34.

Table 3. 34. Analysis of students' responses to Item 11

Item 11: Finding the value of the function for a specific instant

Big idea: Functional thinking

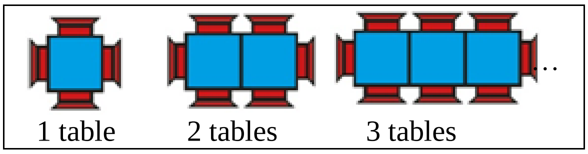
11)A customer ordered breakfast at this cafe and drank 5 cups of tea. How much TL should she pay in total?

	Code	Frequency	Ratio (%)
Correct	4.3=12 TL 20+12=32 TL	119	44.57
	Finding 32 using the equation	42	15.73
	Just the answer of 32	14	5.24
	Total	**175**	**65.54**
Incorrect	5·3=15 20+15=35 TL	24	8.99
	y=20+3x y=20+15 y=35	2	0.75
	4.3=12	14	5.24
	Just the answer of 22	10	3.75
	Just the answer of 15 or 3·5=15	10	3.75
	Unspecified	10	3.75
	Erroneous answers using the equation	7	2.62
	Just the answer of 35	2	0.75
	Total	**79**	**29.59**
Missing responses		13	4.87
General total		**267**	**100**

Similar to the results of previous items, in which students were asked to find a specific instant in a problem rather than constructing the equation, students performed highly in Item 12. The results presented that 64.79% of the students correctly did the task using different ways, such as using equations, arithmetics, or modeling. The students who gave correct responses frequently used modeling or counting the chairs, constructing a pattern. 10.11% of the students preferred using the equation, which belongs to the relationship between the number of tables and chairs. Some students (16.85%) merely wrote the number of chairs without further explanation about their solution; therefore, such responses were coded as partially correct. 15.36% of the students gave incorrect answers because of erroneous operations, wrong direct proportion, and inaccurate calculations while counting the number of chairs, as shown in Table 3.35.

Table 3. 35. Analysis of students' responses to Item 12 [Adapted from Stephens et al. (2017)]

Please answer Items 12, 13, and 14 based on the following problem.
Item 12: Finding the value of the function for a specific instant
Big idea: Functional thinking
In the figure below, there are square tables and chairs placed around these tables. The figure below shows the number of tables attached and the distribution of the number of chairs placed on that tables.



12)Find the total number of chairs placed on the tables when ten tables are brought together.

	Code	F	R(%)
Correct	Finding the result with the equation	27	10.11
	Using arithmetic (10·2=20 20+2=22 or x=4·10=40 40-18=22)	27	10.11
	Drawing the progressive steps	58	21.72
	By counting	59	22.10
	Total	**171**	**64.04**
PC*	Just the response of 22	43	16.10
	Correct result but an incomprehensible explanation	2	0.75
	Total	**45**	**16.85**
Incorrect	Unspecified	36	13.48
	10·4=40	3	1.12
	Calculating the total number of chairs until the 10^{th} step	2	0.75
	Total	**41**	**15.36**
Missing responses		9	3.37
General total		**267**	**100**

**PC: Partially correct*

Item 13 aimed to make students construct the algebraic equation based on the given problem in Item 12. Like in the previous items, which get students to write the algebraic relationship symbolically, students' performance was low in Item 13 as 37.08% correctly constructed the equation. Students' incorrect answers included erroneous equations that did not accurately reflect the algebraic relationship. Only

2.25% of the students confused the symbols of x and y and wrote the equations as x=2y+2 incorrectly. The results are summarized in Table 3.36.

Table 3. 36. Analysis of students' responses for Item 13

Item 13: Finding the rule of function **Big idea: Functional thinking** 13)Write the rule (equation) of the pattern formed between the variables x and y, including x as the number of tables brought together and y as the number of chairs placed on the tables.			
	Code	**Frequency**	**Ratio (%)**
Correct	2x+2=y	92	34.46
	2x+2	7	2.62
	Total	**99**	**37.08**
Incorrect	Unspecified	100	37.44
	x=2y+2	6	2.25
	y=4+2x	3	1.12
	x+2=y	2	0.75
	Total	**111**	**41.57**
Missing responses		57	21.35
General total		**267**	**100**

The results of Item 14 presented that 46.82% of the students could correctly find the number of tables required when 152 chairs were placed. 25.09% of the students preferred to use the equation to solve the number of tables, and the remaining chose to use arithmetic by using operations. Moreover, some students drew the figure for the 75th step to calculate the number of chairs. Incorrect answers showed that students made erroneous calculations using arithmetics or used direct proportions inaccurately (See Table 3.37). Results suggested that students struggled in Item 14 more compared to Item 12. The reason might be related to the item's structure as it was asked for reversed.

Table 3. 37. Analysis of students' responses for Item 14

Item 14: Finding the value of the function for a specific instant

Big idea: Functional thinking

14)Find the number of tables combined if the number of chairs placed on the tables is 152 when some tables are brought side by side.

	Code	Frequency	Ratio (%)
Correct	Correct answers through y=2x+2	67	25.09
	152 ─ 2 = 150 150 : 2 = 75 or 152 – 6= 146 146:2 = 73 73 + 2 = 75	42	15.73
	Drawing the figure and using arithmetic	7	2.62
	152-6=146 146:2=73 73+2=75	3	1.12
	Bu counting one by one	3	1.12
	152:2=76 76-1=75	3	1.12
	Total	**125**	**46.82**
Incorrect	Unspecified responses	64	23.97
	Just 75 response	10	3.75
	Incorrect results through direct proportion (152:4 or x= $\frac{2.152}{6}$)	9	3.37
	Just 76 response	6	2.25
	152:2=76	5	1.87
	152:2=76 76-2=74	4	1.50
	Correct results through an erroneous equation (x=2y+2)	3	1.12
	Incorrect results through an erroneous equation	2	0.75
	Total	**103**	**38.58**
Missing responses		39	14.61
General total		**267**	**100**

In Item 15. students were asked to show the relationship between the time elapsed and the total distance using different representations. Results showed that most students (83.52%) correctly filled the table. Some students (6.37%) wrote the total distance as 100, 100, 100, ... for each time interval, and some confused the symbols' places in the equation. The results are illustrated in Table 3.38.

Table 3. 38. Analysis of students' responses for Item 15

Please answer Items 15, 16, and 17 based on the following problem.

Item 15: Showing functional relationship on the table

Big idea: Functional thinking

A group of friends decided to join an organized trip from Zonguldak to Çanakkale, and they drove the 600 km road at a constant speed of 100 km per hour.

15)Fill in the below table showing **<u>the total distance</u>** traveled from the start to **<u>the end of each hour</u>** during the journey.

Time elapsed (hour)	The total distance (km)
At the end of the first hour	100km

	Code	Code no	Frequency	Ratio (%)
Correct	Filling the table correctly	**1**	**223**	**83.52**
Incorrect	Writing the values as 100.100.100.…	3	17	6.37
	Erroneous x or y values	2	6	2.25
	Missing information in the table	4	1	0.37
	Total		**24**	**8.99**
Missing responses		M	20	7.49
General total			**267**	**100**

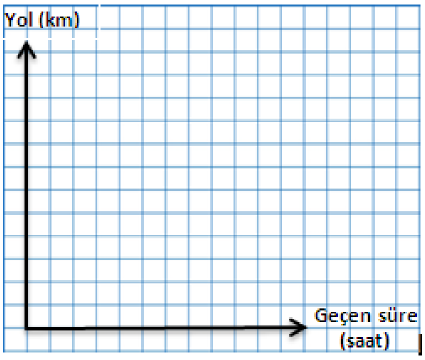
The analysis of students' responses for Item 16 presented that 59.93% of the students could draw the graphic of the algebraic relationship expressed in the problem. 28.09% of the students depicted the graphic erroneously or with missing information (See Table 3. 39). For example, they filled in the values on x and y-axes but did not draw the line showing the relationship, wrote the values of x and y erroneously, did not start the line from zero, and so on. Results showed that students had difficulty constructing the graphic compared to creating the table in Item 15.

Table 3. 39. Analysis of students' responses for Item 16

Item 16: Showing functional relationship on graphic

Big idea: Functional thinking

16)According to the information given in the text, draw the graph showing the relationship between the distance traveled and the elapsed time on the coordinate plane below, with the x-axis showing the elapsed time (hours) and the y-axis showing the distance traveled (km).



	Code	Frequency	Ratio (%)
Correct	Drawing the graph correctly	160	59.93
	Total	**160**	**59.93**
Incorrect	Showing the relationship but not drawing the line	29	10.86
	x veya y değerlerinin yanlış olarak yazılması	21	7.87
	Those who do not start the line from 0	11	4.12
	Drawing the line incorrectly	7	2.62
	Incorrect data and incorrect graphic	6	2.25
	Just the line showing the relationship	1	0.37
	Total	**75**	**28.09**
Missing responses		32	11.99
General total		**267**	**100.00**

In Item 17, 30.34% of the students could accurately write the algebraic relationship between x and y. Although high proportions of students could do the tasks in Item 15 and Item 16. there was a sharp decrease in students' achievement in Item 17 as they were asked to construct the equation based on the given problem. Some students (12.36%) responded that the equation is $t = m \cdot 100$ confusing the places of t and m in the equation. Also, students wrote other erroneous equations, such as m=100+t, 100x+x, 2x=200m, and m=900+t. The analysis of Item 17 is summarized in Table 3.40. It might be inferred that students struggled with the items required to write an equation based on a real-life verbal statement. In contrast, they could easily find the values when asked for a particular instant in a problem situation.

Table 3. 40. Analysis of students' responses for Item 17

Item 17: Finding the value of the function for a specific instant

Big idea: Functional thinking

17)Let the time is represented as t, and the total distance is represented as m during the journey. Write the equation showing the relationship between the distance traveled and the time elapsed.

	Code	Frequency	Ratio (%)
Correct	m=100·t	81	30.34
	Total	**81**	**30.34**
Partially correct	Showing the algebraic relationship correctly using different letters (y=100x)	7	2.62
	Total	**7**	**2.62**
Incorrect	Unspecified responses (1x=100 / 100x+x / 2x=200m /m=900+t /m=10t)	61	22.85
	t=m·100	33	12.36
	m=100+t	5	1.87
	m+100=t+1	3	1.12
	Incorrect using different letters	2	0.75
	Total	**104**	**38.95**
Missing responses		75	28.09
General total		**267**	**100**

3.6.Data Analysis

Qualitative data analysis is the process that starts with the introduction of researchers and the participants to the end of the study to answer the research questions by reducing the data into meaningful parts (Merriam, 2009; Yin, 2003). Merriam (2009) proposed that a researcher should read, prepare and organize the data before starting the data analysis. Therefore, after transcription of the data gathered from teachers through the semi-structured interviews, a preliminary analysis was done by reading the transcriptions and taking margin notes. This study used content analysis to analyze the data collected from mathematics teachers. Krippendorff (1980) described content analysis as a research method to get replicable and valid inferences to provide a picture of the facts, a detailed description and categories of the phenomenon, and new insights. Hsieh and Shannon (2005) also defined qualitative content analysis as “a research method for subjective interpretation of the content of text data through the systematic classification process of coding and identifying themes or patterns” (p. 1278). Grbich (2013) stated that it is possible to examine qualitative data by quantifying and qualitatively investigating. In content analysis, a descriptive approach considers determining the frequencies of codes and the interpretation of the data coding (Downe-Wamboldt, 1992; Morgan, 1993). In contrast, the thematic analysis serves as a detailed qualitative account of data (Braun & Clarke, 2006).

Content analysis might be conducted either with a deductive or inductive approach (Cho & Lee, 2014; Moretti et al., 2011). Both have three main phases: “preparation, organizing, and reporting” (Elo & Kyngäs, 2008, p. 109). In content analyses, the primary purpose is to classify the data into smaller content categories (Weber, 1990; Burnard, 1996). Therefore, there is no specific rule for data analysis (Elo & Kyngäs, 2008). As Elo and Kyngäs (2008) declared, the inductive approach can be used when there is no adequate information about the phenomenon studied in the literature, and there is a requirement to derive the codes from the data. Conversely, deductive content analysis is employed when the structure of the analysis is established based on previous studies. It is helpful if the purpose of the study is “to test a previous theory in a different situation or to compare categories at different time periods” (Elo & Kyngäs, 2008, p.107). It might examine concepts, categories, models, and hypotheses (Marshall & Rossman 1995). If a deductive content analysis is preferred, the next step should be to create a categorization matrix and categorize the data. Depending on the purpose of the study, a structured or unconstrained matrix of analysis might be used to test previous models, theories, mind maps, and literature reviews in deductive cıntent analysis (Kyngäs & Vanhanen, 1999; Polit & Beck, 2004; Hsieh & Shannon, 2005). After creating a categorization matrix, the data is reviewed for content and coded based on the identified categories (Polit & Beck, 2004). Following the principles of inductive content analysis, different classes might be produced within the bounds of an unconstrained matrix. In contrast, in a structured matrix, only aspects that fit the analysis matrix are selected from the data (Patton, 1990). However, if the data does not match the categories, the principles of inductive content analysis might be employed in a structured matrix (Elo & Kyngäs, 2008).

Both inductive and deductive approaches are used in the current study to analyze the data as there are not adequate sources to analyze the data collected, and additional codes might be observed during the present study. If an inductive approach is employed, the next step should be organizing the qualitative data, which includes the stages of "open coding, creating categories, and abstraction" (Elo & Kyngäs, 2008, p.109) are initially followed. Open coding refers to taking margin notes and headings through reading the text and reading the material, again and again, to increase the number of headers until they describe all the aspects of the data (Hsieh & Shannon, 2005). Creating categories refers to collecting the codes into groups of headings and generating the categories from them independently (Downe-Wamboldt, 1992; Burnard, 1991). Lastly, abstraction is developing a general description of a research topic by generating categories (Burnard, 1996; Polit & Beck, 2004). In this study, both the inductive content analysis approach and the unconstraint matrix of the deductive content analysis approach were preferred since they were suitable for the aim of the study.

As researchers suggested, data analysis starts with the preparation phase selecting the unit of analysis that might be a word or theme (Cavanagh, 1997; Guthrie et al., 2004; Polit & Beck, 2004). As Cavanagh (1997) declared, before deciding on the unit of analysis, it is critical to determine what to analyze in detail and consider sampling. Creswell (2009) also described the qualitative content analysis procedure in six steps, arranging and organizing the data for analysis, reading the data, coding the data, producing the themes or the descriptions gathered from the data, interrelating the themes or the descriptions, and interpreting the meaning of the themes or the descriptions. Therefore, in this study, these data analysis steps were taken into consideration by initially analyzing the data and rearranging the analyzed data based on the coding structure of related studies in the literature. The data analysis process of the first research question is explained in the following section.

3.6.1.Data Analysis of the First Research Question

The data based on the first research question was related to teachers' knowledge of fundamental issues on students' understanding and difficulties while learning algebra. To this end, teachers' knowledge of the prerequisite knowledge required to begin learning algebra, students' conceptions and difficulties in algebra, and the strategies teachers used to overcome these difficulties were investigated to get a general image of teachers' knowledge of students' learning of algebra. Firstly, the researcher transcribed and read the interview data to analyze the teachers' understanding of the prerequisite knowledge required to begin learning algebra. Then, the researcher pre-analyzed the data by taking margin notes on the document to identify the codes, as an inductive analysis was done. After pre-analysis, the researcher constructed a table that included the teachers' names and categories to observe the codes regarding teachers and related categories in the second analysis phase. The codes were represented with different colors to make each more visible if the frequency and the distribution of the codes needed to observe. After completing the coding process, the researcher repeated the coding process to ensure the consistency of the analysis and to remove the unnecessary data from the document. Next, the codes were summarized in another document to get an overview.

The second sub-research question explored teachers' knowledge of students' conceptions and difficulties in algebra. Before ADT conducted on students, MSMTs were asked to answer the questions related to their students' learning in algebra. Moreover, they were asked to anticipate their eighth-grade students performances in each item in ADT. Firstly, an inductive content analysis was established to investigate the data gathered from semi-structured interviews. After the codes were obtained, some categories were observed, including related codes. After the data analysis was completed, the researcher conducted a deductive content analysis process (Elo & Kyngäs, 2008) on the same data since the categories of the codes regarding the difficulties of students in algebra that teachers expressed were very similar to the results of the study of Jupri et al. (2014). Therefore, the data were analyzed based on the categories of students' difficulties as given in Jupri et al. (2014) again. After the coding procedure was completed, the codes were labeled in different colors, and categories of the codes were presented in a 5x5 matrix table showing each teacher's statements separately. Lastly, the strategies teachers proposed to overcome students' difficulties in algebra were investigated. Similar to the previous analyses, an inductive content analysis approach was employed. After the researcher analyzed the data and determined the codes, the same procedure was repeated to get the consistency of analysis and produce the categories of codes effectively. The following section will explain the data analysis process of the second research question.

3.6.2.Data Analysis of the Second Research Question

This section presents the data analysis process regarding the data collected through the second research question. Since the second research question includes four sub-questions, the data analysis was conducted based on each sub-question. Before ADT was conducted on students, teachers' predictions about students' performance in ADT were observed through a questionnaire and semi-structured interviews. In the questionnaire, teachers expressed the percentage of students' possible correct and incorrect responses and typical correct and incorrect answers students might give for each task. After the questionnaire, semi-structured interviews were established. In these interviews, they examine their students performances by investigating the analysis of data in ADT. They also asked to interpret students' conceptions, solution paths, difficulties, and errors in each item. In these interviews, MSMTs evaluated the results of analysis of ADT and investigate sample student papers including different types of solution paths and difficulties of students.The data gathered from the questionnaire and semi-structured interviews were analyzed with an inductive content analysis approach. After ADT was conducted on students and the researcher analyzed the test results, semi-structured interviews were done with teachers again to learn their thinking about students' performances in ADT. These interviews were also analyzed using an inductive content analysis approach. In all the analyses, the researcher analyzed the data two times. First, the codes were identified. Secondly, the data analysis procedure was repeated to see whether related codes could be included in particular categories and whether they were compatible with specific categories. After the data analysis process of teachers' predictions and thinking on students' performances in ADT was completed, the data were summarized in a matrix format and compared by the researcher to observe possible similarities and differences.

3.6.3.Data Analysis of the Third Research Question

The teachers' statements on potential sources of students' difficulties were analyzed based on causal attribution theory by using a deductive content analysis approach (Baştürk, 2016; Wang & Hall, 2018; Weiner, 2010). The codes determined in the study of Baştürk (2016) provided a basis for the current research to determine the structure of the coding process, as it was a similar study conducted on pre-service teachers. Moreover, the researcher compared teachers' knowledge about students' understanding and difficulties with their interpretations of students' performances in ADT to see how this knowledge influences their interpretations of students' understanding and difficulties. A 5×3 matrix table was used to know each teacher's statements separately for prerequisite knowledge, predictions, and interpretations of students' ADT performances.

After the researcher completed the analysis, a second coder, who had a Ph.D. in mathematics education, examined the data collected from teachers to determine the interrater agreement. Before she started to analyze the data, the researcher introduced the coding frameworks for the data investigated through a deductive approach and the data examined through an inductive approach. Then, two researchers coded a small portion of the data together to get the second researcher to understand the coding process in more detail. After she completed the coding process, the results of both researchers' analyses were compared to observe at least 80% agreement to ensure inter-coder agreement (Miles & Huberman, 1994). The researchers discussed the disagreement points regarding the analysis until they reached a deal of at least 80%.

3.7.Trustworthiness

Validity and reliability are essential requirements for a qualitative study, including data collection and analysis, presentation, and interpretation of the results (Merriam, 2009; Patton, 2002). However, validity and reliability may not be adequate and feasible for a qualitative study since interpretive conceptions are required (Golafshani, 2003; Seale, 1999). For this reason, researchers offered the terms of *credibility* corresponding to the positivist concept of internal validity; *dependability* referring to reliability, and *transferability,* which might be considered external validity; and *confirmability,* which was an issue of presentation (Creswell, 2009; Gunavan, 2015; Seale, 1999).

Credibility is defined as the "internal validity deals with the question of how research findings match reality. How congruent are the findings with reality?" (Merriam, 2009. p.213). To establish credibility, Merriam (2009) proposed the processes of adequate participation in data collection, triangulation, the position of the researcher, and peer examination of the data. Adequate participation in the data collection, triangulation, and peer review of the data were utilized in this study. Regarding adequate involvement in the data collection, the researcher collected data for one year in the same school with the same teachers. Therefore, the researcher participated in several hours of courses with the participant teachers before and after ADT was done and did semi-structured interviews with each of them for at least one hour. Moreover, different data sources were employed in the study to improve credibility, such as observation field notes, questionnaires, semi-structured interviews, and algebra diagnostic test (Creswell, 2012). Lastly, a second coder who was an expert in mathematics education participated in the data analysis process to establish peer examination of the data. The second issue is transferability in qualitative research, which is slightly different from external validity in quantitative research (Lincoln & Guba, 1985). In qualitative research, the purpose of the studies is to enhance the transferability (Merriam, 2009), whereas, in quantitative studies, the primary goal is to generalize based on the study's results (Lincoln & Guba, 1985; Merriam, 2009). Therefore, the methodology, including the context, participants, data collection process, and data analysis, were described in detail to enhance transferability.

The third issue is ensuring dependability in qualitative research instead of using the term reliability (Lincoln & Guba, 1985). Researchers suggested that all study processes be explained in detail to enable other researchers to repeat the procedures in future studies (Elo & Kyngäs, 2008; Shenton, 2009). As Merriam (2009) suggested, the position of the investigator, peer examination, and triangulation might be employed to improve dependability. In this study, findings were negotiated with a Ph.D. student in mathematics education, and the researcher rearranged the analysis if required. Finally, the term confirmability was proposed by Lincoln and Guba (1985) instead of objectivity in quantitative research. As Shenton (2004) asserted, triangulation, a detailed description of the methodology, and the researcher's assumptions were some of the factors ensuring the confirmability of a study. Triangulation and a detailed description of the methodology were employed in the current study to increase the confirmability of the study.

3.8.Role of the Researcher

Identifying the researcher's role is crucial in qualitative research (Creswell, 2007; Merriam, 2009). In this study, I collected data from the main study in a public middle school I had never been to. Therefore, I worked with each participant for the first time. I conducted informal observations in the courses, semi-structured interviews with teachers, and did ADT on students for approximately two years. For this reason, we became pretty familiar with the participants, and sometimes I felt that I was working as a middle school mathematics teacher in that school towards the end of the data collection process. We established a good relationship with participant teachers, so I could talk to the teachers whenever I wanted throughout the study. Post-interviews were conducted with teachers after one month of classroom observation of each teacher's classes and pre-interviews. Therefore, I felt they were comfortable responding to the interview questions as we had known each other for approximately one year. Although they felt comfortable, they might give their desired responses instead of their actual opinions, which might be referred to as respondent bias

(Creswell, 2007). I observed teachers' algebra teaching for one month to eliminate this threat. They shared their views honestly regarding the teaching and learning process of algebra. That is, their opinions were in line with their actions.

3.9.Ethical Considerations

Official permissions were obtained before the administration of the study from the Applied Ethics Research Center at Middle East Technical University (METU). The approval form of the Human Subjects Ethics Committee is given in Appendix G. Before conducting ADT in the pilot and main studies, I distributed an informed consent form to all participant students. Therefore, they reported their voluntariness before participating in ADT. Moreover, I meet with each teacher separately to inform the study's details and processes. That is, teachers were informed about the study's main purpose, the data collection procedure, and the expectations of the researcher from the participants, became free to discontinue the study at any time they wanted, and the confidentiality of participants' information in the study.

Mertens (2012) declared that researchers should be sensitive while presenting the characteristics of participants while including the study's findings. To eliminate the risk of identifiability, unnecessary information about participants, such as gender and age, should not be shared in the study. Moreover, pseudonyms were preferred in the study to label the teachers instead of their real names. In the next chapter, the study's findings will be explained in detail regarding the research questions.

CHAPTER 4

FINDINGS

This chapter covered the findings in three main sections to address the research questions. The first section presented the conclusions associated with the nature of in-service MSMTs' pedagogical content knowledge regarding common conceptions and difficulties held by students, possible sources of difficulties and errors of students, and the strategies used by in-service teachers to overcome the difficulties and errors of students related to four big ideas in algebra. In-service MSMTs' predictions for students' ADT performance were investigated in the second section. In the last part, the comparison and inferences regarding teachers' predictions of students' performances and students' actual performances, conceptions, and difficulties in ADT were explored in detail. The analysis results for the first research question were summarized in the next part.

4.1. Mathematics Teachers' Knowledge based on Learning of Algebra

This study investigates MSMTs' comprehension of students' conceptions and difficulties related to four big ideas in algebra. In this section, I examined MSMTs' knowledge about students' prerequisite knowledge for learning algebra, preferences for the sources of algebra tasks, ways of reviewing students' conceptions and difficulties, understanding of students' conceptions, difficulties, and errors, and also strategies to overcome the difficulties of students related to four big ideas. Analysis of in-service MSMTs' knowledge started with investigating mathematics MSMTs' knowledge of the students' prerequisite knowledge required for learning algebra.

4.1.1.MSMTs' Knowledge of The Prerequisite Knowledge required by Students to Begin Learning Algebra

One of the purposes of this study was to explore MSMTs' knowledge based on the prerequisite knowledge for learning algebra. Therefore, I asked in-service MSMTs to identify the prerequisite knowledge required to learn algebra. As Welder (2007) stated, what students need to know before entering an algebra classroom should be detailed to get students to perform better in algebra. Therefore, identification of the content is helpful before teaching. Based on the analysis of the MSMTs' statements, they expressed the requirements that students need before learning algebra, namely understanding algebraic key terms, using various forms of numbers, having the capability of doing computation (addition, subtraction, multiplication, and division) with different forms of numbers, the notions of negative and positive, the ability to use graphics, understanding algebraic expressions and knowing the rules while solving algebraic equations, and what is x? (See Table 4.1).

Table 4. 1. The terms indicated by MSMTs related to prerequisite knowledge for learning algebra

Prerequisite knowledge for learning algebra
• Key terms related to algebra
• Integers
• The notions of negative and positive
• The capability of doing computations
• Interpreting graphics
• Using algebraic expressions
• Rules for solving equations
• What is x?
• Change on the x and y-axis (covariation)

First, Ms. Burcu expressed the requirement to comprehend algebra-related key terms as a prerequisite knowledge. However, the participants had no other expression related to the keywords that should be learned for algebra. She stated that “Of course, operations. The meaning of *(the terms)* multiple, more than, less than, and half of something or one-third.” Secondly, Mr. Gürsoy stated the need to learn integers and the terms positive and negative before learning algebra. However, other MSMTs did not express any idea related to the comprehension of numbers, decimals, fractions, etc. Third, MSMTs thought that the most general knowledge for teaching algebra was comprehending different forms of numbers and doing arithmetic operations, as all MSMTs mentioned this necessity. Mr. Gürsoy indicated the need for a concrete understanding of computations with integers and positive and negative terms.

> Students’ math skills for doing operations should be well developed. The ability to do operations with integers should be highly well-developed. Of course, operations. If the student has problems with operations and the concepts of positivity and negativity, it is impossible to continue with algebra.

As Mr. Yücel said, “Addition, subtraction, multiplication, and division. Students also should have to know conducting operations with rational numbers”. Ms. Burcu also talked about the capability of doing operations with fractions and rational numbers. As she stated, "students should know operations with fractions and rational numbers to express those terms in words.” MSMTs generally did not share any knowledge requirements about various forms of integers, rational numbers, exponential numbers, decimals, fractions, etc. We might interpret MSMTs’ statements that they heavily concentrated on making computations with numbers instead of their different forms and meaning, similar to what Ms. Ferhan said:

> First, students should not have any problems with operations. We enter algebra in 5^{th} grade, but we should get students to comprehend problem-solving. We may teach students algebra earlier. What can I say? *(She is considering.)* Students should know operations, express themselves, and understand what they read. If students can understand the text and do procedures, I could say that we construct a sufficient background. After understanding the problem, if students could understand what they read and do operations, they already conceptualize algebra.

The fourth precondition for learning algebra was the interpretation of graphics. Only one MSMT, Mr. Öner, stated the importance of graphics interpretation and how vital it was for learning algebra:

> Indeed, they should have to make interpretations of graphics very, very effectively, which we prepare the background of students in 6th grade and teach in 7th grade. The graphical interpretations are crucial for us during operations with algebraic expressions. If students cannot interpret graphics, there will be nothing to do with their excellent comprehension level of equations.

Mr. Öner also clarified that students should effectively interpret the change in graphics since understanding the meaning of the changes on axes is so crucial by stating, "They should have a deep conceptualization of how the changes on the x-axis and y-axis occur and the meanings of the x-axis and y-axis? Since there are two variables, they should know which unknown represents the y-axis and which is the x-axis.” The other prerequisite knowledge suggested by MSMTs was based on the construction and manipulation of algebraic expressions and equations. Only one MSMT, Mr. Öner, indicated that the priority rules were critical while doing operations. Moreover, he expressed that the regulations were essential for accurately conducting the procedures while solving algebraic operations. Mr. Öner declared that “Priority rules while doing operations…Solving equations is very, very crucial. Knowns are collected on one side, and unknowns are on the other. They should know how to transfer an unknown to the other side very well.” Although he highlighted the importance of comprehending the manipulations in algebraic expressions and equations, he described an operation with an unknown as transferring it to the other side rather than making the same operation on both sides. Furthermore, Mr. Gürsoy also highlighted the significance of writing equations based on mathematical problems:

> It is related to the importance we gave to equations in 7^{th} grade…Since I realized it long ago, equations have been crucial for me, transitioning from mathematical problems to algebraic equations. At the end of the fall semester, I must complete the equations and continue with the rate and ratio topic at the beginning of the spring semester. However, I do not care whether I get behind on the new topic. I can make it somehow but cannot ignore algebraic equation problems.

MSMTs shared no additional statements related to the properties of algebraic expressions and rules for solving algebraic equations. The last prerequisite knowledge suggested by MSMTs was based on understanding and representing functions. Merely Mr. Öner expressed the need to comprehend the variable and provide the need for the transition among words or sentences, abbreviations, and symbols. Moreover, only Mr. Öner identified the importance of understanding what x is and what unknown is, which is crucial for transitioning from arithmetic to algebra. Besides the knowledge before learning algebra, Ms. Ferhan also discussed the necessity of developing students' self-expression skills and understanding a text or problem. There were no statements from other participants based on the development of such skills before learning algebra. The findings showed that MSMTs expressed various prerequisite knowledge types required for learning algebra. MSMTs' statements mainly focus on the computations with numbers and the properties and rules to perform algebraic operations correctly.

All MSMTs specified arithmetic operations as the primary concern for learning algebra. Moreover, two MSMTs highlighted the comprehension of different forms of numbers as prior knowledge of algebra. However, other MSMTs did not discuss various forms of numbers, such as integers, rational numbers, exponential numbers, decimals, and fractions. Instead, they heavily concentrated on just doing computations with numbers in mathematics. Only one MSMT discussed negative and positive integers as prerequisite knowledge for learning algebra. Also, only one MSMT pointed out the importance of understanding the question of 'what is x?' and the process of the transition of verbal, abbreviation, and symbolic notions of algebra, respectively. Besides the knowledge before learning algebra, Ms. Ferhan also discussed the necessity of developing students' self-expression skills and understanding a text or problem. There were no other statements from participants based on the development of such skills before learning algebra. Findings showed that MSMTs expressed various prerequisite knowledge types required for learning algebra. However, MSMTs' statements mainly focused on the computations with numbers and the properties and rules to perform algebraic operations correctly. Based on the results, it might be inferred that MSMTs provided poor information about students' prerequisite knowledge as a basis for learning algebra. They rarely expressed their concerns about students' understanding of negative numbers, rational numbers, the notion of variable, and equivalence. The following section describes MSMTs' knowledge of eighth-grade students' difficulties and errors in algebra.

4.1.2.Mathematics MSMTs' Knowledge of Students' Difficulties and Errors in Algebra

This section investigated MSMTs' knowledge of students' difficulties and errors related to four big ideas in algebra. Some researchers preferred to use the word "difficulties" (e.g., Herscovics & Linchevski, 1994; Jupri et al., 2014; Warren, 2003) or "conceptual difficulties" (e.g., Thomas & Tall, 1991), whereas others called it "errors" (e.g., Booth, 1988) for struggles or problems of the students in algebra. I expressed the struggles or problems of students by the term "difficulties" in the current study. Similar to the study of Jupri et al. (2014), I examined MSMTs' knowledge regarding students' difficulties in algebra under five categories, *applying arithmetic operations, understanding the notion of variable, mathematization, understanding algebraic expressions, and functional thinking,* The first category for students' difficulties and errors was *applying arithmetic operations*. Related to this difficulty, Mr. Yücel mentioned students' struggles while doing operations with integers:

> We have difficulty at most when they first introduced with addition and subtraction of integers. They encounter the concept of negative for the first time. They divide -2x with 2 rather than -2, although we divide both sides with the same number while solving equations.

Based on this difficulty *(dividing -2x with 2 rather than -2),* he indicated that they should prepare students for the examination. Therefore, they should use a practical way of solution:

> In fact, we initially used the balance scale model. Rather than transition to the other side of the equation, we teach it by subtracting the same number from both sides of the equation. However, since our education system is focused on examination, we cannot use this model for each item. Rather than using a balance scale model for each item, we expressed that the number is transformed into the other side of the equation, or we say that we divide both sides by the quotient of x.

Ms. Ferhan also declared a similar error caused by students' insufficient knowledge of numbers and properties of operations. Likewise, as Ms. Ferhan highlighted, students have difficulty getting similar terms together on the same side, especially when there are unknowns on both sides of the equation:

> Here, we can also talk about some of the errors in operations. They have difficulties, they tend to use operations, but some students also have many errors in operations. Especially when there are unknowns on both sides of the equation and gathering them on the same side. Some students may transform the *(quotient of)* 3x as the division on the other side; however, they have difficulty converting *(the quotient of)* $\frac{x}{2}$ as the multiplication on the other side. Rather than multiplication, for example, they convert *(the quotient)* 3 of 3x as -3 to the other side *(she is laughing)*. They do not realize that it is a coefficient. We may experience such difficulties in operations, or they may forget distributive property. Without making the distribution, let us say x + 5 is in the parentheses of 3; they pass the 3 as -3, although they have to distribute it or convert it as a division. There might be such errors, especially when there are unknowns on both sides *(of the equation)*.

Students' struggles in the situations that Ms. Ferhan stated might be caused by the inadequate knowledge of equality and properties of arithmetic operations. For 3(x + 5) = y, the MSMT noted that some students convert 3 on the other side of the equality as -3. We might infer that those students may not understand that 3 is a quotient and that the left-hand side of the equation includes the multiplication of 3 and (x + 5). Alternatively, they may not conceptualize equality; that is why we should divide both sides by three. She noted that they tend to see the equal sign as a symbol that requires "doing something" or separates an answer from the problem rather than being an indicator of equivalence. Although students might misunderstand equality, no MSMT expressed their difficulties on this issue as they did not state its significance and necessity for learning algebra.

Students might also have deficiencies based on the priority rules in algebraic operations or properties of numerical procedures, which refer to commutative, distributive, associative, and inverse properties. The MSMTs also argued about students' difficulties and errors on the distributive property. Mr. Gürsoy stated that:

> The most widespread error is adding a minus sign to the first term of the second algebraic expression only while subtracting two algebraic expressions. That is one of the things I am focusing on, principally, the most widespread error. For example, they distribute but say x-2-x-3 while subtracting x-3 from x-2. The most pervasive error is not distributing the minus sign to the second term while subtracting two algebraic expressions.

Mr. Yücel also mentioned that students have difficulties while doing operations with decimal numbers, although they are successful with other numbers like integers, natural numbers, and so on:

> We are struggling with those concepts. Mainly multiplication and division with decimal numbers related to algebra. Why? They do not use it daily, and it remains in the classroom. I think they have difficulty for this reason. They can do it in school and compute each other to make multiplication and division with decimal numbers. However, the number of students who can do those operations decreases to half the following week. You do not see it in other types of numbers; you do not encounter this while doing operations with natural numbers in word problems. The ratio of students who can do those operations decreases to half; even the number of successful students decreases in decimal numbers.

Mr. Yücel advocated that one of the sources of students' struggles in algebra was their inadequate knowledge of fractions and decimal numbers. He justified that students did not use such numbers daily; therefore, they have difficulty encountering them in word problems:

> I think they are unsuccessful since they do not use it daily. All our questions are related to daily life, but students do not use them this way. Students rarely use fractions and decimal numbers. We had been in Finland, and all students used calculators there. I am considering why; what is the reason? We always give students examples with integers. For instance, we say that the length of one side of a table is 2 meters, but they say 2 meters and 10 centimeters, that is 2,1 cm. They conduct operations with the exact dimensions. They measure and calculate the precise area by using calculators.

Thus, he pointed out the importance of rational numbers while doing algebra. The second category for students' difficulties was *understanding the notion of variable*, a critical concern for learning algebra. Mr. Öner noted that students generally have difficulty understanding what x is and how they construct, use, and interpret the variables and algebraic expressions when confronted with an algebra word problem:

> Students have difficulty conceptualizing the meaning of x and y and the meaning of a when given based on a and b in equation solving. For example, in a word problem, you plant a tree, and there is an initial length of the tree and its length changes by month. Students have difficulty understanding which symbol, y or a-b, represents the length and which shows the month. They make errors at this point at the most. When I write y = 40+3x on the blackboard, they can see that 40 refers to the initial length of the tree, and 3x refers to getting taller by 3 for each month. They can also multiply 6 with 3 and add 40 when they are asked to find the length of the tree six months later. However, they may have difficulty substituting the corresponding values in the equation since they do not conceptualize which one refers to the length or the time as they do not conceptualize or comprehend it.

As Mr. Öner indicated, students could do arithmetic operations when asked to find a specific value in an algebraic equation. However, they might have difficulty substituting the corresponding value in the algebraic expression. We may infer that students experience difficulty transitioning between the problem situation and the symbolic world. Ms. Ferhan also indicated a similar concern related to distinguishing what is the unknown in a word problem:

> They have difficulty with constructing equations at most. I mean, which object do we call x? Because they do not distinguish which object is unknown to them. They may say x for an object and let us multiply with 3 when they see three; however, what is unknown? At first glance, they have difficulty deciding which object they say x and, therefore, finding how many times of something or how much less than something. Indeed, they struggle to decide on the variable at most.

Students generally had difficulty identifying and operating with the variable, as Ms. Ferhan stated. Moreover, Mr. Öner indicated that students tend to use algebraic symbols based on their coordinate axes in the graphics. If a variable was represented on the x-axis, they strictly considered using x as the symbol in the algebraic equation and vice versa. Moreover, as Mr. Öner stated, they had strict beliefs about using symbols for variables. Although they were required to use other characters rather than x and y, they persisted in using x and y since they were accustomed to using them:

> Generally, we label the y-axis as the length and the x-axis as the time. Students try to memorize, and when we exchange the label of the axes, they are confused. They supposed that this is strict and labels of the axes cannot change, the y-axis represent the length, and the x-axis represents the time. For example, if we asked students the algebraic expression of the problem, there is 40 lt gasoline in a car, and the car takes 5 lt gasoline per 100 km, students ask where I can substitute the gas and the time in the equation. Students believe gasoline should be represented on the y-axis and time on the x-axis. Therefore, the equation should be in the form of y equals something. However, as the problem states, the remaining amount of gasoline is represented with symbol a, and time is represented with symbol b. When students are confronted with such a situation, they usually consider representing the y-axis by y and the x-axis by x.

Moreover, Mr. Öner also declared that students' success level identified their comprehension based on the changeability of the symbols. Therefore, he noted that this praxis was based on students' success level rather than other factors such as instruction, textbooks, etc. The MSMT said they generally label the y-axis as the length and the x-axis as the time in the lectures. He stated, "In examinations, we accept such answers as correct since students see them as strict forms. The difference between successful students from other students emerged at this point since successful students conceptualized the difference and broke the mold."

The third category for students' difficulties and errors was *horizontal and vertical mathematization*. Horizontal mathematization involves transitioning between the problem context and the symbolic mathematics back and forth. It consists of schematization, formulation, and visualization of a problem in different forms, converting real-world problems into acquainted mathematical models. On the other hand, the difficulties in vertical mathematization include the processes related to making reconstructions in the mathematical system and working in a symbolic world (Treffers, 1987; Van den Heuvel-Panhuizen, 2000). The activities in vertical mathematics consist of making a combination, formulation, manipulation, proof, and generalization with algebraic models (De Lange, 1987; Treffers, 1987; Van den Heuvel-Panhuizen, 2000). As Ms. Ferhan and Mr. Öner clarified, students also had difficulty with horizontal mathematization while transitioning between problem situations and mathematical symbols since they could not use symbols effectively to represent and operate values of real-world situations. MSMTs also mentioned that one of the main reasons students struggle is the abstract nature of algebra which might be categorized under the categories of understanding the notion of variable and horizontal mathematization. Mr. Yücel expressed the abstract thinking level, which students could not reach before a particular grade level:

> Algebra, x, and unknown are abstract concepts. Students' self-confidence decreases immediately, and negative concerns are raised when introduced to those concepts. Some of the algebra topics were transformed from 6th grade to 7th grade. It was well done since students did not move to the abstract thinking level. We were trying to teach students abstract concepts. Fortunately, most of them were taken into the 7th grade, and some topics were brought into the 8th grade. Students were struggling in algebra in the past. We solved word problems with arithmetic operations, performed inverse operations, and drew boxes. However, some teachers justified giving students equations in 6th grade. I stated that this was absolutely wrong since they did not reach abstract thinking yet; they were at the concrete thinking level.

As Ms. Burcu shared, some students just wrote +4 instead of x+4 for an algebraic expression indicating four more than a number:

> Some students write +4 when it is asked to write four more than a number. +4 is a constant, so it does not mean four more than something; when it is said that more, less, or multiple, it is essential to be able to write the mathematical expression.

In this example, students failed while formulating a word problem into an algebraic expression. Therefore, students' erroneous formulation might be related to the difficulty with horizontal mathematization. Moreover, students' difficulty might be caused by the inability to understand the notion of a variable since they struggled to write the variable in the algebraic expression. However, they could write the constant correctly. Mr. Öner highlighted that students had difficulty understanding the meaning of x. Moreover, he indicated that students struggled, especially while interpreting graphics and solving equations, since students could not establish the sense of x conceptually. He also mentioned graphics and separated them from algebra and noted that students were successful at graphics, but when it was integrated with algebra, it became difficult for students:

> They struggle to interpret the graphics and solve equations. If we could not conceptualize the meaning of x, an abstract concept, in the spring semester of the 7th grade, students already give up at 8th grade in equation solving. Typically, graphics are easy, and everybody attends the lesson willingly in 7th grade. However, some students have difficulties interpreting graphics when integrated with algebra in 8th grade. In brief, that is the truth. Interpreting the graphics and solving equations are the two points students have difficulty with most.

Although graphics were one of the representation forms in algebra, he described it as a separate topic. He might indicate that students successfully represent constants in graphics; however, they failed to interpret the graphics' covariation since it required constructing relationships between variables. I categorized this difficulty as *functional thinking* since it was related to "how two quantities vary simultaneously" (Blanton & Kaput, 2011). Although functional thinking is an essential dimension for learning algebra (Blanton et al., 2015, Blanton et al., 2019), MSMTs gave no additional statements related to students' difficulties based on functional thinking. The other difficulty that students might face is called *understanding algebraic expressions*. Ms. Burcu described a difficulty students experienced based on this issue: "When you say three times something, some children try to write x times three and continue trading with it after a while. 3x, the coefficient is written to the left, although you keep saying it all the time, which is straightforward."

We might infer that students could not realize the equality of x·3 and 3·x. To understand that these two expressions are equal, they need "the encapsulation of the process as an object" without observing the process for particular variable values (Tall & Thomas, 1991, p. 126). Therefore, they could realize that encapsulated objects were the same. I called the difficulty Ms. Burcu indicated a process-product obstacle, as Tall and Thomas (1991) suggested. Thomas and Tall (1991) identified the process-product obstacle as the inability to transition between the process and the product. One could see the process as a product by encapsulating the process as an object. We can accept two encapsulated things as the same if they always give the same product. Therefore, there is no need to follow the process for particular values since the object encapsulates the process. As Ms. Burcu stated, students keep writing x·3 rather than 3x to write 3 multiple of x, although 3x also equals x·3. We might explain this by the deficiency of the understanding that 3x and x·3 were identical products. Students do not use 3x instead of x·3 since they may not realize they are the same products. Therefore, we may infer that students may have the process-product obstacle as they could not encapsulate multiplying x with 3 in different forms.

Moreover, as Ms. Burcu expressed, an additional difficulty that students mostly faced while writing two or more dependent algebraic expressions in terms of a particular variable:

> For example, one *(expression)* is four more than twice the other. OK, they write 2x + 4, but the other is x. You would not be able to write it unless you said x for the other. How did you write that if you did not say x to the other? So, the child gave up there. They equated 2x + 4 to 60 without saying x to the other *(expression)*. They forgot the x, the small one. Alternatively, somebody solved 100 questions *(for each day)* in a couple of days and 150 questions in the remaining days of the same week. One week consists of 7 days. If we wrote x for the days we solved 100 questions, and the remaining days should be 7-x. They could not give the expression of 7-x. When they gave the expression of 7-x, we moved to the number of items solved on particular days. For example, if he solved 100 *(questions)* in a day, 200 in 2 days, 300 in 3 days…What should I do? Multiplication, students also expressed multiplication in the classroom at that moment. Then, we multiply x by 100. OK, those are the days when he solved 100 questions. However, they struggled to say 7-x for the days he solved 150 questions.

This difficulty was also investigated under horizontal mathematization since it requires transitioning from a word problem to a symbolic world. Mr. Gürsoy also gave a similar student struggle as Ms. Burcu did. He noted that students could find the result of the question when they were asked for a specific value. However, they had trouble when they were required to use symbols:

> We have problems with linear equation problems with one unknown. For example, let the sum of Ali’s and Veli’s ages be 30. When I describe students as x to Ali’s age and 30-x for the other, since *(linear)* equations with two unknowns were removed from the curriculum, students begin to struggle so

> much. The children have great difficulty solving this problem with a (linear) equation with one unknown. However, in a situation in which the sum is given as thirty, if you say that one of them is ten years old, they can easily express that the other is twenty years old. Nevertheless, when this returns to x, the problem begins.

Furthermore, he provided a similar example for this difficulty at the next meeting:

> If one of the two quantities that add up to 45 is x, they struggle to write 45-x for the other. For example, they say y immediately *(rather than 45-x)*. Therefore, they are trying to call it with another variable right away. They also have difficulty writing algebraically in examples such as calculating the feet of chickens and cows.

When I asked Ms. Burcu the reason for students' struggle, she confirmed that students most probably did not conceptualize it:

> I guess they think that there cannot be a day like 7-x. The most significant problems are the ones they already had at the beginning. It continues in this way since they begin to do memorization to understand. For example, we create a table. If he solved 150 *(questions)* in one day, then *(we can say that)* he solved 100 in *(each)* 6 days. If he solved 150 *(questions)* in *(each)* remaining 2 days, then *(we can say that)* he solved 100 in *(each)* remaining 5 days. How do we find the five here? Sounds come out of the classroom *(by subtracting it from seven)*. If this is x, here is 7-x. OK, we solved it there. When you move to another example, they become quiet again.

Rather than making a specific inference based on students' cognitive processes or the instructional process of algebra, she indicated some reasons for students' becoming unsuccessful in such instances, which were all related to students themselves, such as becoming lazy or having low-level reading comprehension.

> I do not know why they could not continue in the same way in other instances. Children also do not want to use their brains anymore. I guess they are so used to learning from others. Since they reach everything very quickly through their families and the school, they want everything to be solved on the board and written down in their notebooks. So they are going in this way. It is doubtful how many students you can reach out of forty students in the classroom.

Moreover, Ms. Burcu attached an aspect related to students' difficulties. She asserted that they experience difficulty if their reading comprehension is low by stating, "They have difficulty putting this into a mathematical sentence. Here is where reading comprehension comes into play. Turkish is at work again. How much difficulty they experience in Turkish is reflected in mathematics."

In Figure 4.1, the summary of students' difficulties and errors indicated by their MSMTs was given. To summarize, the MSMTs identified students' typical problems and errors related to applying arithmetic operations, solving equations, and horizontal mathematization. They rarely expressed students' difficulties with negative numbers or different forms of numbers such as rational numbers, decimal numbers, etc. Moreover, they did not share their ideas about students' difficulties regarding the concept of variables and the changeability of the symbols. Furthermore, they rarely expressed the errors students made while making operations with algebraic expressions, such as process-product obstacle, lack of closure obstacle, etc.

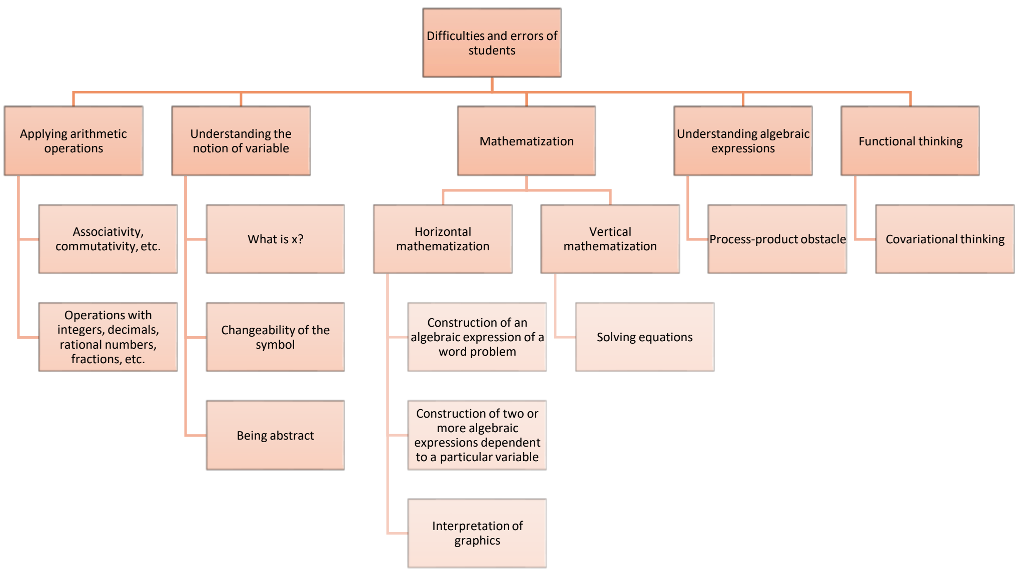


Figure 4. 1. The summary of students' difficulties and errors indicated by MSMTs

Lastly, MSMTs did not share their opinions based on students' difficulties with functional thinking while considering the difficulties students might face in algebra. This section presented in-service MSMTs' knowledge of students' difficulties and errors based on four big ideas in algebra. The next part describes MSMTs' strategies for preventing students' difficulties and errors in learning algebra.

4.1.3.The Strategies Suggested by Mathematics MSMTs to Overcome the Difficulties and Errors of Students

This section investigated in-service MSMTs' strategies to overcome students' difficulties and errors. The data analysis results indicated the strategies suggested by the participant MSMTs: increasing the amount of drill and practice, using concrete examples from daily life, using the tone of voice to emphasize the crucial points, explaining the concept repeatedly, and using activities and materials, making students active participants in learning, and taking into account the individual differences of students. In consideration of teaching methodologies, increasing drill and practice was one of the strategies provided by Ms. Ferhan.

> Some of them need a little more time only. In other words, we may increase concrete examples from daily life. It all passes with time. We might increase the number of practices. Moreover, we may ask about their ideas about solving everyday life problems at school, get them to talk, consider, and help them understand what they read.

Mr. Gürsoy also highlighted the importance of drills and practice. Furthermore, he emphasized using concrete examples to overcome students' difficulties and errors. Similarly, Mr. Yücel also mentioned the use of concrete examples:

> The more we make it concrete, include daily life examples, and create questions, the more we can transcend *(the difficulties)*. Addition and subtraction of integers are not difficult for eighth-grade *(students)*. However, almost a quarter of students still cannot do it. I still explain the addition of -3 and 2 by stating that you are on the third floor; which floor would you go to if you went up two floors to make it more concrete and related to daily life? You try to explain with examples like this, associating it with everyday life by embodying where you will come when you go up 2 floors. However, this process needs to be finished at the 8th-grade level. Therefore, -3 plus 2 equals -1 should be said, and I should have passed there. Unfortunately, we are trying to overcome such difficulties with concrete examples since I feel such missing students.

Moreover, Mr. Yücel highlighted the importance of doing activities with students while teaching algebra. He prepared instructional materials for fifth and sixth-grade students, for example, using algebra tiles while teaching algebra. Mr. Öner also mentioned another way to transcend students' difficulties and misconceptions. He explained the topic repeatedly if students asked the MSMT the points they had trouble with. Ms. Ferhan also added that she got students to devise a plan while solving an algebraic problem. Therefore, they could easily recognize the given and asked information about the situation. Consequently, they could construct an effective solution path for the problem. Ms. Ferhan also mentioned that they might use open-ended items rather than multiple-choice items:

> The child makes operations such as multiplication or division with the numbers *(in the multiple-choice item)*. If she finds one of the choices by doing such operations, she thinks it is OK without considering the item's purpose or reasoning. Furthermore, we should get students to explain the solution path rather than the calculation result. If we focus on the operations of students rather than the correctness of the result, it might be better. Students consider reaching the correct answer among four choices since they always cope with multiple-choice items. Therefore, their primary goal is to find the result instead of concentrating on the problem. For this reason, they miss understanding the problem since they try to answer impetuously. Also, we need to reduce using multiple-choice test items. Instead, we might concentrate on only one problem in a lesson if necessary and talk about it together. We need to change our curriculum completely.

Ms. Burcu also suggested a solution for students' problems while doing algebraic operations, considering as if they were solving an arithmetical word problem:

> Students have difficulty finding the minuend, subtrahend, and difference while subtracting with algebraic terms. I suggest students think as if they are doing subtraction with natural numbers. For example, I propose that they subtract 2 from 10, which results in 8. I always say students never do memorization; find your solution method yourself.

As Ms. Ferhan stated, students gave up thinking about the problem and making rapid calculations to reach the correct answer faster by using four alternatives in the items. As a result, they usually concentrate on the operational process instead of the problem situations. Ms. Burcu also declared that always using multiple-choice items might be one of the reasons for students' failure:

> We conducted a test on seventh-grade students, including open-ended items. Although we explained the content of the test and where the items would be included, they performed low success in the examination. They could guess the correct answer by examining four alternatives, or they could do cheating on their friends on a test including multiple-choice items, but they could not do the same in open-ended items. I wish all examinations would include open-ended items.

Ms. Burcu also noted another aspect related to students' difficulties in mathematics, namely the change of mathematics MSMTs annually for almost every classroom. As she stated, they previously taught in the same classroom throughout the middle school process. Therefore, they knew the characteristics and needs of each student while teaching mathematics. Nevertheless, now, students are mixed and assigned to new classrooms annually. For this reason, MSMTs could not have enough time to observe and identify the students they teach. As Ms. Burcu indicated, the tests, including open-ended items, might help examine students' knowledge, difficulties, and errors; however, they could not conduct such tests anymore. They have run the tests, including multiple-choice items, and students might choose one of the alternatives randomly or by looking at their friends. As a result, such tests could not help them investigate students' knowledge, difficulties, and errors. Mr. Gürsoy mentioned another aspect to get students to learn the points they have difficulty with, using the tone of voice to emphasize the critical issues:

> I watched a video related to rhetoric. In that video, it was said that people concentrate on you if you suddenly begin to talk with a lower voice while talking at a higher volume. I realized that I had done quite the opposite of this. Therefore, I have increased my tone or spoke in a quieter voice than usual when talking about something important. Also, I slow down where I see it as necessary. In problem-solving, I try to explain to students how to read a problem.

Lastly, Mrs. Ferhan indicated, "we might form a peer group among students and get them to help each other." Her strategy might be evaluated as learning from peers. Mrs. Ferhan also mentioned the strategies concerning psychological constructs, including making students active participants in education and paying attention to individual differences. As she clarified, "we might recognize all students and get them to think and talk. Also, we might have them write a simple problem themselves and get their opinion regarding daily life problems." However, she mentioned some obstacles, such as time limitations, the examination system, and crowded classrooms. For this reason, she stated they did not have the opportunity to do such activities. Lastly, she said they should pay attention to students' individual differences. However, as she advocated, they missed medium-level students since they prepared higher-level students for LGS. The strategies proposed by MSMTs are outlined in Figure 4.2.

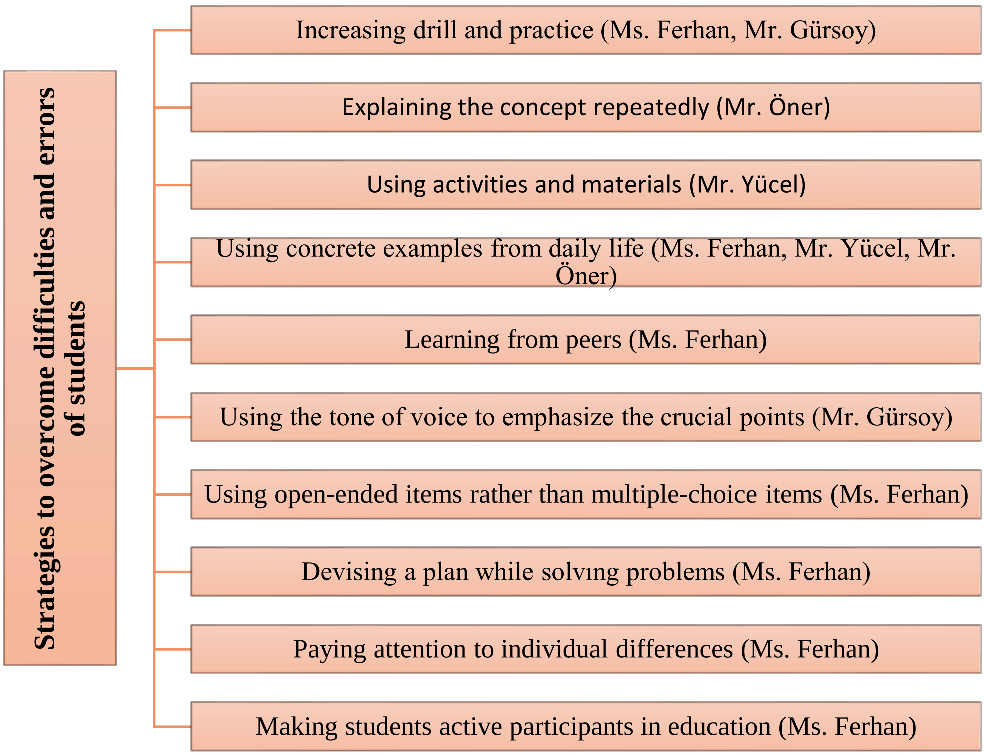


Figure 4. 2. The strategies of MSMTs to overcome the difficulties and errors of students in algebra

We might infer that MSMTs generally provided such methods in which MSMTs were active, and students were passive participants in learning. Their strategies were mainly related to what MSMTs should do while teaching algebra with direct instruction. Moreover, MSMTs described general pedagogical techniques to overcome students' difficulties and errors instead of sharing specific strategies to improve students' algebra learning. In the next part, the findings of the analysis of MSMTs' predictions for their eighth-grade students' performances in ADT will be presented.

4.2.MSMTs' Predictions of Students' Performances in ADT

In this section, I investigated MSMTs' predictions of students' performances in ADT before the test was conducted on students. At first, a questionnaire was conducted on mathematics teachers to get them more familiar with the items by considering the items in ADT, which would be reviewed in pre-interviews, and to learn their predictions for students' performances and preferences for the solution strategies. Next, semi-structured pre-interviews were done with MSMTs to discuss their predictions for students' understanding, difficulties, and errors related to variables and functions in ADT. The results of the analyses of MSMTs' predictions before ADT was conducted were given in the following part.

4.2.1.MSMTs' Predictions about Students' Understanding, Difficulties, and Errors in ADT

A questionnaire was conducted on MSMTs, and a pre-interview was conducted to learn MSMTs' anticipations of students' conceptions, difficulties, and errors in ADT. In the questionnaire, MSMTs were asked how many of their students would answer each item correctly if those items were given to 100 eighth-grade students and what strategies they might prefer to find solutions. The first item was related to equality, and students were asked whether they could show equality without doing multiplication. Mr. Gürsoy, Ms. Burcu, and Ms. Ferhan expressed that most students would correctly answer the first item.

Table 4. 2. Predictions of MSMTs for correct answers of students in Item 1

MSMT	Predictions of MSMTs for the ratio of correct answers
Mr. Gürsoy	0.60
Ms. Burcu	0.80
Ms. Ferhan	0.70
Mr. Yücel	0.10-0.20
Mr. Öner	0.10

**Ratio of correct student responses is 0.50 in ADT.*

Conversely, Mr. Yücel and Mr. Öner asserted that 10-20% of students could correctly answer the item, as shown in Table 4.2. Mr. Öner also stated that even 10% was a high ratio for correct responses given by students. Ms. Ferhan and Ms. Burcu stated that students could factorize or divide both sides with the same number to show that the left and right-hand sides were equal, as presented in Table 4.2. Ms. Ferhan pointed out that students should have a concrete understanding of factorization, which was the initial point they focused on in eighth grade. MSMTs also asserted that students might show equality by doing multiplication on both sides if they could not realize to make factorization. As Mr. Gürsoy identified, students preferred to use the easy and secure way. Even high-achiever students would do multiplication if they were not asked to solve the item without doing multiplication. Mr. Yücel argued that students might also provide a solution shown in Figure 4.3.

Ms. Ferhan	Ms. Burcu	Mr. Yücel
$7 \cdot 22 = 14 \cdot 11$ $1 = 1$	$7 \cdot 22 = 14 \cdot 11$ $14 \cdot 11 = 14 \cdot 11$ or $7 \cdot 2 \cdot 11 = 7 \cdot 2 \cdot 11$	$7 \cdot 22 = 14 \cdot 11$ $7 \cdot 22 = 14 \cdot 10 + 1 \cdot 14$

Figure 4. 3. Possible student solutions given by MSMTs for Item 1

He Yücel stated that he could not understand the item's purpose and the expectation from students until I explained the item in detail. Moreover, he noted that most students would answer it by multiplication since the numbers were one or two digits. Like Mr. Gürsoy, Mr. Öner indicated that students answered the item without multiplication if you asked for an expression including symbols; however, they did not prefer factorization since multiplication was more straightforward. The second item was related to writing an algebraic expression based on a verbal expression. MSMTs generally stated that more than half of the students would correctly identify the algebraic expression of the given statement as 50 - x (See Table 4.3). Only Mr. Gürsoy argued that 30% of the students could answer the item precisely.

Table 4. 3. Predictions of MSMTs for correct answers of students in Item 2 (writing algebraic expression)

MSMT	Predictions of MSMTs for the ratio of correct answers
Ms. Ferhan	0.60
Ms. Burcu	0.60
Mr. Gürsoy	0.30
Mr. Yücel	0.60
Mr. Öner	0.60

**The ratio of correct student responses is 0.68 in ADT.*

Four MSMTs identified x-50 as the incorrect answer that students might give, and they mentioned no other solution that students might provide. Mr. Gürsoy asserted that students struggle when they try to write an algebraic expression based on another algebraic expression by stating that "they had difficulty writing 30-x for an algebraic expression if the other is x. They can state that the age of somebody is 10 and the other's is 20 if the addition of them is 30. However, they cannot identify it when we ask them to write it using x." Mr. Gürsoy criticized their teaching process by stating that they could not provide a solid background for students and described themselves as self-seekers since they conducted the examination-referenced teaching and did not give the rationale behind algebraic expressions. For this reason, as he declared, they lost 25% of students when they began algebraic expressions. In addition, Mr. Yücel added that he highlighted the importance of understanding the problem to overcome students' difficulties. Students were expected to compare two algebraic expressions in the third item and interpret them. In pre-interviews, MSMTs expected higher performance for students in Item 3, as seen in Table 4.4.

Table 4. 4. MSMTs' predictions for students' correct answers in item 3 (Which is larger task?)

MSMT	Predictions of MSMTs for the ratio of correct answers
Ms. Ferhan	0.70
Ms. Burcu	0.30
Mr. Gürsoy	more than 0.50
Mr. Yücel	0.70
Mr. Öner	0.50

**The ratio of correct student responses is 0.17 in ADT.*

As Ms. Ferhan pointed out, students had difficulty with the third item since there were unknowns on both sides. She stated that students struggle more when unknowns exist on both sides rather than just one side of an equation. Ms. Ferhan also mentioned students' difficulties with negative numbers and rational numbers. As she stated:

> Students always want to cope with positive numbers and find the result a positive number. They also do not want to see the result as a rational number. In addition, some students comprehend that only positive numbers are integers. Although we try to overcome this situation, they struggle to consider that the result might also be negative. If we say n is an integer, they automatically claim n is positive. Unfortunately, most of them think in that way. They have worked with positive numbers for many years and introduced negative numbers in the seventh and eighth grades. For this reason, some points may not be understood and thought of by students.

Ms. Ferhan predicted that the students who correctly answered the item could state, "We cannot say whether n is positive or negative and whether it is bigger or smaller than 3." However, the students who expressed invalid answers said that the answer was 3n and did not think in detail. Similarly, Ms. Burcu predicted that 70% of students who gave incorrect responses would express that 3n was greater than n+6. As she claimed, students might think that multiplication was bigger than addition. Also, students might consider that n + 6 was larger than 3n because six was greater than three. Like Ms. Ferhan, Ms. Burcu noted that students struggled with negative and rational numbers. As she pointed out, 30% of the students would state that we could not say which of them was larger. She said students could not interpret this task since they did not study such a particular point and asked if anybody could do it. Ms. Burcu noted that they have a title called the numerical value of algebraic expressions. Under this title, she said that they explain to the child what to do for a specific value of n, and he substitutes it. Such a task exists, but they did not handle such a comparison. Also, Ms. Burcu added that almost half of the students had difficulty with reasoning. Instead, they preferred randomly choosing an alternative to answer an item.

As Mr. Gürsoy stated, more than half of the students could express that they could not compare since it changed based on the values. Also, some students might substitute values into n, which were less or greater than three or equal to three. Then, they might explain that one of them was greater for values less or greater than three, and they were identical for n equals 3. Like Ms. Ferhan, Mr. Gürsoy expressed that students might say n + 6 since six was greater than three, and they might choose 3n since there was more n in this expression. In addition, as Mr. Gürsoy noted, most students might substitute numbers, such as 5, 10, and 20, concluding that 3n was greater than n+6. Mr. Yücel predicted that 70% of the students would correctly answer this item. However, he stated that he did not know if anyone who correctly answered that item could equate two expressions and presented that one was less or greater for a particular value of n. He was not sure if anyone could write that much detail. He thought they could not; they rarely answer it in this way." Like Ms. Ferhan and Mr. Gürsoy, Mr. Yücel predicted that 'n + 6 was the greater' response for students' answers since they made an addition. As Mr. Yücel and Mr. Öner noted, students usually substitute two different values for n to say which one was greater. Similarly, Mr. Öner stated that students did not use inequality and chose 3n because of the quotient 3, or n + 6 because it means six more than something. Also, he claimed that only 10% of students could express three situations based on the value of n.

Table 4. 5. Predictions of MSMTs for correct answers of students in Item 4 (functional thinking)

MSMT	**Predictions of MSMTs for the ratio of correct answers**	
	Item 4a	**Item 4b**
Ms. Ferhan	0.70-0.80	0.60
Ms. Burcu	0.60	most of the students
Mr. Gürsoy	0.40	0.40
Mr. Yücel	0.70	0.70
Mr. Öner	0.40	0.40

**Ratio of correct student responses is 0.34% for item 4a, and the ratio of correct student responses is 0.19 for item 4b in ADT.*

The findings related to Item 4 are presented in Table 4.5. MSMTs claimed that item 4a was familiar to students since they often worked with such tasks in the lectures. Ms. Ferhan added that if you asked to find the smallest number in this item, most students could write it. However, they might have difficulty when asked about the algebraic expression of the verbal statement. MSMTs stated that most students could generate the equation as x+x+1+x+2=84. As Ms. Burcu pointed out, the main reasons for students' difficulty with constructing equations were that they could not concentrate on the verbal algebraic expression and could not read it accurately.

Furthermore, as Mr. Gürsoy and Mr. Öner argued, some students might answer the item by dividing eighty-four by three. Based on their explanation, the average of three consecutive numbers gave the median by indicating the terms x-1, x, and x+1. Mr. Öner identified that they teach students to get the mean of consecutive numbers when asked to find the median. However, as Mr. Gürsoy pointed out, students could rarely give such responses to answer such a task. He added that some students might provide a logically incorrect answer, such as x+y+z=84. It might be correct, but this was not the exact answer we expected since it was not written based on a particular variable. MSMTs generally stated that students correctly answered item 4b. Ms. Burcu, Mr. Gürsoy, Mr. Yücel, and Mr. Öner claimed that the students who constructed the equation in item 4a could also identify what x stands for. Mr. Öner noted that 60% of the students answered item 4b by calculating the middle number as 84:3=28. Mr. Ferhan thought, "If students could not construct the equation, they would not have explained the meaning of the unknown. If they were asked to find the smallest number in this question, most students would be able to find the number." Like Ms. Ferhan, Mr. Yücel considered, "If you ask students, "What is the meaning of the number you find or what does this x mean?" I am sure that there would be some students who could not answer it correctly." Mr. Yücel explained this difficulty as students could not understand the logic of this issue and could not be motivated enough to solve the item. As he stated:

> I do not know what other reasons it might be originated. We are doing our best. Why are they struggling? However, we tell them to substitute the result they found in its place, verify it, and see whether we did it right. If the children comprehend this point, I do not think they will give the wrong answer anymore. The number I found here is x.

As Mr. Yücel noted, he considered that the best way of teaching the meaning of x was to substitute it in its place and observe whether it concluded with the correct result. As he stated, even the students who answered item 4a correctly might give incorrect responses for item 4b. Mr. Gürsoy also asserted that they have no problem-solving linear equations with one unknown. However, they had difficulties understanding the meaning of the terms x, unknown, letter, or symbol. Based on the interviews, MSMTs expected that most students would correctly answer Item 5 (See Table 4.6). The item aimed to investigate students' knowledge of rational numbers and equations.

Table 4. 6. Predictions of MSMTs for correct answers of students in Item 5 (rational numbers

MSMT	Predictions of MSMTs for the ratio of correct answers
Ms. Ferhan	0.80
Ms. Burcu	0.70-0.80
Mr. Gürsoy	most of the students
Mr. Yücel	more than 0.70
Mr. Öner	0.80

** Ratio of correct student responses is 0.40 in ADT.*

As MSMTs' statements illustrated, the difficulties and errors that students might face could be examined under two categories: challenges related to the solution of the equation and the structure of the algebraic expression and problems with rational numbers. Ms. Ferhan, Ms. Burcu, and Mrs. Gürsoy stated that students might make operational errors while solving algebraic equations, such as adding two and eight rather than subtracting eight from two or incorrectly dividing both sides of the equation. Ms. Ferhan declared that most students struggled to divide both sides of 9c = -6 with a correct number; should they divide both sides with minus six or nine? She asserted that if their purpose was to make the unknown isolated at one side of the equation, they should eliminate nine. Therefore, they should divide both sides by nine. As she mentioned, the reason might be related to students' problems with comprehension and making no repetition. Based on the MSMT's statements, she has never discussed understanding equality and equation. Also, she mentioned that students had difficulty with rational numbers since they were familiar with merely natural numbers since primary school. Therefore, they struggled while coping with such tasks. Similarly, Mr. Gürsoy declared that students do not think of negative numbers since they only consider positive numbers while solving questions.

Ms. Ferhan noted that they rarely made such interpretations and added that:

> How can the result equal 2 when you add something to 8? Students have difficulty in these tasks even if there is no unknown in the expression. Moreover, changing the algebraic expression *(from 8+x=2)* to 8-x=10 might do nothing. *(Students have difficulty in understanding)* How can the result equal 10 when we subtract something from 8?

Ms. Ferhan drew attention to operations with negative numbers and explained the reasons for incorrect answers as students' inadequate comprehension. As he said, there were several eighth-grade students with insufficient knowledge related to the topics of sixth grade. Therefore, they could not learn anything in seventh and eighth grades since subjects continued as a chain. She stated that even using distinct letters might affect students' performance. For example, using c rather than x might decrease their performance in this task since they were more familiar with x.

Mr. Yücel also asserted that writing the algebraic equations as $9x + 8 = 2$ and $8 + 9x = 2$ were entirely different situations for students. As he stated, students might have difficulty with the second one. In $9x + 8 = 2$, they quickly wrote $9x = 2 – 8$; however, they could not do the same operation with $8 + 9x = 2$ since the unknown term came after the number. As he stated, they transferred 9x next to 2 and wrote $8 = 2 – 9x$ to solve the equation. Then, they considered how I could subtract 9x from 2 and ultimately gave up doing the task. As he noted, students thought the unknown term must be at the beginning of the algebraic expression. He considered that students' difficulty might be resulted from doing similar tasks in the classroom. Therefore, he suggested that MSMTs solve different examples by changing the placement of unknown terms. In addition, he argued that the number of students would increase if the right-hand side of the equation were greater than nine, and consequently, in which x would be a positive number, such as $9x + 2 = 15$. He was surprised when I mentioned another difficulty students might face: interchanging the right and left-hand sides of the equation, such as $8 + 9x = 2$ and $2 = 8 + 9x$. He stated that this was the same as becoming at two opposite points of a bridge and claimed that primary school teachers were responsible for this problem. As Mr. Öner suggested, students would quickly answer Item 5 by solving the equation if they understood equations in the seventh grade. Like Mr. Yücel, Mr. Öner stated that students with low comprehension levels would not perform well in this task; however, he did not identify the comprehension level needed to succeed. Students who did not understand negative numbers may also consider the item erroneous. Based on the investigation of MSMTs' predictions, only Mr. Gürsoy expressed that students would perform poorly. In contrast, other

MSMTs stated that more than 40% of students would answer Item 6 correctly as presented in Table 4.7. Ms. Ferhan and Mr. Gürsoy argued that students might answer the item by substituting a value into the variable. Ms. Burcu also asserted that it was not something they were unfamiliar with but struggled with. Students were confronted with similar tasks, such as when the side of the square increased by two, how much did its area increase, or how much did its circumference increase?

Table 4. 7. Predictions of MSMTs for correct answers of students in item 6 (functional thinking)

MSMT	Predictions of MSMTs for the ratio of correct answers
Ms. Ferhan	0.50
Ms. Burcu	0.40
Mr. Gürsoy	0.15-0.20
Mr. Yücel	more than 0.80
Mr. Öner	0.70

** Ratio of correct student responses is 0.20 in ADT.*

However, she noted that students had difficulty doing such tasks. She also stated that students might distribute the quotient into the parenthesis and find the increment as six. Mr. Gürsoy declared that he was pessimistic about this issue and expressed that 15-20% of students could answer the item correctly. As he stated, incorrect answers might be related to students' misinterpretation of linear equations, such as if the left-hand side increases by two, the right-hand side should also increase by two. Like Mr. Gürsoy, Mr. Yücel mentioned the same errors students might make. He predicted that more than 80% of the students would answer it correctly. However, as he stated, we should ask the item specifying whether an increased or decreased when b increased by two in the algebraic expression $a = 3b + 4$ since students may not understand what you asked when you said how a 'changed' when b increased by two. As he clarified, their alternatives would be diminished when we asked the item by specifying 'whether it increased or decreased.' Therefore, there would be just two choices for students, either increased or decreased. Mr. Öner also asserted that most students would answer Item 6 correctly. He thought students who gave wrong answers would probably consider that the increment occurred four by four arithmetically since we summed up 3x with 4. Rather than considering the multiplication of 3 and x, they might focus on the summation of 3x with 4 in the algebraic expression. MSMTs concentrated on general operational errors that students might make in Item 6.

MSMTs' previews showed that more than half of the students would correctly solve equations (See Table 4.8). All MSMTs expressed that students might make operational errors, either erroneously transferring minus three to the other side or dividing both sides with two rather than minus two. The most frequently identified error was the first one, adding both sides three, in Item 7a.

Table 4. 8. Predictions of MSMTs for correct answers of students in Item 7 (solving equations)

MSMT	**Predictions of MSMTs for the ratio of correct answers**	
	Item 7a	**Item 7b**
Ms. Ferhan	0.60	0.60
Ms. Burcu	0.60	0.60
Mr. Gürsoy	0.60-0.70	0.60-0.70
Mr. Yücel	0.50	0.50
Mr. Öner	0.60	0.60

**Ratio of correct student responses is 0.54 for item 7a, and the ratio of correct student responses is 0.50 for item 7b in ADT.*

MSMTs provided similar predictions in Item 7b by stating that students might erroneously transfer the knowns and unknowns to different sides of the equality. MSMTs presented no opinion on students' use of parenthesis; however, 20% of students were unsuccessful while distributing the quotient within the terms in the parenthesis.

The MSMTs' predictions for Items 8 and 9 were presented in Table 4.9, two consecutive parts of the same problem of extending a tree by time.

Table 4. 9. Predictions of MSMTs for correct answers of students in Items 8 and 9

MSMT	**Predictions of MSMTs for the ratio of correct answers**	
	Item 8	**Item 9**
Ms. Ferhan	0.50	0.50
Ms. Burcu	0.70	0.70
Mr. Gürsoy	0.30	0.27
Mr. Yücel	0.60	0.60
Mr. Öner	0.82	0.40

**Ratio of correct student responses is 0.42 for item 8, and the ratio of correct student responses is 0.55 for item 9 in ADT.*

Ms. Ferhan and Ms. Burcu stated that students generally could give correct answers by constructing algebraic equations based on the situation. Also, Ms. Burcu expressed that students might count as 20, 30, 40, 50, etc., to get the length of the tree in the eighth month rather than using the algebraic equation. As she noted, students had difficulty in abstract thinking, and predicting students' performance in linear equations was challenging.

As Ms. Ferhan argued, students may not prefer to use equations although they actually could do it:

> Even the best students answer that there is no need to write the equation since they can already do this *(without writing the equation)*. Most students who say this can write it. When you say, let us write the equation anyway, we see that they can write it. However, many have trouble writing equations because they can do their operations without comprehension.

Like Ms. Ferhan, Mr. Yüksel mentioned that some students could not understand the logic and might do memorization. Therefore, they forgot how to construct equations, although they could do it well when they had just learned it. As he stated, they repeatedly did several practices and got feedback from most students about writing equations in the lectures; however, they were unsuccessful. He predicted that most of the students could find the length of the tree; however, students might have difficulty with writing the equation. He argued that students might respond to $y = x + 20$ or $y = 20x$ since they could not understand the problem, and he stated that there was no reason apart from that. Mr. Öner claimed that none of the students solved it by writing the equation if you do not say to construct it. Unlike Mr. Yüksel, he asserted that students did not write equations since solving without equations was easier or did not care about it. He declared that none would use equations if they were not indicated to solve with the equation. They might prefer to calculate it from their minds rather than equations, even in open-ended examinations and LGS. Moreover, some students might miss the initial length of the tree based on his statements.

MSMTs' predictions for the rate of students' correct responses in Items 10 and 11 were presented in Table 4.10. All MSMTs stated that students prefer solving problems without equations since they could solve them using alternative solutions. They argued that the incorrect answer that students might give would be y = 20 + 3x. They might forget to subtract 1 from x and calculate the amount as $y = 20 + 5 \cdot 3 = 20 + 15$.

Table 4. 10. Predictions of MSMTs for correct answers of students in Items 10 and 11

MSMT	Predictions of MSMTs for the ratio of correct answers	
	Item 10	**Item 11**
Ms. Ferhan	0.70	0.70
Ms. Burcu	0.40	0.40
Mr. Gürsoy	0.20-0.30	more than 0.20-0.30
Mr. Yücel	0.60	0.60
Mr. Öner	0.10	0.40

**Ratio of correct student responses is 0.25 for item 10, and the ratio of correct student responses is 0.66 for item 11 in ADT.*

MSMTs generally expressed three main reasons for students' preference for other methods rather than equations: alternative ways were more straightforward and more familiar, students struggled to cope with equations with more than one variable, and students had difficulty using different symbols in algebraic equations. As Ms. Ferhan stated:

> They learn to solve these questions earlier by doing reverse operations. We solve problems without using equations in the sixth grade. When we ask them to solve a problem, they refuse to solve it using equations, although they are very successful at equations. They say we have to do it with equations because we can already solve without them.

Similarly, Mr. Gürsoy, Mr. Yücel, and Mr. Öner declared that students would not construct the equation to solve the problem since they could solve it without using equations, which was more straightforward. Mr. Yüksel identified the reason as students' preference for an easier way, using arithmetics. As Mr. Öner claimed, students did not want to use equations, although they successfully used them. Instead, they would substitute values with unknowns or other methods to solve the problems. Ms. Burcu stated that students could not construct the equation and added that they struggled with equations even with one variable; therefore, they could not do the equations with two variables. Conversely, Mr. Gürsoy declared that one of the most challenging things for students was writing two or more algebraic expressions based on just one variable. As he noted:

> This is always problematic for students. For example, if the summation of two terms is 45, they will have difficulty writing 45 – x for one of them when the other is x. The most frequent problem I have ever seen is writing an algebraic expression based on an unknown used to write the other *(algebraic expression)* in a problem. I do not know how we can overcome this problem. Difficulties arise when writing two algebraic expressions (based on an unknown) in a word problem. They had not had that much trouble explaining the equation with two unknowns *(before the curriculum was changed at this level)*. They could quickly say x and y to write x + y = 45. Nevertheless, now, they can write the algebraic expressions if we give them hints, namely x and 45 – x.

As he stated, students could figure out algebraic terms based on different symbols; however, they had trouble using the same letter to write dependent variables in a word problem. He could not state the reason for this situation and give a solution to overcome this problem. He identified the students' problem as the difficulty of the transition of variables. Students' problems might be described as the difficulty of writing two or more dependent algebraic terms based on a particular symbol in a problem situation. Mr. Öner also mentioned another aspect that students could not understand the changeability of the symbols in algebraic expressions:

> They do not like using even x and y. Therefore, if we add a, b, and c, it becomes more complex. Also, they do not understand that the symbols representing the variables can change. They still use x and y even though we ask them to set up equations with other letters, such as a and t. In addition, they memorize the graphics axes, such as the x-axis always presents the time, and the y-axis presents the length. Although we exchange the x and y-axes in a graphic, they continue to solve the problem without considering which axes represent the time and the length.

MSMTs' predictions for students' performances in Items 12, 13, and 14 were presented in Table 4.11. MSMTs generally expected a higher success in Item 12, and they supposed that the rate of students' correct responses would be similar in Items 13 and 14. Mr. Gürsoy noted that an average student could also solve Item 12 without constructing the equation. As he stated, if a student could write the equation in Item 13, he could also solve Item 14. To build the equation, they would know the table's two points, the left and right ends. Therefore, they could already solve Item 13 if they could solve Item 12. Mr. Yücel also noted that students might have difficulty finding the general rule of the relationship between tables' and chairs' numbers. MSMTs gave examples of students' incorrect answers, such as ignoring the chairs at the left and right end, students' making a wrong ratio for the numbers of chairs, or incorrectly exchanging x and y while substituting the number of tables and chairs. Mr. Öner and Mr. Yücel also stated that students might calculate the result by enumerating the number of tables and chairs.

Table 4. 11. Predictions of MSMTs for correct answers of students in Items 12, 13, and 14

MSMT	**Predictions of MSMTs for the rate of correct answers**		
	Item 12	**Item 13**	**Item 14**
Ms. Ferhan	0.80	0.70	0.70
Ms. Burcu	0.40-0.50	0.40-0.50	0.40-0.50
Mr. Gürsoy	more than 0.29	0.29	0.29
Mr. Yücel	more than 0.40	0.40	less than 0.40
Mr. Öner	more than 0.30	0.30	0.30

**Ratio of correct student responses is 0.65 for Item 12, the ratio of correct student responses is 0.37 for Item 13, and the ratio of correct student responses is 0.47 for Item 14 in ADT.*

MSMTs usually stated that students could fill in the table and draw the graphics correctly. Only Ms. Ferhan talked about an incorrect response that students might give, such as mismarking the line in the graphics beginning from 100 km rather than 0 km. Although some MSMTs noted similar predictions for each item in the questionnaire, as seen in Table 4.12, they all indicated that students would have difficulty constructing the equation in Item 17.

Table 4. 12. Predictions of MSMTs for correct answers of students in Items 15, 16, and 17

MSMT	**Predictions of MSMTs for the rate of correct answers**		
	Item 15	**Item 16**	**Item 17**
Ms. Ferhan	0.90	0.70	0.60
Ms. Burcu	0.80	0.80	0.80
Mr. Gürsoy	0.60	0.60	0.60
Mr. Yücel	0.40	0.40	0.40
Mr. Öner	more than 0.80	0.40	0.05

**Ratio of correct student responses is 0.84 for Item 15, the ratio of correct student responses is 0.60 for Item 16, and the ratio of correct student responses is 0.33 for Item 17 in ADT.*

MSMTs expressed possible erroneous responses of students, such as t = 100m, x = 100y, and m = 100 + t. Mr. Gürsoy explained students' difficulties and errors as students' fatigue. When I asked him for an example of an incorrect response from students, he could not give such an example and stated that:

> Incorrect answer? I do not know. Not incorrect, but they cannot do it. I do not see what erroneous response they can give…The substructure is critical for this topic. Their difficulty in constructing equations is their abstract nature *(of equations)*. It is abstract, and they give up when confronted with x and y. If you told them to use x and y in parenthesis *(rather than the letters; m and t)*, it would also be different for them.

He asserted that using different letters also struggled with students. It might be easier for students to use the letters x and y since they are more familiar with them. Mr. Yücel also mentioned that students preferred to solve without finding the general rule, although we gave them problems with large numbers. He predicted they could not find the general rule and answer Item 17. As he stated, they did not like algebra and forgot what they had learned *(constructing equations)* after two or three months. Like Mr. Yücel, Mr. Öner also noted that most students could answer Items 15 and 16 correctly; however, they could not answer Item 17 since they did not like constructing equations.

4.3.MSMTs' Knowledge of Students' Algebraic Thinking in ADT

In this section, in-service MSMTs' knowledge of students' understanding, difficulties, and errors was investigated by comparing their conceptions regarding students' learning of algebra. Then, MSMTs' knowledge of students' algebraic thinking was compared with their predictions for students' performances in ADT and their interpretations of the students' performances in ADT after the analyses were completed.

4.3.1.Comparison of MSMTs' Knowledge of Students' Conceptions, Difficulties, and Errors with Their Predictions for Students' Performances before ADT and Interpretations of Students' Performances after ADT

MSMTs' responses to the questions in the interviews provided information about their knowledge of students' understanding of algebra. MSMTs' predictions and interpretations of students' responses were investigated based on four big ideas in algebra (Blanton et al., 2015), namely EEEI, generalized arithmetics, variable, and functional thinking (See Table 4.13).

Table 4. 13. Big ideas for learning algebra (Blanton et al., 2015)

Big ideas
1. Equivalence, expressions, equations, and inequalities (EEEI)
2. Generalized arithmetics
3. Variable
4. Functional thinking

4.3.1.1.MSMTs' predictions and interpretations based on EEEI and generalized arithmetic

As presented at the beginning of this chapter, MSMTs provided limited information about the prerequisite knowledge students should have prior to learning algebra in the interviews conducted before ADT. It might be beneficial to investigate MSMTs' interpretations regarding students' performances in ADT on behalf of the prerequisite knowledge MSMTs possess and their anticipations for students' performance regarding particular items in ADT. Thus, valuable conclusions could be drawn regarding MSMTs' knowledge of students' learning in algebra. Concerning the first big idea, *EEEI*, none of the participant MSMTs mentioned the role and importance of equality and the meaning of the equality symbol in algebra as a prerequisite knowledge. In the case of Ms. Ferhan, she just mentioned the capability of doing operations (addition, subtraction, multiplication, and division) as a prerequisite knowledge for learning algebra. She expressed that students had inadequate knowledge about properties of operations, such as commutative, distributive, and associative properties. Although she pointed out the difficulties of students based on manipulating numerical or algebraic terms in equations, she did not state the need for further knowledge of various forms of numbers (e.g., integers, rational numbers, irrational numbers) and students' comprehension of the term 'equality.' She also noted that some students might transfer a term on the other side of the equation, such as writing the equation $3(x + 5) = y$ as $(x + 5) = y – 3$. Such an error might be explained by the inadequate comprehension of dividing both sides by the same number, which requires students to understand 'equality.' However, she described this error as an arithmetic error rather than an insufficient conceptual understanding of 'equality.' Based on the results of Item 5, she argued that students did not consider numbers other than integers and may not think of negative numbers. In Item 7, students were expected to do algebraic manipulations to find the value of x in given equations. Ms. Ferhan appreciated the results in Item 7, stating that 54% was a good indicator of students' success since the negative sign often confused them about how to do the manipulations. Before ADT, she said students did not want to find the result as a rational number, and they assumed that integers were just comprised of positive integers. Therefore, they had trouble when the result was a negative number. She asserted that students had been dealing with positive numbers for long years. However, they were introduced with negative numbers in 7th grade. She thought this issue might be one of the reasons for students' struggle. After Ms. Ferhan analyzed students' performance in ADT, she shared her explanation of doing manipulations with negative numbers:

> We always explain to them to distribute *(the quotient)* within the parenthesis if there exists a minus *(sign)* or a quotient. Furthermore, directly skip the parenthesis if there is nothing in front of the parenthesis. Although we stated, again and again, they make the same errors as if we had not taught them. I do not know what the reason is.

While talking about her way of teaching algebraic manipulations with negative numbers, she skipped some words or used phrases such as 'skip the parenthesis if there were nothing,' which students might learn erroneously. Using such an expression, students may be confused about the meaning of 'nothing' when there is a plus sign or a positive number as a quotient in front of the parenthesis. She added:

> Students could not conceptualize whether the minus *(sign)* belongs to the operation or this *(the quotient).* They have difficulty understanding that these two are interchangeable things. I think this has its source in the objective of addition and subtraction with integers in the 7th grade. Whether it belongs to 7 or the subtraction operation *(in Item 7b)*? That is when students begin to lose interest in math. Students stop or continue learning mathematics after the operations with integers and rational numbers. I guess they cannot make embodied. They cannot understand it, and I do not know why. What kind of a path should we follow?

As she stated, they teach students negative numbers and operations with negative numbers by using daily life examples at the beginning of the topic, such as an elevator or weather forecast problem. However, she noted that they continued with many algebraic manipulations and overwhelmed students with too many algebraic operations. She inferred that they might make each operation more concrete *(by using daily life examples)*. She also stated that students often struggled to construct equations in algebra word problems. They could not learn linear equations in the 8th grade since they could not conceptually understand equations in the 7th grade.

Other MSMTs also frequently mentioned the significance of operations and numbers for learning algebra. Ms. Burcu noted that students should be able to do operations, addition, subtraction, multiplication, and division. Moreover, she highlighted that students should conduct those operations with fractions and rational numbers. They should also be able to use algebra-related key terms such as multiple, more than, less than, and half of something or one-third as prerequisite knowledge. She frequently stated that students had difficulty transitioning from verbal statements to algebraic expressions, especially when trying to construct two dependent algebraic expressions. To illustrate, as she said, while setting up the algebraic expression of the statement 'four more than twice the other', students could write 2x + 4, but they could not specify the other as x. As she noted, if somebody solved 100 questions *(for each day)* in a couple of days and 150 questions in the remaining days in the same week, students may have difficulty writing x for the days they solved 100 questions and (7-x) for the days solved 150 questions. Although she focused on such an essential point about algebraic expressions, she could not provide a concrete explanation for students' difficulties in those situations. She also said that students could quickly identify the number of girls if there were 10 boys in a classroom of 40 students. However, they could not express the number of girls as 40-x if there were x boys in a school of 40. Students could do the tasks with numbers but could not do them when they should use algebraic expressions. She noted that describing the remaining term was fundamental to algebra. In Item 2, Ms. Burcu predicted that 60% of the students correctly responded to the item resulting in 68% in ADT. After ADT, she appreciated the results of Item 2 and stated that some students might also give x-50 erroneously for such tasks. She attributed students' difficulties and erroneous responses in such tasks to an inadequate understanding of arithmetic operations, addition, subtraction, multiplication, and division.

Moreover, she mentioned that the classes were so crowded, and it was challenging to care for each student while teaching mathematics. Based on Item 7, she stated that students often made errors while using negative numbers, as she emphasized in the interviews before ADT. She generally focused on using the negative sign in the interview before ADT. Moreover, she stated that 'they might transfer the term to the other side without changing the sign.' Like other MSMTs, she often uses this term, transferring the term to the opposite side rather than making the same operation on both sides of equality. After I said that students gave incorrect answers while distributing the quotient through the terms in the parenthesis, she was disappointed. They did not show all manipulations about doing the same operation on both sides of the equation as they did in their courses to get students quicker while doing the tasks. MSMTs typically present the solution path of the students for Item 7, as shown in Figure 4.4.

In the case of Mr. Gürsoy, he pointed out the importance of doing operations as prior knowledge for algebra. Moreover, he expressed that students should conceptualize integers and 'positive' and 'negative' terms. Like Ms. Burcu, he identified students' difficulties constructing two dependent algebraic expressions in an algebra word problem. For example, as he stated, they typically struggled to write (45 – x) if the other expression was x in a problem situation. Also, they had difficulty calculating the number of feet on a farm, including chickens and cows.

Item 7a	Item 7b
$-3 - 2x = -9$	$3x + 2 = -7(x - 6)$
$-2x = -9 + 3$	$3x + 2 = -7x + 42$
$\frac{-2x}{-2} = \frac{-6}{-2}$	$3x + 7x = 42 - 2$
$x = 3$	$10x = 40$
	$x = 4$

Figure 4. 4. Typical solution paths of students for Item 7a and 7b written by teachers in the questionnaire before ADT

In such a problem, they directly tried to label the dependent algebraic expression as y and the independent variable as x since it was easier for them rather than writing both dependent and independent expressions based on x. He identified the causes of other errors and difficulties of students as ‘they do not know’ rather than a detailed description. Mr. Yücel also highlighted the importance of the capability of doing operations (addition, subtraction, multiplication, and division) as a prerequisite knowledge for learning algebra. Also, he noted that students should conduct those operations with rational numbers. As he stated, students struggled with the addition and subtraction of integers. He also mentioned errors students typically make while dividing both sides of the equation with the same number. For example, he noted that some students divide -2x to 2 rather than -2 while dividing both sides of the equation with -2. He explained the reason by stating that students forgot the things they had learned since they did not repeat them and drill at home.

Mr. Gürsoy and Mr. Yücel also expressed students' difficulties with negative signs while answering Item 7. Mr. Yücel described the mistake of students while doing the distribution through the parenthesis as a lack of attention. Although he stated the same term ‘transferring the term to the opposite side,’ he noted that students might also have difficulty dividing both sides of the equality with the same number:

> First, we say that we subtract the same number from both sides of the equation. Then we say -3 passes to the other side as plus 3 to be quicker while doing practice. We state that the transfer of one term to the other originates

> from doing the same operation on both sides of the equation. However, it is forgotten, and everyone remembers that −3 passes to the other side as +3. Students made more mistakes while dividing both sides of the equation. The reason is that even teachers solve the task 2x = 6 by just writing x = 3 rather than presenting the division with two on both sides of the equation.

As he added, in such a task, students prefer to answer the question, twice of which number makes 6? If x was a rational number like $-\frac{2}{3}$, they would have more difficulty. Similarly, Mr. Öner noted that students might substitute a value to find x rather than solve the equation. Mr. Yücel also attributed that students forgot such manipulations because they could not associate them with daily life. Like other MSMTs, Mr. Öner also focused on operations and priority rules in those operations. Additionally, he expressed that students should know the statement that 'the known terms were at one side of the equation and the unknowns were at the other.' Apart from other MSMTs, he also concentrated on understanding the meaning of x and unknown and the transition among words or sentences, abbreviations, and symbols in algebra. Mr. Öner also noted that students should effectively interpret graphics, understand the meaning of the interchange between the x-axis and y-axis, and how this change occurred.

Ms. Ferhan and Mr. Öner declared that Item 7b was more difficult since there was x on both sides of the equation, making students struggle more. The most frequent error MSMTs reflected on was using the minus sign while doing operations and solving equations before ADT. MSMTs expressed similar interpretations in the interviews after ADT by stating that the only thing students had difficulty with was rational and negative numbers. MSMTs did not utter other ideas, such as a conceptual understanding of equality and distributive property (See Table 4.14).

MSMTs' statements for Item 1 were investigated considering the big idea of *equivalence, expressions, equations, inequalities (EEEI),* and *generalized arithmetics*. Ms. Ferhan stated that 70% of the students could correctly show equality using *(algebraic)* simplification. Also, as she expressed, students could do this task if they had no gaps related to the factors and multipliers. After examining the results, she appreciated students' performance in Item 1 and expressed that students performed well in this item, although 56% of the students could correctly respond.

Table 4. 14. Summary of MSMTs' statements for Item 7a and 7b

	Prerequisite knowledge	Predictions before ADT	Interpretations after ADT
Ms. Ferhan	• Operations (addition, subtraction, multiplication, and division)	• They might have difficulty with the negative sign.	• Inadequate understanding of the use of parenthesis • Inadequate conceptual understanding of negative sign •
Ms. Burcu	• Operations (addition, subtraction, multiplication, and division) • Operations with fractions and rational numbers • Knowledge of arithmetic terms (e.g., multiple, more than)	• They might have difficulty with the properties of the negative sign and operations with the negative sign	• They always make errors with a negative sign while doing algebraic manipulations
Mr. Gürsoy	• Operations (addition, subtraction, multiplication, and division) • Integers • The terms negative and positive	• They might have difficulty with the negative sign.	• All students can solve equations. • They often make errors while doing operations with a negative sign.
Mr. Yüksel	• Operations (addition, subtraction, multiplication, and division) • Operations with rational numbers	• They might have difficulty dividing both sides with the same number. • They might have difficulty transitioning a term to the other side of equality.	• Students have more difficulty with dividing both sides with a number • If one of the quotients were a rational number, they would have more problems.
Mr. Öner	• Operations (addition, subtraction, multiplication, and division) • Priority in operations	• They might make errors while doing the transition between knowns and unknowns • They might substitute a value to solve the equation.	• 7b is more difficult for them since students might have difficulty when there are unknowns on both sides of the equation. • They might have difficulty using the distributive property for negative numbers

She provided no further explanation based on students' difficulties and errors for Item 1 (See Table 4.15). Similarly, Ms. Burcu predicted that 80% of the students could correctly answer Item 1 and added that most of the students gave such responses, $14 \cdot 11 = 14 \cdot 11$ or $7 \cdot 2 \cdot 11 = 7 \cdot 2 \cdot 11$. After seeing the results for Item 1, she was disappointed and attributed this result directly to students, such as not studying adequately and not doing practice. She expressed, “I do not know whether they learned equality in primary school, but they were taught it in the 5th grade. It is the equality in multiplication that was taught since 5th grade.”

Those statements might indicate Ms. Burcu’s knowledge based on the prerequisite knowledge and conceptual understanding of equality required for students to learn algebra. She could not express when equality was first introduced to students in mathematics. She stated that equality in multiplication is crucial for all students rather than highlighting the ‘equality’ itself. As she described, she did not give students such tasks as interpreting equality. Instead, they asked students how to factorize a number:

> We are talking about the equality of the left-hand and right-hand sides, but we do not focus on such an interpretation. We rarely do. Nevertheless, for example, halving 22 and doubling 7 is the logic of inverse proportionality in 7th grade. So, if one factor is doubled, the other is divided by two. I use it, in general, to get children to use equality rather than *(arithmetic)* operations. It is my typical example: 250 multiplied by 4 is 1000. I make the first one 500. the other 2 again 1000. If you give the logic of inverse proportion so that the result remains constant, the factors are always inversely proportional. One is doubled while the other is halved. I mention it in the lectures. Nevertheless, I am talking about inverse proportion rather than equality.

Based on the expressions of Ms. Burcu, she also gave her students such tasks in the lectures. However, she explained the covariation of 7 and 22 by focusing on the inverse relationship rather than highlighting the equality between two multiplications. Her statement ‘250 multiplied by 4 is 1000’ presented that she reflected the equality as “the answer or result of an arithmetic operation” rather than stating that ‘250 multiplied by 4 equals 1000’ as “an equivalence relation between two quantities” (Asquith et al., 2007, p. 253). She also mentioned a similar relationship in decimal numbers, such as multiplication or division of a decimal number with 10 moves the comma right or left, respectively. That is, there was always an inverse relationship between multipliers. She stated, “In equality, we mostly focus on multiplying and dividing both sides by 2. Moreover, adding to and substructing from 2 on either side. Nevertheless, we mostly use such interpretations in inverse relationship rather than equality.” When I asked her whether it would be beneficial for students to be reminded about equality, she said she would include reminding students' minds about equality in her algebra courses from now on. I deduced that she learned from the results of this item to use in her future algebra classes.

Table 4. 15. Summary of MSMTs' statements for Item 1

	Prerequisite knowledge	**Predictions before ADT**	**Interpretations after ADT**
Ms. Ferhan	• Operations (addition, subtraction, multiplication, and division)	• 70% of the students could do factorization. • They might multiply the numbers if they think it cannot be shown without multiplication.	• The performance of students (56%) is so good.
Ms. Burcu	• Operations (addition, subtraction, multiplication, and division) • Knowledge of arithmetic terms (multiple, more than, etc.)	• 80% of the students could do factorization. • They might multiply the numbers if they think it cannot be shown without multiplication.	• Their performance (56%) is a disappointment. • The knowledge required for the item is similar to the inverse relationship.
Mr. Gürsoy	• Operations (addition, subtraction, multiplication, and division)	• 60% of the students could find the solution without multiplication. • Even high achiever students would conclude the result by multiplication if they were not told not to use it.	• Their performance (56%) is acceptable.
Mr. Yüksel	• Operations (addition, subtraction, multiplication, and division)	• 10-20% of the students could find the solution without multiplication. • Most students find the result by multiplication since the numbers are one or two digits.	• I thought about the successful students while identifying 10-20% of the students before ADT.
Mr. Öner	• Operations (addition, subtraction, multiplication, and division) • Priority in operations	• At most, 10% of the students could find the solution without multiplication. • No students use factorization.	• Student performance exceeded my expectations. • I thought that they preferred to use the easier way.

Based on Item 1, Mr. Gürsoy stated that even high achievers would have found it by multiplying if you did not ask them to solve without multiplication before ADT. He noted that the test results were similar to his predictions for Item 1. He also expressed that one of the most frequently encountered errors in students' responses was misusing the parenthesis. For example, they wrote the expression as x-2-x-3 while subtracting x-3 from x-2, distributing the negative sign to only the first term of the expression. Mr. Yücel also noted that most students would find it by multiplication since they were one or two-digit numbers, and just 10-20 % of them could show it without multiplication before ADT. He also said that he could not understand what was expected from students in Item 1. After I explained to him, he noted that some students might give such an answer: "First, I multiply 14 with 10. then add 14." After ADT was done, he asserted that he might think that only high achiever students could do the task. He did not provide any further explanation based on Item 1. Mr. Öner provided a similar argument to Mr. Yücel by stating that students do the task without multiplication if you give them multiplication of algebraic expressions with unknowns. However, they would not use other solution paths if you gave multiplication of numbers since doing multiplication was more straightforward. He added that he did not think anyone would use factorization such as $7 \cdot 2 \cdot 11 = 14 \cdot 11$. After analyzing ADT results, he appreciated the results since he considered that students would perform an underachievement in Item 1 (See Table 4.15).

To sum up, MSMTs provided narrow information based on the prerequisite knowledge students should have to learn algebraic expressions and equations. To illustrate, none of the MSMTs advocated that students should conceptualize equality as prior knowledge for learning algebra. Therefore, they did not have any interpretation based on students' difficulties and errors related to the inadequacy of knowledge in terms of equality. MSMTs only expressed the need to learn arithmetical operations (addition, subtraction, multiplication, and division) to write and manipulate algebraic expressions and equations. Although they thought students were required to learn operations before algebra topics, they did not express any interpretations related to operations when they observed students' difficulties and errors in setting up algebraic expressions and equations. MSMTs who identified the knowledge of different types of numbers as a prerequisite for learning equations interpreted students' difficulties and errors based on the inadequate knowledge of numbers such as negative numbers and rational numbers. Rather than specific notions related to learning algebra, MSMTs often mentioned other factors unrelated to students' difficulties and errors. The following section will investigate MSMTs' predictions and interpretations of students' performance on variables.

4.3.1.2.MSMTs' predictions and interpretations based on variable

The third big idea was variable, which refers to "symbolic notation as a linguistic tool for representing mathematical ideas in succinct ways and includes the different roles variable plays in different mathematical contexts" (Blanton et al., 2015). The analysis of students' responses to Items 3 and 4 was examined based on the big idea of *variable*. Ms. Ferhan mentioned a crucial point in learning algebra: distinguishing what is the unknown in a word problem. As she stated, students had difficulty discriminating which object was labeled x and solving the problem using accurate manipulations using x. Instead, as she declared, students did random manipulations with the numbers they saw in the problem. Therefore, she stressed that students struggle with the variable at most. Although she highlighted the difficulties students faced with conceptualizing variables, she did not provide an opinion about students' difficulties in understanding variables while making predictions about students' performance in Item 3. Similarly, she did not offer such a statement while considering the students' responses in Item 3 after ADT was performed. Instead, she inferred that "The disparity *(between correct and incorrect answers)* is huge in Item 3. They could not conceptualize being smaller or greater in an algebraic expression. Also, the transfer of one side of the equation to the other side as negative is not well understood." Moreover, she added, "I guess students could not understand the difference and relationship between an inequality and an equation although we thought they understood." Based on her statements, it might be inferred that she attributed students' difficulties and errors to other factors rather than conceptualizing variables in the 3rd item. Before ADT was conducted, she just predicted that students only said 3n was greater, and they did not give further detail. After ADT was implemented, she was surprised when she heard the number of students giving different erroneous answers, such as perceiving n + 6 as 6n and concluding that n + 6 was greater than 3n. Rather than making the same operation on both sides of the operation, she frequently used the phrase 'transmission of one term/side to the other side.' Therefore, this phrase might also be one reason for students’ misunderstandings related to equations. She interpreted that students might have difficulty in Item 3 since there were unknowns on both sides of the equation, making them confused about how to solve the equation.

As she declared, they often had difficulty transitioning a verbal statement to an algebraic expression which was an obstacle for them while solving problems. Finally, she concluded that they should focus on the relationship between different algebraic expressions since students could not understand them. In Item 4, she stated that students were familiar with this task but may have difficulty writing the terms x, x + 1, and x + 2. After the test was conducted on students, she explained the sources of errors students made in ADT. For example, she noted that the students who responded as x + y + z = 84 could not remark on the relationship between the terms in the equation in Item 4a. Also, for the students who responded with x + 2x + 3x = 84, she interpreted that they might have confused the terms ‘consecutive’ and ‘multiple.’ Indeed, students might have struggled to write the algebraic (symbolic) notation of consecutive numbers rather than being confused about ‘consecutive’ and ‘multiple.’ It might be said that they did not know how to write it in the symbolic form. She specified some sources of students’ errors; however, her explanations for students' difficulties were narrow. After seeing the results of Item 4a, she stated that students could not correctly respond to Item 4b if they had trouble with Item 4a. She also added that students’ performance would be better if they were directly asked the value of the small number rather than the meaning of the unknown. As she inferred, MSMTs should solve more examples for students in the classroom. She also criticized asking students to find the value of x rather than the algebraic expression or the meaning of the unknown. Moreover, as she noted, students did not like doing extended operations using their pencils and did not prefer using algebraic expressions. Instead, they use arithmetic solution paths they learned in primary schools, such as inverse operations. Therefore, as she stated, those might be the reasons for their incorrect responses to Item 4.

Ms. Burcu did not offer any prerequisite knowledge that students had related to the concept of variable and any idea on students’ difficulties concerning variable in the interview before ADT. She just stated that students should know the meaning of the terms such as ‘multiple, more than, less than, and one-third.’ She declared that 30% of the students could respond that ‘it cannot be told’ before ADT was conducted. She stated that they had not discussed these types of tasks in the lectures. For this reason, it was impossible to interpret it for students since they did not go into that detail too much. Ms. Burcu predicted that 70% of the students would give the answer $3n > n + 6$ since they often thought that multiplication was greater. Moreover, they would not consider negative numbers, proper fractions, or numbers other than natural numbers. She concluded that at most half of the students could make reasoning on such tasks. After ADT, Ms. Burcu said that she was hopeless when she said 30% before ADT, but she should have been even more desperate after observing students' results in ADT. She stated that:

> We are not doing such tasks in algebra classes. Maybe, I can give students such tasks while teaching algebraic expressions. They cannot answer this item since we are not focusing on that point. There is a topic called the numerical value of algebraic expressions. In this topic, we tell the students what to do when they are given a value to substitute for x, and they substitute it. However, there is no such comparison in algebra classes.

After she examined the responses given by students for Item 3, such as $3n > n + 6$ since 3n was a multiplication, she expressed that students talked nonsense. However, she mentioned the same error in the interviews before ADT by stating that students might consider $3n > n + 6$ since they were prone to think multiplication was greater than addition. Based on another erroneous response, $n + 6 > 3n$, since 6 was more significant than 3, she thought those were the students from the mediocre classroom level. She offered no other idea concerning misunderstandings among students. After she observed the students' responses in Item 4a, she argued that students should have been able to write this *(the algebraic expression)*. As she stated, students' difficulty might stem from their previous learning of dividing 84 by 3 to find the median, and some students tried to solve it without constructing an equation, but they should set it up. When I asked the reason why students had difficulty in writing the equation, although they could find the numerical value of an unknown, she considered quite a long time and stated that:

> They have no trouble finding the numerical value, but they may struggle with comparison because they have never compared two algebraic expressions

> before *(she laughs and considers)*. I suddenly have many ideas in my head. I will prepare and pose such questions to my future students so that they can think about such tasks.

She interpreted both Item 3 and Item 4 and stated that their students successfully found the value of an unknown in an equation; however, they were not good at comparing two algebraic expressions. Therefore, the ratio of the students who could give the correct responses was satisfactory since their students could not complete such a task. As she declared, students began memorizing algebraic processes when they did not understand. Based on Item 4b, she noted that they frequently get students to label the asked expression x in an algebra word problem. To illustrate, if the small number were asked, the small number would be called x; if the medium number were asked, the medium number would be called x.

Although they often specified the solution path for the students and did not get them to use different solutions for the tasks, she complained that students' creativity was weak in constructing such symbolic expressions. As she expressed, students wrote +6 when required to register x+6 since they could not create it in their minds. She also identified mathematics itself as a factor in students' difficulty since it was more difficult than other courses. Based on Ms. Burcu's statements, it might be concluded that she provided weak information depending on students' prerequisite knowledge to learn variable. Moreover, she did not explain students' difficulties and errors in solving items based on the variable.

Like Ms. Burcu, Mr. Gökhan and Mr. Yücel also offered no prerequisite knowledge for students concerning variable while learning algebra (See Table 4.16). Mr. Gökhan did not mention any difficulties students faced related to the variable. Mr. Yücel only stated that algebra, x, and unknown were all abstract concepts. In Item 3, Mr. Gürsoy predicted that more than 50% of the students could express that 'it changes based on the value *(of the unknown)*.' He stated that students might give responses that represented it could not be determined since the variable could refer to multiple values. Moreover, he provided an additional answer that students might give in such a way that they tested a single value for n.

Table 4. 16. Summary of MSMTs' statements for Item 3

	Prerequisite knowledge	Predictions before ADT	Interpretations after ADT
Ms. Ferhan	• Operations (addition, subtraction, multiplication, and division)	• 70% of the students can answer that n might be smaller, equal, or greater than 3. • Some students might say 3n was greater without further explanation.	• The knowledge of the transition of one term to the other side *(of equality)*is inadequate. • They could not understand being smaller or greater in algebraic expressions • They could not understand the difference and relationship between equation and equality. • They had difficulty since there were unknowns on both sides of the equality
Ms. Burcu	• Operations (addition, subtraction, multiplication, and division) • Knowledge of arithmetic terms (e.g., multiple, more than)	• 30% of the students can say that it cannot be displayed. • It is impossible for them to state the three situations for n < 3, n=3, and n > 3. • 70% of the students identified 3n > n + 6 since 3n is a multiplication. • Students cannot make a reasoning.	• We can get students to do such tasks similar to Item 3. • Those students who gave incorrect responses are mediocre-level students. • We do not work on comparing different algebraic terms; instead, we usually substitute a value for the unknown.
Mr. Gürsoy	• Operations (addition, subtraction, multiplication, and division)	• More than 50% of the students can say they cannot determine. • Some students can identify the situations for n < 3, n = 3, and n > 3. • They might say that n + 6 was greater since 6 > 3, or 3n was greater since there are more n in 3n. • Most of the values that students might substitute get 3n to become greater.	• I thought that they could identify it based on the value of n. • We have never mentioned the change in the term based on the variability of the unknown. • We have never focused on the comparison of algebraic terms; instead, we have focused on the sequence of various forms of numbers • Multiplication usually gives greater results for the values they substitute; therefore, they might say that 3n is larger. • Students cannot make interpretations in algebra.

Table 4.16 (continued)

Mr. Yüksel	• Operations (addition, subtraction, multiplication, and division)	• 70% of the students can respond correctly. • They often substitute two values for n to determine which one is greater.	• I thought they could make the comparison since they had learned to transition between verbal and algebraic expressions. • The only missing point of students was that they could not see that one was greater up to a value and the other was greater beyond this value. • We should teach students again after observing their thinking for such tasks.
Mr. Öner	• Operations (addition, subtraction, multiplication, and division)	• 50% of the students can respond correctly. • 10% of the students can determine three situations for n < 3, n=3, and n > 3. • Students express that 3n is greater since its quotient is 3 or n + 6 since it is 6 more than n. • Students substitute different values for n to decide which one is greater.	• How many of them respond to it by substitution? *(he asked several times)* • I expected that most students would answer the task by substituting different values. • The reason might be a poor conceptual understanding of sixth and seventh-grade topics. • The other reason might be a lack of motivation to do those tasks. • There is no problem regarding the conception of variable here. • I am sure they know the variable and could state the relationship between these two terms since they learned inequalities. • There is no such objective in the curriculum, such as a comparison of algebraic terms; instead, we only ask students to specify the values of n if one algebraic term is greater than the other.

Therefore, they might conclude that one of them was greater if n > 3 or n < 3, and they might interpret that they were equal if n = 3. In other words, students' conclusions may vary based on the value they tested. He added that most of the values students substituted made 3n greater since they typically tested values like 5,10. and 20. Mr. Gürsoy also remarked that students might answer as n + 6 was greater since 6 > 3, or they might answer as 3n since there were more n in 3n. After he observed that 16% percent of the students could state that ‘it cannot be determined since it changes based on the value of n,’ he was disappointed. He concluded that they taught students to compare, for instance, rational numbers and squared numbers. However, MSMTs did not mention such an order in algebraic expressions or that n varies according to the value of the unknown. Also, students could not interpret since they did not do such tasks in algebra classes. Therefore, they might answer by substituting a value to n randomly or think that $3n > n + 6$ since multiplication always gives larger results.

Moreover, Mr. Gürsoy stated that students were prone to take greater values such as 10. 11, and 12 while substituting. As a result, they inferred that multiplication was greater since it gave greater results for large numbers. He also pointed out that he was surprised and disappointed since only forty-four students could respond to the item correctly. However, ninety students could pass the examination of science high schools. He concluded that students did not conceptually understand this point and struggled with algebraic expressions and unknowns. Mr. Gürsoy pointed out that they did not give students such tasks and defined themselves as self-seekers since they focused on the points they were responsible for in national examinations. He also noted that students might have difficulty even describing the meaning of 3n and n + 6 as three times something and six more than something. He expressed that:

> The only point I mentioned on this *(variability)* is when describing the constant term. I ask my students why it is constant. To illustrate, for 3n+8, why does the +8 constant instead of 3n? Only three or four students could respond that it depends on the value of n in 3n, but 8 was always 8. I am trying to touch it, but I do not know if I can do it.

As he declared, he explained to students the meaning of being constant. But, he was unsure whether he adequately taught the importance of the variable. Moreover, he mentioned the restrictions depending on the curriculum while teaching algebra to the students:

> Actually, I may not be focusing too much on this subject. It is also about objectives in the curriculum, and we must follow them. The first objective is the addition and subtraction of algebraic expressions. Therefore, I explain it to the students by doing plenty of practice. Also, I have them find the area of a rectangle with a short side n + 6 and a long side 3n-5. So, what is the algebraic expression that gives the perimeter? I continue with such examples based on daily life. But you are right; I have not got them to consider such comparisons.

He continued that he did not know whether there was such an objective related to comparing algebraic expressions in the 6th grade. He expressed that algebraic expressions were the most crucial topic he focused on in the curriculum since students were faced with algebraic expressions in most of the topics they were introduced to in the future. After observing the results for Item 4, he said he would expect higher student performance. He noted that he typically tried to explain these tasks to students by connecting them with arithmetic mean. However, he considered that his way of teaching was not adequate. He was surprised since Item 4 was familiar to students, and even moderate students could answer it correctly. He expressed the answer of $x+x+x=84$ as interesting. He also focused on the response of $3x = 84$ that students typically gave for Item 4a since his solution path could also find this equation: the mean of the smallest and greatest number gave the median. Therefore, multiplying the median by three equaled 84. It might be inferred that MSMTs also get students to solve algebraic problems using methods other than algebraic processes. He noted that it was interesting since they knew what x was while solving the equation; however, they could not express the meaning of x after solving it. He could not see the reason for students' underachievement in this task. After I asked him whether it might be related to an inadequate conceptual understanding of the variable, he remarked that it was related to the variable as in Item 3. As he declared, if I conducted ADT on students two weeks after learning the equation problems, they would perform better in the items depending on the variable. Therefore, he implied that students' difficulty in Item 4 might occur since students forgot the subject. He said that he would focus on the meaning of x after that time and stated that if he had not seen these results, he would not have cared much about this issue since students get used to the subjects after a while. He inferred that the problem was at this point, and the subsequent issues were always about the unknown.

Mr. Yücel predicted that 70% of the students could correctly answer Item 3 and stated that it varied based on the value of n. Although he considered that 70% of the students could correctly answer Item 3, he said they could seldom give such an answer by identifying the reference number 3 and concluding that one was greater if $n > 3$ and the other was greater if $n < 3$. He added that they might say $n + 6$ is greater because 6 was greater than 3. When there was an increment, they said it was greater. After I told him one of the students' erroneous thoughts, multiplication always gave greater results than addition; he was surprised because he had not realized it before in his algebra classes. He predicted that students typically substituted two different values to n and decided which one was greater, such as 3n was bigger when n=10 and n + 6 was bigger when n=2. After observing the results, he expressed that this confused him since he thought that students could determine the outcomes based on different values of n if they could translate from verbal statements to symbolic expressions, as in Item 2. He suggested that interviews might be done with students to investigate their way of thinking behind the responses they gave on the test. After examining students' responses, such as n + 6 was greater since 6 > 3, and 3n was greater since multiplication always gave larger results, he concluded that students tried to make the connection. As he described, the only shortcoming of students was that they could not notice that one was greater for particular values of n and the other was greater for the remaining values of n. Based on the answer of n + 6 > 3n as 6n > 3n, he specified that he could not notice such an error in his classes. He just thought that students might consider n + 6 was bigger since 6 > 3. Finally, he interpreted that they should teach students algebra after observing the results of the analysis in ADT by being aware of the mathematical processes in the inner world of children and the situations where they can make mistakes. Therefore, they could explain it effectively since they know students' shortcomings.

In Item 4, Mr. Yücel was surprised when he saw the results since he thought that students were familiar with such tasks as they did similar practices like box (☐), box plus 1 (☐+1), and box plus 2 (☐+2) in the fifth grade and constructed the symbolic expression of it with x in the sixth grade. He also mentioned multiple-choice items given to students as one of the reasons for their underachievement. Since students were given the alternatives in the item, they got used to solving the item by substituting the values into the equation rather than doing the required algebraic procedures as the items asked for, such as the smallest number or medium number. He highlighted that students had been familiar with such tasks since the fourth grade; therefore, he would expect higher performance from students. One of the other erroneous responses that students gave was x+y+z=84 for Item 4a in ADT. He interpreted that students could not identify the relationship between terms in x+y+z=84. Moreover, he was surprised since students wrote the third variable, z, while constructing the algebraic equation, although they knew algebraic equations with two unknowns at most, namely x and y. He stated that motivation also influenced students' success in mathematics tasks. After examining the students' responses in Item 4b, he inferred that students might not understand the question. If students thought they could not do the task, they skipped it without trying to answer it. He felt that students could do Item 4b if they understood the item. However, he expressed no concern about students' understanding of Item 4b in the interviews before ADT was conducted on students. Moreover, MSMTs' statements showed that they split the classes into three groups, 30% were low-level, 40% were mediocre-level, and 30% were high-level students. They made their interpretations depending on those categories. To illustrate, Mr. Yücel pointed out that mediocre and high-level students should do Item 4 correctly since they did similar tasks in the lectures, including the transformation of verbal statements to symbolic expressions. However, the low-level students could not already do Item 4 because they were unsuccessful. Nevertheless, as he stated, the mediocre and the high-level students could not do it.

Conversely, Mr. Öner highlighted such prerequisite knowledge, what is the unknown, what does x mean, and what does x refer to before learning algebra? He also identified that students should know the rule of transition of the unknowns on one side and knowns to the other, in other words, how an unknown should be passed to the other side. After analyzing the results for Item 3, he was disappointed since he expected that half of the students could do the task by doing a substitution. He reiterated this statement many times that students most probably found the answer by substitution. As he stated, their underachievement resulted from inadequate conceptualization of topics in the 7th grade. He argued that more than 50% of the students already understood the concept of variable. He said that students might give only one value for n rather than substituting two or more values. For this reason, they gave incorrect responses. As Mr. Öner noted, their underachievement in this task was not related to their lack of understanding of the concept of variable. Although he stated that there were no objectives in the algebra curriculum depending on comparing two algebraic expressions, he was sure that students knew the relationship between two algebraic expressions; however, they did not want to cope with this task. Based on Mr. Öner, this was the only reason for students' low performance in Item 3. As he expressed, students were not responsible for such tasks, comparing algebraic expressions. Instead, students were typically asked to find the possible values of the unknown by transferring the knowns and unknowns at opposite sides in an inequality. He concluded that there was no such comparison in the algebra curriculum. If there were such an objective related to comparing different algebraic expressions, they would explain it to the students. However, in the first interview, Mr. Öner clarified that the conceptualization of variable is essential for learning algebra. He did not identify the students' difficulties and could not provide a detailed explanation related to their errors in Item 3.

Based on Item 4, Mr. Öner offered detailed information concerning prerequisite knowledge to do the task and possible difficulties students faced. In the interviews before ADT, he predicted that 40% of the students constructed the equation and 60% of them found the median to respond to the item. The actual ratio of correct responses was 34% for Item 4a. He stated that the results were as expected since approximately 50% of the students could understand the abstract concepts taught in algebra, such as using x to label the unknown. The remaining might not yet understand them, even in the eighth grade. Like Mr. Yücel, Mr. Öner also identified the students who comprised 30% of the class and could not understand anything about mathematics as they were not motivated to learn it. He argued that Item 4a was one of the algebra's most practical tasks to transition between concrete operations and abstract procedures. Furthermore, as he stated, they used such tasks frequently when they moved to abstract topics in the seventh grade. He mentioned that the current algebra curriculum was well modified as the previous one was so intensive, especially for the sixth and seventh grades. He highlighted the importance of integers for learning algebra by stating that the transition of some objectives concerning integers to sixth grade was beneficial for students. Also, he appreciated moving equations to seventh grade in the mathematics curriculum. After observing the results of Item 4b, he said he would expect them to respond correctly to this item. He concluded that students could construct the equation but did not know the meaning of the unknown they used since they did memorization. As he noted, only in this way could it be explained. After I asked him whether students' difficulty might be related to the conceptual understanding of variables, he expressed that they focused on the meaning of variables in the seventh grade by stating what x is and why we call it x.

Nevertheless, they did not express the sense of x in the eighth grade since there was no objective concerning the meaning of x in the eighth grade. Also, as he stated, students were result-oriented while doing the tasks. For this reason, he considered that students could not answer Item 4 correctly and might focus on just doing the job with a shortcut solution. Based on the results, MSMTs provided restricted explanations for students' difficulties and errors in Item 3 and Item 4. Ms. Ferhan, Ms. Burcu, and Mr. Gürsoy declared that they would focus on the meaning of x in their future classes. In contrast, Mr. Yücel and Mr. Öner clarified that students' difficulties and errors might be related to their lack of understanding of the task or forgetting the topic rather than their inadequate understanding of the concept.

In summary, MSMTs identified limited information related to the prerequisite knowledge for learning the concept of variable. MSMTs expressed that they would not expect such a high disparity between correct and incorrect answers in Items 3 and 4 as they thought that students were familiar with these items. After observing the results, they noted that they frequently asked students about the numerical result of an algebraic process rather than asking about the variable's meaning, or they asked students to solve an inequality instead of comparing different algebraic expressions. In addition, they noted that students had difficulty transitioning from verbal to symbolic expressions, especially for situations in which at least two related variables were included. However, they could not provide prior knowledge for students and an interpretation of the reasons for students' difficulties. None of the MSMTs expressed the meaning of variable as a crucial factor for learning algebra. Therefore, they did not mention that the lack of knowledge of variable might be one of the reasons for students' difficulties experienced in ADT. At the end of the study, MSMTs inferred that they should be careful and spend more time getting students to conceptualize variables before teaching algebra. The progressive part examined MSMTs' predictions and interpretations based on functional thinking.

4.3.1.3.MSMTs' predictions and interpretations based on functional thinking

The last big idea was *functional thinking* which refers to covariational relationships of quantities and reasoning about those relationships through verbal statements, algebraic notation, tables, and graphs (Blanton et al., 2015). Students' responses for Item 6, Item 8–9, Item 10-11, Item 12-13-14, and Item 15-16-17 in ADT were investigated based on the big idea of functional thinking. In Item 6, Ms. Ferhan predicted that half of the students could answer the item correctly, and students responded to the task by substituting a value for the unknown. To illustrate, they stated that let b = 2, then $3 \cdot 2 = 6$ and let b = 4, then $3 \cdot 4 = 12$. Since $12 - 6 = 6$, they answered as 6. In the pre-interview, she stated that students might respond as 'It increases two times.' She explained the reasoning behind this prediction as follows:

> Children often focus on numbers instead of understanding what they read. When they see 2 in the problem, they say if b increases by 2, a will increase by two times. I thought they might make this mistake because they frequently tried to answer immediately with the numbers they saw without understanding. So, they focus on the numbers they see and state that it increases by two times.

As she highlighted the importance of reading comprehension in the first interview, she remarked on additional explanations based on students' low performance depending on their inadequate reading comprehension skills. After observing the students' responses, she inferred that students could not conceptually understand, although they could do the required operations in algebra. She noted that they could perform the operations but could not explain how they changed. She concluded that the main reason was their motivation to learn mathematics and prejudice towards mathematics. Since they did not know why they learned algebra, they began memorization rather than doing interpretation, functional thinking, and reasoning. As she pointed out, they just memorized it until they passed the exam. Based on students' answers who found the result by substitution, she stated that students usually want to make it concrete while doing the tasks. After I asked her whether this might be related to students' inadequate understanding of linear equations and their presentation with graphics, she asserted that they might have been late to teach students the representations of linear equations as they gave this topic to students in the spring semester of the eighth grade. She added that MSMTs might introduce students to algebraic expressions earlier. As she noted, students did not prefer to use algebraic expressions while solving algebra problems since they already did it with the methods they had already learned in primary school. Based on Item 6, Ms. Ferhan expressed that students could do algebraic operations but could not interpret such an interchange since they memorized and did not conceptually understand.

Based on Items 8 and 9, Ms. Ferhan predicted that half of the students could give the correct answer to Items 8 and 9, but they might not prefer to use equations since they could solve the problem by already known methods. She noted that students typically could give $y = 20 + 10x$ and $y = 20 + 10 \cdot 8 = 100$ as the correct answers to Items 8 and 9. After ADT, she appreciated students' performance, with 41% correct responses. However, she stated that the number of correct answers would be lower if I conducted the test after one month; they learned linear equations. As she asserted, students often forgot the topics they learned in mathematics, but she did not know the reason. Moreover, she argued that their abstract thinking skills, which might develop at different times for different students, were also crucial for their learning. She suggested that minor changes might be made in the curriculum considering the levels of students' abstract thinking skills. She criticized the mathematics curriculum by stating that the spiral structure of the curriculum might be ineffective in some situations. As she said, they gradually taught various subjects to the students.

In contrast to Mr. Öner, she suggested that students should be taught the topics more intensively when introduced to them first rather than given them in pieces. She also asserted that students preferred the solution they were familiar with rather than the new one. Changing their habits would have become problematic if they had adopted a method. Different solution paths might be shown to them simultaneously while teaching algebra, and MSMTs might get students to choose which one they prefer. For this reason, she claimed that students reject using algebraic expressions and equations while solving problems since they were familiar with arithmetic solution paths from primary school. She stated that they were late in teaching equations and offered that algebra might be taught to students much earlier in primary school; therefore, they were getting used to it in middle school. She also added that they must move fast because they have a large number of objectives they have to do. To illustrate, she stated that she wanted to spend a lot more time on equation problems, but there was not enough time to do that. As she said, they learned many new things and forgot when they did not repeat the previous ones. As she suggested, subject repetitions could be made in the curriculum occasionally. In Items 10 and 11, Ms. Ferhan predicted that 70% of the students correctly responded to the items. Also, she provided a typical solution that students might give, as shown in Table 4.17.

Table 4. 17. Students' typical correct and incorrect answers given by Ms. Ferhan

	Typical correct answers	**Typical incorrect answers**
Item 10	$y = 20 + 3(x - 1)$	$y = 20 + 3x$
Item 11	$y = 20 + 3(5 - 1)$	$y = 20 + 5 \cdot 3$
	$= 20 + 3 \cdot 4 = 32$	$y = 20 + 15$

Like Items 8 and 9, she stated that students might prefer to use the solution paths taught in the sixth grade rather than an equation since they rejected using equations as they already could solve the problems with the methods they learned. In contrast to the prediction of Ms. Ferhan, only 26% of the students could write the equation; however, 65% of them could find the bill that must be paid if five glasses of tea were drunk. She said there was a significant difference here, which might be related to their inadequacy in conceptual understanding of the meaning of x. Also, she stated that students might be asked the given and requested information in the problem while solving problems. So that students became aware of the result they got at the end of the algebraic manipulations. After observing the results, Ms. Ferhan discussed the need for self-criticism in their teaching. As she noted, students tried to do the algebraic manipulations quickly by writing fewer numbers and operations and memorizing the procedures, which might be concluded with the entrance of technology into our worlds.

In Item 12, she offered similar views to Items 10 and 11. She predicted that 80% of the students could respond to Item 12 and 70% could correctly answer Items 13 and 14 before ADT. However, results suggested that 65% of the students could answer

Item 12, and 37% and 47% could correctly respond to Items 13 and 14, respectively (See Table 4.18).

Table 4. 18. Students' typical correct and incorrect answers given by Ms. Ferhan

	Typical correct answers	**Typical incorrect answers**
Item 12	$2n + 2 = 2 \cdot 10 + 2 = 22$	$2n = 2 \cdot 10 = 20$
Item 13	$y = 2x + 2$	$y = 2x$
Item 14	$152 = 2n + 2$	$152 = 2n$
	$150 = 2n \quad n = 75$	$n = 76$

Based on the students' difficulty writing the equation in Item 13, she expressed that students had trouble with reading comprehension. If they understood what they read, they would be more successful writing the equation depending on an algebra problem. In Item 14, she stated that students had difficulty reasoning as they should think of the situation in reverse. She suggested that patterns and equations might be switched in the curriculum since students must know the equation to write the rule of patterns. She argued that students were prejudiced about equations, and most did not use them for this reason.

Ms. Ferhan predicted that 90% of the students could answer Item 15, 70% of the students could answer Item 16, and 60% of the students could answer Item 17 correctly, while the results of ADT were as follows 84% for Item 15, 60% for Item 16, and 30% for Item 17, respectively. Although Ms. Ferhan made a close prediction of the results of Item 15 and Item 16, there was a significant difference between her predictions and the actual test results in Item 17. Students' typical correct and incorrect responses given by Ms. Ferhan were represented in Table 4.19. After observing the results, she stated that they should concentrate on the topic of equations more. She claimed that students were familiar with graphics but just learning to make a table. She concluded that they should give algebra to the students much earlier and make it more concrete by identifying the meaning of variables in each task.

Table 4. 19. Students' typical correct and incorrect answers given by Ms. Ferhan

	Typical correct answers	**Typical incorrect answers**
Item 15	They could correctly fill the table.	They may write 600 km for the first hour.
Item 16	They could correctly draw the graph.	They may start the line showing the linear relationship on the graph at 100 km.
Item 17	y = 100x	x = 100y

Based on the analysis of interviews with Ms. Burcu, she predicted that 40% of the students could respond to Item 6 correctly. She also declared that students were familiar with such tasks:

> We ask such questions to students. It is not something they are unfamiliar with but have difficulty with. For example, it is asked, if the side of the square increases by 2, how much does its area increase, or how much does its circumference increase?

As she offered, students typically answered the task as a = 3(b + 2) + 4, distributed the quotients as a = 3b + 6 + 4, and found a = 3b + 10. She estimated they have no difficulty finding the numerical value but may have trouble making comparisons because they had never compared the two algebraic expressions. She noted that they should know this was a linear equation and respond using this knowledge. In the post-interview, she offered that students could solve such tasks in algebra classes, such as finding the change of the perimeter when the short side decreased by two, and the long side increased by four in a rectangle, by substitution than by doing the required algebraic manipulations. She also stated that the most challenging point for students was correctly writing two sides of an equation with the same units. She illustrated with an example:

> Let the length of one side of a square is 3x + 4, and the area of the square is 25. Students struggle to write the equation as $(3x + 4)^2 = 25$. Instead, they might get the length of one side equal to the area, although they should write it in the form of area equals area *(rather than the length of one side equals area)*. Also, they might erroneously write an equation in which the left side represents the number of students while the right side presents the number of tables.

Although this comment highlighted the importance of understanding equality and constructing equations, she focused on the capability of writing equality with the same units in general. Ms. Burcu also mentioned another difficulty for students: they may not build the equations in an algebra word problem; however, they might solve the problem quickly if the same problem is given with numbers rather than unknowns.

> Lets we have 50 and 100 cents in a money box. If we have twenty-four coins, we should write that x + y = 24 and 50x + 100y =5000. Although I tried to explain them by saying that if there is one 50 cent, there are twenty-three 100 cents, if there are two 50 cents, there are twenty-three 100 cents, etc. This issue is always about primary school. When you put five instead of x in the problem, they can multiply it by 50. However, when it turns out x, they cannot consider multiplying it by 50.

In Items 8 and 9, Ms. Burcu predicted that 70% of the students could give the correct answer; however, 42% of the students in Item 8 and 55% in Item 9 could correctly solve the items in ADT. In the pre-interview, she noted that students had abstract thinking problems. Furthermore, she added that their predictions might not hold in linear equation problems. Students' typical correct and incorrect answers given by Ms. Burcu were illustrated in Table 4.20. After investigating the results, she stated that students had problems with abstract thinking and lacked the motivation to learn mathematics. She inferred that she should do such tasks in the classroom while teaching algebra. She offered no further explanation based on the results of Items 8 and 9.

Table 4. 20. Students' typical correct and incorrect answers given by Ms. Burcu

	Typical correct answers	**Typical incorrect answers**
Item 8	$y = 20 + 10x$	$y = 20 + 5x$ or
Item 9	$x = 8 \quad y = 20 + 10 \cdot 8 = 100$	Incorrect solutions based on
	or	the wrong equations
	20.30.40.50.…,100	

In Item 10, she predicted that 40% of the students could do the task, and they did not write the equation of the given situation in Item 10 . She stated that students had difficulty even in equations with one unknown, so they had more problems in equations with two unknowns. Moreover, she asserted that students might have trouble in those tasks since they were familiar with such expressions while solving equation systems, such as the sum of two values and twice Ayşe's age is equal to triple Ali's age. Students' typical correct and incorrect answers in Items 10 and 11 given by Ms. Burcu was presented in Table 4.21.

Table 4. 21. Students' typical correct and incorrect answers given by Ms. Burcu

	Typical correct answers	**Typical incorrect answers**
Item 10	$y = 20 + 3(x - 1)$	$y = 20 + 3x$
Item 11	$y = 20 + 3(5 - 1)$	$y = 20 + 3 \cdot 5$
	$= 20 + 3 \cdot 4 = 32$	$y = 20 + 15 = 35$

She predicted that 40-50% of the students correctly responded to Items 12, 13, and 14. Similar to previous items concerning functional thinking, students performed lower in Item 13, which required the construction of the equation based on the algebra word problem. In Item 12, 65% of the students could respond to the item correctly. As Ms. Burcu explained, students were successful in Item 12 since they did not need to use an unknown while solving the problem. She stated, "When the letter is involved, they think they cannot do it; they cannot overcome this obstacle. They do not realize that if they can do Item 12, they can also do Item 13." She implied that students often wanted to cope with memorized formulas or rules so that they could do memorized operations rather than constructing equations or finding the relationship. As she noted, it was impossible for a student who wrote +6 for the expression six more than a number to set up this equation. She expressed that students had trouble comprehending algebraic operations, such as substituting 5 to a to find the perimeter of a square whose circumference was represented as 4·a and the length of whose one side was 5 cm. Also, as she identified, students had difficulty understanding different forms of numbers, such as negative numbers, squared numbers, and exponential numbers. She noted that students had trouble comprehending that the minus sign belonged to the number while doing operations with negative numbers. Like Ms. Ferhan, she concluded that students had trouble using equations while solving problems since they were familiar with the methods they had learned previously, such as reverse operations in primary school. Students' typical correct and incorrect answers in Items 12, 13, and 14 given by Ms. Burcu was presented in Table 4.22.

Table 4. 22. Students' typical correct and incorrect answers given by Ms. Burcu

	Typical correct answers	**Typical incorrect answers**
Item 12	---	---
Item 13	y = 2x + 2	y = 3x or y = 2x
Item 14	152 = 2x + 2 150 = 2x x = 75	y = 2 · 152 + 2 y = 306

Ms. Burcu stated that 80% of the students could correctly answer Items 15, 16, and 17. Also, she noted that students typically would give m=100t as the correct answer and might give m=100+t as an incorrect answer. After investigating the results of Items 15, 16, and 17, she interpreted that their predictions came true since they expected that one-fourth or one-fifth of the classroom could conceptually understand such tasks. Moreover, she made some inferences based on the students' responses to ADT. Firstly, she noted that she noticed students' trouble comprehending x with the help of the results of ADT. Also, she expressed that ADT tasks were not similar to those from textbooks or students' examinations. Furthermore, students were familiar with the multiple-choice rather than open-ended items as in ADT. Therefore, as she said, they might be surprised when they see the items in ADT. She pointed out that:

> Let the sum of three consecutive even natural numbers is 84. If you had asked what the smallest number is, most students could solve it. However, they were unfamiliar with such questions in ADT. Students are interested in the result rather than the solution path. If they cannot write the equation, they can find the result by substituting the choices given in a multiple-choice item but cannot do it in those items.

As she expressed, she noticed that they should teach some points in algebra differently. To illustrate, students should comprehend the manipulations done in Item 1, and she clearly stated that students should consider such tasks while doing algebra. As she said, if students were asked to express whether 7 · 22 and 14 · 11 were equal, they could directly state that they were equal. However, if they were asked why two terms were equal, they could not explain it. That is, the way the question was asked was also crucial at this point. She usually explained the problems to the students using natural numbers to cope with this difficulty. To illustrate, if there were a rational number, an exponential number, a squared number, or an unknown in a problem, she changed the value or unknown with a natural number to make students understand the problem at first. However, as she reported, this way of explaining the situation did not work as the results of ADT illustrated. As she noted, one reason for students' underachievement was not doing sufficient practice after learning the topic.

Mr. Gürsoy guessed that 15-20% of the students could respond to Item 6 correctly, and he stated that he was disappointed depending on the students' performances. As he predicted, some students could see that 'a' would increase by six if 'b' increased by two, or they might substitute a value to answer the task. Based on the incorrect answers, as he expressed, they might state that 'a' would increase by two if 'b' increased by two since if one side was increased by two, then the other side also increased by two as it was equality. As he stated:

> There is a variation in this item. If you asked the value of a when b equals 2, I would say that the ratio of correct answers is 70%. They might think that let b=1, then b=3, and let b=5, then b=7. However, those values are different; therefore, students might have confused.

He concluded that they were not good at abstract concepts, such as identifying the relationship between algebraic expressions and understanding the meaning of x. As teaching just the eighth-graders, he stated that he did not explain the relationship of algebraic expressions since he expected that students should learn in the seventh grade. The other dimension regarding the underachievement of students in this issue was the crowdedness of the classrooms. Since his classrooms were too crowded, with approximately forty students in each classroom, he pointed out that it was difficult to control each student's learning conceptually. He suggested that an objective might be included in the curriculum, such as 'students will be able to understand the meaning of the variable. If he explained the solution of Item 6, he would offer a daily life example from the classroom as he stated:

> If I explained the change of the value of ‘a’ in the equation of a = 3b + 4, I would get fifteen students in three groups of five. Then, I would get four more students outside of these groups. Afterward, I would add two students to each group except for the group of four. Next, I would present the operations (3·5) + 4 and (3·7) + 4, respectively. Moreover, I would ask them how did it change? However, I believe that there were more important points that I should teach my students.

Based on Item 6, specifically functional thinking, he noted that this part was undoubtedly crucial. Still, he was unsure if he would mention it because he believed more critical points should be taught to students, such as the meaning of x and unknown, the concept of the term, and the remaining parts of algebra that were more crucial for him. He concluded he would definitely focus on the concept of variable in his algebra classes; however, he would not mention the covariation between variables. He said no one had taught him the interchange of variables, but he had learned it over time. Therefore, as he claimed, students could also understand it over time. For example, if you ask Item 6 to a high school student, each student could respond correctly. As he declared, some concepts became more reasonable after a while. For this reason, he was not planning to teach the interchange between two variables:

> To be honest, I will not explain it *(covariation between variables)*. I will concentrate on the concept of variables in 7th grade right now. I just wrote in my mind that I would focus on the interpretation of the variable. I will focus on reasoning while solving equations, but I may not consider that. The item might also be essential, but I have some priorities, and this is not one of them. If there were no examinations, I would teach everything.

After I asked him whether the development of functional thinking might be helpful while solving linear equations to understand the interchange between two variables, he stated that when you asked in this way, it makes sense; the change in the result when the variable increased was also significant. That is, he changed his mind and said it could be better if we explained the covariation between two variables and the meaning of the variable. He also mentioned the crowdedness of the classrooms and the assignment of the students to the MSMTs randomly for each year as the factors negatively affecting their teaching process.

Based on Items 8 and 9, Mr. Gürsoy foresaw that 30% of the students correctly responded to Item 8, and 90% of the students, who correctly responded to Item 8, could find the length in the eighth month. He claimed that students might answer as 80 cm for incorrect answers, omitting the initial size. Also, he predicted that everyone who could write the equation Item 8 could correctly answer Item 9. After examining students' results in ADT, he was satisfied with the results since the students performed better than his expectation. Like other MSMTs, he asserted that students do not prefer using equations while solving algebra problems. He was surprised when we talked about the students who did not prefer using equations while solving the problem for a particular value, although they could set up the equation.

In Items 10 and 11, Mr. Gürsoy predicted that 20-30% of the students could correctly respond. Furthermore, he considered that the ratio of students who could correctly answer would be higher in Item 11 since it could be solved without an equation. When we talked with him after ADT, he noted that he would say 10% for his current 8th-grade students since he was becoming increasingly despaired. He stated that students could find x but did not know what they found. Similarly, he clarified that they encountered such cases in the greatest common divisor and least common multiple tasks. For example, they could find the greatest common divisor of 50 and 60 as 10 in a mathematical word problem. After that, when I asked them what 10 referred to, they just said the greatest common divisor rather than expressing its meaning as the length of one of the equal parts of two blocks of 50 cm and 60 cm. Like other MSMTs, he argued that 30% of the students were prejudiced toward mathematics. For this reason, they could not understand, although MSMTs did their best. Moreover, he believed that mathematical skills and background were also required to be successful in mathematics. He asserted that if a student could not understand mathematical concepts efficiently in elementary school's second and seventh grades, it would be challenging to succeed in future mathematics topics.

Mr. Gürsoy predicted that approximately 29% of the students could accurately answer the task in Items 12, 13, and 14, and he stated that they practiced this type of task in algebra classes. He considered that students could solve this item by using the equation, and students who could write the equation could find the number of tables in the last part. Since students should be aware of the chairs at the left and right ends, he expected that students could calculate the number of tables by subtracting two if they could write the equation correctly. As he noted, students use a direct relationship as four chairs for one table and eight for two tables if they give incorrect answers. When he examined the results of ADT, he appreciated it and said that the results were better than expected. Since this question was above average in terms of difficulty, he stated that the results were excellent. When I asked why the students might not have preferred to solve using equations, he did not express a reason but noted that the solution path they used was more complicated than solving with equations.

Mr. Gürsoy presumed that 60% of the students could correctly answer Items 15, 16, and 17. He noted that the abstract nature of algebra caused students' difficulties as they gave up learning when confronted with x and y. He added that the symbols used in the tasks also affected their performance. For example, their success would increase if x and y were used rather than m and t as they were more familiar with x and y. When I asked him about the incorrect answers students might give, he could not give a specific example but stated that they would give wrong answers. He also stated that students' lack of motivation was a critical concern for students' struggle in these tasks. Moreover, he mentioned students' backgrounds as a factor in learning algebra. After observing the results, he interpreted that students' performance was directly related to the importance and time MSMTs gave to equations in the seventh grade. He repeatedly mentioned the importance of examples of transition between algebra word problems and algebraic equations. Based on the results, he noticed that he should spend more time on algebra topics and not hurry up to continue with the new topic. He asserted that students memorized the solution paths and were unaware of the results they found at the end of the solution process. When he asked students, "what did you find with that result?" he noticed that students did not consider the meaning of the result. Moreover, he criticized himself in terms of such students who solve the tasks by randomly doing arithmetic operations without understanding and considering the problem. He expressed that he did not ask these students how they found that result and did not revise such incorrect results. He advocated that the classrooms were so crowded and it was difficult to care about such students among forty students. He noted that his focus was on the successful students, and he cared about these students in general. Mr. Yücel argued that 80% of the students could respond to Item 6 correctly. As he stated:

> Students cannot understand…the variation *(he considers)* when you asked that how does it change? If you wondered whether ‘a’ would increase or decrease, they would see that it was related to mathematics and respond that it would increase. When the item asks whether the value of ‘a’ increases or decreases rather than how it changes, the students’ choices decrease, and their job becomes easier. There will be two options: increase or decrease, and if ‘b’ has increased, it is more likely that ‘a’ will increase.

This comment presented that his students might be unfamiliar with such tasks that required making interpretations of algebraic expressions and equations since he advocated that students should be directed rather than allowed to interpret in such situations. Like other MSMTs, he forecasted that students might say “a was also increased by two” as an incorrect answer. After observing the correct responses of students, he identified the solutions of students done with algebraic manipulation, such as $a = 3(b+2) + 4 = 3b + 6 + 4$, as high-level thinking, and with substitution, such as substituting 1 and 3 for b respectively, as a reasonable and concrete approach. Based on the answers of “a increases since b increases,” he stated that students thought superficially, and it would be worse if they said that “a increases proportionally with b.” He interpreted that it was interesting regarding the incorrect response of students, namely, “a would be equal to 5b + 4 if b increased by two”. He provided no further explanation concerning such students’ erroneous responses. He stated that students were knowledgeable about the topic, but it was lacking. He also asked the researcher, "If you were teaching the lesson, where do you think I should pay attention the most while teaching?” He stated that the solution of Item 6 might be explained to students from easy to complex, such as students might be asked to respond the change of a for $a = 3b$ without +4. After substituting a particular value for b to see the change on a, students could be asked to answer the change of a when b increased by two in $a = 3b + 4$. He offered that they could teach students the meaning of x by showing that it varied. To illustrate, in a consecutive number problem, the terms might be written as x, x–1, and x–2, in which x refers to the smallest number. Then, the terms can be written as x–1, x, and x+1, in which x refers to the middle number. However, as Mr.

Yücel identified, they had limited time and many objectives they should teach students. For this reason, explaining the concepts to students in such detail was not easy.

Based on Items 8 and 9, Mr. Yücel noted that 60% of the students could correctly do the tasks. He declared that students might have difficulty writing the equation. He expressed its reasons as students' lack of understanding of constructing an equation and forgetting the concepts they learned. As he predicted, students could find the length of the sapling for a specific month but might have difficulty constructing the equation. After observing the results, he said that students avoid using equations since they do not like algebra and equations. He expressed that students preferred doing arithmetic operations to equations since they were more concrete, whereas algebraic operations were abstract. Moreover, as he said, students tried to solve the items quickly since the current examination system makes them think and solve them faster.

In Items 10 and 11, Mr. Yücel estimated that 60% of the students could correctly answer the item and would answer using arithmetic since it was easier than setting up equations. Moreover, he noted that students might answer 20+3.5=35 if they forgot the first tea, which was free. After observing the ADT results, he focused on the difference in students' performance in Item 10 (25%) and Item 11 (65%). He noted that this was an important issue we should have considered and explained it by stating that students did not like equations. However, MSMTs explained equations effectively to students by pursuing all steps. He noted that very few students would solve this by setting up equations. He also pointed out that they need more time to teach algebra with activities in the classroom. As he said, students forget the concepts they learned since they cannot do activities in mathematics classes.

Mr. Yücel anticipated that approximately 40% of the students could correctly answer Items 12, 13, and 14. He presumed there would be more accurate results in Item 12 than in 13. Moreover, as he asserted, students performed better in Item 13 compared to Item 14. He guessed that students might struggle to write the algebraic rule given in the problem. Moreover, they use the pattern to find a specific value rather than setting up the equation in Item 14. When he analyzed the results in Items 12, 13, and 14, he said that the frequency of the correct responses was so high. He saw the decline in students' correct answers when he investigated the results for Item 13, which asked about the equation. After that, he asked my opinions on the decrease in students' responses rather than interpreting the issue. Mr. Yücel asked me what to do at this point:

> What should we do? Now that they answered this question, they can solve it and understand it logically. They can answer it correctly. However, I want them to be able to do that using the equation as well, but here we fail. Nevertheless, they have solved the question: How should I evaluate it? Is it missing or not? So how should we look?

He suggested that a further study might examine students' preferences for solution paths while solving algebra word problems. Students might be observed to investigate whether they use equation or arithmetic operations while solving a problem. He wondered if students needed to learn equations as they had already solved problems with arithmetics. Also, he argued that solving with arithmetic operations was more straightforward for students; therefore, they preferred solving with this path. Mr. Yücel noted that the seventh grade in algebra was the most critical level since more abstract concepts were beginning to be taught. He mentioned that since the time given for algebra topics was insufficient, he has increased the hours allocated to algebra topics. Therefore, like Mr. Gürsoy, he was late for the next subject. For this reason, he stated that curriculum experts should increase the time given to algebra topics in the curriculum.

In Items 15, 16, and 17, Mr. Yücel guessed that 40% of the students could correctly respond to the tasks. He predicted that students could accurately fill the table and draw the graphic but would struggle to set up the equation based on the problem. He noted that students did not like algebra. He said they would not use the general rule if I used huge numbers in the problem. He asserted that students probably said they could not do Item 15. After examining the results of ADT, he interpreted that students' could do the tasks with visual images; however, their performance decreased when x and y were included in the tasks. He said that the only explanation for this result was the abstract nature of algebra. Lastly, he observed that students could write 100 km for 1 hour and

200 km for two hours as they saw visually on the table. However, students felt tortured when they must write the relationship as y = ax + b. After he analyzed the results, he inferred that they should spend more time teaching students to construct equations as students could correctly do the tasks, including tables and graphics, but they were unsuccessful in doing the tasks which required writing the equation in an algebra word problem. Finally, he concluded that these interviews contributed to him and changed his point of view positively related to teaching algebra.

Mr. Öner predicted that 70% of the students could correctly answer Item 6 by stating that ‘a’ increased by two. Based on the incorrect answers, he presumed they might answer as four since there was a term +4 in the equation. For this reason, students might think that ‘a’ increased four by four and, therefore, could say that the answer was 4. After observing the results and responses of students, he stated that his predictions did not come true. He expected students to solve the item by substituting a value for b since they should do it this way. He added that students might have difficulty since there were two unknowns in the equation. If the item included only one unknown, such as 8 = 3b + 4, they should subtract 8 – 4 = 3b by transforming 4 to the other side at first and then divide 4 by 3 to find the value of the unknown. However, as he inferred, they might confuse which unknown they should substitute a value to observe the change. All MSMTs concluded that students had difficulty creating the algebraic expression of a relationship and preferred arithmetic solution paths rather than using algebraic expressions. In the next section, hypothetical reasons for students' difficulties and errors in algebra were investigated based on MSMTs' statements regarding their students.
In Items 8 and 9, Mr. Öner predicted that 60% of the students solved the item using the equation, and 40% of the students who inaccurately responded did not use the equation. He also noted that students prefer doing calculations or doing them mentally instead of setting up the equation. He added that they did not like setting up equations since solving problems without equations was more straightforward. Also, they do not want to solve items with equations even in the examinations. For incorrect answers, he just stated that students might not consider the initial length of the sapling. After observing the results, he interpreted that students could do this task since students could see the image of the problem situation as a concrete object. After he saw that

approximately 28% of the students solved the item using the equation, he asserted that he expected such an outcome since students mostly preferred calculating the result by using the ratio on the graphic.

In Items 10 and 11, Mr. Öner predicted that 40% of the students could correctly respond to the task. He asserted that students who would solve the task with equation correctly would not be more than 10%. Moreover, more than half of the students could not set up the equation. In Item 11, he noted that several students would respond as 5 · 3 + 20 since they forgot the first glass of tea. He asserted that students did not like setting up equations since solving with arithmetics was more straightforward. As he clarified, students always asked such questions: "Why do we learn equations when there is a shorter way?" and "How do we use it in the future?" He stated that he responds to students' questions as "how do you answer such an item in the examination if they ask you the equation rather than the result?" That is, it might be inferred that he could not provide students with a strong reason other than examinations for the requirement of learning equations. He also mentioned another difficulty that he observed in students, the difficulty of understanding the changeability of the symbols in algebraic expressions. For example, he wants to get students to write an equation in an algebra problem and ask students to write the linear equation using the symbols of a and t. He observed that students continued to use y and x to write the equation rather than the specified characters as they were used to them. He noted that they were not aware of the changeability of algebraic symbols. Also, as he added, they did not prefer even using x and y. Therefore, everything would get mixed up if they included a, b, and c. when he observed his own statements, he was surprised when he read that "students who would solve the task with equation correctly would not be more than 10%." and asked, "Is this my statement?" Then, he declared that nobody would struggle with equations if there were a more concrete solution path. He added that they specified when they asked students to set up the equation in an algebra problem. If not, nobody would use the difficult way *(constructing an equation)* when there is an easier solution path *(using arithmetic)*. Although he expressed that students did not prefer equations while solving algebra problems several times, he could not provide a concrete reason for students' preferences.

In Items 12, 13, and 14, Mr. Öner presumed that more than 30% of the students could correctly respond to Item 12, and 30% accurately answered Items 13 and 14. He guessed that several students would calculate the number of chairs by counting. Moreover, as he noted, students might struggle to write the equation presenting the relationship between the numbers of tables and chairs. Also, students might write erroneous equations since they did not consider the sides where tables came together. After examining the results, he advocated that if students could quickly solve the problem without establishing the equation, they would most likely not establish the equation if it was not specified in the question. He argued that such students continued with their concrete knowledge and did not accept the abstract information for the responses in which students found the result by counting the chairs or tables or drawing the figure for progressive steps. He appreciated that 25% of students used the equations while solving the problem, as he presumed that students did not use equations if they were not asked to do it. His statements showed that he perceived equations as an entirely abstract phenomenon, whereas he considered arithmetic operations concrete procedures. Therefore, he interpreted that students could solve the problem using arithmetic operations since it was concrete and students had been familiar with them since primary school. However, students could not solve it using equations since they were abstract, and students rejected abstract knowledge.

Mr. Öner anticipated that more than 80% of the students could correctly fill the table in Item 15, 40% could draw the table in Item 16, and 5% could write the equation in Item 17. He inferred that finding the result on the graphic was easier for students. However, as he said, students could not learn to construct equations since equations were abstract. Moreover, he highlighted that students usually asked where they would use equations in the future and why they must learn them. As Mr. Öner remarked;

> Students ask why we use x when there is an alternative way to do it. For example, in Item 13, why do we set up an equation when there is an alternative way *(arithmetic operations)*, subtracting 6 from 152, dividing by 2, and then adding 2? They always ask what will happen if we find out *(using arithmetic)*. Does it not work if we solve it normally? Is it better when we set up an equation?

As he expressed, he motivated his students by stating that they would use x in modern mathematics in high school. If students did not know what x would do in their future life, they did not want to learn. Lastly, he concluded that he agreed with the students. In other words, not everyone has to learn algebra if they would not continue with mathematics-related professions. Like Mr. Gürsoy and Mr. Yücel, he just stated the reason for students' difficulty in constructing equations as the abstract nature of algebra. He did not provide a further reason for students' struggle in functional thinking.

In conclusion, MSMTs noticed students were not good at writing an algebraic expression of a given situation in algebra word problems. However, MSMTs gave limited information based on students' prerequisite knowledge and performances on different algebraic representations such as tables, graphics, and equations. Moreover, before and after ADT, MSMTs did not express any information related to the covariation of the variables, which was crucial for learning functional thinking. After they investigated the results of ADT, they noticed that students had difficulty constructing equations, whereas they could solve the problems using arithmetic operations. They noticed students' low performance in setting up equations; however, they could not express the reasons for students' difficulties. They just inferred that they should spend more time on this topic and make students practice more.

This section compares MSMTs' predictions and interpretations of students' performances in ADT. The results showed that MSMTs allocated students' difficulties and errors with some factors that were not mainly related to students' cognitive processes in learning algebra. In addition to factors related to students' algebra learning, MSMTs expressed such factors for students' difficulty in algebra: students-related, instructional process-related, task-related, and social environment-related. Therefore, the following part will further investigate the factors MSMTs expressed based on students' difficulties and errors in ADT.

4.4.MSMTs' Causal Attributions for Students' Difficulties in ADT

Based on post-interviews with mathematics MSMTs, I analyzed MSMTs' hypothetical reasons for students' difficulties and errors in algebra tasks. I investigated the causes of students’ difficulties based on MSMTs’ causal attributions (Wang & Hall, 2018; Weiner, 2010). The findings revealed that mathematics MSMTs most frequently related students' difficulties and errors with student-related factors, including effort, learning, understanding, innate math skills, and motivation. MSMTs often thought that students' difficulties were related to student-related factors if they felt they had already completed similar tasks at school. Because the students were already familiar with the tasks, they interpreted that their failure could be attributed to the students themselves. MSMTs typically link students' difficulties and errors to students’ cognitive processes, effort, and motivation. Apart from student-related factors, MSMTs also described instructional process-related factors related to the teaching process, curriculum, and tasks. The last attribution was task-related factors, such as task difficulty and item structure (See Table 4.15). Firstly, the codes were created based on MSMTs' comments about the causes of students' difficulties and errors in algebra tasks. Then, the codes were assigned to particular themes.

The statements of MSMTs related to student-related factors were investigated in four dimensions, namely, students’ cognitive processes, students’ effort, students’ math skills, and students’ motivation. The most frequently observed code was students’ cognitive processes. *Students’ cognitive processes* refer to MSMTs’ quotes about students' learning, the conceptualization of the topics, and difficulties and obstacles in learning algebra. MSMTs typically pointed out the reasons for failure in students’ cognitive processes in Items 3, 4, 5, and 6 in ADT, which required algebraic reasoning. Moreover, they often expressed difficulties in students’ cognitive processes in Items 10-11, Items 12-13-14, and Items 15-16-17, which included the transition from verbal to algebraic expressions in algebra word problems. Based on Item 3, Ms. Ferhan stated, “They can write the expression, but it means that they have not fully grasped the comparison and relationship, so we need to dwell on this issue a little more.” Table

4.23 presents the distribution of each participant's attributions for students' difficulties.

Table 4. 23. The distribution of mathematics MSMTs' causal attributions for students' difficulties in algebra

Causal attribution of MSMTs	Ms. Ferhan	Ms. Burcu	Mr. Gürsoy	Mr. Yüce	Mr. Öner	Total
Student-related factors						**86**
• Students' cognitive processes	✓	✓	✓	✓	✓	40
• Students' effort	✓	✓	✓	✓	✓	22
• Students' math skills	✓	✓	✓		✓	6
• Students' motivation	✓	✓	✓	✓	✓	18
Instructional process-related factors						**32**
• Related to the teaching process	✓	✓	✓	✓		15
• Related to the curriculum	✓	✓	✓	✓	✓	14
• Related to the high-stakes tests	✓		✓		✓	3
Task-related factors						**15**
• Task difficulty	✓	✓	✓	✓	✓	11
• Structure of the item	✓	✓		✓	✓	4
Family and classroom environment-related factors	✓	✓	✓	✓		**6**

Thus, it was accepted as one if one MSMT expressed one or more statements related to a particular causal attribution. The analyses of the items based on MSMTs' causal attributions for students' difficulties were summarized in Figure 4.5. Each MSMT's statements regarding each causal attribution were accepted as one.

Causal attribution of teachers	**Item 1**	**Item 2**	**Item 3**	**Item 4**	**Item 5**	**Item 6**	**Item 7**	**Items 8&9**	**Items 10& 11**	**Items 12,13, &14**	**Items 15,16, &17**	**Total**
Student-related factors	**4**	**4**	7	**12**	7	**11**	**4**	**9**	**11**	7	**10**	**86**
Students' cognitive processes	1	1	5	5	4	5	2	4	5	3	5	40
Students' effort	2	1	1	3	1	4	1	2	2	3	2	22
Students' math skills	1	-	-	-	1	1	1	-	1	0	1	6
Students' motivation	-	2	1	4	1	1	-	3	3	1	2	18
Instructional process-related factors	**1**	**0**	**4**	**6**	**1**	**6**	**2**	**4**	**4**	**2**	**2**	**32**
Related to the teaching process	-	-	2	3	1	3	2	1	2	-	1	15
Related to the curriculum	1	-	2	3	-	2	-	2	1	2	1	14
Related to the high-stakes tests	-	-	-	-	-	1	-	1	1	-	-	3
Task-related factors	**1**	**0**	**0**	**4**	**0**	**0**	**2**	**0**	**1**	**4**	**3**	**15**
Task difficulty	-	-	-	2	-	-	2	-	1	4	2	11
Structure of the item	1	-	-	2	-	-	-	-	-	-	1	4
Family and classroom environment-related factors	**1**	**1**	**0**	**1**	**0**	**2**	**0**	**0**	**0**	**0**	**1**	**6**

**MSMTs'statements related to each causal attribution were counted as one.*

Figure 4. 5. Number of MSMTs who expressed particular causal attributions for students'difficulties in each item in ADT

Ms. Burcu and Mr. Öner mentioned students' inadequate understanding of algebraic expressions and memorization. Ms. Burcu declared, "The children memorize when they do not understand anymore. Somehow they memorize identities, like finding the square of the sum of two terms, so that is not the point." Mr. Öner also noted, "This means that the children have memorized the equation in this manner, that they understand how to set up the equation but have no idea about x. *(He considers.)* Yes, but we can explain it in this way." Mr. Gürsoy also stated that students' concerns stemmed from a lack of understanding of the unknown concept since all future topics would be built around it. MSMTs frequently said that students learned to solve problems with arithmetics in primary school. Since they were used to employing arithmetics, they did not prefer algebraic expressions and equations while solving problems. Similarly, Ms. Ferhan declared, "It is difficult to change a child's habits once they have become accustomed to them. If they associate it with those methods in primary school, they do not want to do it with the new knowledge in secondary school. It is challenging for us to change their habits." As Ms. Ferhan explained, students were resistant to learning a new method rather than the method they were accustomed to. All MSMTs stated similar concerns about students' use of procedures other than algebra while solving problems. Ms. Burcu also noted a similar example related to students' preferences for the solution in Item 4: "Students frequently answer this question like this *(dividing 84 to 3)*. Because, based on their prior knowledge, they know that dividing 84 by 3 will yield the middle number."

Moreover, she thought that students' abstract thinking might not develop in the same way in all students. Students might need even a couple of months since their ages were also essential in learning algebra. Ms. Ferhan also stated, "I am not sure why they forget so much. Indeed, it is something that should not be forgotten after learning. We want it to be that way, but I cannot see it in my students." As she noted, students' forgetting the things they learned was one factor related to their cognitive processes. Lastly, MSMTs' statements expressed that students' difficulties were also associated with the inadequate conceptualization of negative and positive numbers. Mr. Gürsoy and Ms. Ferhan asserted that students could not conceptualize the meaning of the minus sign, whether it was a negative sign or a subtraction symbol. As Ms. Ferhan noted, students could not understand that they might be used interchangeably.

Similarly, Ms. Burcu pointed out, "When you asked for six more than a number, they wrote +6. What is +6? Six more of what? They are unable to develop it in their minds." Therefore, it might be inferred that students could not create algebraic expressions in their minds, even though MSMTs thought they were so simple.

The second most frequently encountered causal attribution was *students' effort*. MSMTs proposed that students made little effort to understand the subject. Ms. Burcu pointed out, "We believe that most students understand it in class, but they do not support it at home, do not repeat, and do not practice." Moreover, as Mr. Öner noted, "50% of the students know that n is variable. I keep 30% of the students apart. I do not think they care after telling them what it is several times." That is, he despaired from these students who were from that 30% part. MSMTs also stated that students did not like performing detailed operations with a pencil. They preferred to make operations mentally and in a straightforward manner. As a result, when solving an algebra word problem, they construct short solutions without writing an equation. Based on Item 1, Mr. Öner declared, "Children generally prefer the easy way because multiplication is simpler in this process. I believe multiplying these two numbers would make it much easier to demonstrate equality." Like Mr. Öner, Ms. Ferhan noted, "They answered, 'I do not like using the pencil. Mentally, I can do it. So, why should I create the equation?' when we ask students to solve a problem using equations."

Moreover, MSMTs asserted that students did not cope with algebraic expressions if they could do the tasks using already-known methods. As Ms. Ferhan stated, they did not prefer to use equations when a more practical and familiar way can be used to solve a problem:

> Students have preconceived notions about how to solve problems. They do not want to solve a problem using a new one if they have previously learned it well. They also do not want to solve by writing with paper and pencil. Many of my students think of it in this manner. For example, if you pose a problem, brilliant students will be able to solve it. However, you can see that they do not use equations because they do not use their pencils. They cannot use equations; however, they can solve the problem by applying their arithmetic knowledge, such as performing an inverse operation, which they learned in primary school. They do not prefer to use equations when solving a problem because they can solve it more practically.

Like Ms. Ferhan, Mr. Öner clarified, "Nobody uses an equation when there is a more concrete solution. If students can do it easily without establishing an equation in such questions, they almost certainly do not write the equation, as it is not stated in the question.” Mr. Öner also said that students were result-oriented and did not like lengthy, detailed operations. All they wanted to do was solve the problem quickly and complete it. MSMTs also stated that students had prejudices toward mathematics. They wondered why they had to learn algebraic expressions as they could already manage the tasks with known solution paths. Therefore, such statements related to MSMTs' causal attributions for students’ difficulty in algebra were investigated under the code *students’ motivation*. Mr. Yücel emphasized that students lacked the self-confidence to learn mathematics:

> Some students are now completely isolated *(from mathematics)*. Even if the question is simple, they are biased. They say we cannot do it and either leave it blank or say something random. I observed the following: they simply wrote the numbers in the question and left it blank. They could also make a random addition using the numbers in the problem. They are scribbling something because some students have entirely lost their self-esteem.

Like Mr. Yücel, Ms. Ferhan stated that students were more isolated from mathematics if they arrived at secondary school with preconceived notions about mathematics. She stated, “If a student came to secondary school with prejudice, it is difficult to break it. If students could not do it in primary school, they became convinced that they could not do it in secondary school either.” She also mentioned a similar concern related to students' prejudice using equations by saying, “The equation itself creates prejudice in many students. While solving equations, we cannot break this prejudice for mathematics in many students. For this reason, even those who can do it do not want to *(use equations)*.”

MSMTs also stated that students did not like using equations and, according to students, equations were unnecessary while solving algebra word problems. As Ms. Ferhan and Mr. Yücel expressed, students did not prefer to use equations since they did not like them. Instead, they would like to use alternative solution paths in more familiar ways. Ms. Ferhan implied that students asked such questions: “Why are they forcing us to use equations, and why do we have to deal with them?” Since it was meaningless to them, they did not want to understand it. Mr. Yücel and Mr. Öner also supported this view by stating that students did not prefer to use equations, although they could understand and manipulate algebraic expressions and equations:

> They do not want to cope with abstract *(concepts)*. When I explain abstract topics to my students, they ask why we use x while there are other solution paths. For example, in Item 12, why do we create an equation when we can simply subtract 6 from 152, divide by 2, and add 2? They frequently ask, ‘Where will we use it? What will happen if we find out x? Isn’t it all right to solve it normally? Is it better to set up an equation and find it there?’ in algebra.

As Mr. Öner’s quotes illustrated, students were generally subject to using algebraic expressions and equations while solving algebra word problems. Instead, they usually preferred solving problems by using typical solutions they had been familiar with since primary school. As a result, MSMTs attributed their students' algebra difficulties to their prejudice toward mathematics and lack of self-confidence in using algebraic expressions and equations. One of the other student-related attributions was *students’ math skills*. Students’ creativity, functional and abstract thinking, and reasoning skills were investigated under students’ math skills. MSMTs stated that some students had more potential or were more talented to do mathematics. Therefore, they were more successful than others. Such statements were investigated under students’ math skills. Also, Ms. Burcu proposed that students’ creativity and imagination were inadequate and continued to decrease in mathematics. Moreover, Ms. Ferhan stated, “making interpretation, functional thinking, analyzing in children *(were insufficient)*. Children are far from interpreting and do not want to do this.” She declared that students were deprived of such math skills and had difficulty learning algebra. In addition, as all MSMTs implied, students struggled with abstract thinking, and it was more convenient for them when the concepts and tasks became more concrete.

Secondly, MSMTs also attributed students’ failures to instructional process-related factors. MSMTs' causal attributions to the instructional process were investigated under the topics of the teaching process, objectives (curriculum), tasks, and high-stakes tests. MSMTs frequently mentioned teaching process-related factors related to students’ difficulties and errors. Those factors were called as *teaching process*. Ms. Ferhan and Mr. Gürsoy admitted that they should focus more on the meaning of algebraic expressions based on the teaching process as they realized they did not do so adequately after examining students' results in ADT. Mr. Gürsoy declared that:

> I have only mentioned it when I explain the constant term. Why do we say it is constant? For example, I write 3n+8. Why does +8 constant instead of 3n? I am trying to touch it too, but I do not know if I can do it adequately. I may not be able to study effectively on it. I solve examples from everyday life but do not create *(such a comparison)*.

He added, "We usually teach the order in rational and square root numbers. However, we have not mentioned such an order in an algebraic expression or that n varies depending on *(the value of)* the unknown.” As he said, they did not give students such tasks to compare different algebraic expressions. Moreover, he noted that his students generally could not perform well in such tasks that required interpreting the comparison of two algebraic expressions. Instead, they would try to find the answer by making a guess or substituting a value for the unknown. Additionally, based on Item 4, Mr. Gürsoy mentioned that he taught students to find the arithmetic mean when asked to find the middle number in a problem with even or odd consecutive numbers. He stated that “It is typical. While solving equations with even and odd consecutive numbers, I also try to explain the arithmetic mean to students. I tried to make a point about it, but I guess it was insufficient.”

Instead of subtracting two from x and adding two to x, he highlighted this solution: finding the arithmetic mean. His intention could be to solve the problem quickly with a shortcut solution. As a result, he might get students to focus on calculating the mean rather than constructing the algebraic expression based on the situation. Except for Item 4b, MSMTs believed that students were familiar with such tasks. However, as they all admitted, they had not asked students what an unknown refers to in an algebraic equation meant. Related to Item 4b, Mr. Öner stated that:

> We teach it in the seventh grade, but we do not repeat it in eighth grade, such as thinking about ‘what is x’ and ‘why do we call it x’? We always explain that we call the smallest number x. The children understand how to write

> equations. They do not need to think about such questions in eighth grade because they have already conceptualized and understood them.

He believed they did not need to ask students the meaning of the unknown in an algebraic expression. They did not need to repeat it in the eighth grade because they already knew it. Furthermore, he stated that he always has students call the smallest number as x to find the value of x, as other MSMTs did, which may result in memorization while solving problems. Ms. Burcu talked about a similar solution path that she suggested for students:

> We work on similar problems. The equation is sometimes asked directly rather than the smallest, middle, or greatest number. If a problem asks for the middle number, we teach students to label it as x. If they are asked to find one of the values rather than the equation, we tell them to use the label x to represent any value they want. Then, they can subtract 2 from x and add 2 to x to form the remaining algebraic terms. However, if the problem asks for the middle number, we say the middle one should be labeled as x. Alternatively, if the problem asks for the greatest number, the greatest value should be labeled as x.

Ms. Burcu's quote could provide insight into her teaching while completing such a task in the classroom. Although she stated that students could use x to find the result for any value, she directed them to use the prompted solution path. If the problem asked for the median, students should label it as x, and if asked for the greatest one, students should mark it as x, and so on. Students might be influenced by such a statement to use only one solution path and memorize the solution. Although Ms. Burcu directed students on how to label the unknown in an algebra word problem, she expressed her concerns about memorization of algebraic expressions and their relationships rather than conceptualizing the meaning of algebraic expressions. Ms. Burcu noted, "When the children are unable to understand, they begin to memorize. Memorize the identity, find the square of adding two terms, and remember it. That is not correct. The children are beginning to memorize, and after that, they will be unable to learn anything."

As she stated, rather than doing memorization in algebra, students should conceptually understand what they did. As a general rule, they should not memorize a solution path. Instead, they could develop their own solutions while solving an algebraic problem.

Although MSMTs have complained about their students' memorization, MSMTs may also force students to remember such solution paths. Lastly, Mr. Gürsoy charged themselves with teaching mathematics with an examination-based approach. They taught the topics, did the tasks, and asked questions based on the ones covered in the national examination. For this reason, he considered that they could not provide effective mathematics instruction for their students since they were self-seekers to make students perform better in the examination. Therefore, he criticized their way of teaching from this perspective.

Curriculum-related attributions were also one of the instructional process-related causal attributions claimed by MSMTs. According to MSMTs, the elementary school mathematics curriculum lacked objectives related to algebra. Ms. Ferhan declared, “In primary school, we do not use algebraic expressions. As a result, they do not want to solve the problem with an algebraic expression by establishing an equation because they can do it by using the reverse operation and other methods.” She claimed that students should be taught algebraic expressions earlier because they are likelier to use strategies learned in elementary school. She proposed to introduce students to algebra much earlier:

> This is something we are unable to provide for children. What should we do? I believe we are a little behind schedule in delivering algebraic expressions. I am not sure if starting it in 5th grade is possible. We might use a box to represent the unknown in 4th grade. Because the students associate it with those approaches in primary school, they do not want to do the same thing with new knowledge in secondary school.

MSMTs also noted that the middle school curriculum lacked questions like "what is x?", "Why do we employ x?”, and "What is the significance of x?" Ms. Burcu stated, “We would focus on such issues if such objectives were included in the curriculum.” She contended that such subjects should not be taught to students because they were not included in the curriculum. MSMTs also mentioned that the curriculum's time restriction was another factor while teaching algebra. As Mr. Yücel and Ms. Burcu stated, there was not enough time to teach algebra adequately since they required more time to make students practice more and conduct activities in algebra. Ms. Burcu pointed out, “We are going too fast because we need to catch up on the syllabus. The curriculum might be more flexible. For example, this month, we have to finish this topic before moving on to the next one." All MSMTs stated that they spent more time on algebra than written in the curriculum. Therefore, they suggested increasing the hours allocated for algebra in the curriculum.

MSMTs' some of the attributions were related to the tasks in ADT conducted on students. Therefore, those statements were coded as *task-related* attributions, which were investigated under two topics, *task difficulty* and *structure of the item*. MSMTs mentioned that students had difficulties since algebraic expressions were abstract and unfamiliar to students based on task-related failures. Also, students did not like doing operations with rational numbers and algebraic expressions. Such challenges were investigated under the sub-category of *task difficulty*. All participant MSMTs asserted that students did not like algebra as much because of its abstract nature. Moreover, they were unfamiliar with such letters until middle school. Ms. Burcu declared, "Students think they cannot overcome the obstacle when letters are involved. Algebra is one of the most enjoyable subjects, but it is too abstract for them. We are always dealing with x's and y's in algebra."

Other MSMTs also commented on the abstract nature of algebra and students' unfamiliarity with algebraic symbols. Another notion that MSMTs mentioned frequently was the term 'unknown' and the use of unknown to construct an algebraic expression. MSMTs stated that students successfully understood and solved a problem using proportional thinking, doing operations, and substituting a value for the unknown. However, they were not good at making transitions from verbal statements to algebraic expressions, even in moderate tasks. Ms. Burcu pointed out, "Students struggle to construct an equation with even one unknown. Therefore, they are struggling with two unknowns much more." Similarly, Mr. Yüksel confirmed that:

> Students could do arithmetic operations but struggled when x's and y's became involved in the processes. Writing $y = ax + b$ feels like torture to the child. Here, the student can see and write that 100 kilometers for the first hour, 200 kilometers for the second hour, and 500 kilometers for the third hour. The child struggles to write the same relationship with an equation. They have not gotten used to it.

Therefore, it might be inferred that conceptualizing the term ‘unknown’ and transitioning between different forms of algebra were complex tasks for students, as MSMTs noted. MSMTs also mentioned that students struggled when the task included rational, irrational, or exponential numbers. As Mr. Yücel said, “The incorrect answers would be more if the quotient of x were $-\frac{2}{3}$ *(in an equation)* since dividing both sides by the coefficient of x disappears in this situation *(when the quotient is not an integer)*.”

Ms. Burcu also gave a similar example about students' difficulty with numbers other than natural numbers. She stated that students would have more problems if the task included exponential or square root numbers. She also expressed that students were unfamiliar with ADT tasks since they were not the same as those they encountered in the textbooks, quizzes, and examinations. Therefore, students were confused. As she continued:

> If you asked the value of the small number, most students would give the correct answer. The question was asked differently from what they used to. They are interested in finding a value *(at the end of the solution)*. It is not critical for them to use equations to find the answer.

Another task-related attribution observed in MSMTs’ statements was related to *the structure of the item*. This attribution included the factors about the construct of the items. Firstly, Mr. Yücel criticized the education system by stating:

> Everything has recently begun to be measured on multiple-choice tests. The children become wholly accustomed to the testing system. There are usually options in those questions, and they choose one. It asks for the median, the largest, or the smallest number. Students begin by assuming that one of them is the small number, then proceed to find the correct answer by substituting the others in order.

Mr. Öner expressed similar concerns about high-stakes tests, including multiple-choice items. He added that he did not like it, but they had to use it because of the educational system. According to Mr. Öner, it might be necessary to abolish the examination system entirely and replace it with teaching just focusing on objectives. In conclusion, MSMTs' causal attributions for students' difficulties were analyzed under three main categories, student-related, instructional process-related, and task-related causal attributions, as summarized in Figure 4.6.

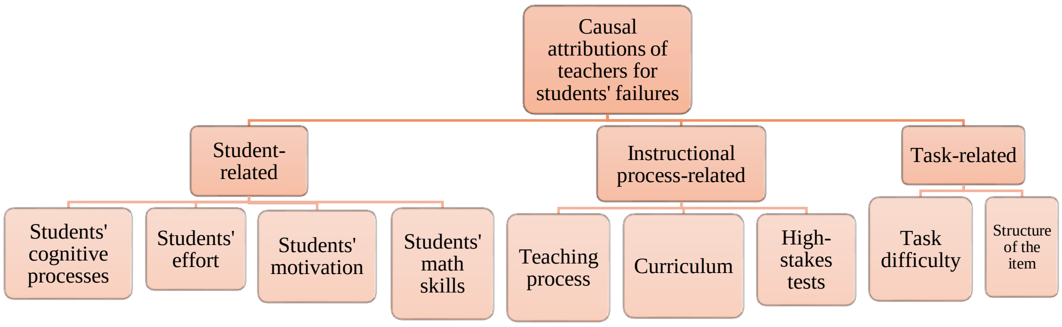


Figure 4. 6. MSMTs' causal attributions for students' difficulties

This section analyzed MSMTs' causal attributions: student-related, instructional process-related, and task-related attributions. Analysis of MSMTs' statements presented that student-related attributions were the most commonly encountered attributions for students' difficulties and errors in algebra. Especially the causes regarding students' cognitive processes and effort were the most frequently observed attributions MSMTs expressed. The second most typical causal attribution was instructional process-related causal attributions, mainly the teaching process and curriculum. Based on MSMTs' statements, students' cognitive processes were not just one factor affecting their performance as there were various factors such as students' effort, motivation, teaching process, and curriculum. In the following part, the results are discussed in more detail in light of the studies from the literature.

CHAPTER 5

CONCLUSIONS AND DISCUSSION

The purpose of this study can be explained fourfold. The first purpose of this study was to examine MSMTs' knowledge regarding their students' conceptions, difficulties, and misconceptions in algebra. The second purpose of the study was to observe how MSMTs anticipate their students' performance in particular tasks related to algebraic thinking. The third purpose of the study was to investigate how MSMTs interpret students' conceptions and difficulties by examining their results in particular algebra tasks. The last purpose of the study was to unpack MSMTs' causal attributions for students' difficulties in particular algebra tasks. This chapter included three sections, each discussing the findings of the study and concluding the results. The first section explained MSMTs' knowledge of the prerequisite knowledge required by students prior to learning algebra and students' difficulties and errors in algebra classes. The second section discussed the findings related to MSMTs' predictions and interpretations regarding students' performances in ADT. Subsequently, the third section presented MSMTs' causal attributions for students' difficulties in ADT and algebra in general. The implications and assumptions of the study and recommendations for further research studies follow these sections.

5.1.MSMTs' Knowledge of Students' Conceptions, Difficulties, and Errors

This section will present some conclusions regarding the MSMTs' knowledge of eighth-grade students' conceptions, difficulties, and errors in algebra classes and the ADT. Findings showed that MSMTs could provide limited information regarding the prerequisite knowledge students should have prior to learning algebra. Most of the MSMTs expressed that constructing and solving equations, solving algebra problems, and factorization and identities were examples of the most challenging tasks in mathematics. The MSMTs identified operations and negative numbers as prerequisite knowledge before learning algebra. However, they provided rare or no information associated with such notions of equivalence, variable, and covariational thinking. MSMTs could analyze students' algebraic thinking in their algebra classes and the tasks in ADT; however, they could not explain the underlying reasons for students' difficulties in particular notions.

MSMTs were initially asked to identify the prerequisite knowledge for algebraic thinking of eighth-grade students. They expressed that students should know operations prior to learning algebra. Two MSMTs mentioned rational numbers, and only one mentioned the importance of the comprehension of integers for learning algebra. Gallardo (2001) emphasized that extending students' comprehension from natural numbers to integers was crucial to achieving algebraic competence to solve equations and problems. Although MSMTs refer to operations with numbers, none talk about the number sense, which is also critical for algebraic reasoning (Asquith et al., 2007). Schifter (1999) noted that "to think through what multiplication does, why $18 \cdot 12$ is equivalent to $18 \cdot 10 + 18 \cdot 2$ [then when that student] enters an algebra class, having had such an opportunity…he will understand why $(a + b)(c + d)$ does not equal $ac + bd$" (p. 75). Asquith et al. (2007) clarified that students should practice, apart from memorizing the procedures and rules in algebra.

Although the MSMTs pointed out students' memorization in algebra, they did not mention the requirement for such a transition from numeric operations to algebraic expressions. Only one MSMT mentioned the need to transition from verbal expressions to the symbolic notation of algebraic representations, which was expressed as the rhetorical stage (use of words or sentences to represent algebraic statements), syncopated stage (use of abbreviations to represent algebraic statements), and symbolic stage (use of symbols to express quantities, operations, and relationships and doing manipulations using those symbols based on well-understood rules) in the study of Katz (2007). None of the MSMTs mentioned the relational understanding of the equal sign for algebraic reasoning (Stephens et al., 2013). When they were asked about whether their middle school students were ready to learn the topics related to algebra, two MSMTs stated that they were ready, two MSMTs noted that some of them were ready, but some others did not, and one MSMT specified that his students are not ready to learn algebraic topics.

MSMTs also talked about students’ difficulties in algebra classes. Ms. Burcu described students’ repeated errors and tendency to write the quotients in explicit multiplication instead of writing as an implicit multiplication, such as using x·3 rather than 3x. We might infer that students could not realize the equality of x·3 and 3x. To understand that these two expressions are equal, they need “the encapsulation of the process as an object” without observing the process for particular variable values (Tall & Thomas, 1991, p. 126). Therefore, they could realize that encapsulated objects were the same. For this reason, the difficulty Ms. Burcu indicated might be called a process-product obstacle, as Tall and Thomas (1991) suggested. The researchers identified the process-product obstacle as the inability to transition between the process and the product. They noted that one could see the process as a product by encapsulating it as an object. Two encapsulated things could be perceived as the same if they always give the same product. Therefore, there is no need to follow the process for particular values since the object encapsulates the process. As Ms. Burcu stated, students keep writing x·3 rather than 3x to write three multiple x. We might explain this by students’ deficiency of the understanding that 3x and x·3 were identical products. Therefore, we may infer that students may have the process-product obstacle as they could not encapsulate multiplying x with 3 in different forms. Ms. Burcu expressed it as a simple operational process that students should already understand since she may not be aware of such an obstacle students might face. Erbaş (1999) found that students had non-mastery skills in operations with literal expressions, including parentheses. He added that students had difficulties with cross multiplication and multiplication over parentheses in solving equations. In this study, 20% of students were unsuccessful at using parentheses while solving algebraic equations. However, MSMTs shared no idea about students' difficulties using parentheses in algebra.

As Mr. Gürsoy and Ms. Burcu explained, students had difficulty expressing the addend x as one of the addends, although they could determine (45 – x) in a problem of “the summation of two numbers is 45.” Mr. Gürsoy said this was the most common challenge he faced with his students. He explained that the problem was related to the difficulty of writing two algebraic expressions in terms of the same unknown. Ms. Burcu also mentioned this issue by stating that students had difficulty writing the expression “the sum of two numbers is sixty, one of which is four more than twice the other.” using symbols. As she stated, students especially struggled to identify the addend x, although they could determine the other addend as 2x + 4. She highlighted that students could not realize that x was one of the addends, although they could write the addend 2x + 4. Therefore, students incorrectly wrote the expression as 2x + 4 = 60. She also provided a similar example: if a student read 100 pages for some days of the week and read 150 pages for the remaining days, they had difficulty calling the number of days in which he read 100 pages as x, and he read 150 pages as (7 – x). As Ms. Burcu added, they could not write the expressions 100x and 150(7 – x) since they could not realize that they should do multiplication. This issue might also be related to students’ inadequate understanding of the multiplicative relationship (Blanton & Kaput, 2004).

Blanton and Kaput (2004) studied to improve early graders’ capacity for functional thinking, and they used the example of finding the relationship between the number of dogs and their eyes. They observed that second-grade students could see the multiplicative relationship by stating that “If you double the number of dogs, you get the number of eyes,” which indicated that students could express how quantities corresponded as they could use the term “double” (Blanton & Kaput, 2004, p. 140-141). It means that the number of dogs should be ‘doubled’ to acquire the number of eyes. The researchers also described t-charts as the most common path to organize and track the data. Ms. Burcu also explained that she got students to construct a t-chart to identify such a problem: if Ayşe read 150 pages in one day, she read 100 pages in six days; if Ayşe read 150 pages in two days, she read 100 in five days, and so on. Therefore, she expressed that, by generalizing, she showed students they should do subtraction to find (7 – x) if the number of the remaining days was x. Students’ these difficulties might be related to the inadequate understanding of how the covariation of two quantities occurred and how their relationship could be represented symbolically as a functional correspondence, such as finding the number of eyes of n dogs by writing “the number of eyes = 2*n*” (Blanton & Kaput, 2004, p. 141).

Consequently, as Blanton and Kaput (2004) proposed, students' difficulty might be caused by an inadequate understanding of recursive patterns, such as counting by threes, and multiplicative relationships, such as finding double or triple a quantity. Ms. Burcu stated that although students understood it, they could not do the same procedures in another task. She claimed that this might be related to students' memorization, lack of effort, and lack of motivation. Despite the MSMTs identifying students' problems with writing two algebraic expressions in the same unknown, they could not explicitly express the underlying reasons for these difficulties. In general, MSMTs'statements presented that they could identify the difficulties that students might have in algebra, but they could not explicitly express the reasons for their difficulties and errors. MSMTs' anticipations for students' performance in ADT and their interpretations of students' results will be discussed in the next section.

5.2.MSMTs' Predictions and Interpretations based on Students' Algebraic Thinking, Difficulties, and Errors in ADT

This section will present the conclusions regarding the findings of MSMTs' predictions and interpretations of students' performances in ADT. Based on the findings, MSMTs acquired varying degrees of success when they tried to predict how students would solve the tasks in ADT. They could anticipate students' possible solutions and how many students could do the tasks in the items, which included doing simple translations from verbal statements to algebraic expressions, solving equations, and demonstrating given data on the table or graphic. However, MSMTs could not predict students' responses in the tasks related to EEEI, variable, and functional thinking.

Variable. MSMTs' predictions of students' algebraic thinking regarding the concept of variable were not mainly aligned with students' actual responses to the task in contrast to the study of Asquith et al. (2007). Asquith et al. noted that students' most prevalent misconception shared by MSMTs was that multiplication always ends up with larger results compared to addition. Similarly, three MSMTs anticipated the same misconception in the pre-interviews, whereas one of the MSMTs was surprised when he heard about this misconception in the current study. Like the study of Asquith et al., three MSMTs predicted that students test various values for n and observe which one is larger. None of the MSMTs considered students' inadequate comprehension of the letters as variables that referred to an obstacle to solving the task, as few MSMTs mentioned this obstacle in Asquith et al. (2007). Only one MSMT predicted that some students could use the strategy to determine which expression was larger when n was less than 3, n was equivalent to 3, and n was greater than 3. Approximately 15% of the students could do the task using this strategy, in contrast to the study of Asquith et al. (2007). Ms. Ferhan noted that most students would say that 3n was larger since they perceived that all the numbers should be positive, and they did not consider numbers other than positive integers, such as rational numbers. Asquith et al. (2007) found that sixty-seven percent of the $3n$ responses were not justified or clarified explicitly, and forty-four percent of the $n + 6$ responses explained that six was greater than three. The researchers indicated that students tended to concentrate on the numbers included in the task instead of the operations if they had difficulty understanding algebraic expressions. In the current study, students performed similarly to Asquith et al. (2007). A substantial number of students responded that 3n was larger as multiplication gave greater results, and some responded that n + 6 was larger since 6 > 3. Apart from the study of Asquith et al., a significant number of students who used substitution with a single value or more responded that one of the terms was larger. This result might correspond with their MSMTs' tendency to use substitution to solve such tasks as they typically offered substitution of different values to n as a common solution path of students for this task. MSMTs' explanations presented that they heavily concentrated on substituting single or multiple values to n to determine which one was greater. Also, they mentioned students' incorrect thinking related to the comparison of 3n and n+6, such as 3n since multiplication always gives greater results or n+6 since 6 is greater than 3. Asquith et al. (2007) found more discrepancies when they compared the MSMT predictions with student performance in "which is larger task" than "the literal symbol interpretation task" (p. 259). In this study, the discrepancies between MSMTs' predictions and students' performances were significant in Item 3.

EEEI and generalized arithmetic. Based on the pre-interview results for Item 1 (equivalence task) in ADT, three MSMTs anticipated that most students could show equivalence without multiplication with a relational-structural conception (Stephens et al., 2013). Moreover, two MSMTs predicted that few students could do it without multiplication, and most students adopted a relational-computational conception (Stephens et al., 2013). MSMTs noted that multiplication was easier for students than relational-structural strategies such as factorizing the numbers. As the ADT results showed that approximately 30% of the students adopted a relational-structural strategy (e.g., "$7 \cdot 22 = 7 \cdot 2 \cdot 11$ or $7 \cdot 2 \cdot 11 = 14 \cdot 11$" or "One is multiplied by 2 when the other is divided by 2.") and half of the students used a relational-computational strategy (e.g., "22 equals the multiplication of 11 and 2; 14 equals the multiplication of 7 and 2."), MSMTs' estimations were not aligned with students' performance in the equivalence task generally. Stephens et al. (2013) found that most third, fourth, and fifth-grade students had an operational conception of the equal sign, which stimulated students to "do something" such as computing or calculating, and students struggled to recognize the underlying structure of equality (p. 174). In this study, MSMTs' expressions indicated that all MSMTs concentrated on the multiplication operation or factoring the numbers accurately instead of conservation of equivalence. Like their students, MSMTs mainly presented relational-computational arguments related to the solution of Item 1, such as "They should directly consider 16 multiplied with 3 when they see 48" or "Some students may multiply 14 with 10, then add 14 to the result". Moreover, Mr. Öner stated that "Students would find it without multiplication if you asked them the same task including unknown terms. However, nobody solves it with factorization since multiplication is more straightforward with numbers." Although the item asked students to show the equivalence, Mr. Öner used the word 'find' (e.g., the result) rather than 'show' or 'present' (e.g., the equality) while explaining the solution paths of students. His preference for this word might be interpreted as he might focus on students' finding a result instead of showing the equivalence in the task. Therefore, it might be inferred that the MSMTs also have a relational-computational conception while teaching the equal sign. Researchers found that most pre-service teachers were unaware of students' misconceptions about the operational thinking of the equal sign (Alapala, 2018; Isler & Knuth, 2013; Stephens et al., 2013). Also, pre-service teachers concentrated more on computational than structural thinking (Stephens, 2006). Similar to those studies, the results of this study suggested that participant MSMTs could not anticipate students' conceptions and difficulties regarding the equivalence and the relational understanding of the equal sign.

Although there were such student responses using substitution in Item 3, they were not observed as frequently as MSMTs predicted. One of the MSMTs noted that most of the values students substituted make 3n larger since students tend to substitute positive numbers such as 5, 10, and 20. Ms. Burcu noted that there were unknowns on both sides of the equation; therefore, students had more difficulty. Based on this issue, Gallardo (2001) stated that a *didactic cut* occurred when doing a transition from the expression Ax + B = C to Ax + B = Cx + D. Although it was enough "to invert" or "undo" the underlying operations for Ax + B = C, it was inadequate "to invert" the operation in Ax + B = Cx + D (Gallardo, 2001, p. 127). Instead, it was required to operate with the unknown. There were two similar tasks regarding these equations in Item 7 in ADT, –3 – 2x = –9 and 3x + 7 = –7 (x – 6). Students demonstrated similar performances in two tasks, respectively; 54% and 50% of the students responded correctly. The most typical error MSMTs mentioned was the use of the minus sign. MSMTs mainly stated that students might forget to change the minus sign while transferring the term to the other side *(of the equal sign)*. Ms. Ferhan noted that students had difficulty understanding whether the minus sign referred to the operation or belonged to the number. She noted that students struggle to realize that these two are interchangeable things, and this was one of the most common difficulties of students. She argued that this was a crucial point for students that they either stop or continue learning mathematics. Based on students' solutions, Mr. Gürsoy also clarified that students would typically consider transferring –3 to the other side instead of adding +3 to both sides while solving the task –3 – 2x = –9. Moreover, as Mr. Yücel said, students had more difficulty with the division since MSMTs directly wrote x = 3 for 2x = 6 without showing the division of both sides by 6. MSMTs' anticipations showed that they mainly concentrated on students' errors related to using the minus sign correctly and transforming the terms correctly to solve the algebraic equations in Item 7.

Based on the algebraic expressions, including operations on both sides of the equal sign (e.g., 4x – 5 = 3x + 7), Knuth et al. (2005) emphasized the importance of the relational view of the equal sign when students have to deal with algebraic equations. The researchers noted that a relational view of the equal sign got students to understand

that the equivalence relation was conserved through the transformation process while solving the equation. They clarified that this was one of the most complex ideas that several students had difficulty with, and it was not a focal point of typical instruction. Steinberg et al. (1990) also found that several eighth and ninth-grade students did not comprehend equivalent equations. The researchers found that many students understood how to do transformations while solving equations but could not determine the equivalence of two equations. As Knuth et al. (2005) inferred, the problem might be associated with an inadequate understanding of mathematical equivalence. This study showed no remarkable difference in students' performances for the two tasks in Item 7. Although the researchers suggest that students might have more difficulty conserving equivalence while coping with algebraic equations, including unknowns on both sides, MSMTs attributed students' difficulties and errors to their inadequate knowledge of negative numbers and how to operate with the quotient of the variable.
In contrast to Asquith et al. (2007), MSMTs' predictions of students' performance related to understanding the variable were not aligned with students' actual responses in ADT. MSMTs rarely figured out students' misconceptions about variable as an obstacle to doing algebra tasks like Item 3 and Item 4, similar to Asquith et al.'s study. MSMTs noted that Item 4a was a familiar but challenging task for students. They considered that most students could write the expression as $x + x + 1 + x + 2 = 84$, and some students might answer $3x = 84$, which might be incorrect for $x + x + x = 84$. MSMTs identified limited examples of students' incorrect answers and could not express particular ideas for the underlying reasons behind these inaccurate answers, such as the inadequate conception of the variable. For example, Ms. Burcu noted that students might make such errors since they could not understand the problem instead of providing a more specific explanation for their errors. Also, Mr. Öner clarified that students might not solve this task since they had no motivation to learn mathematics. As Mr. Gürsoy anticipated, many students wrote $x + y + z = 84$ for the algebraic expression, which was not algebraically incorrect but unexpected as students were expected to write the expression using one variable.

Based on the results of Item 4b, as Mr. Gürsoy and Mr. Öner said, they taught students to calculate the average to find the median value in an algebra problem. They noted that students should divide 84 by 3 to find the median value. To illustrate, Mr. Öner asserted that most students could correctly write the algebraic expression x + x + 1 + x + 2 = 84; however, they mostly found the solution by dividing 84 by 3 in the pre-interview. These statements might show that some MSMTs directed students to use arithmetic instead of algebraic expressions for solving such tasks. Moreover, Ms. Ferhan highlighted that students would find the result if they were asked to find the value of the small number. As she noted, they were unsuccessful since they were asked to express the meaning of the unknown. MSMTs anticipated that most students could express what the unknown refers to in Item 4b. Although Mr. Yüce predicted that some students might not express what the result or x stands for, he could not explain their struggle explicitly. He asserted that he taught students to substitute the result for the variable to ensure they correctly solved the task, and students would become successful if they understood substitution.

MSMTs' predictions could not accurately mirror students' performance in Item 5, related to using rational numbers in algebraic equations. Although MSMTs anticipated that most students would respond to the item correctly, 40% of MSMTs could explain why the student's thinking was not correct. MSMTs noted that students would solve the equation to explain their thinking. Ms. Burcu stated that students struggle with such tasks although there was no unknown in the expression, such as 8 + … = 2. As she expressed, if the expression became 8 – … = 10. it would be more difficult for them to understand how the result could increase when we subtract a value from the minuend. MSMTs attributed students' difficulty to inadequate comprehension of the procedures while solving an equation, such as transfer of the addend to the opposite side or dividing both sides of the equation by the quotient.

Moreover, they expressed inadequate knowledge of negative numbers. Ms. Burcu noted that students had difficulties since they forgot or did not repeat the topics they had learned enough. In addition, she thought students could not conceptualize rational numbers and only recognize integers. Ms. Burcu and Mr. Yücel highlighted that even using distinct letters might affect students' performance, such as using c instead of x might decrease their performance as they were more familiar with x. Mr. Yücel said that students' performance would increase when 9x + 8 = 2 was changed to 9x + 8 = 12 or 9x + 8 = 24 since x became a positive number. Although he identified this example, he could not explain the underlying difficulty of students. Mr. Yücel also argued that 8 + 9x = 2 and 9x + 8 = 2 were quite different for students. He clarified that students struggled when the addend with an unknown was written after the numerical addend since students tend to transfer the second addend to the opposite side of the equal sign. As he expressed, students memorized to transfer the second addend to the other side of the equal sign, such as they would write 8 = 2 – 9x to solve the equation 8 + 9x = 2. They considered that the unknown addend should always be on the left in an algebraic expression. Tall and Thomas (1991) mentioned a similar concern related to the cognitive conflict between the natural language and algebra's symbolic world. As they stated, natural language and algebra were written and read from left to right in most civilizations, which might be problematic in algebra. To illustrate, students often read 2x + 5 from left to right, but they read 5 + 2x from left to right as 'five plus two x,' which was computed from right to left to calculate '2x' before adding with 5. This difficulty was called a *parsing obstacle,* the changeable sequence of algebraic processes in contrast to the natural language (Tall & Thomas, 1991). Therefore, the difficulty of students mentioned by Mr. Yücel might be associated with the parcing obstacle, which originated from the contradiction between the natural language and the symbolic world of algebra. Mr. Yücel noted that students' difficulty might be caused by inadequate practice with different forms of equations. That is, MSMTs might get students to practice reversing the places of unknown terms in algebraic equations. During the conversation, Mr. Yücel was surprised when he heard about students' difficulty in understanding that 9x + 8 = 2 and 2 = 9x + 8 were the same equations. Instead of focusing on equivalence, he explained it with a metaphor, becoming at two ends of a bridge, and noted that waiting on the reverse sides of the bridge would not change the situation. He expressed that primary school teachers were responsible for this difficulty.

Functional thinking. MSMTs anticipated that students would perform more in functional thinking tasks, although students demonstrated lower performance than their predictions in ADT. Based on MSMTs' predictions in Item 6, students would mainly think that, as an equation, an increased by two since b was increased by two. Ms. Burcu expressed that this task was not something they were unfamiliar with but struggled with. As she noted, students coped with similar tasks in algebra classes but had difficulty, such as when the side of the square increased by two, how much did its area increase, or how much did its circumference increase? Mr. Yüce argued that students should be asked whether it would decrease or increase instead of asking how it would change since they might have difficulty talking about variation. Mr. Öner also noted that students might consider increment four because of the added four in the equation instead of focusing on the variables. Based on the results, only Mr. Gürsoy could anticipate students' performance accurately by stating that approximately 20% of the students could correctly respond to Item 6.

MSMTs noted that students would not prefer solving the task using the function rule even if they could construct it. As they mentioned, students considered that using the function rule to solve algebra questions was useless since there were more practical ways that they were used to employ. Mr. Öner said, "students did not solve the items with equations unless you told students to use them because they did not care about it, and arithmetic solution paths were more straightforward than equations." For this reason, MSMTs often anticipated that students would succeed in the items that could be solved with arithmetic solution paths. Nevertheless, as they stated, students might be unsuccessful in some items that required writing the function rule. Based on students' results in functional thinking tasks, there was a sharp decrease in students' performance in writing the rules compared to solving the problem for a specific instance.

MSMTs often expressed that students had problems with abstract thinking; therefore, they might have some problems while doing these tasks. They provided superficial explanations for students' difficulties in finding the rule of the function as students' loss of motivation, doing memorization, and not enjoying algebra. In addition, MSMTs could provide examples of students' incorrect answers; however, they could not explicitly describe their reasons. For example, Mr. Yüce asserted that students might incorrectly give the responses of $y = 20x$ or $y = 20 + x$ in Item 8. He described the reasons for these incorrect responses as students' not understanding the topic efficiently or memorizing. As he noted, although they did several practices in the classroom, they forgot it after a while. In Item 9, MSMTs predicted that students might give incorrect answers: $y = 30x$, $y = 20 + 5x$, $y = 20x$, and $y = 20 + x$. Only Ms. Ferhan mentioned the incorrect answer y = 20 + 10x = 30x, which might be associated with the parsing obstacle explained by Tall and Thomas (1991). Therefore, students might read 20 + 10x as 20+10, which results in 30, and they concluded that y equals 30x. Mr. Öner and Mr. Gürsoy mentioned another difficulty for students in writing algebraic equations using symbols other than x and y. For example, when Mr. Öner asked students an algebra problem and asked them to write the equation using the letters a for the length and t for the time, students usually wrote the equation as y = 3x + 50 instead of a = 3t + 50. As Küchemann (1978) presented that students preferred to use x instead of prescribed symbols such as a, b, and t in the tasks, Mr. Öner noted that students could not understand the changeability of the letters, which showed the variables. As he stated, while they did not even prefer to use x and y, it got complicated when MSMTs added letters such as a, b, and c.

MSMTs did not mention the covariation between variables crucial for learning functional thinking. In their study, Şen-Zeytun et al. (2010) observed that teachers perceived functions as correspondence relationships instead of covariational structures. They also concluded that teachers' anticipations of students' reasoning abilities were limited since their anticipations could not go beyond their own thoughts related to the problem. In pre-interviews, Mr. Gürsoy argued that he had more important things to teach than the task in Item 6, which examined the covariation of two variables. As he expressed, the notions of "term" and "variable" in algebraic expressions were more crucial. As a result, he was not planning to teach such covariance between variables to students. He advocated that no one taught him the covariance of variables in the past; therefore, students could learn it independently as time progressed. Asquith et al. (2007) suggested that 'exposure' and 'understanding' were two different things experts should consider. Although the students were exposed to tasks often related to covariational thinking, they might not understand it well even if they were exposed several times. After Mr. Gürsoy observed students' results in ADT, he said that he would definitely concentrate on the concept of variable more in his algebra classes, but he may not focus on the covariation of variables in a functional relationship. In addition, he added that this task was not unimportant, but he should teach more critical topics that would be included in the national examination. He argued that the covariation of variables was not included in the objectives of the mathematics curriculum. Blanton et al. (2011) said functions were crucial in developing algebraic thinking. As they stated, functions improved students' meaningful understanding of symbolic notation by making students consider the relationship between quantities. Blanton and Kaput (2004) highlighted that the emphasis on finding patterns in single variable data sets might hinder the emphasis on functional thinking in the following elementary school years. It might be inferred that such thoughts may limit his teaching in improving students' functional thinking. Like his predictions of students' performance, he provided narrow interpretations for students' actual results in functional thinking tasks in ADT and gave limited examples of students' incorrect responses to the tasks. Although his anticipations were close to the students' actual performances, he could not explicitly describe why students struggled to write the equations in functional thinking tasks. He just provided superficial explanations like the abstract nature of algebra and students' prejudice towards x and y.

MSMTs could accurately predict students' performances in the tasks, including solving simple equations and algebra problems with arithmetic. Conversely, MSMTs' predictions were not aligned with students' responses in the tasks related to a relational understanding of equivalence, understanding the meaning of variable, and functional thinking. Erbaş (2005) found that although teachers were aware of students' thinking and difficulties 'knowing that,' their knowledge was narrow and sometimes problematic regarding 'knowing why' and 'knowing how' as they could not express the reasons behind students' thinking and difficulties. Although MSMTs could identify students' difficulties regarding corresponding items, they expressed the underlying reasons for students' difficulties superficially. The next part will discuss the reasons given by MSMTs for eight grade students' difficulties in algebraic thinking based on causal attributions.

5.3.MSMTs' Causal Attributions for Students' Difficulties in Algebraic Thinking

This part will demonstrate the conclusions and discussions regarding MSMTs' causal attributions for students' difficulties in ADT. MSMTs expressed potential reasons for students' difficulties while learning algebra in the current study. Wang and Hall (2018) highlighted that teachers' causal attributions might influence their instructional behaviors that significantly impact students' academic performance, behavior, and motivation. Therefore, the reasons offered by MSMTs for students' failure were investigated based on the causal attribution theory of Weiner (1985, 2010). Like the study of Bozkurt and Yetkin-Özdemir (2018), this study found that teachers usually tended to identify attributions for failure. For this reason, teachers' causal attributions for students' difficulties were investigated in the current study. Similar to the results of previous studies (Baştürk, 2012; Medway, 1979; Wang & Hall, 2018), it was observed that MSMTs frequently attributed students' difficulties to student-related factors, including their cognitive processes, effort, innate math skills, and motivation. This study showed that MSMTs typically associated students' difficulties in algebraic thinking with students' cognitive processes, which was an external, stable, and uncontrollable factor for MSMTs (Wang & Hall, 2018; Weiner, 2010). MSMTs advocated that students' failures were mainly related to students since they were already familiar with the tasks. They might have considered that studying a subject in algebra classes was adequate for students, as their statements presented. Other attributions MSMTs mainly talked about were students' effort, an internal, unstable, and controllable factor, and students' motivation, an internal, unstable, and uncontrollable factor (Wang & Hall, 2018; Weiner, 2010).

Medway (1979) and Baştürk (2012) found that pre-service teachers most frequently attributed students' difficulty or success to the innate math talent, which was an external, stable, and uncontrollable factor for teachers (Wang & Hall, 2018; Weiner, 2010). In contrast to the results of these studies, the least observed student-related attribution mentioned by MSMTs was the students' math skills. Glasgow et al. (1997) observed that students who attributed their failures to uncontrollable factors demonstrated lower classroom engagement. As the researcher noted, if teachers associated students' failure with factors such as a lack of innate talent, their students might consider it the same way. He added that if teachers believe in such uncontrollable factors as a barrier to success in mathematics, they may not perform much effort for the untalented students. In this study, MSMTs frequently expressed cognitive process-related attributions, which was also an uncontrollable factor. During the interviews, MSMTs frequently described one-third of the students in the classroom as unsuccessful students while talking about their performances for almost all items. Moreover, they noted there was nothing to do for these students as they were already unsuccessful and could not learn mathematics. MSMTs' those statements for unsuccessful students could also be interpreted as uncontrollable factors. It might be inferred that MSMTs would do nothing to improve these students' performances as they thought they had no control over them. For example, Mrs. Gürsoy stated that he was interested in improving the performance of high-achiever students in the classroom, and his sole purpose was to take them further. These statements might mirror his view about the students who were not among these students. He might consider that only successful students could learn mathematics and go beyond, and he could do nothing for the remaining students since he had no control over low achievers and there was nothing to do with them.

MSMTs also attributed students' difficulties to instructional process-related factors. Teaching process-related attributions were the most repeatedly observed attributions, similar to the study of Bozkurt and Yetkin-Özdemir (2018), which were often internal, unstable, and controllable factors for teachers, such as inadequate lesson content and lack of addressing key points. Therefore, it might be inferred that teachers considered some points were under their control while teaching algebra, such as focusing on critical concepts (e.g., understanding the concepts of variable and relational understanding of the equal sign, solving daily life examples, and improving students' functional thinking by focusing on covariation). MSMTs' second most frequent instructional process-related causal attribution was related to objectives (curriculum) which might be identified as external, stable, and uncontrollable factors for teachers. For example, MSMTs advocated that the curriculum did not comprise the objectives related to identifying the notion of the variable in an algebraic expression or comparing the magnitude of two different algebraic expressions with the same unknown, such as n + 3 and 6n. Also, as they asserted, students do not use algebraic symbols in the primary school mathematics curriculum. Bozkurt and Yetkin-Özdemir (2018) showed that teachers attributed the failures to themselves more as their study was a lesson study and teachers designed their instruction themselves collaboratively. Conversely, based on the current study results, MSMTs attributed students' failures to student-related factors more. Lastly, teachers seldomly attributed students' failures to the high-stakes tests, which was an external, stable, and uncontrollable factor for MSMTs. Ms. Burcu noted that the examination system stressed them, and they could not teach math as they wanted, such as focusing on just a few questions or rarely doing activities in algebra classes.

Moreover, MSMTs advocated that students might have difficulty in algebra since the tasks were complex and the items were open-ended in ADT, something they were not used to until that time. Therefore, the third attribution MSMTs made was task-related attribution, which was an external, unstable, and uncontrollable factor for MSMTs, including task difficulty and structure of the item. For example, Ms. Burcu said, “In Item 4a, if you asked what is the smallest number when the summation of three numbers equals 84, most would respond correctly. Nevertheless, item 4a was not a task they were familiar with.” Lastly, MSMTs rarely expressed attributions related to family and social environment, which were external, unstable, and uncontrollable factors for them, such as the crowdedness of the classes or the students' nutrition.

Results showed that MSMTs associated students’ difficulties with external factors more. It can be argued that MSMTs mostly considered the reason for students’ difficulties is because of students themselves. Furthermore, MSMTs mainly attributed students’ difficulties to stable and uncontrollable factors. It might be inferred that MSMTs dominantly considered that they had no control over the factors that make students unsuccessful, such as students’ cognitive processes, lack of motivation, inadequate math skills, and content of the objectives in the curriculum. As researchers stated, attributions significantly affect teachers’ expectations for students’ future academic performances (Clarkson & Leder, 1984; Peterson & Barger, 1985). Also, as Wang and Hall (2018) asserted, teachers’ causal attributions might influence their instructional behaviors that significantly impact students’ academic performance and motivation. MSMTs stated that they would be more attentive based on particular points, such as the concept of the variable or covariational thinking. However, their attributions might indicate that students’ low performance was mainly related to factors other than their teaching in general. Therefore, they might continue to teach algebra similarly, as most of the factors that cause students' difficulties are beyond the responsibilities of teachers.

5.4.Implications of the Study

The current study contributes to the literature on pre-service mathematics teachers, in-service mathematics teachers, teacher educators, and policymakers related to teachers' knowledge of students' algebraic thinking in the variable concept, equivalence and equations, and functional thinking. Findings suggested that all MSMTs mainly highlighted the significance of having the capability of doing computation (addition, subtraction, multiplication, and division) with different forms of numbers as a prerequisite for success in algebra. Moreover, some mentioned understanding algebraic key terms, using various forms of numbers, the ability to use graphics, understanding algebraic expressions, and knowing the rules while solving algebraic equations. Based on the findings and conclusions, none of the MSMTs talked about the relational understanding of the equal sign, understanding the notion of variable, proportional reasoning, and covariational thinking prior to higher-level algebra topics. Although the main purpose of algebra was to represent a general relationship or procedure, to solve a wide range of problems using those representations, and to produce new representations based on the known ones (Booth, 1986), it might be inferred that the participants of this study perceived algebra as doing manipulations with symbols and solving equations to solve the problem in general. Teachers' knowledge regarding the prerequisite knowledge to learn algebra might be a precursor related to their algebra perceptions. Similar to the results of Tanışlı and Köse (2013) and Stephens (2006) on pre-service mathematics teachers, it might be inferred that participant MSMTs required improvement related to their SMK of equivalence and the relational understanding of the equal sign. The deficiency of their knowledge in SMK might prevent teachers from determining students' difficulties, errors, and misconceptions, as reported by Boz (2004). Therefore, they might improve their PCK regarding students' thinking in equivalence and the relational understanding of the equal sign.

An important implication of this study was that MSMTs failed to analyze the students' algebraic thinking at some points, similar to the study of Tanışlı and Köse (2013). MSMTs could explain students' algebraic thinking processes related to the big ideas of equations and generalized arithmetic. However, they provided limited explanations for the causes of students' erroneous thinking and difficulties in the big ideas of variable, equivalence, and functional thinking. However, they could not discuss students' thoughts in a detailed manner. They usually focused on students' algebraic practices of doing operations, using numbers, doing a substitution, and solving equations. To illustrate, teachers focused on students' capability of doing factorization. Next, they interpreted that students' difficulties and errors were caused by inadequate knowledge of negative numbers or deficiency in equation-solving procedures. Although these factors also some of the causes of students' difficulties, the relational understanding of equal sign and equivalent equations was crucial in dealing with equations as students could observe the conservation of equivalence relations throughout the transformation process (Knuth et al., 2005; Steinberg et al., 1990). Based on the findings of this study, it might be claimed that a teacher development program might help improve MSMTs' SMK and PCK to have teachers notice the importance of the meaning of equal sign and comprehension of equivalence to be successful at constructing and solving equations.

Moreover, teachers sometimes provided irrelevant explanations when they could not understand or express students' thoughts. For example, they stated that students preferred memorization of the rules or were not working hard instead of providing the underlying reasons for students' difficulties and errors in particular concepts. Those results might be associated with teachers' inadequate knowledge of students' algebraic thinking. In this study, the inadequacy in MSMTs' knowledge of students' algebraic thinking might be attributed to the deficiency of previous instruction given to teachers focusing on students' algebraic thinking. Based on the findings and conclusions of this study, it might be suggested that professional development programs for mathematics teachers might be conducted to improve teachers' knowledge regarding the big ideas of algebra and middle-grade students' difficulties, errors, and misconceptions in algebra reported by the literature studies. Since pre-service teachers were future teachers, similar revisions might be helpful for their improvement regarding the algebraic thinking of students. Based on The Council of Higher Education (CoHE, 2018), teacher education programs were revised, and "Teaching Algebra" was assigned as a must-course for one semester for pre-service elementary mathematics teachers as it was included as a part of

"Methods of Teaching Mathematics" course in the previous mathematics teacher education programs. Including "Teaching Algebra" as a must-course in teacher education programs was an excellent start to enrich teachers' knowledge of students' thinking in algebra. As pre-service teachers are future teachers, they might have similar conceptions and needs to MSMTs in this study. Tanışlı and Köse (2013) proposed that teacher education programs might offer elective courses for pre-service teachers to improve their PCK, especially the knowledge of students' learning of mathematics and students' misconceptions. Pre-service teachers could also experience various algebra tasks and particular solutions for students to broaden their knowledge about students' algebraic thinking, difficulties, and misconceptions in "Teaching Algebra" courses and additional elective courses in elementary mathematics teacher education programs.

The literature review showed the scarcity of studies examining MSMTs' causal attributions in algebra (Shores & Smith, 2010; Wang & Hall, 2018). Regarding causal attributions, this study might serve the related literature on two aspects, student-related attributions and instructional process-related attributions of MSMTs. Findings suggested that teachers associated students' difficulties more with external and uncontrollable factors. MSMTs mainly attributed students' failure in algebraic thinking to student-related factors, primarily cognitive process-related factors such as inadequacy of students' understanding, lack of motivation, and inadequate math skills. Researchers argued that attributions significantly affect teachers' expectations for students' future academic performances (Clarkson & Leder, 1984; Peterson & Barger, 1985). As Glasgow et al. (1997) asserted, if teachers attributed students' failures to such uncontrollable factors, they might not perform much effort into the untalented students since they thought they had no control over them. According to those findings, further studies might be conducted to observe causal attributions of MSMTs regarding student-related factors involving MSMTs and their middle-grade students as participants in the study. Moreover, different types of data collection tools, such as interviews and classroom observations, might be utilized to collect detailed data from both middle school students and their MSMTs to test the validity of MSMTs' arguments regarding external and uncontrollable factors for students' failures in algebra.

Based on MSMTs' attributions, the second factor for students' difficulties in algebraic thinking was instructional process-related factors, including teaching process-related, curriculum-related, and examination system-related factors. One of the most frequently observed instructional process-related attributions was curriculum-related attributions which were external, stable, and uncontrollable factors such as the duration of the courses and objectives in the mathematics curriculum. MSMTs argued that there was no emphasis on such notions of variable, equivalence, and covariational thinking. Moreover, as they stated, they were not expected to do such tasks, considering the meaning of a variable in an algebraic expression, comparing two algebraic expressions with the same unknown, and observing the covariation between the quantity of two variables in a function. These findings proposed that future studies which investigate the content of the current middle school mathematics curriculum (MoNE, 2018) in terms of big ideas of algebra (Blanton et al., 2015; Blanton et al., 2019) might provide helpful information for MSMTs, researchers, and curriculum developers.

5.5.Recommendations for the Further Research

In this section, the implications related to the present study will be given. This study concentrated on in-service MSMTs' knowledge of middle school students' conceptions and difficulties in algebra. This study focused on the knowledge of features of learning mathematics dimension of mathematical knowledge for teaching model (Carrillo-Yañez et al., 2018). During the administration process of the study, I observed that it was difficult to discriminate the two dimensions of PCK: the knowledge of features of learning mathematics and knowledge of mathematics teaching, as their borders have not been determined yet (Hill et al., 2007). In further studies, it might be recommended that researchers might focus on both dimensions to observe teachers' knowledge of students' algebraic thinking.

Based on teachers' statements about students' understanding of the equal sign, MSMTs could not anticipate students' difficulties regarding the meaning of the equal sign. Alapala (2018) illustrated that pre-service MSMTs improve their capability of anticipating students' misconceptions regarding the operational thinking of the equal sign after having instruction on this issue. Results also suggested that MSMTs' predictions were aligned with students' performance in the tasks which required simple algebraic procedures, solving a problem using arithmetics, and solving an equation. However, they often failed to predict students' task performance, including understanding the variable, comparing different algebraic expressions, and functional thinking. In addition, they could not explicitly express the reasons for students' difficulties. For in-service MSMTs, teacher training programs including students' correct and incorrect solution paths might be a good point to improve students' algebraic thinking, as presented in the studies of Tirosh (2000), Tanışlı and Köse (2013), Alapala (2018), and Didiş-Kabar and Amaç (2018). Therefore, professional development programs might be offered for in-service MSMTs to broaden their knowledge and awareness of crucial points in students' algebraic thinking, such as relational understanding of equivalence, understanding of variable, and different types of functional thinking (Blanton et al., 2015; Blanton et al., 2019; Blanton & Kaput, 2004; Stephens et al., 2013).

It was observed that MSMTs' thoughts were changed in the post-interviews compared to the pre-interviews. For example, they were unaware of students' needs to understand the concept of variable. After they observed the students' low success in the tasks related to the big idea of the variable in ADT, they expressed that they should focus on the variable more in their algebra classes. They stated that they would have continued not to dwell on this issue if they had not seen these results. Moreover, although a participant teacher stated that there was no need to focus on the covariation of variables in functions, he changed his mind and expressed that he would concentrate on the covariation of variables in functions from that time after he investigated the students' performance in ADT. Although there was no intervention in this study, some changes in MSMTs minds were observed after the interviews. In further case studies, using observations, teachers' thoughts and behaviors might be investigated to see the possible changes before and after the interviews. As teachers' conceptions are reluctant to change (Thompson, 1992), intervention studies might also be conducted on teachers to observe how their thoughts and behaviors change by using particular tasks and corresponding correct and incorrect student solutions regarding students' algebraic thinking.

5.6.Assumptions and Limitations of the Study

It might be stated that MSMTs and I, as the researcher, had developed a trusting relationship as we had known each other for approximately one year during the interviews and informal observations. Therefore, it might be assumed that MSMTs confidently shared their ideas with the researcher throughout the study. The Algebra Diagnostic Test (ADT) was conducted to eighth-grade students at the end of the algebra topic, and it was assumed that all students from the public school had similar instructions regarding algebra. Also, all the students were conjectured to take all the algebra courses they should have completed. Furthermore, the ADT was applied to students at two different times because of the administrative issues in Pilot Testing I and II. Thus, it was assumed that none of the students who saw the items earlier shared them with those who had not taken the test yet. Considering this issue, the final ADT was applied to all eight grade students simultaneously. Therefore, the possibility of students sharing the items with other students was eliminated.

In addition to the assumptions, some points might be accepted as the study's limitations were presented as follows. The first issue that might be considered a limitation was that the data was gathered from five MSMTs in a public middle school in Zonguldak. The study's findings might have been different if other MSMTs had participated from different schools or schools from other cities. The participant MSMTs were mainly teaching eighth-grade students throughout the study. If other MSMTs were selected for the study whose students were distributed differently regarding grade levels, different findings might have been observed at the end of the study since the participant MSMTs usually tend to consider eighth-grade students as they were teaching eighth graders for several years. Since the participant MSMTs were selected for the study by purposeful sampling, they might not represent other MSMTs. For this reason, the findings might be presumed less generalizable to different situations. However, as the study had a qualitative nature, generalization of the findings was not the goal of the study.

Another issue was related to data collection regarding students' performances in ADT. The data gathered from students was collected at once and limited to one data collection tool, the ADT, including big ideas of equivalence and equation, variable, generalized arithmetic, and functional thinking. Therefore, the findings of this study regarding students' algebraic thinking were limited to the big ideas of algebra included in ADT. The final part of this chapter presented some recommendations for further research in the next section.

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APPENDICES

A. PILOT TEST I

Sevgili öğrenciler,

Bu test cebir konularını kapsayan 14 sorudan oluşmaktadır. Teste verdiğiniz cevaplar, bilimsel bir çalışma kapsamında 7. sınıf öğrencilerinin cebir öğrenme alanındaki bilgilerini değerlendirmek amacıyla kullanılacaktır. Testin süresi 40 dakikadır. Başarılar☺

Öğrencinin adı ve soyadı:

1) Aşağıdaki eşitliğin doğruluğu, 7 x 22 ve 14 x 11 çarpma işlemlerinin sonucu bulunmadan gösterilebilir mi? Nedenleriyle açıklayınız.

$$7 \times 22 = 14 \times 11$$

2) *"50 dakikalık bir sınavda, sınav başladıktan x dakika sonra kalan süre"* ifadesinin cebirsel olarak yazımı nasıldır?

3) n bir rasyonel sayı olmak üzere, 3n ve n+6 ifadelerinden hangisinin daha büyük olduğunu söyleyebilir misiniz? Cevabınızı kısaca açıklayınız.

4) "*Ardışık 3 çift doğal sayının toplamı 84'tür.*" ifadesine karşılık gelen denklemi yazınız. Bu denklemdeki bilinmeyenin neyi ifade ettiğini açıklayınız.

5) a = 3b + 4 olduğuna göre, b'nin değeri 2 arttırılırsa a'nın değeri nasıl değişir?

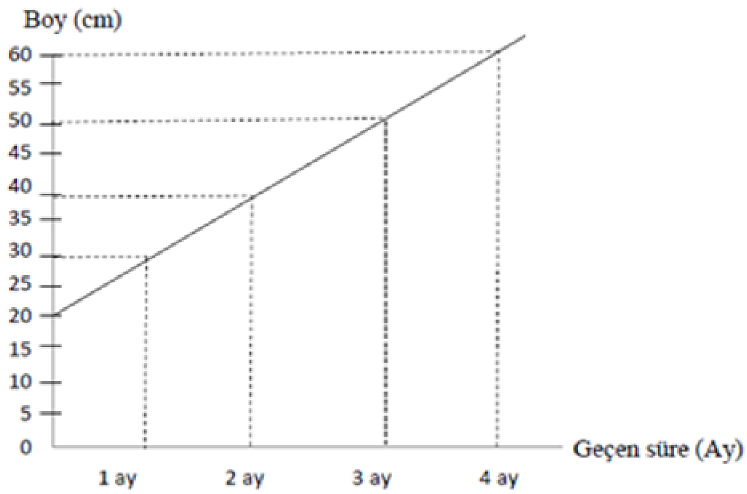
6) Öğretmen, sınıfta 8 + 9c = 2 şeklindeki bir denklemi çözmek için tahtaya yazmıştır. Bu sırada, sınıftaki öğrencilerden biri denklemin hatalı olduğunu, 8 ile bir sayının toplamının 2'ye eşit olamayacağını söylemiştir. Sizce bu öğrencinin düşüncesi doğru mudur? Nedenleriyle açıklayınız.

7) $-3 - 2x = -9$ denklemini işlem adımlarını sırayla göstererek çözünüz.

8) Bir kafede, kahvaltı ücreti 20 TL olarak belirlenmiştir. Bu kafede kahvaltı sipariş eden müşterilerinden, içtikleri ilk çay için ücret alınmamakta; sonraki her bir çay siparişi için ise 3 TL ücret alınmaktadır.

a) Buna göre, bu kafede kahvaltı sipariş eden bir kişinin içtiği çay ve ödediği toplam ücret arasındaki ilişkiyi gösteren denklemi yazınız.

b) Bu kafede toplam 5 bardak çay içen bir müşteri toplamda ne kadar ücret ödemiştir?

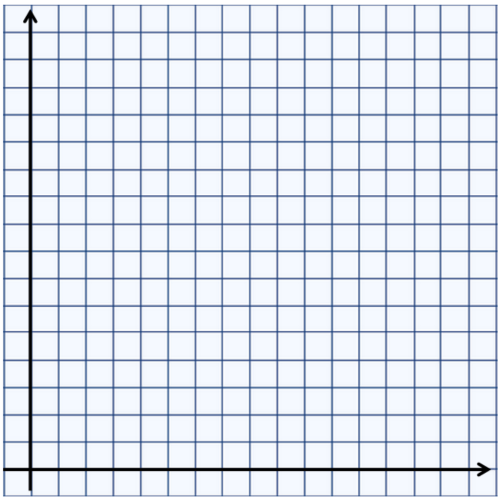


9) Yukarıdaki doğrusal grafik bahçeye dikilen bir fidanın aylara göre uzama miktarını göstermektedir. Grafiğe göre; her ay eşit miktarda büyüyen bu fidanın dikildikten 8 ay sonraki uzunluğu kaç cm olur? Bu sorunun çözümündeki işlem adımlarını sırayla gösteriniz.

10) Bir arkadaş grubu Zonguldak'tan Muğla'ya düzenlenecek olan bir geziye katılmaya karar vermiş ve 800 kilometrelik yolu saatte 100 km sabit hızla gitmişlerdir.

a) Gidilen yolu x (km), zamanı t (saat) ile gösterecek şekilde her bir saat sonunda gidilen yolun tablosunu oluşturunuz.

b) Tablodaki verilerin grafiğini x-ekseni zamanı, y-ekseni gidilen yolu gösterecek şekilde aşağıdaki koordinat düzlemi üzerinde çiziniz.



a) Grafik üzerindeki noktaları aynı doğru üzerinde birleştirmek doğru olur mu? Neden?

b) Verilen t değerlerinden yola çıkarak x değerlerini bulmanızı sağlayacak olan genel kuralı sözel olarak ifade ediniz.

c) Verilen t değerlerinden yola çıkarak x değerlerini bulmanızı sağlayacak olan genel kuralı ifade eden denklemi yazınız.

d) Yazdığınız denklemi kullanarak, $\frac{3}{5}$ saatte gidilen yolu bulunuz.

B. PILOT TEST II

Sevgili öğrenciler, bu test cebir konularını kapsayan 11 sorudan oluşmaktadır. Teste verdiğiniz cevaplar, 7. sınıf öğrencilerinin cebir öğrenme alanındaki bilgilerini değerlendirmek amacıyla kullanılacaktır. Testin süresi 40 dakikadır. Başarılar☺

Öğrencinin adı ve soyadı:

1) Aşağıdaki eşitliğin doğruluğu, 7×22 ve 14×11 çarpma işlemlerinin sonucu bulunmadan gösterilebilir mi? Nedenleriyle açıklayınız.

$$7 \times 22 = 14 \times 11$$

2) *”50 dakikalık bir sınavda, sınav başladıktan x dakika sonra kalan süre”* ifadesinin cebirsel olarak yazımı nasıldır?

3) n bir rasyonel sayı olmak üzere, 3n ve n+6 ifadelerinden hangisinin daha büyük olduğunu söyleyebilir misiniz? Cevabınızı kısaca açıklayınız.

4) “*Ardışık 3 doğal sayının toplamı 84’tür.*” ifadesine karşılık gelen denklemi yazınız. Bu denklemdeki bilinmeyenin neyi ifade ettiğini açıklayınız.

5) $a = 3b + 4$ olduğuna göre, b’nin değeri 2 arttırılırsa a’nın değeri nasıl değişir? Lütfen cevabınızı nedenleriyle açıklayınız.

6) Öğretmen, $8 + 9c = 2$ denklemini tahtaya yazar ve öğrencilerin çözmesini ister. Bu sırada, sınıftaki öğrencilerden biri denklemin hatalı olduğunu, 8 ile bir sayının toplamının 2’ye eşit olamayacağını söyler. Sizce bu öğrencinin düşüncesi doğru mudur? Nedenleriyle açıklayınız.

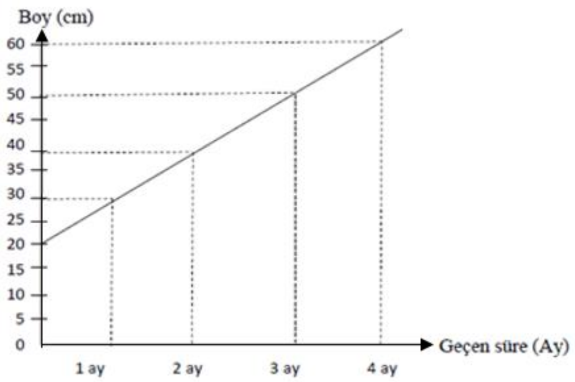
7) $-3 - 2x = -9$ denklemini, işlem adımlarınızı sırasıyla göstererek çözünüz.

8) Bir kafede, kahvaltı ücreti 20 TL olarak belirlenmiştir. Bu kafede kahvaltı siparişi veren müşterilerden, içtikleri ilk çay için ücret alınmamakta; sonraki her bir çay siparişi için ise 3 TL ücret alınmaktadır.

a) Buna göre, bu kafede kahvaltı siparişi vermiş olan bir kişinin içtiği çay ve ödediği toplam ücret arasındaki ilişkiyi gösteren denklemi yazınız.

b) Bu kafede kahvaltı siparişi vermiş ve 5 bardak çay içmiş olan bir müşteri toplamda ne kadar ücret ödemiştir?

9)



Yukarıdaki doğrusal grafik bahçeye dikilen bir fidanın aylara göre uzama miktarını göstermektedir.

a) Grafikte x-ekseni geçen süreyi (ay), y ekseni de fidanın boyunu (cm) göstermek üzere; her ay eşit miktarda büyüyen bu fidanın boyu ile geçen zaman arasındaki ilişkiyi gösteren denklemi yazınız.

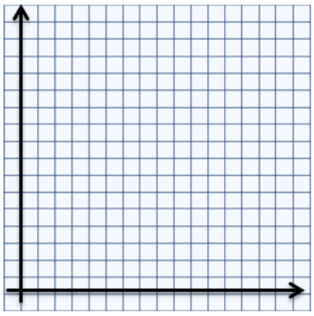
b) Grafiğe göre; her ay eşit miktarda büyüyen bu fidanın dikildikten 8 ay sonraki uzunluğu kaç cm olur? Bu sorunun çözümündeki işlem adımlarını sırayla gösteriniz.

10) Bir arkadaş grubu Zonguldak'tan Muğla'ya düzenlenecek olan bir geziye katılmaya karar vermiş ve 800 kilometrelik yolu saatte 100 km sabit hızla gitmişlerdir.

e) Yolculuk boyunca **her bir saat sonunda** gidilen **toplam yolun** tablosunu oluşturunuz.

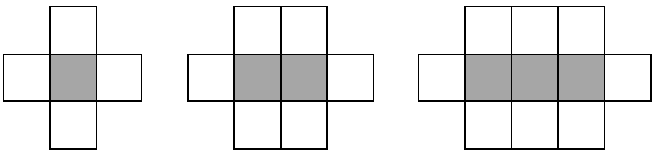
Zaman (saat)	Yol (km)

f) a şıkkında oluşturduğunuz tablodaki verilerin grafiğini x-ekseni zamanı (saat), y-ekseni gidilen yolu (km) gösterecek şekilde aşağıdaki koordinat düzlemi üzerinde çiziniz.



g) Yolculuk boyunca geçen zamana t, gidilen yola da m denilirse, gidilen yolun zamana göre değişimini ifade eden denklemi yazınız.

11) Aşağıdaki şekilde, yan yana getirilen kare masalar (gri renkli) ve sandalyeler (beyaz renkli) bulunmaktadır. Yan yana getirilen kare masalara yerleştirilen sandalye sayılarının, masa sayılarına göre dağılımı aşağıdaki gibidir. Buna göre,



1 masa 2 masa 3 masa

a) 10 tane masa yan yana getirildiğinde, o masaya yerleştirilen toplam sandalye sayısını bulunuz.

b) n tane masa ile o masaya yerleştirilen sandalye sayısı arasında oluşturulan örüntünün kuralını (denklemini) yazınız.

c) Bir miktar masa yan yana getirildiğinde, o masaya toplam 152 tane sandalye yerleştirildiğine göre, birleştirilen masa sayısını bulunuz.

C. ALGEBRA DIAGNOSTIC TEST

Sevgili öğrenciler, bu test cebir konularını kapsayan 17 sorudan oluşmaktadır. Teste verdiğiniz cevaplar, 8. sınıf öğrencilerinin cebir öğrenme alanındaki bilgilerini değerlendirmek amacıyla kullanılacaktır. Testin süresi 50 dakikadır. Başarılar☺

Öğrencinin adı-soyadı:

Şube: 8/…

1) Aşağıdaki eşitliğin doğruluğu, $7 \cdot 22$ ve $14 \cdot 11$ çarpma işlemlerinin sonucu bulunmadan gösterilebilir mi? Nedenleriyle açıklayınız.

$$7 \cdot 22 = 14 \cdot 11$$

2) *"50 dakikalık bir sınavda, sınav başladıktan x dakika sonra kalan süre"* ifadesinin cebirsel olarak yazımı nasıldır?

3) n bir tam sayı olmak üzere, 3n ve n+6 ifadelerinden hangisinin daha büyük olduğunu söyleyebilir misiniz? Lütfen cevabınızı kısaca açıklayınız.

4) "*Ardışık 3 doğal sayının toplamı 84'tür.*"

a) Yukarıdaki ifadeye karşılık gelen denklemi yazınız.

b) Yazdığınız denklemde, harfle ifade ettiğiniz bilinmeyenin neyi belirttiğini açıklayınız.

5) Bir öğretmen derste 8 + 9c = 2 denklemini tahtaya yazar ve öğrencilerin denklemi çözmesini ister. Bu sırada, bir öğrenci denklemin hatalı olduğunu, 8 ile bir sayının toplamının hiçbir zaman 2'ye eşit olamayacağını söyler. Sizce bu öğrencinin düşüncesi doğru mudur? Lütfen cevabınızı nedenleriyle açıklayınız.

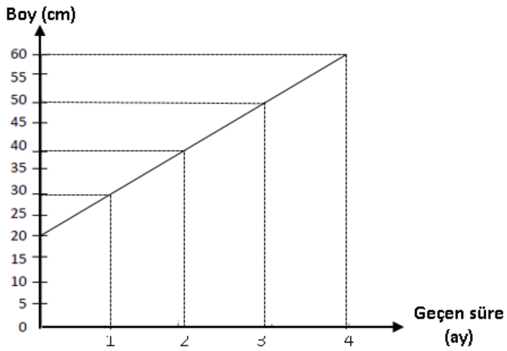
6) a = 3b + 4 olduğuna göre, b değeri 2 arttırılırsa a değeri nasıl değişir? Lütfen cevabınızı nedenleriyle açıklayınız.

7) Aşağıda a ve b şıklarında verilen denklemleri, işlem adımlarınızı sırasıyla göstererek çözünüz.

a) $-3 - 2x = -9$ b) $3x + 2 = -7(x - 6)$

8. ve 9. soruları aşağıdaki grafiğe göre cevaplayınız.

Aşağıdaki grafik bir bahçeye dikilen ve her ay eşit miktarda uzayan bir fidanın aylara göre uzama miktarını göstermektedir.



8) Grafikte x-ekseni geçen süreyi (ay), y-ekseni de fidanın boyunu (cm) göstermek üzere; bu fidanın boyu ile geçen süre arasındaki ilişkiyi gösteren denklemi yazınız.

9) Grafiğe göre; bu fidanın dikildikten 8 ay sonraki uzunluğu kaç cm olur? Bu sorunun çözümündeki işlem adımlarını sırasıyla gösteriniz.

10. ve 11. soruları aşağıdaki metne göre cevaplayınız.

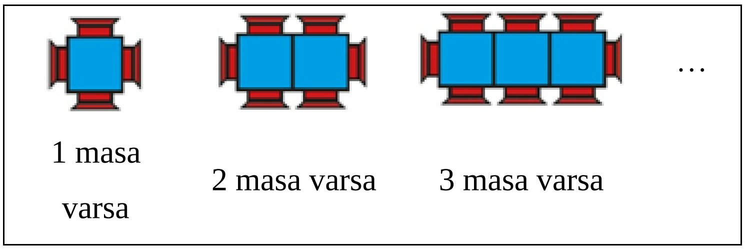
"Bir kafede, kahvaltı ücreti 20 TL olarak belirlenmiştir. Bu kafede kahvaltı siparişi veren müşterilerden, içtikleri ilk çay için ücret alınmamakta; sonraki her bir çay siparişi için ise 3 TL ek ücret alınmaktadır. "

10) Bu kafede kahvaltı siparişi vermiş olan bir kişinin içtiği çay sayısı x bardak, ödediği toplam ücret ise y TL olmak üzere, x ve y değişkenleri arasındaki ilişkiyi gösteren denklemi yazınız.

11) Bu kafede kahvaltı siparişi vermiş ve toplam 5 bardak çay içmiş olan bir müşterinin toplamda kaç TL ücret ödemesi gerekmektedir?

12., 13. ve 14. soruları aşağıdaki metne göre cevaplayınız.

Aşağıdaki şekilde, yan yana getirilerek birleştirilen kare biçimindeki masalar ve bu masaların çevresine yerleştirilen sandalyeler bulunmaktadır. Masalara yerleştirilen sandalye sayılarının, masa sayılarına göre dağılımının görünümü aşağıdaki şekilde verilmiştir.



12) 10 tane masa yan yana getirilerek birleştirildiğinde, masalara yerleştirilen toplam sandalye sayısını bulunuz.

13) Yan yana getirilerek birleştirilen masa sayısı x, masalara yerleştirilen sandalye sayısı da y olmak üzere, x ve y değişkenleri arasında oluşan örüntünün kuralını (denklemini) yazınız.

14) Bir miktar masa yan yana getirilerek birleştirildiğinde, masalara yerleştirilen sandalye sayısı 152 olduğuna göre, birleştirilen masa sayısını bulunuz.

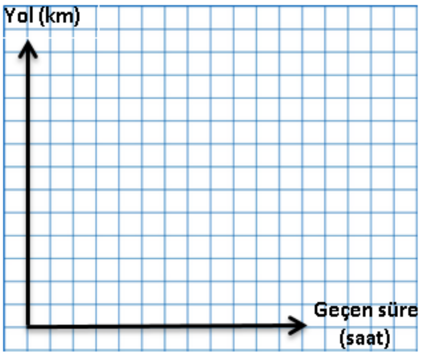
15., 16. ve 17. soruları aşağıdaki metne göre cevaplayınız.

"Bir arkadaş grubu Zonguldak'tan Çanakkale'ye düzenlenecek olan bir geziye katılmaya karar vermiş ve 600 kilometrelik yolu araçlarıyla saatte 100 km sabit hızla gitmişlerdir."

15) Yolculuk boyunca, **her bir saatin sonunda** yolculuğun başından itibaren **gidilen toplam yolun** tablosunu oluşturunuz.

Geçen süre (saat)	Gidilen toplam yol (km)
1. saatin sonunda	100 km

16) Metinde verilen bilgilere göre, x-ekseni geçen süreyi (saat), y-ekseni gidilen yolu (km) göstermek üzere, gidilen yol ve geçen süre arasındaki ilişkiyi gösteren grafiği aşağıdaki koordinat düzlemi üzerinde çiziniz.



17) Yolculuk boyunca geçen süre t, gidilen toplam yol m ile gösterilmek üzere, gidilen yol ile geçen süre arasındaki ilişkiyi gösteren denklemi yazınız.

D.A QUESTIONNAIRE FOR TEACHERS BASED ON ALGEBRA DIAGNOSTIC TEST

Öğretmenin adı ve soyadı:

Sevgili öğretmenler, bu anket 6., 7. ve 8. sınıf cebir konularının bir kısmını kapsayan ve 8. sınıf öğrencilerine uygulanacak olan 8. Sınıf Kavramsal Cebir Testi'ne yönelik düşüncelerinizi öğrenmeyi amaçlayan sorulardan oluşmaktadır. 8. Sınıf Kavramsal Cebir Testi'nde, 8. sınıf öğrencilerinin cebir öğrenme alanındaki *eşitlik, değişken, denklem ve doğrusal denklemler* konuları ile ilgili kavrayışları ve zorluklarını incelemek amacıyla kullanılacak olan sorular bulunmaktadır. Testteki soruların cevaplanması için öğrencilere 60 dakika süre verilecektir. Lütfen soruları yalnızca dersine girdiğiniz 8. sınıf öğrencilerini düşünerek cevaplayınız. Katılımınız için çok teşekkür ederim.

Nurbanu Yılmaz

Soru 1) *Aşağıdaki eşitliğin doğruluğu, $7 \cdot 22$ ve $14 \cdot 11$ çarpma işlemlerinin sonucu bulunmadan gösterilebilir mi? Nedenleriyle açıklayınız.*

$$7 \cdot 22 = 14 \cdot 11$$

a) Yukarıdaki soru için öğrencileriniz tarafından verilebilecek olan doğru veya yanlış cevaplar neler olabilir? Sizce doğru veya yanlış cevap veren öğrencileriniz, bu sorunun çözümü için hangi çözüm yollarını kullanmış olabilirler?

b) Yukarıdaki sorunun okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

Soru 2) *"50 dakikalık bir sınavda, sınav başladıktan x dakika sonra kalan süre" ifadesinin cebirsel olarak yazımı nasıldır?*

a) Yukarıdaki soru için öğrencileriniz tarafından verilebilecek olan doğru veya yanlış cevaplar neler olabilir?
b) Yukarıdaki sorunun okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

Soru 3) *n bir rasyonel sayı olmak üzere, 3n ve n + 6 ifadelerinden hangisinin daha büyük olduğunu söyleyebilir misiniz? Lütfen cevabınızı kısaca açıklayınız.*

a) Yukarıdaki soru için öğrencileriniz tarafından verilebilecek olan tipik doğru veya yanlış cevaplar neler olabilir? Sizce doğru veya yanlış cevap veren öğrencileriniz bu sorunun çözümü için hangi yöntemleri kullanmış olabilirler?
b) Yukarıdaki sorunun okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

Soru 4) *"Ardışık 3 doğal sayının toplamı 84'tür."*

a) *Yukarıdaki ifadeye karşılık gelen denklemi yazınız.*

) *Yazdığınız denklemde harfle ifade ettiğiniz bilinmeyenin neyi belirttiğini açıklayınız.*

a) Öğrencilerinizin yukarıdaki sorudaki a şıkkı için verebileceği doğru veya yanlış tipik cevap örnekleri neler olabilir? Lütfen örneklendirerek açıklayınız.
b) Sizce öğrencileriniz b şıkkında sorulmuş olan bilinmeyenin neyi ifade ettiğini açıklayabilirler mi?

c) Yukarıdaki sorunun okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

Soru 5) *Öğretmen, c bir rasyonel sayı olmak üzere, 8 + 9c = 2 denklemini tahtaya yazar ve öğrencilerin denklemi çözmesini ister. Bu sırada, sınıftaki öğrencilerden biri denklemin hatalı olduğunu, 8 ile bir sayının toplamının 2'ye eşit olamayacağını söyler. Sizce bu öğrencinin düşüncesi doğru mudur? Nedenleriyle açıklayınız.*

a) Öğrencilerinizin yukarıdaki soru için verebileceği doğru veya yanlış tipik cevap örnekleri neler olabilir?
b) Yukarıdaki sorunun okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

Soru 6) *a = 3b + 4 olduğuna göre, b'nin değeri 2 arttırılırsa a'nın değeri nasıl değişir? Lütfen cevabınızı nedenleriyle açıklayınız.*

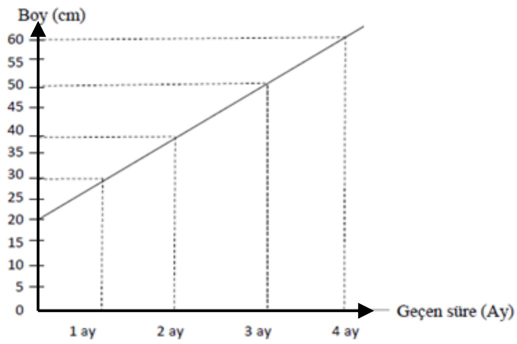
a) Öğrencilerinizin yukarıdaki soru için verebileceği doğru veya yanlış tipik cevap örnekleri neler olabilir?
b) Yukarıdaki sorunun okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

Soru 7) Aşağıdaki denklemleri, işlem adımlarınızı sırasıyla göstererek çözünüz.

$-3 - 2x = -9$ $\qquad\qquad$ $3x + 2 = -7(x - 6)$

a) Öğrencilerinizin yukarıdaki soru için verebileceği doğru veya yanlış tipik cevap örnekleri neler olabilir?
b) Yukarıdaki sorunun okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

Aşağıdaki grafik bir bahçeye dikilen fidanın aylara göre uzama miktarını göstermektedir. 8. ve *9. soruları aşağıdaki grafiğe göre cevaplayınız.*



Soru 8) Grafikte x-ekseni geçen süreyi (ay), y ekseni de fidanın boyunu (cm) göstermek üzere; her ay eşit miktarda büyüyen bu fidanın boyu ile geçen zaman arasındaki ilişkiyi gösteren denklemi yazınız.

Soru 9) Grafiğe göre; her ay eşit miktarda büyüyen bu fidanın dikildikten 8 ay sonraki uzunluğu kaç cm olur? Bu sorunun çözümündeki işlem adımlarını sırasıyla gösteriniz.

a) Öğrencilerinizin yukarıdaki 8. ve 9. sorular için verebileceği doğru veya yanlış tipik cevap örnekleri neler olabilir?
b) Yukarıdaki 8. ve 9. soruların okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

10. ve 11. soruları aşağıdaki metne göre cevaplayınız.

"Bir kafede, kahvaltı ücreti 20 TL olarak belirlenmiştir. Bu kafede kahvaltı siparişi veren müşterilerden, içtikleri ilk çay için ücret alınmamakta; sonraki her bir çay siparişi için ise 3 TL ücret alınmaktadır. "

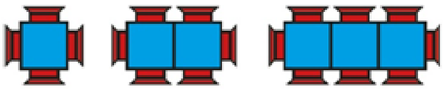
Soru 10) Bu kafede kahvaltı siparişi vermiş olan bir kişinin içtiği çay sayısı x, ödediği toplam ücret ise y olmak üzere, x ve y değişkenleri arasındaki ilişkiyi gösteren denklemi yazınız.

Soru 11) Bu kafede kahvaltı siparişi vermiş ve 5 bardak çay içmiş olan bir müşteri toplamda ne kadar ücret ödemiştir?

a) Öğrencilerinizin yukarıdaki 10. ve 11. sorular için verebileceği doğru veya yanlış tipik cevap örnekleri neler olabilir?
b) Yukarıdaki 10. ve 11. soruların okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

13., 14., 15. ve 16. soruları aşağıdaki metne göre cevaplayınız.

Aşağıdaki şekilde, yan yana getirilerek birleştirilen kare masalar (gri renkli) ve bu masaların çevresine yerleştirilen sandalyeler (beyaz renkli) bulunmaktadır. Yan yana getirilerek birleştirilen kare masalara yerleştirilen sandalye sayılarının, masa sayılarına göre dağılımı aşağıdaki gibidir. Buna göre,



1. adım 2. adım 3. adım

Soru 13) Yan yana getirilerek birleştirilen masa sayısı ve her bir adımdaki sandalye sayısı arasındaki ilişkiyi gösteren aşağıdaki tabloyu doldurunuz.

Adım sırası	Her bir adımdaki masa sayısı	Her bir adımdaki sandalye sayısı
1. adım	1 masa	4 sandalye
2. adım		
3. adım		
4. adım		
5. adım		

Soru 14) 10 tane masa yan yana getirilerek birleştirildiğinde, o masaya yerleştirilen toplam sandalye sayısını bulunuz.

Soru 15) Yan yana getirilerek birleştirilen masa sayısı x, o masaya yerleştirilen sandalye sayısı da y ile gösterilirse, x ve y değişkenleri arasında oluşan örüntünün kuralını (denklemini) yazınız.

Soru 16)Bir miktar masa yan yana getirilerek birleştirildiğinde, o masaya toplam 152 tane sandalye yerleştirildiğine göre, birleştirilen masa sayısını bulunuz.

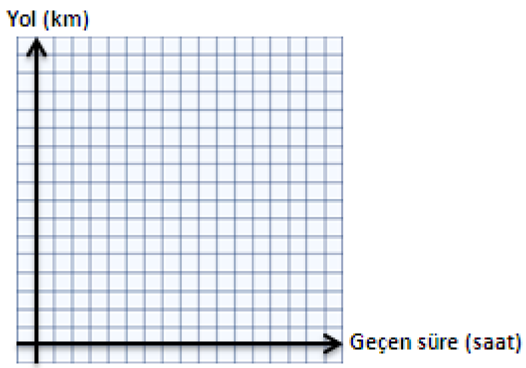
a) Öğrencilerinizin yukarıdaki 13., 14., 15. ve 16. sorular için verebileceği doğru veya yanlış tipik cevap örnekleri neler olabilir?
b) Yukarıdaki 14., 15. ve 16. soruların okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

17., 18. ve 19. soruları aşağıda verilen metne göre cevaplayınız.

"Bir arkadaş grubu Zonguldak'tan Çanakkale'ye düzenlenecek olan bir geziye katılmaya karar vermiş ve 600 kilometrelik yolu saatte 100 km sabit hızla gitmişlerdir."

Soru 17) Yolculuk boyunca **her bir saat sonunda** gidilen **toplam yolun** tablosunu oluşturunuz.

Soru 18) Metinde verilen bilgilere göre, x-ekseni geçen süreyi (saat), y-ekseni gidilen yolu (km) gösterecek şekilde, gidilen yol ve geçen süre arasındaki ilişkiyi gösteren grafiği aşağıdaki koordinat düzlemi üzerinde çiziniz.



Soru 19) Yolculuk boyunca geçen süre t, gidilen toplam yol m ile ifade edilirse, gidilen yolun zamana göre değişimini gösteren denklemi yazınız.

Geçen süre (saat)	Gidilen toplam yol (km)
1. saatin sonunda	100 km

a) Öğrencilerinizin yukarıdaki 17., 18. ve 19. sorular için verebileceği doğru veya yanlış tipik cevap örnekleri neler olabilir?

b) Yukarıdaki 17., 18. ve 19. soruların okulunuzda 8. sınıfta öğrenim gören (her başarı seviyesinden öğrencinin içinde bulunduğu) 100 öğrenciye sorulduğunu düşününüz. Soruyu doğru veya yanlış olarak cevaplamış olan öğrenciler için, belirttiğiniz her bir çözüm yolunun kullanılma yüzdelerini belirtebilir misiniz?

E.SEMI-STRUCTURED INTERVIEW QUESTIONS (BEFORE ALGEBRA DIAGNOSTIC TEST)

1) Sizce, cebir öğrenimine başlamadan önce öğrencilerin hangi konuları iyi derecede bilmeleri gerekmektedir?
   - Cebir konusuna geçmeden önce öğrencilerinizin bu konu(lar)da yeterli bilgiye sahip olduğunu düşünüyor musunuz?
2) Cebir konusunu anlatırken hangi kaynaklardan yararlanıyorsunuz? Bu kaynakları nasıl kullanıyorsunuz?
3) Bugüne kadar elde ettiğiniz deneyimlere dayanarak, sizce öğrenciler cebir öğrenme alanındaki hangi noktalarda zorlanıyorlar?
   - Öğrencilerin zorlandıkları noktaları nasıl belirliyorsunuz?
   - Konu anlatımında ve problem çözümünde bu zorlukları gidermek amacıyla kullandığınız herhangi bir yöntem var mı?
4) Daha önceki deneyimlerinize göre, öğrenciler cebir konusunda genellikle ne tür hatalar yapıyorlar?
   - Öğrencilerde bulunabilecek olan kavram yanılgıları ve öğrencilerin yapabilecekleri hataları belirlemeye yönelik kullandığınız bir yöntem var mıdır?
5) 8. *Sınıf Kavramsal Cebir Testi* ile ilgili olan aşağıdaki soruları, lütfen her bir test sorusu için sırasıyla cevaplayınız.
   - Testteki her bir soru için, öğrenciler tarafından verilebilecek olan *tipik* bir cevap örneği sizce nasıl olur?
   - Öğrencilerinizin hangi soruları doğru veya sıra dışı bir şekilde cevaplayacağını düşünüyorsunuz? *(Lütfen doğru cevaplayabilecek olan öğrencilerin yüzdesini her bir soru için belirtiniz.)*
   - Bu testte, öğrencilerinizin zorlanacağı veya hatalı cevaplar verebileceği sorular var mıdır?

- Cevabınız evet ise, öğrencilerin testteki hangi sorularda ve bu soruların hangi kısımlarında zorluk yaşayacağını düşünüyorsunuz?
- Neden, öğrencilerin zorluk yaşayabileceklerini düşündünüz?
- Sizce öğrencilerinizin sorularda hata yapabileceği noktalar neler olabilir? *(Lütfen soruları hatalı cevaplayabilecek olan öğrencilerin yüzdesini her bir soru için belirtiniz.)*
- Bu hataların sebepleri neler olabilir? Lütfen, sırasıyla her bir soru için açıklayınız.

F. SEMI-STRUCTURED INTERVIEW QUESTIONS (AFTER ALGEBRA DIAGNOSTIC TEST)

1) Öğrencilere uygulanan 8. *Sınıf Kavramsal Cebir Testi* 'ndeki sorulara öğrenciler tarafından verilen cevapların içerik analizi sonuçları şu şekildedir *(öğretmene, öğrencilere ait cevapların tümünün analiz sonuçları gösterilir).*

- Bu sonuçlar hakkında ne düşünüyorsunuz?
- Sizce, öğrenci cevaplarının analizlerinde görülen bu hataların sebepleri ne(ler) olabilir? *(3. sorunun sonuçlarını göstermek gibi spesifik sorular da sorulabilir. Öğrencilerin yüzde ... kadarı şöyle yapmış, sizce neden?)*
- Bu hatalar nereden kaynaklanıyor olabilir? Lütfen açıklayınız.

2) Öğrencilerin çoğu tarafından doğru olarak cevaplanacağını düşündüğünüz, fakat aksi şekilde sonuç elde edilen bir soru var mı?

- Eğer varsa, sizin düşüncenizin aksine, öğrenciler tarafından bu soruya/sorulara verilen hatalı cevaplarının çoğunlukta olmasının sebebi sizce ne olabilir?

3) Öğrencilerin cevaplarında gözlenmiş olan bu zorlukları ve hataları gidermeye yönelik bir öneriniz var mıdır? Varsa nelerdir? Lütfen, sırasıyla her bir soru için açıklayınız.

- Öğrencilerin sahip olabileceği bu zorluk ve hataları sınıfınızda da gözlemlediğiniz oluyor mu?
    - Cevabınız evet ise, bu zorluk ve hataları gidermeye yönelik önerileriniz var mıdır?
- Belirttiğiniz önerileri gerekli gördüğünüz durumlarda sınıfınızda uygulayabiliyor musunuz?
    - Evet ise, bu önerileri nasıl uyguladığınızı lütfen açıklayınız.
    - Hayır ise, uygulayamama sebeplerinizi lütfen açıklayınız.

G. APPROVAL OF THE METU HUMAN SUBJECTS ETHICS COMMITTEE

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
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19 ARALIK 2018

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof.Dr. Erdinç ÇAKIROĞLU

Danışmanlığını yaptığınız Nurbanu YILMAZ'ın **"8. Sınıf Öğrencilerinin Değişken, Eşitlik ve Doğrusal Denklemler Konularındaki Öğrenmeleri ve Zorlukları ile İlköğretim Matematik Öğretmenlerinin Bu Konudaki Bilgilerinin İncelenmesi"** başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülerek gerekli onay **2018-EGT-204** protokol numarası ile araştırma yapması onaylanmıştır.

Saygılarımla bilgilerinize sunarım.

Prof. Dr. Tülin GENÇÖZ
Başkan

Prof. Dr. Ayhan SOL
Üye

Prof. Dr. Ayhan Gürbüz DEMİR (4.)
Üye

İZİNLİ
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Üye

Doç. Dr. Emre SELÇUK
Üye

Doç. Dr. Pınar KAYGAN
Üye

Dr. Öğr. Üyesi Ali Emre TURGUT
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H. CURRICULUM VITAE

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EDUCATION

Degree	Institution	Year of Graduation
MS	Middle East Technical University Secondary Science and Mathematics Education	2014
BS	Middle East Technical University Elementary Mathematics Teacher Education	2011

WORK EXPERIENCE

Year	Place	Enrollment
2011- Present	Zonguldak Bülent Ecevit University	Research Assistant

PUBLICATIONS

Journal Papers

Karataş, İ., Pişkin-Tunç, M., Demiray, E., Yılmaz, N. (2016). Öğretmen adaylarının matematik öğretiminde teknolojik pedagojik alan bilgilerinin geliştirilmesi. *Abant İzzet Baysal Üniversitesi Eğitim Fakültesi Dergisi, 16*(2), 512-533.

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Pekbay, C., & Yılmaz-Tığlı, N. (2021). Barbie bungee jumping: bir stem etkinliği örneği. *Gazi Eğitim Bilimleri Dergisi, 7*(3), 261-288.

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Yılmaz, N., Çetinkaya, B., & Erbaş, A. K. (2014). Kamu personeli seçme sınavı ilköğretim matematik öğretmenliği öğretmenlik alan bilgisi testinin bir analizi. *Paper presented at Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi (UFBMEK),* Adana, Türkiye.

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Yılmaz, N., & Çakıroğlu, E. (2022). Mathematics teachers' knowledge of students' understanding of algebraic expressions and equations. *Paper presented at the 45th Conference of the International Group for the Psychology of Mathematics Education (PME),* Alicante, Spain.

I. TURKISH SUMMARY / TÜRKÇE ÖZET

ORTAOKUL MATEMATİK ÖĞRETMENLERİNİN SEKİZİNCİ SINIF ÖĞRENCİLERİNİN CEBİRSEL DÜŞÜNMELERİ İLE İLGİLİ BİLGİLERİ

1. Giriş

Cebir, okul matematiğinde önemli bir geçiş noktası olarak tanımlanmaktadır (Donovan vd., 2022). Blanton vd. (2011) cebiri "matematiksel yapıyı ve ilişkileri özlü biçimlerde ifade etmek için işlemleri, değişkenleri ve sayıları birleştiren matematiksel bir dil" olarak tanımlamıştır (Blanton vd., 2011, s. 67). Araştırmacılar cebirsel düşünmenin matematik eğitiminde çok önemli bir yere sahip olduğunu dile getirmektedir. Sembolleri manipüle etmek ve cebirsel prosedürleri hatasız bir şekilde kullanmak için kuralları öğrenmeye değil, cebirsel düşünmeyi geliştirmeye vurgu yapılmaktadır (Asquith vd., 2007; Cai ve Moyer, 2008; Hodgen vd., 2018; Kieran, 2004). Cebirsel düşünmenin özünde iki ana tema vardır: "genelleme yapmak" ve "matematiksel fikirleri temsil etmek, problemleri temsil etmek ve çözmek için sembolleri kullanmak" (Carpenter ve Levi, 2000, s. 5). Öğrencilerin cebir öğrenirken yaşadıkları zorluklar, öğrencilerin matematikten soyutlanmalarına ve erken yaşlarda matematik öğrenmekten vazgeçmelerine neden olmuştur (Kaput, 2002). Öğrencilerin cebirdeki zorluklarını ve kavram yanılgılarını gösteren pek çok çalışma bulunmaktadır (Alibali vd., 2007; Carraher ve Schliemann, 2007; Kieran, 1992; Kilpatrick vd., 2001; Knuth vd., 2005; Knuth vd., 2006; Sfard, 1991). Öğrencilerin cebirde karşılaştıkları güçlükler: eşittir işaretinin işlemsel anlamı (Kieran, 1981), genelleştirilmiş ifadeler yerine cevaplarda belirli niceliklere vurgu yapılması (Booth, 1984), aritmetikte sayı ve işlemin temel özellikleri ile ilgili bilgi eksikliği (MacGregor, 1996) ve nicelikler arasındaki ilişkiyi gösteren değişken gösterimini anlama eksikliği (Bednarz, 2001) bu çalışmalarda bahsedilen zorluk ve hatalardan bazı örneklerdir.

Booth (1988), öğrencilerin cebirde yaşadıkları zorlukların, öğrencilerin aritmetiği yeterince anlayamamasından veya geçmişte aritmetik ile ilgili eksikliklerin giderilmemiş olmasından kaynaklanabileceğini belirtmiştir. Filloy ve Rojano (1989), bunu daha somut olan aritmetik süreçlerden daha soyut olan cebirsel düşünmeye doğru evrime dayalı olarak “bir tür düşünceyi diğerinden ayıran bir dönüm noktası” olarak tanımlamıştır (s. 19). Benzer şekilde, Herscovics ve Linchevski (1994) aritmetik ve cebir arasındaki bilişsel boşluğun varlığını öğrencilerin bilinmeyenle veya bilinmeyenle ilgili işlem yapamamalarından söz ederek tanımlamıştır. Cebir için önemli olan bir diğer konu ise değişken kavramını anlamaktır (Blanton vd., 2015; Stephens, 2005; Usiskin, 1988). Araştırmacılar değişkenin değişen bir nicelik, genelleştirilmiş bir sayı ve bir parametre olması gibi anlamlarının kapsamlı bir yaklaşımla ele alınması gerektiğini vurgulamışlardır (Blanton vd., 2015; Usiskin, 1988). Ayrıca, Jupri vd. (2020), öğrencilerin cebirsel yeterlik göstergelerinden birinin de sembollerin kavramsal bir şekilde anlaşılması olduğunun altını çizmiştir (Bokhove & Drijvers, 2010; Jupri vd., 2020; Skemp, 1976; Bokhove & Drijvers, 2010; Jupri vd., 2020). Pek çok çalışma, öğrencilerin nicelikleri ve bu niceliklerin ilişkilerini göstermek için değişken gösterimini kullanmakta zorlandıklarını göstermiştir (Bednarz, 2001; McNeil vd., 2010; Stephens, 2005; Vergnaud, 1985). Öğrencilerin zorluk yaşadıkları diğer noktalar denklem çözümü ve sözel bir ifadeyi sembolik bir ifadeye dönüştürmektir. Kenney ve Silver'ın (1997) çalışmasına göre, on ikinci sınıf öğrencileri basit cebirsel denklemleri çözmekte, sözelden sembolik temsillere geçmekte ve çözümleri için akıl yürütmelerini paylaşmakta ve gerekçelendirmede zorluk yaşamışlardır. Kieran'ın (1992) öğrencilerin anlama eksikliğinin üstesinden gelebilmek için çoğunlukla kuralları ve prosedürleri ezberlemeye başvurduklarını ve bu durumun onlar için cebirin temeli olduğunu söylemiştir.

Öğrencilerin cebir konusundaki zorluklarını gidermek ve cebiri tüm yaş grupları için kavramsal olarak anlaşılır bir hale getirmek için araştırmacılar bir cebir reformu çağrısında bulunmuşlardır. Cebir reformu, cebiri geleneksel bir lise dersi olmaktan çıkarıp, okul öncesi dönemden lise matematiğine kadar uzanan sürekli bir yol olarak düşünmektedir (Asquith vd., 2007). Blanton ve ark. (2015), tipik ilköğretim matematik müfredatının ve geleneksel öğretimin, öğrenciler için “ilkokulun somut, aritmetik muhakemesi” nden “ortaokul ve sonrası için gerekli olan karmaşık, soyut cebirsel muhakeme”ye olan geçişi yeterince gerçekleştiremeyeceğine dikkat çekmiştir (Blanton vd., 2015, s.76). Bu nedenle, araştırmacılar matematik öğretmenlerinin öğrencilerin cebirsel düşünmelerini teşvik edecek durumları tanımasını zorunlu kılarak, matematik öğrenimi ve öğretiminde reformlar yapılması çağrısında bulunmuşlardır (Asquith vd., 2005; Carpenter vd., 2003; Kaput, 1998). Aritmetik ve cebirsel muhakeme arasındaki bağlantıyı güçlendirmek için genişletilmiş bir müfredatın geliştirilmesi ve öğretmen bilgisinin zenginleştirilmesi gerekmektedir (Asquith vd., 2007).

Öğretmen bilgisi ile öğrencilerin öğrenmesi arasında güçlü bir bağlantı vardır (Carpenter vd., 1988; Carpenter vd., 1989; Franke vd., 1998; Hill vd., 2005). Bu nedenle, öğretmen bilgisi, sınıf uygulamalarının temel bir özelliğini oluşturur (Borko ve Putnam, 1996). Shulman'ın (1986) Pedagojik İçerik Bilgisini (PAB) tanımladığı gibi, “konu bilgisinin ötesine geçerek öğretim için konu bilgisi boyutuna giden” (s. 9), bilgi olan öğretmenlerin öğrencilerin düşüncelerine ilişkin bilgisi, bir PAB'nin temel bileşenidir (Ball ve Cohen, 1999; Kazemi ve Franke, 2004). Bu nedenle, öğretmenlerin öğrencilerin cebirsel düşünmesine ilişkin bilgisi, ilkokul sınıflarının somut, aritmetik akıl yürütmesinden lise matematiği ve ilerisi için gerekli olan daha karmaşık, soyut cebirsel akıl yürütmeye önemli bir geçişin olduğu orta sınıflarda çok daha derinlemesine çalışılması gerekmektedir (Asquith vd., 2007; MEB, 2018).

Cebir hem öğretimi hem de öğrenimi zor bir konu olarak tanımlanmıştır (Stacey vd., 2004; Watson 2009). Stump ve Bishop (2002), "matematik eğitimini reforme etmeye ve geliştirmeye kendini adamış matematik öğretmen eğitimcileri için en büyük zorluklardan birinin, sınıf öğretmeni ve ortaokul matematik öğretmen adaylarının cebirsel muhakeme için bir takdir geliştirmelerini sağlamak" (s. 1903) olarak ifade etmiştir. Araştırmacılar bunu matematik reformunun temel taşı olarak açıklamışlar ve öğretmenlerin öğrencilerin cebirsel muhakemelerini geliştirmede en önemli faktörlerden biri olduğunu vurgulamışlardır (Blanton ve Kaput, 2005; Kaput, 1998).

Nathan ve Koedinger (2000b) matematik öğretmenlerinin müfredatı yorumlama ve uygulamalarının temel olarak öğretimle ilgili bilgi ve inançlarından (Ball, 1988; Borko vd., 1992; Clark ve Peterson, 1986; Raymond, 1997; Thompson, 1984), öğrencilerin öğrenmesinden (Ball, 1988; Carpenter ve diğerleri, 1989; Fennema vd., 1992; Romberg & Carpenter, 1986) ve matematikten (Cooney, 1985; Raymond, 1997) etkilendiğini ifade etmiştir. Kaiser vd. (2017) özellikle son birkaç yılda matematik öğretmenlerinin bilgisine odaklanan pek çok büyük ölçekli araştırma olduğundan bahsetmiştir (Ball ve Bass, 2000; Blömeke vd., 2014; Bruckmaier ve diğerleri, 2016; Kunter vd., 2013). Shulman (1986), bir konuyu anlamayle ilgili öğretmenlerin ihtiyaç duyacağı bilgiyi, "bir şeyi bilmek" ve "nedenini bilmek" olmak üzere iki başlık altında ifade etmiştir. Bu iki tür bilgi, öğretmenlerin öğrencilerin düşünme biçimlerine ilişkin bilgilerini araştırırken çok önemlidir (Even ve Tirosh, 1995). 'Bir şeyi bilmek' öğrencilerin bir konu hakkındaki düşünme biçimlerine ve ortak kavramlarına ilişkin araştırmaya dayalı veya deneyime dayalı bilgi olarak tanımlanabilir. 'Nedenini bilmek' ise altta yatan kavramların potansiyel nedenleri hakkındaki bilgi olarak tanımlanmıştır. Bu nedenle, her iki boyut da öğretmenlerin düşüncelerini tanıması ve yorumlaması için çok önemlidir.

"İlkokul matematiğinin somut, aritmetik akıl yürütmesinden lise matematiği ve ötesi için gerekli olan, giderek daha karmaşık, soyut cebirsel akıl yürütmeye önemli bir geçişe işaret eden bir dönem" olan ortaokul seviyesindeki öğrencilerin cebirsel düşünmelerine ilişkin ortaokul matematik öğretmenlerinin bilgisiyle ilgili daha fazla araştırmaya ihtiyaç duyulmaktadır." (Asquith vd., 2007, s. 251). Ortaokul matematik öğretmenlerinin öğrencilerin cebirdeki düşüncelerine ilişkin bilgilerini araştıran çalışmalar olmasına rağmen (Asquith vd., 2007; Baş vd., 2011; Li, 2007; Tanışlı ve Köse, 2013; Putnam vd., 1992; Stephens, 2006), bu konu hakkında yapılacak olan yeni çalışmalara halen ihtiyaç duyulmaktadır (Asquith vd., 2007; Borko ve Putnam, 1996).

Shulman (1986), öğrencilerin kavrayışlarının altında yatan potansiyel sebeplerin nedenini bilmenin, öğrencilerin cebirsel düşünmelerinin nasıl olduğunu bilmek kadar önemli olduğunu ifade etmiştir. Öğretmenlerin, öğrencilerin zorlukları ve hatalarının neler olduğunu ifade edebildikleri (örn., Stump, 2001), ancak, öğretmenlerin mesleki bilgisinin bir şeyin nedenini bilme yönüyle ilgili olarak, bu zorluk ve hataların nedenlerini ifade etme konusunda ise başarısız olmuşlardır (Erbaş, 2004). Bu nedenle, öğretmenlerin öğrencilerinin cebirdeki performanslarının altında yatan nedenler, olası kaynaklar, konusundaki bilgilerinin detaylı bir incelemesini yapmak faydalı olabilir.

Nedensel yükleme teorisi (Weiner, 1985, 2000, 2010), bireylerin kendi performanslarını ve başkalarının performanslarını nasıl algıladığı ve bu nedensel yüklemelerin bireylerin duygularını, bilişlerini ve davranışlarını eğitim bağlamında nasıl etkilediği ile ilgili kapsamlı bir teorik çerçeve sunar (Wang ve Hall, 2018). Bu teori aynı zamanda, öğretmenlerin öğrencilerin zorluklarını ve mesleki stres faktörlerini ve nedensel yüklemelerinin öğretim davranışlarını, öğrencilerle etkileşimlerini ve duygusal iyi oluşlarını nasıl etkilediğini açıklar (Wang ve Hall, 2018). Bazı araştırmacılar öğrencilerin cebirdeki zorluklarını öğrencilerin gelişimsel eksikliklere veya yetersiz bilişsel gelişimine bağlamaktadır (Collis, 1975; Filloy & Rojano, 1989; Herscovics & Linchevski, 1994; Kuchemann, 1981; MacGregor, 2001). Kişilerin kendisine bağlı faktörlere atfedilen zorlukların dışında (örneğin, öğrencilerin bilişsel süreci, motivasyonu ve matematik becerileri), öğretim kalitesi, şans veya çevresel koşullar gibi kişilerarası faktörler de öğrencilerin başarısına veya zorluklarına atfedilebilir. Wang ve Hall (2018), yetmiş dokuz ilişkilendirme çalışmasını incelemiş ve öğretmenlerin öğrencilerin performansını önceki öğrenme deneyimleri ve önceki öğretmenler gibi dış ve kontrol edilemeyen faktörlere dayalı olarak açıkladığını öne süren bazı çalışmalar olmasına rağmen, öğretmenlerin genellikle öğrencilerin başarısızlığını öğrencilerin kendileriyle ilgili faktörlere bağladıklarını bulmuşlardır (Rolison ve Medway, 1985). ; Hall vd., 1989; Bertrand ve Marsh, 2015). Örneğin, erken cebir yaklaşımı üzerine yaptıkları çalışmalara göre, Carraher ve Schliemann (2007) öğrencilerin yaşadığı zorlukların aritmetiğin, genellikle temel matematiğin öğrencilere nasıl tanıtıldığına ilişkin eksikliklere atfedildiğini belirtmişlerdir.

Araştırmacılar, nedensel yüklemelerin öğretmenlerin öğrencilerin gelecekteki performanslarına ilişkin beklentilerinde çok önemli bir rolü olduğunu öne sürmektedir (Clarkson ve Leder, 1984; Peterson ve Barger, 1985). Weiner'in (2000, 2010) nedensel yükleme teorisine dayanarak, öğretmenlerin öğrencilerinin performansının nedenlerini tahmin etme biçimleri, onların duygularını etkileyebilir ve bu da onların öğretimdeki davranışlarını tahmin edebilir. Bu nedenle, öğretmenlerin öğrencilerinin performansı hakkındaki düşünceleri hakkında bilgi sahibi olmak ve sınıf davranışlarını tahmin etmek için ortaokul matematik öğretmenlerinin öğrencilerin performansına yönelik nedensel yüklemelerini araştırmak yararlı olabilir. Ortaokul matematik öğretmenlerinin öğrencilerin zorlukları ile ilgili nedensel yüklemelerini incelemek, öğretmenlerin öğrencilerin performansları hakkındaki düşünceleri ve öğrencilerin cebirsel düşünmeleri hakkında ne bildikleri hakkında fayfalı bilgiler sağlayabilir. Bu bağlamda, çalışmada ortaokul matematik öğretmenlerinin sekizinci sınıf öğrencilerinin cebir öğrenme alanındaki kavrayışları, zorlukları ve hataları konusundaki bilgileri ve öğretmenlerin öğrencilerin cebir konusundaki zorlukları ile ilgili nedensel yüklemeleri incelenmiştir.

1.1. Çalışmanın Amaçları ve Araştırma Soruları

Çalışmanın ilk amacı, ortaokul matematik öğretmenlerinin öğrencilerin cebir öğrenmedeki kavrayışları, zorlukları ve hatalarıyla ilgili bilgilerini incelemektir. Çalışmanın ikinci amacı, ortaokul matematik öğretmenlerinin, öğrencilerin eşitlik ve denklem, genelleştirilmiş aritmetik, değişken ve fonksiyonel düşünme alanlarındaki performanslarına ilişkin tahminlerini ve yorumlarını incelemektir. Çalışmanın son amacı, matematik öğretmenleri tarafından dile getirilen, öğrencilerin eşitlik ve denklem, genelleştirilmiş aritmetik, değişken ve fonksiyonel düşünme alanlarındaki zorluklarının ve hataların nedenlerini incelemektir. Sonuç olarak, bu amaçlar göz önünde bulundurularak çalışmada aşağıdaki araştırma soruları ele alınacaktır:

1. Ortaokul matematik öğretmenlerinin öğrencilerin dört büyük fikir ile ilgili kavrayışları hakkındaki pedagojik alan bilgisinin doğası nasıldır?
    1.1. Ortaokul matematik öğretmenlerinin cebir öğrenmeye başlamak için gerekli gördüğü ön koşul bilgiler nelerdir?
    1.2. Ortaokul matematik öğretmenlerinin, 8. sınıf öğrencilerinin dört büyük fikir ile ilgili kavrayışları ve zorluk yaşadıkları noktalar hakkındaki bilgileri nasıldır?
    1.3. Ortaokul matematik öğretmenleri, 8. sınıf öğrencilerinin dört büyük fikir ile ilgili yaşadıkları zorlukların üstesinden gelebilmek için hangi stratejileri kullanmaktadır?

2. Ortaokul matematik öğretmenlerinin öğrencilerin cebir öğrenmesi ile ilgili bilgisi, cebir tanılama testindeki 8. sınıf öğrencilerinin kavrayışları ve zorluklarıyla hangi ölçüde uyumludur?

2.1. Ortaokul matematik öğretmenlerinin 8. sınıf öğrencilerinin cebir tanılayıcı testindeki kavrayışları ve zorluklarıyla ilgili tahminleri nelerdir?

2.2. Ortaokul matematik öğretmenlerinin tahminleri, öğrencilerin cebir tanılayıcı testindeki dört büyük fikir ile ilgili performansıyla karşılaştırıldığında nasıl bir sonuç ortaya çıkmaktadır?

2.3. Ortaokul matematik öğretmenlerinin öğrencilerin cebiri öğrenme bilgisine ilişkin bilgisi, tanılayıcı cebir testindeki 8. sınıf öğrencilerinin kavrayışları ve zorluklarıyla ilgili yorumlarını nasıl etkilemektedir?

3. Ortaokul matematik öğretmenleri öğrencilerin cebir performansını etkileyen faktörleri nelere dayandırmaktadır?

1.2. Çalışmanın Önemi

Öğrencilerin cebir öğrenme konusunda yaşadıkları güçlükler ve cebirin eğitim ve istihdamda gelecekteki fırsatlar konusunda bir öncül olması (Asquith vd., 2007; Ladson-Billings, 1998; Moses ve Cobb, 2001; National Research Council [NRC], 1998) matematik eğitimi araştırmacılarını bir cebir reformu çağrısı yapmaya yönlendirmiştir (Kaput, 1995, 1998; Olive vd., 2002; Stacey ve Mac Gregor, 2001). Araştırmacılar, cebiri okul öncesi dönemden lise düzeyine kadar uzanan bir konu haline getirmek için okul cebirinin yeniden kavramsallaştırılmasının gerektiğini ifade etmişlerdir (Asquith vd., 2007). İlköğretim sınıflarında cebirsel akıl yürütmenin dahil edilmesi sayesinde cebir, sembolik işlemlerde iyi bir düzeyde yapmaktan çok, okul öncesinden liseye tüm öğrenciler için erişilebilir bir konu olarak algılanmaya başlamıştır (Asquith vd., 2007; Carpenter ve Levi, 2000; Schifter, 1999).

Third International Mathematics and Science Study (TIMMS) sınavlarında, öğrencilerden cebirsel modelleri kullanmaları ve ilişkileri açıklamaları, iki nicelikten biri formülde verildiğinde diğerini belirleme gibi cebirsel işlemleri açıklamalarını isteyen gerçek yaşam problemlerini çözmeleri istenmiştir. Ayrıca, bir değişkenin değeri değiştiğinde diğer değişkenin değerindeki değişimi gözlemlemek için doğrusal denklemler ve fonksiyonları içeren problemleri çözmeleri istenmiştir (Mullis vd., 2020). Türkiye'deki sekizinci sınıf öğrencilerinin cebir puanlarında yıldan yıla kademeli olarak artan bir performans gözlemlense de (MEB, 2014; MEB, 2016; MEB, 2020), sekizinci sınıf öğrencilerinin TIMMS 2019'daki cebir sorularına verdiği yanıtların analizi, Türkiye'deki sekizinci sınıf öğrencilerinin cebir puanlarının ortalama matematik puanlarının altında olduğunu göstermiştir (MEB, 2020). Bu nedenle, uluslararası sınavların sonuçları, Türkiye'deki sekizinci sınıf öğrencilerinin cebir performanslarını geliştirmeleri için desteklenebileceğini göstermiştir.

Carpenter ve meslektaşları çalışmalarında öğrencilerin başarısı ile öğretmenlerin öğrencilerin cebirsel düşünmelerine ilişkin bilgileri arasında güçlü bir ilişki olduğunu göstermişlerdir (Carpenter vd., 1988; Carpenter vd., 1989; Franke vd., 1998). Carpenter vd. (1989), öğretmenlerin öğrencilerin cebirsel düşünmelerine daha aşina olması gerektiğini savunmuştur. Bu deneysel çalışmalar, öğretmenlerin öğrencilerin cebirsel düşünmelerine odaklanan mesleki gelişim programlarına katılan öğretmenlerin öğrencilerinin daha iyi performans gösterdiğini ortaya koymuştur. Asquith vd. (2007), ortaokul matematik öğretmenlerinin öğrencilerin cebirsel düşünmeleri ile ilgili bilgilerine ilişkin sınırlı sayıda çalışma olduğunu ifade etmiştir. Ayrıca, öğretmenlerin öğrencilerin cebirsel düşünmesiyle ilgili bilgilerini ortaya çıkarmak, öğretmenlerin kendi cebirsel bilgilerine ilişkin ipuçları sağlayabilir (Ball vd., 2008). Bu nedenle, öğrencilerin cebir öğrenme alanındaki performansını artırmak ve cebirsel muhakemelerini genişletmek için ortaokul matematik öğretmenlerinin öğrencilerin cebirsel düşünmesine ilişkin bilgilerinin araştırılması faydalı bir adım olabilir. Bu nedenle, bu çalışma, ortaokul matematik öğretmenlerinin öğrencilerin cebirsel düşünmeleri ile ilgili bilgisine ve öğrencilerin cebirdeki zorluk ve hatalarına dayalı öğretmen bilgisi literatürüne katkıda bulunabilir.

“Etkili matematik öğretimi, öğrencilerin ne bildiklerini ve neye ihtiyaç duyduklarını anlamayı ve bu noktaları iyi bir şekilde öğrenmeleri için onları zorlamayı ve desteklemeyi gerektirir” (NCTM, 2000, s. 16). Araştırmacılar, ortaokul matematik öğretmenlerinin de "cebirsel akıl yürütmenin zengin ve bağlantılı yönleri konusunda çok az deneyime sahip olduklarını" vurgulamaktadır (Blanton ve Kaput, 2005, s. 414). Araştırmalar, ortaokul matematik öğretmenlerinin ve matematik öğretmeni adaylarının, öğrencilerin cebir öğrenme alanındaki kavrayışlarını belirleme ve öğrencilerin cebirdeki güçlüklerinin ve kavram yanılgılarının altında yatan nedenleri öngörme konusunda eksiklikleri olduğunu göstermiştir (Alapala, 2018; Asquith vd., 2007; Dede ve Peker, 2007; Didiş -Kabar ve Amaç, 2017; Gökkurt vd., 2016; Li, 2007; Stephens, 2004, 2006; Li, 2007; Şen-Zeytun vd., 2010; Tanışlı ve Köse, 2013; Tirosh vd., 1998). Ortaokul matematik öğretmenlerinin, öğrencilerin cebirsel akıl yürütme bilgileri ile bu öğrenme alanındaki güçlüklerinin ve kavram yanılgılarının altında yatan nedenlere odaklanan sınırlı sayıda çalışma bulunmaktadır (Asquith vd., 2007; Şen-Zeytun vd., 2010; Tirosh vd., 1998).

Blanton vd. (2011), cebirsel düşünmenin öğretiminin, matematikteki diğer temel konularla benzer şekilde çoğu öğretmen adayının üniversitedeki standart matematik eğitimi derslerinde deneyimlediklerinin ötesine geçen özel bir bilgi gerektirdiğini öne sürmüştür (NCTM, 2000, s. 17). Bu çalışma, ortaokul matematik öğretmenlerinin öğrencilerin cebirsel düşünmelerini, zorluklarını ve hatalarını nasıl tahmin edip yorumladıklarına, öğrencilerin düşünmelerini tahmin etmek ve yorumlamak için hangi noktalara önem verdiklerine ve öğrencilerin cebirsel düşünmedeki performanslarına dayalı olarak hangi çıkarımlarda bulunduklarına ilişkin bilgiler sağlayabilir. Bu çalışmanın bulgularına dayanarak, ortaokul matematik öğretmenlerinin öğrencilerin cebirsel düşünme bilgilerine ilişkin ortaokul matematik öğretmenlerine, öğretmen eğitimcilerine ve matematik eğitimi araştırmacılarına yönelik faydalı bilgiler ve çıkarımlar önerilebilir.

Öğrencilerin yazılı çalışmaları, öğrencilerin matematikteki kavrayışlarını ve zorluklarını yorumlamak ve bu zorluklara müdahale etmek için etkili bir araçtır (Grosman vd., 2009; Jacobs ve Philipp, 2004). Doerr (2004), öğretmenlerin cebir öğretmeyi nasıl öğrendiklerinin ve kendi uygulamalarını nasıl anladıklarının, kendi kültürel bağlamlarında araştırılması gerektiğini ifade etmiştir. Ancak, verilerin öğretmenlerden kendi öğrencilerinin yazılı çalışmaları yoluyla toplandığı sınırlı sayıda çalışma bulunmaktadır (Asquith vd., 2007; Stephens, 2004, 2006; Tirosh vd., 1998). Bu nedenle, öğrencilerin eşitlik ve denklem, genelleştirilmiş aritmetik, değişken, ve fonksiyonel düşünme sorularına verdikleri yanıtları kullanarak veri toplamak faydalı olabilir. Bu nedenle, öğrencilerin belirli sorulara yönelik çözümleri için önceden hazırlanmış örnekleri kullanmak yerine, sekizinci sınıf öğrencilerinin cebirsel düşünmelerini ve ardından ortaokul matematik öğretmenlerinin kendi kültürel ortamlarında öğrencilerin cebirsel düşünmelerine ilişkin bilgilerini araştırmak için tanılayıcı cebir testi geliştirilmiştir. Bu çalışmanın, ortaokul matematik öğretmenlerinin cebirsel akıl yürütme soruları aracılığıyla sekizinci sınıf öğrencilerine dayalı olarak geri bildirim almalarını sağlayacağı düşünülmektedir. Bu şekilde öğretmenler, tahminlerini öğrencilerinin gerçek performanslarıyla karşılaştırabilirler. Böylece, öğrencilerin çözümlerini nasıl tahmin etmeleri ve yorumlamaları gerektiği ve öğrencilerin cebirsel düşünmelerinin ve cebirde yaşadıkları zorluklarının öğretmenlerin tahminlerinden hangi noktalarda farklılaştığı konusunda gözlem yapma şansına sahip olabilirler. Ayrıca, öğretmenler, öğrencilerinin daha önce fark etmedikleri kavrayışlarını, zorluklarını ve hatalarını fark etme fırsatına sahip olabilirler.

2. Alanyazın Taraması

Ball vd. (2008), öğretmen bilgisi modellerinde alan bilgisi ve pedagojik alan bilgisini öğretmen bilgisinin iki ana boyutu olarak yapılandırmıştır. Pedagojik alan bilgisi üç alt boyuttan oluşur: içerik ve öğrenci bilgisi, içerik ve öğretim bilgisi ve müfredat bilgisi. Modelde içerik ve öğrenci bilgisi, öğrenciler için hangi ondalık sayıların zorlayıcı olduğunu düşünmek gibi, öğrencilerin düşünmesi ve öğrenmesi hakkında öğretmenlerin bilgisini ifade eder. Hill vd. (2008), içerik ve öğrenci bilgisinin matematikte öğrencilerin düşünmesine ve öğrenmesine odaklanarak pedagojik alan bilgisine çok önemli bir temele katkıda bulunduğunu vurgulamıştır. İkinci olarak, içerik ve öğretim bilgisi, öğrencilerin matematiksel kavramlardaki güçlüklerine öğretmenlerin nasıl tepki vermesi gerektiğini bilme ile ilgili boyuttur. Modelin son boyutu, içeriğin öğrencilerle nasıl paylaşılması gerektiğine ilişkin öğretmenlerin bilgisi üzerine yapılanan müfredat bilgisidir.

Carrillo-Yañez vd. (2018), Ball ve arkadaşlarının öğretmen bilgisi modelinden yola çıkarak öğretme ve öğrenmeye ilişkin pedagojik alan bilgisinin sırasıyla matematik öğretimi bilgisi, matematik öğrenmenin özellikleri bilgisi ve matematik öğrenme standartları bilgisi olarak adlandırılan üç alt alanı belirlemiştir. Matematik öğrenmenin özellikleri bilgisi, öğrencilerin matematik prosedürlerini, öğrencilerin kullandığı stratejileri ve farklı terminoloji türlerine ilişkin bilgileri içerir. Ayrıca bu bilgi boyutu, matematik öğrenimine ilişkin duygusal bir yönü de içerir (Hannula, 2006). Yani, matematik kaygısının farkındalığı (Maloney vd., 2013) ve matematik öğrenirken öğrencilerin motivasyonunu etkileyen faktörler bu boyut altında incelenir. Araştırmacılar bu boyutu "matematiksel öğrenme teorileri, matematik öğrenmedeki güçlü ve zayıf yönler, öğrencilerin matematiksel içerikle etkileşim yolları ve matematik öğrenmenin duygusal yönleri" olarak özetlemektedir (Carrillo-Yañez ve diğerleri, 2018, s. 247). Bu nedenle, bu çalışmada Carrillo-Yañez vd.'nin (2018) öğretmen bilgisi modelinin matematik öğrenmenin özellikleri bilgisi alt boyutuna odaklanılmıştır.

Çalışmada, öğretmenlerin öğrencilerin performanslarını yorumlamaları, öğrencilerin güçlüklerinin ve hatalarının olası nedenlerine ve öğrencilerin zorluk yaşadıkları noktalara dayalı olarak öğretmenlerin yorumları ve çıkarımları incelenmiştir. Nedensel yüklemeler, bireylerin gelecekteki başarı beklentilerini, davranışlarını ve duygularını etkileyebilir (Graham ve Williams, 2009; Weiner, 1992, 2000). Bu nedenle, öğretmenlerin dile getirdiği öğrencilerin zorluk ve hatalarının nedenleri, öğretmenlerin algılanan yeterliklerinin ve öğretimsel kararlarının habercisi olabilir. Wang ve Hall (2018), nedensel yükleme teorisinin, öğretmenlerin öğrencilerin zorluklarını ve mesleki stres faktörlerini nasıl algıladıklarını ve öğretmenlerin atıflarının öğretmen-öğrenci etkileşimlerini ve öğretim davranışlarını nasıl etkilediğini incelemeye yardımcı olduğunu belirtmiştir. Bu nedenle, bu çalışmada öğretmenlerin öğrencilerin başarısızlıklarına ilişkin nedensel yüklemeleri de araştırılmıştır.

Blanton vd. (2015), Kaput'un (2008) ve Shin vd.' nin (2009) çalışmalarından yola çıkarak cebir öğrenme alanında beş büyük fikir tanımlamıştır. Araştırmacılar beş büyük fikri "(a) denklik, ifadeler, denklemler ve eşitsizlikler; (b) genelleştirilmiş aritmetik; (c) fonksiyonel düşünme; (d) değişken; ve (e) akıl yürütme (Blanton ve diğerleri, 2015, s. 43). Literatürdeki çalışmalar öğrencilerin cebirde aritmetik, değişken kavramı, eşittir işaretinin kavramsal anlamı ve fonksiyonel düşünme gibi konularda zorluklar yaşadığını göstermiştir (Asquith vd., 2007; Blanton ve Kaput, 2011; Blanton vd., 2017; Herscovics ve Linchevski, 1994; Linchevski ve Herscovics, 1996; Stephens, 2003). Öğretmenlerin öğrencilerin düşünme bilgileri ile cebirdeki başarıları arasındaki ilişkiyi araştıran birçok çalışma bulunmaktadır (Asquith vd., 2007; Baş vd., 2011; Even ve Tirosh, 1995; McCrory vd., 2012; Stephens, 2006; Şen-

Zeytun vd., 2010; Tanışlı ve Köse, 2013; Tirosh vd., 1998). Even ve Tirosh (1995), öğretmenlerin sadece öğrencilerin sahip olabileceği belirli kavram yanılgılarına ilişkin bilgileri değil, aynı zamanda bu tür kavram yanılgılarının neden ortaya çıktığını da ifade edebilmeleri gerektiğini ifade etmiştir.

Literatürdeki araştırmalar incelendiğinde, öğretmenlerin öğrencilerin değişken (Asquith vd., 2007; Tanışlı ve Köse, 2013), eşitlik (Asquith vd., 2007; Stephens, 2007; Tanışlı ve Köse, 2013), ilişkisel düşünme (Stephens, 2007) ve kovaryasyonel muhakeme yeteneği (Şen-Zeytun vd., 2010) konularında zorluklarını ve kavram yanılgılarını belirleme konusundaki bilgilerinin sınırlı olduğu sonucu ortaya çıkmıştır.

3. Yöntem

3.1. Araştırma deseni

Bu çalışmada ortaokul matematik öğretmenlerinin, öğrencilerin eşitlik ve denklem, genelleştirilmiş aritmetik, değişken ve fonksiyonel düşünme konularına ilişkin kavrayışlarına ilişkin bilgilerini keşfetmek olduğu için nitel bir araştırma metodolojisi olan durum çalışması araştırma deseni olarak kullanılmıştır. Bu çalışmada iç içe geçmiş tek durum deseni kullanılmıştır (Yin, 2003). Analiz birimleri ortaokul matematik öğretmenlerinin bilgisi ve bu öğretmenlerin 8. sınıfta öğrenim görmekte olan öğrencilerinin cebirsel düşünmeleridir.

3.2. Çalışmanın Bağlamı ve Katılımcılar

Bu çalışmanın katılımcıları bir devlet okulunda görev yapmakta olan beş ortaokul matematik öğretmenidir. Bu öğretmenlerin 8. sınıfta öğrenim görmekte olan öğrencilerinden elde edilen veriler, öğretmenlerin öğrencilerin cebirsel düşünmeleri konusundaki bilgilerini incelemek amacıyla kullanılmıştır. Çalışma, Türkiye'de Batı Karadeniz Bölgesi'nde bulunan bir devlet ortaokulunda gerçekleştirilmiştir. Okulda yaklaşık iki bin beş yüz öğrenci öğrenim görmektedir ve bu öğrencilerin 620'si sekizinci sınıf öğrencileridir. Değişken kavramı ve fonksiyonlar ilk olarak ortaokul matematik müfredatına dayalı olarak ortaokulda öğretildiği için (MEB, 2018), ortaokul, ilkokuldaki aritmetik akıl yürütmeden lisedeki karmaşık cebirsel akıl yürütmeye geçişi sağlayan bir dönemdir. Bu nedenle araştırma için ortaokul öğrencileri tercih edilmiştir. Sekizinci sınıf öğrencileri, ortaokul cebir öğreniminin son döneminde oldukları için, 8. sınıf öğrencileri, öğrencilerin cebirsel düşünme ve güçlükleri hakkında inceleme yapabilmek amacıyla çalışmaya seçilmişlerdir.

Araştırmacı verileri zenginleştirebilmesi ve verilerin toplanacağı okulun kolay ulaşılabilir olması için araştırmada amaçlı örnekleme yöntemi kullanılmıştır. Şehirdeki diğer devlet ortaokullarına kıyasla seçilen okulda çok sayıda öğrenci ve öğretmen bulunmaktadır. Araştırmanın veri toplama sürecinde çok sayıda sınıf içi gözlem ve görüşme yapıldığı için il merkezine yakın bir devlet ortaokulu tercih edilmiştir. Ayrıca, devlet okulundaki katılımcı öğretmenleri seçmek için de amaçlı örnekleme içinde ölçüt örnekleme yöntemi tercih edilmiştir. Çalışma, ortaokul matematik öğretmenlerinin 8. sınıf öğrencilerinin cebirsel düşünme konusundaki bilgilerini araştıracağından, çalışmaya sadece 8. sınıflara ders veren ortaokul matematik öğretmenleri davet edilmiştir. Ortaokul matematik öğretmenleri çalışmaya katılmayı kabul ettikten sonra, katılımcı öğretmenlerin tüm 8. sınıf öğrencileri çalışmaya davet edilmiştir. Sonuç olarak, Türkiye'de Batı Karadeniz Bölgesi'ndeki bir devlet ortaokulunda beş ortaokul matematik öğretmeni ve 620 sekizinci sınıf öğrencisi çalışmaya katılmıştır. Son olarak ilçedeki diğer iki devlet ortaokulunda pilot çalışmalar yapılmıştır. Geçerlik ve güvenirliğe yönelik tehditleri ortadan kaldırabilmek için, ana araştırma, pilot çalışmaların yapıldığı okullardan farklı bir okulda gerçekleştirilmiştir.

3.3. Veri Toplama Süreci

Çalışmanın ilk kısmı, 8. sınıf öğrencilerinin cebirsel düşünmelerini incelemek ve cebir öğrenme alanında yaşadıkları zorlukları araştırmaktadır. İkinci bölüm ise, ortaokul matematik öğretmenlerinin öğrencilerinin cebir ile ilgili sorulardaki performanslarına ve yaşadıkları zorluklara dayalı bilgilerini araştırmaktır. Bu nedenle, veri toplama prosedürü iki aşamaya ayrılmıştır: öğrencilerin cebirsel düşünmelerini ve zorluklarını araştırmak ve öğrencilerin cebirsel düşünmeleri ile ilgili ortaokul matematik öğretmenlerinin bilgilerini incelemektir. Bu doğrultuda;

sınıf gözlemi, yarı yapılandırılmış görüşmeler, tanılayıcı cebir testi ve öğretmenler için tanılayıcı cebir testine dayalı bir anket veri toplama aracı olarak kullanılmıştır. Öğrencilerin cebirsel düşünmelerini incelemek ve zorlandıkları noktaları belirlemek amacıyla hazırlanan testi geliştirmek için, sınıf gözlemi, öğretmenlerle yarı yapılandırılmış görüşmeler ve ilgili literatürden faydalanılmıştır.

Test oluşturma prosedürü sırasında, Blanton vd. (2015) ve Kaput (2008)'un çalışmaları dikkate alınmıştır. Blanton vd. (2015), Kaput'un (2008) içerik dizilerine ve erken cebire ilişkin literatüre (Blanton vd., 2011; Carraher ve Schliemann, 2007) dayalı olarak beş büyük fikir belirlenmiştir. Bu beş büyük fikir, eşitlik, cebirsel ifadeler, denklemler, eşitsizlikler; genelleştirilmiş aritmetik; fonksiyonel düşünme; değişken; ve orantısal muhakemedir. Bu çalışma ilk dört büyük fikir olan eşitlik, cebirsel ifadeler, denklemler, eşitsizlikler; genelleştirilmiş aritmetik; fonksiyonel düşünme üzerine odaklanmıştır. Orantısal düşünme matematik eğitiminde çok geniş bir alana sahip olduğundan, bu çalışma esas olarak ilk dört büyük fikre odaklanmıştır. Tanılayıcı cebir testi hazırlanırken her bir büyük fikir ile ilgili maddeler teste dahil edilmeye çalışılmıştır.

Öğrencilere tanılayıcı cebir testini uygulamadan önce katılımcı öğretmenlerle öğrencilerinin cebirsel düşünmeleri ve tanılayıcı cebir testinde öğrencilerinin performanslarını tahmin etmeleri üzerine yarı-yapılandırılmış görüşmeler yapılmıştır. Tanılayıcı cebir testinin öğrencilere uygulanması ve öğrenci cevaplarının incelenerek analiz edilmesinden sonra katılımcı öğretmenlerle öğrencilerinin testteki performanslarını değerlendirmeleri üzerine tekrar yarı-yapılandırılmış görüşmeler yapılmıştır.

3.4. Veri Analizi

Bu çalışmada, matematik öğretmenlerinden toplanan verileri analiz etmek için içerik analizi yöntemi kullanmıştır. Merriam (2009), bir araştırmacının veri analizine başlamadan önce verileri okuması, hazırlaması ve düzenlemesi gerektiğini ifade etmiştir. Bu nedenle yarı-yapılandırılmış görüşmeler yoluyla öğretmenlerden elde edilen veriler yazıya döküldükten sonra, yazılar okunarak ve kenar notları alınarak bir ön analiz yapılmıştır. Bu çalışmada verileri analiz etmek için hem tümevarım hem de tümdengelim yaklaşımları kullanılmıştır. Bu çalışmada hem tümevarımsal içerik analizi yaklaşımı hem de tümdengelimli içerik analizi yaklaşımının kısıtsız matrisi çalışmanın amacına uygun olduğu için tercih edilmiştir. Araştırmacıların önerdiği gibi, veri analizi, bir kelime veya tema olabilecek analiz biriminin seçilmesiyle hazırlık aşamasıyla başlar (Cavanagh, 1997; Guthrie vd., 2004; Polit ve Beck, 2004). Creswell (2009) ayrıca nitel içerik analizi prosedürünü, analiz için verileri düzenleme ve organize etme, verileri okuma, verileri kodlama, verilerden toplanan temaları veya betimlemeleri üretme, temaları veya betimlemeleri birbiriyle ilişkilendirme ve temaların veya açıklamaların anlamlarının yorumlanması. Bu nedenle bu çalışmada öncelikle verilerin analizi yapılarak ve analiz edilen veriler literatürdeki ilgili çalışmaların kodlama yapısına göre yeniden düzenlenerek bu veri analiz adımları dikkate alınmıştır.

3.4.1. Birinci Araştırma Sorusunun Veri Analiz Süreci

İlk araştırma sorusuna dayalı veriler, öğrencilerin cebir öğrenirken karşılaştıkları güçlükler ve anlamaları ile ilgili temel konularda öğretmenlerin bilgileri ile ilgilidir. Bu amaçla, öğretmenlerin cebir öğrenmeye başlamak için gerekli olan ön koşul bilgisine, öğrencilerin cebirdeki kavramlarına ve zorluklarına ve bu zorlukların üstesinden gelebilmek için öğretmenlerin kullandıkları stratejilere ilişkin bilgileri araştırılmıştır. İlk olarak, araştırmacı, görüşme verilerini yazıya dökmüş ve okumuştur. Tümevarımsal bir analiz yapıldığından, kodları belirlemek için belge üzerinde kenar notları alarak veriler önceden analiz edilmiştir. Ön analizin ardından araştırmacı, ikinci analiz aşamasında öğretmenlere ve ilgili kategorilere ilişkin kodları gözlemlemek için öğretmenlerin adlarını ve kategorilerini içeren bir tablo oluşturmuştur. Kodların her birinin daha görünür olması için farklı renklerle temsil edilmiştir. Araştırmacı, kodlama işlemini tamamladıktan sonra, analizin tutarlılığını sağlamak ve gereksiz verileri belgeden çıkarmak için kodlama işlemini tekrarlamıştır. Daha sonra, genel bir bakış elde etmek için kodlar başka bir belgede özetlenmiştir.

İkinci alt araştırma sorusu, öğretmenlerin cebirdeki öğrencilerin kavramlarına ve zorluklarına ilişkin bilgilerini araştırmaktadır. İlk olarak yarı yapılandırılmış görüşmelerden elde edilen verileri incelemek için tümevarımsal içerik analizi yapılmıştır. Kodlar elde edildikten sonra ilgili kodlar da dahil olmak üzere bazı kategoriler gözlenmiştir. Verilerin analizi tamamlandıktan sonra, öğretmenlerin cebirde öğrencilerin güçlüklerine ilişkin kodların kategorileri başka bir araştırmanın (Jupri vd., 2014) sonuçlarına çok benzer olduğu için araştırmacı aynı veriler üzerinde tümdengelimli bir içerik analizi süreci (Elo ve Kyngäs, 2008) gerçekleştirmiştir. Bu nedenle veriler, Jupri vd.'nin (2014) çalışmasında verilen öğrencilerin zorluk kategorilerine göre tekrar analiz edilmiştir. Kodlama işlemi tamamlandıktan sonra kodlar farklı renklerde etiketlenmiş ve kodların kategorileri her öğretmenin ifadesini ayrı ayrı gösteren 5x5 matris tablosunda sunulmuştur. Son olarak, öğrencilerin cebirdeki zorluklarını aşmak için öğretmenlerin önerdiği stratejiler incelenmiştir. Önceki analizlere benzer şekilde, tümevarımsal bir içerik analizi yaklaşımı kullanılmıştır. Araştırmacı verileri analiz edip kodları belirledikten sonra, analiz tutarlılığını sağlamak ve kod kategorilerini etkin bir şekilde oluşturmak için aynı işlem tekrarlanmıştır.

3.4.2. İkinci Araştırma Sorusunun Veri Analiz Süreci

Bu bölümde, ikinci araştırma sorusu aracılığıyla toplanan verilere ilişkin veri analiz süreci sunulmaktadır. İkinci araştırma sorusu üç alt soru içerdiğinden, veri analizi her bir alt soru üzerinden yapılmıştır. Tanılayıcı cebir testi öğrencilere uygulanmadan önce, öğretmenlerin öğrencilerin testeki performansına ilişkin tahminlerine yönelik veriler bir anket ve yarı yapılandırılmış görüşmeler yoluyla toplanmıştır. Ankette öğretmenler, öğrencilerin olası doğru ve yanlış yanıtlarının yüzdesini ve öğrencilerin her bir soru için verebilecekleri tipik doğru ve yanlış yanıtların yüzdesini ifade etmişlerdir. Anketin ardından yarı yapılandırılmış görüşmeler yapılmıştır. Anket ve yarı yapılandırılmış görüşmelerden elde edilen veriler, tümevarımsal içerik analizi yaklaşımıyla analiz edilmiştir. Tanılayıcı cebir testi öğrencilere uygulandıktan ve araştırmacı test sonuçlarını analiz ettikten sonra, öğrencilerin Tanılayıcı cebir testindeki performansları hakkındaki düşüncelerini öğrenmek için öğretmenlerle tekrar yarı yapılandırılmış görüşmeler yapılmıştır. Bu görüşmeler tümevarımsal içerik analizi yaklaşımı kullanılarak analiz edilmiştir. İkinci olarak, ilgili kodların belirli kategorilere dahil edilip edilemeyeceğini ve belirli kategorilerle uyumlu olup olmadığını görmek için veri analiz prosedürü tekrarlanmıştır. Öğretmenlerin tanılayıcı cebir testindeki öğrencilerin performansları hakkındaki tahminleri ve düşüncelerine ilişkin veri analizi süreci tamamlandıktan sonra, veriler bir matris formatında özetlenmiş ve olası benzerlik ve farklılıkları gözlemlemek için araştırmacı tarafından karşılaştırılmıştır.

3.4.3. Üçüncü Araştırma Sorusunun Veri Analiz Süreci

Öğretmenlerin öğrencilerin karşılaştıkları güçlüklerin potansiyel nedenlerine ilişkin ifadeleri, tümdengelim içerik analizi yaklaşımı kullanılarak nedensel yükleme kuramına dayalı olarak analiz edilmiştir (Baştürk, 2016; Wang ve Hall, 2018; Weiner, 2010). Baştürk'ün (2016) çalışmasında belirlenen kodlar, öğretmen adayları üzerinde yapılan benzer bir çalışma olması nedeniyle kodlama sürecinin yapısını belirlemeye yönelik mevcut araştırmaya temel oluşturmuştur. Ayrıca araştırmacı, öğretmenlerin kavramsal yüklemelerinin, öğrencilerin anlama ve güçlüklerine ilişkin yorumlarını nasıl etkilediğini görmek için, öğretmenlerin öğrencilerin kavrayışları ve güçlüklerine ilişkin bilgilerini öğrencilerin tanılayıcı cebir testindeki performanslarına ilişkin yorumlarıyla karşılaştırmıştır. Ön koşul bilgisi, tahminler ve öğrencilerin tanılayıcı cebir testi performanslarının yorumlanması için her öğretmenin ifadesini ayrı ayrı görebilmek amacıyla 5×3 matris tablosu kullanılmıştır.

Araştırmacının analizi tamamlamasının ardından, matematik eğitiminde doktorasını yapmış olan ikinci bir araştırmacı, öğretmenlerden toplanan verilerin tümünü analiz etmiştir. Araştırmacı, bahsedilen araştırmacının verileri analiz etmeye başlamasından önce, tümdengelim yaklaşımıyla incelenen veriler ve tümevarım yaklaşımıyla incelenen verilerle ilgili kodlama çerçevelerini tanıtmıştır. Ardından, ikinci araştırmacının kodlama sürecini daha ayrıntılı olarak anlaması için iki araştırmacı verilerin küçük bir bölümünü birlikte kodlamıştır. Daha sonra ikinci araştırmacı verilerin kodlanmasının tamamlanmasından sonra, kodlayıcılar arası uyumu sağlamak için her iki araştırmacının analizlerinin sonuçları en az %80 uyum gözlemlemek için karşılaştırılmış ve araştırmacılar arası uyumun %90 olduğu sonucu elde edilmiştir (Miles ve Huberman, 1994). İki araştırmacı tarafından farklı şekillerde kodlanan kısımlar iki araştırmacının uzlaşacağı bir şekilde tekrar kodlanarak, veri analizi süreci tamamlanmıştır.

4. Bulgular ve Tartışma

Bu bölümde, çalışmadan elde edilen bulgular, birinci araştırma sorusu çerçevesinde, öğretmenlerin; öğrencilerin cebir öğrenmeden önce sahip olması gereken ön koşul bilgiler, cebiri kavrayışları ve cebirde yaşadıkları zorluklar göre, ve bu zorlukların nasıl giderilebileceği şeklinde özetlenmiştir. İkinci araştırma sorusuna yönelik elde edilen veriler, öğretmenlerin öğrencilerinin tanılayıcı cebir testindeki performanslarına ilişkin tahminleri ve öğretmenlerin öğrencilerinin cebir kavrayışları ile ilgili bilgisinin, bu teste yönelik tahminleri ve test sonucundaki yorumlarının karşılaştırılması şeklinde sınıflandırılmıştır. Son olarak, üçüncü araştırma sorusu çerçevesinde, öğretmenlerin öğrencilerinin tanılayıcı cebir testinde zorluk yaşadıkları noktalara ilişkin nedensel yüklemelerine ilişkin analizler yapılmış ve sunulmuştur.

Ortaokul matematik öğretmenlerinin öğrencilerin cebir öğrenmeden önce sahip olması gereken ön koşul bilgisine ilişkin sınırlı miktarda bilgi sağlayabildiği gözlemlenmiştir. Bütün öğretmenler, cebir öğrenmeden önce dört işlemin iyi bir şekilde öğrenilmesi gerektiğini ifade etmiştir. Ayrıca, öğretmenler negatif sayıları cebiri öğrenmenin bir ön koşul bilgisi olarak tanımlamıştır. Bununla birlikte, eşittir işaretinin kavramsal anlamı, değişken kavramı ve kovaryasyonel düşünme kavramlarından çok az miktarda bahsetmişlerdir. Ortaokul matematik öğretmenleri öğrencilerin cebir derslerindeki cebirsel düşüncelerini ve tanılayıcı cebir testindeki yanıtlarını analiz edebilmiş; ancak öğrencilerin belirli kavramlarda yaşadıkları güçlüklerin altında yatan nedenleri açıklayamamışlardır. Öğretmenler öğrencilerin cebiri öğrenirken ezber yapmaya yatkın olduklarından bahsetmiş, fakat sayısal işlemlerden cebirsel ifadelere geçişin öneminden bahsetmemişlerdir. Sadece bir öğretmen sözel ifadelerden cebirsel temsillerin sembolik notasyonuna geçişi Katz'ın (2007) çalışmasında bahsedilen retorik aşama (cebirsel ifadeleri temsil etmek için kelimelerin veya cümlelerin kullanılması), aksak (syncopated) aşama (cebirsel ifadeleri temsil etmek için kısaltmaların kullanılması) ve sembolik aşama (miktarları, işlemleri ve ilişkileri ifade etmek için sembollerin kullanılması ve bu sembolleri kullanarak iyi anlaşılmış kurallara dayalı manipülasyonlar yapılması) şeklinde ifade etmiştir. Öğretmenlerin hiçbiri cebirsel muhakeme için eşittir işaretinin kavramsal olarak anlaşılması gerektiğinden bahsetmemiştir (Stephens vd., 2013). Ortaokul öğrencilerinin cebir ile ilgili konuları öğrenmeye hazır olup olmadığı sorulduğunda iki öğretmen hazır olduklarını, iki öğretmen bazılarının hazır olduğunu, bazılarının olmadığını, bir öğretmen ise hazır olmadıklarını belirtmiştir.

Gürsoy, öğrencilerinin "İki sayının toplamı 45'tir." ifadesinde (45–x)'i bulabilmelerine rağmen x'i toplanan terimlerden biri olarak ifade etmekte zorlanmışlarını dile getirmiş ve öğrencilerinde en sık karşılaştığı zorluğun bu olduğunu söylemiştir. Buradaki sorunun iki cebirsel ifadeyi aynı bilinmeyen cinsinden yazmanın zorluğu olarak açıklamıştır. Burcu da öğrencilerin "İki sayının toplamı 60'tır, bu sayılardan biri diğerinin iki katından dört fazladır." ifadesini sembolik olarak yazmakta zorlandıklarını belirterek bu konuya değinmiştir. Belirttiği gibi öğrenciler özellikle x'i bulmakta zorlanırken, diğer ifade olan (2x + 4)'ü belirleyebilmişlerdir. Öğretmenler, öğrencilerin iki cebirsel ifadeyi aynı bilinmeyen cinsinden yazma problemlerini tespit edebilmelerine rağmen, bu zorlukların altında yatan nedenleri açıkça ifade edememişlerdir. Genel olarak öğretmenlerin ifadeleri, öğrencilerin cebirde karşılaşabilecekleri güçlükleri belirleyebileceklerini, ancak öğrencilerin güçlüklerinin ve hatalarının nedenlerini açıkça ifade edemediklerini ortaya koymuştur.

Bulgulara dayanarak, öğretmenler, öğrencilerin tanılayıcı cebir testindeki soruları nasıl çözeceklerini değişen derecelerde doğru şekilde tahmin etmişlerdir. Öğretmenler, öğrencilerin sözel ifadelerden cebirsel ifadelere basit çeviriler yapma, denklemleri çözme ve verilen verileri tablo veya grafik üzerinde göstermeyi içeren sorular için öğrencilerin olası çözümlerini ve bu soruları kaç öğrencinin yapabileceği konusundaki tahminleri gerçek öğrenci cevaplarına yakın niteliktedir. Ancak, öğretmenlerin eşitlik, değişken ve fonksiyonel düşünme ile ilgili sorulara öğrencilerin verebileceği yanıtlar konusundaki tahminleri gerçek öğrenci cevaplarından farklı niteliktedir. Tanılayıcı cebir testindeki eşitlikle ilgili 1. soruda , üç öğretmen çoğu öğrencinin ilişkisel-yapısal bir anlayışla çarpma olmadan eşitliği gösterebileceğini öngörmüştür (Stephens vd., 2013). Ayrıca, iki öğretmen, çok az öğrencinin soruyu çarpmayı kullanmadan yapabileceğini düşünmüş ve çoğu öğrencinin ilişkisel-hesaplamalı bir anlayışı benimsediğini ifade etmiştir (Stephens vd., 2013). Öğretmenler, çarpma işlemini yapmanın öğrenciler için sayıları çarpanlara ayırmak gibi ilişkisel-yapısal stratejilerden daha kolay olduğunu belirtmişlerdir. ADT sonuçları, öğrencilerin yaklaşık %30'unun ilişkisel-yapısal bir strateji benimsediğini göstermiştir (örn. "Biri 2 ile çarpılırken diğeri 2'ye bölünür."). Ayrıca, öğrencilerin yarısı ilişkisel-hesaplamalı bir strateji kullanmıştır (örn. "22 eşittir 11 ve 2'nin çarpımı; 14 eşittir 7 ve 2'nin çarpımıdır."). Bu nedenle, öğretmenlerin tahminlerinin öğrencilerin eşitlik ile ilgili gerçek performanslarıyla uyumlu olmadığı sonucuna ulaşılmıştır. Araştırmacılar, çoğu öğretmen adayının, öğrencilerin eşittir işaretine ilişkin işlemsel düşünme konusundaki yanılgılarından habersiz olduğunu bulmuşlardır (Alapala, 2018; Isler ve Knuth, 2013; Stephens vd., 2013). Ayrıca, öğretmen adayları yapısal düşünmeden daha çok sayısal düşünmeye odaklanmışlardır (Stephens, 2006). Bu çalışmalara benzer şekilde, bu çalışmanın sonuçları, katılımcı matematik öğretmenlerinin öğrencilerin eşitlik ve eşittir işaretinin ilişkisel olarak anlamlandırılmasına ilişkin kavrayışlarını ve zorluklarını tahmin edemedikleri gözlenmiştir.

3n ve n+6 cebirsel ifadelerinin karşılaştırıldığı üçüncü soru ile ilgili ön görüşme verilerinin analizi, öğretmenlerin öğrencilerinin değişken kavramına ilişkin kavrayışlarının, Asquith vd.'nin (2007) diğerlerinin çalışmasının aksine, öğrencilerin testteki gerçek sonuçlarıyla uyumlu olmadığını göstermiştir. Öğretmenlerin açıklamaları, iki cebirsel ifadeden hangisinin daha büyük olduğunu belirlemek için n yerine tekli veya çoklu değerler koymaya odaklandıklarını göstermiştir. Asquith vd.'nin (2007) aksine, öğretmenlerin öğrencilerin değişken kavramını anlama performansına ilişkin tahminleri, öğrencilerin tanılayıcı cebir testindeki gerçek yanıtlarıyla uyumlu çıkmamıştır. Tanılayıcı cebir testindeki dördüncü sorunun a şıkkı ile ilgili olarak, öğretmenler, sorunun öğrenciler için tanıdık ama zorlayıcı bir soru olduğunu belirtmişlerdir. Asquith vd.'nin çalışmasına benzer şekilde, öğrencilerin üçüncü ve dördüncü sorularda zorlanmalarının sebebini değişken kavramıyla ilgili olabileceğine nadiren değinmişlerdir. Öğretmenler, öğrencilerin tanılayıcı cebir testindeki fonksiyonel düşünme sorularında yüksek bir performans sergileyeceğini düşünmüştür. Fakat, öğrencilerin performansı öğretmenlerin tahminlerinden düşük olmuştur. Fonksiyonel düşünme sorularında öğretmenler, öğrencilerin bir soruyu cevaplandırabilseler bile fonksiyon kuralını kullanarak çözmeyi tercih etmeyeceklerini belirtmişlerdir. Öğretmenler öğrencilerin teste verdikleri cevapları inceledikten sonra, öğrencilerin cebir sorularını çözmek için fonksiyon kuralını kullanmanın gereksiz olduğunu düşündüğünü ve kullandıkları diğer yöntemlerin onlar için çok daha pratik olduğunu dile getirmişlerdir. Bu nedenle, fonksiyonun kuralını bulabiliyor olsalar bile kullanmayı tercih etmediklerini dile getirmişlerdir. Öğretmenler, öğrencilerin soyut düşünmede sorun yaşadıklarını sıklıkla dile getirmiştir. Öğrencilerin fonksiyonun kuralını bulmada yaşadıkları güçlüklere, öğrencilerin motivasyonlarının kaybolması, ezber yapmaları ve cebirden zevk almamaları gibi açıklamalar getirmişlerdir. Ek olarak, öğretmenler öğrencilerin yanlış cevaplarına örnekler verebilmiş; ancak bu yanlış cevapların nedenlerini detaylı bir şekilde açıklayamamışlardır. Öğretmenler, fonksiyonel düşünmeyi öğrenmek için çok önemli olan değişkenler arasındaki kovaryasyondan ise bahsetmemişlerdir. Şen-Zeytun vd. (2010), öğretmenlerin fonksiyonları, kovaryasyonel yapılar yerine birbirine karşılık gelen ilişkiler olarak algıladıklarını gözlemlemiştir. Ayrıca öğretmenlerin, öğrencilerin muhakeme yeteneklerine ilişkin beklentilerinin, problemle ilgili kendi düşüncelerinin ötesine geçemediği için sınırlı olduğu sonucuna varmışlardır.

Blanton vd. (2011), cebirsel düşünmenin geliştirilmesinde fonksiyonların çok önemli olduğunu ifade etmiştir. Belirttikleri gibi, fonksiyonlar, öğrencilerin nicelikler arasındaki ilişkiyi düşünmelerini sağlayarak öğrencilerin sembolik gösterimi anlamlı bir şekilde öğrenmelerine katkı sağlar. Blanton ve Kaput (2004), tek değişkenli veri setlerinde örüntü bulmaya yapılan vurgunun, sonraki ilkokul yıllarında fonksiyonel düşünmeye vurgu yapılmasına engel olabileceğini vurgulamıştır. Öğretmenler, öğrencilerin performansıyla ilgili tahminleri gibi, öğrencilerin tanılayıcı cebir testindeki fonksiyonel düşünme sorularındaki gerçek sonuçları için yeterli açıklamalar yapamamışlardır. Öğretmenlerin beklentilerinin öğrencilerin gerçek performanslarına yakın olmasına rağmen, öğrencilerin neden fonksiyonel düşünme sorularında fonksiyonun kuralını yazmakta zorlandıklarını açık bir şekilde açıklayamamışlardır. Genellikle, bu durumu cebirin soyut olmasına ve öğrencilerin x ve y'ye karşı önyargılarının olmasına bağlamışlardır. Öğretmenler, öğrencilerin aritmetik işlemleri yapabilmesi, basit denklemlere ve cebir problemlerine ilişkin çözümleri ile ilgili performanslarını doğru bir şekilde tahmin etmiştir. Fakat, öğretmenlerin tahminleri, eşittir işaretinin anlamı, değişkenin anlamı ve fonksiyonel düşünme ile ilgili sorularda öğrencilerin yanıtlarıyla uyumlu değildir. Öğretmenler öğrencilerin ilgili maddelere ilişkin zorluklarını belirleyebilmelerine rağmen, öğrencilerin yaşadıkları zorlukların altında yatan nedenleri yeterli düzeyde ifade edememişlerdir.

Wang ve Hall (2018), öğretmenlerin nedensel yüklemelerinin, öğrencilerin akademik performansını, davranışlarını ve motivasyonunu önemli ölçüde etkileyen, öğretim davranışları üzerinde etkili olabileceğini dile getirmiştir. Bu nedenle, öğretmenlerin öğrencilerin başarısızlığı ile ilgili sundukları nedenler, Weiner'in (1985, 2010) nedensel yükleme teorisine dayanarak araştırılmıştır. Bozkurt ve Yetkin-Özdemir'in (2018) çalışmasıyla benzer şekilde, bu çalışma da öğretmenlerin genellikle başarısızlıklara yönelik nedensel yüklemeler yapma eğiliminde olduklarını ortaya koymuştur. Bu nedenle, bu çalışmada öğretmenlerin öğrencilerin cebirde deneyimledikleri zorluklara yönelik nedensel yüklemeleri incelenmiştir. Literatürdeki çalışmaların sonuçlarına benzer şekilde (Baştürk, 2012; Medway, 1979; Wang ve Hall, 2018), öğretmenlerin öğrencilerin zorluklarını sıklıkla öğrencilerin bilişsel süreçlerine, çabalarına, doğuştan gelen matematik becerilerine, ve motivasyonlarına bağladıkları sonucuna ulaşılmıştır. Bu çalışma, öğretmenlerin öğrencilerin cebirsel düşünmedeki zorluklarını çoğunlukla, öğretmenlerin dışında olan, istikrarlı ve kontrol edilemez bir faktör olan öğrencilerin bilişsel süreçleriyle ilişkilendirdiğini göstermiştir (Wang ve Hall, 2018; Weiner, 2010). Öğretmenler, öğrencilerin bu tanılayıcı cebir testindeki sorulara aşina oldukları için öğrencilerin başarısızlıklarının esas nedeninin öğrencilerin kendisi ile ilgili olduğunu savunmuştur. Öğretmenlerin ağırlıklı olarak bahsettiği diğer nitelikler, içsel, istikrarsız ve kontrol edilebilir bir faktör olan öğrencilerin çabası ve içsel, istikrarsız ve kontrol edilemeyen bir faktör olan öğrencilerin motivasyonudur (Wang ve Hall, 2018; Weiner, 2010). Medway (1979) ve Baştürk (2012), öğretmen adaylarının öğrencilerin zorluklarını veya başarılarını en sık, öğretmenler için dışsal, istikrarlı ve kontrol edilemez bir faktör olan doğuştan gelen matematik yeteneğine bağladıkları bilgisine ulaşmışlardır (Wang ve Hall, 2018; Weiner, 2010). Literatürdeki çalışmaların sonuçlarının aksine, bu çalışmada öğretmenler tarafından bahsedilen öğrenci ile ilgili en az gözlemlenen nitelik, öğrencilerin matematik becerileridir.

Sonuçlar, öğretmenlerin öğrencilerin cebirde yaşadıkları zorlukları dış etkenlerle daha fazla ilişkilendirdiğini göstermiştir. Öğretmenlerin, öğrencilerin yaşadıkları güçlüklerin çoğunlukla öğrencilerin kendilerinden kaynaklandığını düşündükleri söylenebilir. Ayrıca öğretmenler, öğrencilerin zorluklarını çoğunlukla sabit ve kontrol edilemeyen faktörlere bağlamıştır. Öğretmenlerin ağırlıklı olarak öğrencilerin bilişsel süreçleri, motivasyon eksikliği, yetersiz matematik becerileri, programdaki kazanımların içeriğinin yetersiz olması gibi, öğrencileri başarısız kılan faktörler üzerinde kontrollerinin olmadığını düşündükleri söylenebilir. Araştırmacıların belirttiği gibi, nedensel yüklemeler öğretmenlerin öğrencilerin gelecekteki akademik performanslarından beklentilerini önemli ölçüde etkiler (Clarkson ve Leder, 1984; Peterson ve Barger, 1985). Ayrıca, Wang ve Hall'un (2018) iddia ettiği gibi, öğretmenlerin nedensel yüklemeleri, öğrencilerin akademik performansını ve motivasyonunu önemli ölçüde etkileyen öğretim davranışlarını etkileyebilir. Bu çalışmada, öğretmenler, değişken kavramı veya kovaryasyonel düşünme gibi belirli noktalara dayalı olarak daha dikkatli olacaklarını belirtmişlerdir. Bununla birlikte, öğretmenlerin nedensel yüklemeleri, öğrencilerin düşük performansının genel olarak öğretim dışındaki faktörlerle ilgili olduğunu gösterebilir. Bu nedenle, öğrencilerin zorlanmasına neden olan faktörlerin çoğu öğretmenlerin sorumluluklarının dışında olduğundan, öğretmenlerin cebiri benzer şekilde öğretmeye devam edeceği sonucu çıkarılabilir.

Bu çalışmanın, öğretmenlerin öğrencilerin eşitlik, ifadeleri denklem, genelleştirilmiş aritmetik, değişken ve fonksiyonel düşünme büyük fikirlerine yönelik cebirsel düşünmeleri ile ilgili bilgilerine ilişkin matematik öğretmen adayları, matematik öğretmenleri, öğretmen eğitimcilerine yönelik literatüre katkıda bulunabilir. Bu çalışmada, öğretmenlerin hiçbiri daha üst düzey cebir konuları için gerekli olabilecek eşittir işaretinin kavramsal bir şekilde anlaşılması, değişken kavramının anlaşılması, orantısal düşünme ve kovaryasyonel düşünme hakkında düşüncelerini dile getirmemişlerdir. Tanışlı ve Köse (2013) ve Stephens'ın (2006) matematik öğretmeni adayları ile yürüttükleri çalışmaların sonuçlarıyla benzer şekilde, katılımcı öğretmenlerin eşittir işareti ile ilgili alan bilgileri ve eşittir işaretini kavramsal anlamayla ilgili iyileştirmeye ihtiyaç duydukları sonucuna varılabilir. Boz'un (2004) da belirttiği gibi alan bilgisi konusundaki bilgilerinin eksikliği, öğretmenlerin öğrencilerin zorluklarını, hatalarını ve kavram yanılgılarını belirlemelerini engelleyebilmektedir. Bu nedenle, öğretmenler, öğrencilerin eşittir işareti konusundaki düşünceleri ve eşittir işaretinin öğrenciler tarafından kavramsal olarak anlaşılmasına ilişkin pedagojik alan bilgilerini geliştirebilirler.

Bu çalışmanın önemli bir çıkarımı, Tanışlı ve Köse'nin (2013) çalışmasına benzer şekilde, öğretmenlerin öğrencilerin cebirsel düşünmesini bazı noktalarda analiz edememesidir. Öğretmenler, öğrencilerin genelleştirilmiş aritmetik, cebirsel ifadeler ve denklemler ile ilgili cebirsel düşünme süreçlerini açıklayabilmektedir. Ancak, öğretmenler, öğrencilerin hatalı düşünmelerinin nedenlerine ve eşitlik, ifadeler, denklem ve fonksiyonel düşünme gibi büyük fikirlerde yaşadıkları zorluklara ilişkin sınırlı miktarda açıklamalar sunabilmişlerdir. Öğretmenler, genellikle öğrencilerin işlem yapma, sayıları kullanma, yerine koyma ve denklem çözme gibi cebirsel uygulamalarına odaklanmışlardır. Bu konular aynı zamanda öğrencilerin zorluk yaşadıkları faktörlerden bazıları olsa da, eşittir işareti ve eşdeğer denklemlerin ilişkisel olarak anlaşılması denklemleri çözümündeki en önemli noktalardan biridir (Knuth vd., 2005; Steinberg vd., 1990). Bu çalışmanın bulgularına dayanarak, bir öğretmen geliştirme programının, öğretmenlerin denklem kurma ve çözmede başarılı olmak için eşittir işaretinin anlamının ve denkliği kavramanın önemini fark etmelerini sağlamak için Öğretmenlerin alan bilgileri ve pedagojik alan bilgilerini geliştirmeye yardımcı olabileceği düşünülmüştür.

Literatürde, öğretmenlerin cebirdeki nedensel atıflarını inceleyen çalışmaların yetersiz olduğu ifade edilmiştir (Shores ve Smith, 2010; Wang ve Hall, 2018). Bu çalışma, nedensel yüklemelerle ilgili olarak, matematik öğretmenlerinin öğrenciyle ilgili yüklemeler ve öğretim süreciyle ilgili yüklemeler olmak üzere iki yönüyle ilgili literatüre katkı sağlayabilir. Bulgular, öğretmenlerin öğrencilerin zorluklarını daha çok dışsal ve kontrol edilemeyen faktörlerle ilişkilendirdiğini göstermiştir. Öğretmenler, öğrencilerin cebirsel düşünmedeki başarısızlıklarını temel olarak öğrenciyle ilgili faktörlere, öncelikle öğrencilerin anlama yetersizliği, motivasyon eksikliği ve yetersiz matematik becerileri gibi bilişsel süreçle ilgili faktörlere bağlamıştır. Araştırmacılar, nedensel yüklemelerin öğretmenlerin öğrencilerin gelecekteki akademik performanslarına ilişkin beklentilerini önemli ölçüde etkilediğini savunmuştur (Clarkson ve Leder, 1984; Peterson ve Barger, 1985). Glasgow vd. (1997), öğretmenlerin, öğrencilerin başarısızlıklarını bu tür kontrol edilemeyen faktörlere bağlamaları halinde, yeteneksiz öğrenciler üzerinde kontrollerinin olmadığını düşündükleri için yeterince çaba gösteremeyebileceklerini ileri sürmüştür. Bu bulgulara göre, öğretmenlerin, öğrencilerle ilgili faktörlere ilişkin nedensel atıflarını gözlemlemek için matematik öğretmenlerinin ve ortaokul öğrencilerinin dahil olduğu yeni araştırmalar yürütülebilir. Ayrıca, öğretmenlerin, öğrencilerin cebirdeki başarısızlıkları için dış ve kontrol edilemeyen faktörlere ilişkin argümanlarının geçerliliğini test etmek için hem ortaokul öğrencilerinden hem de bu öğrencilerin öğretmenlerinden ayrıntılı veri toplamak amacıyla görüşmeler ve sınıf gözlemleri gibi farklı veri toplama araçları kullanılabilir.

Öğretmenlerlerin öğrencilerin cebirdeki xorlukları ile ilgili nedensel yüklemelerinden bir diğeri ise öğretim süreci, müfredat ve sınav sistemiyle ilgili faktörlerdir. Öğretim süreciyle ilgili en sık gözlemlenen yüklemelerden biri, matematik müfredatındaki derslerin süresi ve kazanımlar gibi dışsal, sabit ve kontrol edilemeyen faktörler olan müfredatla ilgili yüklemelerdir. Öğretmenler, görüşmeler sırasında ortaokul matematik müfredatında değişken, eşitlik ve kovaryasyonel düşünme kavramlara vurgu yapılmadığını savunmuştur. Ayrıca cebirsel bir ifadede değişkenin anlamını göz önünde bulundurarak, iki cebirsel ifadeyi aynı bilinmeyenle karşılaştırarak ve bir fonksiyonda iki değişkenin niceliği arasındaki kovaryasyonu gözlemlemek gibi etkinliklerin müfredatta olmadığını belirtmişlerdir. Bu bulgular, mevcut ortaokul matematik müfredatının (MEB, 2018) içeriğini cebirdeki büyük fikirlere (Blanton vd., 2015; Blanton vd., 2019) göre inceleyen çalışmalar için yol gösterici olabilir. Ayrıca, gelecekte yürütülebilecek olan bu tür çalışmaların, öğretmenlere, matematik eğitimi araştırmacılarına ve müfredatı geliştiren uzmanlara yönelik yararlı bilgiler sağlayabileceği düşünülmektedir.

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