CLASSROOM MATHEMATICAL PRACTICES OF THE SEVENTH GRADERS ABOUT RATIO AND PROPORTION CONCEPTS

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ABSTRACT

CLASSROOM MATHEMATICAL PRACTICES OF THE SEVENTH GRADERS ABOUT RATIO AND PROPORTION CONCEPTS

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The purpose of this study is to examine the Classroom Mathematical Practices (CMPs) of seventh-grade students with regard to ratio and proportion concepts through the implementation of a specific instructional sequence. The study was conducted using an educational design research methodology and was based on the Classroom Hypothetical Learning Trajectory and related instructional sequence developed by Stephan et al. (2015). The instructional sequence was mainly carried out by an experienced middle-grade mathematics teacher in a public school in Yenimahalle, Ankara with the support of design team. The data was collected through classroom observations, teacher interviews, field notes, and student and teacher documents. The observations mainly took place over 34 lesson hours. The argumentation components described by Toulmin (1958), taken-as-shared ideas, and the CMPs embedded in the classroom discussions were analyzed using a three-phase documentation approach to establish the CMPs (Rasmussen & Stephan, 2008). The study found five CMPs that enhanced student learning: reasoning about discrete/continuous objects and the rule of the ratio; linking and iterating composite units; covariation among composite units in a ratio table; representing ratio and proportion symbolically; and adapting strategies for comparing non-equivalent ratios. These practices demonstrate various aspects of proportional reasoning and suggest that the instructional sequence has the potential to improve the teaching of ratio and proportion concepts in terms of unitizing, linking and iterating composites representing, data organizing, conducting operations, comparing the ratios, applying build-up and multiplicative relations, creating, and analyzing equivalent ratios. This study contributes to the ongoing effort to define an instructional design for helping seventh-grade students understand ratio and proportion.

Keywords: Classroom Mathematical Practices, Ratio, Proportion, Educational Design Research, Proportional Reasoning.

ÖZ

ORAN VE ORANTI KAVRAMLARI HAKKINDA YEDİNCİ SINIF ÖĞRENCİLERİNİN SINIF İÇİ MATEMATİKSEL UYGULAMALARI

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Bu çalışmanın amacı, eğitimde tasarı tabanlı araştırma yöntemi kapsamında belirli bir öğretim dizisi aracılığıyla yedinci sınıf öğrencilerinin oran ve orantı kavramlarına ilişkin Sınıf İçi Matematiksel Uygulamalarını (SMU) araştırmaktır. Çalışma boyunca oran ve orantı kavramları için varsayıma dayalı öğrenme rotası ve buna dayalı olarak geliştirilmiş öğretim dizisi uygulanmıştır (Stephan vd., 2015). Öğretim dizisi, Ankara'nın Yenimahalle ilçesindeki bir devlet okulunda çalışan deneyimli ortaokul matematik öğretmeni tarafından tasarım ekibinin desteğiyle de gerçekleştirilmiştir. Veriler, yedinci sınıf öğrencilerinin sınıf oturumları, öğretmen görüşmeleri, alan notları ve öğrencilerin ve öğretmenlerinin belgeleri aracılığıyla toplanmıştır. Sınıf gözlemi ve uygulama 34 ders saati sürmüştür. Sınıf tartışmaları içinde yer alan argümantasyonlar içerisindeki SMU'ları oluşturan paylaşılmış fikirler, Toulmin tarafından oluşturulan argümantasyon modeli ve üç aşamalı sınıf içi matematiksel uygulamalar analizi ile çözümlenmiştir. Bu çalışmada öğrenci öğrenimini kolaylaştıran beş SMU şunlardır: süreksiz /sürekli nesneler ve oran kuralı hakkında akıl yürütme; bileşik birimler arasında bağlantı ve yineleme; oran tablosu içinde bileşik birimler arasındaki ilişkili değişim (covariation); oran ve orantının simgelerle gösterilmesi; eş olmayan oranları karşılaştırmak için strateji geliştirme. Bu uygulamalar, orantısal akıl yürütme ile ilgili birçok boyutunu işaret etmekte ve öğretim dizisinin oran ve orantı kavramlarının öğretim kalitesini iyileştirme potansiyeline işaret etmektedir. Bu çalışma, yedinci sınıf öğrencilerinin oran ve orantı kavramlarını anlama yolunda nasıl bir öğretim tasarımının tanımlanacağı konusunda fikir verebilir.

Anahtar Kelimeler: Sınıf İçi Matematiksel Uygulamalar, Oran, Orantı, Eğitimde Tasarı Tabanlı Araştırma, Orantısal Akıl Yürütme

To Hope…

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LIST OF ABBREVIATIONS

A#	Activity #
A1Q1	Activity# Question #
CCSSI	Common Core State Standards Initiative
CMP	Classroom Mathematical Practice
EDR	Educational Design Research
HLT	Hypothetical Learning Trajectory
HSF	Horizontal Scale Factor
INC	Lesson Hour for Inclusion Students
LCM	Least Common Multiple
LED	Launch-Explore-Discuss
MoNE	Ministry of National Education in Turkey
NCTM	National Council of Teachers of Mathematics
TAS	Taken-As-Shared Ideas
VSF	Vertical Scale Factors
ZPD	Zone of Proximal Development

CHAPTER 1

INTRODUCTION

Mathematical reasoning is essential for mathematical teaching and learning. In the early years of elementary education, the curricular focus is to develop children’s additive mathematical thinking in primary grades. As their education proceeds, the focus is expected to shift from additive to multiplicative thinking (Fernández et al., 2012; Jiang et al., 2017; van Dooren et al., 2009). This transition requires complex knowledge and skills, which have been reported to pose several challenges. Students may struggle to differentiate between multiplicative and additive thinking (de Bock et al., 2007; Modestou & Gagatsis, 2007; van Dooren et al., 2009), may overuse additive thinking in proportional situations (Misailidou & Williams, 2003; Tourniaire & Pulos, 1985), or may overuse multiplicative thinking in additive situations (Modestou & Gagatsis, 2007; van Dooren et al., 2009) as reports revealed.

Proportional reasoning is the form of mathematical reasoning involving a sense of covariation and the ability to make multiple comparisons in relative terms (Lesh et al., 1988), as well as the utilization of other knowledge and skills such as fractions and linear functions (Ruiz & Valdemoros, 2002). Proportional reasoning provides a foundation for abstract thinking within the scope of mathematical reasoning (Che, 2009; Dole et al., 2015; Karplus et al., 1983). For example, proportional reasoning is infused in every part of life, such as nutrition (Thornton et al., 2020), economy and marketing (Raghubir & Greenleaf, 2006), medicine (Arndt et al., 1991), and the like. Proportional reasoning was claimed to develop during the formal stage of human life (Inhelder & Piaget, 1958). It affects students’ higher education levels (Tabart et al., 2005) as well as their understanding of other disciplines because achieving competency can be considered a milestone within elementary mathematical concepts (Lesh et al., 1988).

Despite its significance both in daily life and in higher mathematics, difficulties in understanding the web of knowledge and skills representing proportional reasoning have been frequently reported (Cramer & Post, 1993a; Fujimura, 2001; Lamon, 1993; Lesh et al., 1988; Misailidou & Williams, 2003; Nabors, 2003; Noelting 1980b; Tourniaire & Pulos, 1985; van Dooren et al., 2009). Furthermore, many adults may not demonstrate competency in proportional reasoning (Lamon, 2012). Even preservice and in-service teachers may need help reasoning proportionally (Cramer & Post, 1993a, 1993b). Karplus and colleagues (1983) collected data from 13–15-year-old students in seven countries related to proportional reasoning, and 75% of these students had still not advanced to the formal stage of proportional reasoning.

Comprehensive studies about rate, ratio, and proportion have displayed that there is still a need to advance in research about proportional reasoning and its acquisition (Behr et al., 1992; Lamon, 2007; Tourniare & Pulos, 1985). As a result, a body of knowledge has been developed to enlighten those studying future instructional materials. Although it has been advocated, creating learning environments that are sensitive to the requirements of the learners takes much work. Steffe (1994) emphasized two assumptions that a child and teacher make in the classroom for arithmetical operations: the student uses their own methods, and the teacher uses their own methods. Conventional research was deemed useless for these plans to enhance, transform, or renovate the current educational system (Edelson, 2002). Instructional designs originating from research and practice should be used to create a rich learning environment (Ben-Chaim et al., 2012; Lamon, 2007; Tournaire & Pulos, 1985). Considering this, teaching through different instructional methods and structural changes in this comprehensive web of concepts must be encouraged in a classroom learning environment since proportional reasoning is not a unitary construct. All in one design which encourages the effective combinations of successful instructional methods, tools, and learning environments have been implemented lately for the development of proportional reasoning to document the successful and unsuccessful practices of the implementation.

Since Brown (1992) introduced it, educational design research has become a paradigm for systematically investigating instructional materials, programs, curricula, or tools in response to the emergent features of educational settings (Design-Based Research Collective, 2003). Educational Design Research (EDR) seeks out a "working" learning environment and describes a set of milestones and measurements to check the design's "working" nature. To document a working nature of teaching proportional reasoning with its all aspects can give feedback about the literature based on the working nature of their implications. To reach this kind of conclusions through educational design research, if the design is about development of a mathematical concept, there are some basic assumptions about learning environment.

Learning theories can be used pragmatically through careful coordination of them to benefit in an optimum way for learning (Simon, 2009). Social constructivism and sociocultural views are two standpoints that have affected the educational processes in teaching and learning mathematics. Two views were described in the activities, and their shared ideas were handled carefully by the researchers (Cobb et al., 1996; Cobb & Yackel, 1996; Gravemeijer, 1999; Yackel, 1995). They elicited both perspectives and put an emphasis on their lenses for learning in terms of social interactions and individual cognition (Cobb et al., 1996). Adaptation of sociocultural and constructivist perspectives in care can be achieved with a question addressed by Confrey "The question is not what is the relationship between the two theories, but what relationship between the perspectives, given the problem one seeks to study" (1995, p. 202).

Making use of multiple perspectives is suggested considering their advantages and constraints based on the research focus (Simon, 2009). The reflections of constructivist and sociocultural approaches in education are observed both in studies and in learning environments. The complementary role of both approaches in learning is expressed by researchers (Cobb, 1995; Cobb & Bauersfeld, 1995; Cobb & Yackel, 1996; Simon, 2009). Moreover, the emergent perspective has been utilized by mathematics educators around the world to inform a variety of research efforts (see Akyüz, 2014; Bowers et al., 1999; Cobb et al., 2001; Hershkowitz & Schwarz, 1999; Partanen & Kaasila, 2015; Rasmussen et al., 2004; Rasmussen et al.; 2015; Stephan & Akyuz, 2012; Yackel, 2000). These studies presented above generally declared several constructs leading their research design. Similar to other learning theories, the mechanism working behind the emergent perspectives should be defined to see its constraints and benefits. Individual and social learning developing together requires a set of understanding related to learning, interactions, tools, and teachers' role. Learning environments both providing tools for conveying meanings and learning opportunities for sharing with the students are reflections of the emergent perspectives.

In an inquiry-based mathematics classroom, two facets of students' social relationships appear to be essential for fruitful small-group work: The first involves creating a taken-as-shared meanings for mathematical communication, while the second involves participating in conversations that actually entail mathematical argumentation (Cobb & Bauersfeld, 1995). According to these criteria, exchanges where one student explains their thoughts do not effectively result in learning opportunities for either student. Instead, it is crucial to take into account the different sorts of interaction that students engage in when evaluating the impact that specific activities, like explaining, can have on their mathematical progress.

EDR and classroom mathematical practices collected from EDR have recently been in frequent use (see Ayan-Civak, 2020; Gravemeijer et al., 2003; Stephan & Akyuz, 2012; Şahin Doğruer, 2018; Stephan, 2015). These studies sought to fill the gap between realized (practice) and proposed (theory) learning processes through the demonstration of a potential learning path of the students after implementation in real classrooms (Cobb et al., 2003a). A mass of information was collected through the execution of the instructional tasks related to students' engagement with the tasks, experiences, and reflections of the design team (Stephan, 2015). In the field of design research, studies about the mathematical practices of a community presented mathematical ideas as a product of collective learning and provided feedback for the continuous development of the instruction throughout teaching experiments on that topic (see Akyüz, 2014, 2016; Cobb, 2003; Cobb et al., 2001; Stephan & Akyuz, 2012; Stephan & Rasmussen, 2002). Through evaluating all stages of learning and teaching in terms of theoretical background and practical usability, students build their mathematical practices within a collective classroom activity, which are taken-as-shared ideas built on a shared understanding of a concept (Cobb et al., 2001; Plomp, 2013). That means students are observed to internalize, organize, and reinvent a new model for specific mathematical content and for learning that content in a social environment throughout the development of classroom mathematical practices, thanks to the contribution of individual and social perspectives to understanding learning. Classroom mathematical practices may provide insights into understanding the tasks and the nature of the instructional design by providing feedback to check its mechanism.

1.1. Significance of the Study and Research Questions

There are several issues that this study takes its strength from. First, proportional reasoning and its components was investigated within a compact design—ratio and proportion instructional sequence and hypothetical learning trajectory developed by Stephan et al. (2015) so that the task and learning outcome conformity was documented. The development of proportional reasoning is typically emphasized in middle-grade curricula (see Adjiage & Pluvinage, 2007; Ben-Chaim et al., 2012; CCSSI, 2010; MoNE, 2013, 2018) when students are eligible to conduct formal operations according to Piagetian theory (Fujimura, 2001; Inhelder & Piaget, 1958). However, it continues to prove problematic for students (Cengiz & Rathouz, 2018; Çalışıcı, 2018; Fernández et al., 2012; Jensen, 2018; Jiang et al., 2017; Karplus et al., 1977; van Dooren et al., 2009) and teachers (Ellis, 2013; Ledesma, 2011) because of various reasons. From the students' perspective, proportional reasoning has been described as a web of knowledge and skills in which they feel lost. At the end of the teaching process, students are expected to conduct procedures related to proportional reasoning. However, there is a variety of implicit knowledge and skills that students must gain before achieving procedural competency in proportional reasoning. To illustrate, they have to discriminate between proportional and nonproportional situations (de Bock et al., 2007; Modestou & Gagatsis, 2007; van Dooren et al., 2009), discrete and continuous quantities (Boyer et al., 2008; Fernandez et al., 2012), inverse ratio situations, types of problems (Lamon, 1993), understanding units and unit of units (Battista & van Auken Borrow, 1995; Behr et al., 1992; Lamon, 1993), and the like. These knowledge and skills are required for the development of proportional reasoning (Lamon, 2020).

In this web of connections, students need coherent guidance from simple to complex during instruction. Thanks to recent research, various combinations of the teaching-learning process have tested empirically and reviewed systematically. However, educational research should not be isolated from practice and real classroom environment. To achieve this, research knowledge needs to be transformed into useful knowledge for teaching and learning. Educational design research enabled researchers to conduct effective interventions so that the findings may be transferred from experimental and isolated classrooms to the average classrooms operated by and for average students and teachers (Brown, 1992). Through the use of the children's real-life experiences, they socially construct, represent, argue, and refute their own systems of learning to create a shared understanding of ratio and proportion. In the end, the trend in the literature is to use some predetermined teaching episodes in order to optimize the teaching and learning ratio and proportion within a collaborative, active learning, problem-solving environment (Akyüz, 2014; Ayan-Civak; 2020; Dixon et al., 2009; Stephan & Akyuz, 2012; Stephan & Rasmussen; 2002; Şahin Doğruer, 2018; Uygun, 2016).

Second issue, this study revised and tested conjecture hypothetical learning trajectory (HLT) in different context, Turkey because Stephan and colleagues (2015) developed this HLT and ratio and proportion instructional sequence in USA. They assumed various learning outcomes and evolution of proportional reasoning as Simon identified as “a hypothetical learning process—a prediction of how the students’ thinking/understanding will evolve in the context of the learning activities” (1995, p. 35). From this perspective, this study implicitly compares children from two countries within the learning process.

Third issue is the emergent perspective which explains the individual and classroom learning to base the view of community development. In the macro view, *the community exists with its* *members' contributions, its members exist with the community's contributions, and so does the classroom in micro view*. The emergent perspective, therefore, is a paradigm case molding the social (social constructivism perspective) and individual (psychological perspective) views to describe learning. Both views are equally important and complementary (Cobb & Bauersfeld, 1995). Thus, *focusing on classroom progress* other than individual development led to the integration of a teaching model that increased the number of participations, the opportunity to share the knowledge, emergent tools, misconceptions, and the like so that the students were encouraged to discuss, refute, change, and improve their ideas in the classroom environment.

The emergent perspective originated from symbolic interactionism, in which the development of meaning was not just intrinsic but also required interaction so that the interpretation may occur (Blumer, 1969; Cobb & Yackel, 1996; Yackel, 2000; Yackel & Cobb, 1996). Within this context, mathematical learning also emerged through individual construction and social interaction (Cobb & Bauersfeld, 1995; Yackel & Cobb, 1996); most of the time, their interactions cannot be crystal clear. Social interaction did not refer to students' state of being in the classroom but engaging in the learning activities through changing, abandoning, and retaining their ideas (Yackel, 2000). On the other hand, the psychological perspective considers the individual contribution to the collective learning processes (Cobb & Yackel, 1996). Cobb and Bauersfeld stated, "Neither an individual student's mathematical activity nor the classroom microculture can be adequately accounted for without considering the other" (1995, pp. 9–10). This statement focuses on the claim that individual development might be related to the emergence of collective activity which emerges over time (Bowers & Nickerson, 2001). These two aspects of learning are considered complementary and inseparable.

Classroom mathematical practices representing collective content-specific ideas or strategies which involve students' interpretations of the mathematical tasks in an instructional sequence that generates interactive engagement during the activity characterized by dynamic reasoning about the mathematical entities (Bowers et al., 1999; Cobb et al., 2001, 2011). The relationship between educational design research and classroom mathematical practices originates from the teaching-learning interaction. In other words, educational design research seeks to optimize the design for the sake of the successful learning outcomes or performances. In particular, a classroom mathematical practice is a product of a classroom's normative way of reasoning, not just of a learner (Bowers et al., 1999). It is advantageous to create a learning environment supporting mathematical comprehension and reasoning, which will make it possible to arise taken-as-shared ideas to form classroom mathematical practices (Cobb et al., 2001, 2011; Stephan et al., 2003). They provide content-specific information about knowing, reasoning, explaining, and persuading others by justification related to the mathematical classroom community which creates a collective argumentation environment providing opportunities to explore informal material and move to more formal mathematics by engaging in the processes of negotiation, collaboration, and discussion (Gravemeijer et al., 2000; Streefland, 1991).

The focus of this study is what Lamon (2007) proposed, a methodology of educational design research to investigate proportional thinking in problem-solving situations; to document the instructional sequences, growth of ideas, and the breadth and depth of students' understanding. Lamon (2007) demonstrated a need for a design underpinning a hypothetical learning trajectory. Stephan and colleagues (2015) developed a trajectory for teaching ratio and proportion by considering the needs of the field. The scaffold underlying the teaching process and hypothetical learning trajectory (HLT) consisting of phases offers a rich learning environment. One of the purposes of HLT is to design an instructional plan for a particular mathematical concept, promoting teaching and learning of that topic (Simon & Tzur, 2004). The significance of this study is to revise the HLT through both the teacher's and students' involvement. Testing and revising the HLTs and instructional sequence for the classroom may show successful and unsuccessful practices related to ratio and proportion concepts. The strength of the methodology originates from its retrospective analysis demonstrating "What works" by considering "how, when, and why" it works (Cobb et al., 2003). Therefore, the connection between theories—their reflection on the design and testable conjectures—and implementation practices brings new forms of learning. In this way, domain-specific instructional theories are constructed through the unfolded description of the progression of the students' ideas within a context and these ideas contributed to the domain-specific instructional theory (Cobb, 1999). Educational design research methodology was employed to investigate domain-specific learning of ratio and proportion concepts with an implication of proportional reasoning development. Specifically, the design team produced materials, sources, and a rich learning environment with a variety of tasks and instructional approaches, showing their effectiveness with a layered description (Confrey, 2006). For that reason, students' learning process produces a model for learning by considering its relation to teaching.

Teachers need guidance on best practices for organizing learning within the analogy of a system of corridors, not as rigid curricular sequences but as intellectual spaces through which students' progress (Confrey, 2006). NCTM (2000) reported that the majority of mathematics teachers labor alone, with little encouragement to develop their practices. These practices can be improved, however, when teachers discuss their pedagogical methods with peers and other professionals. Students' experiences were reflected and collected through different curricular and instructional designs (Battista & van Auken Borrow, 1995; Ben-Chaim et al., 1998; Szetela, 1970). The journey to better teaching of the ratio and proportion concepts covers several combinations of the utility of tools (Szetela, 1970), the utility of new instructional methods (Jackson & Phillips, 1983; Jitendra et al., 2009; Şen et al., 2021), the specific focus of a concept in proportional reasoning (Fernández et al., 2012; Thompson, 1994), integration of the technology (Abrahamson & Cigan, 2003; Adjiage & Pluvinage, 2007; Tajika, 1998) and utility of all formerly mentioned components in one design (Ayan-Civak, 2020; Battista & van Auken Borrow, 1995). Selecting among the best teaching and learning practices equipped with the required materials and tools to produce a design for ratio and proportion requires a systematic investigation of teaching and learning.

The learning difficulties that have arisen about ratio-proportion have been compiled in general and awareness of these difficulties is important for understanding the concept of ratio (Karagöz Akar, 2014). Based on this, instructional designs that emphasize the concept of ratio, which students develop simultaneously with the ratio table, can produce successful results in terms of learning (see Abrahamson & Cigan, 2003). Likewise, it is thought that knowing the situations within which the use of tools such as ratio tables is appropriate will affect the efficiency of the teaching process. The conceptualization outlined to distinguish between known and unknown quantities and to relate these to the two types of quantities of that relationship across the two multiplicative comparisons appears to be a component of the answer to this issue. In this study, a teaching sequence designed to support each other simultaneously and mutually was applied in learning the ratio-proportion topic of the concept and the tool (Stephan et al., 2015). The aim is to present applications that can increase the efficiency of learning (see Abrahamson & Cigan, 2003; Ayan-Civak, 2020; Dole, 2008; Ercole et al., 2011; Middleton & van den Heuvel-Panhuizen, 1995) and teaching by addressing the situations in favor of the formation of the learning ratio and proportion concept.

To summarize, the trend in the literature is to use some predetermined teaching episodes in order to understand the teaching and learning of the rate, ratio, and proportion within a collaborative, active learning, problem-solving environment. The aforementioned studies integrated partial or combined models of beneficial research elements in their research designs. The trends in educational design involving research-based beneficial elements of teaching and learning provided a description of a rich learning environment. Based on the collection of literature, Lamon suggested focusing on the differentiation of the problem-solving tasks involving direct proportional situations (increasing and decreasing), inversely proportional situations, integer and noninteger problems, and combinations of four operations (Lamon, 2020). In summary, variation in the tasks is assumed to bring issues related to semiotics, number values, proportion types, ratio types, and imageries, which leads the research focusing on students' experiences from rich learning environments and the transfer of these experiences to develop into other tasks and mathematical topics.

With the help of the study as well as teachers and researchers together, the mathematical learning areas should be organized according to the nature and sequence of the mathematics to be taught (Simon, 1995). Ben-Chaim et al. (2012) developed a model to increase the knowledge and skills of teachers based on a theoretical perspective on the teaching ratio and proportion method. Their model targets the required knowledge for a teacher to teach ratio and proportion concepts. The model was designed based on the findings of the research and practice they have conducted. Abrahamson and Cigan (2003) focused on teaching ratio and proportion to fifth graders with integer problems through a journey from repeated addition to multiplicative practices. Adjiage and Pluvinage (2007) also developed a learning environment for middle graders with six ratio situations and three registers (fraction, decimal, and linear scale) with software support for linear scale. Jitendra and colleagues (2009) designed an instructional intervention to meet the diverse needs of students in classrooms using studies from both special education and mathematics education. Each instructional design contributed to the learning of ratio and proportion concepts in terms of their research focus but still, there is a need to understand their contributions holistically in actual classrooms. Through pragmatically combining these research-based elements with a focus on developing proportional reasoning, new research can be created with the focus of iterating its results to the next implementation in such a way that an optimum design emerges.

Design studies aim to investigate the compelling practices of the teachers in a local environment that teachers always experience and to provide details and specifications about classroom interactions (Confrey, 2006). Lamon (2020) studied this context and offered some implications about how to design an effective ratio and proportion instruction considering the types of questions (comparison, missing value), numbers (decimals, larger numbers), and problems combining four operations. Instructionally based research and development based on the cognitive dimensions of proportional reasoning and their connections to semantic problem types may be the most useful route for discovering the best ways to facilitate more sophisticated student thinking in this domain while allowing the students to build on and extend their informal knowledge. (Lamon, 1993). In this respect, the present study may help improve the quality of teaching in mathematics by providing ideas that teachers construct through practical learning tasks for their teaching (Confrey, 2006).

1.2. Purpose of the Study

The purpose of the present study is to investigate the mathematical practices of seventh-grade students regarding ratio and proportion concepts through a particular instructional sequence which takes its origin from the emergent perspective in which mathematical practices change parallel to students' cognitive development. The teacher was the initiator and a guide within the teaching and the learning process; on the other hand, the students were responsible for their individual learning.

For this reason, this study used an already-designed hypothetical learning trajectory for ratio and proportion concepts. The research questions are given below:

> Which classroom mathematical practices of the seventh-grade students emerged through the implementation of the ratio and proportion instructional sequence within an educational design research environment?

Although an emergent perspective guided the theoretical background of this study, the focus is on classroom mathematical practices such that extraction of individual learning does not take place in this study. However, the influence of the particular meaning within mathematical practices is made visible through retrospective analysis. As mentioned in the research question, this study aims not to develop a new HLT or instructional sequence but to test and revise the already set HLT for ratio and proportion concepts throughout students' mathematical practices and teaching process—this HLT is embedded in the ratio and proportion instructional sequence. The basic scenario of the sequences starts with an imaginary alien nightmare that prepares students for the context of the "aliens" instructional sequence. The problems within the sequences were structured from simple to complex, from concrete to abstract form, and they were designed based on the principles of the Realistic Mathematics Education approach. Furthermore, the research did not aim to extract the students' individual learning. The majority of the instruction was delivered via carefully sequenced tasks scheduled to bring up significant themes. Critically essential parts of each day's lesson included the opening and closing remarks, where the team summarized the day's events and invited feedback from the students. Continual performance-based evaluations were utilized to track changes in students' progress.

1.3. Definition of the Terms

Additive (constant difference) strategy is called "default" strategy, and this strategy may disappear after systematics instruction which includes appropriate build-up and multiplicative strategies (van Dooren et al., 2009).

Additive (constant difference) strategy is called the "default" strategy, and this strategy may disappear after systematics instruction which includes appropriate build-up and multiplicative strategies (van Dooren et al., 2009).

Argumentation refers to the "whole activity of making claims, challenging them, backing them up by producing reasons, criticizing those reasons, rebutting those criticisms, and so on" (Toulmin et al., 1984, p.14).

Classroom Mathematical Practice represents collective content-specific ideas which involve students' interpretations of the mathematical tasks in an instructional sequence resulting from interactive engagement with the activity to give meaning (Bowers et al., 1999; Cobb et al., 2001/2011). A classroom mathematical practice is a product of a classroom's normative way of reasoning, not just of a learner. They are taken-as-shared ideas without direct access to an individual's cognition (Simon, 1995). They are built in a classroom community

Collective Activity is "... normative ways of reasoning of a classroom community" (Rasmussen & Stephan, 2008, p. 195).

Constant of Proportionality refers to the invariant relationship between composite units representing the same ratio. Its mathematical model is described by Lamon, $\frac{y}{x}$ = k, k representing the constant of proportionality in direct ratio situations; on the other hand, the constant of proportionality changes into $k = y.x$ in inversely proportional situations (2020, p.4). In this study, instead of this representation, the constant of proportionality emerged as the rule of the problem, which is provided explicitly in the alien episodes and became explicit in the mixture and proportion tasks. The students needed to identify or extract the rule in missing value problems or comparison problems. Since they used a ratio table as a formal tool, they used a constant of proportionality on the table.

Composite Unit represents units of units considering two related entities as a single entity (Kilpatrick et al., 2001). For example, in this study, “three aliens eat one food bar” becomes “three aliens per food bar” unit. The ratio representation of this composite unit is 3:1.

Educational Design Based Research describes a research perspective for systematically investigating the *design* of these instructional materials, programs, curricula, or tools in response to the emergent features of the setting (Design-Based Research Collective, 2003) within an effective teaching and learning environment (Stephan et al., 2015).

Hypothetical Learning Trajectory (HLT) was identified as “a hypothetical learning process—a prediction of how the students’ thinking/understanding will evolve in the context of the learning activities” (Simon, 1995, p. 35). That is, children follow a path within the learning process. The teacher anticipates the big ideas and generates priority for the instruction. HLT contains an instructional sequence where teachers align the activities with their goals. Based on this HLT was defined by Stephan and colleagues (2014) as the route that the classroom is anticipated to travel throughout the engagement with the sequenced tasks. However, this anticipated trajectory is a collection of social negotiation but not actualized by the teacher and the classroom student (Stephan, 2015).

Proportion represents a mathematical statement in which two equal ratios convey the same constant factors relating to the two quantities (Cramer et al., 1993; Langrall & Swafford, 2000). The symbolic representation is demonstrated as $\frac{a}{b} = \frac{c}{d}$.

Proportional reasoning is being able to conduct argumentation and multiplicative comparisons between linked entities and having a constant relationship (Lamon, 2012, p. 3). Rational numbers, ratios, unit ratios, proportionality, and multiplicative comparisons are parts of proportional reasoning. Proportional reasoning involves a group of skills about knowing exact words and symbols, scaling up and down, understanding and applying relationships between composite units in situations involving simple direct and inverse proportions, and differentiation between additive and multiplicative situations (Lamon, 2020; Lesh et al.,1988).

Ratio represents “an ordered pair that conveys the relative sizes of two quantities” (Lamon, 2020, p.31).

Ratio and Proportion Instructional Sequence developed by Stephan and colleagues (2015) in USA, consisting of various tasks including ratio, rate, proportion, and percent concepts. This instructional sequence involves students' activity sheets, teacher's guide, and hypothetical learning trajectory for proportional reasoning.

Rate represents rate as "a reflectively abstracted constant ratio" (Thompson, 1994, p. 191).

Taken as shared ideas describes the discrepancy between the individual's interpretations and leading to discussion as a new learning opportunity (Cobb et al., 1992b). They also stated, "Instead, the taken-as-shared basis for mathematical activity evolved as they each made adaptations which eliminated perceived discrepancies between their own and others' mathematical activity while pursuing their goals." (p.118). Toulmin's argumentation schema was found eligible to investigate the taken-as-shared forms of argumentation that constitute the mathematical practices (Cobb et al., 2011; Krummheuer, 1995, 2007).

CHAPTER 2

LITERATURE REVIEW

Since the aim of this study is to document the classroom mathematical practices of the seventh graders, this chapter proposes the literature underlying this scene. The headings are ordered from general to specific issues in the case of ratio and proportion teaching and learning. As a required mathematical skill, proportional reasoning concepts and their development in learners are explained in order to understand the classroom mathematical practices on the ratio and proportion concepts. This is the primary purpose of the study. Due to its dual connection with teaching, current practices related to teaching ratio and proportion are also explained.

2.1. Proportional Reasoning and Fundamental Concepts

Proportional reasoning is mathematical reasoning involving a sense of covariation and the ability to make multiple comparisons in relative terms (Fielding-Wells et al., 2014; Post et al., 1988). Lamon (2020) highlighted that proportional reasoning is an *umbrella term* more than rational numbers and its contexts. The reasoning is being able to conduct argumentation and multiplicative comparisons between entities which are linked and have a constant relationship (Lamon, 2012, p. 3). Rational numbers, ratio, unit rates, proportionality and multiplicative comparisons are parts of proportional reasoning. Proportional reasoning involves a group of skills about knowing exact words and symbols, scaling up and down, understanding and applying relationships between composite units in situations involving simple direct proportions and inverse proportions and differentiation between additive and multiplicative situations (Lamon, 2020; Lesh et al., 1988).

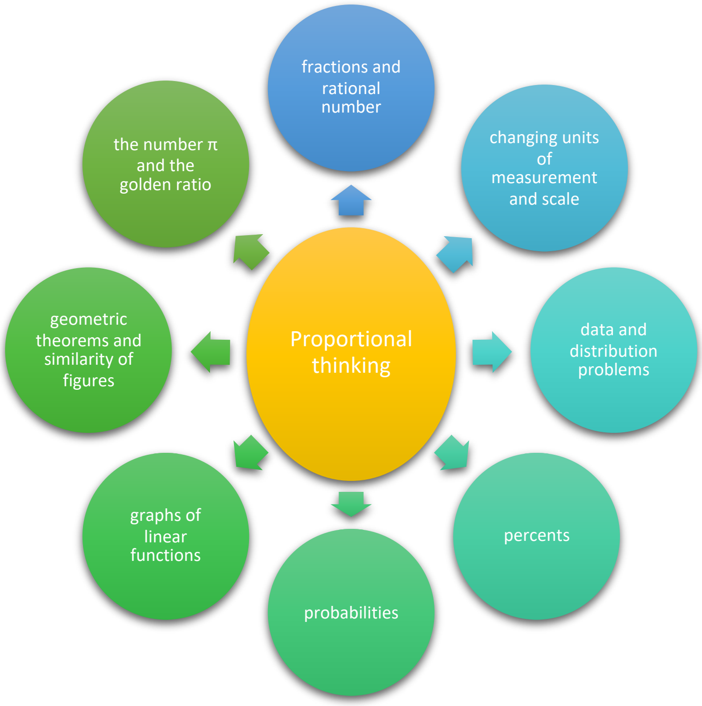


Figure 2. 1. Proportional reasoning between disciplines and mathematical topics

Figure 2.1 represents the relationship of proportional reasoning with other disciplines and mathematical topics, though its utility cannot be limited to this representation. Behr and colleagues (1992) investigated proportional reasoning within the context of the different meanings of fractions: part-whole, divisor, ratio, and probability. They emphasized that students could develop proportional reasoning with the help of instructional strategies that include students' qualitative reasoning, and enlisted the deficiencies related to the multiplicative structure of the proportion, consistent with the study of Tournaire and Pulos (1985). The point Behr and colleagues (1992) highlighted about the relationships among proportionality, fractions, and rational numbers was further investigated by other studies (Davis, 2003; Lamon, 2007; Nabors, 2003). The common claim of Nabor (2003) and Davis (2003) was that by dealing with fractional tasks it was possible to develop proportional reasoning. In her comprehensive study, Lamon revisited this idea and added to her hypothesis that "… However, understanding the larger concept of proportionality comes about later, through interaction with mathematical and scientific systems that involve the invariance of a ratio or of a product." (2007, p. 640). It helps to differentiate between the different meanings of fractions especially with part-to-whole and ratio (van de Walle et al., 2016). Thus, the development of proportional reasoning is required for the development of knowledge and skills in other mathematical and interdisciplinary studies.

On the other hand, within the core of proportional reasoning there lie a variety of concepts and skills that are crucial for the development of proportional reasoning. These were summarized by Lamon (2020) in a web-like diagram to demonstrate the complexity and relationships between the ways of thinking.

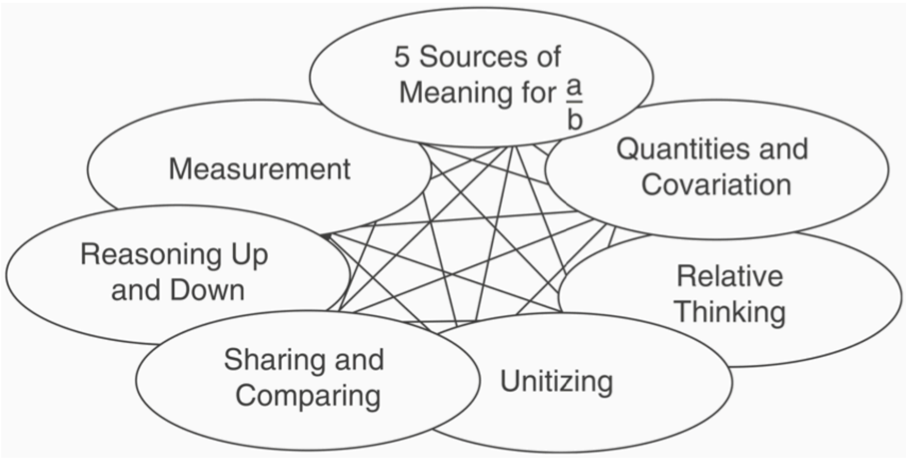


*Figure 2. 2.*Thinking skills diagram defined by Lamon (2020, p. 10)

This web showed that our thinking about proportional reasoning may go reciprocally and interrelatedly between these constructs of knowledge and skills, which shows nonlinearity of thinking, and it took many years to advance them (Lamon, 2020). The concepts within them lead us to master each way of thinking step by step.

Proportional reasoning is beyond doing multiplication and applying mathematical procedures. It includes a variety of concepts to be a successful proportional thinker. One of them is about seeing an *invariant structure* in the problem situation and transforming it accordingly by the *covariation* of the two linked quantities given as the ratio. In other words, invariant relationship refers to the relationship between the two measure spaces remaining the same, on the other hand, covariation refers to the variation of the two measures in the same ratio (Lamon, 2020). Harel et al. related invariance of ratio with taste and tried to focus on "…the taste of a mixture will not vary when the volume of the mixture varies" (1994, p. 341).

Invariance of a ratio may be sustained under ideal conditions, and this could be discussed also during the instruction and not taken for granted (Harel et al., 1994).

Units and the unitizing process are one of the crucial elements of proportional reasoning (Behr et al., 1992; Confrey, 1994; Lamon, 1994, 1996, 2002, 2020). Children start unitizing with their fingers and the base ten system of numbers. As they develop further, *unitizing and norming* is required for them to develop multiplicative strategies, which is "the ability to construct a reference unit or a unit whole, and then to interpret a situation in terms of that unit." (Lamon, 1994, p. 94). It is an ability to consider quantities in distinct groups (Lamon, 2002). Norming is an expression similar to unitizing, considering "the reconceptualization of a system in relation to some fixed unit or standard" (Steffe & Cobb, 1988). For example, it requires construction of units as a chunk of units 2 (15 eggs-box) or 3 (5 alien-group). Lamon (1996) investigated unitizing processes of children from the fourth to sixth grade level through a cross-sectional study, in which students' partitioning strategies were developed into a more complex level. Lamon developed didactic strategies for children to improve their sophisticated thinking of units. Through unitizing, a child can preserve relationships in iteration and create units of units (composite units) until the child sees a ratio as the composite unit. Composite unit represents units of units considering two related entities as a single entity (Kilpatrick et al., 2001). For example, in this study, "three aliens eat one food bar" becomes "three aliens per food bar" unit. The ratio representation of this composite unit is 3:1. Within the unitizing process, students turn singleton units into composite units. From this aspect, unitizing by forming composite units and iterating them may be a foundation for transition from addition to multiplication (Singh, 2000).

Another issue, the tasks related to proportional reasoning requiring children's intuitive knowledge of *relative thinking* as *qualitative reasoning principles* for proportion situations (Behr et al., 1992). They explained:

> Qualitative reasoning in a proportion situation then involves qualitative reasoning about a rational number or a ratio a/b or a product a·b. Of concern is the qualitative question of the direction of change, or no change in a/b or a·b as a result of combinations of qualitative changes of increase, decrease, or no change in a and b. (Behr et al., 1992, p. 299).

Relative thinking is called a bridge between additive and multiplicative thinking both qualitative and quantitative proportional reasoning, which requires understanding of multiplicative change (Lamon, 1993). Qualitative reasoning situations involve the questions "if a or b changes, how does b or a change respectively?" expecting the response of "increases, decreases, or stays the same”. This thinking gives information about the problem solver related to whether they can determine the direction of the increase, decrease or no direction and there is no need for numerical values (Cramer & Post, 1993a). Freudenthal provided various examples of qualitative reasoning that a kindergartener may confront, including “tasting sweeter chocolate”, “a flea jumping higher than a man”, “the further the distance is the more expensive the flight is" (2002, p. 194) and this could become more sophisticated with the words “much, very much, a bit” in addition to more and less (Freudenthal, 2002). Furthermore, more sophisticated thinking resulted in quantitative thinking that describes the relationship between the components a and b (Chi & Glaser, 1982). This implied that qualitative reasoning may correlate with the future application of quantitative proportional reasoning (Lamon, 1993). Qualitative reasoning questions the multiplicative relationships so that the situation is invariant or covariant (Lamon, 2007). Heller and colleagues (1990) integrated qualitative reasoning (increasing, decreasing, stays same) for the indirect development of proportional reasoning in word problems. Qualitative problems should be included with missing value and comparison problems to develop proportional reasoning (Cramer & Post, 1993b; Heller et al., 1990; Noelting, 1980a).

2.1.1. Ratio Concept

Defining the ratio concept has been quite complicated due to the field that the concept defined: science or mathematics (Lamon, 2007). From the scientist’s perspective, Freudenthal (1973, 2002) defined ratio as *external* or *between* and *internal* or *within* systems in terms of the component of ratio being in scientifically different “measure” spaces such as volume and mass for density. Measure spaces were explained as “different sets of objects, different types of quantities, or different units of measure” (Lamon, 2007, p.634).

According to Lamon (2007), considering the food bar and alien relationship (One food bar feeds 3 aliens, 4 food bars feed 12 aliens) could also represent the within or between relationship although the measure spaces were not defined in science as Freudenthal (2002) assumed.

1 food bar : 4 food bars = 3 aliens : 12 aliens (internal or within)

1 food bar : 3 aliens = 4 food bars : 12 aliens (external or between)

Although the researchers focused on different aspects to define ratio such as its symbolic representation (Jackson & Phillips, 1983) or required skill (Ben-Chaim et al., 2012); a comprehensive description of ratio is “…a number that relates two quantities or measures within a given situation in a multiplicative relationship (in contrast to a difference or additive relationship)” (van de Walle et al., 2016, p. 31). A ratio equipped with two entities interrelatedly and multiplicatively represents a single entity even if these entities may be quantified in different units. Symbolic representation of ratio can differ: a:b (colon notation), a/b (division notation), or ab (fraction notation) (Billstein et al., 2016). Its verbal representation may change into “a to b”, “a for b”, “a per b”, or “for every b there are a” (Lamon, 2012).

Table 2. 1

Various Definitions of the Ratio Concept

Definitions of Ratio	**Source**
“Ratio is a function of an ordered pair of numbers or magnitude values.”	Freudenthal, 2002, p. 179
“the speed, shape, or other characteristic whose specification leads to a constant ratio relationship is called a 'rate' or *intensive* variable, to distinguish it from extensive variables such as the length, time, weight, or other quantitative description of the extent of an object or event”	Karplus et al., 1983, p.1983
“ $\frac{a}{b}$ or a:b”	Jackson & Phillips, 1983, p.338
“The notion of ratio is that of comparison of two quantities”	Behr et al., 1992, p. 298
“Ratio is a comparative index that conveys the abstract notion of relative magnitude.”	Lamon, 1995, p.169
“A ratio describes an underlying relationship among a set of proportions; it is an expression of how a comparison between numbers can “stay the same” while the individual numbers change.”	Confrey & Carrejo, 2005, n.p.
“Ratio is the quantification of a multiplicative relationship that is calculated by dividing (or multiplying) one quantity by another. The multiplicative quantifier is determined by dividing (or multiplying) two magnitudes.”	Ben-Chaim et al., 2012, p.25
“A ratio is a number that relates two quantities or measures within a given situation in a multiplicative relationship (in contrast to a difference or additive relationship).”	van de Walle et al., 2016, p. 18
“A ratio, denoted as a/b, $\frac{a}{b}$, or a:b, where a and b are rational numbers, is a comparison of two quantities.”	Billstein et al., 2016, n.p.
“A ratio is an ordered pair that conveys the relative sizes of two quantities.”	Lamon, 2020, p.31

Different representations and their common points with fractions may confuse students’ understanding of ratio (Lamon, 2007) due to the perspective that fraction has regarded the starting point for ratio (Streefland, 1991). Students who perceive a ratio as a fraction tended to reduce the ratio to lower terms and then use an additive strategy, adding (or subtracting) the numerator of the reduced ratio to the original numerator of the ratio and the reduced denominator to the original denominator to find other equivalent fractions. These students consistently described the ratios as fractions, using words like "It is twenty-four fortieths" and "It is three-fifths." (Middleton & van den Heuvel-Panhuizen, 1995). This leads students into a wrong imagery and part-whole relationship between the numbers.

Fractions represent one of the meanings of ratio. A ratio can be used to represent part-whole relationships in that the ratio concept is introduced as a part-whole fraction imagery in which the two entities are identical and represent a part or subset of a unit or a set (Kennedy et al., 2008). A ratio can represent the comparison of the parts of a set or a unit, which is called part-part relationship and does not represent a fraction. A fraction can be considered a ratio in the part-whole meaning that fractions represent, however this relationship is not reciprocal (Lamon, 2020). To make the distinction between two concepts, Lamon (2020) followed a strategy about notations: using a part-whole relationship for fraction and using other meanings' colon notation. Another meaning for ratio comes from understanding the relationship between each entity and can be understood as "per" quantities (Lamon, 2020) or "quotients" (van de Walle et al., 2016). Further, Confrey and Carrejo (2005) claimed that defining ratio based on division or multiplication may lead students to make incorrect deductions about the ratio concept. They suggested that proportions are the generators of ratio through focusing on the invariant characteristics of a ratio within a proportional situation (Confrey & Carrejo, 2005).

2.1.2. Rate v. Ratio Concept

Rate is another concept directly related to proportional reasoning. According to Thompson (1994), rate and ratio concepts are confused in every field of education, not only in schools. The ratio as rate meaning is more complicated than the other meanings because the literature has different explanations for rate. The difference between ratio and rate was originated from the comparison of quantities that belong to the same or different measure spaces (Vergnaud, 1988). Thompson (1994) claimed that ratio and rate represented the same entity, and differences come from the situations of the problem that they represented. When a very particular situation is considered, ratio is used. However, if that ratio is one that is extendible to a broader range of situations, it is a rate (Karplus et al., 1983). The *extensiveness* and *intensiveness* issues were considered by Kaput and West (1994). They explained the term "extensive quantities" in a problem statement consisting of numbers and referents, where the referent identifies the measure such as length or area. A "referent" can describe entity, situation, or event being considered (Kaput & West, 1994, p.239). They also added discrete units to a special case for measure. Another differentiation was made by Karplus et al. (1983) according to the attribute of the variable's *intensiveness* and *extensiveness*. Intensive variables such as speed or orange juice flavor (Noelting, 1980a, 1980b) represented a constant ratio differentiated from extensive variables such as width and length (Karplus et al., 1983). In that sense, Behr et al. added that intensive quantity, "a ratio of quantities from different measure spaces to which a common name, such as density, is applicable." (1992, p. 298); nevertheless, they explained rate in terms of referencing time such as speed. It is quite common to see conceptualizations of rate in terms of examples such as "speed as miles per hour" (see Table 2.2). Table 2.2 involves a list of definitions related to rate.

Table 2. 2

Various Definitions of the Rate Concept

Definition of Rate	Source
"the speed, shape, or other characteristic whose specification leads to a constant ratio relationship is called a 'rate' or intensive variable"	Karplus et al., 1983
"Rate is used to denote a comparison between elements in two different measure spaces (e.g., 3 laps/12 minutes)"	Heller, et al., 1990, p. 389
"rate is considered to be a ratio in which the reference quantity is time"	Behr et al., 1992, p. 298
"relatively abstracted constant ratio"	Thompson, 1994, p.192
"A rate is usually a ratio representing two different sets or measures so that one of the terms of the ratio is 1, such as miles per hour (1 hour), heartbeats per minute (1 minute), or $6.75 per pound (1 pound)."	Kennedy et al., 2008, p.338
"A rate is a ratio between two measurements with different units. Relationships between two units of measure are also rates—for example, inches per foot, milliliters per liter, and centimeters per inch."	van de Walle et al., 2016, p.454
"if that ratio is one that is extendible to a broader range of situations, it is a rate."	Lamon, 2020, p.31

Table 2.2 summarizes the changes in the definition of rate. Its definition was fictionalized according to the comparison with ratio as a tradition due to defining rate as more complex than ratio (Lamon, 2020). From the contemporary perspective, a rate can be considered as a quality indicator in that a combination of two distinct measures gives its meaning through the relationship, they constructed such as the cost of one lemon. Thompson (1994) discussed the

authorities' perspective related to the difference on rate and ratio and concluded that ratio and rate concepts were no longer related to its dimensions. Thompson also stated, "A rate as a reflectively abstracted constant ratio, symbolizes that structure as a whole, but gives prominence to the constancy of the result of the multiplicative comparison." (Thompson, 1994, p.192).

2.1.3. Proportion Concept

Similar to the ratio concept, proportion is also involved informally in the primary grades' curricula within the learning outcomes of problem solving, and the students started to learn ratio and proportion in middle grades conceptually and procedurally in middle grades. A proportion is constructed on a mathematical structure involving two or more multiplicatively linked equal ratios. The proportion may be constructed with the equality of two ratios including similar measure spaces (within state) or two distinct measure spaces (between-state) (Noelting, 1980b). Some definitions focus on the semantics of a proportion (see Table 2.2 Jackson & Phillips, 1983). According to Noelting (1980), if a and c are the quantities or measures from the same space, and b and d are the quantities or measures from the same space, then proportion can be expressed as $\frac{a}{c} = \frac{b}{d}$ or a:c = b:d (within state) in addition to $\frac{a}{b} = \frac{c}{d}$ or a:b=c:d (between state). Additionally, a proportion may involve more than two covaried ratios (Kennedy, 2008) provided that the ratio of one quantity to the other stays invariant (Lobato et al., 2010).

There are two proportional situations: direct and inverse. Table 2.2 describes the direct proportional situations qualitatively in that when one of the quantities increases/decreases in a ratio then the other quantity also increases/decreases to the same extent (multiplicatively). On the other hand, inversely proportional situations can be qualitatively described where when one of the quantities increases/decreases in a ratio then the other quantity decreases/increases to the same extent (multiplicatively). Ben-Chaim et al. (2012) provided the semantic structure of direct and indirect (inverse) proportions:

> "In mathematical notation, this means that four variables, a, b, c, and d (a ≠ 0, b ≠0, c ≠0, d ≠0) will form a proportional relation in the following two situations:
> 1. When a / b = c / d. This is direct proportion: the quotient of the two parts of the ratio, a and b, is constantly equal to that of c and d.
> 2. When a × b = c × d. This is indirect proportion: the product of the two parts of the ratio, a and b, is constantly equal to that of c and d." (2012, p. 34)

Table 2. 3

Various Definitions of the Proportion Concept

Definition of Proportion	Source
"The two different types of ratios making up a proportion: within-state ratios and between-state ratios. In a proportion, say a/b = c/d"	Noelting, 1980b, p.334
"Proportion as $\frac{a}{b} = \frac{c}{d}$ or a:b = c:d"	Jackson & Phillips, 1983, p.338
"a proportion is a statement of equality of two ratios, i.e., $\frac{a}{b} = \frac{c}{d}$"	Tournaire & Pulos, 1985, p.181
"A proportion is the statement that two ratios are equal in the sense that both convey the same relationship."	Langrall & Swafford, 2000, p. 255
"A proportion is an equality between two or more ratios, such as $\frac{2}{3} = \frac{4}{6}$."	Kennedy et al., 2008, p. 335
"When two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor."	Lobato et al., 2010, p. 11
"A proportion is a statement that two given ratios are equal."	Billstein et al., 2016, n.p.

As a notice, it should be added that a and c are in the same measure space and b and d are in the same measure space. Additionally, direct proportion may represent direct or indirect linear relationship between two variables. Furthermore, Karplus et al. (1983) described the direct proportion as a linear functional relationship such that y= mx, in which m represents the ratio or rate in the problem situation.

2.2. Reported Children's Strategies and Difficulties

The literature reports that children use several strategies to solve proportional reasoning tasks. Proportional reasoning is not only related with proportional situations but also related with non-proportional situations as well. Identification of the correct strategy to solve a proportional situation is an ability to be a proportional thinker (Lamon, 1993). There are some strategies students performed correctly and incorrectly but the difficulties of the students can be summarized as:

- Inability to distinguish between proportional and non-proportional situations (incorrect strategy use) (van Dooren et al., 2003)

- The tendency to use additive or absolute thinking in a multiplicative proportional situation (misuse) (Cramer & Post, 1993a; Lamon, 1993; Lesh et al., 1988; Misailidou & Williams 2003; Noelting, 1980b; Tourniaire & Pulos, 1985),
- The tendency to use multiplicative thinking in additive thinking situations (Van Dooren et al., 2010),
- Inappropriate utility of algorithms and mistakes in calculations (Lamon, 1993; Misailidou & Williams, 2003; Nabors, 2003).

2.2.1. Additive or Absolute Thinking

Additive (constant difference strategy) is called "default" strategy and this strategy may disappear after systematic instruction which includes appropriate build-up and multiplicative strategies (van Dooren et al., 2009). This strategy can be described as "the relationship within the ratios is computed by subtracting one term from another, and then the difference is applied to the second ratio" (Tournaire & Pulos, 1985, p. 186). Students may calculate the result additively in a missing-value problem, but on the other hand, the equal ratios may also create a wrong expectancy of additive thinking due to the same difference such as 4:6, which has a difference of 2, and 6:9 which has a difference of 3 (Abrahamson & Cigan, 2003). The students may conclude that 4:6 and 6:9 ratios are not proportional. Children who prefer additive comparisons rather than multiplicative may have some difficulty solving complicated problems which require using multiplicative problems (Boyer et al., 2008). Built on this, ratio, and proportion teaching focuses on students' differentiation of multiplicative and additive thinking. If there are nonproportional situations in the problem context such as the relationship between the length of a tree and age or the relationship between two people.

Tournaire and Pulos (1985) reviewed the literature about proportional reasoning about forty years ago by considering experimental studies after the 1950s. They categorized the students' errors and correct strategies and elaborated on the issues of what makes students confused while reasoning about proportionality. The list included students' use of the additive strategy as an error strategy or fall-back strategy to solve complicated problems consisting of non-integer ratios, unfamiliar order of the missing value, big size of the number, different units in the same question (external ratios), or the existence of a mixture problem (Tournaire & Pulos, 1985). Remarkably, Lamon (1993) categorized additive thinking as a nonconstructive strategy, and it is referred to as an erroneous strategy that is typical for younger students with little or no experience with the multiplicative relations in proportional situations (Abrahamson & Cigan, 2003).

There is, in other words, a tendency for students to approach proportional situations additively instead of multiplicatively (Cramer & Post, 1993a; Lamon, 1993; Lesh et al., 1988; Misailidou & Williams, 2003; Tournaire & Pulos, 1985). Moreover, students are in favor of additive approaches as default strategy because students are uncertain about the next steps to take, they typically rely on methods that involve adding things together (Kennedy et al., 2008).

2.2.2. Build-up strategies

Build-up strategies is an elementary approach involving repeated addition of quantities and is frequently visible during childhood and adolescence (Hart, 1981). Build-up strategies are regarded as informal proportional strategies in which reasoning patterns aid the solution of missing-value problems without employing cross-product algorithms (Kaput & West, 1994). These have been claimed to lead successful solutions in simple, non-integer ratios (Hart, 1984; Tournaire & Pulos, 1985). According to Kaput and West (1994) three types of informal proportional reasoning were available and two of them were related to build-up strategies: coordinated build-up/build-down processes and abbreviated build-up/build-down processes using multiplication. In the coordinated build-up/build-down processes, the number values were increased or decreased with double skip counting for each unit coordinately.

For seven silver, there is four china;
for fourteen silver, there is eight china;
for twenty-one silver, theren is twelve china;
for twenty-eight silver, there is sixteen china;
for thirty-five silver, there is twenty china.

Silver	7	14	21	28	35
China	4	8	12	16	20

Silver	China
7	4
14	8
21	12
28	16

Figure 2. 3. Examples of coordinated build-up processes (Kaput & West, 1994, p. 246)

As given in Figure 2.3, the coordinated build-up strategy focuses on students' repeated addition for all the units described in the task. On the other hand, there is a transition process from additive to multiplicative reasoning. Its impact can be seen in the build-up strategies as well. Students started to abbreviate their way of representation of composite numbers. Kaput and West described this process as being "[a]ssociated with the more efficient handling of incrementing-decrementing process using multiplication and division, there seems to be an abstraction of the unit-forming process away from the semantically organized referents and toward the pure numerical values of the quantities involved." (Kaput & West, 1994, p. 248). Students realized the relationship between fourteen silver and twenty-eight silver was the multiple of two, and in conclusion they abbreviated the strategy and multiplied eight by two as well. Both build-up processes include mini processes in terms of conceptualization and computation (Kaput & West, 1994). In the conceptualization process, children are required to present a comprehension of all the units and numerical values and their relationships. The difference between the coordinated and abbreviated build-up strategy emerges during the coordination part:

Table 2. 4

Coordinated and Abbreviated Build-up/down Processes (Kaput & West, 1994)

Coordinated build-up/down processes	Abbreviated build-up/down processes
- "Increment/decrement or skip counting until the third given quantity is reached… - Identification its corresponding element of the other quality as the problem's solution." (p. 247)	- "Divide the total given quantity by the quantity per unit to obtain the number of units. - Multiply the number of units by the corresponding quantity per unit to determine total unknown quantity." (p. 249)

Table 2.4 describes the differences between two processes. Additionally, there can be adjustments of the units and given number of materials in the abbreviated process. Children started to re-form units as illustrated below:

> For seven silver, there is four china
> For fourteen silver, there is eight china
> Four twenty-eight silver, there is sixteen china

Build-up strategies are considered transitional methods that lead to proportional reasoning (Ercole et al., 2011) but do not mean achievement of proportional reasoning (Lamon, 2005, p. 100). They may appear spontaneously during instruction independent of instruction (Hart, 1984) especially in solving missing-value problems (Lamon, 1993). Similarly, Smith (2002) argues that the correct use of build-up strategies to equal the situations in the second situation is still additive, not yet multiplicative.

It was also claimed that part-part-whole problems may basically result with the solution of build-up strategy or ratio table, which also may not be evidence of proportional reasoning even students are eligible to reason proportionally in other tasks (Lamon, 1993). The emphasis is put on the students' selection of the simplest method which satisfies the problem task's requirement therefore various problem tasks can be sequenced to elicit proportional reasoning efficiently instead of selecting one or two only (Lamon, 1993).

2.2.3. Multiplicative Strategies

As described in the web of proportional reasoning, each concept puts multiplicative thinking at the core of definitions. In the end, to be a competent proportional thinker, proportional reasoning requires multiplicative strategies. Students who cannot reason multiplicatively can be understood to be thinking at a concrete level. Teaching strategies that were conducted were applied in seventh grade in general since the students are supposed to be in their formal stage, and proportional reasoning develops during the formal stage (Inhelder & Piaget, 1958). In order to reason multiplicatively (proportionally) in the formal stage, they must reason at a more abstract, or operational level (see Kaput & West, 1994; Lamon, 1993; Langrall & Swafford, 2000; Nabor, 2003). Therefore, it is considered as the foundational thinking for ratio and proportion. It was defined as "… the ability to iterate abstract composite units" by Battista and van Auken Borrow (1995, p. 3) for developing ratio understanding. Because children do not comprehend that proportions indicate multiplication or multiplicative reasoning, they (and many adults) frequently struggle with proportional reasoning. Instead, they believe that many proportional situations show addition or additive relationships.

2.2.3.1. Cross-product algorithm

Cross-product algorithm is defined as the *syntactic manipulation* of formal algebraic equations (Kaput & West, 1994) and can also be called the traditional proportion algorithm (Cramer & Post. 1993a) or cross-multiplication method to be used to solve proportion problems (Lesh et al., 1988). It was also regarded as a frequently used formal instructional strategy provided in the textbooks and instructional designs with multiplicative missing value problems (Cramer & Post, 1993a; Kaput & West, 1994). From this perspective, although it looks like the whole picture, using a cross-product algorithm represents one of the procedural knowledge and skills. To set-up a proportion equation, one needs first to identify and distinguish the quantities involved. Then writing the equation within measure or between measures, these two pairs make up the two equal ratios, hence one must be able to match them for cross-multiplication.

According to Smith (2002), students avoid using cross-multiplication although it is instructed because conducting the procedural process does not match the process in build-up strategies and it lacks meaning without conceptual development, but it has an important role when solving proportion problems, especially when viewing the relationships between the non-integer numbers. Smith and colleagues noted that "Once solid conceptions of proportionality have been developed, cross multiplication can be introduced as an efficient algorithm for solving any missing-value proportion problems - especially those for which the numbers in the problem make intuitive strategies difficult or cumbersome to apply" (Smith et al., 2002, p. 150). This strategy was offered after the discovery of students with several tools. For example, Abrahamson and Cigan (2003) declared that the proportion quartet in a table is promising to support students' recognition of the diagonal relationship among four quantities so the cross-product algorithm. Before that the students need to study the multiplicative number relationship among the quantities.

Furthermore, one of the studies conducted by Cramer and Post (1993a) revealed that preservice teachers generally tend to use cross-product algorithms for additive situations which do not involve proportional understanding. They claimed that this issue may be the result of a superficial understanding of proportion and overuse of memorized rules. In a missing-value problem, many students use this method to find the fourth value in the problem but just a few may differentiate the proportional and nonproportional situations (Cramer & Post, 1993a). In order to prevent this strategy from being primarily perceived as a rote operation, teachers must make connections between the traditional algorithm of cross multiplication and the prior knowledge of their students. In order to prevent this strategy from being primarily perceived as a rote operation, teachers must make connections between the traditional algorithm of cross multiplication and the prior knowledge of their students (Ercole et al., 2011). Moreover, as reported by van Dooren and colleagues (2009), students overuse proportionality due to school and everyday experience, and these researchers link between a deficient understanding of mathematical concepts and proportionality, recalling previous experiences about the problems.

2.2.3.2. Within and Between Strategies

Within and between strategies are used in proportional situations, which differ in terms of different measure spaces. A proportion can be expressed within or between state ratios (a:c = b:d or a:b = c:d on the condition that a and c are within the same measure space, b and d are within the same measure space). Based on this, there are two perspectives for within and between strategies. Tournaire and Pulos (1985) reported two strategies as the correct ones. In multiplicative strategies, the developed constant relationship within one ratio is extended to the other ratio. These multiplicative strategies can occur within the numerator and denominator of a ratio and can also occur between the numerators or denominators between two ratios. The first perspective focused on the relationship within a ratio so that within the missing value problems two distinct measure spaces are organized within the same ratio (a:c = b:d). The assumption is that there is a multiplicative relationship within two quantities or measures in the same ratio, which is called the within ratio strategy (Lamon, 1994). In a proportion quartet, this constant relationship within one of the ratios is preserved and transferred to the other ratio. On the other hand, the between ratios strategy is described within the frame of the perspective that the multiplicative relationship between the two ratios in a proportion is extracted from the quantities of the same measures (relationship between a and b or c and d in terms of a:c = b:d).

Apart from Lamon's claim (1994) in a proportion quartet, there can be a relationship between the two ratios of the quantities of the same measures, which is also called the scalar method. There is no dimension in the scalar method (Vergnaud, 1994). The within strategy involves a division operation with the quantity or measure in the same measure system provided that the proportion type (a:c = b:d) involves two quotients where a division is first conducted inside the each state (i.e measure space) (Noelting, 1980b). The second perspective considered the measure spaces. When the two kinds of ratios are compared, the quantities in the ratio can be in the same nature such that both of them are countable such as the number of trees, the number of people, or an amount of dough, an operation referred to as the scalar method. On the other hand, when the nature of the quantities changes in the same ratio, the ratios are called external, and the method is external (Tournaire & Pulos, 1985). To summarize, these relationships, magnifying the scale factors within the same measure space, are referred to as scalar ratios (Lamon, 1994; Lesh et al., 1988; Vergnaud, 1994), internal ratios (Freudenthal, 1973), within measures ratios (Lamon, 1994), or between ratios (Karplus et al., 1983; Noelting, 1980a, 1980b).

Alternately, the link between the quantity of the first measure space and the second measure space expressed with a multiplicative relationship is called the between-state ratio. The concept of proportion itself can be thought as a variation among its equivalence class by means of is *a multiplication of a within-state ratio* by a *between-state operator* (Lamon & Carrajo, 2005). It is thus the combination of both types of ratios. What establishes a multiplicative relationship between a and b in a:b is not finding how many bs are in a or vice versa, but rather the act of preserving the a:b relationship by acting multiplicatively on both the quantities (Lamon & Carrajo, 2005). As a summary, multiplicative comparison of different measure spaces are called external ratios (Freudenthal, 1973), between measures ratios (Lamon, 1994), functional relationships/rates (Lamon, 1994; Lesh et al., 1988; Vergnaud, 1994), across-measure spaces (Nabors, 2003), or within ratios (Karplus et al., 1983; Noelting, 1980a, 1980b).

In either perspective, the emphasis is put on the factor of change and its constancy. This factor can be referred to as vertical or horizontal (Stephan et al., 2015), but the point is whether it is within the same measure space or between distinct measure spaces. Whether students use vertical or horizontal may change according to the integer and non-integer aspect of the numerical values. Vergnaud expressed this with a question: consumption of flour is on average 3.5 kg per week for ten people for require floor 50 people (1994, p. 49) and double proportion table in that two relationships were demonstrated on the table.

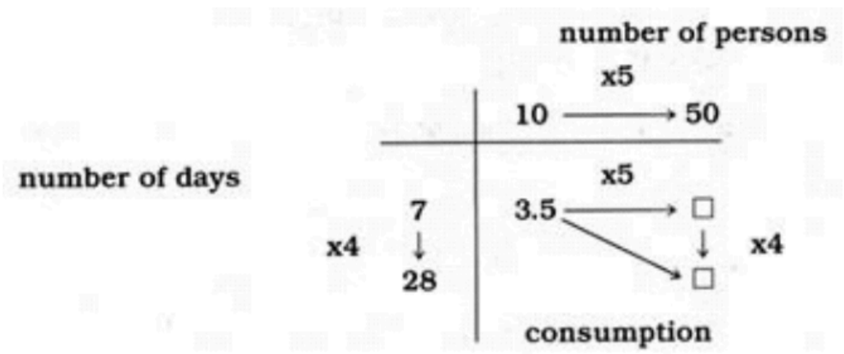


Figure 2. 4. Double proportion example showing scalar and functional relationship

The second concept of ratio will be called the ratio relationship concept. Students who perceived the ratio as a relationship between two specific numbers - here, a part-whole relationship - did not necessarily reduce the ratio to lower terms. This means that students generated equivalent ratios by multiplying the numerator and denominator of the ratio by a common factor. They understood that when multiplying by a common factor, the relationship between the two numbers remained constant - that 24/40 is the same ratio as 48/80, a transformation by a factor of 2, for example. They had great difficulty, however, in finding "in between" instances wherein the multiplying factor was not a whole number - that 24/40 is the same ratio as 27/45, a transformation by a factor of 1.125, for example. relating factors in a proportion in a qualitative manner, rather than applying quantitative strategies (Lesh et al., 1988); inappropriate use of algorithms, such as cross multiplication (Lesh et al., 1988; Nabors, 2003); and incorrect build-up/pattern building (Lamon, 1993; Misailidou & Williams, 2003).

2.2.3.3. Unit rate strategy

What establishes a multiplicative relationship between a and b in a:b is not finding how many bs are in a or vice versa, but rather the act of preserving the a:b relationship by acting multiplicatively on both the quantities. Pursuing these transformations until one of the two quantities equals 1 produces a unit ratio (Lamon & Carrajo, 2005). In other words, unit rate strategy is to find "how many for one?" (Cramer & Post, 1993b; Ercole et al., 2011). This strategy is also called unit factor strategy (Kaput & West, 1994). From the perspective of the linear relationship, in the function rule y = mx, m represents not only the constant factor but also the unit rate. Although it is a useful strategy for unitizing, the unit ratio strategy may not be handy for the students. Lamon (2002) explained this issue based on pricing comparison in the supermarket, the cost of one kg of a detergent or one unit in general. The cost in the supermarket is not whole numbers, there are decimals, and conducting long division is not to be reasonably expected. Even if it is done, the result may possibly be wrong (Lamon, 2002).

Unit rate strategy is also related with partitive and quotative division problems. Partitive division problems require the number of objects in one of the partitioned small groups within the total number of the whole group; whilst the quotative division is called subtractive division, wherein the number of objects is repeatedly subtracted to reach the target number (Lamon, 2020). Quotative division reflects the measurement each group gets, whereas partitive division reflects the equal sharing issue and is related to the rate concept (Lamon, 2020). In the teaching experiment study of Lo and Watanabe (1997), they demonstrated the capabilities of a fifth-grade student, Bruce, in proportional reasoning tasks, revealing that Bruce had difficulty transitioning from quotative to partitive division strategy. Based on this, it may be concluded that students' strategies for the calculation may differ, and that this may create difficulty in their understanding.

2.3. Development of Proportional Reasoning and Its Assessment

Although the mystery of the brain and its effect on human thought remains unsolved, neuroscientific studies continue to enlighten educational issues within the field of mathematics. During the twentieth century, the development of human thought was the subject of genetic epistemology, the study area of Jean Piaget, which explains the origins of knowledge and its place in the cognitive development of people. He focuses on learning as a product of cognitive activities of the individual and this cognition can only be developed by its owner. While focusing on cognitive capabilities, Piaget's clinical interviews and observations revealed that children achieved and failed to do several mathematical tasks at certain ages. Throughout empirical investigations with Inhelder, Piaget concluded several tasks related to proportional reasoning in the last stage of cognitive development, the formal operational stage (11 years-old and onward). Inhelder and Piaget (1958) described proportions and proportionality issues (a few "ratio" as a word) within mathematics and sciences, which were the concepts that a child in the formal operational stage could achieve successfully. Based on their claims, studies were conducted to understand the validity of the stages in terms of proportional reasoning (teaching of mathematics has still been influenced by this perspective). Lovell (1966) continued studying Piaget's stages and found that the stages are nearly consistent under circumstances undefined in their reports (Inhelder & Piaget, 1958). On the other hand, Shayer et al. revealed that their students' capabilities between 9–14-year-old children did not match the capabilities described in the preformal and formal operational stages (1976). One-fifth of the children were reported to reflect formal operational thought. Does this mean that even the middle grades of mathematics are not suitable for proportional thinking? The researchers started to look beyond the age and stage relationship and instead focus on the process skills and operations a human can achieve (i.e., Domingo et al., 2021), which were described by Inhelder and Piaget (1958). In other words, studies gathered around competence or mastery in proportional reasoning and evaluation of the extent to which learners are proportional thinkers in terms of framing knowledge and skills. Even young learners can perform several skills (Lamon, 1993) as opposed to the claims of Inhelder and Piaget's study (1958).

Noelting (1980a, 1980b) investigated stages in the development of proportional reasoning in terms of ratio comparison problems, matching them with Piaget's cognitive development stages. He used an orange juice experiment comparing mixtures of varying numbers of glasses of orange juice and water and increased the complexity of the problems in each stage. The criterion for attaining a stage was given as success or failure at distinct items and stages were differentiated through non-parametric tests. For example, the capabilities of a student in the preconceptual stage was to identify the elements in the figures; in the intuitive (preoperational) level it was to use strategies for comparison of terms; in the concrete operational level it was to use strategies related to joint multiplication or division of terms and yielding equivalence classes; in the formal operational stage it was to use strategies related to operation on operation after previous equivalences were reconstructed. Noelting (1980a) concluded that each stage was reconstructed on the former strategy used in the previous stage by the participants and developmental stages involved qualitative and quantitative changes. Noelting's descriptions of the stages included them being filled with specific performances of the students. Distinctively, Harel and colleagues (1994) studied the notion of the constancy of taste as the intensive quantity in mixture problems. They used Piaget's constancy of taste approach (Piaget & Inhelder, 1967) to understand students' rate schema in terms of taste and concluded that an apparent understanding of "taste" cannot be taken for granted in terms of its invariant structure based on the multiplicative change in the volume of the mixture (Harel et al., 1994). In the meantime, Thompson and Thompson (1994) reported on the teaching experiment investigating speed as rate (epigenesis of speed, Piaget & Duckworth, 1970), concluding that the shortages of language constrained by arithmetical operations, numbers, and procedures express a new concept, one that is nontechnical. Thompson and Thompson (1996) continued to study with teachers and concluded that teachers may struggle teaching conceptual constructs to their students even if they are capable of proportional reasoning. Although the Piagetian conceptualization of proportional thinking has influenced the mathematics education field, these studies can be summarized as concluding that proportional reasoning cannot develop naturally and requires special instructional strategies to advance (Sowder et al., 1998).

The studies have been turned around the capabilities of proportional thinkers in several stages in contrast to age and stages relationship. These capabilities change according to time and experience. Unlike Noelting (1980a, 1980b), Cramer and Post (1993a) described the capabilities of a proportional thinker as a whole to make a person competent in proportional reasoning with a special focus on knowing and applying mathematical characteristics, solving strategies of proportional and nonproportional situations, and solving quantitative and qualitative tasks. Although they directed the focus on proportional/nonproportional and qualitative/quantitative situations, the need to empirically consider the capabilities of the students to describe mastery in proportional reasoning continued.

Similarly, Lamon (1993) started to conceptualize proportional reasoning in terms of what *a child can and cannot do* with a study considering a hypothesis on problem types and students' level of sophistication for proportional reasoning. In the level of sophistication two domains were described: nonconstructive (additive thinking and lack of mathematical arguments) and constructive (mathematical arguments from pictorial to symbolic). She concluded that two issues were essential for the transition from a nonproportional to a proportional thinker: *relative thinking and unitizing*.

Table 2
Sixth-Grade Students' Strategies for Solving Ratio and Proportion Problems

Strategies	Characteristics
	Nonconstructive strategies
Avoiding	No serious interaction with the problem
Visual or additive	Trial and error or Responses without reasons or Purely visual judgments ("It looks like....") or Incorrect additive approaches
Pattern building	Use of oral or written patterns without understanding numerical relationships
	Constructive strategies
Preproportional reasoning	Intuitive, sense-making activities (pictures, charts, modeling, manipulating) and Use of some relative thinking
Qualitative proportional reasoning	Use of ratio as a unit and Use of relative thinking and Understanding of some numerical relationships
Quantitative proportional reasoning	Use of algebraic symbols to represent proportions with full understanding of functional and scalar relationships

Figure 2. 5. Level of sophistication in student thinking and strategies (Lamon, 1993, p. 46)

Until students develop the ability to think relatively in multiple contexts, they consistently struggle to attain a qualitative understanding of proportional reasoning within each semantic category. This is likely because students cannot grasp the functional and scalar connections inherent in proportions without first recognizing the multiplicative nature of situations involving ratios and proportions. Therefore, it is crucial for students to appreciate the importance of making relative comparisons in the problem situations presented to them, as these situations can serve as reference points for understanding proportion.

Based on Lamon's study (1993), Langrall and Swafford (2000) remodeled her level of sophistication and categorized the capabilities of the children into four levels (Level 0-3). While Lamon grouped trial-error strategies or random guessing into nonconstructive strategies context, Langrall and Swafford included them in the Level 0 from which students use additive thinking and no proportional reasoning. Students at Level 1 use some concrete materials such as pictures and models. Additionally, they detailed "understanding of numerical relationship" (Lamon, 1993) with build-up strategies and put qualitative thinking into Level 1 (informal reasoning). Level 2 thinkers develop utility with materials or models and use some numeric calculations. Level 3 formal proportional reasoner is capable of constructing a proportional quartet, using procedural strategies (cross-product algorithm), and demonstrating invariant and covariant relationships (2000).

Understanding students' cognitive strategies while solving ratio and proportion tasks demonstrates a way for reasoning to describe multiplicative structures. Kaput and West (1994) also proposed a three-level description of proportional reasoning involving coordinated build-up/down processes (Level 1), abbreviated build-down processes (Level 2) and lastly unit factor approaches (Level 3). While transitioning from Level 1 to Level 3, the operations conducted become more sophisticated and require learners to conduct multiplicative actions. Based on Kaput and West's (1994) framework, Nabor (2003) advanced these levels with the addition of cognitive schemes of operation. All these studies have shown that even if there are qualitative differences between these conceptualizations of proportional reasoning, it can be concluded that the influence of age level and experience cannot be avoided to evaluate the proportional reasoning capability of a pupil.

2.4. Teaching Ratio and Proportion Concepts

Besides individual development, the development of proportional reasoning may be enhanced through the combination of several strategies. Students' experiences were reflected and collected throughout different curricular and instructional designs (Battista & van Auken Borrow, 1995; Ben-Chaim et al., 1998; Szetela, 1970). The journey to better teaching of the ratio and proportion concepts covers the utility of tools (Szetela, 1970), utility of new instructional methods (Jackson & Phillips, 1983; Jitendra et al., 2009; Şen et al., 2021), specific focus of a concept in proportional reasoning (Ferrandez et al., 2012; Thompson, 1994), integration of the technology (Abrahamson & Cigan, 2003; Adjiage & Pluvinage, 2007; Tajika, 1998) and utility of all formerly mentioned components in one design (Ayan-Civak, 2020; Battista & van Auken Borrow, 1995).

2.4.1. Enhancement of the Instructional Materials

There are instructional strategies reporting the importance of learning environments for the development of proportional reasoning. The focus might be either one component of proportional reasoning or development of a framework for teaching proportional reasoning holistically.

Heller and colleagues (1990) investigated qualitative reasoning about rates and the extent to which students use their rational number skills to solve problems. Integrating qualitative reasoning (increasing, decreasing, stays same) for the indirect development of proportional reasoning in word problems was suggested for further studies. On the other hand, Lawton (1993) took attention to contextual issues given in word problems with an implication that the physical representation of the figures might affect students' relative thinking of ratios and proportion. Ferrandez-Reinisch (1985) focused only on inverse proportional situations and designed a training experiment accompanied by within-variable quantification. Pre-test and post-test results have shown the expected effect on the development of students' quantification process. Lachance and Confrey (2001) examined the effectiveness of an instructional program designed based on decimal instruction considering the assumption that fractions, percents, and decimals can be explored from the broad conceptual field of ratio in terms of multiplicative thinking. Distinctively, they concluded that extensive work on the multiplicative field within the context of ratio may be helpful for the learning of other relevant fields such as decimals (Lachance & Confrey, 2001). Correspondingly, Lehrer and colleagues (2001) described a set of design experiments that took place in primary school classrooms. The researchers provided initial assistance to students in constructing mathematical models that utilized concepts of ratio and similarity. Once these models were established, the students were able to further explore and examine various materials' weight, volume, and density. Ratio concept at the macro level has been used to construct understanding of other topics.

Ben-Chaim et al. (1998) conducted an experimental study with seventh grade students (traditional group and inquiry-based intervention group) to teach proportions and rational numbers. Students in the intervention group constructed their sense-making tools and explanations. Using collaboration to develop an understanding of the rate problems and integrating inquiry-based learning also reported as significant to the students related with improvement in their reasoning skills (Ben-Chaim et al., 1998). Ben-Chaim and colleagues (1998) studied the hypothesis that a collaborative learning environment designed by rate and density problems would create an effect on students' achievement in proportional reasoning. With the collection of research knowledge, Ben-Chaim and colleagues (2012) developed a theory-based model on the method of teaching ratio and proportion. Their model targets the required knowledge for a teacher to teach ratio and proportion concepts, and it contains four components (see Figure 2.6). Authentic activities based on realistic situations are to be presented in the core of the model considering rate, ratio, scaling, and indirect proportions. The aim of the core in Figure is to prepare students for relevant concepts of proportional reasoning based on students' prior knowledge. There are also three supporting units which form the instructional principles. Ben Chaim and colleagues (2012) reported that this teaching model is helpful for preservice and in-service teachers to understand the knowledge and practice so that they can solve problems in the diagnostic test successfully. To this end they suggest several other strategies and sophisticated explanations to solve problems.

It is important to provide teachers guidelines regarding effective strategies for arranging corridors which does not aim to establish a fixed curriculum sequence, but rather to create intellectual areas that facilitate students' progress (Confrey, 2006). In this aspect, there are also several studies focusing on the combination of various research-based reports in the benefits of development of proportional reasoning. Battista and van Auken Borrow (1995) reported the transitions of the students from episode to episode while dealing with ratio and proportion problems considering their construction of meanings to the composite units. While developing higher skills of proportional reasoning they abstracted knowledge related to fractions and ratio. Additionally, there is a sequence to develop ratio and proportional reasoning (Battista & van Auken Borrow, 1995). An experimental study with seventh grade students in American schools was conducted to investigate the difference between two different curricular designs in terms of proportional thinking by Ben-Chaim and colleagues (1998). The Connected Mathematics Project classroom outperformed the traditional classroom, in which the ratio and proportion were integrated with the topic rational numbers and collaborative problem-solving environments and reported little building up strategies from students, unlike what Tournaire and Pulos (1985) observed. The steps to develop build-up strategy were provided by Nabors (2003) in designed teaching episodes. Nabors (2003) conceptualized proportion, rate and ratio with fractional reasoning. One of the findings has showed the effect of the teacher while solving the problems in the direction of discouraging the use of algorithms (Nabor, 2003).

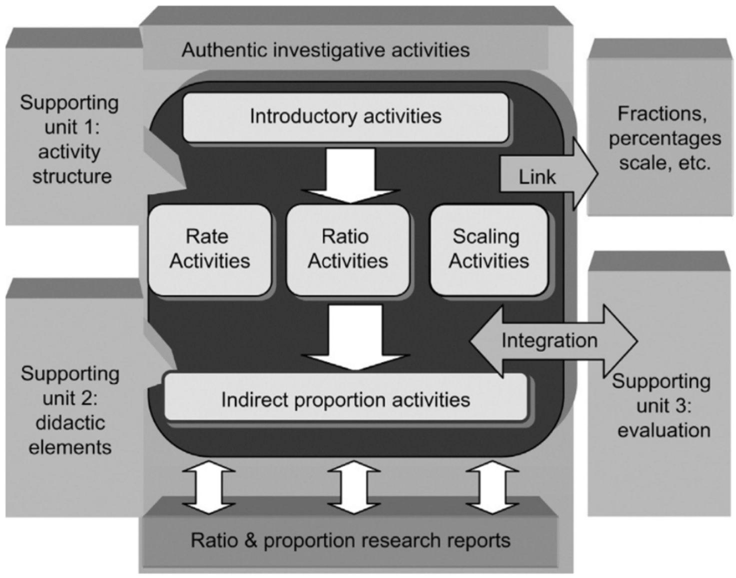


Figure 2. 6. A Model for Teaching Ratio and Proportion (Ben Chaim et al. 2012)

In terms of proactively teaching, Bowers and Nickerson (2001) studied prospective mathematics teachers and found that in order to establish an effective interaction with the students, instead of asking conceptual explanations, it is better for teachers to expect from students to explain what they saw or experienced in that particular activity since they may not have the formal language to give conceptual explanations. Steffe (1994) expected the teaching experiments would provide a learning environment that involved mathematical interactions. In brief, her perspective focuses on students' prior knowledge being constructed through mathematical interactions.

To summarize, the trend in the literature is to use some predetermined teaching episodes in order to understand the teaching and learning of rate, ratio, and proportion within a collaborative, active-learning, problem-solving environment. The aforementioned studies integrated partial or combined models of beneficial research elements in their research designs. The trends in educational design involving research-based beneficial elements of teaching and learning provided a description of a rich learning environment. To encourage students to think about relationships and develop more efficient problem-solving skills, it is advisable to use diverse types of questions, numerical structures, and quantities. By incorporating a variety of challenges, students will be prompted to move beyond their familiar strategies and explore new ways of thinking. Based on the literature, Lamon (2020) suggested focusing on differentiation of problem-solving tasks involving increasing and decreasing direct proportional situations, inversely proportional situations, integer and noninteger problems, and combinations of the four operations.

In terms of proactively teaching, Bowers and Nickerson (2001) after studied with the prospective mathematics teachers stated that for setting an effective interaction with the students, instead of asking conceptual explanations, the teachers are strongly recommended to expect from students to explain what they saw in that particular activity or experienced since they may not have the formal language to give conceptual explanations. Steffe (1994) expected the teaching experiments would provide a learning environment that involved mathematical interactions. In brief, her perspective focuses on that students' prior knowledge will be constructed through mathematical interactions.

To summarize, the trend in the literature is to use some predetermined teaching episodes in order to understand the teaching and learning of the rate, ratio, and proportion within a collaborative, active learning, problem solving environment. The aforementioned studies integrated partial or combined models of beneficial research elements in their research designs.

The trends in educational design involving research-based beneficial elements of teaching and learning provided a description of rich learning environment. It is wise to vary the kinds of questions, quantity structures, and numbers so that students are forced beyond their comfort zone, so that they must think about relationships and adopt increasingly efficient solutions. Based on the collection of literature Lamon suggested to focus on the differentiation of the problem-solving tasks involving increasing and decreasing direct proportional situations, similarly inversely proportional situations, integer and non-integer problems, combinations of four operations (Lamon, 2020).

2.4.2. Problem Types Used for the Teaching of Proportional Reasoning

There are different perspectives on how to investigate the different types of problems used in proportional reasoning (Kaput & West, 1994; Karplus et al, 1983; Lamon, 1993). The problem types can be differentiated according to the strategies required to solve them (Lamon, 1993). The types of proportional reasoning can be matched with problem types in which the students can use several ways of reasoning. Karplus et al. (1983) defined three types of reasoning by considering students' solution strategies within comparison problems: between, within, and the other. They explained, "[t]he rationale for the first term is that the initial operation combines the values of corresponding variables between two uses of the linear function. The rationale for the second term is that the initial operation combines values of the two variables within one application of the linear function. The choice of the third term is self-explanatory" (Karplus et al., 1983, p. 221). Each problem type provides a challenge which is related to proportional reasoning. For example, a comparison problem challenges students' understanding of the relationship of the ratio as a single entity and comparing those single entities or mixture problems provides a challenge to determine the most, the best, and the least of a new element that they created after a mixture of an intensive entity (orangeyest) (Tournaire & Pulos, 1985).

One of them focuses on the students' procedural strategies. Word problems are used to convey understanding for proportional reasoning. Whether a problem is one-step (Carpenter & Moser, 1982) or multistep (Quintero, 1983) may determine its level of difficulty for the students. Word problems that cannot be solved by a routine application of a single arithmetic operation are defined as multistep word problems. The structure of the problem was given as "There are a x's for every b y's. There are c y's. How many x's are there?" (Quintero, 1983, p. 103). Students' difficulties came from familiarity with one-step word problems, data organization, and selection of the appropriate strategy (Quintero, 1983).

Similarly, Lamon (1993) categorized the problem types semantically to organize students' difficulties based on them. Four categorizations (*well-chunked measures, part-part-whole, associated sets, stretchers and shrinkers*) were used in the study and students' level of sophistication in thinking was investigated (Lamon, 1993). Well-chunked is related to the understanding of transition of *two extensive measures into one intensive measures*; part-part whole is related to the subsets of a set; "associated sets" indicates an association of two different variables by means of problem context; lastly stretchers and shrinkers is related to scaling up and down of the continuous quantities multiplicatively (Lamon, 1993, p.43). Based on four types of problems the researcher investigated the changes in thinking of the students. For example, while part-part-whole recalls informal strategies such as build-up/down processes which do not require higher order thinking skills provided that larger number values are not used, associated sets require thinking in composite units (Lamon, 1993). Stretchers and Shrinkers problems are utilized in similarity reasoning of geometry problems by Chapin and Anderson (2003), who examined the students' transition from the levels of sophistication through these tasks in which students learned to use scalar and functional methods.

Although each categorization turned into another concept (i.e., well-chunked measures: rate problem), these categorizations influenced other research. Kaput and West (1994) conducted a teaching experiment with 138 students from the upper middle grade level. They used a computer-supported instructional sequence consisting of four units. They investigated students' problem-solving processes and concluded that students performed poorly during stretchers/shrinkers problems. Similarly, Langrall and Swafford (2000) used Lamon's categorization, used the problems below and conducted their studies with 4^{th}-8^{th} grade level students.

Part-part-whole

Mrs. Jones put her students into groups of 5. Each group had 3 girls. If she has 25 students, how many girls and how many boys does she have in her class?

Associated sets

Ellen, Jim, and Steve bought 3 helium-filled balloons and paid \$2 for all 3 balloons. They decided to go back to the store and buy enough balloons for everyone in the class. How much did they pay for 24 balloons?

Well-known measures

Dr. Day drove 156 miles and used 6 gallons of gasoline. At this rate, can he drive 561 miles on a full tank of 21 gallons of gasoline?

Growth (stretching and shrinking situations)

A 6" × 8" photograph was enlarged so that the width changed from 8" to 12". What is the height of the new photograph?

Figure 2. 7. Types of proportional problems

In this perspective, there is also another categorization for the problem types: missing-value and ratio-comparison problems (Cramer et al., 1993; Karplus et al., 1983; Lamon, 2020; Lesh et al., 1988). Missing-value problems involve three related quantities and asking to find the fourth (missing value) quantity. Middle school mathematics textbooks were claimed to have this type of question as a tradition (Cramer et al., 1993). On the other hand, it is considered as a difficult concept due to the involvement of two number or quantity comparisons (e.g., Hart, 1981; Kaput & West, 1994; Karplus et al., 1983; Lesh et al. 1988; Tourniaire & Pulos, 1985). Comparison problems provide "four quantities forming two ratios to determine whether they are proportional, smaller or larger" (Lamon, 2020, p. 117). Comparison problems involve tasks such as "Which one is the most sweet?" Lesh et al. (1988) reported that four categories (missing-value, comparison, qualitative prediction, and qualitative comparison problems) influenced students' difficulties in terms of numerical relationship and problem contexts. Based on this, Cramer and Post (1993) reported qualitative prediction and comparison problems as an instructional strategy.

Table 2. 5

Illustration of the Problem Types (Hilton et al., 2013)

Problem type	Example/Description
Non-proportional: Constant	A group of children sings a song. If we double the number of children, how long will it take to sing the song?
Non-proportional: Additive	Two children run around a track at the same pace. One child starts two laps before the other. How far will the second have run when the first completes a given number of laps?
Missing value; associated sets; part-part-whole	If my recipe requires 10 cups of flour for 4 cups of sugar, how much flour will I need if I use 6 cups of sugar?
One- or two-dimensional scale	If double the length and width of a diagram, what will happen to the area of the diagram?
Familiar (well-chunked) rate; Inverse relationships	If a child runs at a certain pace and it takes a given amount of time, what will happen to the time if he runs at twice the pace?
Rate; translation of representations	Linking a verbal problem about speed and time with a graphical representation of the situation described
Relative thinking; associated sets	Given data about the number of children who choose an activity and the total children in each of two classes, identify which activity was relatively the most popular
Inverse proportion	If it takes 6 people 3 days to paint my fence, how many will it take to paint the fence in 2 days?

There are also non-proportional situations as problem types which also help students develop their way of thinking in the following terms: composite units, for example, those applied to rates; representation-related problems; and measurement-related abilities (Lesh et al., 1988, p. 9). Identifying the appropriate type of reasoning to apply in a given situation can be challenging for students. Van Dooren and colleagues (2005) proposed that students tend to generalize the range of situations where proportional thinking is suitable. As a result, they frequently rely on proportional reasoning even in situations where it is not necessary. The researchers identified three types of non-proportional problems where students commonly use proportional reasoning inappropriately: constant, linear, and additive situations (as detailed in Hilton and colleagues, 2013).

Students may not differentiate the direct and inverse proportional situations accordingly. Direct and inverse proportionality situations have different issues to consider in terms of proportionality and it is recommended that they be integrated into the instructional strategies. In the case of inverse proportionality, children understand that the more a given variable increases, the more the other is bound to decrease (Ferrandez-Reinisch, 1985). Multiplicative increases or decreases by the same factor from both quantities in the ratio is one situation to consider for direct proportional situations, and just looking at an increase and decrease may mislead students (Lamon, 2020). The growth of a tree is related with its age, but this is not a direct proportional situation. On the other hand, in the case of inverse proportional situations, students can establish a relationship between the two variations involved, nevertheless, how this relationship between two variations is is what makes the issue problematic (Ferrandez-Reinisch, 1985). In a direct proportion, the direction of change in the related quantities is the same; both increase or decrease by the same factor. Reversely, in an inverse proportional situation, the bound is established increasing by the same factor for one of the linked quantities and decreasing the other quantity with the same factor.

2.4.3. Problem Contexts (Semantics) and Attribute of the Units

Studies have testified that variations in contexts and modes of presentation of individual problems sometimes affected students' responses (Ben-Chaim et al., 1988; Cramer & Post, 1988; Lamon, 1993; 1996; Lo & Watanabe, 1997; Tournaire & Pulos, 1985). Similarly, Tournaire and Pulos (1985) reviewed the literature about proportional reasoning by considering experimental studies conducted since the 1950s. They categorized the students' errors and correct strategies and elaborated on the issues of what makes students confused while reasoning about proportionality. The list included students' use of the additive strategy as an error strategy or fall-back strategy to solve complicated problems which consist of non-integer ratios, unfamiliar order of the missing value, big size of the number, different units in the same question (external ratios), and the existence of the mixture problem (Tournaire & Pulos, 1985). Problem contexts can be investigated under four headings: existence of non-integer ratios, discrete or continuous variables, order and the complexity of the ratio.

One of the task related difficulties that a child confronts is whether the ratio is an integer or non-integer. In their teaching experiment, Lo and Watanabe (1997) portrayed the tasks from simple to complex: 2:6= 8:?= ?:21 (easy), 9:12= 21:?= ?:40 (hard) in that the relationship between the variables can be explained by integers for an easy one (within ratio times 3, between ratios times 4) contrary to a difficult one (within ratio times 4/3, between ratios 7/3).

While students are using within and between strategies, the existence of a non-integer scale factor influences students' selection of functional or scalar method for solving the problem (van Dooren et al., 2009). They had great difficulty, however, in finding "in between" instances wherein the multiplying factor was not a whole number - that 24/40 is the same ratio as 27/45, a transformation by a factor of 1.125, for example. relating factors in a proportion in a qualitative manner, rather than applying quantitative strategies (Lesh et al., 1988); inappropriate use of algorithms, such as cross multiplication (Lesh et al., 1988; Nabors, 2003); and incorrect build-up/pattern building (Lamon, 1993; Misailidou & Williams, 2003).

The presence of discrete and continuous variables within the tasks may influence students' strategies. Lamon (1996) employed differentiation between the partitioning tasks in terms of discrete and continuous aspects of the variable. Discrete units cannot be partitioned into its section-units, carrying the same properties with the whole, whilst continuous units can be partitioned into subsections in that the smaller units preserve the properties of the whole unit. For example, a human or an egg is a discrete variable and homogenous mixture, whereas water or pizza are dissectible continuous units.

There is evidence that students' ability to make sense of the ratio can be related with their matching and counting skills, and that therefore, ratios involving discrete quantities can be easily represented (Lamon, 1993). Warren and Cooper (2007) identified students' initial experience as difficulty in breaking the repeating pattern into its discrete repeats. The teacher's emphasis of breaking patterns into parts not only allowed students to identify the repeats, but also to begin to discuss the structure of one repeat, two repeats and so on, and the similarities and differences between these differing repeats. This involved the development of common class words (e.g., 'repeating part') with which to describe the problems (Warren & Cooper, 2007).

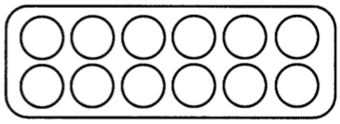
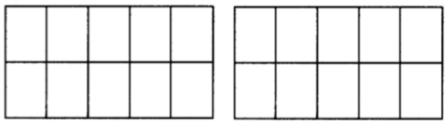
Characteristics	Identifier	Task
Discrete Subsets separable Array form	eggs	You have the carton of 12 eggs pictured below, and 3 people who want to eat them for breakfast. 
Discrete Subsets separable Composite form	gum	You have 5 packs of gum and 4 people. (A pack of gum has 5 sticks of gum inside.)
	cola	You have 8 six-packs of cola and 3 people.
	juice	You have 2 six-packs of juice and 4 people.
Continuous Elements dissectible Like items	pepperoni pizza	You have 4 pepperoni pizza pies and 3 people.
	chocolate chip cookies	You have 4 chocolate chip cookies and 3 children.
	oatmeal cookies	You have 4 oatmeal cookies and 6 children.
Continuous Elements dissectible Unlike items	4 pizzas	You have 1 cheese pizza, 1 mushroom pizza, 1 sausage pizza, and 1 pepperoni pizza for 3 people.
	3 meals	You have 3 Chinese dinners (1 pork, 1 beef, and 1 chicken) and 6 people to eat dinner.
	4 meals	You have 4 Chinese dinners (1 pork, 1 beef, 1 chicken, and 1 seafood) and 3 people for dinner.
Continuous Subsets separable Prepartitioned	candy	You have the 2 candy bars shown below and 5 children. 

Figure 2. 8. Lamon partitioning tasks related to discrete and continuous (1996, p.174)

Lastly, order and complexity affected students' performance. As Tournaire and Pulos (1985) stated, how a word problem context is structured matters. The complexity of ratio also matters in terms of the place of the missing value, the answer requiring the biggest number misbelief, and the involvement of internal/external quantities, mixture or rate problems (Tournaire & Pulos, 1985), The contexts involve research-based features such as recipes (Brinker, 1998). According to Brinker, Recipes can assist students in comprehending their solutions and may additionally prevent them from incorrectly applying standard algorithms that they had learned previously.

2.4.4. Tool Integration for the Enhancement of Teaching and Learning

It is recommended for teachers to implement various tools in order to make mathematics learning efficient and to enrich teaching (MoNE, 2013, 2018; NCTM, 2000). These tools involve concrete materials, mathematical models, or visualization through computer applications.

Concrete materials describe the hands-on tools for a specific purpose to develop mathematical understanding. Szetela (1970) developed a calculator-based instruction involving realistic data gathering activities to help seventh-grade students' overcome difficulties in learning ratio concept. Calculators were assumed to help them calculate realistic data for ratio. The results were not statistically significant, but the mean difference was reported. Another implementation focused on the vocabulary and symbols orientation in the classroom to improve teaching ratio and proportion (Jackson & Phillips, 1983). The list of vocabulary and symbols was provided to the classroom as the classroom routine, and the results were found to be statistically significant. In the same manner, Thompson and Thompson (1994) focused on the language, conversation, and discourse in terms of teaching rate, a special focus on conceptual understanding. They designed the teaching experiment through problems constructed on speed as rate with a medium of software.

In the literature, the transition from online education and computer-supported education has shown its effect on teaching of ratio and proportion concepts. Tajika (1998) developed a ratio word problem software to teach ratio in a computer-supported learning environment and reported that a computer supported environment improved the development process of the mental model for ratio. Similarly, Abrahamson and Cigan (2003) focused on teaching ratio and proportion to fifth graders with integer problems through a journey from repeated addition to multiplicative practices. Nabors (2003) studied the computer enhanced constructivist teaching experiment with four seventh grade students aiming to improve their rate, ratio, and proportion concepts through documenting their cognitive strategies. Adjiage and Pluvinage (2007) also developed a learning environment for middle graders, consisting of six ratio situations and three registers (fraction, decimal, and linear scale) with software support for linear scale. They conducted an experimental study and found a statistically significant effect of the intervention in terms of better acquisition and use for solving proportion problems. Jitendra and colleagues (2008) designed an instructional intervention to meet the diverse needs of students in classrooms using studies from both special education and mathematics education. It entails specific problem-solving strategies linked to particular types or classes of problems (e.g., ratio and proportion).

For mathematical models, vector spaces, graphs, and ratio tables were frequently mentioned in terms of their influence on the development of proportional reasoning. Confrey and Carrejo (2005) used vector spaces to demonstrate the invariant structure within the ratios of the proportion; on the other hand, they used number line for fractions. They also (2005) used a toolbox and represented vertical scale factors and horizontal scale factors as vertical multiplication and horizontal multiplication. This binary vector implementation was also suggested by Behr et al. (1992) to be used in teaching ratio, proportion, rate, and intensive quantity. Similarly, Cramer and Post (1993a) transit proportional learning from number patterns to graphical representation.

The ratio table is accepted as a conceptual and computational tool employed in both conceptual and procedural learning of ratio and proportion (Ercole, Frantz, and Ashline 2011; Lamon, 2012; Middleton and van den Heuvel-Panhuizen 1995). The ratio table can be defined as a mathematical model that allows students to represent fractions and ratios (Brinker, 1998) and can be used in the teaching of a variety of topics such as decimals, percents, and fractions except for ratio and proportion. Tables of values have long been acknowledged for their contribution to students' mathematical understanding (Warren, 1996). The structure of the ratio table, which emphasizes learning outcomes such as distinguishing units, recognizing the binary number relationship in the ratio, and comprehending the multiplicative relationship between the units by completing the spaces in the cells, supports the understanding of the ratio-proportion issue (Abrahamson & Cigan, 2003).

Constructing a table helps to identify the numerical relationship between the two quantities (Cramer & Post, 1993a). The ratio table is a tool that builds these connections in a way that allows students to develop an understanding of rational numbers - and as such, it is a good alternative to cross multiplication (Middleton & van den Heuvel-Panhuizen, 1995). It enables the development of an understanding of the successive manipulation of numbers to maintain the relationship between two quantities in the ratio and it is a conceptual tool for understanding equivalent ratios. As such, it supports proportional thinking (Middleton & van den Heuvel-Panhuizen, 1995; Abrahamson & Cigan, 2003; Ercole et al., 2011). As a challenge, it further seems to be the case that one can create the equation without this understanding.

The learning difficulties that have arisen around ratio-proportion have been largely compiled and awareness of these difficulties is important for understanding the concept of ratio (Karagöz Akar, 2014). Likewise, it is thought that knowing the situations regarding the use of tools such as ratio tables will affect the efficiency of the teaching process. Dole (2008) expressed students' first impressions about the ratio table, such as that the ratio table was boring and time-consuming, and that the ratio table consisted of filling in the blanks, and it was emphasized that a lot of practice should be done by the students. A ratio table presented independently of the development of the concept of ratio in the student can make learning difficult, so when and how the tools are used can give information about the efficiency of teaching. Sozen-Ozdogan et al. (2019) also outlined that even the learning of the ratio table needs to be a development process and suggests a community contribution to explore all of its aspects. Based on this, instructional designs that emphasize the concept of ratio, which students develop simultaneously with the ratio table, can produce successful results in terms of learning (see Abrahamson & Cigan, 2003). Dole (2008) reflected on some issues related to efficient utility of ratio table as a tool:

- Drawing the ratio table in a neat way is not the main focus of the learning. Regular lines and regular columns are not one of the learning outcomes.
- Predetermined table may have lots of cells and each cell does not need to be filled with numbers. Students are allowed to increase and decrease the number of cells in a ratio table to reach a solution.
- Calculations in the cells of the ratio table might not have to follow an order. That is, the numbers might be from smaller to bigger or vice versa.
- In the essence of each calculation, there is multiplicative relationship.

Ratio table is considered to be one of the structured methods for solving problems efficiently and solving noninteger problems (Ercole et al., 2011). The important thing to note is that the ratio table supplies documentation - a record of how students solve ratio problems - and this documentation is a small window into their understanding of what a ratio is. In creating equivalent ratios, students use a strategy that fits the way they initially perceive the ratio. Those who perceive it as a part-whole relationship tend to use the additive strategy because they see the underlying fraction as the operator. Those who perceive it initially as a ratio tend to use the multiplicative strategy because it preserves the original numeric relationship intact. This differentiation offers powerful opportunities for assessment.

CHAPTER 3

METHOD

The present study aimed to find the answer to the research questions through *Educational Design Research* (EDR) methodological framework within the classroom teaching experiment setting. In this chapter, the properties of EDR and the rationale are discussed first. As one of the EDR designs, the classroom teaching experiment is the guide for this research. Characteristics of the methodological framework intertwined with characteristics of the intervention (van den Akker, 1999) are explained throughout this chapter, which creates a favorable environment for the emergence of classroom mathematical practices under the Design Phases title, which consists of four phases. The first phase includes the description of the conjectured local instructional theory and preparation for the intervention; on the other hand, the second phase covers the implementation of the intervention and data collection during the intervention. The third phase mentions the data analysis processes in detail. Finally, the trustworthiness of the study is provided. In summary, this chapter describes the research method design and its procedures, the design itself (with its brief history), field and classroom settings, implementation of the design, analysis of the mathematical practices, and the processes that supported the insurability of the data.

3.1. Educational Design Research (EDR)

Educational design research describes a research perspective for systematically investigating the *design* of these instructional materials, programs, curricula, or tools in response to the emergent features of the setting (Design-Based Research Collective, 2003) within an effective teaching and learning environment (Stephan et al., 2015). In accordance with the significance of the study, there has been a need to investigate the ratio and proportional context with its real stakeholders; seventh graders, their teachers, and field experts in mathematics education within the common understanding of teaching-learning effectively without resisting to change the context, the discourse, materials and the like.

The dynamism in this kind of educational design is best captured in the frame of educational design research from many aspects. First, the design research paradigm partially emerged out of the shortages of the other research methodologies in which the need for a dynamic and live classroom environment was mainly ignored (Brown, 1992). The highly interventionist structure of the design research aims to overcome this drawback by adjusting the research plan before, during, and after the study is conducted (Plomp, 2013). The instructional materials and the proposed learning process are realized in the actual classroom setting, and potential learning pathways emerge in the design research (Cobb et al., 2003).

Design refers to "what is designed to promote learning or solve an educational problem (module, unit, tools, classroom culture, organizational infrastructure)" (Bakker, 2018, p. 15). With the introduction of Brown (1992), design research became a paradigm for systematically investigating the *design* of these instructional materials, programs, curricula, or tools in response to the emergent features of the setting (Design-Based Research Collective, 2003). The *design* of this study is the ratio and proportion instructional sequence developed by Stephan and colleagues (2015) and stakeholders' interaction with this context for active learning and teaching. Furthermore, the active participation of teachers and researchers as stakeholders in EDR directly influences the iterative and process-oriented part of this research, as Plomp (2013) highlighted, "involvement of practitioners". The teacher's participation in developing the researcher's initial plan and intentions is required to achieve a more feasible learning environment responding to the learners' needs. In brief, the current research design molds the teaching and learning processes by considering the research-based collection of information through systematical enactment of the proposed ratio and proportion instructional plan with the help of stakeholders to understand the interaction of teaching and learning, which supports emergent learning activities of the learner as the center of teaching and learning that is consistent with the aim and the research question of this study.

It is possible to encounter many labels to describe EDR, such as design research, developmental research, or design experiment (Bakker, 2018; van den Akker et al., 2006). Among various names, educational design research was selected for this study due to the reason for differentiating the research method from the other terminologies utilized in other fields to avoid confusion (McKenny & Reeves, 2014). Although different namings and descriptions have been used, there are common characteristics of design research (van den Akker et al., 2006; Bakker, 2018; Wang & Hannafin, 2005) guiding this study, which were presented briefly below:

Relevant research nourishes a design of a study that is improved by scientific standards developed by educators (Bakker, 2018). One of the fundamental characteristics of EDR that came into prominence is *reflexivity*, which aims to develop the relationship between theoretical background and practical utility of the *design* in a real-world context with stakeholders (Cobb et al., 2001; McKenny & Reeves, 2014). Therefore, the current study's design, a learning environment that conjectures fostering proportional thinking, was developed based on the research tree growing on the development of proportional reasoning as described in the Design Phase in detail. Not only the learning of the concepts but also the teaching model, the way of delivering the idea, the structure of "from simple to complex" design of the context, and the method of informing teachers about the instructional sequence have been constructed based on existing theories that brought the conjectured local instruction theory of this study.

Educational design research aims to develop an intervention in a real-world setting (van den Akker et al., 2006, p. 5), which provides opportunities for the design team to appraise educational improvements in their natural context. Intervening in what naturally happens is manipulating the elements of the learning environment according to a theoretical model in a systematic way (Bakker, 2018) that makes educational design research interventionist. Design research involves systematic educational interventions that are cyclical in nature (Plomp, 2013). The main aim is not only to test the initial instructional theory but to revise, develop, and evaluate it within each design cycle. This iteration is to balance the researcher's intention and utility of it in the classroom. Micro-cycles and macro-cycles are the essential components of the cyclical process in design research. In each micro-cycle, the research team conducted thought experiments on the instructional sequence (Plomp, 2013). Micro-cycles are described as the processes in which some actions are taken to develop optimized learning situations for the sake of conjectured learning goals enlisted below:

- "anticipation of the mental activities of the students throughout the planned instructional activities before the implementation,
- checking the correspondence of hypothesized learning and actual learning that emerged during implementation,
- reconsideration of the potential or revised activities." (Gravemeijer & Cobb, 2013, p. 82).

Initiation of each mini-cycle begins with considering the effective and ineffective interventions based on the past analyses of the classroom activities; therefore, these interventions depend on each other (Cobb et al., 2003; Design-Based Research Collective, 2003). Daily enactment of the activities in this study was part of the micro-cycles in which the students’ engagement, conjectured learning goals, and tasks had been revised throughout the enactment of ratio and proportion instructional sequence. The teacher and the researcher had been together in the class throughout the implementation. They intervened in the ongoing process after a small talk related to the lesson's learning goal. As the hypothetical learning trajectory suggested, they selected students with distinct solutions to lead the classroom to the learning goal. The decision to continue or finish the task was also the routine of these daily cyclical processes. Each week, the researcher summarized the one-week process to the expert who suggested possible pathways for the weekly challenges and revised the ongoing process, which is also described in the Implementation Phase in detail. Another process in educational design research that encompasses micro-cycles and retrospective analysis is called macro-cycle (Bakker, 2018; Gravemeijer & Cobb, 2013). Moreover, ill-founded instructional sequence attempts are determined, revised, and documented based on the teaching-learning environment's conjectured and actualized configuration. In the end, improved instructional sequence are prepared for the next macro-cycles. Throughout this dissertation, the level of one macro-cyclic process is schematized with a thick description of the design and design research.

Based on the characteristics mentioned above, the educational design research approach was the guide for the design of this study. Ratio and proportion instructional sequence as the design developed by Stephan and colleagues (2015) implemented throughout the process consisting of the instructional design. As the nature of this study, the focus is the collective mathematical practices of the classrooms. Therefore, it has its own procedures, as described in detail below.

3.2. Classroom Design Study

This study draws its strength from *classroom-based analyses* as Cobb et al. described (2001). Design research can be conducted in various settings, such as one-on-one design experiments (teacher-experimenter and student), classroom design experiments, preservice teacher development experiments, in-service teacher development studies, and school district structuring experiments (Cobb et al., 2003). A classroom teaching experiment has been named a classroom design study (Cobb et al., 2016) which investigates the means of learning in a classroom environment with a design team (Cobb et al., 2003; Rasmussen & Stephan, 2008). Educational design research cannot be enacted entirely in the real classroom without documenting learners' collective activity throughout the instruction. By implementing a proposed research-based design, potential learning pathways emerge in the design research (Cobb et al., 2003). Therefore, EDR is to produce solutions for the problems by investigating not only the summative evaluation at the end of the intervention but also the analyses of the whole teaching-learning process (Plomp, 2013).

The classroom teacher was suggested to be a design team member, and the teacher proactively conducted the teaching process (Stephan, 2015). One of the aims is to use students' crucial mistakes to eliminate them by modifying their schemas while constructing knowledge to overcome these mistakes (Steffe & Thompson, 2000). Mainly, a classroom teaching experiment involves a sequence of teaching episodes, and they are used to improve and test an initial instructional theory by analyzing collective classroom activity (Cobb, 2003).

Initial learning pathways, mathematical tools, and a teaching cycle for the ratio and proportion instructional sequence were developed by Stephan and colleagues (2015) to support students' improvement proactively in proportional reasoning. This initial design is open to the changes offered by a design team based on students' feedback and design team throughout the implementation. For the systematic analysis, classroom-based design research phases (Stephan, 2015) provided an outline to identify the logs of the research in sequential order as provided in Figure 3.1, and this figure is explained for this study in a more detailed way under the following phases

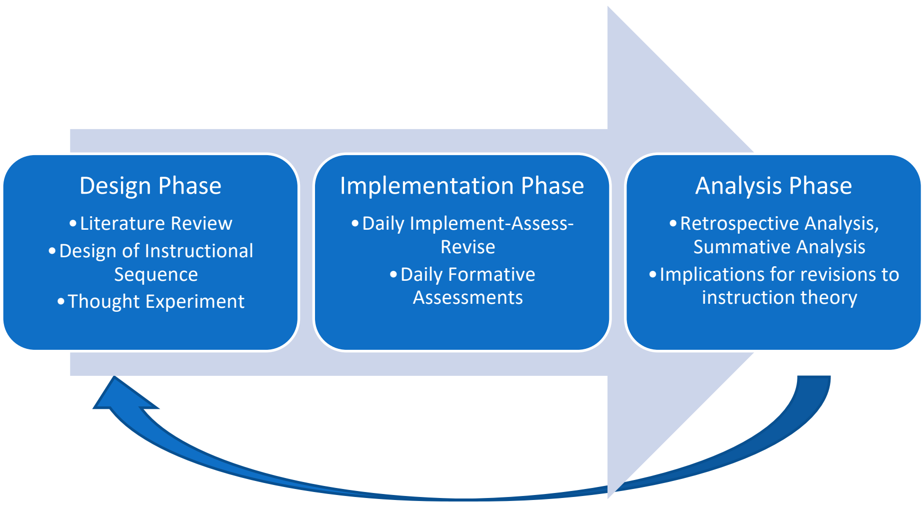


Figure 3. 1. Classroom-based design research phases (Stephan, 2015)

3.3. Design Phase (1): Conjectured Local Instruction Theory and Preparation

A design phase in a classroom teaching experiment describes the preparation of the *design* of the study (Bakker, 2018; Gravemeijer & Cobb, 2013), which refers to the ratio and proportion instructional sequence developed by Stephan et al. (2015). The sequence was constructed based on the research-based information collection on the ratio and proportion concepts. In this part, both conjectured local instruction theory for this design and preparation for the classroom teaching experiment are explained in detail to understand fundamental elements of the learning environment, which may be called *learning ecology* (Gravemeijer & Cobb, 2013) of this ratio and proportion context and the preparation for the design study.

3.3.1. Local Instruction Theory for the Instructional Sequence

Conjectured local instruction theory involves learning goals, instructional activities, tools, and learning processes (Gravemeijer, 2004); therefore, it is open to adaptations thanks to the educational design research which contributes to its development (Gravemeijer & Cobb, 2006) through the investigations on the conjectured and enacted instructional plans (Cobb et al., 2003). This part describes the knowledge accumulation for the conjectured local instruction theory based on the literature review and thought experiment.

3.3.1.1. Realistic Mathematics Education (RME)

Realistic Mathematics Education is described as the domain-specific instruction theory through which the learning environments are designed in a *realistic* context so that the students will develop a formal and general understanding of the knowledge gradually by using consequently former knowledge that has been reorganized employing each context (van den Heuvel-Panhuizen, 2014). *Realistic* was explained as situations that can be imagined by the students (van den Heuvel-Panhuizen, 2014); in other words, “experientially real for students” (Gravemeijer et al., 2003, p.52). The RME is a reaction to traditional teaching approaches in which the students apply what their teachers introduce symbolic manipulations and algorithms before making sense of them (Cobb et al., 2008). What is promising is that experientially real situations support students’ informal ways of engagement with the activities through speaking, symbolizing, and the like (Cobb et al., 2008).

The origin of RME dates back to Hans Freudenthal and colleagues and the foundation of the Institute for the Development of Mathematical Education at Utrecht University in the 1970s, also known as IOWO. The name of the Institute was changed after his death to the Freudenthal Institute for Science and Mathematics Education in the Netherlands (called Freudenthal Institute in short), where Freudenthal developed his ideas related to Realistic Mathematics Education by starting “Mathematics in Primary School (Wiscobas)” Project with other didactitians: Edu Wijdeveld, Fred Goffree and Adri Treffers (van den Heuvel-Panhuizen, 2014). RME constructs emerged from the need of Freudenthal (1968) to redefine the fundamental problem of mathematics as the problem of teaching mathematics and to redefine teaching mathematics by emphasizing the importance of the rediscovery of mathematical entities in learning (Freudenthal, 2002) because it is the *human activity* (Freudenthal, 1973) of mathematizing the reality (Freudenthal, 2002). His claim was in the direction of perceiving mathematics not as a mental object but as “the backbone of our cognitive structures” (Freudenthal, 2002, p. x).

In this study, the ratio and proportion instructional sequence was constructed based on the heuristics proposed by Stephan et al. (2014) for learners to engage in collective mathematical activity to develop an individual and social way of mathematical understanding based on RME. These design heuristics explain the base to come together with the content-specific research-based knowledge.

The first heuristic is the *guided reinvention* which directly involves the teachers’ guidance, carefully sequenced problems, and domain-specific tools along a learning path defined by the contributions of the teacher and the students (Stephan et al., 2014). It is for students to reinvent the concepts themselves (Gravemeijer, 2004). The students developed various ways of solving the problems throughout the ratio and proportion instructional sequence. They challenged their own ideas each time. They set their own ratio table as tools, and they preferred to use vertical or horizontal scale factors based on their own needs to reach a goal that was predetermined before the classroom and formed with the emergent needs of the students. With the help of a discussion environment, students are always allowed to share/argue or refute their ideas. Apart from introducing the ratio table as the formal tool, as given in the detailed findings, students experimented with their ideas in each argument and found an efficient solution.

To propose a realistic context that encourages students to initiate the development of mathematical structures is the main aim of the RME. His studies continued with RME, *didactical phenomenology* concept is one of the heuristics used by Freudenthal (2002) to organize the teaching and learning of the mathematical entities into a sequence so that students may develop the mathematical structures through rich contexts. It is to make the students responsible for their own mathematical learning, not the receivers of the ready-made mathematics. The activities were designed to make them imagine the situations and match the context with the mathematical explanations. Therefore, this heuristic is about *experientially real sequences* that influence task selection to make the learners relate the context with their initial mathematical descriptions and to learn mathematical reasoning more sophisticatedly (Stephan et al., 2014). To illustrate, the context of “alien and food bar” forced them to think about sharing the food bars as “aliens are not sliceable, but the food bars are” in other words, if the student did cut the aliens, then the feeding the alien context would be ignored, and the problem was unsolvable. While solving the tasks, mathematical values were evaluated with their contextual situations. To propose a realistic context that encourages students to initiate the development of the mathematical structures through RME refers in this study to a domain-specific instructional theory focusing on *realistic situations* in a broader context than real-life situations.

Another heuristic of RME is *emergent modeling* which involves mathematizing as the process of organizing and reorganizing knowledge through the processes of doing mathematics, such as (modeling) them (Freudenthal, 1991). *Emergent models* also define the tools suggested by the design team or developed by the students to encourage them to reason to make the tasks more sophisticated (Stephan et al., 2014). The teacher and the researcher gradually introduced formal and informal tools to foster proportional reasoning during the study. The ratio table is one of the tools offered by Stephan and colleagues (2015). It is a table in which the numerical values are organized in a pattern so learners can create their own table (Dole, 2008; Middleton & van den Heuvel-Panhuizen, 1995). Suggested tools and students’ drawing models were anticipated before the study.

Treffers (1987, 1993) further investigated mathematizing which is categorized as horizontal and vertical. In the horizontal mathematizing part, the learners are to engage with the integrative themes, including real-life situations so that they are able to develop and use mathematical tools and procedures for problem-solving (Treffers, 1987, 1993); in other words, describing the themes in mathematical terms (Gravemeijer & Doorman, 1999). In the vertical mathematizing part, students transfer the situations into symbols, connect them with the mathematical topics or use specific strategies; in short, they are modeling the situations (Treffers, 1987, 1993) and reaching a higher level of mathematics and expanding the mathematical reality (Gravemeijer & Doorman, 1999). Thus, students transform their own informal knowledge into an acknowledged model with the help of mathematical reasoning (Gravemeijer, 2004, p. 117). Horizontal mathematization represents the mathematical approach to a problem situation; on the other hand, vertical mathematization represents the transformation of this mathematical approach into abstract structures through formulating or generalizing. Thus, horizontal and vertical mathematization processes are not dichotomies but complete each other (Rasmussen et al., 2005).

For this educational design research, RME heuristics were not independent of emergent perspective, and three adaptations that Cobb and colleagues (2008) made to RME theory have been valid throughout the process regarding the social aspect of mathematics learning for the established mathematical practices. RME provided a perspective to investigate students' reasoning from their experientially real views through contextual problems, visuals, and tools. To illustrate, there were discussions to mathematize the ideas within the context, such as "the limit of the aliens' stomach to eat food bar is the rule of the composite units (one food bar for three aliens)." Additionally, horizontal mathematical situations were embedded within the activities to help students change imageries into their collective mathematical representations accordingly. Next, there was a vertical mathematization process from "linking composite units" to "comparing ratios" through activities designed to develop these representations into valid, more abstract mathematical solution systems.

3.3.1.2. Hypothetical Learning Trajectory (HLT)

The discussion of the different meanings of the constructivist epistemology between radical and social constructivism emerged from the questions related to the priority of cognitive and social processes for mathematical understanding. Simon (1995) proposed that individual cognition could not be learned separately from classroom interaction in a classroom environment. He claimed a lack of a model describing how to teach mathematics within an interactive environment. His idea focused on the influences of teachers, instructional materials, and classroom interaction on students' learning. Therefore, he defined a learning environment that consists of the teacher's knowledge, hypothetical learning trajectory, and assessment of students' knowledge as the main elements of the mathematics learning cycle (see Figure 3.2).

As shown in Figure 3.2, Simon (1995) described Hypothetical Learning Trajectory (HLT) as the crucial part of the Mathematical Learning Cycle designed based on the social constructivist perspective. There are three components in it: the teacher's learning goal which defines the direction, the teacher's plan for the learning goals, and the teacher's hypothesis of the learning process which Simon identified as "a hypothetical learning process—a prediction of how the students' thinking/understanding will evolve in the context of the learning activities" (1995, p. 35). That is, children follow a path within the learning process. The teacher anticipates the big ideas and generates priority for the instruction. HLT contains an instructional sequence where teachers align the activities with their goals (see Figure 3.2). HLT was defined by Stephan and colleagues (2014) as the route that the classroom is anticipated to travel throughout the engagement with the sequenced tasks. However, this anticipated trajectory is a collection of social negotiation but not actualized by the teacher and the classroom student (Stephan, 2015).

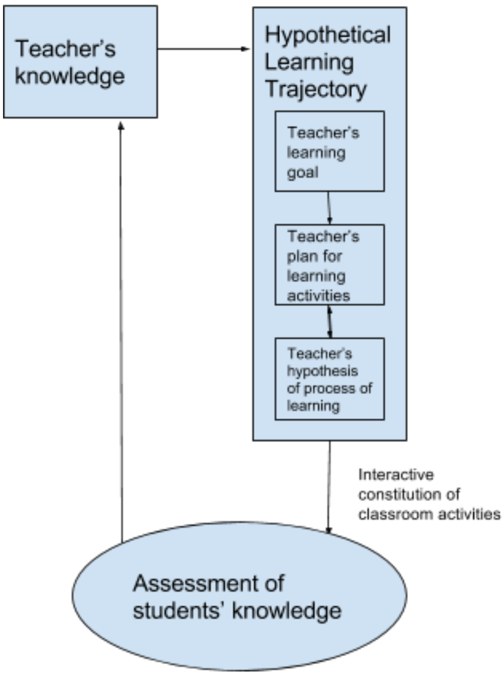


*Figure 3. 2.*Mathematical learning cycle (Simon, 1995)

3.3.1.3. Ratio and Proportion Hypothetical Learning Trajectory

As stated in the Literature Review Chapter, the current study was built upon prior research regarding children's development of ratio and proportion concepts by considering the learning goals described in Simon's mathematical learning cycle (1995). Stephan and colleagues (2015) designed the ratio and proportion instructional sequence around HLT considering Common Core State Standards (CCSSI, 2010), the table guiding the instructors in designing and implementing the instructional sequence. There are specific titles: *Big Ideas*, *Tools/Imagery*, *Possible Topics of Discourse,* and *Activity Pages.* Big Ideas are the "backbone" of this instructional sequence which was offered by Battista and van Auken Borrow (1995) and developed by Stephan et al. (2015) as specific to the ratio and proportion context. These big ideas can be considered as phases of the Hypothetical Learning Trajectory. Nine phases are already described for learning ratio and proportion concepts, such as "linking composite units." (see Appendix G for detail). On the other hand, the *Tools/Imagery* part introduces possible tools and models related to the content teaching for the design team. For this instructional sequence, informal symbolizations or figures and ratio tables are some examples of imagery and tools used in the classroom teaching experiment. Lastly, *Possible Topics of Discourse* are proposed as a kind of support for the instructors to create interactions among the content, the students, and the teacher. This described HLT is equipped with teaching through problem-solving and RME. Tasks involve questions that students can imagine and make sense of. Furthermore, initial problems were accompanied by visual representations to create taken-as-shared ways of reasoning (Reinke & Casto, 2022).

The ratio and proportion hypothetical learning trajectory involves eight segments, and each segment and activity include various concepts and ideas related to the ratio and proportion context. The segments are not considered as levels of the students and do not reveal a linear order of learning. On the other hand, they represent anticipated reasoning skills for learning ratio and proportion context from simpler to more complex advances of the students, as explained below.

The trajectory started with *linking composite units* to deal with the ratio and proportion problems, which was critical to developing higher-order thinking (Battista & van Auken Borrow, 1995; Stephan et al., 2015). They used different linking strategies, as seen in Figure 3.4 which provided the rule of Activity 1: one food bar (on the left) feeds three aliens (on the right). The first task was suggested for the students to develop those links between the units, and the pictures supported them to develop this skill by drawing. They used drawing in other activities as well in a more complicated way. Figure 3.4 (b) revealed a typical solution for "how many food bars will be enough to feed the aliens?" by drawing. On the other hand, Figure 3.4 (c) shows a distinct solution that became a discussion topic based on the discourse, "The food bar can be broken, can the aliens and the rule be broken?" to make sense of the rule. In this context, being continuous or discontinuous of a unit influenced the students' reasoning. A sliceable food bar in a rectangular form led students to share it equally among the aliens. Thanks to equal-sharing imagery, they explored the unit ratio strategy simultaneously.

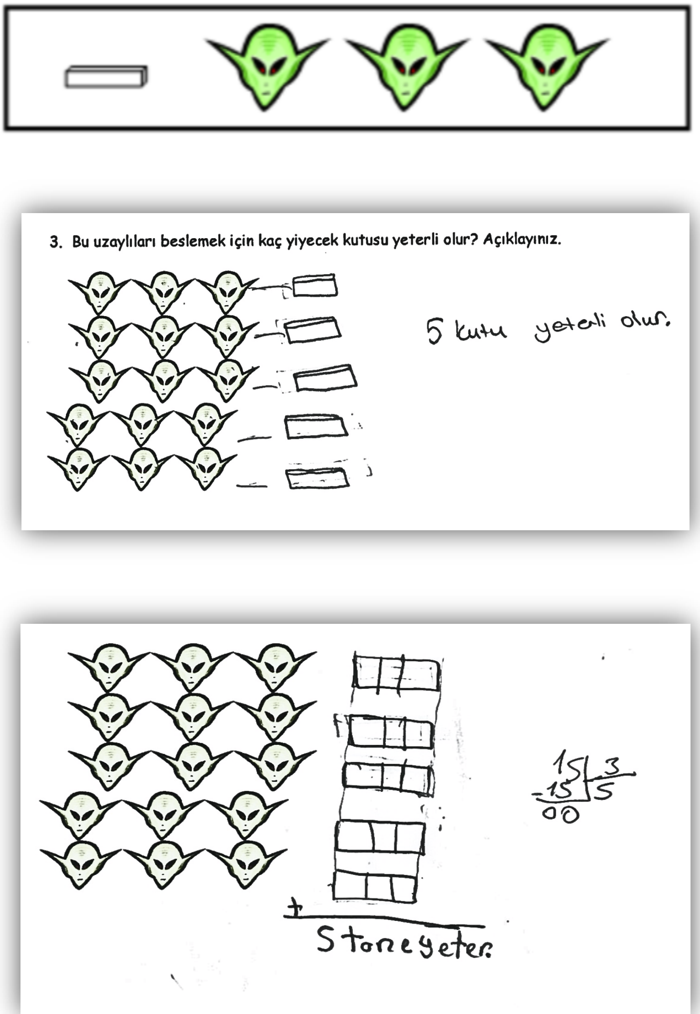


Figure 3. 3. (a) The rule of the task, (b) Common linking strategy of composite units, (c) A start for unit ratio

The second segment of the HLT covers Activity 2-4, aiming at *the iteration of the composite units*. The first segment was still valid to make sense of the iteration within this context. The students tried to organize the numbers, and the iteration was embedded in their solutions explicitly or implicitly. Since the number value in the questions increased, the students moved to other models, such as the ratio table strategy rather than the drawing they frequently used in the first segment of HLT. The ratio table was not evolved smoothly, but its utility was advanced by the students during the classroom discussions. Like the first segment of the HLT, this part of the instructional sequence informed the teacher about the LED phases in detail (as an example, see Figure 3.5). These guides helped the researcher and the teacher organize the students' answers while visiting the students throughout the explore phase. The students' responses were gathered around four main strategies: drawing, division/multiplication algorithm, unit ratio strategy, and ratio table strategy. The second activity was to lead the emergence of the ratio table; however, throughout the implementation process ratio table did not emerge as an answer through the second activity in which the teacher and the researcher decided to talk about the ratio table in the third activity that focuses on the structure of the ratio table. The discussions are shaped around "How to organize the units" and "What is a ratio table?" "What makes a ratio table a ratio table?" "What is the relationship between the columns."

Big Mathematical Idea(s): The idea of this page is to encourage students to link two composites together and to begin to organize these links when there are large quantities involved.

Rationale: Students need to find a way to organize the links as they increase in size. A ratio table should be introduced from students' work on this page.

LAUNCH: As students to write something down to show how they solved each problem. They need to make sure they feed the aliens appropriately so the intergalactic war does not start.

EXPLORE: 10 minutes

DISCUSSION: Sequence the solutions by having a student who did the division strategy first. The student likely will not be able to explain why they divided. Move to the students who drew the picture, table second, additive reasoning (incorrect) next. Ask students what is common and different about the picture and table strategy. They will likely say that they both take a long time to create but the table is quicker. In fact, name the table method as a ratio table and acknowledge that it is quicker than drawing pictures but that it represents the picture in a more organized way. Do not play up the division strategy today. Let students know that they can use ratio tables or pictures to justify their thinking on future problems since those are the ones that seem to make sense to most students.

Figure 3. 4. Instructional guide of the second activity (p.2)

The third segment of the HLT is about developing build-up strategies (long table and short table), which is embedded in the third and fourth activities. The context of the activities is still alien feeding for the sake of saving the Earth. Iterating composite units are the main idea of this segment. During the implementation, the students slowly internalized using a ratio table as a strategy; therefore, Activity 4 lasted four hours. The researcher and the teacher discussed the models of students' iteration through various problem-solving strategies and their efficacy in solving the latter problems with larger quantities or decimal representation of numbers. The intent was to create a need to develop a time-efficient model for data organization. The students created emergent models, as seen in Figure 3.6. The rule is given in Figure 3.6 (a), and the quantity is larger than the other questions in Activity 4. Short and long ratio tables were used as models to describe the relationship between and within the units of a given ratio. A student discovered the relationship between the units of the ratio and represented it through a short ratio table. Still, she had difficulty organizing the data, as seen in Figure 3.6 (b). On the other hand, a long ratio table was also used by the students who could not see the multiplicative relationship between the number values, as given in Figure 3.6 (c). Iteration of number values and skip counting were helpful to show the repeated addition aspect of multiplication.

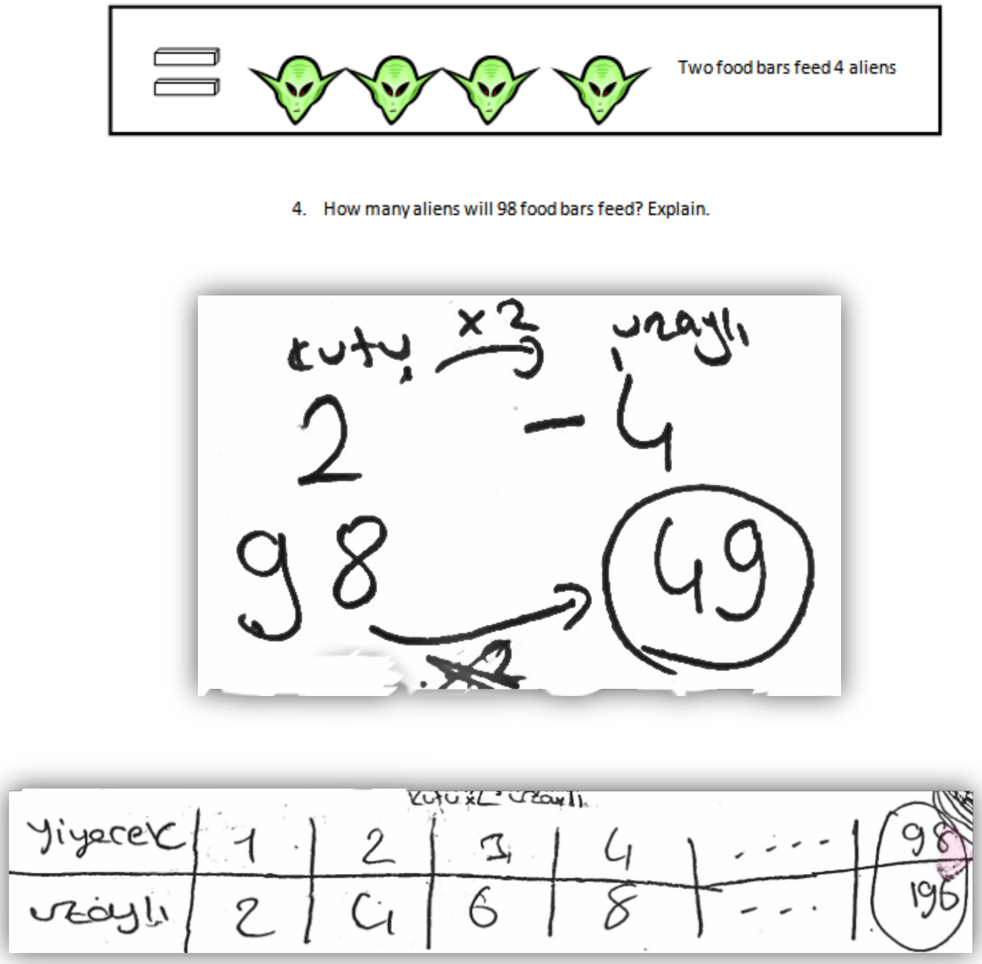


Figure 3. 5. (a) The rule of A4, (b) a short ratio table, (c) a long ratio table

The fourth segment of the HLT is about comparing additive and multiplicative reasoning within the long and short tables, which is conjectured to be discussed mainly in Activity 5. The alien feeding rule is “two food bars feed three aliens,” but this time, four predefined horizontally extended ratio tables are serviced for students to reason about the solution strategies within these tables. There is a long ratio table using build-up strategies and a short ratio table using scale factors. Moreover, there is a ratio table using additive thinking within the variables. During the enactment, the students compared each solution strategy and tried to find the best way. Nearly half of the class had difficulty understanding the scale factors meaningfully. The instructional sequence supported multiplicative thinking but achieving this thinking as a collective activity in the classroom was significant. Therefore, based on their answers, the long ratio table (see Figure 3.6.a) was more concrete than the short ratio table (see Figure 3.6.b). Long and short ratio tables comparison continued in the latter activities.

The fifth segment of the HLT focuses on structuring ratios multiplicatively. This segment introduces the ratio table as a formal tool in Activity 3. Shortening the ratio table evolves from a long to a short ratio table. In the long ratio table, the students investigated build-up and abbreviated strategies. They used skip counting and iterated each unit with a correct number. Teacher Merve introduced the short ratio table using ellipses (see Figure 3.6.c). After that, the students explored the number relationships, patterns, and scale factors (horizontal and vertical) within and between the units through the ratio table. While experiencing the number relationships, they developed a sense of multiplication step by step. Within this respect, even the build-up strategy underlined the repeated addition meaning of multiplication.

The sixth segment of the HLT is to create equivalent ratios, which is conjectured to be discussed starting from Activity 8. Although not introduced yet, students are reasoning proportionally every time to find a missing value in a table because reasoning proportionally means to set two ratios equivalent. Scale factors within the ratio tables created a basis for representing equivalent ratios as symbolic representation, "$\frac{a}{b} = \frac{c}{d}$". The students previously used drawing (pictorial representation), arithmetical operations (division, multiplication), unit ratio (fraction imagery), and a ratio table (a formal tool), all of which did not include the symbolic representation of equivalent ratios. Missing value and comparison problems reinforced the students' understanding of the equivalent ratios. The students determined the most helpful scale factor based on whether the ratio was decimal or integer. Determining two proportional ratios and finding missing values in a proportionality were also discussed under this segment.

The seventh segment of the HLT is to analyze the equivalent ratios after students learn to create equivalent ratios. In this respect, the students explored the scale factors in the equivalent ratios similar to the ratio table. They concluded that the scale factors in the ratio table could be transferred to proportionality. Moreover, fraction imagery helped reduce a ratio while finding the equivalent ratios. Although this segment was formally mentioned in Activity 10 and 13, it was simultaneously developed with the sixth segment throughout the discussions.

The eighth segment of the HLT is to focus on comparing ratios. Apart from missing value problems, comparison problems are also significant in helping students develop proportional thinking. Till Activity 17, missing value problems are dominant in the tasks. The students were engaged the most with the missing value problems. Comparison problems were not easy to adapt to at first. They investigated the meaning of the ratio context while representing (1:5 or 5:1). Therefore, fraction imagery (especially the simple fraction imagery) became a challenge without considering the meaning of the ratio (the ratio of boys to girls or the ratio of girls to boys) while comparing the tasks. In this respect, the students compared the ratios using a common numerator and denominator, considering their meanings.

Linking composite units	• Connecting given the pictures of units and making a group
Iterating linked composites	• Iterating the groups to make many groups of the composite units
Developing Build-up Strategies	• Developing strategies by using ratio table tool, drawing to produce efficient models, elaborating on the organization of the data , development of the ratio table
Additive vs. Multiplicative reasoning	• Studying with larger quantities and models to develop multiplicative understanding as a need, development of the ratio table
Structuring ratios multiplicatively	• Studying on the ratio table and its structure in terms of vertical scale factor and horizontal scale factor
Creating equivalent ratios	• Vertical and horizontal scale factors with a focus on equivalent fractions
Analyzing equivalent ratios	• Reducing ratios
Comparing ratios	• Finding common numerators or denominators; size of the scale factors; unit ratios
Comparing rates	• Difference between a ratio and rate; unit rates; common denominator and numerators

Figure 3. 6. A summary of HLT

Additionally, the classrooms might need more profound knowledge about the topic. In the case of classroom 7/X, the students needed more to discuss inverse ratio and cross-product algorithms, which were also learning outcomes of the national mathematics curriculum. In the instructional sequence, questions related to the inverse ratio were expressed implicitly on the topic. Teacher Merve brought the case to the fore and said, "Think about the tasks we have done so far (referring to the tasks in the activities which consider direct ratio relationship), and there are cases such as four workers finishing the painting of the walls in two days. What about eight workers? Is this case similar or different from the formers?". She spent time discussing these topics to make the students aware of the proportional relationships. These additions and changes, most of which emerged during the enactment of the instructional sequence, made the teaching-learning experience special for the classroom.

3.3.1.4. Instructional Tools as Conveying Mathematical Meaning

A tool can be simply explained as 'it is something you use or create to do something' (Monaghan & Trouche, 2016). Tools can be in different forms such as social, technological. From social perspective, interactions can be established through communication conveying the meanings through language as the cultural tool (Vygotsky, 1979). Along with interactions, classroom community constructs their knowledge based on using their models or already assigned models as their ancestors have done in their own societies (Monaghan, 2016). In learning environments, as students use tools throughout time, the ways in which they are incorporated into a solution and the solution process change (Walkerdine, 1998). Even tool utility can be taken as shared and developed throughout the learning process (Gravemeijer, 2004; Sozen-Ozdogan et al., 2019). A tool representing children's thinking is transferred to formal mathematical models through students' engagement with the activities (Cobb, 2000; Gravemeijer, 2004).

How to represent a tool in a classroom environment can be explained from the perspective of multiple representations of mathematics, which can be considered as the mathematical language which comes from society, children's ordinary concepts coming from themselves turn into mathematical language through their interaction in the society (Steele, 2001). Multiple representations can be considered as a list of tools used in the instructions. How mathematical knowledge should be expressed has been discussed in several studies (Behr et al., 1992; Dienes, 1969; Lesh, 1979, 1981; Pape and Tchoshanov, 2001; van Someren et al., 1998). Among these studies, the multiple representation model developed by Lesh (1981) is one of the models frequently used in the literature on mathematics education (Olkun and Toluk-Uçar 2012; van de Walle et al., 2016).

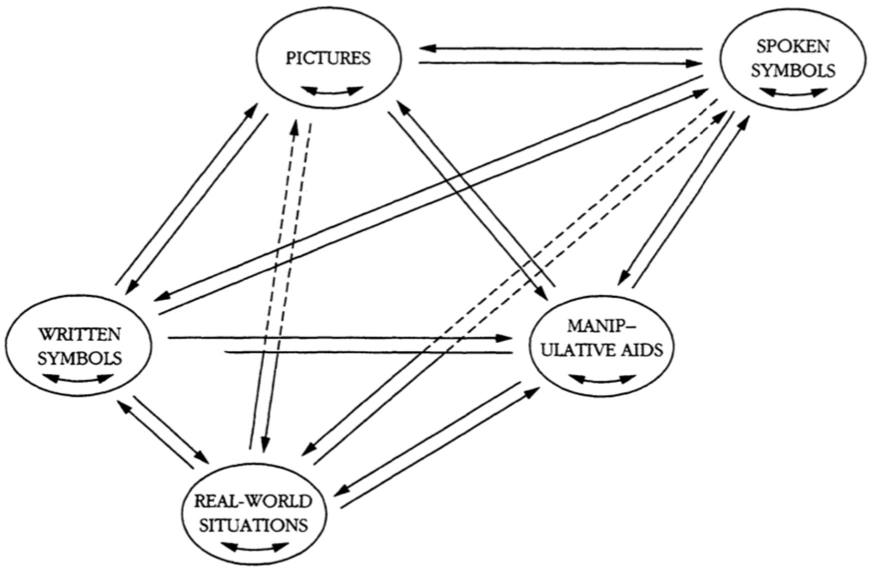


Figure 3. 7. Lesh's renovated model for multiple representation (Behr et al., 1982, p. 330)

In this model, five different forms of representation of mathematical knowledge that can interact with each other are mentioned: (1) written symbols (2) pictures (3) verbal symbols (4) concrete objects and (5) real-world situations. This diagram was referred as translations among modes of representation (Behr et al., 1992; Lesh, 1981). Modes of representation created several questions based on the differences of pictures, manipulatives and real-world situations. Lesh differentiated them:

> "For example, distinctions between a real world situation and a manipulative model arise because the models usually involve less "noise" (i.e., attributes that are irrelevant to the concept they are intended to embody), and the models are typically used in a symbolic way to represent many different real world situations. The primary distinction between manipulative models and pictures derives from the actions which are an integral part of the models - but which are difficult to incorporate into static pictures." (1981, p.247)

According to the multiple representation model, in order to learn mathematics in a meaningful way, it is envisaged that students express mathematical concepts with pictures, concrete objects and symbols based on real life situations, and also explain their mathematical thinking processes using verbal language (Olkun & Toluk-Uçar, 2012). Representations, which include area, linear, and set models "are grounded in the way that contextual problems are solved by the students and are grounded through the tools used to represent each model (Gravemeijer & Stephan, 2002, p. 148). The tools, or physical shape, are used as a way for students to represent a solution and solution process. As a result, there are various ways in which students may incorporate tools when solving problems.

It is recommended to use different forms of representation of knowledge, and multiple representation is emphasized in the curriculum of many countries (see Common Core State Standards Initiative [CCSSI], 2010; Ministry of National Education [MoNE], 2020) and multiple representation is considered as an important process skill (National Council of Teachers of Mathematics [NCTM], 2000). To illustrate, multiple representations are reported as crucial for rational numbers that students who can represent rational numbers in a variety of representations and move between these representations gain a deeper comprehension of the concepts (Post et al., 1993). Therefore, tool utility during the instructional sequence and hypothetical learning trajectory is strictly recommended in such a way that students have opportunities to work with various tools (Gravemeijer, 2004). It is especially important that mathematical knowledge is handled not only in verbal language, but also in these five different representations shared above, in supporting children with different levels of language proficiency in learning mathematics. The common belief that children who are not educated in their mother tongue will need less support in areas that require numerical thinking, such as mathematics, have changed with recent studies (Gottlieb & Ernst-Slavit, 2013). In the acquisition of mathematical knowledge, language is not just a tool; language can act as a basic mechanism that provides the process of mathematical concept formation (Radford, 2000). For this reason, lecturing with multiple representation educational materials (Gottlieb & Ernst-Slavit, 2013), does not limit the communication only to verbal communication, but being aware of body language signs (Villegas, 2002), subject-based mathematical terminology. Children's learning is supported by methods such as identifying academic words and communicating with students by emphasizing and highlighting these words during the lesson (Olkun & Toptaş, 2007), emphasizing the academic language of mathematics through the use of visual elements (Gottlieb & Ernst-Slavit, 2013). To summarize, the use of multiple representations (concrete materials, body language, virtual materials, representations with drawings, verbal and symbolic representations) is a win, especially for students with disadvantaged conditions.

3.3.2. Preparation for the Design

Before starting the implementation, some preparations were conducted to get the teacher and the classroom used to the researcher and research environment.

3.3.2.1. Field and Classroom Settings

This study took place in a public lower secondary school (middle-grade 5^{th}-8^{th} grade) in the district of Yenimahalle, in Ankara. One thousand eleven students enrolled in total were distributed in 31 classrooms in 2017. There was also one computer laboratory, one science laboratory, and one library for multi-purposes. There were 33 students per classroom on average. As a notice, this was quite a large class size based on the OECD report about the average class size in Turkey, which reported 26 students per class in public lower secondary schools in 2017 (OECD, 2020). Apart from that, 51 teachers were responsible for many disciplines, six of whom were elementary mathematics teachers. They also gathered in the staff room during the breaks and used this place for one-to-one teaching of the inclusion students. Each teacher had at least three inclusion students' responsibilities individually and approximately 21 lesson hours' workload per week; each took forty minutes, with a ten-minute break between consequent lessons. Additionally, the school administrator had graduated in industrial technology education and had a PhD from the same program. As far as the researcher observed, he encouraged teachers to complete their graduate degrees and attend the projects supported by the National Science Foundation in Turkey, TUBİTAK. Two assistant administrators were working with the administrator. The teachers said that they were happy with the current administrative staff.

Two elementary mathematics teachers participated in the study. One of them, Zehra, was an experienced elementary mathematics teacher with ten years of teaching experience. She graduated from ODTÜ Elementary Education PhD Program, conducted distinct instructional sequences with her previous seventh-grade classes, and attended "Design Research in Education" at METU Faculty of Education. The other teacher, Merve, had volunteered to participate in the study, and she had never been experienced in such an instructional sequence before and had never heard about design research. At school, Merve was responsible for the mathematics lessons in three classrooms: 7/X, 7/Y, and 8/V in the 2016-2017 fall semester. The weekly schedule of Teacher Merve is provided in Table 3.3. Abbreviation INC represented the study hour of inclusion students with Teacher Merve.

Table 3. 1

Weekly Schedule of Teacher Merve

	8.30-9.10	9.20-10.00	10.10-10.50	11.00-11.40	11.50-12.30	13.10-13.50	14.00-14.40
Monday	7/Y	7/Y	7/X	INC1			
Tuesday	8/V	8/V	INC2	7/Y	7/Y		
Wednesday	7/X	7/X	8/V	8/V			
Thursday	7/X	7/X	7/Y	INC3	8/V		
Friday	8/V	8/V	7/X	7/X		7/Y	7/Y

We studied with two classes, 7/X (including 20 boys and 18 girls) and 7/Y (including 22 boys and 16 girls), which were overcrowded compared to the other classes in the school. The sizes of the two classes were 38 in total. Demographic information of the classroom is presented in Table 3.4. The physical setting of the classroom included one smartboard integrated with two whiteboards, one cupboard, and desks.

Table 3. 2

Demographic Information of The Two Classrooms

	7/X	7/Y
The Number of the Students	38	38
Boys	20	22
Girls	18	16
Academic Achievement in Mathematics	72, 63	72, 40

Each classroom had at least one inclusion student, and their cognitive level might change from student to student. There were two inclusion students in 7/X and one inclusion student in 7/Y. The inclusion students in 7/X, Damla, and İsmet could study the topics of middle-grade mathematics with help and extra care. On the other hand, the inclusion student in 7/Y, Atıf, could not read, write, and count and had no cardinality or one-to-one correspondence skills for mathematics. It was tough to engage with him throughout the lesson. He hardly understood the social norms of the classroom, but responsible teachers and the staff in the center of guidance and counseling were taking care of him during their one-to-one studies, ten hours in a week in total at most. Atıf also attended a private school where the curriculum was designed according to the mental retardation of the children who participated in one-to-one sessions with special education experts. As far as the researcher observed and talked with students and the teacher Merve and Zehra, it might be true that Atıf changed the atmosphere of the classroom 7/Y. He always needed extra care from the teacher in the classroom. He might scream, stand up, have a fight with other friends, hug them all of a sudden, and the like, which changed the flow of the course. This information helped us select primary data from 7/X because Atıf's situation influenced the time management in 7/Y.

Based on the observation and interview protocols at the beginning of the lesson, two distinct learning environments emerged due to the instructional sequence implementation. The previous classroom teaching-learning routine was based on the primarily telling-explaining relationships as described in Table 3.5. The students had a LED experience for the first time. The role of the teacher changed to allow students' engagement in mathematical situations and reveal their way of mathematical thinking through the LED cycle. Apart from the researcher, the teacher guided the classroom learning with her questions. The researcher and the teacher actively participated in daily planning (Cobb & Yackel, 1996). Each student had their mathematical reality. These realities reflected what students said and did while engaging in mathematical activity. The role of the researcher was to model these "students' mathematics" and to present the "mathematics of the students" (Steffe & Thompson, 2000). That is, the students individually shaped their understanding of mathematics through interaction with the social environment. The focus was on students' ways of thinking, and interactive mathematics communication was at the center of this method.

Ratio and proportion concepts started at the sixth-grade level as offered by the middle-grades national mathematics curriculum (MoNE, 2013). These concepts were mainly covered in the seventh grade with respect to the other grades in the lower secondary level. Seventh grade was suitable to implement the instructional sequence because of its content and schedule. Along with these, there were some expectations from the students to maximize the efficiency of the classroom discussion. The implementation started with an agreement reached at the beginning of the instructional sequence (see Figure 3.8). These obligations were documented by Cobb et al. (1988) and suggested for the mathematics learning environment by Stephan and colleagues (2015). It was noted that the classroom environment should encourage students to explain their reasoning; ask questions when they do not understand; critique and understand the reasoning of others and use mistakes as sites for learning opportunities (Stephan et al., 2015). Cobb et al. (2001) considered this kind of blended social and sociomathematical norms normative ways to conduct a helpful classroom environment.

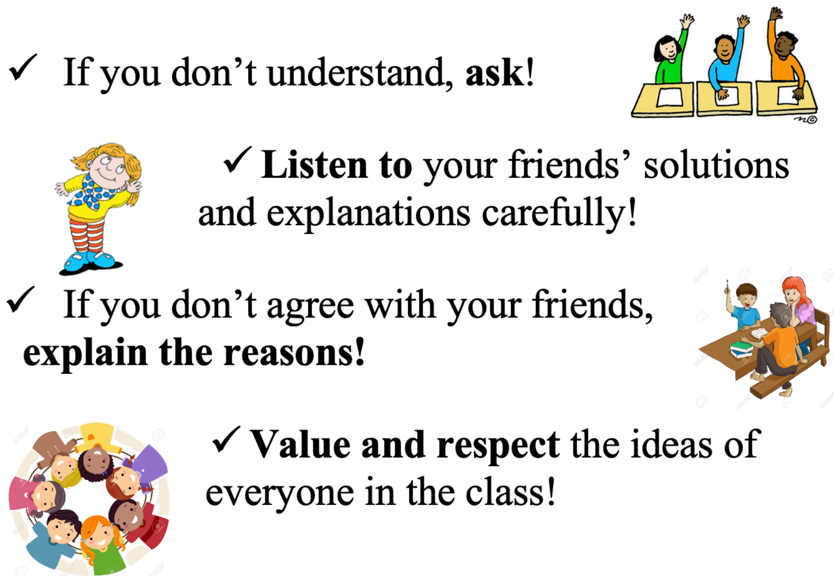


Figure 3. 8. Agreement on the classroom rules

Previously, Teacher Merve was asked if it was possible to provide such an environment for this research. She explained that the students already agreed and interiorized the idea of asking when they did not understand, which the researcher also observed before the implementation. Before the instructional sequence started, one hour was used to discuss the rules throughout the lessons to maximize the learning opportunities. For learning, "Listening to each other" social norm was not wholly achieved. According to a small focus group interview study with 7/Y students for a list of elements of an effective mathematics classroom, they admitted first that listening to each other is essential. However, they also ironically agreed that they could never achieve this altogether. They were not motivated to listen to other friends' solutions to learn. Additionally, the students were not aware of their responsibility for their own learning. Peer-to-peer interaction for learning was emphasized. They needed to be approved of their solution method/result by the teacher/researcher as the only source of knowledge.

3.3.2.2. Two Classroom Settings

There were reasons to conduct ratio and proportion instructional sequence within two classrooms. First, Teacher Merve was responsible for two seventh-grade classrooms that year, and both classes were volunteers to be a part of this project. Therefore, two classrooms participated in the ratio and proportion instructional sequences and were involved in the entire process. Additionally, the researcher suggested selecting one of the classrooms as the guide for the implementation process, and the implementation started one week before the other class's implementation so that the emergent models and experience that teacher gained throughout the instructional sequence became an opportunity for the rich learning environment for both classrooms. Although the average academic achievement of the classrooms was similar for the two classrooms, they had different models and reasoning; therefore, the pace of the collective classroom activity differed throughout the process. Although 7/Y was selected randomly as the guide, 7/X completed 23 activities at the end of the semester, and 7/Y completed 16 activities (see Appendices E and F). In the beginning, the teacher had suggested 7/X as the primary classroom because the students' communication was better than 7/Y, which was also predicted to influence the pace of the instructional sequence. Therefore, the classroom progress of 7/Y was slower than 7/X due to the students' dynamism, as expected. However, the teacher and the researcher experienced the instructional sequence twice, and in some cases, they connected the emergent models of the students from two classrooms.

3.3.2.3. Preparation with the Design Team

Before starting the research, the researcher participated in a meeting with Dr. Stephan and Dr. Akyuz (content specialists) about the instructional sequence's fundamentals and details. Some readings were suggested to understand the whole ratio and proportion of the instructional design process. Based on the readings and guidance, the researcher reflected on the procedure of being the participant-observer in a classroom and making the teacher one of the designers of the classroom learning trajectory. Therefore, the researcher was in the field before the implementation began (for the agenda, see Appendices E and F). In the table, "the topics related to Geogebra" is written. The purpose of the present study was first to investigate the mathematical practices of seventh-grade students regarding the ratio and proportion in technology-assisted classroom settings. Theoretically, integrating technology with the ratio and proportion of instructional sequence could be helpful for classrooms with smartboards and tablets. Based on this, we studied Geogebra with Teacher Merve together, and she also supported the utility of the idea of discovering the linearity of proportionality and shrinking and extending the pictures, which made the six lesson-hour in total. But practically, there were some points we needed to consider based on the observations and discussions with the teacher and my advisor, Dr. Akyuz:

- The routine teaching-learning procedure was based on Initiate-Response-Evaluation (IRE) interaction. Students were not used to Launch-Explore-Discuss (LED) interaction. There was a lack of resources such as a documentation camera to reflect on the board so that students might show the explanations they did on the computer. That led to different anticipated explanations from the students than Stephan and colleagues (2015) suggested.

- The computer laboratory did not have a sufficient internet connection facility for an effective instructional sequence enrollment. The computers were checked to see whether they had enough performance to conduct sample alien Geogebra activity. The researcher had a small talk with the teacher responsible for computer education. Using these computers would be hard for the students to perform instructional sequences with Geogebra. It required high performance for using a computer, which those PCs would not achieve.
- Tablets were not delivered to the school as the administrators expected before the study.
- Some phases required software other than Geogebra to deliver them as a tool.

We did not want to add new challenges like dealing with technological issues such as connection, viruses, or slow-running PCs. Apart from that, it was more important to increase the participation rate, be used to LED interaction, and listen to each other for learning. The accessible literature also made us a little bit confused about GeoGebra integration for proportional thinking in the phase of introduction. All in all, we postponed technology integration for further studies. We continued with Teacher Merve studying the issues based on mathematical practices, ratio table tool, cross-multiplication algorithm, linking composite units and ratio and proportion hypothetical learning trajectory. Instructional sequences prepared by Stephan and colleagues (2015) adapted to the Turkish context by considering various issues because the objectives of the activities designed by CCSSI (2010) were similar to the seventh-grade mathematics curriculum in Turkey (MoNE, 2013).

- Most of the contexts were present in the original order.
- There were no misplaced graphics, incomplete texts, or incorrect options.
- The translations were inserted precisely (correct spelling, no missing words).
- The graphics were printed colorful and correctly as in the original.
- Changes in proper nouns were a necessary adaptation for the current study. These included the names of people, cities, and official titles (if students were unfamiliar with these notations).
- Adaptations in mathematics and science notations were accepted, provided that they consider units of measurement, decimal notation, place value notation, and time

> (Maxwell, 1996). The original instructional sequence used imperial measure, which was substituted for the metric standard. As Maxwell stated, "…this was acceptable only when the values did not also need to be changed. For example, it is acceptable to change "six bags of flour, each weighing 10 lb," to "six bags of flour, each weighing 10 kg," but not to "six bags of flour, each weighing 22 lb"." (1996, p. 5).

The content of the instructional sequences and activity sheets were adapted based on the criteria above and the teachers' suggestions related to the Turkish context so that the students did not feel unfamiliar with the unmathematical situation, which might affect their perspective on the instructional sequences. Teacher Merve and Zehra checked them, commented on the changes, and recommended some ideas before the study and during the implementation. The teachers and the researcher had a copy of the teacher's instructional sequences throughout the classroom sessions. Although the role of Teacher Zehra was mentioned throughout the method chapter, it was important to summarize here that she checked out all the print-out materials, tasks, visited the classroom a couple of times and gave feedback about the instruction, evaluated the data analysis as peer debriefer.

3.4. Implementation Phase (2)

The second phase of the classroom teaching experiment describes the enactment of the ratio and proportion instructional sequence. *The implementation phase* provided snapshots of a Hypothetical Learning Trajectory, which guides the enactment of the teaching-learning environment (Bakker, 2018, p. 60), and how the data collected is another focus of this phase. An ongoing analysis informs the process of adapting and revising the conjectures of classroom events, which Stephan (2015) named the daily implement-analyze-revise cycle. How the ratio and proportion of instructional sequence were enacted are described here. This process was realized by adjusting the next activity based on students' previous interpretations, as suggested by Gravemeijer (1994). Because the main purpose of this study is to investigate the classroom mathematical practices of the seventh-grade students related to the ratio and proportion concepts, emerging ideas and actualized hypothetical learning trajectory were documented daily; therefore, the initial design was improved based on them. To conduct this analysis, we focused on some issues and the changes we made with the teacher and the expert (Advisor of the researcher). They are complementary to retrospective analysis. Instructional sequences were modified based on the needs of the students and the teacher's demands relating to the context of the tasks, a student workload, ratio table as a tool, and social norms. This was one of the actions in the implementation phase.

Implementation in 7/X lasted 34 hours and 23 activities; on the other hand, implementation in 7/Y lasted 39 hours and 16 activities. There were several reasons for this differentiation. Classroom 7/Y revealed the need for the understanding of fractions. While studying ratio tables in Activity 9, most students failed to calculate the unit ratio, vertical scale factor, and horizontal scale factor. Therefore, the teacher and the researcher prepared a worksheet to remind the procedural knowledge for the fractional operations. While conducting the tasks, the teacher and the researcher focused more on the problems involving fractional operations. Another reason was that the big ideas were achieved with more discussion than expected. This situation was also related to the potential of the classrooms in which students were to learn how to discuss, learn from each other, share ideas, and agree and disagree mathematically.

Classroom enactment required taking action to make a better learning environment for the ratio and proportion. While enacting, different solution strategies were encouraged to enrich the discussion environment, and the teacher and the researcher created some challenges to discuss based on the classroom needs. In these two cases, the students represented different needs. Teacher Merve conducted the instructional sequence along with the researcher; she was not only the implementor but also responsible for the flow of the discourse based on the goals and emergent ideas of the students; the teacher facilitated the whole class discussions and developed the tasks to make it more efficient for that classroom. All activities progressed according to the sequence, which was predetermined. However, the classroom decided on the pace of the instructional sequence.

3.4.1. Instructional Sequence and Its Implementation

HLT was planned around the current study's phases, starting from linking composite units to the vertical/horizontal scale factors. Aligned with the design research process requirements, HLT was constructed on some conjectures of the researcher and the participating teacher. Based on the taken-as-shared ways of reasoning of the participations, the conjectures were aligned with the anticipated hypothetical learning trajectory.

Before starting the implementation, the researcher matched *big ideas* of the ratio and proportion of instructional sequence and *learning outcomes* for the national annual plan for elementary mathematics education for seventh grade. The teacher Merve and Zehra revised the table to describe to what extent this sequence was in harmony with the national mathematics curriculum (MoNE, 2013). Stephan and colleagues (2015) already defined the conjecture development of big ideas as given in Table 3.3 which shows the matrices of the two outcomes tentatively based on the dominant features of the activities. This table was developed to understand to what extent students would deal with the objectives of the national curriculum while engaging in big ideas of the activities. Apart from line graphs and inverse ratio situations, the activities provided most of the national curriculum suggested.

Table 3. 3

Activity Matrix That Represents Big Ideas and National Learning Outcomes (NLO)

NLO / Big Ideas	Find the missing term in the second pair so that it conveys a similar relationship with the first pair	Find the missing term that one of the quantities in the second pair is equal to	Examines real-life situations, tables, or line graphs to determine whether two quantities are proportional.	Represent direct proportional relationship between two quantities as tables or equations.	Identify the constant of proportionality for simple direct ratios and interpret.	Solve the problems related to the direct and inverse ratio situations
Linking composite units	Activity 1-2	-	-	-	-	Activity 1-2
Iterating linked composites	Activity 2-4	Activity 4	-	-	-	Activity 2-4
Build-up strategy Abbreviated Build-up Strategy	Activity 3-4	Activity 4	Activity 3	Activity 3-4	Activity 3-4	Activity 3-4
Additive v. multiplicative reasoning	Activity 5	-	Activity 5	-	Activity 5	Activity 5
Structuring ratios multiplicatively	Activity 3-7	Activity 3-7	Activity 3-7	Activity 3-7	Activity 3-7	Activity 3-7
Creating equivalent ratios	Activity 8-10	-	Activity 8-10	Activity 8-10	Activity 8-10	Activity 8-10
Creating equivalent ratios	Activity 11 & 14-16	-	Activity 11 & 14-16	Activity 11 & 14-16	Activity 11 & 14-16	Activity 11 & 14-16
Analyzing equivalent ratios	Activity 11-13	-	-	Activity 11-13	Activity 11-13	Page 1 & Page 14
Comparing ratios	-	-	-	Activity 17-22	Activity 17-22	Activity 17-22
Comparing rates	-	-	--	Activity 22-24	Activity 22-24	Activity 22-26

The activities were already designed and ordered based on the big ideas, and there were 39 activities, and 26 of them were listed based on the big ideas in the given instructional sequence (Stephan et al., 2015). Activity 24-39 is designed based on the percent and percentage concepts following ratio and proportion concepts. All activities were adapted into Turkish, and it was planned to implement the first 26 activities. Although the implementation started one week earlier than the suggested annual mathematics education plan, various issues occurred throughout the implementation, as seen in Appendix E, which made progress stay behind schedule. The implementation process lasted seven weeks, with 23 activities at the end of the semester. The process did not continue in the spring semester of 2017.

A rich discussion environment is one of the enlightened commonalities described by the studies supporting both the individual and social dimensions of learning (see Knapp, 2019; Stephan & Akyuz, 2012). Such class discussions provide a crucial learning environment where student knowledge is contradictory, refuted, enriched, or altered in mathematics learning and teaching (Forman, 1996; Nathan and Knuth, 2003). Nevertheless, learning environments involving whole-class discussions are highly recommended for such environments rather than studying in groups or pairs (Waring, 2009). Approaches that are only followed through books and include a narrative-listening relationship (Prawat, 1992) may not meet the needs of learning environments where the social aspect of mathematics learning is enriched. Launch-Explore-Discuss teaching model, which deals with learning in two dimensions (social and individual), has been a tool in the discussion-centered teaching of the ratio-proportion issue, which is also directly suggested by Stephan and colleagues (2015) and embedded in the ratio and proportion instructional sequence.

There are other models similar to the Launch-Explore-Discuss teaching model. Launch-Explore-Summarize is one of them suggested in the 6^{th}–8^{th} -grade mathematics textbooks in the United States (Lappan et al., 2014). Summarize part involves the discussion of the topic and a brief conclusion of the task, which partially differs from Discuss Phase of the LED model, which highlights the debate of the students dominantly. LED model is used in the project called Vision Mathematics Vision Project (MVP, 2019), which is carried out by a team of academicians and educators providing mathematics materials and professional development services for teachers; it is seen as a teaching cycle that iterates for each task (MVP, 2017).

Each phase of the LED model suggests different roles for the teacher and the students. Therefore, it is vital to examine the teaching model stages closely. *Launch* phase can be carried out using a warm-up activity to attract students' attention or a reminder activity if it is to be continued through previous studies. In this way, the subject of the mathematics course is introduced at this stage (Stephan et al., 2015), and the student is motivated (MPV, 2017, n.d.). For example, before implementing the instructional sequences, the classroom watched a video about aliens coming to the Earth. The teacher told a story focusing on the emergent food needs of the aliens who invaded Earth. The tasks included the rules “one food bar feeds three aliens” and the problems considering these rules. In the *Explore* phase, the students studied the activity sheets or the main tasks, such as problems and activities, and the instructors allowed them to explore their ideas. The student should have enough time to do the tasks; the teacher must not give a direct solution when the students are stuck, and the teacher should direct the students to their peers to discuss and find the answers together (Stephan et al., 2015). In the explore phase, the teacher's primary role is collecting, selecting, sorting information about students' ideas/solutions/models, and preparing for the *Discussion* phase (MPV, 2017; Stephan et al., 2015). In the discussion environment, the teacher invites the volunteer students to a place where their friends can see their works to determine their correctness after each solution and/or idea (Stephan et al., 2015). Students are expected to make a joint decision, but the teacher has an essential role if everyone accepts the wrong idea or if no idea comes from the students. Therefore, the discussion phase is critical to make the emergent models come to the surface and make the students present them. Table 3.4 was organized based on the suggestions for teachers.

Table 3. 4

Launch-Explore-Discuss Teaching Model (MPV, n.d.; Stephan et al., 2015)

The Phases of LED	Planning Lesson	Teacher, Researcher Role in Teacher-Student Interaction
Launch	Be prepared for possible student ideas; anticipate!	Prepare activities that support students' reasoning and problem-solving skills! Use and connect different mathematical representations!
Explore	Follow students' ideas! Choose specific ideas!	Support productive struggle for learning.
Discuss	Sort the selected ideas! Connect different ideas!	Ask purposeful questions! Allow for mathematically meaningful discourses! Uncover students' ideas and use them as evidence!

As described above, Teacher Merve was informed about LED and its nature to create a suitable learning environment for the instructional sequence. As previously observed by the researcher, Teacher Merve was given feedback about productive struggle and uncovering the students' ideas, which rarely occurred before the study. Teacher Merve improved herself when practicing the behaviors shown in Table 3.4. She gave the students time to think and discuss their ideas. Teacher Merve and the researcher followed the students' solutions during the explore phase and noted them according to the suggestions of the ratio and proportion instructional sequence. Teacher Merve and the researcher frequently came together during explore phase and discuss phase to improve the learning as recommended in the sequence. Additionally, Teacher Merve revoiced the students' ideas to summarize an argumentation process. Simultaneously, the researcher was responsible for the equipment (activity sheets, cameras, word-hypothesis-theory walls, and the like). She also took field notes and met with Teacher Merve before, during, or in the end of the activities in order to evaluate the daily mini-cycles. While Teacher Merve was active in improving the design for the instruction, the researcher was a participant observer during the enactment. The researcher helped the teachers and the students to make the teaching-learning process efficient

3.4.2. Data Collection

Data were collected from mainly the classroom sessions and the students' activity sheets. Moreover, in the participant observation study, unstructured interviews with Teacher Merve were conducted to evaluate the daily cycles.

Two cameras videotaped each class session throughout the instructional sequence implementation. While one of them videotaped the implementation from behind the classroom, another videotaped it from the front view (see Figure 3.10). The whole class sessions were recorded from the beginning to the end. The front camera remained next to the teacher's desk and focused on the whiteboard. The other was dynamic, and the researcher controlled the camera by focusing on the students coming to the board. Therefore, the teacher, the students, and their interactions were recorded to investigate the students' mathematical practices related to the ratio and proportion context. The researcher attended the mathematics class as a non-participant observer before the study started to understand the school climate and get to know teacher-student interaction. As suggested (Fraenkel et al., 2011)., being familiar with the school context was a helpful procedure for the researcher to enrich the field notes and memos about the students and the school staff.

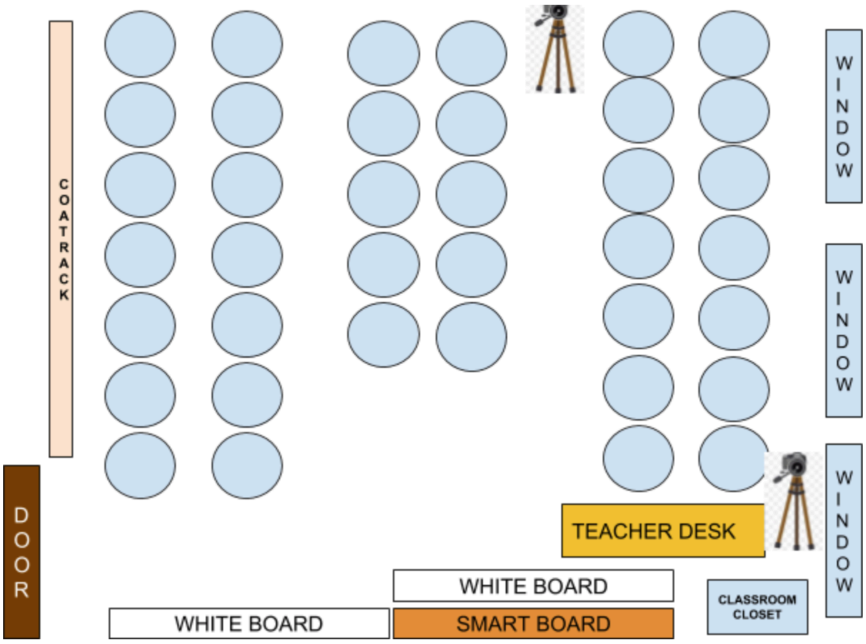


Figure 3. 9. Classroom plan 7/X

The researcher and Teacher Merve attended all the classroom sessions. 34 lesson hours were spent to enact the instructional sequence in 7/X; on the other hand, 39 lesson hours were spent in 7/Y (see Appendices E and F). Each session had whole-class discussion parts in which the meaning-making was collectively done. The video recordings consisted of these argumentations and screenshots.

23 activity sheets were collected for each student throughout the instructional sequence. The students were asked to fill out the activity sheets in the launch phase of LED. These sheets were introduced as the notebook in which they could make whatever necessary changes they needed for the questions. The duration of the activity was determined based on the students' performances. If the task was not finished at the end of the class session, the researcher collected the sheets so they would be ready for the next session to study. There were a variety of reasons to collect these documents. First, they supported students' explanations as another visual representation. Second, some students did not prefer to talk in front of the classroom, and it was hard to follow each student's contribution to the whole class's progress by investigating the activity on the whiteboard. Therefore, the activity sheets reflected the collective activity. Third, the students' might have used a distinct method on the sheet than they represented on the whiteboard. That means even those students themselves could challenge their ideas. Fourth, the students might have consistently used their ideas on the

activity sheet even if they did not insist on their strategies throughout the discussions. These reasons helped fill the gaps left in the video.

As a participant-observation study, the interviews were not the dominant data-gathering strategy, but it was employed in conjunction with classroom sessions and participant observation (Bogdan & Biklen, 2007). Teacher Merve and the researcher shared their observations through the instructional sequence during daily cycles. They informally came together before the new activity, during the activity, and at the end of the activity. Those meetings were about what was happening in the classroom sessions regarding the big ideas in the instructional sequences and objectives of the national curriculum. On the other side, students' errors, misconceptions, invented strategies, argumentations, emerging teaching strategies, and their influence on each activity were discussed basically during the activity. After each class session ended, there was an informal conversation about planning for the following classroom session. In addition, Teacher Zehra participated in these meetings several times to understand the classroom progression and offered some contributions.

Another data-gathering strategy was to take field notes during the daily cycle. In her notes, she recorded the students' learning activity, attendance, contribution, potential mathematical practices, and unexpected activities in the class. Several of them were discussed with Teacher Merve in order to evaluate the effectiveness of the instructional sequence to reach the Big Ideas with several attempts. These researchers' notes were called *reflective field notes* by Bogdan and Biklen (2007). They helped the researcher understand the specifics of the classroom discussions.

Proportional reasoning is related to multiplicative thinking and the ability to differentiate between multiplicative and additive reasoning situations accordingly (Cramer et al., 1993; Lamon, 1993). The students support these skills using other skills related to the other domains in mathematics, such as fractions, decimals, and scaling (Lamon, 2012). Stephan and colleagues (2015) designed the instructional sequence to develop proportional reasoning considering these research-based elements, as mentioned in the literature chapter. Therefore, there needed to be a diagnostic test that involved those structures in evaluating the progress that the students performed. Until now, a formative assessment of the teaching-learning process was described. Additionally, pretest and posttest were also applied as summative assessments.

Hilton and colleagues (2013) developed a two-tiered instrument to measure the students' proportional reasoning before and after the teacher training intervention. They prepared the test considering proportional and non-proportional reasoning situations and selected the problems developed by Lamon (1993) and van Dooren et al. (2005). These problem types and examples are provided in Appendix C. In this context, this instrument was suitable for measuring students' development in proportional reasoning regarding missing value problems, comparison problems, and proportional and nonproportional situations.

The items of the instrument were selected and adopted mainly from Hilton and colleagues (2013) two-tier diagnostic instrument and Lamon's examples (1993). Selected items were translated into Turkish and transformed into an open-ended structure in order to investigate the students' reasoning for solving the problems. (see Table 3.5). Two mathematics teachers and one graduate mathematics education student revised the problems. Because the feedback was mainly related to the format of the instrument, there were minor changes related to the context and representation of the problems.

Table 3. 5

Investigation of the Test Items

Name of the Item	Adapted from	Mathematical Structure	Changes
1. Running	Hilton et al. (Running laps)	Non-proportional: Additive	Names
2. Cake	Hilton et al. (A Sticky mess)	Missing value; associated sets; part-part-whole	"Sticky mess" word
3. Bagel	Lamon (Balloons) (1993)	Missing value; associated sets; part-part-whole	Currency
4. Sport	Hilton et al. (End of term activities)	Relative thinking; associated sets	End of the term activities
5. Washing soap	Hilton et al. (Washing Days)	Relative thinking; Associated sets	Pictures
6. Choir	Hilton et al. (Sing Song)	Nonproportional: Constant	No change
7. Crayons	Hilton et al. (funky music)	Missing value; associated sets; part-part-whole	"iPod" word Name
8. Mixture	Hilton et al. (three cups)	Relative thinking; associated sets	No change
9. Profit	Lamon (1993)	Well-known measures	No change
10. Rectangle	Hilton et al. (drawing insects)	Growth problem/scale	"Area" word Pictures

This instrument was implemented for the seventh-grade and eighth-grade students to understand their learning, covering the required knowledge and skills for mastering proportional reasoning. For the pilot study, it was first implemented with eighth-grade level students. After appropriate changes were done, the pretest was given to 7/X and 7/Y one week before the instructional sequence started. Posttest were also applied immediately after the study ended.

Pre-test and post-test were applied to both classrooms. The students who were not available that day were given the test another day. Moreover, Teacher Merve embedded several questions in the second mathematics lesson midterm as an assessment. It was her motivation to prepare and apply for this midterm. She explained that she wanted to see what was happening in students' minds regarding the ratio table and "proportional reasoning with aliens." This was another source of knowledge to see students' learning and crosscheck the results with the post-test results so that this may give an idea about the reliability of the post-test results. After the study ended, the researcher continued to visit the children and the teacher to inform them about the pre-test and post-test results.

In the instrument, van Dooren et al. (2005) categorized the items as easy and difficult regarding the range of numbers. The criteria for this categorization were 1-40 numerical values for the easy items and 1-100 for the difficult variants. Similarly, the test also involved values from easy and difficult categorization. The instrument requires decimal and integer results as the instructional sequence was designed to develop non-integer relationships. Another property of the test is that it started with the nonproportional situation to eliminate the expectancy that "each problem is related to proportionality." Students had 40 minutes per lesson-hour to finish the test, and they were asked to write their way of reasoning and calculations.

3.5. Data Analysis Phases (3): Qualitative and Quantitative Data

Both quantitative and qualitative methods were used to answer the research question complementarily. The qualitative part included multilevel analysis in which *the analysis in progress* and *retrospective analysis* was conducted (Cobb et al., 2003). It was such a complicated process that there needed to be some tools and strategies that helped to analyze the data in an empirically grounded method. After all data were gathered, the retrospective analysis was conducted to reveal a complete picture of the process so that mathematical practices with effective and ineffective strategies were provided (Cobb et al., 2003) for the current study and future designs. In this part, the data analysis process is explained in detail.

Since analyses in progress aimed to describe the justification of the enacted instructional sequence, the information on the analysis in progress was in the *Implementation Phase* part. On the quantitative part, the pre-test and post-test analyses were presented accordingly.

3.5.1. Emergent Perspective

Psychological constructivism and sociocultural views are two perspectives that have affected the educational processes as a way of teaching and learning mathematics. Psychological constructivism focuses on learning as the individual's cognitive processes. On the other hand, the sociocultural way of learning discusses the social dimension as the primary source of learning because knowledge is transmitted through sociocultural practices (Cobb et al., 1996; Simon, 1995). Cobb (1994) provided an example from mathematics education practices to distinguish between the views such as participation practices of the students and individuals' sensory-motor development. The model for learning mathematics in this research was based on the emergent perspective known as social constructivism coordinating the constructs from these two paradigms (Stephan, 2003). This model magnifies the progress of learning in person and community. The emergent perspective originated from symbolic interactionism, in which the development of meaning was not just intrinsic but also required interaction so that the interpretation may occur (Blumer, 1969; Yackel, 2000; Yackel & Cobb, 1996). Within this context, mathematical learning also emerged through not only individual construction but also social interaction (Cobb & Bauersfeld, 1995; Yackel & Cobb, 1996). Social interaction did not refer to students' state of being in the classroom but engaging in the learning activities through changing, abandoning, and retaining their ideas (Yackel, 2000). On the other hand, the psychological perspective considers the individual contribution to the collective learning processes (Cobb & Yackel, 1996). Cobb and Bauersfeld stated, "Neither an individual student's mathematical activity nor the classroom microculture can be adequately accounted for without considering the other" (1995, pp. 9–10). This statement focuses on the claim that individual development might be related to the emergence of the social activity which emerges over time (Bowers & Nickerson, 2001). These two aspects of learning are considered complementary and inseparable.

The ratio and proportion instructional sequence was designed to increase social interaction for the sake of individual and community learning. The essential elements of the learning ecology were informal and formal tools, the teaching cycle (Launch-Explore-Discuss), and peer-to-peer and teacher-to-peer discourse support. Emergent perspective flourished the idea of supporting students through social interaction while constructing their knowledge by abandoning the idea of leaving the children alone with the teacher-made knowledge.

Classroom interaction is one of the important aspects of emergent perspective. Argumentation is a collection of classroom interaction activities consisting of making claims, challenging them, backing them up by producing reasons, criticizing those reasons, rebutting those criticisms, and so on (Toulmin, 1984, p.14). In the shared classroom community, these incidents involved the interactions between the students and the teacher. Each incident included an argumentation process explained by Krummheuer: "an argumentation usually contains a sequence of statements, each of which plays a different role in the emerging argument" (1995, p.239). In the classroom discussions, children's mathematical practices lies in the public discourse in a condition that an argument in an argumentation cannot always refer formal logic such as mathematical proofs or logic (Krummheuer, 2000; Toulmin, 1984). These argumentations were investigated in detail based on the model created by Toulmin (1958, 2003) to analyze these different roles of the statements in the arguments. It is a kind of scaffold of a main argument. Toulmin's argumentation schema was eligible to investigate the taken-as-shared forms of argumentation that constitute the mathematical practices (Cobb et al., 2011; Krummheuer, 1995, 2007). In an argumentation situation, a participant may demonstrate his/her rationality or irrationality to the process through reasoning in advocate of or against the proposed idea. Krummheuer and Yackel (1991) presented examples of mathematical argument that created social interaction from the emergent perspective focusing on the cognitive conflict. In the further studies, Krummheuer (1991) concluded that

(a) The possibility of learning was rather given when the students developed different interpretations of the problem which they further pursued in their cooperative problem-solving attempts.
(b) Learning was also facilitated when the students tried to compare these alternative interpretations argumentatively.
(c) Learning of the individual took place when these different interpretations generated a cognitive conflict and the comparative argumentations of the interaction help him to cope with this internal conflict (1991, p.263)

The standards of argumentation established in an inquiry classroom are such that the teacher and students typically challenge explanations that merely describe the manipulation of symbols. Further, acceptable explanations appear to carry the significance of acting on taken-as- shared mathematical objects (Cobb et al., 1992). For this respect, Voigt (1996) studied on this issue to place social interaction within learning. Voigt put an emphasis on the relationship between mathematical meanings and emphasis on mathematical processes. Consequently, the teacher and students seem to be acting in a taken-as-shared mathematical reality, and to be elaborating that reality in the course of their ongoing negotiations of mathematical meanings.

3.5.2. Interpretive Framework

Students' interactions and argumentations may be actualized from teacher to student or students to student in a small group study or whole class discussion. It was reported that small group study increases the learning opportunities (Cobb, 1995; Yackel et al., 1991). Cobb (1995) described small group study both its constraints and opportunities that cognitive and social process presented: first, children's cognitive capabilities could be limited, and they might establish several learning opportunities while studying in pairs; second, selected opportunity may constrain other learning opportunity and children's cognitive capabilities. In small group works students are expected collaboratively to make sense of the mathematical situation by explaining their reasoning, listening to the others in the group (Yackel, 1995). Similar processes are valid in the whole class discussion. It creates a space for every individual or group to repeat the social and sociomathematical norms in the whole class discussion to constitute the classroom mathematical practices. Therefore, it is no surprising that individual learning formed within the discussion and groups development formed by the individual contributions, so these two are interrelatedly dependent to each other.

The mathematics learning environment involves highly complex learner and teacher interactions. While analyzing these collective activities, interpretive framework, modeling the emergent perspective, is used to organize the teacher's and students' way of interaction (Cobb & Yackel, 1996). This framework considers psychological (individual) and social (a classroom community) aspects of mathematical learning, as shown in Table 3.6.

Table 3. 6.

Interpretive Framework for Mathematical Activity (Cobb et al., 2001)

Social Perspective	Psychological Perspective
Classroom social norms	Beliefs about our own roles, others' roles, and the general nature of mathematical activity in school
Sociomathematical norms	Mathematical beliefs and values
Classroom mathematical practices	Mathematical interpretations and reasoning

Each aspect consists of three components (Cobb et al., 2003). Although the focus of the current study is classroom mathematical practices of the students, each construct is described briefly in Table 3.6. The constructs within each aspect are ordered from general to specific learning activities. Psychological perspective consists of individual beliefs related to the constructs of teaching and learning in general and specific to mathematics, which focuses on individuals' knowledge of and about mathematics, and mathematical activities, individual understanding of other's roles, personal beliefs, values, interpretations, and reasoning (Cobb et al., 2001). Basically, it focuses on individual understanding related to mathematical knowledge and the learning environment.

On the other hand, the social perspective highlights the construct "norms," which are described as "…regularities in interaction patterns and, as such, are interactively constituted by the classroom participants, including the teacher and the students." (Stephan et al., 2014, p. 42). Social norms are kind of taken as shared regularities to support learning such as explaining and justifying solutions in whole class discussions (Cobb, Yackel & Wood, 1995). In that, they are the taken-as-shared behaviors do not have to be specific for a discipline, moreover, these norms sustain classroom cultures characterized by explanations, justifications and argumentation (Cobb et al., 1992). These are like communication social skills including explaining one's mathematical thinking to peers, listening to and attempting to make sense of the peer's explanations, challenging explanations that do not seem reasonable, justifying interpretations and solutions in response to challenges. and agreement on an answer and. ideally, a solution method (Cobb, 1995). In the current study, the instructional sequence created a learning environment that reinforced the students to provide contextually relevant mathematical reasoning. Talking about irrelevant situations was a social norm to avoid. All in all, social norms were formed around the social-interaction rules of mathematical activity.

Yackel and Cobb (1996), who focused on the study of conditions that create opportunities for learning mathematics, identified another class of norms that are about the actual process by which students and teacher contribute. They called such norms sociomathematical, to designate the classrooms social constructs specific to mathematics that individuals negotiate in discussions to develop their personal understandings. Of equal importance, sociomathematical norms can be described as distinct, sophisticated, efficient solutions for a mathematical activity (Bowers et al.,1999). They express the students' ideas which the classroom participants count as acceptable mathematical explanations (Yackel & Cobb, 1996). During this study, the students negotiated what an acceptable solution was and identified what constituted a different solution to a given problem with the help of Teacher Merve and the researcher. Social and sociomathematical norms may not be specific to that mathematical topic, but they help learn mathematics. The emergent perspective places face-to- face classroom social norms against general beliefs, sociomathematical be- liefs against mathematical beliefs and values, and classroom mathematical practices against mathematical conceptions. Sociomathematical norms is taken as shared way of social learning specific to the mathematics, which also increases learning opportunities of mathematics (Yackel & Cobb, 1996). Herschkowitz and Schwarz (1999) focused on sociomathematical norms in their Project CompuMATH. They reported three examples from sociomathematical norms about evidence, a good hypothesis, and acceptable explanations.

During negotiation, discussion and argumentation process, students share their ideas, which may become another student's reasoning (Steele, 2001). Classroom mathematical practices are students' interpretations of the mathematical tasks in an instructional sequence resulting from interactive engagement with the activity to give meaning (Bowers et al., 1999; Cobb et al., 2001/2011). They are taken-as shared ideas that are not already decided forms of reasoning (Cobb et al., 2001) and do not have to operate in identical ways (Steffe & Thompson, 2000). In other words, since the way of students' construction of their knowledge is constrained with their perception, giving meaning to that interaction, and participation, students may have distinct conceptions (Yackel, 1995). Classroom mathematical practice emerges from these ideas that students in the same classroom understand the idea and react to it, although the reasoning may be different from theirs (Rasmussen & Stephan, 2008). The process of emergence of mathematical practices in a taken-as- shared way is beneficial to develop mathematical practices and also mathematical understanding and individuals' reasoning. These practices represent the ways of understanding, reasoning, explaining and convincing others by justifications in a way that mathematical classroom community make them taken-as-shared for the particular mathematical content by specific mathematical tasks or ideas (Cobb et al., 2011; Stephan, Bowers & Cobb, 2003).

Therefore, unlike social and sociomathematical norms, classroom mathematical practices are content-specific ideas developed throughout the instructional sequences and documented to identify the students' mathematical development (Bowers et al., 1999). They are taken-as shared ideas that are not already decided forms of reasoning (Cobb et al., 2001) and do not have to operate in identical ways (Steffe & Thompson, 2000). Classroom mathematical practice emerges from these ideas that students in the same classroom understand the idea and react to it, although the reasoning may be different from theirs. Within this context, this study aims to investigate the emerging classroom mathematical practices molded with the social and sociomathematical norms throughout the ratio and proportion instructional sequence enactment.

Aforementioned three constructs were all taken-as-shared ideas, strategies or behaviors influencing each other interrelatedly so that an optimum learning environment is created. Under the guidance of the teacher, these three constructs can be formed in favor of effective teaching and learning. In that case, not only students develop their learning but also their understanding of mathematics (Cobb et al., 1997). The development of classroom mathematical practices is supported by the reinforcement of such learning environments' formation of social and sociomathematical norms (Stephan & Akyuz, 2012). Apart from that, there is a growing body of research providing mathematical practices of the students from different grade levels ans different mathematical topics (Akyuz, 2014; Ayan Civak; 2020; Dixon, 2009; Roy; 2008; Stephan & Akyuz, 2012; Stephan & Rasmussen; 2002; Şahin Doğruer, 2018; Uygun, 2016).

While these emergent ways of reasoning likely shape students' individual ways of reasoning, they do not determine what students learn. Scholars working from the emergent perspective have always maintained that the relationship between normative ways of reasoning and students' ways of reasoning is indirect and reflexive (Cobb, 1995; Cobb et al., 2003; Rasmussen & Stephan, 2008). This means that as students participate in the social processes involved in creating emergent mathematical practices, this affects their personal conceptions, but a researcher should not assume that all students construct identical conceptions as a result of participation in the classroom.

3.5.3. Qualitative Analysis

After the classroom teaching experiment ended, all data were analyzed holistically and retrospectively to investigate the mathematical practices of the seventh-grade students. For the retrospective analysis, MAXQDA© 2022 was used to organize the data, thanks to METU's shared programs. The classroom discussions were not conversations that went fluently, smoothly, and one by one. Sometimes, more than three students started to talk simultaneously as an illustration, or one incident can be evaluated in another lesson hour. How this kind of complicated discussion is analyzed is the topic of the study developed by Rasmussen and Stephan (2008). To document the collective activity of the classroom community, they developed a three-phase method that was suggested for investigating students' normative ways of reasoning (Figure 3.10).

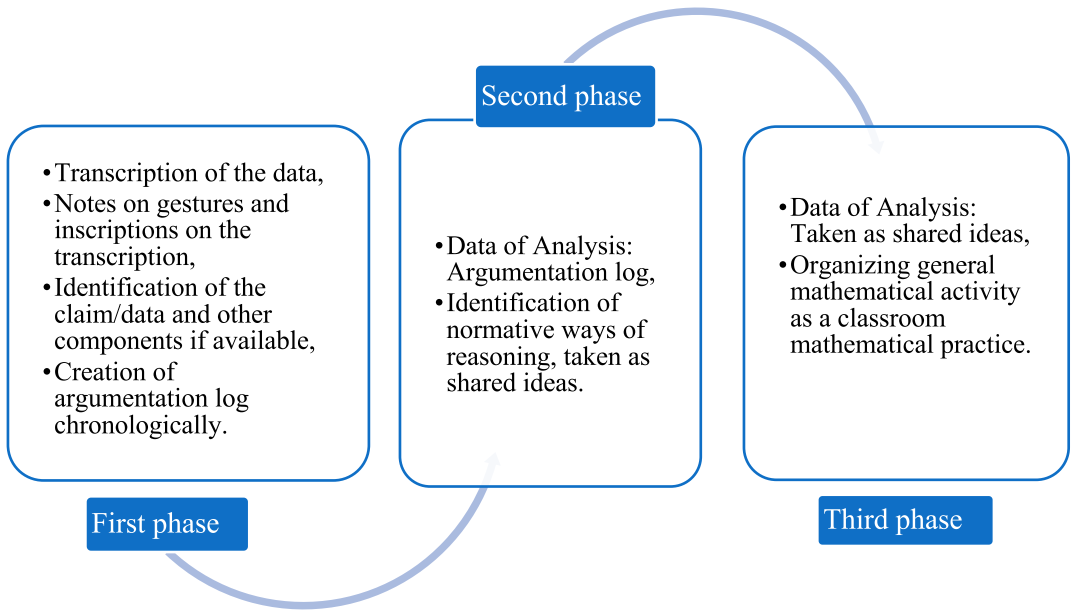


Figure 3. 10. Collective Activity Documentation *(Rasmussen & Stephan, 2008)*

The three-phase analysis organization has similarities with the constant comparison inquiry described first by Glasser and Strauss (1967) within the grounded theory approach. Constant comparison method aims to create a developmental theory in terms of the generation of theories of process, sequence, and change of organizations, positions, and social interaction. Cobb and Whitenack (1996) defined constant comparison inquiry as the theory grounded in the data analysis and lying on the conjectures and refutations that emerged in the classroom. Constant comparison analysis requires investigating similarities and differences of the data by constantly comparing them (Glasser & Strauss, 1967). Therefore, it was found suitable to examine the pattern of interactions of the students and teacher around a mathematical activity through a similar analysis method (Cobb et al., 2011). Constant comparison inquiry is an inductive method of theory development (Glasser, 1965) from descriptive to conceptual understanding (Butler-Kisber, 2010). At that point, the constant comparison inquiry differs from the three-phase method due to its data generation (Cobb et al., 2011). For this research, a frame comes from the initial analysis and is used to compare this frame with the new patterns that emerged in educational design. In constant comparison inquiry, the themes and categories were inductively devised anew with the data of that study (Cobb et al., 2011).

3.5.3.1. First Phase of the Retrospective Analysis

Data were chronologically transcribed from the video recordings (from Activity 1 to Activity 23). Two recordings from the two cameras were transcribed, and both were merged. These transcriptions were also enriched by means of the pictures taken from the videos, student activity sheets, field notes, and meetings with the teacher. The incidents in the transcripts were divided chronologically and categorized descriptively at the beginning. They consisted of refutations, conjectures, and revisions of the individual reasoning as it was suggested (Cobb et al., 2011). The data consisted of these collections of incidents in the first phase.

In the shared classroom community, these incidents involved the interactions between the students and the teacher. Each incident included an argumentation process explained by Krummheuer: "an argumentation usually contains a sequence of statements, each of which plays a different role in the emerging argument" (1995, p.239). In the classroom discussions, children's mathematical practices related to ratio and proportion context were embedded in these argumentations. These argumentations were investigated in detail based on the model created by Toulmin (1958, 2003) to analyze these different roles of the statements in the arguments (see Figure 3.9). It is a kind of scaffold of a main argument. Toulmin's argumentation schema was eligible to investigate the taken-as-shared forms of argumentation that constitute the mathematical practices (Cobb et al., 2011; Krummheuer, 1995, 2007).

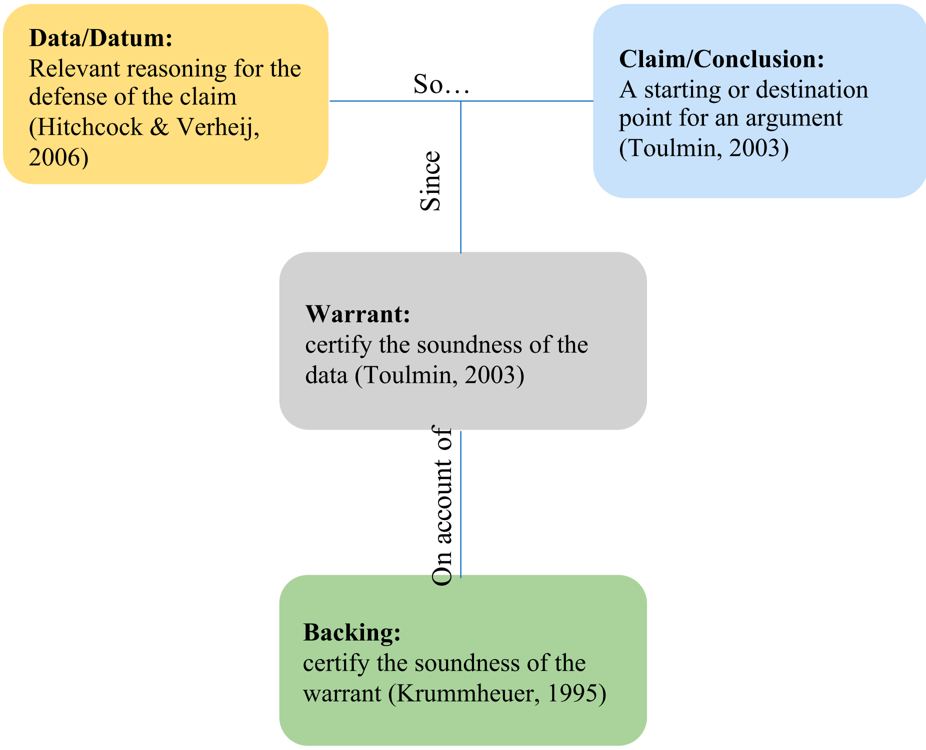


Figure 3. 11. The argumentation schema developed by Toulmin (2003)

While analyzing the arguments, there are several issues to consider. Toulmin (2003) describes the field-invariant and field-dependent aspects of an argument. Field-invariant part refers to the argumentation layout, which is the same in all fields; on the other hand, field-dependent aspect of an argument emphasizes different features of a field (van Eemeren et al., 2019). An example of a field-invariant aspect is the procedural nature of an argument. According to Toulmin (2003), a question is formed and asked to start the argumentative procedure. Possible solutions emerge, and the participants compare them to find "necessarily" or "the best solution". Besides, a field-dependent variable defines whether the condition of an argument is valid within that field (Toulmin, 2003). Krummheuer (1995) provided a framework for argumentations in mathematics education by using Toulmin's model. In this aspect, field dependency offers an area for developing a systematic investigation of arguments within its own context.

In the current study, the incidents happened in the classroom discussion where all students, Teacher Merve, and the researcher were involved. Each incident started with a question in the activity sheets and with an answer of the students to the question (field-invariant aspect of an argument). The students came up with a *claim* (C), and each incident evolved around this claim, producing a chain of arguments at the end (see Figure 3.9). *A claim* is one of the components in the argumentation structure described by Toulmin (1958, 2003), also called a *conclusion* (Krummheuer, 2007). Toulmin, Rieke, and Janik defined claim as the "…assertions put forward publicly for general acceptance" (1984, p.29). Similarly, a claim in this study refers to the starter or conclusion of arguments, generally stated as the result of the questions directed to the students during the activities. The validity of the claims is to be questioned throughout the argumentation process. It must be supported or refuted with several components of the argumentation model in need of the community. A claim might become *a conclusion* or destination point within an argumentation process. Metaphorically, the class tries to build working electrical circuits and tests them through argumentation. For example, there is a question in the instructional sequence asking, "Are there enough food bars (4) to feed nine aliens if the rule is that three aliens eat one food bar?" (See Figure 3.9). Berk claimed that there were more than enough food bars. His conclusion created a path to be enlightened with more details. The conclusion was detected in the transcripts as the answer to the questions.

Another component that provides ground for the claim is *data* (D) which is a way of delivering relevant reasoning for the defense of the claim (Hitchcock & Verheij, 2006). In the current study, verbal and written expressions, references to the previous solutions, mathematical algorithms, definitions, and the like became the *data* of the arguments. It points to the fact on which the claim is based (Toulmin, 2003). In the analysis of this study, it was not easy to differentiate the datum from other elements in the argumentation layout. Still, the nature of the study provided a frame for the components. When the students answered (a claim) to the question, the class asked, "how did he/she find it?". The following question accompanied an explanation of the strategy. For example, Berk replied to this question, "I multiplied four by three and got 12. Nine is the total number of aliens in the problem". He verbally explained his strategy on his activity sheet, calculated the numbers, and used a multiplication algorithm to conclude. Still, his verbal explanation was insufficient to reach a conclusion (See Figure 3.9).

At this point, a *warrant* (W) is used to reinforce the relationship between data and claim. In the structure of the arguments, warrants are used to "certify the soundness of the data" (Toulmin, 2003, p. 92). Practical aspects of assessing arguments outside mathematics can be considered to differentiate data and warrant (van Eemeren et al., 2019). That is, each classroom discussion has its own nature of argumentation based on the mathematical topic and social learning of the classroom. In this study, warrants emerged to explain a datum. For example, Berk verbally explained “doing a calculation,” as given in Figure 3.9. While the instructional sequence continued, there was no need to add a warrant explicitly because it turned out to be a normative way of reasoning.

There is another construct in the schema, “backing (B)” of the warrant. Warrant and backing are two structures that might have been confused just like the others. A backing is used to certify the soundness of the warrant (Krummheuer, 1995). In this study, a backing is generally proposed, provided that the challenging component of the argument is a warrant. Rasmussen and Stephan (2008) differentiated a backing from a warrant as the answer to "Why should I accept your argument (the core) as being sound mathematically?" (p. 197). Backings, therefore, function to give validity to the argumentation. The class compared nine aliens given in the problem and 12 aliens found in the datum, and enough food bar conclusion was emphasized because 12 is larger than nine as *Backing,* implicitly.

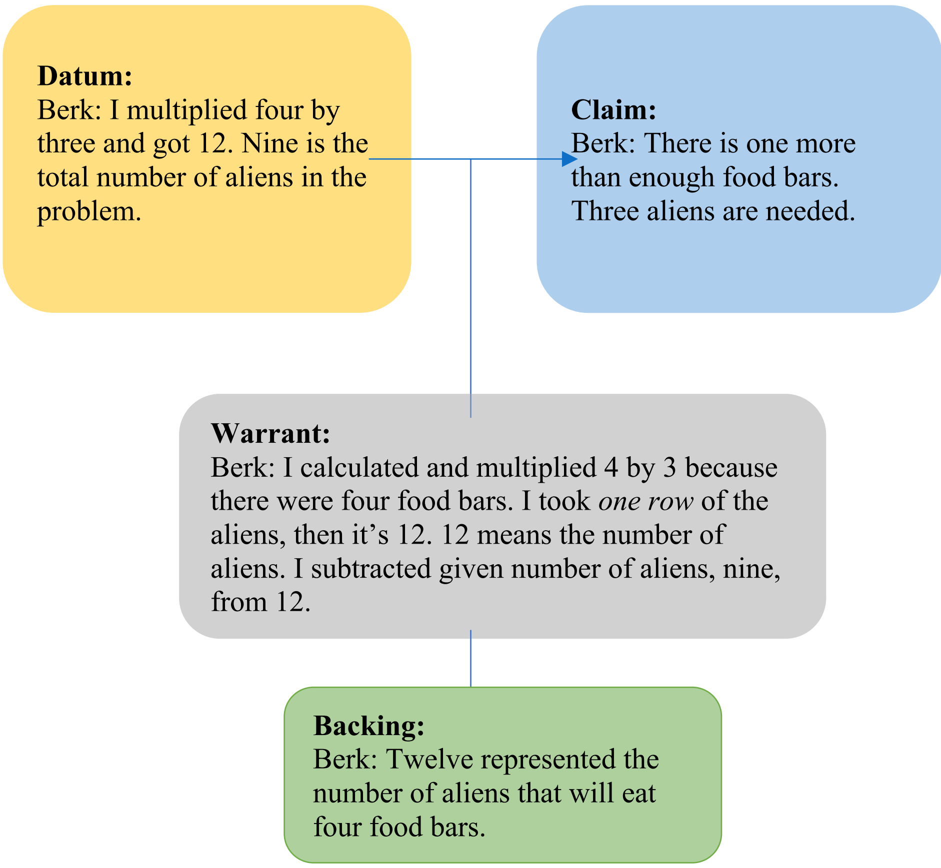


Figure 3. 12. An example of a part of the argumentation

In argumentation, a claim can be either justified, refuted, or unevaluated. The degree of value of the warrants may change from argument to argument; sometimes, it might be so strong that each person in the discussion agrees on the claim; however, some warrants were insufficient to explain a claim. At this point, Toulmin (2003) provided two terms for explicit references of relevant qualifications and conditions to support a warrant: Qualifier (Q) and Rebuttal (R). A qualifier is to define to what extent our conclusion is satisfied, such as "probably" or "necessarily" (Hitchcock & Verheij, 2006, p. 2), which did not explicitly emerge in this study. On the other hand, a rebuttal is a statement of refutation that can be against a datum; a claim; a warrant; a bridge from a datum to a claim; a bridge among a warrant, a datum, and a suit (Hitchcock & Verheij, 2006), which increased the quality of discussion. The rebuttal may provide an alternative claim or counterevidence for a claim (McNeill & Martin, 2011). In this study, rebuttals were one of the learning tools to question the understanding of phases of the instructional sequence. The rebuttals were easier to identify compared to warrant and backing. In Figure 3.9, Berk clarified the meaning of 12, indirectly its relationship with the claim, and summed the argument up. Another student rebutted the backing and the warrant relationship and said, "You need to tell exactly how many food bars are required". That is, 12 was not the exact answer. The relationship between 12 and enough food bars should have been constructed according to the class. The student accepted the preposition of the rebuttal and produced a new backing and told, "three food bars required for nine aliens; therefore, one food bar left". This rebuttal was an example of dismissed rebuttal and did not emerge in the argumentation layout (Figure 3.9). Some rebuttals refuted an argument and became a datum of a new argument. This incident is one of the analyzed arguments noted in the argumentation log.

After the transcriptions were complemented with field notes and activity sheets, as Glaser and Strauss (1967) suggested, the parts of Toulmin's argumentation schemes (1958, 2003) were identified as the argumentation patterns in the incidents. Two researchers studied the parts of the schemes to reach an agreement about missing points, disagreements and to crosscheck the points (Glaser & Strauss, 1967; Rasmussen & Stephan, 2008). At the end of the phase, in the case of agreement, the argumentation patterns formed the log of argumentation in which they were ordered chronologically (see Appendix H, an example of an argumentation scheme). This kind of argumentation log became the unit of analysis for the second phase. Two argumentation logs were created based on the needs. One of them gathers the evolution of a specific understanding under a code, such as drawing strategy or unit ratio, because an idea may not grow in the same lesson hour. On the other hand, other argumentation log helps to see all the arguments holistically in a chronologically linear order.

3.5.3.2. Second Phase of the Retrospective Analysis

The researchers investigated the argumentation log in the second phase in terms of taken as shared ideas and potential taken as shared ideas and cross-compare consecutive lessons by several criteria described below.

> *C1*: If backings or warrants are initially needed to establish a claim but later become no longer necessary, the way of reasoning is considered normative (dropping of) (Rassmussen & Stephan, 2008).

The dropping of a warrant or a backing in the argument means that they no longer appear explicitly in a discussion, but the classroom has already constructed what a claim implies. Therefore, there may be no challenge to the components of the argument, which may end up with only a datum and a claim. Similarly, a student challenge an argumentation, which is later rejected; the argument is self-evident and considered a normative way of reasoning (Rasmussen and Stephan, 2008). To sum up, mathematical ideas are accepted in that they no longer need to be justified.

> *C2*: If a piece of information shifts the function it plays in the argument, a way of reasoning is established as normative (e.g., shifting a claim to a datum) (Rassmussen & Stephan, 2008).

A class can be considered the mathematical idea widely accepted when any of the four components of an argument—data, warrant, claim, or backing—shift their functions in subsequent arguments and remain uncontested (or, if contested, the challenges are dismissed). For example, when students use a previously justified claim as an unchallenged justification (the data, warrant, or backing) for future arguments, we conclude that the mathematical idea expressed in the claim becomes a part of the group's normative ways of reasoning (Rasmussen and Stephan, 2008). In conclusion, mathematical ideas are accepted in that they can be used to support new ideas under consideration.

> *C3*: Repeated use of an idea as a datum or a warrant is a third way to establish normative ways of reasoning (Cole et al., 2012).

Cole and colleagues (2012) developed the last criterion to define repeated emergence of a datum or a warrant in arguments so that a way of reasoning has become a standard way of reasoning in the class. In this study, the students carried a mathematical idea in an argument they had learned previously to another argument, which was evidence of normative ways of reasoning.

> *C4*: Dropping of an incorrect mathematical idea in further discussions is a fourth way to establish normative ways of reasoning. A rebuttal shifts into an alternative claim.

Lastly, rebuttals also contributed to identifying taken as shared ideas. Some rebuttals eliminated an idea that did not occur in the subsequent lessons. The students learned what not to use as mathematical argument components. Another issue was that a rebuttal shifted its position from a datum or a new claim to be discussed. Simultaneously, a rebuttal helped the students identify a mathematical idea's limitation. Rebuttals were also used in several studies to understand the quality of argumentation (Erduran et al., 2004), to describe the argumentation practices of the students (Evagorou et al., 2020), or to carry forward an argument into a more productive discussion (McNeill & Martin, 2011). Although rebuttals have been handled more in science education, the structure of a mathematical argumentation is also suitable for rebuttals, as given in the current study's findings.

Table 3. 7

An Example of A Comprehensive Mathematical Chart For Activity 1

Big Idea	**Tools/ imagery**	**Possible Topics of Discourse**	**Planned Big Idea**	**Possible Topics of Discourse (Expanded)**	**Emerging Ideas (CMP, Strategies, Errors, Misconceptions)**
Linking composite units	Connecting pictures of aliens to food bars	If the rule is that 1 food bar feeds 3 aliens, it can't be broken if we add more food bars	The idea of this page is to encourage students to link two composites together.	The rule can not be broken. What about the food bar and the aliens? (Let's discuss the attribute of the units) How many food bars were enough for one alien? How to share one food bar among three aliens? How to use this ratio in the operations?	The number of rows: Is that important? What is the meaning of division and multiplication operation in terms of the problem context

The arguments with their components were analyzed in the second phase based on these four criteria and sequenced chronologically. Although these practices do not guarantee every student's learning in the classroom setting, they provide an overview of collective mathematical learning in a class (Cobb, 1998). With the complementation of social and sociomathematical norms, classroom mathematical practices framed the social perspective used for qualitative analysis during this study.

3.5.3.3. Third Phase of the Retrospective Analysis

The argumentation log of as-if-shared ideas was organized under a general mathematical activity of the ratio and proportional reasoning. Chronologically sequenced argumentations were reorganized according to the relevant argumentation content. A mathematical practice may not be completed in one daily cycle or may be completed in a spontaneous activity hour. These arguments were gathered to reach the development progress of mathematical practices.

There were mainly 16 taken-as-shared ideas identified in the analysis. There were subcategories or repetitions of an idea. Main categories were enriched with related subcategories. Repetitions were also used as evidence for mathematical practice becoming widespread in the classroom. These strategies and ideas created everyday mathematical activities which constituted classroom mathematical practices for the ratio and proportion instructional sequence (Rasmussen & Stephan, 2008). In line with these, the documentation of these arguments was gathered to present the progress of mathematical practices. In this respect, five classroom mathematical practices were produced in the current study: the ratio and proportion reasoning on reasoning about discrete/continuous units and the rule of the ratio (CMP 1), linking and iterating composite units (CMP 2), covariation among composite units within ratio table (CMP 3), analyzing ratio and proportion in symbolic representation (CMP 4), and adapting strategies for comparing non-equivalent ratios (CMP 5).

3.5.4. Quantitative Analysis

Quantitatve data collection instruments consisted of pre-test and post-tests which were mentioned in the Data Collection Instruments. A paired-samples t-test was used to analyze the scores of the students before and after the implementation of the instructional sequence because this statistical analysis was suitable to investigate the group means of the participants at two different times for the current study, before and after participation in the instructional sequence implementation.

All hard copy data gathered from the tests were scanned for each student, and a list of tables was created for each student to analyze the qualitative and quantitative changes in the results. The quantitative change was analyzed by SPSS 24 software provided by TED University. The scores of the 7/X and 7/Y were calculated for the pre-test and post-test accordingly. It is noteworthy to say that the sample sizes of 7/X and 7/Y were 38, but the inclusive students were not involved in the sample sizes since they were provided differentiated instruction. Therefore, the sample size for 7/X decreased to 36. On the other hand, the sample size of 7/Y decreased to 33 due to one inclusive student and the absence of the four students from the pretest.

The null hypothesis of this study was that there is no statistically significant difference between the results from the pretest and posttest ratio and proportion administrations of the tests. The means of mathematics achievement scores were the same for the two groups. Likely, the groups' comparison showed that there was no significant difference between the scores gathered through tests, which represented the students' prior knowledge.

3.6. Trustworthiness of the Study

In the current educational design-based research, qualitative methods came into prominence in light of the study's research questions. There have been some concerns about the trustworthiness of qualitative research methods and educational design-based research. Therefore, there are some precautions, guidelines, and suggestions in order to improve the quality of data collection and analysis processes (Guba, 1981; Shenton, 2004). From the beginning, a route was followed to ensure the *reliability* and *validity* of the findings. Instead of *reliability* and *validity* terms, qualitative terms were described first in the current study in order to ground what had been done *before data collection* till the end of the *data analysis* phases respectively.

While entering the field, ethical issues are considered at first, including consent, harm, privacy, and deception that the researcher considered (Fraenkel & Wallen, 2006). Before the study began, the researcher talked with the administrators and informed them about the study. After their approval, the researcher contacted the volunteer teacher to discuss the study and the data collection processes. Prior to the data collection, approvals were given through the ODTU Ethics Committee and the Ministry of National Education, respectively. It was to confirm that no physical and psychological harm was predicted for the participants during the research (Fraenkel & Wallen, 2006). These approvals were valid till the data collection ended. After that, the verbal consent of the students was taken before the study, and parent consent forms were collected. Moreover, previously the teacher informed the parents of the children about this study verbally at the parents' meeting. The researcher's mobile phone number and email address were available in the consent forms, and one of the parents provided more information through mobile phone.

Before starting to collect data, children were informed by the researcher that it is not obligatory to participate in the research, which was also written in the voluntary participation form. Moreover, they had the right the withdrawal anytime. Confidentiality or privacy is another concern of the researcher. The identity of the participants was not shared. Each child was given a pseudonym. Students asked questions related to the study, and the researcher explained the study's primary aim. Deception is not the case for this study. They wondered about whether video recordings would be sent to their parents and whether others would watch video recordings. The researcher assured that the recordings were only allowed to watch by the design team if it was required. The classroom teaching experiment started with the teacher and the two classrooms in December 2016 and continued till February 2017.

Credibility corresponds to internal validity for quantitative research. It is to show how credible the inferences were for qualitative research (Creswell & Miller, 2000), in other words, the *truth value* of the inferences (Lincoln & Guba, 1985). Some strategies were followed to ensure the credibility of the inferences that the researcher followed (Creswell, 2012; Creswell & Miller; 2000; Şimşek & Yıldırım, 2011). Creswell (2012) and Guba (1981) recommended using multiple resources such as activity sheets, video records, teacher interviews, and field notes as it was done with this study. Firstly, a methodological triangulation was carried out between classroom observations and the documents gathered from the students during this study (microcycles, videos, student activity sheets, and tests). They allowed the researcher to cross-check and form themes among data sources to enhance the study's credibility (Creswell, 2012; Shenton, 2004).

One of the main advantages of design-based research for the study's credibility is a prolonged engagement of the researcher with the teachers and students (Gravemeijer and Cobb, 2013). In the current study, the researcher was in the field for one semester, from September 2016-February 2017. The researcher's presence before the data collection helped her organize the design's implementation well and reach key school people where to get the technical help needed, such as using a smartboard. Moreover, it helped researchers and stakeholders in the school get used to each other. During seven weeks of implementation, the teacher and the researcher were in contact before, during, and after the interventions at the end of each week. Moreover, the researcher contacted domain-specialist Dr. Akyuz as a debriefer to inform her about the process and to take measures related to the implementation flow. In the middle of the intervention, Teacher Zehra was invited to observe and give feedback on the classroom's engagement. She provided a verbal report and written keywords after the lesson based on the loud voices in the classroom, which was also the researcher's concern that she studied throughout the implementation.

Additionally, Guba (1981) offered to use *member checks* and *peer debriefing* through the involvement of the relevant stakeholders (teachers, experts) in the data collection and analysis process. Although the mathematical practices were constructed through the views of mainly Teacher Merve, she was also provided the overall findings on whether she approved the correctness and appropriateness of the presentation of the content given in the findings in which they were asked to collaborate to check the accuracy of their described experiences and themes. The issues were not understandable in video records and transcriptions, and the researcher took permission from the participants to apply their help to eliminate those ambiguous points. Teacher Merve and Teacher Zehra, as peer debriefers, helped bring out points missed and added issues they have run across in their data collection, observation, and crosscheck points. (Butler-Knisbet, 2010). For example, Teacher Zehra in the design team was informed about the system constructed for the retrospective data analysis and asked to evaluate a specific part of the data analysis (CMP 1). The results were consistent with the researcher who continued analysis by using the same system. Microcycles conducted with Teacher Merve guided identifying the main template for the classroom mathematical practices, which was another concern related to the consistency of the findings, in other words, dependability of the results. That is, the emergent changes are dependable on the changes associated with the components of the designed study (Guba, 1981), not due to accidental errors (Gravemeijer, 1994). In the current study, dependability concern was satisfied through the thick description of the methodology (Shenton, 2004), addressing what was done in the field and evaluating the effectiveness of the inquiry process. The methodology of the current study was described in detail to provide evidence for the dependability.

Confirmability of research is related to the data produced. The term was born from the concern of the *burden of neutrality* in qualitative research (Guba, 1981). That is, all decisions throughout the study were made considering answers to the research questions with the help of a research framework that was a collection of human intellectualities, not the researchers' preferences. Furthermore, since video recording in the classroom was another concern for the neutrality of the classroom (it may affect the role of the teachers and the students in the classroom), the researcher had been in the classroom approximately one month before the study began with the cameras to make students and teachers get used to the camera and the participant observer in case any camera effect may occur during the field. There were two cameras in the classroom and stood still in the same place, but the focus of one of the cameras was changed dependent on the students who spoke, explained, or showed in the Discussion phase of LED.

The last consideration is the generalizability (external validity) of the study. Due to the selection of the participants and the qualitative nature of the research, it was not appropriate to generalize the results to the entire population (Englander, 2012). Instead of generalizability, there are other concepts for qualitative research, transferability. They define the responsibility of a qualitative researcher as discussing the possibility of transferring the findings of the research to different settings (Şimşek & Yıldırım, 2011). That is, the context of one of the studies is transferrable to another, provided that there are essential similarities between the two contexts (Guba, 1981) by preserving the individual characteristics of the contexts (Gravemeijer & Cobb, 2013). Freudenthal (1991) highlighted the transferability of the findings through the consciousness of the cyclic process and reporting this process. The documenting methodology developed by Rasmussen and Stephan (2008) was used to analyze the collective activity of the students. Rasmussen and Stephan (2008) gave two different examples from different mathematical examples by using the same methodology for other grades. One of them is the measurement topic with the first graders, and the second one is the differential equations topic with the high school students. The study revealed that the methodology was successful for different contexts in mathematics. It is noteworthy that although this study does not aim to be generalizable, it is transferable provided that the assumptions of the methodology are satisfied. Therefore, a thick description of the data collection and analysis is one way to increase the transferability probability (Guba, 1981). Gravemeijer and Cobb (2013) described this process also as a means for credibility. They recommended documenting all these processes involving the conjectures and refutations so that the study's findings can be justified by tracking back and forward, which is called a zigzag between the previous and latter conjectures and refutations (Glaser & Strauss,1967).

3.7. Limitations

There are some limitations to the current study. First, being educational design research, the study's findings are not generalizable but transferrable when the assumptions are satisfied. This means that the study can be conducted in other schools or countries to investigate the commonalities and differences in the findings.

The second limitation would be conducting the study based on only one macrocycle with two classrooms. The pilot study started one week before the main study started to investigate the validity and applicability of the ratio and proportion instructional design, which was planned to inform the main study. Throughout the process, the main group left behind the pilot group in the instructional sequence of the study. Although there were subtle differences between the groups' normative ways of reasoning, their classroom mathematical practices were similar. This issue also provided a scenario for the second macrocycle in terms of repetition and differentiation of the ideas, reasoning, and strategies in terms of ratio and proportion. Cobb et al. (2003) emphasized the iterative nature of design research, but they underlined those macro-cycles should not have to be repeated by illustrating the study of Cobb et al. (2003) which they only used micro-cycles by using revised learning activities in the following lesson.

The third limitation of the study would be the restrictions of implementation. The classrooms were crowded, and the implementation process was conducted as planned, but the treatment of social context was restricted to the social norms and mathematical practices that were previously established and established during face-to-face classroom interactions. This, to some degree, shaped the emergence of the classroom mathematical practices, but the researcher and Teacher Merve worked together to overcome the disadvantages by taking some measures such as participation of every student or more care for the slow learners.

CHAPTER 4

FINDINGS

This research aimed to examine the seventh graders' classroom mathematical practices (CMPs) that facilitated collective student learning regarding ratio and proportion concepts and contributions of the instructional sequence to CMPs. The learning environment was constructed on the Launch-Explore-Discuss teaching model in which an argumentative classroom discourse was encouraged, and classroom mathematical practices were built collectively. The instructional sequence consisted of activities that were designed based on a hypothetical learning trajectory (HLT) and its phases to develop students' understanding of ratio and proportion through a transition from simple to complex. Each activity consisted of experientially real problems. Teacher Merve and the researcher were in the classroom to help and guide students to facilitate their learning. The research questions given below were the topic of this chapter:

1. Which classroom mathematical practices of the seventh-grade students emerged through the implementation of ratio and proportion instructional sequence within an educational design research environment?

Based on these research questions, the findings were presented qualitatively and quantitatively. The practices reflecting the classroom community's HLT and taken-as-shared ideas (TAS) were presented in five main categories. Data were gathered mainly from classroom discussions, field notes, and visuals captured from the classroom discussions. Moreover, the data were enriched with the students' artifacts and with the contributions of Teacher Merve. Data were molded complementarily through the lens of the interpretive framework. Documented data were analyzed and presented by using the argumentation layout of Toulmin (2003), transferred by Krummheuer (2007) to investigate classroom mathematical practices, and enriched by Hitchcock and Verheij (2006) in the case of rebuttal. Instructional sequences were also described in detail based on phases and conjectures. The schedule for the activities is presented in Appendix E, which shows the timetable for when these activities took place. It should be noted that a few activities were not mentioned in the findings due to the repetition of reported normative ways of reasoning. In addition to qualitative descriptions of the practices, quantitative results were delivered by analyzing the students' pre-test and post-test scores to reveal the change in the students' performance before and after ratio and proportion instruction sequence implementation.

4.1. CMP 1: Reasoning About Discrete/Continuous Objects and Rule of the Ratio

The launch of the instructional sequence was very significant not only in establishing a joint base for the development of ratio and proportion content but also in establishing an effective learning environment. There were 22 activities applied, 10 of which were episodes developed around feeding aliens based on linking composite units. The link was also called the *rule of the activity* in the classroom. The first activity is the start of feeding-aliens' episodes. The rule is "One food bar feeds three aliens." Each rule is presented in the activity with a pictorial representation of the composite units as external mediators. Diversely, the first episode was full of pictorial representation based on the conjecture linking the units and exploring other solution strategies. This task aimed to develop an understanding of the relationship between the units: one food bar and three aliens. The students' reasoning while representing this link was the focus of the class discussion. While studying composite units, the students developed an awareness of the conservation of the rule step by step. The class not only agreed on the solution method but also practiced refutation and rebuttal conditions for the arguments. They used agreed ideas and did not use the refuted ideas, becoming a classroom mathematical practice. The first practice established through the students' participation in the reasoning about the dimensions of problem context involved investigating the units' attributes.

4.1.1. TAS 1 Reasoning About Discrete/Continuous Quantities within Ratio

The first activity started with the researcher's question of whether there were enough food bars (2) for the aliens (9). (see Figure 4.1). The students were expected to use the rule to constitute composite units. However, before linking the composite units, students developed an understanding related to the units' attributes and the problem context. Although it seemed an easy task for the students, they were encouraged to engage with this activity, ending in more than one lesson hour. All the activities were reflected on the smartboard and were present during the lesson. The teacher also drew the questions on the whiteboard. Therefore, in explore phase, Teacher Merve drew the aliens' pictures. Given pictures of the food bars in the activity were redrawn by the students when they came to the board.

Figure 4. 1. Activity 1 Question 1 (A1Q1)

We started with a self-evident explanation provided by Ferit, drawing an imaginary line between the teacher's drawing of aliens and food bars (Warrant) as anticipated by the instructional sequence previously, with a claim that there were more aliens than food bars required. He claimed that three aliens were too many. The students agreed on the solution method of drawing a line between aliens and food bars as guided by the visuals in their activity sheets (see Figure 4.2). There was no backing occurred. This representation did not get any objection either from the students or the teacher. Most of the solutions were based on this kind of drawing, followed by a small explanation by the students and represented from the students' activity sheets in Figure 4.3. However, mathematically incorrect solutions, ideas, and models were also observed during this activity's explore phase. The motivation of this intervention was to make the students talk about their ideas even if they were wrong. Therefore, one of the different ideas was carried to the board to make others' ideas surface. Although it looked like an obvious problem for the students regarding their previous learnings, they could transfer invalid knowledge from other mathematical content.

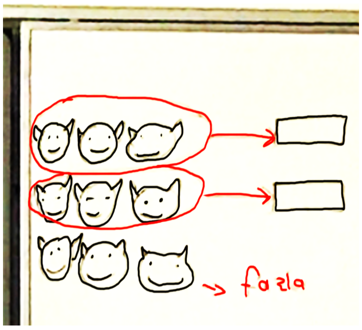


Figure 4. 2. Teacher Merve's drawings and Ferit's answer

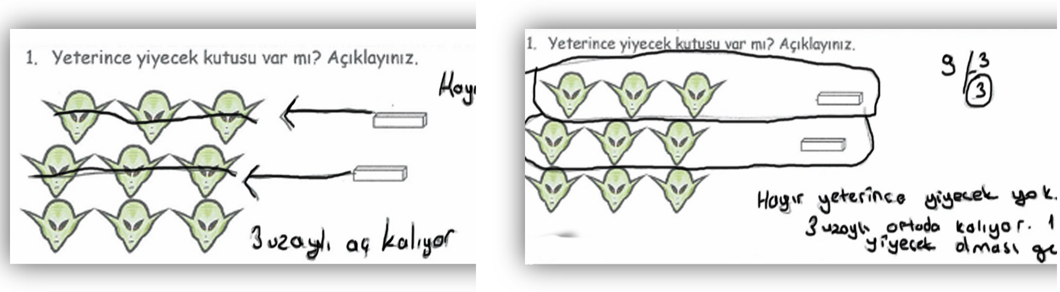


Figure 4. 3. Different representations captured from students' activity sheets

Hereupon, Teacher Merve asked, "Is there anyone claiming that they did it in a different way" to encourage students to retrieve distinct correct or incorrect models for discussion after Ferit's accepted solution. Eray denied Ferit's claim, asserting enough food bars (*Rebuttal and New Claim*). He explained that there were three groups of three aliens, the first of which was fed with the first food bar, and the second group of which was fed with the second food bar (*Warrant for new claim*). His implicit datum was that the rule of the question had not yet become apparent to form his strategy. He divided the third group of aliens one by one first. He distributed one of the aliens in this group to the first food bar, the third alien to the second food bar, and then the second alien in the group left (*Warrant*). What he did with the second alien was to divide it into two parts, and he matched the halves of the alien into the two food bars. At first, Teacher Merve did not understand, and she asked him to explain his solution method on the board. Eray drew everything as the same as Ferit did on the board in the beginning. As seen in Figure 4.4, the last three aliens were divided into two groups of one and a half aliens, and two food bars were also distributed to these two groups. As seen, he transferred the skills he achieved from the previous mathematical content, fractions. He performed a successful equal sharing among the food bars in case of fractional issues. Nevertheless, he was missing two crucial points: the rule and the context of the problem.

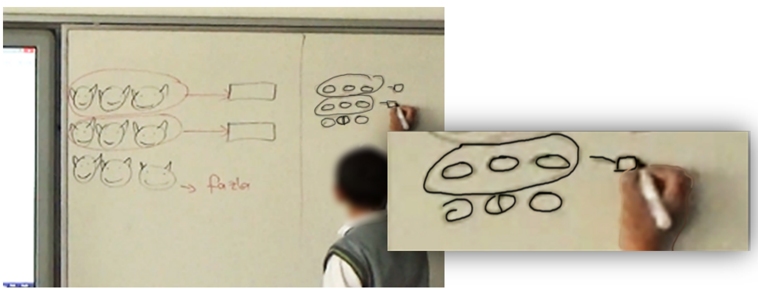


Figure 4. 4. Eray's explanation for A1Q1

Teacher Merve: Okay, Eray, tell us what you said by drawing it on the board.

Eray: Teacher, I will draw all of them. These are the aliens…(showing his drawing on the board) *(Datum)*

Teacher Merve: They are also food boxes *(Datum)*

Eray: There are two boxes here, teacher; they also eat a box, as you said, teacher. Ma'am, I am sending this here, here, and this in a symmetrical way, this here and this there. *(Warrant)*

(From the class, "but… [objection]" voices rise)

Teacher Merve: How would you send it[alien] there?

Eray: By sharing… *(Warrant)*

Teacher Merve: Do you think it is true? How do you cut and split the alien? *(Rebuttal-1)*

Berk: If you cut the alien, how will it be fed? He dies then. *(Rebuttal-2)*

After the students listened to Eray's explanation, many of them showed their refusals aloud for his idea of cutting aliens (*Rebuttal*). In this context, the class found cutting an alien impossible because the aim was to feed them, not to slice. They immediately intervened in his understanding of cutting aliens which was about the context of the problem. On the other hand, the class did not discuss the rule, which was another concern Eray avoided. Still, the link between the composite units was done unconsciously and needed to be addressed from several perspectives. One of the students, Berk, added, "If you cut the alien, how will it be fed? He dies then." After this rebuttal, Eray withdrew his claim. The rebuttal was about cutting an alien, not about the rule which Eray broke. The students were totally opposed to cutting aliens; therefore, they refused to evaluate the solution method of Eray, and in the end, he did not object to his peers' refusals. Then, his argument was refuted both by himself and her friends. Therefore, the rebuttal was given for the datum of *sharing* the aliens. Contextual issues emerged first, then the rule of the task.

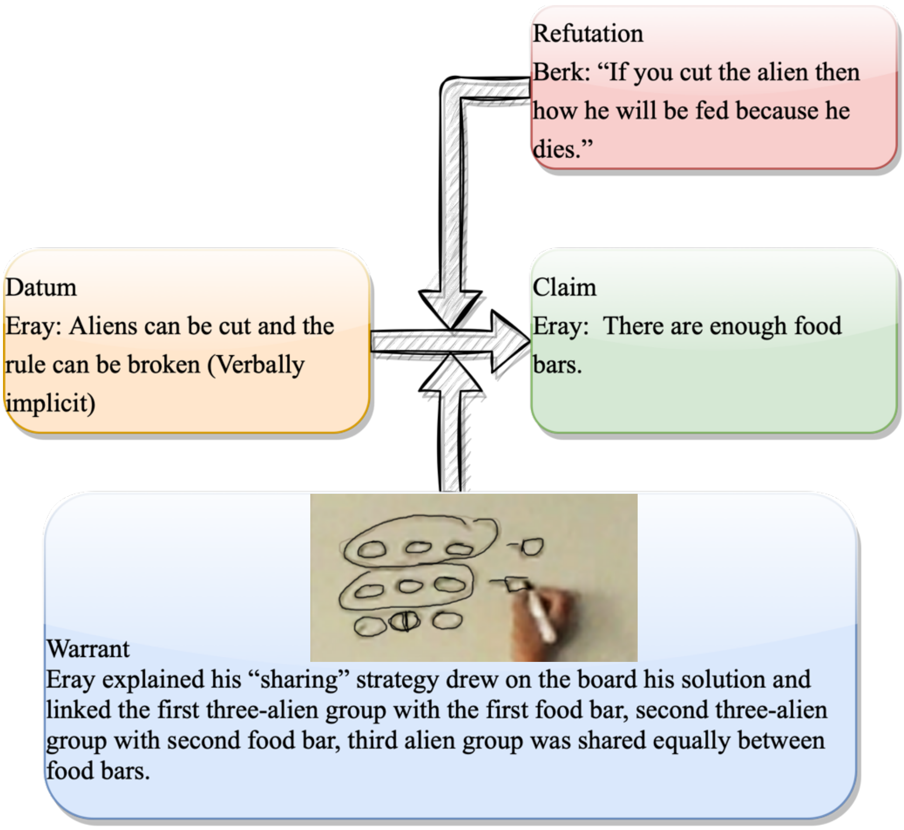


Figure 4. 5. Argumentation scheme for the rebuttal "Slicing an alien

As given in Figure 4.5, slicing aliens was not an acceptable argument, and it was not used as a datum in further discussions, but this argumentation was transferred to other discussions. In which context was it possible to slice the variables? If an alien could be sliceable, was it also possible to solve this question like this? Thus, this refutation needed to be discussed in deep. Before a healthy experience linking the units, the instructional sequence first conjectured to understand the problem's rule. However, understanding the units' attributes came first before discussing the rule. This activity aimed to foster linking composite units for understanding the rule and transfer this link to the other questions in the same context. This rebuttal became a trigger for questioning the relevant ideas related to the attribute of the units and a deterrent for questioning the other unrealistic ideas with the problem context in advance.

4.1.2. TAS 2 Reasoning About Invariant Structure of Ratio

In the second question of Activity 1 (see Figure 4.6), there is a pictorial representation of the food bars and aliens again. Previously, students did not mention the conservation of the rule while slicing the aliens. The question asked about the links between the aliens and the food bars. This time, there were more food bars than the number of groups of aliens. The rule was again, “One food bar feeds three aliens.” A similar problem-solving process during the whole-class discussion occurred in the second question. The students seemed to follow the previous solution paths, but the teacher’s questioning and students’ distinct answers enriched the classroom discussion.

Figure 4. 6. Activity 1 Question 2 (A1Q2)

Similar to the steps followed by Ferit, Ahmet used his fingers to show the connection between the food bars and aliens to solve the next question. Ahmet said that “That group eats this box, that group eats this box. The following group also eats this box again (Showing one food bar left). That's how I did it.” *(Warrant and Claim)*. All the students in the class accepted this warrant, and it was similar to Ferit’s solution in A1Q1, except that the students did not request any backing. There was also no question about this representation. In addition to this solution, the students had other strategies that needed to be discussed in the classroom. Berk was also a volunteer to show his work on the board. He first explained his solution verbally, and next, he carried out an algorithm to find the solution on the board. Berk was eager to share his solution because he thought that his solution was different than the others’. Both drawing and calculation strategies were the same as the answer to the first question, but it took a while for students to reach a common understanding of their solutions.

> Berk: (By showing on the sheet) Ma’am, I multiplied this with this [4 food bars with 3 aliens] to get 12. I subtracted 9 out of 12. Ma’am, there are 3 left, so one more bar, teacher. 3 aliens are missing, 1 food bar is more than needed, teacher. (*Datum, Warrant, and Claim*)
>
> Teacher Merve: Did you do it via doing calculations? (Warrant)

Berk: Yes, Ma'am. (*Warrant*)

Teacher Merve: Can you write that process on the board?

Berk: Teacher, first I multiplied 3 by 4 and found 12. There are 9 aliens in total, Ma'am. (*Warrant*)

Teacher Merve: So why did you multiply 3 by 4?

Berk: Ma'am, there are 4 food bars. I took a row of aliens, teacher, 12. There are 9 out of 12 in total, Ma'am, I took it out. *(Warrant)*

Teacher Merve: What does that "12" you found mean?

Berk: Well, the number of aliens, Ma'am. *(Backing)*

Teacher Merve: That alien number?

Berk: Yes!

Berk explained that his solution was about "doing calculation". He started analysis from the given number of total food bars, four. He used multiplication and subtraction respectively, to support his claim. His warrant became more apparent with Teacher Merve's questions. Teacher Merve revoiced what Berk said in the discussion, and the class agreed on this solution strategy.

When Teacher Merve asked why he multiplied four by three, it was the first time he used the name *row* to express the *three-alien figures on a line* given in activity one. *The row*, in other words, was a critical word only for this activity, which we realized its importance later in the whole discussion. Since Berk's solution did not elicit a reaction from other students, the teacher asked the questions to make his argument clearer about his solution. After all backings, the answer was clear for the teacher and Berk. This dialogue between the teacher and Berk framed a shared understanding (Figure 4.7) for a ground of solution strategies described further.

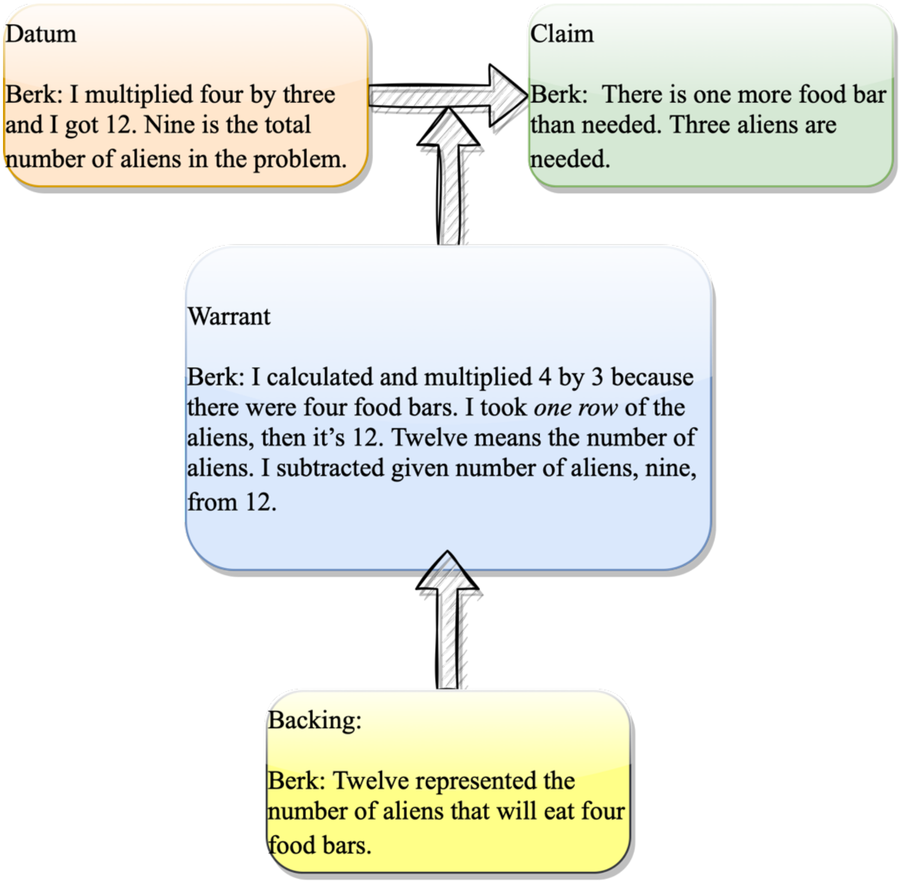


Figure 4. 7. Argumentation scheme for A1Q2 developed by Berk

As Figure 4.7 represents, the students did not add anything to this argumentation, but there was a point where Berk used "one row" in his backings. In addition, it did not get the attention of Teacher Merve and the researcher at that time. In what follows, as a general norm for the intervention, she continued to ask whether there was a different solution method for this question. Cansu had something different in her mind. She had already achieved linking the units; however, throughout the intervention, she was one of the several students who tried to find a solution method that would work for other problems as well. In other words, she was a student who developed a second solution on her own so that it was unique and generalizable. She developed her own explanation for A1Q1, but it was unclear. However, the retrospective analysis made her answer observable during A1Q2. She tried to explain that she calculated the result using *a row (Datum)* in A1Q2.

> Cansu: Ma'am, for one row, for one row, there are three in a row. If there are three in a row, there are three rows. *(Datum)*
>
> Teacher Merve: It is the same as Berk's.

Teacher Merve confirmed that her solution was proper, but she admitted that it was similar to Berk's. At this time, Cansu insisted on her answer and improved her reasoning in A1Q2. It was her second attempt at this distinct strategy. She explained her solution by giving the data in which she used *rows*. Teacher Merve highlighted by asking, "The rows?". Cansu felt the need to explain the number of rows. According to Cansu, dividing the number of aliens (9) into three brought us the number of rows. Surprisingly, nine aliens were evenly distributed in three rows, as provided in the problems (see Figure 4.6), which made her answer correct.

> Teacher Merve: How did you solve this question, Cansu?
>
> Cansu: Ma'am, I divided the number of aliens by the number of rows. (*Datum*)
>
> Teacher Merve: By the number of rows?
>
> Cansu: Ma'am, Teacher, that is, when nine is divided by three, there are three. There are nine [aliens]. There are three rows, Ma'am (*Warrant*).
>
> Batu: The row is not important here. (*Rebuttal*)

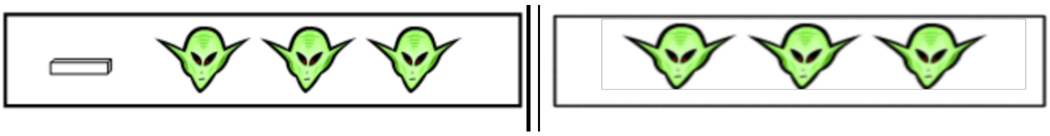


Figure 4. 8. The rule of Activity 1 (left) and one row of aliens (right)

The questions in the first activity have the same visuals, and the total number of aliens is grouped three by three for aliens (iteration of the rule). Each line has only three aliens and one food bar, as given in the rule (see Figure 4.8). The visuals in A1 led several students to discover *the row* concept. First, Berk implicitly described its structure in detail by representing three aliens on a line in the whole class discussion. After Berk's description, Cansu changed the rows into a concept, namely *the number of rows of the alien groups* as composite units. Such kind of students' deductions from the visuals was not conjectured while getting prepared for the instruction. Therefore, this misconception was presented to understand Cansu's perspective better in the whole class discussion and avoid potential students' errors that emerged due to the design.

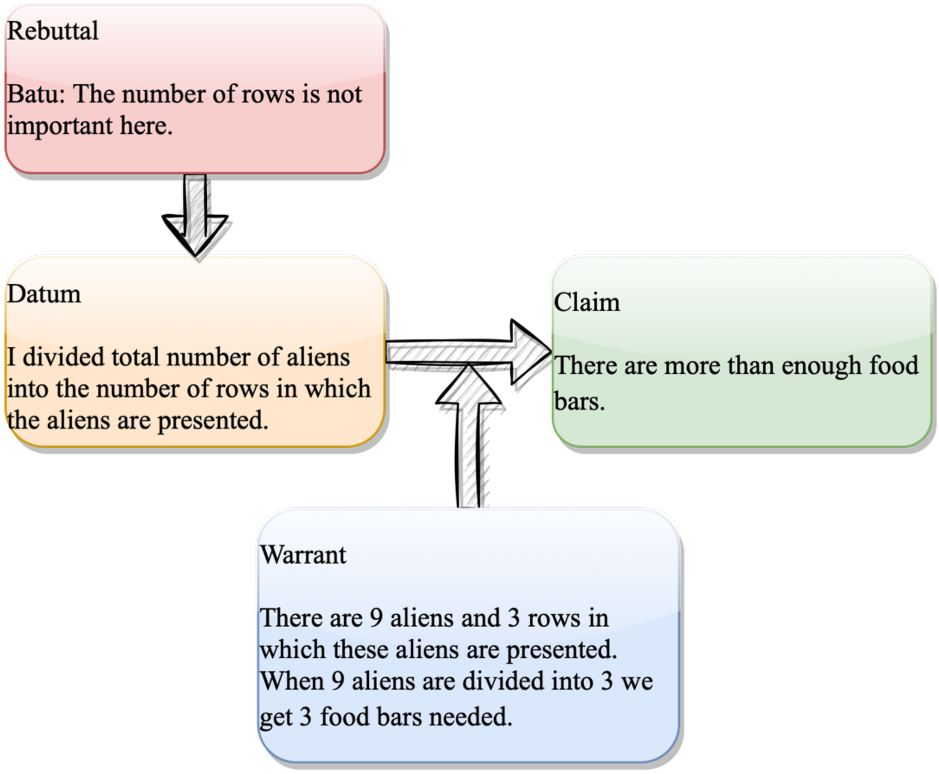


Figure 4. 9. Argumentation scheme for A1Q2 by Berk with path 2

In Figure 4.9, the datum of Cansu became a conclusion of the argument that needed further discussion. She explained that three aliens in a group represented one row, and there were three rows in the question. So, there should be three food bars. However, she concluded that there were more than enough food bars because there were four food bars. Batu directly rejected the idea of *the number of rows of the alien groups*. Based on the visuals in A1, the same misconception could emerge among the other students. Therefore, several instructional strategies were developed simultaneously after a small discussion between Teacher Merve and the researcher. In the whole-class discussion, Teacher Merve selected the mathematical examples carefully to create counterarguments for Cansu's datum, which would enrich Batu's rebuttal. Teacher Merve started changing the visuals of the question and continued with the "what if?" questions, "What if I drew the visuals unsymmetrical like this (drawing new figures on the board, see Figure 4.10), would the number of rows matter?". We realized that each row was the iteration of the rule three times, and it could be either a coincidence or on purpose, about which we did not have any information on the instructional sequence. This situation could shadow the meaning of the composite unit of one food bar with three aliens; therefore, it was investigated through discussion.

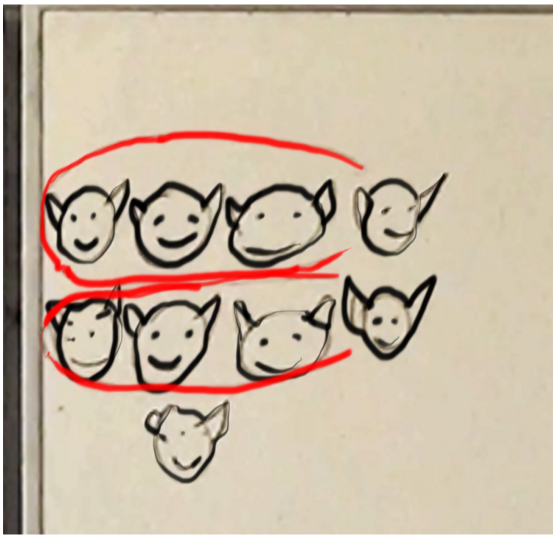


Figure 4. 10. Teacher Merve's first attempt at the counterargument

As seen in Figure 4.10, Teacher Merve distributed two aliens from the bottom row to the first two rows to change the number of rows and the number of aliens in a row. She opened the red line representing the three-alien group. Now, the rows were not symmetrical or equally distributed among each row. The highlighted issue here is the grouping of the object according to a rule. It was a question of whether Cansu would use either a grouping of three aliens or a grouping of rows.

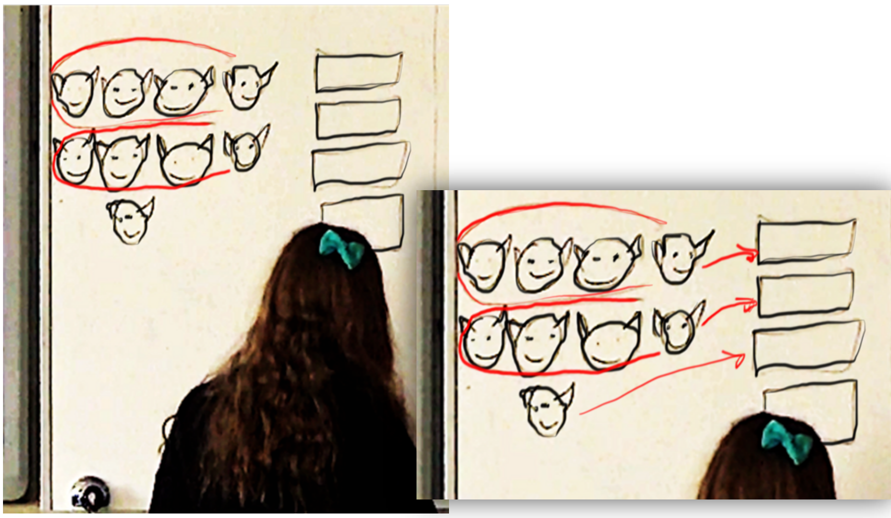


Figure 4. 11. Cansu's matching of four food bars

Teacher Merve proposed a case that recreated the total number of aliens in the same number of rows (see Figure 4.11) and asked whether the number of rows mattered. At that moment, Cansu made up her mind and said that she would solve the problem differently at this time. She explained, “now I would do it by matching the aliens one by one.”. Teacher Merve further questioned “matching one by one” and asked for a better explanation of her solution (dividing by the number of rows). Teacher Merve tried to learn how Cansu decided to divide the number of aliens into the number of rows. Cansu divided nine aliens by three. Teacher Merve explicitly hinted to Cansu whether she used grouping and interrogated how she would solve *unsymmetrically* distributed nine aliens in three rows. Cansu had a fixed idea that the “dividing the number of rows” model worked and insisted on the same result, 3. It was a good start for the discussion of fixing the number of rows into three. In pursuit of this, Teacher Merve went beyond this situation and made two rows of nine aliens unsymmetrically and wondered how Cansu would solve this case (see Figure 4.12).

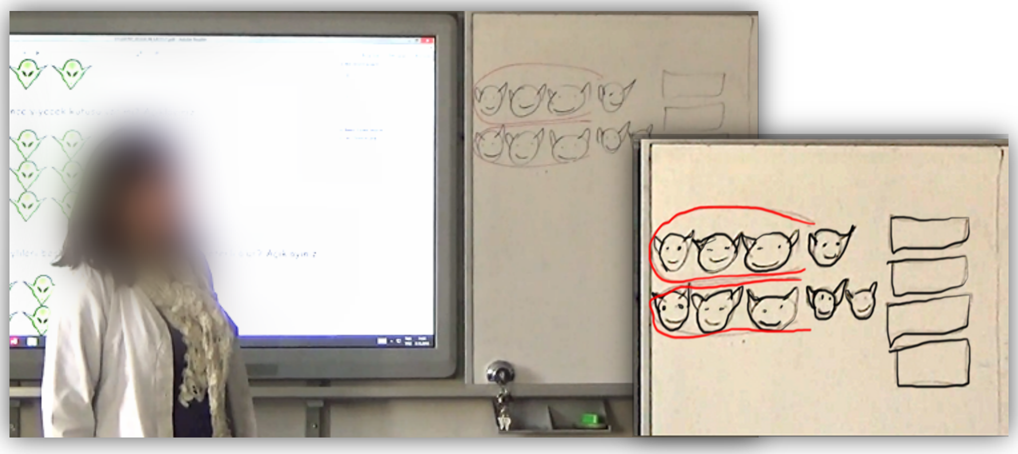


Figure 4. 12. Teacher Merve’s second attempt at the counterargument

Cansu was so confused that she found the same result, three, by dividing nine into two rows. She was not dividing nine into two rows, but she realized that the situation would not change. Some students refused this result, and Teacher Merve said that nine aliens could not be divided into two due to the *asymmetrical* grouping of the aliens with food bars. At this time, Teacher Merve opened the discussion for the whole class, showed the redrawing, and asked how the students would solve this problem in this case. Although the distribution of the aliens changed after Cansu’s misconception, the other students did not change their solution method. For example, after Teacher Merve selected Alp to clarify his understanding of this issue, Alp wrote path 1 to answer the problem on the board. He agreed with the conclusion that three food bars would be enough for nine aliens. Teacher Merve asked, “is dividing the number of aliens into the number of rows” a solution method that always works? Arda said that it did not always work, and the other students agreed with his answer. Teacher Merve asked the class how they would solve this case. They kept doing their previous solutions even if the distribution of the aliens had changed. Berk's rebuttal became a new claim, and Teacher Merve negates Cansu's datum with new scenarios involving the number of aliens in the row did not equal the number of aliens in the rule. The row explanation is valid based on the condition, but the row case was refused since it was not a generalizable method according to the rebuttal condition. Cansu ignored the rule by misjudging the visuals (the row and food bar matching).

> Teacher Merve: Let's say it's two rows; for example, I took this [alien standing in the third row as in Figure 4.11], put it up, it was four by five, what would you do with it? [See Figure 4.12]
>
> Cansu: Ma'am, again, when nine is divided by two, it becomes three.
>
> Researcher: What are your opinions?
>
> Teacher Merve: Nine cannot be divided by two as a whole.
> .
> Most of the students: (Loudly) WE DO NOT AGREEE….
> .
> Researcher: Why don't you agree?
>
> Teacher Merve: Tell us the reason. Just yes, look, your friend said that I grouped the rows because they are symmetrical. So she said to group the food bars with the rows above. Could this be a solution strategy that always works?
> The class: NOOOOOOOOOO...
>
> Teacher Merve: For example, if your teacher had given you the number of aliens in the rows as four to five…
>
> Arda, Alp, and several students: It doesn't work.
>
> Teacher Merve: It doesn't work. How would you do it? Alp, come here. How would you do it, for example?
>
> Alp: Ma'am, I wrote the explanation [He calculated].
>
> Teacher Merve: Okay, explain it to us.

In this argumentation process, Teacher Merve and the students refuted Cansu's idea, which became a rebuttal for the argumentation scheme in Figure 4.9. The condition was described, "The row number can be used as a strategy if the total number of aliens was distributed evenly and according to the rule". Batu's rebuttal argument was moved to the claim level by Teacher Merve, and a significant amount of time was spent making this case clear. Nodding Teacher Merve's and the other students' rebuttal, Cansu agreed on the number of rows due to symmetrically ordered visuals. Additionally, Teacher Merve said that it was essential to question the sustainability of the strategy. Cansu's strategy was only coincidentally useful for A1Q2, and the first counterargument was given as in Figure 4.13.

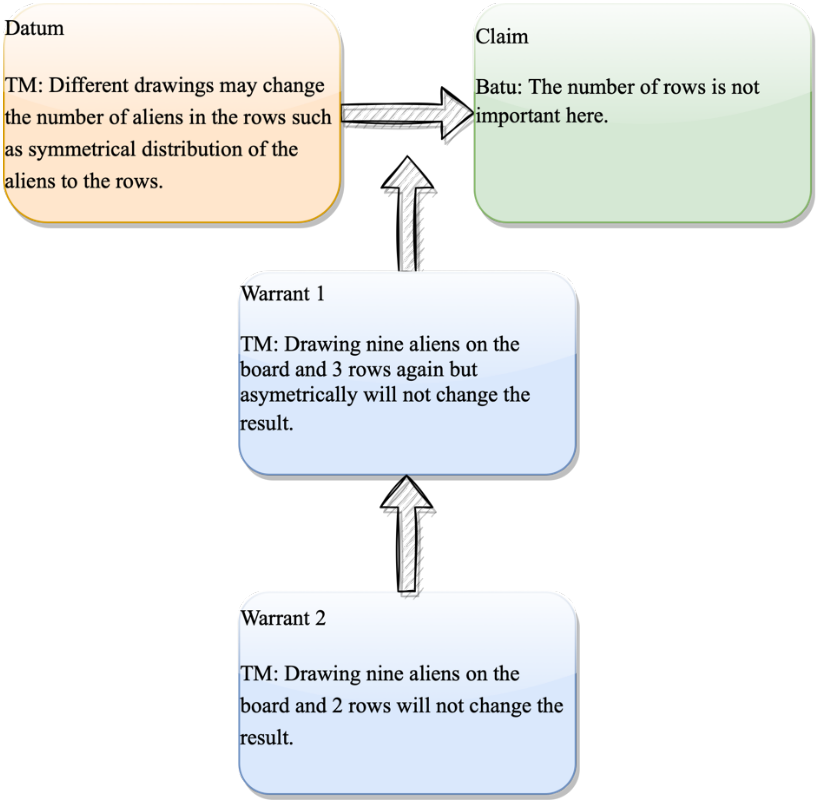


Figure 4. 13. Argumentation scheme for Batu's rebuttal

Teacher Merve wanted to learn the students' agreement about this question. Then, Buket voluntarily came to the board and merged *warrant 2* of Teacher Merve as *the datum*. Buket concluded, "I added the number of aliens in two rows, five and four, together, nine in total. As one food bar feeds three aliens, nine aliens need three food bars after dividing nine aliens into three." Buket emphasized that she came to the board to show that the number of rows was unimportant, but the total number of aliens was. She explained how she did solve the problem after the number of rows had changed. In this case, not the order of the visuals, but the rule was emphasized implicitly for the linking composite units.

4.1.3. TAS 3 Reasoning About Continuous Units and Invariant Structure of Ratio

At the beginning of the feeding-aliens' episodes, Teacher Merve introduced the story behind these episodes, as instructed previously. Teacher Merve described each unit one by one. The aliens were so hungry that they were invading the Earth, and humanity found a *food bar* to ease their hunger and live peacefully. This food bar was a package containing a mixture like a chocolate bar. This imagery and the attempt to slice aliens raised other issues in the students' minds. Was a food bar a piece of mixture to be sliced homogenously?

Previously, Buket solved the A1Q2 by calculating to show that the number of rows did not have any place in this calculation, and this case was done. After that, Arda provided another solution. He was eager to share the "relevant idea" and carried out a different perspective. He was okay with the solution methods of his peers; however, he was still searching for a different result, a solution method, reasoning, and the like. Arda repeated the statements of his peers at first, "Three aliens ate one food bar, nine aliens ate three food bars." and continued that one food bar was left. He added that he would divide this left food bar into three aliens, as seen in Figure 4.14. First, he distributed one food bar for each group of three aliens and then gave a piece from the left food bar to the last alien of each group. He did not dive into the details about how to divide the food bar then. Eray's "slicing the aliens" case was still vivid for him. On the other hand, his peers did not have the same opinion as him. He was a very dedicated student finding a different solution, discussing something, and talking about his ideas, whether it was rational or not. Because of that, his friends had already developed a prejudice against him. When Arda spoke of something, the other students either did not listen to him or objected to his ideas. It is noteworthy that Arda, according to Teacher Merve, was a very brilliant and hardworking student. However, he had a terrible experience with family issues. Therefore, he reflected on this experience with such behaviors in the classroom. However, during this instructional sequence, he contributed many times, triggering several questions to discuss in the classroom.

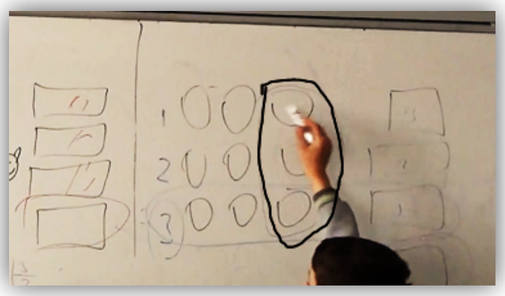


Figure 4. 14. Arda's representation of the different grouping

Berk declined Arda's statement same that it was not possible to share the last food bar because they were already shared by three alien groups *(Rebuttal).* He repeated path 2 to support his argument. *Unlike in the rebuttal for slicing the alien, this time came the rebuttal for the conservation of the rule, not for slicing the food bar.*

Conservation of the rule discussion continued with the solution of Evren. Teacher Merve was very interested in Evren's solution because Evren did not often participate in the discussions before the instructional sequence. Although Evren ignored the rule of the activity, the teacher encouraged him to come to the board to participate. Evren reoriented to Arda's explanation, saying that he could divide the last food bar into three equal parts and give each piece to three aliens in each group. Berk again refused to provide more than enough food bars to the aliens. In addition, Berk treated Evren's and Arda's solutions as if they came up with the same solution. Highlighting Berk's statement, the researcher said that there should always be one food bar left. Teacher Merve attempted to differentiate the two solutions. She emphasized that Evren shared the food bars equally among three groups of aliens, but Arda gave one food bar to one alien from each group. Most students agreed with Berk's solution, several followed Evren's arguments, and some confused students had not decided on the correct solution. Two ideas had important messages about proportional reasoning. Conservation of the rule for linking composite units, slicing food bars but not aliens, and equal sharing of the food bar were the main emerging issues. Therefore, the teacher and the researcher decided to listen to both arguments in detail.

The classroom started slicing the units with Eray whose offer was not accepted since the aliens would die if they were cut. They now tried to cut another unit given in the rule, the food bar. It was deduced from the students' discussion that they assumed the food bar as a homogenously distributed edible material, not a meal box or something. Therefore, no one refuted dividing the food bars into equal pieces. If the food bar were assumed to be a meal box, it would not be sliced again due to the equal sharing issue. Evren just added the division of the left food bars into *three*, starting another idea for the unit ratio, and this way of thinking was still open to discussion. As Evren rarely shared his mathematical ideas, Teacher Merve appreciated his willingness to share his ideas. Moreover, he was not the only one to share the left food bar. So far, how to divide the left food bar into three was not still resolved. At that time, Zenan was the one who furthered Evren's idea. She equally divided the last food bar into three pieces and distributed each piece into the three alien groups, as presented in Figure 4.15 below.

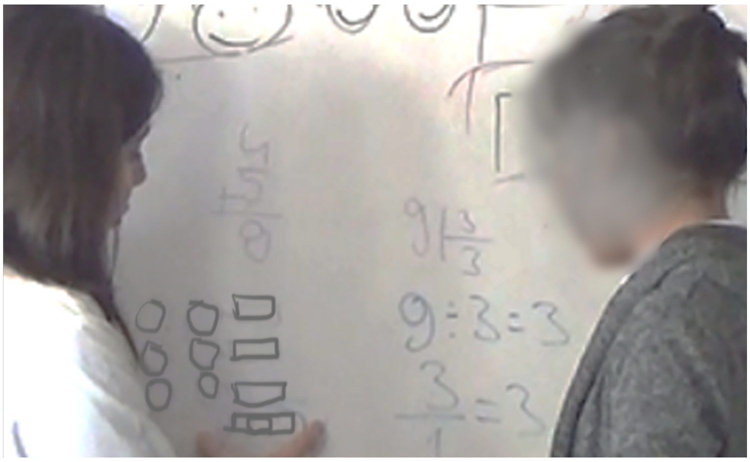


Figure 4. 15. Teacher Merve and Zenan slicing the food bar into three

As expected, her peers did not grasp her reasoning, and Teacher Merve interrogated the ideas. Throughout the process, Zenan generally needed support for previous mathematical knowledge and skills in four arithmetical operations, fractions, and reasoning. Nonetheless, she engaged in the activities frequently and tried to solve the problems. The teacher asked Zenan to explain in detail how she concluded that Evren was right. With the help of Teacher Merve, Zenan said that the last food bar could also be divided into three, and each alien group could take four smaller parts. She tried to show this verbal explanation with division operation. After dividing nine by three, she divided the quotient three by one (right side of Figure 4.15). Teacher Merve realized Zenan's reasoning was not sufficient. Hereupon, Teacher Merve told her to express the pieces as fractional representations (See Figure 4.16). This recurrent teacher decision-making process emerged frequently based on the students' reflections during the instructional sequence.

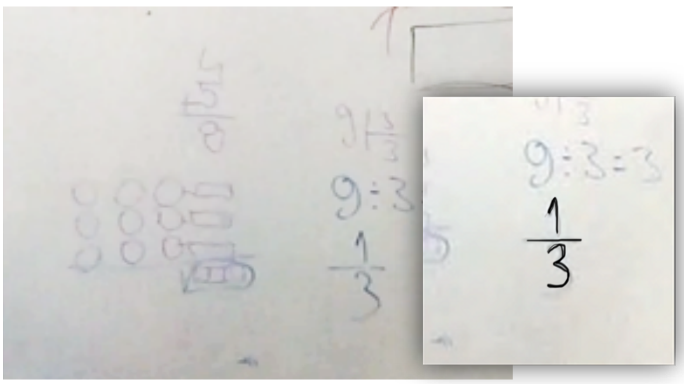


Figure 4. 16. Zenan's showing each piece

Dividing equal pieces of each food bar was a good start for talking about unit ratio; however, it was out of context at that moment. The students explored slicing the food bar, but the rule of the mathematical problem needed to be discussed. At that moment, Berk again objected to Zenan's distributing pieces of the left food bar and repeated that the aliens were full of food bars and thus could not eat one more piece. He defended his thoughts to the fullest. Zenan, Evren, and Eray, according to Berk, were wrong. Teacher Merve again brought this defense to the classroom discussion. Fatma drew attention to the representation of *1/3*. She still focused on Evren's idea and added another block to Zenan's discussion. Fatma said that they needed to divide one piece of the food bar into three again to share each piece with an alien. Zenan found four pieces for an alien group, and Fatma found 1/3 and 1/9 pieces of food bar per alien. Drawing on Zenan's already cut pieces of food bars, Fatma said that these food bars were already divided into three for three aliens (See Figure 4.17). Furthermore, she said they divided one small piece of food bar (1/3 food bar) into three so that the number of food bar pieces was equally shared with an alien, not for three aliens in one group.

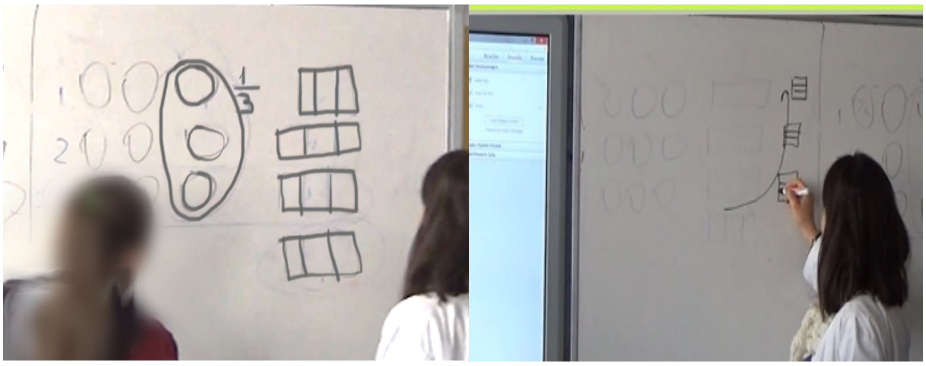


Figure 4. 17. Fatma and Teacher Merve representing the 1/3 of 1/3 of one food bar

After Fatma's explanation, Zeynep and Deren refused the idea of sharing the last food bar with the alien groups. They advocated Berk's idea. On the other hand, some students opposed the idea of Berk. There were two different ideas, and there was no agreement in the classroom. This discussion remained and was not resolved by the teacher to enable the students to discuss Activity 2. It was appropriate to cut the food bar *mathematically* and *contextually*. We put this discussion on the hypothesis wall as two different claims to make the students think about the unresolved debate. There were Berk's claim on the left and Evren's claim on the right side (see Figure 4.18). The students were queued under this idea. One-third of the class advocated Evren's idea, while two-thirds supported Berk's idea (See Figure 4.18).

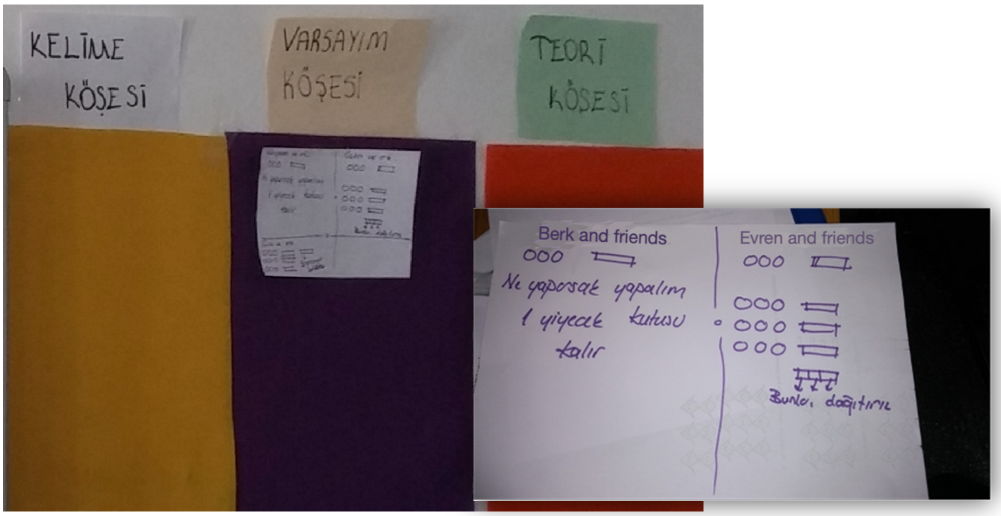


Figure 4. 18. Two ideas were hung on the hypothesis wall[1]

While the students were discussing at the end of the lesson, the teacher said this was a kind of discussion specific to this question. Even Evren was on the side of Berk, but he liked the way of cutting off the food bar correctly. It was understood that their problem was not the rule but the processing of their previous knowledge.

The argumentation so far can be summarized as given in Figure 4.19. At the beginning of the classroom, Teacher Merve and the researcher already talked about the students' knowledge related to fractions because the context required essential knowledge and skill related to this topic. The students needed some practice and a reminder for fractions. After the lesson ended, Teacher Merve and the researcher talked about this context, and Teacher Merve said that she used this part of the discussion as if it was a practice for fractions. While Fatma was calculating on the board, the students also did fractional operations. Fatma multiplied 1/3 with 1/3 and found 1/9 as one small piece for each alien. Evren was quite confused about this tiniest piece of food bar since he could not imagine it. Teacher Merve divided one-third of the piece into one-third to make it more concrete with the rectangular area model (see Figure 4.17).

[1] Word wall, hypothesis wall and theory wall were ordered respectively on the wall in the classroom as it was recommended by Stephan et al. (2015). These kind of reminder tools may help students regulate their collective knowledge. New terms, concepts, definition were hanged on the word wall (yellow board); claims which were not concluded by the classroom were hanged on the hypothesis wall (purple board); the claim which was accepted by the classroom was hanged on the theory wall (orange board). These applications unfortunately were lasted 3 weeks due to the other classroom coming after the session of 7/X ended. The two classrooms were using the same classroom as before noon and after noon. It was not possible to carry those walls since they had materials hanged on them.

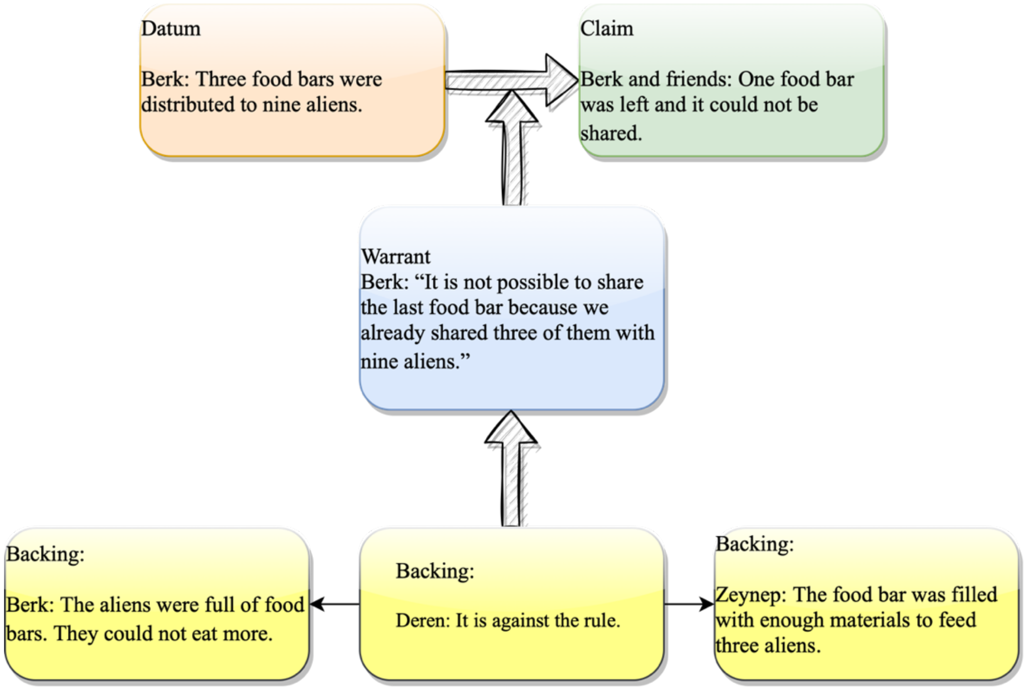


Figure 4. 19. Argumentation scheme for Batu's rebuttal

After this case was left open, the class moved to the third question in Activity 1 (A1Q3), as shown in Figure 4.20. In the problem, 15 aliens are drawn three by three in five rows, and it asks, "how many food bars are needed to feed these aliens? Explain." As the structure of the third question resembled the first two problems, the students' solutions were repeated in the whole-class discussion. Moreover, these solutions represented acquired strategies throughout Activity 1.

Figure 4. 20. Third question in Activity 1 (A1Q3)

First, Egemen expectedly solved the problem by drawing food bars across the alien groups. So far, the students did not get tired of drawing all the units because the number values were small. However, Egemen used his time effectively and completed the previous drawing of his peers with two pairs of three-alien groups (See Figure 4.21). He concluded that according to the rule, there should be five food bars to feed the aliens in the picture according to the rule. The students accepted Egemen's solution without any counterargument. He solved this by completing the picture with food bars given in the problem. The other students continued to raise their hands to explain the process in detail because Egemen did not explain his solution. The students also transferred and merged the idea of cutting one food bar into three equal pieces and showed it on the board as an addition to the solution of Egemen.

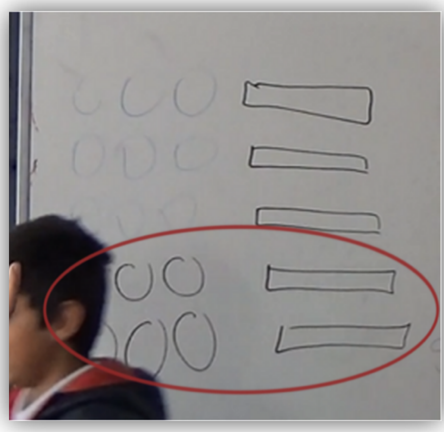


Figure 4. 21. Egemen's solution for A1Q3 through drawing

In this part, the students transferred the idea of slicing the alien to slicing the food bar. They questioned the possibility and the conditions which make *slicing* mathematically correct and experientially real in the context of the problem. This search led them to a significant learning experience. The issue of *slicing the aliens that* was molded with previous experiences unfolded, *slicing the food bar* issue. It did not happen immediately after the experience with slicing aliens. Several collective activities caused this issue to emerge. Second, the students preferred calculating algorithms by updating and developing the solution each time. Ayten used the connection of aliens and food bars with 1/3 ratio. In previous explanations, 1/3 represented a piece of food bar for an alien. Ayten represented 1/3, "One food bar for three aliens", indicating that their understanding of unit ratio was cooperatively upgraded.

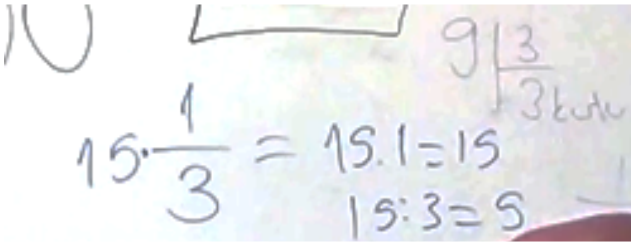


Figure 4. 22. Ayten's solution strategy for A1Q3

Ayten used path 1 and started with the given total number of aliens (15). She multiplied it by 1/3 (See Figure 4.22). She found one-third of the aliens to reach the total number of food bars. This solution method is different than solely dividing 15 aliens into three. It includes the knowledge gathered from the previous discussions.

4.1.4. Summary of Classroom Mathematical Practice 1

Classroom Mathematical Practice 1 (CMP 1) consists of three ideas of seventh-grade students foregrounding the discrete/continuous attributes of units and the presentation of the problem context through the linkage between composite units emerged through the whole class discussion of Activity 1. These three normative ways of reasoning are around (1) Discrete units in composite units cannot be reduced for covariation due to problem context; (2) Invariant structure of ratio is independent from the random grouping of objects; (3) Maintaining invariant structure of the composite units, continuous units in composite units can be reduced.

Table 4. 1

A Brief Summary of the Criteria for CMP1

CMP 1	CMP Criteria
TAS 1	C3: Drawing arrow C4: Rebuttal "Slicing an alien"
TAS 2	C3: Doing calculations C4: Rebuttal "Using row is not important"
TAS 3	C4: "Rule should be conserved" C3: TAS1C4 Transferring "Slicing the food bar"

While digging the ideas with Teacher Merve, their perspectives on the problems became recognizable. Additionally, each idea was formed by datum, claim, warrants, backings, and rebuttals. Ideas also involved both anticipated and unanticipated student contributions. Anticipated ones were dominantly using algorithms, drawing lines between aliens and food bars. On the other hand, unanticipated ideas enriched the lens used for the instructional sequence. Specifically, these ideas connected students to discover the attributes of the units in terms of discrete and continuous (TAS 1 and TAS 3) and the illustrations representing the problem context (TAS 2), which were not among the anticipated student thinking in the instructional sequence. Each idea influenced the emergence of the other ideas, and they were dominantly formed by the relationship between the objects. They contributed to develop reasoning about invariant structure of a ratio (rule of the problem), reasoning about discrete/continuous objects within composite units, and grouping a number of linked objects. These TASs were also helpful to build social and sociomathematical norms for other classroom practices because students started to understand the ways of mathematically meaningful discussion and warn other's whether their ideas were wrong or not.

4.2. CMP 2: Linking and Iterating Composite Units [2]

The ratio and proportion instructional sequence was designed to allow students first to explore proportional situations within an experientially real context (e.g., aliens and food bars) and move progressively towards more abstract proportional reasoning. Activities 1-4 were to develop linking and iterating composite units that the students had used during the instructional sequences. They explored how to organize the data, and these four activities strengthened them through discussions focusing on mathematically correct and time-efficient reasoning to solve the problems. These activities encouraged the students to produce ideas for the organization of the data and refutation of the blur, inexplicable arguments. The questions were posed to make the students willingly organize the quantities in some structured ways. Data organization started from reasoning about composite units through drawing the links on the illustrations/pictures. Composite units and its existence in the various data organization strategies created a complexity about the identification of the composite units and their iteration. The students' got benefit from other data organization strategy and molded them to explain the iteration. The students created distinct solutions from the beginning of the instructional sequence. Each solution method was seen in activity sheets and on the board several times. Students' ideas were dynamic, as observed in the classroom discussions and

[2] Part of this mathematical practice was published as a book chapter.

activity sheets. The same student's solution on the board might even change. As represented below, the practices of data organization strategies were dissolved from the integrated discussions in the classroom. In each question or problem, it was possible to see several solution methods simultaneously as a datum, warrant, rebuttal, or backing. The second practice established through the students' participation in the distinct linking process after reasoning about discrete/continuous quantities.

4.2.1. TAS 1 Linking and Iterating Composite Units Through Drawing

With its visuals and related questions, the first activity encouraged students to use drawing units to make concrete their reasoning. The incipient three activities gave other ideas to students for withdrawing or continuing with this strategy that was constructed as the collective classroom activity. In other words, students' drawing strategies evolved during the instructional sequence. The students who used the drawing strategy mainly started drawing all the units on the board or their activity sheets. For example, Egemen directly drew the rule and iterated that rule by drawing five times until he got 15 aliens in total. Repeated additions accompanied by a verbal representation of skip counting; 3, 6, 9, 12, 15. Drawing the initial ratio and repeating it became his warrant and helped Egemen explain his result (see Figure 4.23). No disagreement has occurred during this solution method. Drawing the rule and iteration became one of the normative ways of reasoning.

> Egemen: Hocam (Ma'am or Teacher), since the rule is "one food bar feeds three aliens", I drew it like that (while drawing on the board). *(Datum)*
>
> Teacher Merve: Okay, Egemen! Can you explain how you found the result?
>
> Egemen: The rule is "one food bar feeds three aliens". Every three aliens eat one food bar (Drawing ellipses around each of three aliens on the board to show five groups of three); therefore, the result is five food bars needed in total. *(Warrant and Claim)*

Egemen provided one of the typical solution strategies: "to identify the rule of the ratio for the problem and model it," "to iterate that model by drawing," and "find the missing value". The students eventually determined that drawing out all aliens and food bars was not a time-efficient solution for ratio and proportion questions. The students called this kind of solution a *drawing* strategy, and they did classify similar solutions in the drawing strategy set. Therefore, when Teacher Merve asked for any other different solutions/results, another drawing strategy solution did not show up.

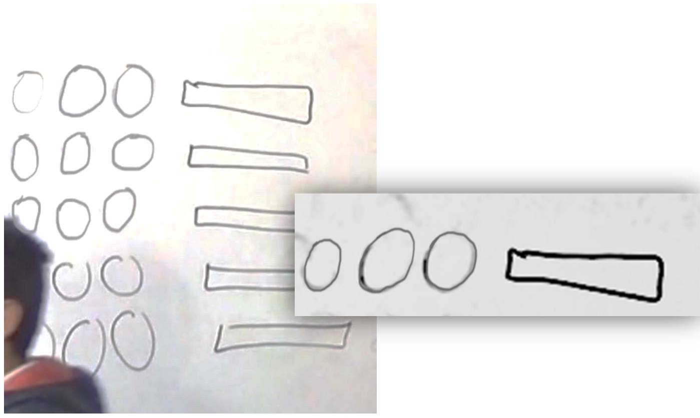


Figure 4. 23. Egemen's iteration of the rule for A1Q4

In the second activity, the number values in the total number of aliens and the total number of food bars were increased. We were looking for a new data organization strategy for the students. Drawing each value on the board required extra time. Therefore, the learning environment and instructional activities provoked the students to develop more efficient solution strategies for the next time. In Activity 2, the rule of the ratio was "one food bar feeds three aliens" again, and A2Q1 asked, "Will 12 food bars be enough for 36 aliens? Explain." This time, the numerical values given in the activities were increased, and the comparison problem, instead of a missing value problem, was used to provoke the students to develop different strategies. They were efficiently iterating drawings to find a logical solution. At this time, one of the students, Burcu, provided a unique solution practiced by the other students in the following questions. When she said that she used the drawing on the board, her peers thought that she would use the strategy of Egemen. They objected to drawing 36 aliens one by one and updated their strategies accordingly.

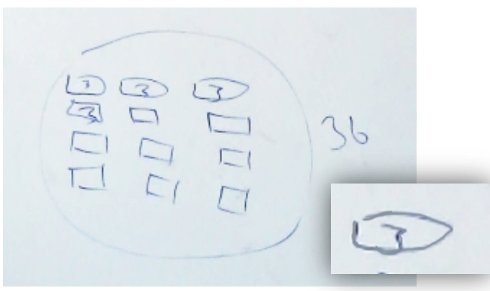


Figure 4. 24. Burcu's drawing for A2Q1

Burcu: Teacher! Look at it here, please! When we do all three by three, here is a total of 36. That's enough [draws 12 rectangles to stand for 12 food bars and places the numeral 3 in each; see Figure 4.24]. *(Datum and Claim)*

Teacher Merve: I see! You did it like, "I did not draw the aliens, and I wrote the numbers in it (food bars)," so when you said 3, 3, 3, it was 36 in total. Okay, nice. *(Warrant)*

Instead of drawing aliens, Burcu drew the food bars and wrote the number of aliens in the rule inside each rectangle. She modeled the rule at first with one rectangle instead of two different pictures (circle and rectangle) and repeated the model 12 times to test whether 12 food bars were enough for 36 aliens (see Figure 4.24). She set aside the 36 aliens and tried to reach the total number of aliens as the missing value by skip counting. As a result, there were 12 food bars drawn on the board. She compared this result with the problem, multiplied it by 3, and concluded, "That's enough". Her way of iteration led her to organize by iterating. This solution was not rejected, and again several students used this solution strategy in the following questions, as provided in Figure 4.25.

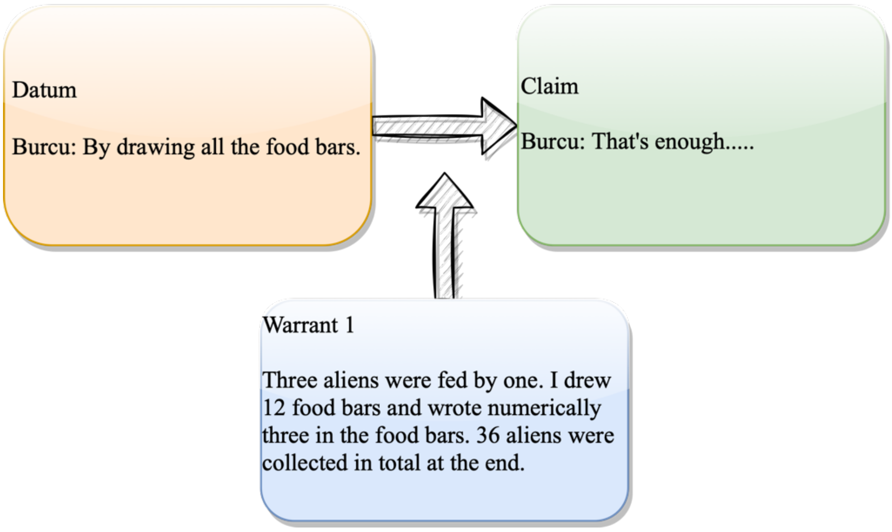


Figure 4. 25. Argumentation scheme for Burcu's drawing solution

In Figure 4.25, the rest of the class accepted remodeling the rule as the new unit and its iteration process because of its time efficiency with respect to drawing all the units. This organization method and explanations were repeated in the further instructional sequence. Fourth question of Activity 2 (A2Q4) focuses on a comparison problem with eight food bars and 20 aliens. Can represented his solution and explained his reasoning.

Can: She says 8 food boxes are enough to feed 24 aliens because they say 3 aliens in one box, 24 people will be fed here. Ma’am, 8 food bars feed 24 people, normally because it says 20 people here; it will feed more than enough, Ma’am. *(Datum and Claim)*

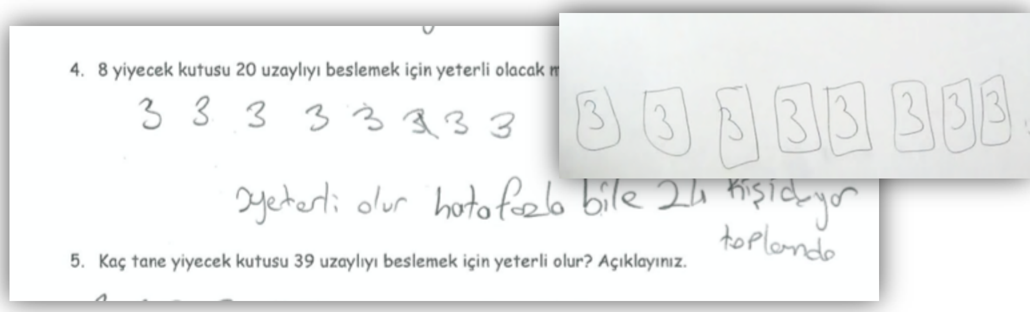


Figure 4. 26. Can’s solution for A2Q4 before and after Burcu’s explanation

As seen in Figure 4.26 (left figure), Can iterated number symbol “3” for eight times before Burcu’s explanation and just wrote that “it would be enough. it is even more than needed.” In his explanation, similarly, Can made the model, iterated it eight times as given the number of food bars, and computed how many aliens were needed for eight food bars. Lastly, he compared 24 aliens with 20 aliens to be fed as given in the problem. He concluded that the aliens were already full of eight food bars. With his explanation and changing model, Burcu’s datum was accepted and transferred by Can from A2Q1 to A2Q4.

After the students studied the same rule in the two consequent activities, “one food bar feeds three aliens,” it was pretty attractive for them to see a different rule, “one food bar feeds five aliens,” in Activity 3 (see Figure 4.27). They started testing their strategies. The first problem was “How many food bars are needed to feed 30 aliens?” (A3Q1). As a suggestion for the teacher provided by the instructional sequence, Teacher Merve selected a student, Eray, who used a drawing strategy.

1. How many food bars are needed to feed 30 aliens? Explain

2. How many aliens can you feed with 5 food bars? Explain

3. Using a table show how many food bars you would need to feed 70 aliens?

4. How many food bars do we need to feed 35 aliens? Explain.

Figure 4. 27. Activity 3

Eray again drew all the composite units as seen in Figure 4.28. He started drawing five-by-five alien figures, so each row had five aliens. For every five-alien group, he added one food bar near it. He iterated this rule till the total number of aliens was 30. At the end of this iteration, he counted all the food bars, and the result was six. As seen, Eray transferred the strategy in the first activity's visuals of the first activity and his friends' explanations. There was no disagreement about his warrant which was already accepted in the previous activities, but we started to focus on the time-efficiency part of the solution. The students also talked about the issue, "it will take a long time." Larger numbers became a source of motivation for the students to develop their models or adopt new models to reduce the time for a solution, as revealed by Burcu and Can's solutions. Moreover, they did not leave iterating the unit by drawing repeating patterns since drawing was mentioned as a "more concrete solution" initially, and a few students presented multiplicative understanding with justifiable reasoning. Unit iteration made the students feel safe with their justifications for the solution. Using repeating patterns by drawing was not enough to achieve multiplicative thinking. The learning environment was designed to reinforce students' thinking toward proportional reasoning with the new challenges in the following activities.

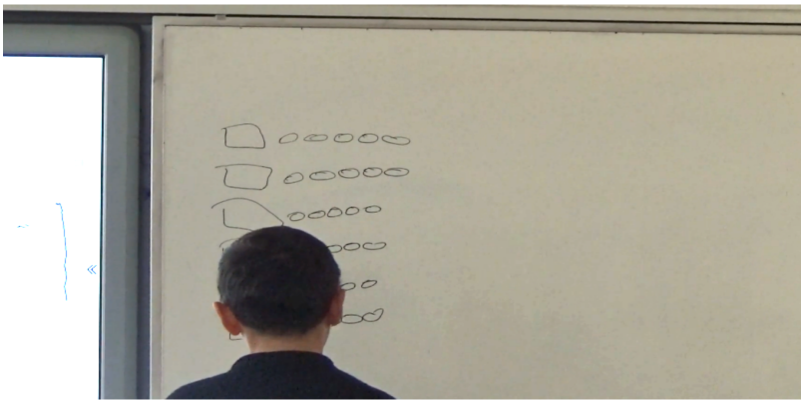


Figure 4. 28. Eray using drawing all the units

Drawing all the units did not become the subject of the discussion after Eray's solution. Meanwhile, in the second question of the third activity (A3Q2: How many aliens can you feed with one food bars?) Ahmet again came up with a solution similar to Burcu and Can's answers (see Figure 4.29). He used a startup strategy for ratio table tool in his activity sheet, but he changed his solution on the board. He was influenced by Burcu and Can. He drew the food bar, wrote the number of aliens eating one food bar in the rule, iterated them five times, and calculated through repeated addition by five.

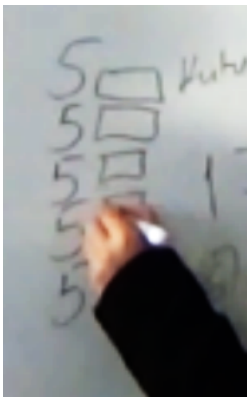


Figure 4. 29. Ahmet's semi-symbolic drawing for A3Q2

Can evolved Burcu's strategy into a *more symbolic way,* as represented in Figure 4.30. This time, he drew neither the food bar nor the aliens; instead, he only wrote the number values of each unit. In his model, he used the equation sign to express "feeding" action. "Five aliens eat one food bar" rule was represented by number values written in a square, and Can repeated this rule five times to express five groups of food bar-alien ratios. Both units were represented in numerical symbols now. This was the last point that the drawing strategy observed in this sequence.

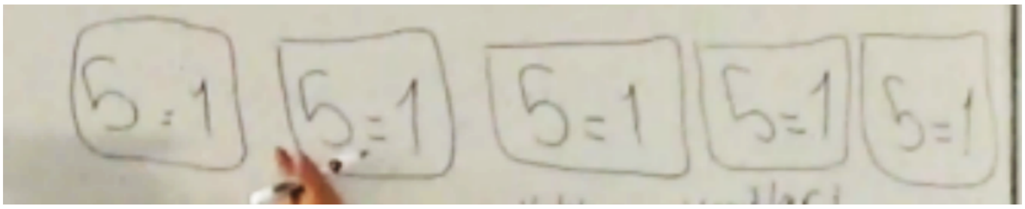


Figure 4. 30. Can's remodeling of Burcu's strategy for A3Q2

At the end of the fourth activity, Teacher Merve observed the utility of drawing in a different context. The drawing was used as a backing for the division algorithm by Eray. He was transferring the strategy of Ahmet. A4Q1 asks, "will 10 food bars be enough for 20 aliens?" with the rule, "two food bars feed four aliens". Eray divided 10 food bars into two and multiplied it by four, as given on the left of the board (see Figure 4.31). He described the division algorithm but thought he could not explain it, and added drawing strategy as a backing for his claim.

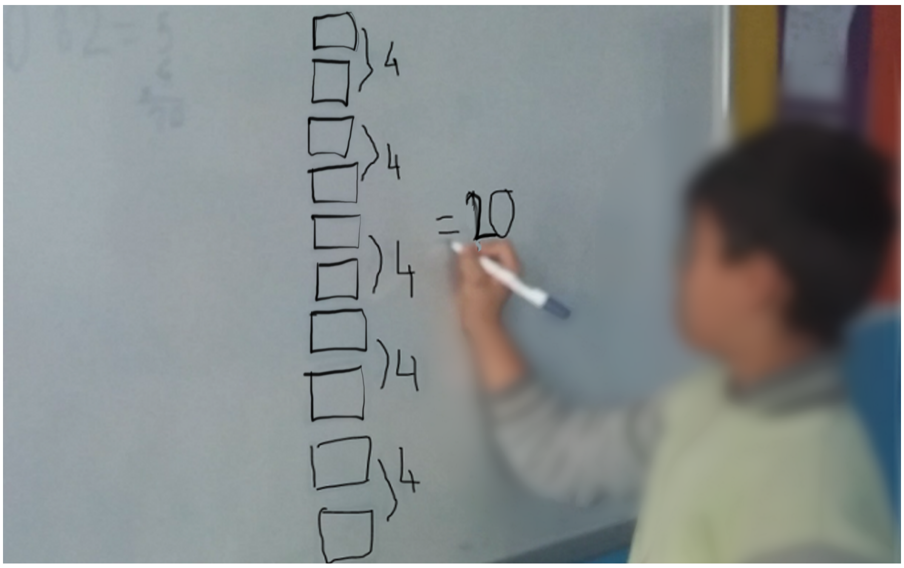


Figure 4. 31. Eray using drawing as a backing for division algorithm

In the second question of the fourth activity (A4Q2), the teacher and the researcher observed that drawing was also used as a backing for the unit ratio strategy. The rule is "two food bars feed four aliens," and Alp reduced the rule to "one food bar feeds two aliens". His solution represented a synthesis of Ahmet's and Can's solutions. Alp transferred the equation symbol from Can and the numerical symbol of aliens with the food bar figure from Ahmet to show the composite links (see Figure 4.32). It was also evidence of normative way of reasoning.

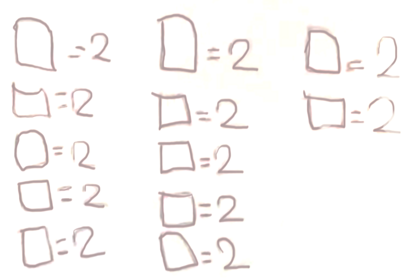


Figure 4. 32. Alp using drawing as a backing for unit ratio in A4Q2

During A4, the drawing strategy evolved into a symbolic representation since the number of food bars in the rule increased. All the efforts were to make the solution time efficient. Instead of drawing the figures, writing numerals was easy and fast. Can's solution is given as an example in Figure 4.33. The numbers on the left side represented the food bars. He calculated the food bars in every composite unit by skip counting by two. The number on the right side represented the number of aliens and increased four by four.

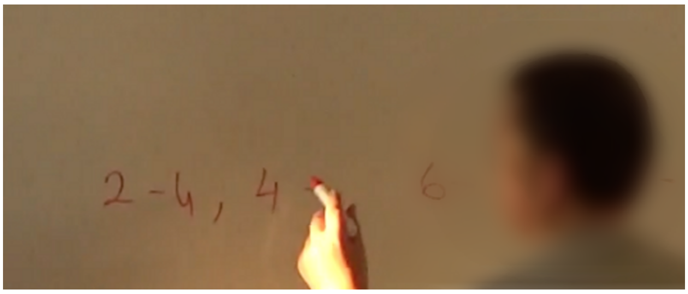


Figure 4. 33. Can's solution for A4Q1

In further activities, students found new ways to organize their ideas as prompted by both a change in instructional activities and classroom discussion. Each representation had its own language, and the students added their remarks each time. Drawing strategy neither fostered multiplicative thinking nor provided time efficiency. Nevertheless, the drawing strategy became a backing for several questions in the further activities and it enabled unit ratio and ratio table strategies to occur smoothly. Additionally, the drawing strategy encouraged several students (Evren, Lale, and Acar) to come to the board or to talk about their ideas. According to Teacher Merve, this was a rare situation, and she always supported their courage in the classroom with the help of this strategy.

4.2.2. TAS 2 Reasoning about Algorithms for Composite Units

In classroom 7/X, the teacher and the researcher observed frequently doing multiplication and division operations to solve the questions in the beginning. The instructional sequence also reported doing multiplication and division as a solution strategy. Although they were conducting the correct procedural skills for doing algorithms, they could not explain their answers clearly. As frequently seen in the students' activities, the teacher and the researcher gave some time to understand the students' reasoning about these algorithms. The components of the operations had their meanings according to the context of the problem.

Teacher Merve asked for different answers for A1Q1, which became a social norm for the intervention. The reasoning of the four students was based on similar calculations, but they claimed that they used different strategies. Teacher Merve asked Batu his solution. He expressed that his answer was about *multiplication (Datum)*.

> Batu: Ma'am, but I made it multiplied. (Datum)
>
> Teacher Merve: Okay. Tell me, how did you do it?
>
> Batu: For example, there are 2 boxes here. If three aliens eat one box, we multiply 2 by 3, 6. There are 9 aliens, and we will subtract 6 from 9. Three aliens left. *(Warrant and Claim)*

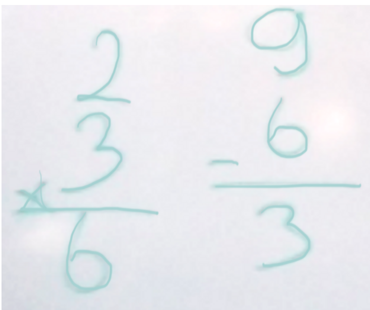


Figure 4. 34. Doing by multiplication by Batu

In his explanation, Batu represented his understanding of the rule in Figure 4.34. He first multiplied two food bars with three aliens to find how many aliens in total could be fed with two food bars *(Warrant)*. Since nine aliens were waiting to be fed, three aliens remained *(Claim)*. Indeed, there was silent consent about the answer for A1Q1—that is, one more food bar was needed, or there were not enough food bars to feed the three groups of aliens *(Claim)*. The students tried to show how to reach these results through *operations/algorithms.* In this argument, Batu used calculations starting from the total number of food bars. Both strategies on the students' papers were sampled in Figure 4.35 below.

Figure 4. 35. Strategies for solving A1Q1 by doing algorithms

The students either selected path 1 or path 2 according to the selection of the total number of food bars or the total number of aliens. Path 1 is a solution in which the students used the given total number of aliens and the number of aliens in the rule. On the other hand, path 2 is a solution that the students used the total number of food bars to multiply by the number of aliens in the rule. They multiplied them to find the exact number of aliens to be fed. Batu's explanation for path 2 was accepted as transparent, and no more questions were asked.

Fatma and another student came up with the explanation of path 1 after Batu. Fatma, a hardworking student with above average in mathematics according to Teacher Merve, was very encouraged to show her work. Fatma claimed that she solved the question differently and there should be three food bars. She explained that she divided the total number of aliens (nine) into the number of aliens in the rule (three); three aliens remained (*Datum and Claim*). Her explanation was not clear at first. The teacher invited her to the board to explain her solution better. Fatma used path 1 to show her answer; according to her, the solution method was different. She concluded, "There should be three food bars." by representing the quotient "three" (*Datum*). For further steps, she calculated how many food bars were needed to feed nine aliens. One of her friends agreed with the solution saying that she solved the question the same way as Batu.

> Fatma: Ma'am, I did it like this…There are nine things… we need three food bars. I divided [it] by three and there were three left (They presented the quotient as the answer of the problem) *(Data and Claim)*

Additionally, Teacher Merve confirmed her solution, and there wasn't any reaction to her solution. The students' activity sheets also represented that Fatma's solution strategy was one of the normative ways of reasoning and needed to be discussed in detail. While Teacher Merve was testing the classroom whether the misconception of *the number of rows* continued, Buket came to the board to show that the number of rows did not matter to solve A1Q2 (see Figure 4.6). Buket added four aliens in the first row to five in the second row to find the total number of aliens. As one food bar fed three aliens, Buket concluded that three food bars were needed after dividing nine aliens by three. This operation resulted in three in the quotient, meaning how many three-alien groups there were in the nine aliens if the divisor was three in the given rule. In this context, *the quotient does not directly represent the number of food bars needed.* As seen in Figure 4.36, one operation was missing: "3 (quotient representing the number of groups) × 1 (food bar needed for each group) = 3 (food bars needed in total)." Another possibility was that the divisor could be three aliens per food bar (3/1 composite units). In this case, the quotient could be a food bar. This possibility seems more acceptable and meaningful in terms of proportional reasoning in order to reinforce the utility of the constant of proportionality. All the students and Teacher Merve had already accepted this practice. As each alien group matched with one food bar, the missing operation did not catch anyone's attention. Moreover, the students had different understandings, especially when the numbers were the same. Which component represented a food bar or alien in the division operation confused the students because they did not understand the meaning of the operation. This kind of data organization in the multiplication and division algorithm did not encourage students to write the name of the variables. Teacher Merve, at the end of the explanation to the students, revoiced the students' explanation on the board to make the meaning of the operation clear for the rest of the class.

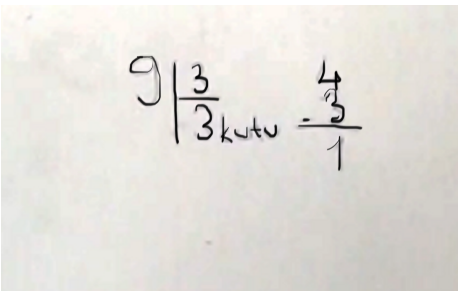


Figure 4. 36. Buket's solution strategy for A1Q2

After the first activity, Teacher Merve and the researcher continued questioning the students' understanding of division and multiplication algorithms. They conducted these algorithms to explain their reasoning about the solutions that students were eager to use. Activity 2 consists of five questions with the same rule, “One food bar feeds three aliens,” but the number values are larger than the first activity. The first three questions involve comparison problems in which the number of aliens is three multiples of the number of aliens, but the rest of the questions are not. A2Q1 is presented in Figure 4.37.

1. Will 12 food bars be enough to feed 36 aliens? Explain.

Figure 4. 37. First problem in the Activity 2 (A2Q1)

Doing operations even without its incomplete structures was accepted and applied by the students. Next, Simge was the one who came to the board, and she conducted the division algorithm. She divided 36 into 12, her datum supporting the conclusion. The representation of the number three in the quotient of the division operation was unclear, owing to her explanation without a warrant. The rest of the students did not add anything, but the teacher and the researcher continued questioning her reasoning.

> Simge: I divided it by 12 because there are 36 aliens. It's already 3 out. (Datum and Claim)
>
> Teacher Merve: You divided 36 by 12; you did a correct operation, but if you could explain it, why did you divide it for us? What is 36? What is 12? What is the result?
>
> Simge: Ma’am, 36 is the number of aliens... 12 is the number of food bars. When I divide by 12, one bar for all three aliens is enough... (Datum)
>
> Teacher Merve: So when you say something is the number of aliens.
>
> (My teacher is wrong, students raise their hands to add, share their opinions, etc.)
>
> Teacher Merve: Wait for a second. She says three she found showed that 12 food bars were enough. All right, but why does it show that when 3 comes out, it's enough?
>
> Simge: It says in the question that it shares one bar with 3 aliens. (Warrant)

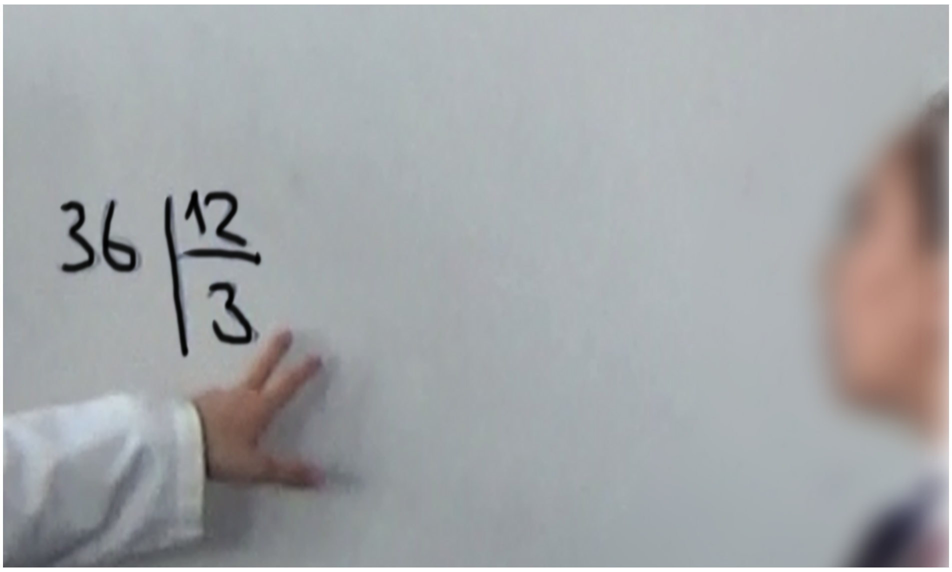


Figure 4. 38. Simge was explaining her answer

In Figure 4.38, Simge explained the dividend, divisor, and quotient. She said that 36 was the total number of aliens given in the problem; 12 was the total number of food bars. What she found represented the three aliens provided in the rule. In this comparison problem, Simge matched the quotient with the rule and made a comparison with them. She concluded that it was enough to explain why the given number of food bars was enough for the aliens. Both Buket and Simge provided the quotient as the claim in these questions. These practices were also changing according to the type of questions. Dividing the total number of aliens by the total number of food bars gives the number of aliens per food bar, which is different than path 1 and path 2. Simge connected the links between the total number of units and compared it with the rule of the question. Her explanation satisfied the class, but her explanation was not enough and needed some backing (see Figure 4.39). Division algorithm without naming the units made it harder to interpret the result. *Quotient did not directly provide a food bar or aliens, but it is a composite unit representation, a ratio.* The classroom was slowly moving towards this understanding together.

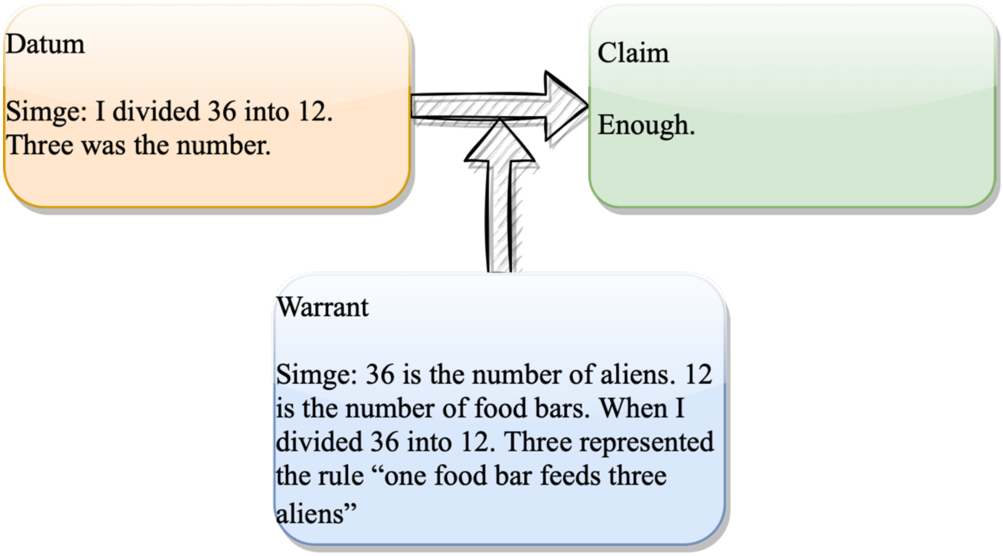


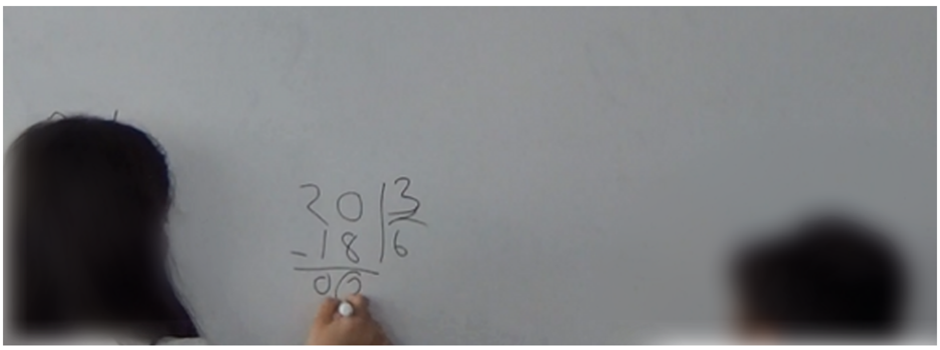
Figure 4. 39. Simge’s argumentation scheme of division algorithm for comparison problem

The first three questions in the Activity 2 were comparison problems, and the total number of aliens was three times as much as the number of food bars. What if the comparison question is not three times as much as the number of food bars? A2Q4 was very appropriate to test the students' understanding of the acceptance of Simge’s warrant and to elicit more ideas about division algorithm. The question asks, “Will eight food bars be enough to feed 20 aliens?”. Egemen again came to the board; this time, he used the division algorithm. He put the dividend as the total number of aliens given in the food bar, the divisor as the three aliens given in the food bar, and the quotient as the warrant of the “enough food bar” claim.

Teacher Merve: 4 is okay. Will 8 food bars feed 20 aliens? Is it enough? What about 8 food bars?

Egemen: We will divide 20 by 3 to feed 20 aliens. No matter how many… Enough. (Datum and Claim)

Teacher Merve: (Shows the remaining 2 on the board and asks what are these 2?) Now what is this remainder 2? What does it represent?



Egemen: Ma'am, well... Aliensss… Number of food bars. (Warrant)

Teacher Merve: Look, you divided 20 aliens into three, and there were 6 of them.

Egemen: 6 bars. 6 bars feed 18 aliens. 2 aliens are left as remainder. (Warrant)

Teacher Merve: So you said 6 boxes feed 18 aliens (Teacher showing on the board). (And the remainder of the division by 2) This is the number of aliens left.

Egemen came up with a solution. He offered to divide 20 aliens into three *(Datum)*. He concluded that eight food bars were enough for 20 aliens *(Claim)*. Currently, there was a remainder, two. Teacher Merve insisted on the meaning of remainder and the quotient. It was mentioned that *three* in the divisor meant three aliens in the rule. For the remainder part, Egemen was confused and said that the remainder was the food bar left. He was confused because he might think there were eight food bars in total; the required food bars were six in the quotient, and two food bars would remain. The remainder of the division operation made him confused. When Teacher Merve intervened and asked Egemen the meaning of six in the quotient, he changed his statement and added that six in the divisor represented the required food bars. Although the divisor did not directly represent the number of food bars, he repeated Buket's warrant that the quotient was the food bar needed. This assumption was repeated during the instructional sequence, accepted, and practiced like this by the students several times. The teacher wrote Egemen's explanation on the board, as given in Figure 4.40.

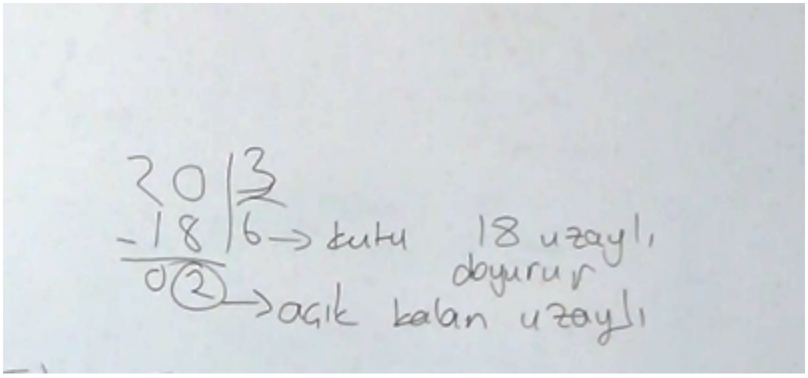


Figure 4. 40. Teacher Merve wrote the explanations.

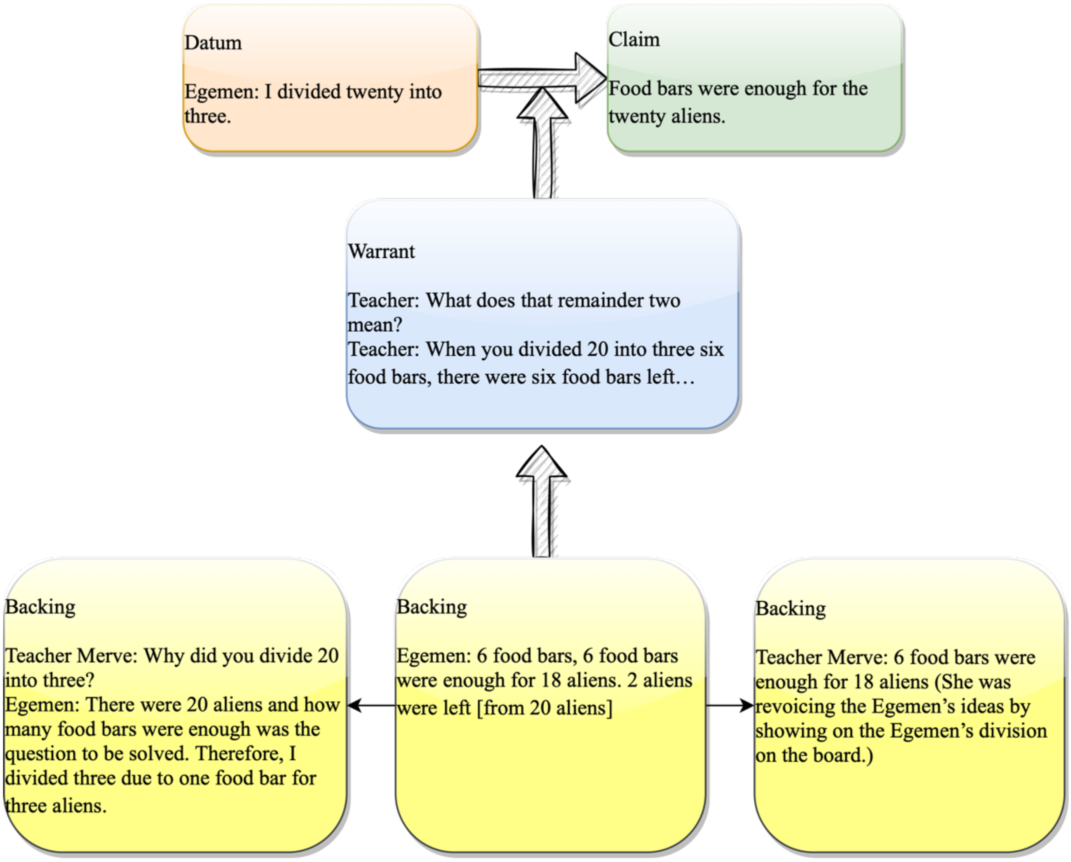


Figure 4. 41. Argumentation scheme for Egemen's division algorithm

A2Q4 is a comparison problem for ratio. Egemen did not explicitly put what kind of comparison he did, as seen in the argumentation scheme (see Figure 4.41). Six food bars were enough for 18 aliens; how to feed two aliens remaining? Buket's and Simge's practices involved only one operation, and the quotient represented the result. Likewise, Egemen showed the quotient as a result. In the cases of Simge and Buket, the total number of aliens was multiple of the total number of food bars given in the problem. However, Egemen's case was not the exact multiple, non-integer ratios. Although the quotient did not provide a direct result for those three situations, the students accepted Buket and Simge's representation of the quotient. This means several students realized the difference between Egemen's and the others' solutions. According to the class, Egemen should explain what six and two meant in the quotient and the remainder, respectively, regarding the problem context. There were two aliens left to be fed. Batu, Nazan, and Berk added some explanations for this discussion. The answers to these questions were the other backings for the argumentation scheme of Egemen.

Batu tried to explain, "six food bars were found, eight food bars were given in the problem. It is impossible to say six food bars were enough". Teacher Merve connected this discussion to the point that eight food bars were given in the problem. Egemen found six food bars, but the class waited for a detailed explanation for the "enough food bar" claim. Batu supposed that the quotient should be equal to eight to say six food bars were enough. Path 2 (starting from food bars) was more logical from his perspective. He multiplied eight given food bars with three aliens in the composite units[3]. He concluded that 24 aliens would be fed eight food bars, so four more could be fed eight food bars. Several students approved of this strategy because they also used it. In the end, Egemen's and Batu's solutions led to the emergence of a question. Berk interpreted that two aliens remained hungry in the Egemen's solution. On the other hand, extra four aliens needed to feed with eight food bars in Batu's solution. Which one was the correct solution? Two solution strategies were correct: eight *food bars were enough to feed 20 aliens*, but the left food bars or left aliens confused their thinking. In this situation, another path occurred, path 3. Akın divided the total number of aliens (20) by the total number of food bars (8). He found four as the remainder and two as the quotient. Teacher Merve requestioned the meaning of the quotient, the remainder, and the reason to follow path 3. Akın said that four were the food bars and two were the aliens remaining (see Figure 4.42), which became a recurring situation in the algorithms misguiding the students to reason proportionally. Immediately, Zeynep objected to this reasoning and claimed the reverse: four were the aliens remaining, and two were the aliens to be fed. The idea of comparing the quotient with the proportionality constant did not emerge, and this new path did not make sense in the class.

As it was mentioned in the former paragraphs that A2Q4 is a comparison ratio problem. Path 1 and path 2 did not completely satisfy the students in the comparison problem with non-integer ratios. Moreover, path 3 was emerged as well but its argumentation remained incomplete. The students transferred the same reasoning for path 1 and path 2 successfully with the integer ratio questions without any backing and warrant when they needed this strategy. A template for solution with *doing calculation* was formed while advancing in the activities. It was accepted clear that when student said "I solved with doing calculation" or "I calculated" which became a datum to accept the claim. In such kind of argument, the conversation followed the explanation of which pathway he/she followed as warrant and claim

---

3 The students who used path 2 accepted that the given number of food bars were multiplied with the number of aliens in the given rule as in path 1 and the result gave the number of aliens required, In this question, the result became 24 "food bars X aliens" as the units of the result.

sometimes datum and claim were accepted as enough for the argument. These argument templates can be considered as normative ways of reasoning due to dropping of the backings and even warrants for the explanation.

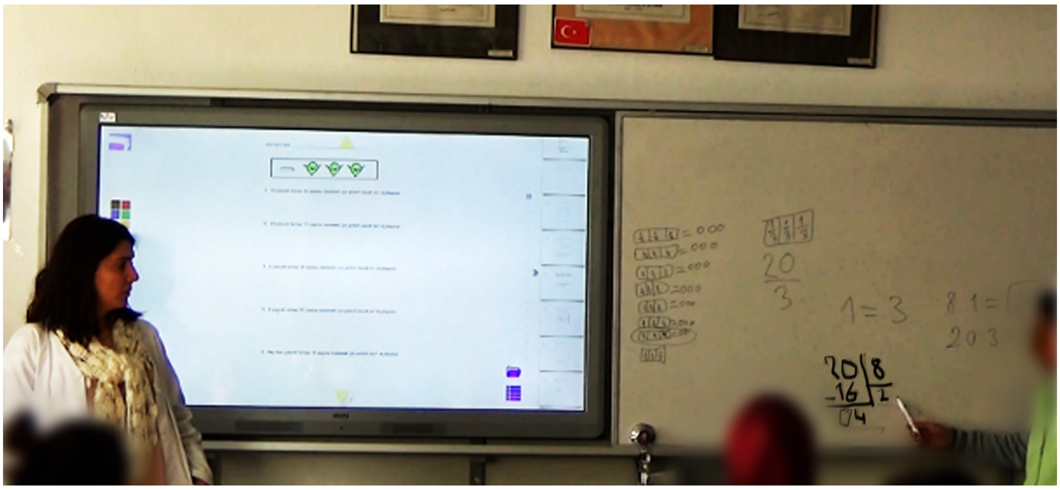


Figure 4. 42. Akın's showing path 3 for A2Q4

As mentioned in the former paragraphs A2Q4 is a comparison ratio problem. Path 1 and path 2 did not completely satisfy the students in the comparison problem with non-integer ratios. Moreover, path 3 emerged as well, but its argumentation remained incomplete. The students successfully transferred the same reasoning for path 1 and path 2 with the integer ratios without any backing and warrant when they needed this strategy. A template for a solution by *doing calculations* was formed while advancing the activities. It was accepted that when a student said, "I solved by doing calculation" or "I calculated," it became a datum to accept the claim. In such an argument, the conversation followed the explanation of which pathway they followed as warrant and claim sometimes might turn into datum and claim. This situation was accepted as enough for the argument because these argument templates can be considered normative reasoning due to the dropping of the backings and even warrants for the explanation.

In another practice, students did not need to show which component represented what, but it was unclear for most students. Teacher Merve further questioned how the ideas and the similar number in the components confused students. To organize the data, it was necessary to name the unit verbally, through drawing, or in writing. In this respect, the division algorithm did not help the students in 7/X. Teacher Merve's intervention and revoicing the arguments made the students reorganize and requestion their understandings. Moreover, we frequently experienced the utility of paths 1 and 2 in further activities with the same scheme, especially in the missing value problems. Nevertheless, the division algorithm did not wholly help students to investigate the comparison problem further. Still, they transferred *the slicing the food bar idea* here as the unit ratio strategy, which is explained under the third strategy heading below.

4.2.3. TAS 3 Linking and Iterating Composite Units Through Unit Ratio Strategy

As described in the discussion of slicing a food bar, the students explored finding a piece of food bar per alien to feed and representing them as fractions. Since the rule was the same in the second activity, they transferred this knowledge to the fourth question in A2. As described in the division strategy, Batu and Egemen doubted path 1 or path 2, in which some aliens or food bars were left. What do the remainders mean, and how to organize this new data? Arda proposed slicing the food bars, which already became a taken-as-shared idea. The students put a name for this strategy, *sharing*. Arda tried to explain the doubt through equal sharing of food bars, in other words, unit ratio, how many food bars were needed for one alien. Arda started to divide one of the food bar figures into three parts on the board. Teacher Merve added that there were eight food bars. Arda began drawing a box on the board and divided that box into three (see Figure 4.2.3.1). The conversation is given below.

> Teacher Merve: Arda, do your own solution on the board and let us see! (Arda starts to do it on the board. He draws a box on the board and divides it into three.)
>
> Teacher Merve: What is the reason to divide one by three, I mean that one-third? (Teacher Merve shows the number on the board)
>
> Arda: One-third of a piece feeds an alien. (*Datum*)
>
> Teacher Merve: Hahh! So, you are saying one eats one-third per alien.
>
> (Teacher and researcher walk between them)
>
> Buket: (turning to the teacher) What Arda did makes sense, Ma'am.
>
> Researcher: What are you doing, Arda, now?
>
> (Arda draws all the food boxes (eight pieces) and first divides them into three, then places 20 aliens on edge.)
>
> Arda: We are dividing this food bar, Ma'am, for three aliens. Ma'am, one alien, gets one piece. Likewise, two aliens over there. (*Warrant*)
>
> Researcher: How many aliens did you draw there?
>
> Arda: 20
>
> Researcher: You drew a total of 20 aliens. So how many food boxes did you make?
>
> Arda: 8 of them.
>
> Researcher: What happened next? Are there two (food boxes) left here?

Arda: Two (showing a full box of food and the remaining food bar) remain. Accordingly, my teacher eats one-third of it, my teacher. That piece remains, Ma'am. (*Claim*)

Arda drew all the units one by one and divided each rectangular form of the food bar into three according to the rule. He gave one piece to each alien. Six food bars were given to 18 aliens. Two aliens remaining were fed with two pieces, and one piece was left from the one food bar. All in all, one whole and one piece of food bars were not eaten. He combined drawing and unit ratio to solve the dilemma between Egemen and Batu. This case was an example of the exploration of the unit ratio and transfer of the idea, *slicing the food bar with respect to the rule* as the datum (see Figure 4.43).

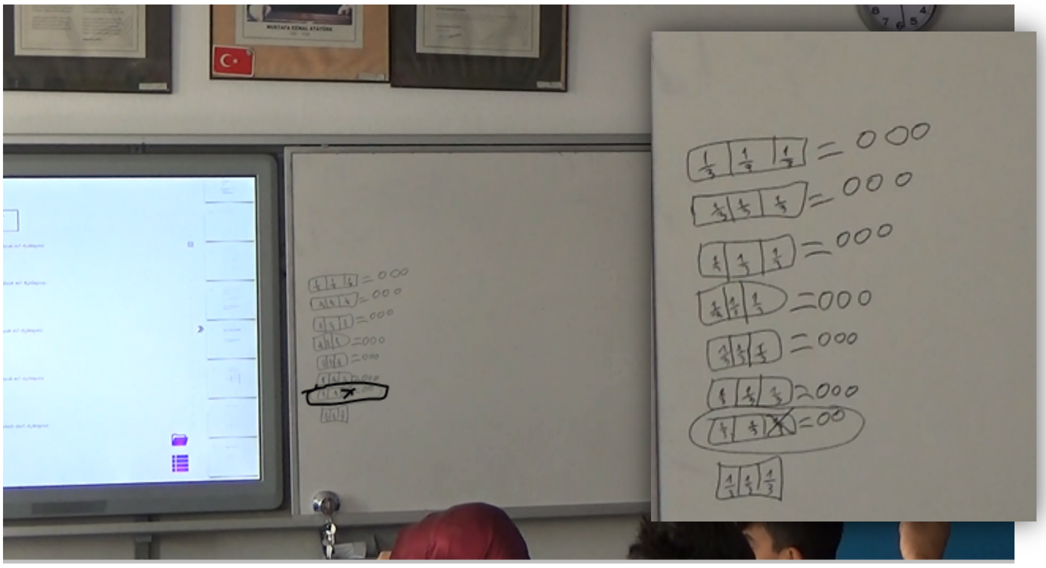


Figure 4. 43. Arda's unit ratio modeling on the food bar figure

To further explicate the construct that Arda created with the help of Evren and Fatma's previous contribution, Ercan added symbolic representation to Arda's explanation as a warrant. He preferred using four operations to solve problems all the time. He was one of the students who performed well at explaining his solution. He transferred the idea of equal sharing that Arda did as a warrant into a symbolic representation. Without drawing, Ercan conducted this solution by multiplying (see Figure 4.44). He repeated the same process for slicing the food bars. Fatma also symbolized the little piece of food bars as 1/3; similarly, Ercan represented one food bar as 1/3 and multiplied 20 aliens by 1/3 to find the total number of food bars required for 20 aliens. He crosschecked his solution by counting the pieces of Arda's drawing and concluded that it was enough by showing the left pieces on the board. It was the backing of Arda's explanation with the unit ratio. Teacher Merve concluded that Ercan did the same way as Arda, but they used different strategies. Both were constructed at the very beginning of the sequence.

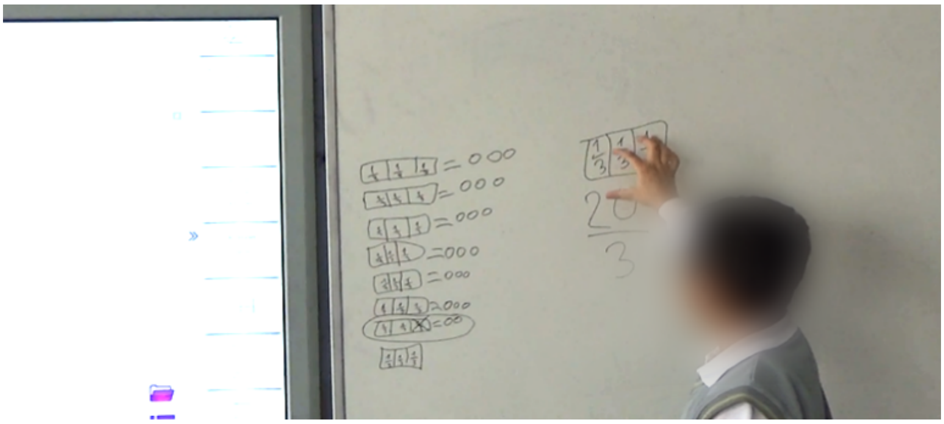


Figure 4. 44. Ercan's calculation of 20/3 food bars

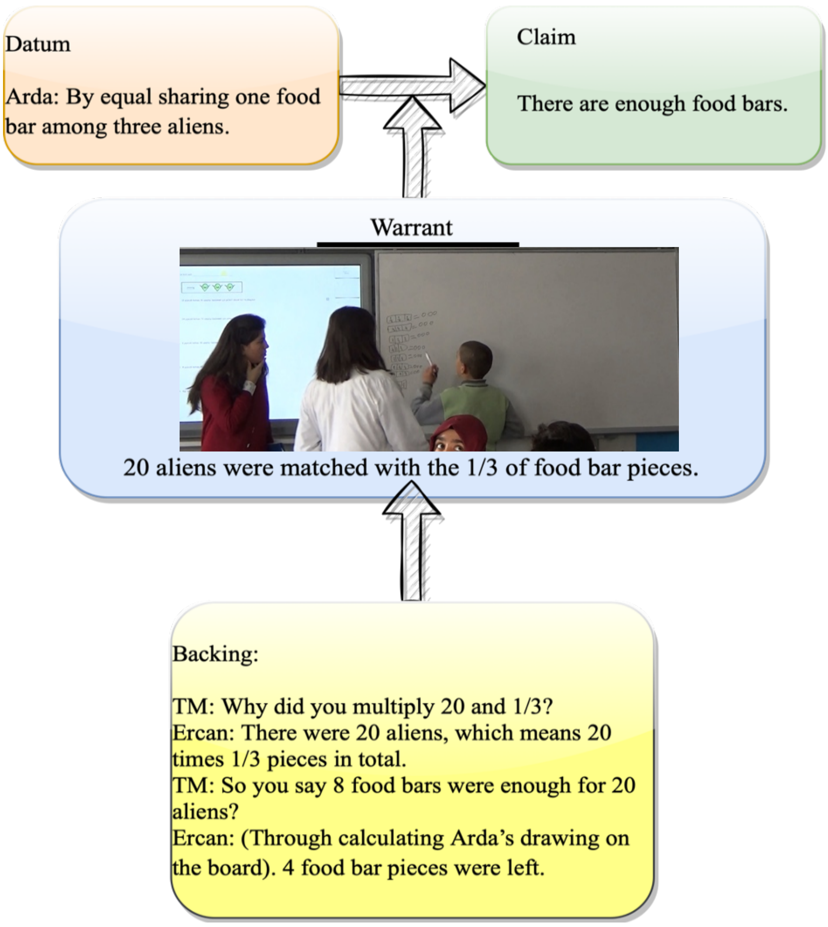


Figure 4. 45. Argumentation scheme for Arda and Ercan's sharing of food bars

The unit ratio strategy emerged when the given numbers were non-integer ratios. Therefore, the unit ratio strategy did not emerge in the third activity since all the questions involved exact multiples, integer ratios. In A4, the rule changed, and it became, “two food bars feed four aliens”. As the reader notices, the number value of the food bars in the rule was *one* for aliens. The rule was given in the unit ratio, the number of aliens per food bar, the reverse of which is the number of food bars per alien. The students in this classroom investigated unit ratio with the number of food bars per alien strategy. The new context in A4 laid the groundwork for the other unit ratio representation.

1. Will 10 food bars be enough to feed 20 aliens? Explain.

2. Will 12 food bars be enough to feed 22 aliens? Explain.

3. How many aliens can 14 food bars feed? Explain.

4. How many aliens will 98 food bars feed? Explain.

5. How many food bars are needed to feed 16 aliens? Explain.

Figure 4. 46. Activity 4

The students built on new ideas in the explore phase of the LED teaching cycle. When they realized the rule, they identified by thinking aloud that if two food bars were enough for four aliens, then one food bar would be enough for two aliens. The rule directly guided them to make some reductions as a strategy. Our question in mind was whether students would connect dividing one food bar into pieces or two food bars so that, in each case, the same amount of food bar per alien would be used. The students finished the problems in seven minutes. Teacher Merve started again with A4Q1, “will 10 food bars be enough to feed 20 aliens?”. Apart from the students’ solutions with calculation mistakes (we checked them in the explore phase), the class responded as “enough”, which became a typical response for a comparison problem with integer ratios. In the discussion phase, as the students compared different ways of solving the question, they found a chance to discuss a shared way of reasoning. Therefore, Teacher Merve asked the students if any solutions were using any of the four strategies. During explore phase, Teacher Merve and the researcher observed that there were unit ratio strategies and decided to move on with this idea. Zeynep was one of the students who performed well in mathematical reasoning in the classroom. She solved the problem on the board and described what she did. She used the rule in the given frame (similar to Ahmet’s drawing in Figure 4.29), and she reduced the number of aliens related to the food bar, so the result became one food bar feeding two aliens. Afterward, she found the number of aliens that each food bar feeds, one food bar for two aliens. She used path 2 and started calculating from the given food bar. Two, as the multiplier in the multiplication operation, represented the number of aliens, and 10 represented the number of food bars provided. This idea was widespread idea among the students. As the division algorithm strategy mentioned, two represented “two aliens per food bar”. Accordingly, she calculated how many aliens could be fed with 10 food bars. The result was 20 aliens (see Figure 4.47).

> Zeynep: I first gave a form like this on the top (showing the rule). Then two aliens go to a bar: (Continues to solve on the board).
>
> Teacher Merve: Why did you double up with it [10]?
>
> Zeynep: Because, Ma’am, I said that if two bars are enough for 4 aliens here, 2 aliens will go to one bar. Since there are 10 food bars, I multiplied by the number of 2 aliens, which also feeds.

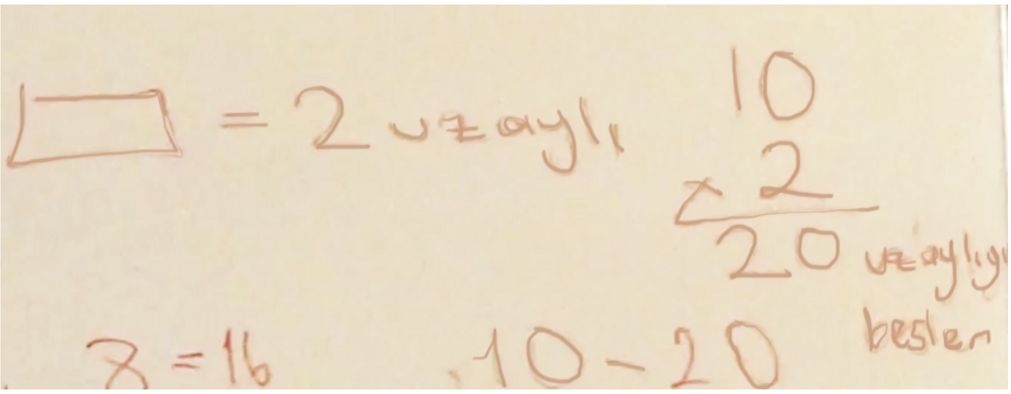


Figure 4. 47. Zeynep’s unit ratio strategy

While the argumentation scheme above focused on sharing (see Figure 4.45), Zeynep's argumentation focused on the implicit reduction of the composite units (see Figure 4.47). Furthermore, the class did not label Zeynep's argumentation as sharing. Arda found the number of food bars per alien; on the other hand, Zeynep found the number of aliens per food bar. Two ways were actively used during the instructional sequence.

The class reflected on Zeynep's solution as difficult because of the implicit reduction of the rule. Therefore, this issue was discussed in the following questions, which had the potential to simplification of the rule. To illustrate, A4Q2 asked, "Will 12 food bars be enough to feed 22 aliens?", which is also a comparison problem with non-integer ratios. Teacher Merve was in search of detailed explanations of this issue and called up. Alp came up with a claim, "It is already enough, " meaning there were more food bars than required. The class agreed on this idea. How they reached this idea started with the attempt of Alp through drawing. He used Zeynep's drawing strategy for the rule and iterated the composite units 12 times as the number of total food bars in the problem (see Figure 4.32). As a result, he used the number of food bars as the base and found how many aliens were required to count by two.

> Researcher: Do you guys follow Alp's strategy?
>
> Alp: Ma'am, a bar feeds 2 aliens, so we need to collect it. 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24 or even increasing.
>
> Teacher Merve: You say one of them is increasing.

Simultaneously, Lale wondered about the differences between her and Alp's models by showing her solution on the activity sheet to the researcher. The researcher encouraged Lale to show her solution on the board (Figure 4.48). She was a student who believed that she could not perform any mathematical activity as she explicitly described in a small talk with the researcher. Lale's question was a good sign for us as Teacher Merve, and the researcher set out to make all students engage in the activities in a mathematically meaningful way. The question of Lale was carried to the classroom discussion, and she presented her model on the board. Lale used the rule as the way given in the activity sheet and iterated it. She did not reduce the rule. She drew the food bars but none of the aliens, a mathematical practice gained from simplifying the drawing. In the end, she used a build-up strategy. The researcher asked about the difference between Alp's and Lale's solutions to the class. Fatma said Alp used the "Exact Rule". Alp simplified it as one food bar for two aliens. Based on this description, the researcher asked if there was any distinction between the solutions with or without reduction.

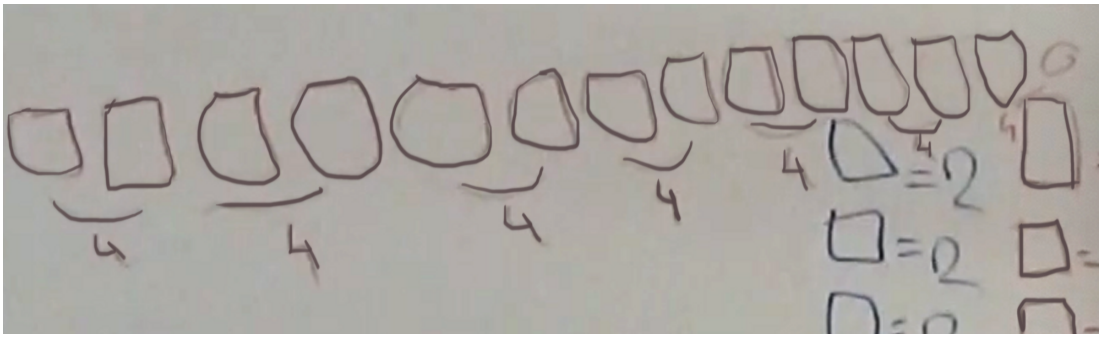


Figure 4. 48. Lale's solution for A4Q2

A five-minute small discussion was conducted based on this problem. The students were fond of both claims. The students who advocated "they are different (*Claim1*)" presented the following data: "Alp divided the rule into two small groups, Lale used the "Exact Rule" and "Alp's solution is easier", "Alp's solution is simplified." On the other hand, the students who advocated "they are the same (*Claim2*) presented the following data: "They found the same result", "They used the same solution method" as *data*. The students did not reject the other group's ideas, and they accepted each datum. In summary, several students focused on the numeral difference of the rule for the two strategies, and they concluded that the result was the same. However, when the easiest solution was asked, all of them agreed that the exact rule was "not easy to draw" and the simplified rule was "easy and practical". Essentially, they were comfortable simplifying but not changing the numbers in the given rule. This question directed the students to think about the concept of unit ratio as it was conjectured. The students continued using the unit ratio frequently. The unit ratio strategy became a helpful tool for the utility of the ratio table strategy, which is explained in the following strategy.

4.2.4. TAS 4 Linking and Iterating Composite Units Through Ratio Table Tool[4]

Ratio and proportion instructional sequence recommends students to use ratio table as a formal tool for the organization of data and the exploration of the proportional reasoning. In the first two activities, the students explored data organization with informal tools such as drawing, and algorithms as mentioned above. Unlike these organization strategies developed by the

[4] In Classroom 7/Y, the ratio table development occurred in a smooth way. 7/Y was a classroom in which the drawing strategy was mostly preferred in the first quarter of the activities compared to the 7/X. They evolved their solution with respect to the efficiency of time and effort. The details about ratio table strategy development for 7/Y can be found in the article: Sozen-Ozdogan, S., Akyuz, D., & Stephan, M. (2019). Developing ratio tables to explore ratios. *The Australian Mathematics Educational Journal (AMEJ), 1*(1), 16-21.

students, the emergence and construction of the ratio table did not emerge from the classroom 7/X in a natural way[5]. The students were familiar with using tables but within different mathematical contexts. With the guidance of educational design research and the directions provided by the instructional sequence, discussions about ratio table started emerging in the third activity. Teacher Merve and the researcher guided students to construct ratio table effectively through comparing and rebuilding in their solution strategies.

Before moving on to Activity 3, several students showed evidence of the multiplicative relationship within variables. Cansu presented her way of solution several times, although she was not competent in data organization yet. As illustrated in Figure 4.2.4.1, she wrote the rule in the first row, “1- 3,” for A1Q1. She interpreted this relationship with a curvy arrow, “three is *three times* larger than one”. In the question, 15 aliens were given, and the number of required food bars was asked. She wrote the number of aliens, 15, to the left side, under the number one (representing the food bar). She concluded that if the relationship between the rule was three multiples, the number of aliens must be divided into three. This was the example of emergent ideas of knowledge organization given in the question, so it led to the basics of ratio table and cross multiplication. Although Cansu’s contribution was noteworthy, it needed some reorganization of the units and a better representation of the multiple relationships between them. Moreover, she used the horizontal relationship between the quantities of different units, which was also a display of emergence for the ratio table (See Figure 4.49). Some students agreed with this solution method but did not contribute to it. Teacher Merve and the researcher understood that the students were unfamiliar with this representation. Berk even said that “I am not sure about its correctness, Ma’am”, and “Well, I think it is not correct.” What he was opposed to was not the idea of “three times larger” but the representation of the concept. At that point, this discussion remained an unevaluated claim.

The rule of the third activity was “One food bar feeds five aliens,” and the third problem stated, “Using a table, show how many food bars you would need to feed 70 aliens.” While they were solving this question the students asked, “What does using table mean? How can we do this question by drawing table?”. We guided them to do the problem what they knew about a table from their previous experiences. Their imageries about ratio table were differentiated. Teacher Merve and the researcher decided to focus on the principles of ratio table in this lesson hour.

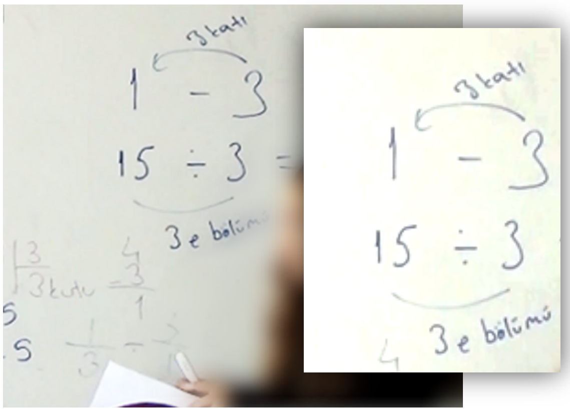


Figure 4. 49. Cansu’s data organization method for A1Q3

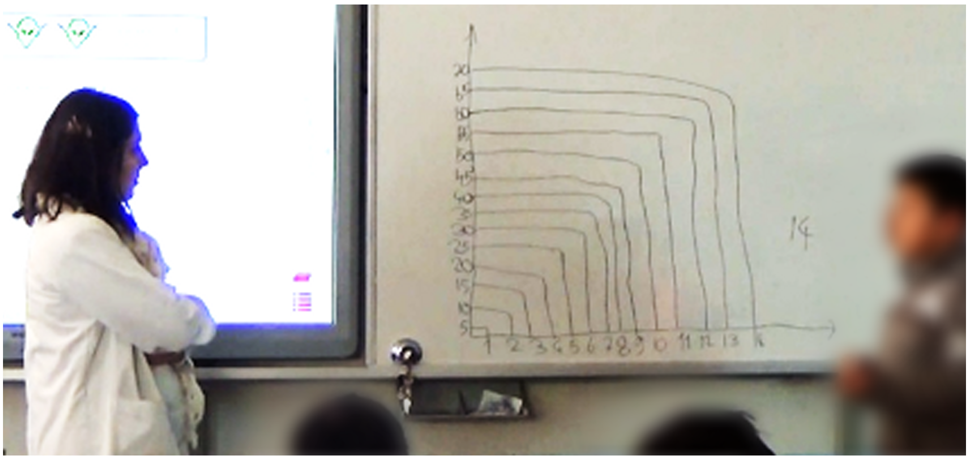
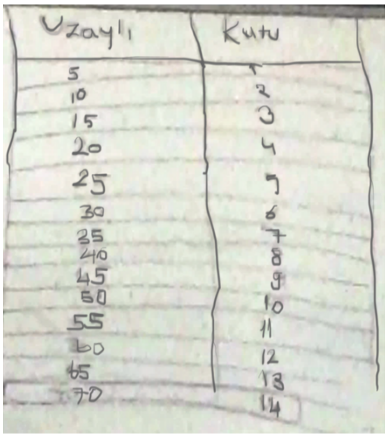


Figure 4. 50. Eymen’s imagery about the ratio table for A3Q2

The issue in Figure 4.50 revealed that students may have different imageries about the ratio table. Eymen used graphical representation, and several students admitted that it was a graphical representation. Eymen continued to explain his drawing at the same time. For five aliens, one food bar was needed. He spent three minutes drawing this graph. He drew 14 food bars and 70 aliens on the axis. At the end of the drawing, Teacher Merve asked the whole class whether it was a table. Teacher Merve and the researcher saw several examples of this representation in the explore phase. Therefore, Teacher Merve spared some time to talk about this discussion. Eymen found the correct answer, and the graph provided the linear relationship of the ratio. However, Teacher Merve focused on the similarities between the ratio table and asked, “Can you understand what this graphical representation shows?”. Students said that the names of the axis must be written. Which axis represented which data was a common point for a ratio table and graphical representations? Specifically, the students had to decide how many unique quantities to include in a table (i.e., aliens and food bars) and then debated different ways to organize the quantities in a table (e.g., deciding whether to write these names in the table).



Alien	Food Bar
5	1
10	2
15	3
20	4
25	5
30	6
35	7
40	8
45	9
50	10
55	11
60	12
65	13
70	14

Figure 4. 51. Deren's long table representation

One of the students, Deren, represented her answer on the board after Eymen (see Figure 4.51). Deren created her table extended vertically because, being the tallest person of 7/X, she could easily use the top of the whiteboard, which could be seen from everywhere in the classroom. Since the shape of the space given in the activity sheet and the whiteboard was rectangular (length is bigger than width), a horizontal table was more practical to extend to the right of the table. This change in the form of the ratio table created by the physical conditions for writing did not influence students' mathematical ideas. We could see horizontal ratio tables in further examples. As in Deren's example, two columns showed the names of the values: aliens and food bars respectively. Her reason to start with the "alien" column originated from the question with 70 aliens and asking the missing value food bars. She began writing the numbers under the "alien" column, and then she wrote the numbers under the "food bar" column. Under those names, the numbers were placed in an increasing order. The "alien" column started with five and increased by five in the column. In the other column, namely "food bar", she increased one by one starting from 1. The increase in the numbers was decided by the rule given in the problem, and she developed two different growing number patterns in the table. She used a long build-up ratio table by using addition with constant number values. Repetition of the numbers was an evolution of thinking in the classroom to use growing patterns. When the students used repeating patterns in previous questions, they also added the numbers in each step, saying it verbally, but this time pattern was more explicitly visible for the discussion.

They were not iterating the model of the units, but this time they did not iterate the constant difference.

Deren's table was drawn vertically. Since it was the first time, she used a table that would lead their friends as we expected. Teacher Merve looked at the other students' solutions to check for distinct models. To create an opportunity to compare different models, Teacher Merve and the researcher encouraged the other students to solve the problems. Some students used drawing strategies in this problem. The strategy did help students find the result—they were counting on and writing the result. Nevertheless, these strategies were prone to miscalculation. The students adapted this new tool to their strategies. Ferit was one of those who used drawing strategies and adapted them into a kind of "table". Ferit came to the board claiming that this long table was not satisfying for him and presented his *new* solution (see Figure 4.52). Ferit created 3 × 5 cells and put five into each cell. Out of 15, he filled 14 cells with 5s to reach 70 aliens. The first question was whether there were enough 5s in the cells; two students counted the number of cells, and then Ferit counted them. He said that "…I added five [5]s, and then, teacher, I counted five [5]s, and here is how many [food bars] were needed."

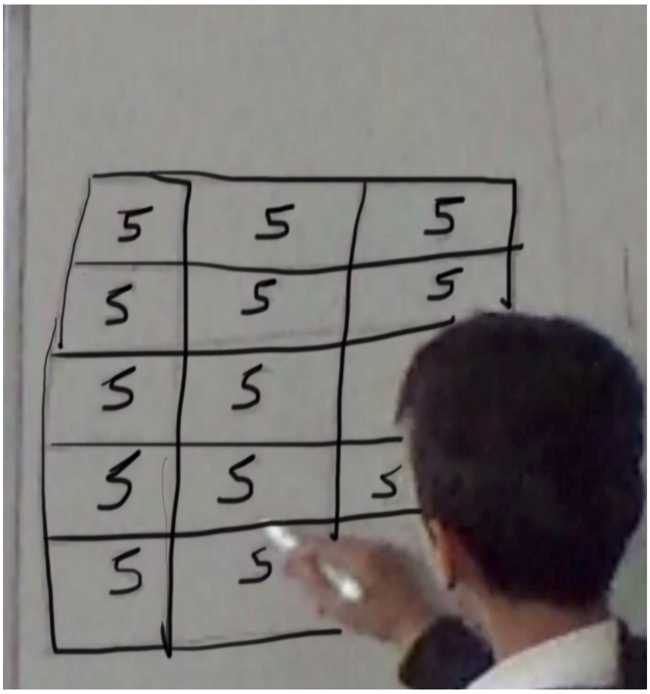


Figure 4. 52. Ferit's adaptation drawing strategy into ratio table for A3Q3

Teacher Merve asked if it was a table representation or not. Students did not accept this representation as a table. Teacher Merve asked what makes a model a table representation.

This discussion would shape their understanding of the ratio table for further questions. Zenan repeated that there should be a nametag for the value on the table (as Deren represented). This representation brought a new discussion of how to draw a ratio table and its properties again. After the whole class agreed that *there were names of the unit on the table, there should be an answer to the problem*, Alp asked, “How do we know the question just by looking at Ferit’s solution?”. The students tried to turn Ferit’s solution into a “table” but could not. Then, Teacher Merve asked, “Does Deren’s solution make a table, then?” All the students agreed that it was a suitable table based on their explanations about the properties of a table. Teacher Merve also probed students to consider another difference between the two representations. Ferit used “5” as the number of aliens in the rule, and Deren used “5” as the constant difference between the aliens in two consecutive rows and, additionally, “1” for the constant difference between the food bars in two consecutive rows. In this way, Teacher Merve called students’ attention to another pattern within the table.

This discussion took a lot of time due to drawing the graphical representation on the board. Teacher Merve and the researcher agreed on the idea of feeling the need for time efficiency in solutions. The way the students used the ratio table supported their use of repeated addition and multiplication as well. To reinforce students’ multiplicative thinking, Teacher Merve and the researcher used the term “short table” by highlighting how Deren and Ferit spent lots of time-solving this problem with addition. In the growing patterns with the addition given above, each column was handled, yet not dependent on each other. Using the long build-up strategy, the focus was on one column at a time, the correctness of the pattern, which was the increase between the rows in columns: 5, 10, 15, …,70 and 1, 2, 3,…, 14 respectively. However, there was also another relationship between the two columns in the same row. This situation was enlightened by the students’ different solutions in the whole-class discussion.

> Teacher Merve: Look, Deren wrote aliens and food bars (showing the columns), then she found five, five aliens first and then found the number of food bars in return, is it true, Deren?
>
> Deren: Yess!
>
> Researcher & Teacher Merve: Is there anyone who did it shortly?
>
> Researcher: Cansu, can you show us your explanation? [In the Explore Phase of the Activity, the researcher saw Cansu’s short form of explanation]
>
> Cansu came to the board and drew a table.

At the end of the activity, Cansu proposed a different representation of solving the problem and showed her result to Teacher Merve first. Teacher Merve announced that Cansu had another solution. She was also a student who adapted her strategy into a ratio table like Ferit. Her performance in A1Q3 (see Figure 4.49) changed with the labels of the quantities provided in a table (see Figure 4.53). As shown in the Figure, Cansu repeated her solution for A1Q3 and initially wrote the given number of aliens in the wrong place, but she found the multiplicative relationship irreversibly. When the researcher asked about this situation, she immediately realized her mistake and changed it.

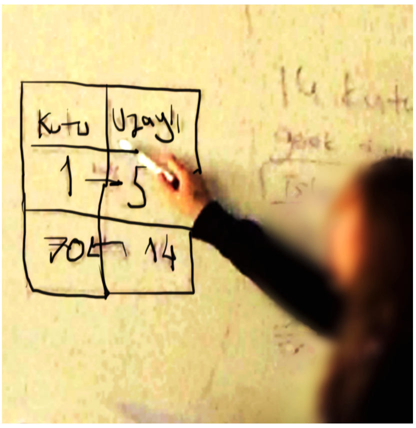


Figure 4. 53. Cansu's adaptation of ratio table for A3Q3

In the explore phase of LED, the researcher saw Cansu's explanation involving a short table for the problem. The researcher and Teacher Merve decided to call her to the whiteboard in order to introduce a short table to her peers after Deren's table. The advantage of the short table is that it is a more efficient way to represent repetitive iterations of the ratio and can also begin to support multiplicative relationships within a ratio and between equivalent ratios. In other words, the short table can better support multiplicative interpretations of the patterns represented in the table. Cansu drew a vertical table and used "alien" and "food bar" as the names for the columns, similar to Deren. She explained that she put the values in the first row according to the rule: one for the food bar and five for the aliens. She explored the relationship between the two cells/units in the same row five times (See Figure 4.54). The constant relationship between the cells in the same row was the ratio between two units: food bars to aliens. She expressed this relationship with an arrow from left to right, as shown in Figure 4.54. The arrow's direction was also meaningful because it meant division with five if the arrow was directed from right to left, as given in the second row in Figure 4.54. She divided 70 aliens by five and found 14 as a result. All in all, this short table was tiny, time-efficient, and represented a multiplication relationship, as suggested, but the classroom needed some time and practice to internalize this process. While internalizing, the students started using multiplication as a more efficient algorithm between two units.

> Researcher: Cansu, can you explain to us how you found it?
>
> Cansu: Ma'am, one food bar feeds 5 aliens *(Datum)*. This is 5 times the number of food bars. Therefore, 70 (aliens) will be 5 times the number of food bars. Teacher, if 5 aliens are fed with one food bar, 70 aliens then… (calculates, divides 70 into 5, and writes the result 14 in the empty cell in the table)
>
> Deren: This is very short; mine is long.

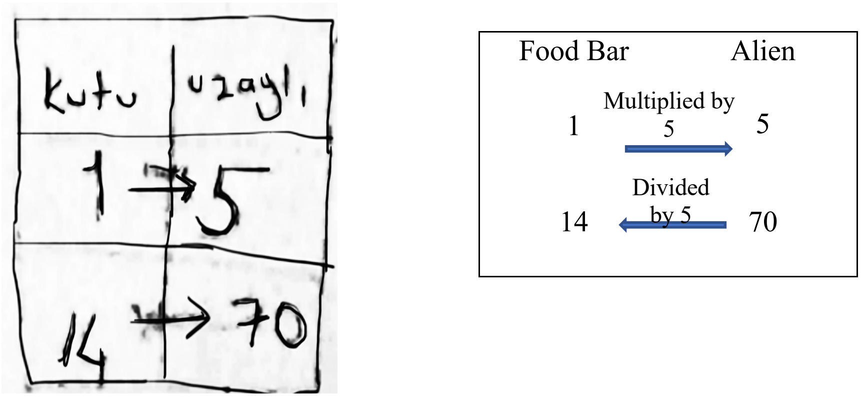


Figure 4. 54. Cansu's short table representation for A3Q3

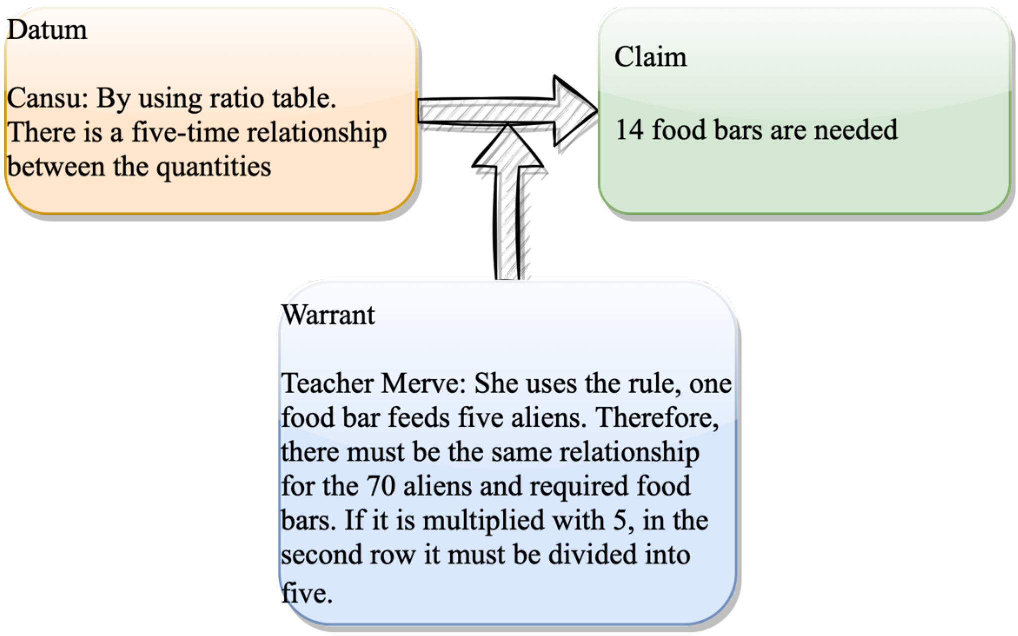


Figure 4. 55. Cansu's short table argumentation with teacher

The evolution of the students' solution strategies continued for the ratio table, and they were doing experiments on their activity sheets. The goal of the fourth activity was to support students in curtailing their additive patterns and using a short table. Only a few students created a short table, so Teacher Merve continued asking questions during the whole class discussion. In Activity 4, the rule was "two food bars feed four aliens". It was the first time the rule did not directly represent a unit ratio regarding the number of aliens per food bar. The third question was, "How many aliens can 14 food bars feed? Explain.". In the previous activities, it was observed that Burcu used repeating patterns. After the whole-class discussion of Activity 3, Burcu evolved his solution by representing the multiplicative relationship between the two units, as given in Figure 4.56. This could reflect Cansu's solution with the multiplicative relationship between the composite units.

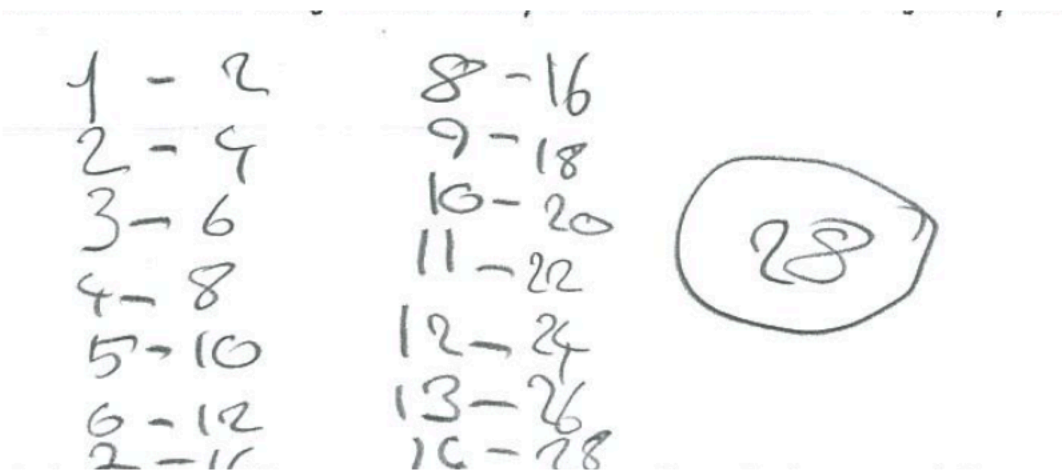


Figure 4. 56. Burcu's representation of the relationship between two units for A4Q3[6].

First, Burcu wrote the name food bar since the number of food bars was given in the problem, and the number of aliens was the missing value. Burcu listened to the previous discussion about finding the unit ratio and used the multiplicative relationship as "one food bar feeds two aliens". The number written on the left part was the number of food bars, and the number written on the right part was the number of aliens. The line between the cells represented the relationship between the numbers. She started from one food bar to two food bars. The number of aliens was two times greater than the number of food bars. Although she did not draw any arrow kind of symbol to represent the relationship between the alien and the food bar, she followed an order from left to right "one food bar feeds two aliens, two food bars feed four

[6] This picture was taken from Burcu's activity sheet because the picture taken from the video is not clear.

aliens …". Using this calculation, she replicated this rule of the pattern until she reached number 14 in the food bar part. This time, she used the idea that two was a constant multiplier between the numbers, not independent units. As in Deren's explanation in Figure 4.51, Burcu used considerable time and space on the whiteboard to draw the table at that length.

Teacher Merve asked the students, "Is there any other way to save time to solve this problem?" in order to prepare them for a subsequent problem, "How many aliens will 98 food bars feed? Explain." This time, the number of aliens was the largest among all, and the hope was that the students would curtail drawing the aliens and food bars more quickly. Teacher Merve got an affirmative result from the students: "it is not a question to use drawing". On the other hand, the multiplication strategy was still new to the students. Still, the classroom welcomed Cansu's strategy (see Figure 4.57), and they did not ask any further questions about the strategy. Cansu was confused about the scale factor, and she concluded that "49 aliens" was the result, which was half the number of the food bar. Her friends warned her about this mistake, and she immediately corrected it. The feedback for the claim "49 aliens" but not for "the strategy" was evidence of multiplicative understanding behind this strategy.

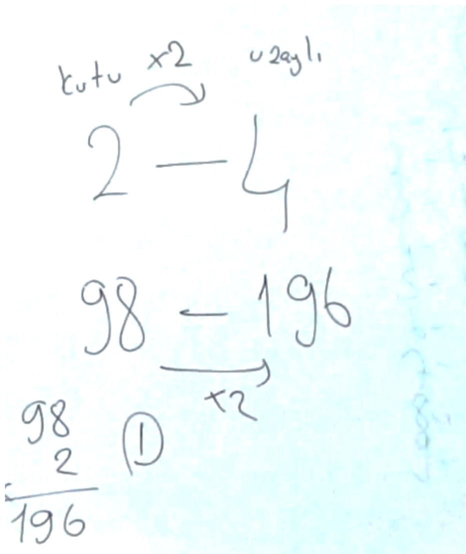


Figure 4. 57. Cansu's explanation of the multiple relationship

> Teacher Merve: Well, I'll say something. Suppose you took an exam; you had a limited amount of time. How would you do that? Were you dealing with ratio table, dealing with drawing shapes, how would you do it? Ercan?
>
> Ercan: I would use division and multiplication directly as well. But I used the table this time.

Teacher Merve: You would do division and multiplication! Okay, you are saying this is more practical. So, is it necessary to create a long table after understanding its essence? Making a table may provide a more in-depth explanation to understand the problem, yes, but no need for writing all numbers; for example, we can make the table short.

Several students in the classroom: I did it like this!

Teacher Merve: Can we put dots in between and make the table short? Well, I'll show you something ... Guys, look here. Food bars, number of aliens… Suppose I will make a ratio table. How many aliens can be fed with one food bar?

Most students did not prefer drawing all the units, but their limited experience with ratio tables did not lead them to a time-efficient solution. As an illustration, when Teacher Merve asked for a time-efficient solution, Ercan admitted that he would use multiplication or division for an optimum solution. However, several students could not recognizably select either multiplication or division operations to find a correct answer since some of them even claimed the result as “98” and “49”. It was not surprising to come across those results because division or multiplication might mislead the students, as in Strategy 2. The ratio table was a data organization tool that could help students overcome this situation. Teacher Merve was moving step by step and trying to fill the gap between Burcu’s, Deren’s, and Cansu’s solution strategies. With the explanation of Teacher Merve, she made it more concrete to help the rest of the students create a connection between building up with repeated addition and shortening the table with multiplication. Teacher Merve replaced the number of food bars in the first-row one by one and then filled the second row with the number of aliens by multiplying the number of food bars by two, as given in Figure 4.58. She filled the cells by question-answer with the whole class. By emphasizing this constant relationship among two units, food bars and aliens, with the dots and 98 food bars, the students automatically multiplied 98 by two and found the result of 196. This was one of the relationships within the table between two distinct units. What about the relationship between the two same units?

After filling the gaps within the ratio table through question-answer with students, Ercan came to the board and displayed his short ratio table on his activity sheet to the teacher. Teacher Merve emphasized that he realized that the number of food bars was two times the number of aliens. Previously, Ercan was fond of multiplication or division, but after a whole-class study with a short ratio table, he developed another optimum solution. In the first column, he used the multiplication symbol “×” and “2” representing multiplicative relationships of “two times” between the number of aliens and food bars. In the first row, he used the number of food bars and the number of aliens in the second row, as given in Figure 4.59.

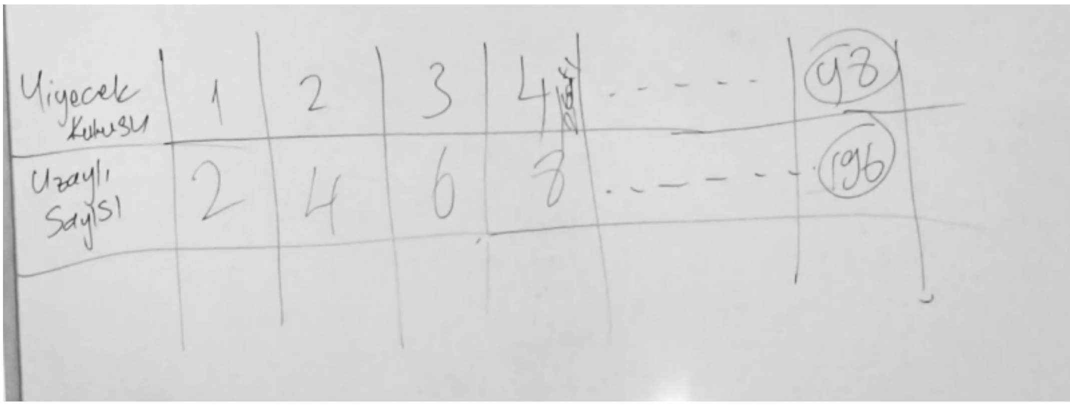


Figure 4. 58. Teacher Merve's representation of short table

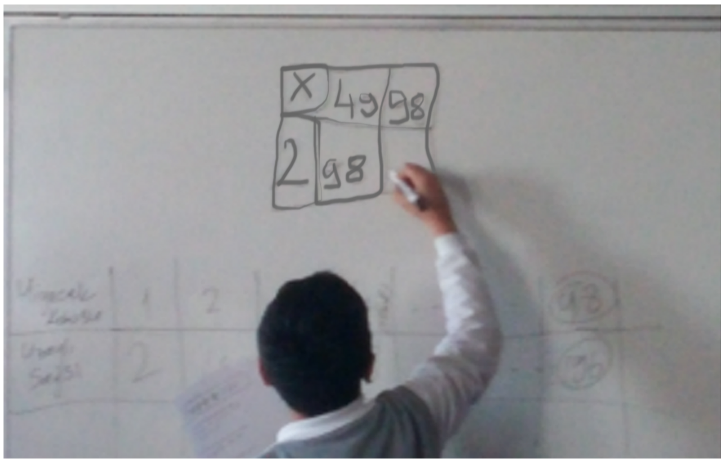


Figure 4. 59. Ercan's short ratio table representation for A4Q4

The last problem of the fourth activity (A4Q5), one of the students realized another constant relationship between the variables, and it became another strategy for solving the problems. A4Q5 asks, "how many food bars are needed to feed 16 aliens?". The number values again became smaller, but Teacher Merve insisted on solutions using an efficient data organization tool. This time, several students found a pattern among the same units. Nur brought up this issue in the discussion, as given in Figure 4.60. If the multiplicative relationship between the two cells of the food bars was four, this relationship between the number of aliens was the same as food bars, four. This constancy was sustained throughout the table, and it showed the variety of relationships among the variables given in the tables. Nevertheless, the students were facing disequilibrium and they were not sure about this magic for now. As a strategy, ratio table became a classroom mathematical practice, but its properties were still needed to be explored.

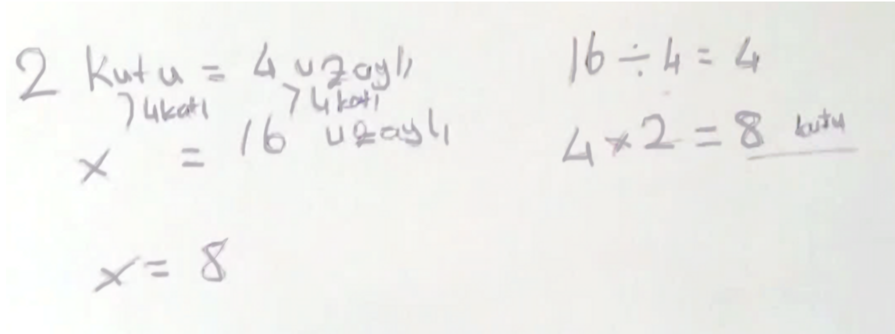


Figure 4. 60. Nur's representation of the multiplicative relationship between the same units

4.2.5. Summary of Classroom Mathematical Practice 2

In the previous four strategies, the class developed their ways of reasoning to organize the data for linking and iterating composite units given in the activity sheets during eight class sessions. These four ideas came to the front as taken-as-shared strategies to constitute important part of data organization in missing-value problems. Each time they faced different challenges (problems involving operations of whole numbers and non-integer numbers, missing value problems, comparison problems, number value of the units in the rule different than one), they tested the capabilities of their informal tools. As the formal tool, ratio table was introduced with the help of the instructional sequence and the students' interaction with the tool was observed.

Table 4.2 represents not only the summary of CMP 2, but also chronological emergence of the ideas. Classroom Mathematical Practice 2 consisted of four ideas from seventh-grade students who explored linking composite units and iterate them in various situation. These four normative ways of reasoning are (1) Composite units can be represented and iterated in their pictorial or symbolic forms; (2) Division and multiplication algorithms require linking correct units for missing-value and comparison problems; (3) The unit ratio can be created to reconstruct composite units (unitizing); (4) Ratio table is composed of iteration of composite units. It took time to deal with ratio table. Nonetheless, smooth adaptation of the students to the ratio table tool was the focus of the Activity 3. Therefore, Teacher Merve listened to all solution strategies other than ratio table in the discuss phase of the LED teaching cycle. Each TAS became a normative way of reasoning through four criteria of CMPs. Each taken-as-shared strategy was transferred to another classroom mathematical practice.

Table 4. 2

A Brief Summary of the Criteria for CMP 2

CMP 2	CMP Criteria
TAS 1	C3: Accepting drawing strategy C3: Upgrading drawing strategy to semi-symbolic C3: Upgrading drawing strategy to symbolic C2: Shifting “drawing strategy” datum into backing (looking back of a solution) C2: Dropping of warrant for drawing strategy
TAS 2	C3: Accepting algorithms path 1 and path 2 (TAS1C3 continues) C1: Dropping of warrant and backing for “Doing calculations” as datum for integer ratio questions.
TAS 3	C3: Transferring CMP1 TAS 3 C3 into food bar unit C3: Accepting unit ratio strategy “alien per food bar” & “food bar per alien”
TAS 4	C1: Developing ratio table strategy from symbolic drawing strategy (TAS1C3) C3: Transfer of using algorithms (TAS 2)

4.3. CMP 3: Covariation Among Composite Units Within Ratio Table

As a formal tool, ratio tables gradually took part in the students’ activity sheets and their solutions on the board. There were arithmetical attempts for several students to internalize ratio tables and explore the multiplicative relationship in tiny steps. On the other hand, several students studied scale factors among the numbers and have used them effectively since then. In each situation, the students accepted the solution strategies mentioned in CMP 2. They molded them with the ratio table tool to better understand the relationship between the already organized composite numbers. In CMP 3, the students explored how to use vertical and horizontal scale factors conveniently, and their misconceptions during instructional sequence emerged and were eliminated. Besides, the cross-product algorithm strategy was carried to the classroom discussion, and so were the inversely proportional situations, which were not conjectured. Additionally, the conditions for selecting the scale factors and ratio table were influenced by the students’ previous experience with decimals. As dealing with ratio and proportion context was in line with other mathematical concepts, the instructional sequence also conjectured difficulty in using decimals. Contributions of the taken-as-shared ideas to covariation among composite units are described in this classroom mathematical practice in detail.

4.3.1. TAS 1 Connecting Additive to Pre-Multiplicative Thinking through Build-up Strategies[7]

At the end of Activity 6, students reasoned about how to link the composites and iterate them. The class had nearly completed the development of three strategies: drawing, algorithm, and unit ratio; however, the ratio table was still in need of development. For example, it was also reported that skip counting, and long table were very common in table construction. The students did not improvise the short ratio table, in other words, the multiplicative relationship among the numbers. They were good at skip counting and not tired of writing all the numbers. Relatedly, the numbers in rows representing the aliens and food bars were mostly filled independently. That is, several students filled the first row and then the second row in the horizontally extended table. Only one or two relationships were checked, which could bring some mistakes to the table. For them, long was still safer than confusion with the numbers.

In CMP 2 Strategy 4, the situation was as follows: on the one hand, there was the solution of Cansu's novel solution in which she used a short ratio table and multiplication. She could use both vertical and horizontal scale factors. She advanced this method from the beginning and tried to explain it to the class. In each activity, she gradually increased her advocates. On the other hand, Burcu advocated drawing, but she left this strategy and preferred using a long ratio table. Her method, skip counting in the ratio table, was accepted by the students who also used drawing. The long ratio table became a transition model for understanding multiplicative structures for the short ratio table.

Burhan's example in Figure 4. 61 represented a common long ratio table which was extended horizontally on the board. Filling the cells was done with skip counting within the same row. Just drawing this ratio table and filling the cells through skip counting became the datum for the claim. This datum was not questioned in detail because he used skip counting while filling the cells. The drawing represented his datum and claim. One of the students, Buket, watched him write the numbers and realized the scale factor between food bars and aliens. She added spontaneously that each column had a scale factor of three, representing that a long ratio table

[7] This snapshot about number patterns and long ratio table development was explained in detail in the book chapter: **Sozen-Ozdogan**, S., Akyuz, D., & Stephan, M. (2022). Chapter 6 Patterns and relationships within ratio contexts: Students' emerging ideas through ratio tables. In P. Jenlink (Ed.), Mathematics as the science of patterns: Making the invisible visible through teaching (pp. 99-125). Information Age Publishing.

might be a helpful tool for dealing with number patterns and multiplicative relationships[8]. This emergence of number relationships occurred several times when the long ratio table was used and offered as a strategy to gain some time.

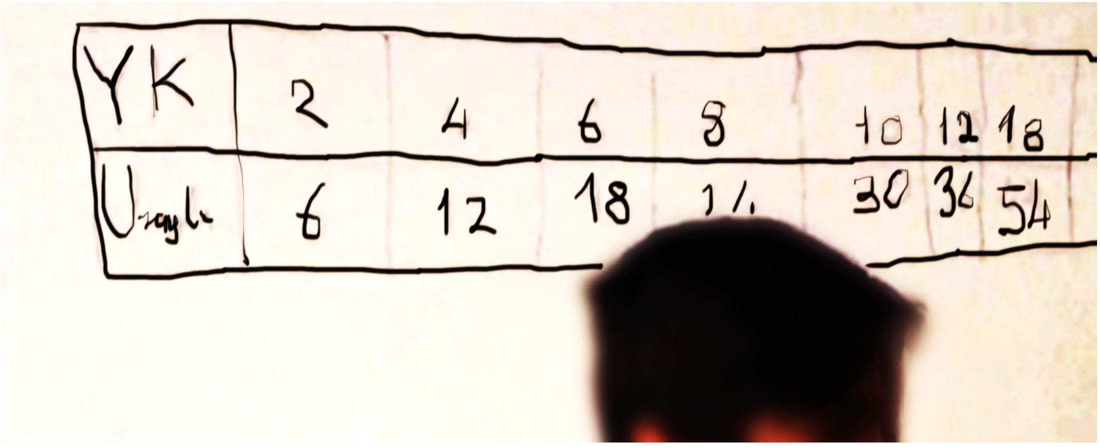


Figure 4. 61. A common build-up strategy used by Burhan

Alp also drew a long ratio table in the same activity, which was not surprising since he was fond of drawing and conducting multiplication/division together. Surprisingly, he was filling the cells by using the multiplicative relationship between aliens and food bars one by one.

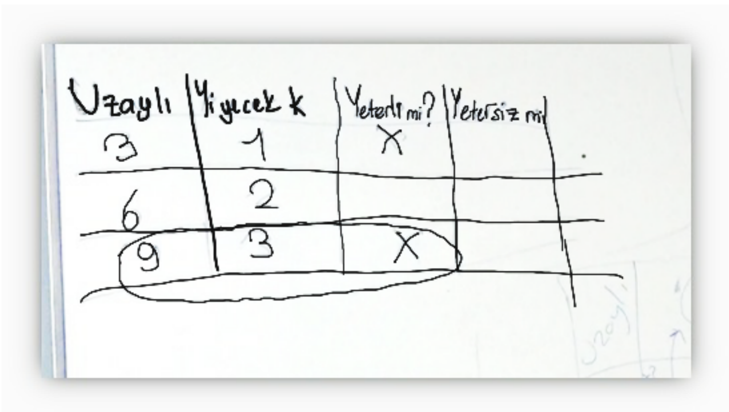


Figure 4. 62. Alp's vertically extended, long ratio table

Alp used a vertically extended long ratio table. He started from one food bar and multiplied with three to write the number of aliens in the same row. This process continued for rest of the

---

[8]8 This snapshot about number patterns and long ratio table development was explained in 8

rows. His build-up strategy involved "vertical" scale factor (VSF), but he also preferred developing the table row by row for the food bar and he calculated the numbers by multiplying with three. As seen in Figure 4.62, it was a vertical ratio table and this time it should not be called as VSF since this could lead him a misconception. Nevertheless, he was not aware of using a scale factor (development of naming scale factors is handled in CMP 3 Idea 3). What he proposed as data was filling the cells correctly. Although Teacher Merve provided a strategy for shortening a ratio table (see Figure 4.58), it did not become a practice during the instructional sequence.

Additionally, some students tried combining different number groups by iterating with the correct numbers. The questions in the activities were in line with each other most of the time. The result of the question might help find the result of the other question. Akın was one of the students who explored this kind of relationship within Activity 9 between questions four and five (see Figure 4.63).

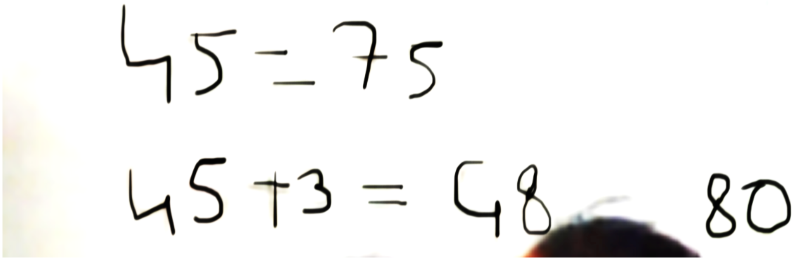
4. Use a table to find how many food bars will feed 75 aliens.

5. Use a table to find how many aliens can be feed with 48 food bars.

Figure 4. 63. Questions four and five in Activity 9

The claim for question four was 45 with the rule, "three food bars feed five aliens". By using this result, Akın claimed that he could find the result. He found the difference between 45 and 48 food bars and added five aliens to 75 aliens given in question four. He wrote, "if 45=75, 45+3=75+5". The left side of the equation represented the number of food bars, and the right side represented the number of aliens. He iterated three for food bars and five for the aliens.

> Teacher Merve: There is a different strategy. Akın, tell me your strategy.
>
> Akın: Ma'am, in the other question, we made 45, or 45 was equal to 75 aliens, Ma'am. I added three to 45.[45] Plus three equals 48. So, it's our food bars. With this reasoning, I found that 48 is 80. (Added on top of the previous explanations) (Datum)



(Datum and Claim)

> Teacher Merve: What did you add five to? Shall we do this, Ali?
>
> Ahmet: Teacher, where did we find these seventy-five?
>
> Teacher Merve: From the previous question, Akın found 45 food bars to feed seventy-five aliens. Then he says here; he only added three, my teacher says. He said, "We'll add five here; it's eighty," he said. That's how he found it. Looking at the previous question... He looked, and we found 45 food bars in the previous question. So, I find the number of food boxes in this question. I'm feeding five aliens for every three. He said, "I add five aliens for every three lunch boxes; I add five to seventy-five and get eighty." So, he found it based on the previous question (Warrant)

There was no rejection of this claim, and Akın used a distinct build-up strategy without filling all the cells in the ratio table, but this kind of strategy was rarely used during the instructional sequence. Filling cells within the ratio table considering build-up strategies and correct iteration of numbers using previous claims were also evidence of understanding the ratio concept. Flexibility with the numbers provided them an exploration area for the scale factors among the numbers and drawing a long ratio table slowly left its place to a short ratio table.

4.3.2. TAS 2 Reconstructing Meaning of A Scale Factor as A Multiplicative Operator

Classroom mathematical practices might be constructed and deconstructed during the learning process. As explained in CMP 1, the classroom already agreed on the idea that "aliens cannot be sliced". There was a time that they thought, "What does a half-alien pictorial representation mean?". Previously, they concluded that half of an alien was not suitable for the problem context, and aliens would die. While practicing the strategies that they developed during the sequential activities, the students came across a confusing question in Activity 6. A6Q2 asks, "How many aliens does one food bar feed?". The rule is given in Figure 4.64. This time, the unit ratio includes a decimal number. Teacher Merve encouraged the class to use more organized and time-efficient solutions such as ratio tables. In their activity sheets, the students mainly used drawing in a semi-symbolic way since the number values were smaller. However, Teacher Merve started to invite the ones who used ratio tables as a strategy to the board. There were students who used drawing as well among those coming to the board. It was already conjectured that some students would argue they could not answer this question through any strategy because it did not make sense to feed a half-alien. Furthermore, the activity suggested that teachers praise this thinking and say that aliens were allowed to be partially full of from now on.

Figure 4. 64. The rule in Activity 6 (A6)

An alien is not a sliceable variable in the problem context in which aliens should be fed as given in the rule so that the world can be saved. Suddenly, in the second question, students came across with "half-alien" concept. In this question, "one to two and a half" is a mathematically correct statement for the rule "two to five". Teacher Merve wanted students to represent decimal scale factor or unit ratio, which was food bar pieces for one alien. There should be one food bar for two and a half aliens, but for one alien, there should be two-fifths of a food bar in reverse. Before this lesson hour, the classroom had studied this question and had already created an efficient solution. Here, Teacher Merve was trying to summarize the solution of the other students, and Esra was attempting to connect each solution strategy. Esra was confused and claimed that the result of A6Q2 was three. Her datum was that one food bar should feed three aliens instead of two aliens and a half. This reflected the conclusion discussed in the first activity, "one alien cannot be sliced". She warranted the data by offering two equal pieces to two aliens and offering the left piece to the third alien, which meant three aliens in total. Therefore, she concluded that one food bar was enough for three aliens as she thought in real life context. When Teacher Merve looked at Esra's drawing in Figure 4.65, she saw that Esra drew three aliens even though she knew mathematically two and a half of an alien would be fed. She was in the middle of how to express her previous learning and this situation. Teacher Merve questioned the statement that Esra constructed.

Figure 4. 65. Esra's drawing on her activity sheet

Teacher Merve: Esra, please come here. How did we do in the second question?

Esra: Ma'am, this is how I did it. There is one food bar. We divide one into three, Ma'am. (Datum and Claim)

Teacher Merve: Why did you divide it into three? 2.5 aliens eat one of them. Where did you find "one food bar is enough for 2.5 aliens"?

Esra: Teacher, one food bar satisfies two of the aliens. (Shown in Figure 4.65). This food bar is given to this alien; this food bar is given to this alien. The other alien (represented by half) eats it too. (Warrant)

Teacher Merve: Does anyone agree with what Esra is doing? She split a food bar into three. You divided it into three equal parts, one of which will be half-fed so that the others will be fully fed. So, think about it this way, I give you one loaf of bread as a whole, you are full, and your other friend is also full with one piece of bread. What about giving half of the bread? Would they be full again? As if there is a logic error somewhere? (Rebuttal 1) Okay, I'll say something. Can anyone figure this question out for Esra to understand? Ahsen, let's see, he did it with a shape. Let's see where your mistake is. Esra, let's see what the difference with Ahsen's is.

Ahsen: Teacher, I said if 5 aliens eat two bars, then I thought, how many aliens can eat a bar. After that, I found 2.5 aliens because I divided the remainder in half here. For half of 5 I found 2.5. (Rebuttal 1 continues)

Teacher Merve: You say two and a half aliens are fed.

Esra: Teacher, how can an alien be half-fed?

Teacher Merve: So it's half full, not full, you see?

Teacher Merve questioned the equality of the pieces distributed to the three aliens in the discussions above. If two aliens were full of pieces, the third alien would be hungry with the left of the food bar. To better understand this issue, Teacher Merve called one of Esra's peers, Ahsen, who rebutted Esra's claim of *three* aliens. Ahsen concluded that if five aliens ate two food bars, she must reduce the rule by two to find the number of aliens for one food bar. Based on this data, she deduced two and a half aliens for one food bar as the new claim. Esra wondered, "How can an alien be fed half?". Teacher Merve explained that an alien would be half full in this context, and the peers exhibited their own understanding of "half of an alien" representation. The rest of the students continued to represent with four strategies. Ahmet expressed half-fullness through drawing strategy (see Figure 4.66).

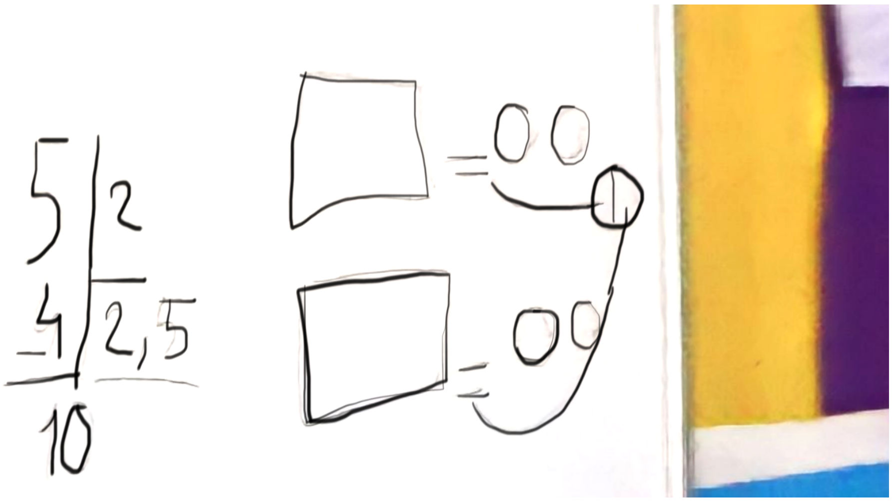


Figure 4. 66. Ahmet's "half-alien" representation through drawing strategy

Ahmet explained that one alien ate half from one food bar and half from another. The circular areas represented the stomach this time. The meaning changed contextually. The decimal scale factor was experientially real for Ahmet with this representation. In addition to Ahmet's explanation, Teacher Merve guided the classroom to study the context of unit ratio in detail because decimals and rational numbers are also interrelated with this topic. Fatma came to the board to show the unit ratio for one alien to a food bar. This was offered as another alternative to "half full-alien". Fatma's datum and claim were similar to Ahmet's. However, Fatma used two representations: pictorial and symbolic. Unlike Ahmet, Fatma first divided one food bar into five equal pieces, distributed them among five aliens one by one (*Warrant*), and repeated this action twice for the second food bar, as given in Figure 4.67 (*Backing*). In total, two pieces were distributed to one alien. She also explained it symbolically through fractions. She iterated 1/5 at five times to represent a whole food bar. Each 1/5 piece of food bar was distributed to each alien. 2/5 of the food bars illustrated the piece shared with one alien.

Figure 4.68 summarizes the argumentation going through about finding the unit ratio. Fatma and Ahmet directly refuted Esra's claim. Rebuttal became a new claim, and their solution strategies became new data and warrants, which were accepted by the class. CMP 1 Idea 1 was conditionally rehandled and reconstructed for decimal scale factors. This was an issue of consensus among the class. Noticeably, decimal and repeating decimal scale factors were calculated in several ways, as represented in Figures 4.66 and 4.67. The students, nevertheless, did not feel the need to use a ratio table in this question (A6Q2). In the retrospective analysis, it might be concluded that till now, the students' experiences revealed that they used ratio tables in large numbers to make their increase concrete. This time, there was a different process; they needed to simplify the rule and reduce the numbers by two. This first-time experience did not lead them to use a ratio table. In the end, students did not question half of an alien. To conclude, aliens cannot be sliced, but they can be half-fed, which became one of the normative ways of reasoning.

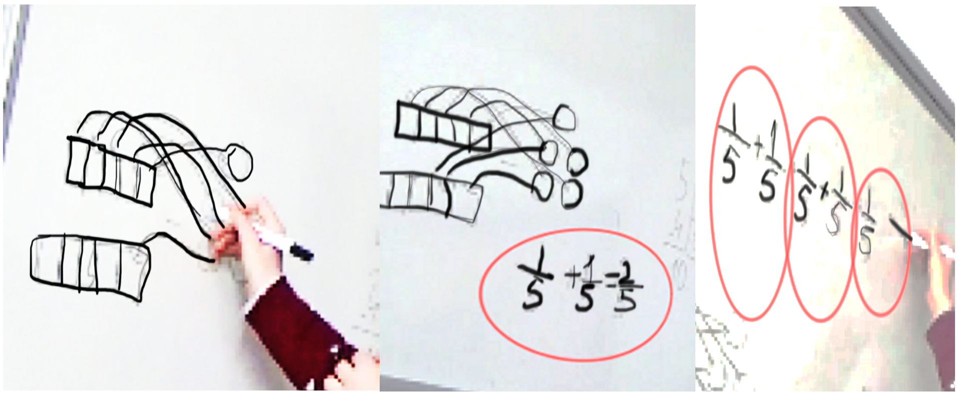


Figure 4. 67. Fatma's "half-alien" representation in a chronological order

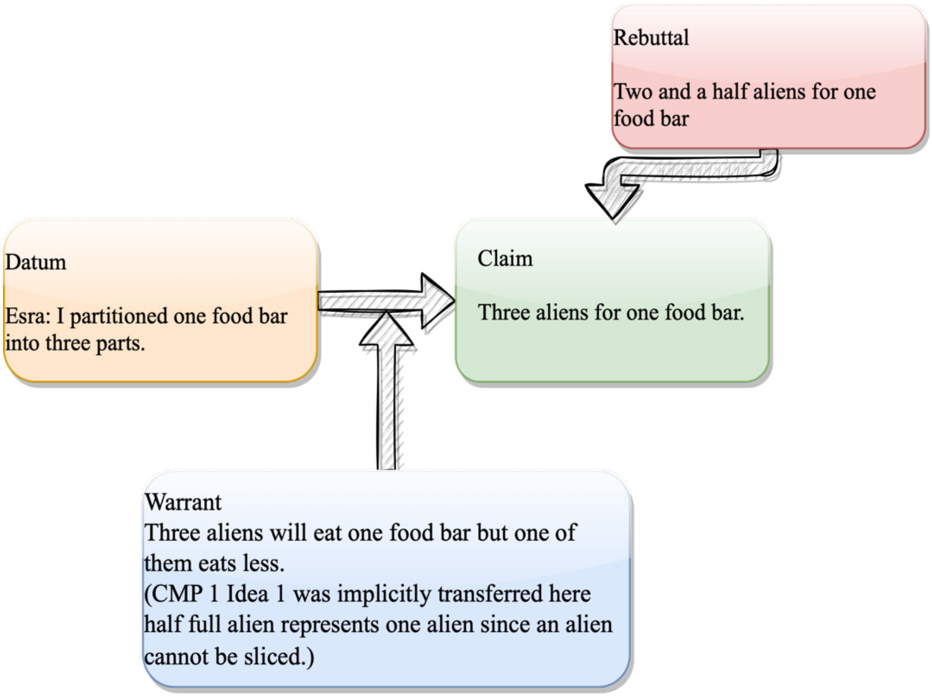


Figure 4. 68. Rebuttal for Esra's Claim

Activity 8 focuses on the decimal and repeating decimal scale factors. The rule of the activity was given in an unfamiliar way in a ratio table. They first tried to define the exact rule. They agreed on “two food bars feed six aliens”, and no one had trouble in making that multiplicative relationship of the first and second ratio tables was whole-number scale factors (see Figure 4.69). The third and fourth ratio tables were tough ones with respect to the others due to decimal scale factors. In the third and fourth ratio tables, Teacher Merve reinforced the experience they gained from A6Q2. She asked for fractional representations of solutions in them. Ercan was a volunteer this time

1. Fill in the missing values in each table below.

Food Bars	2	1
aliens	6	?

Food Bars	2	7
aliens	6	?

Food Bars	2	?
aliens	6	7

Food Bars	2	?
aliens	6	1

Figure 4. 69. Rule of the activity was implicitly given in Activity 8

Ercan first conducted a division algorithm and found repeating decimals $2,\overline{3}$ and $0,\overline{3}$ respectively. As a solution strategy, the students quickly grasped the “multiples of three” multiplicative relationship and applied it in the first two tables. Unlikely, the rest of the tables asked for the number of food bars required. Ercan divided the given number of aliens by three. To make repeating decimals experientially real in the problem context, Ercan transferred it to the drawing strategy and showed it as presented in Figure 4.70.

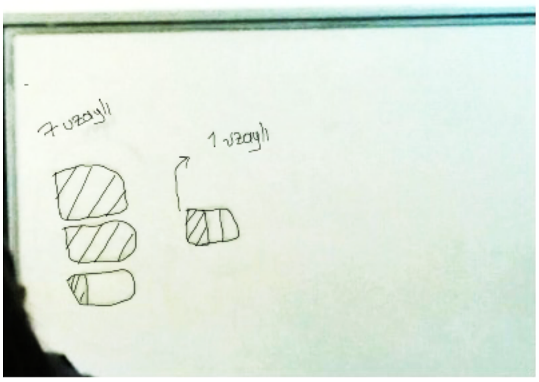


Figure 4. 70. Ercan’s drawing for repeating decimals, $2,\overline{3}$ (A8Q1)

Several students objected to his drawing. He shaded the first two whole food bars and one-third of the last food bar. He did not explicitly show the partitioning on the food bars pictures for 7/3. Therefore, several students did not correlate 7/3, 2, and drawing. Teacher Merve realized that the students immediately found how many aliens one food bar feeds on the first table. On the other hand, the unit ratio, one in the alien part, was not a familiar question. Relatedly, ratio table representation was still not easy to understand for decimal scale factor meaningfully. Furthermore, Teacher Merve and the researcher agreed that the students were weak in fractions and needed to be encouraged to use different models as the instructional sequence offered.

Teacher Merve wanted to be sure about understanding the unit ratio to show them 7/3. Based on collective experiences, she led the piece for one alien and continued iteration seven times for seven aliens with the help of the students. After the students felt relieved about Ercan's representation, Teacher Merve concluded that the multiplicative relationship should not necessarily be integer numbers. On the contrary, it may be decimals or fractions, which also represent an amount of something. She continued, those three kilograms of flour also show an amount; two whole and one-third kilograms of flour show an amount. The number representations just described the quantity of the object. Although the units' attributes were not conjectured within the instructional sequence, their influences were also reflected in other activities.

Activity 11 was the first activity after the aliens' episodes. It includes two discontinuous variables: humans. There are given ratios for teachers and children as variables in a day nursery called "Tiny Tots". The students were expected to solve the problems by using those ratios. This context helped question the units' attributes in the case of the decimal scale factor, which was accepted to calculate the ratio. Still, the context of the problem was also involved in making meaning of "half-full aliens". While discussing, Teacher Merve developed an effective questioning to emphasize CMP 1 again. She asked a question similar to alien-food bar composite units. Aliens, teachers, and children are discrete variables. However, the meaning of half of a variable may change according to the problem context. To illustrate, can we feed half of a teacher if an alien can be half-fed? Since this issue was discussed in a different context earlier, Teacher Merve expected them to transfer the knowledge of the attributes of the units into this problem. The students already knew that they could not slice aliens and humans. Thus, they had to solve this problem on the condition that aliens and humans would be alive as living organisms.

Teacher Merve: Okay, okay. If the staff-child ratio is one-fifth, that means there is a need for one staff per every five children. Let's suppose that there are 10 children in this age group, that is, in the 0-36 month-old age group; how many people do you have to employ?

Most of the students: Two…

Teacher Merve: Two... What about 15 [children]?

Most of the students: Three.

Teacher Merve: Three, right? What if there was a number in between [10-15 children]? For example, what if it was twelve instead of fifteen?

Zenan: Teacher, I think there would be two again.

Teacher Merve: She says it will be two again.

Arda: No, no. Ma'am, it wouldn't be like that. Staff would take care of a child again, my teacher, because that one of them is looking at those two, Ma'am. That sounds too much.

Teacher Merve: We would employ one more, he says. If there were twelve, the other would take care of the other two.

Arda: Yes, Ma'am.

Berk: Teacher, can I say something? Our teacher says there is a staff for one out of five children. Teacher, you are right, but for example, there will be two personnel if there are twelve children. It can't be two and a half, Ma'am.

Teacher Merve: What about those who say there is three staff for twelve children?

Several students together: Yes, Ma'am.

Teacher Merve: You say that two are employed, and the other two people have to take care of it anyway.

Akın: Teacher, what if we employ half and two personnel?

Teacher Merve: How is that? Can you employ a man and a half?

Arda: No, Ma'am. Can someone work half a day, Ma'am?

Teacher Merve: There is no such thing as he has to work full-time.

The discussion above summarized the ideas of the whole classroom about the required number of teachers for 12 children with a ratio of one to five. If the scale factor was an integer number, the process of finding the result was smooth. Teacher Merve brought out whether the number of children was between 10 and 15. The underlying question is about the decision-making process for the result, whether it should be the exact decimal number or rounding the numbers into one larger number or one smaller number. Teacher Merve highlighted maintaining the minimum staff-to-child ratios. There were students who claimed that two teachers would be enough for 12 children, reasoning that two teachers would be enough for the last two children. Some students argued that three teachers would be enough for 12 children because there should be one more teacher for the last two children. In addition, Akın asked, "what about two and a half teachers?" and Arda emphasized that half means half of the working hours. Children's reasoning was in line with real life, but the capacity given as the ratio needed to be emphasized here. As a result, three teachers were required to care for 12 children. There was experientially real reasoning for this question, but the problem context covers the full-time job, so we did not consider the idea that 2,5 teachers mean that one of the teachers will study half a day.

This context was also constructive in understanding the relationship between the units' attributes and procedural operations with ratio. While discussing, Teacher Merve developed excellent questioning in the process. She asked a question similar to alien-food bar composite units. First, she wondered about missing values considering integer scale factors and got fluent results from the students. On the other hand, she said the non-integer value of the number in ratio. Since the units' attribute was discontinuous, it was impossible to divide each unit. Therefore, the way of solving the problem was discussed by the students here. They already knew they could not slice aliens, and humans were also living organisms who should be alive when this problem is alive. They did not attempt to divide the humans. Since the previous practices were transferred here and evolved into a new understanding, this transfer showed a new normative way of reasoning. If the number of children is not the exact multiple of the number of personnel, choosing the larger number is considerable. As an example, given by the teacher, three teachers are needed for 12 children.

The discussion was reflected on the further activities in which there was a question about "how many graphic calculators can a school buy if it can spend $2500 for one calculator of $80?" (A15Q3). Nazan was unsure about her calculation since there was remainder in the division problem. Non-integer contexts made the students think for the second time as given in Figure 4.71.

Arda intervened in Nazan's solution and explained that the school had a budget of $100 so that the school could buy one more calculator. The decimal number in the quotient was meaningless. That is, a person cannot buy half of a calculator, which is a discontinuous variable. Teacher Merve added that it was impossible to cut the calculator out to buy, providing two solutions: you purchase either one or none. Throughout these discussions, the students practiced questioning the attribute of the variables to give meaning to the result. The meaning of the decimal scale factors helped evaluate the result/claim.

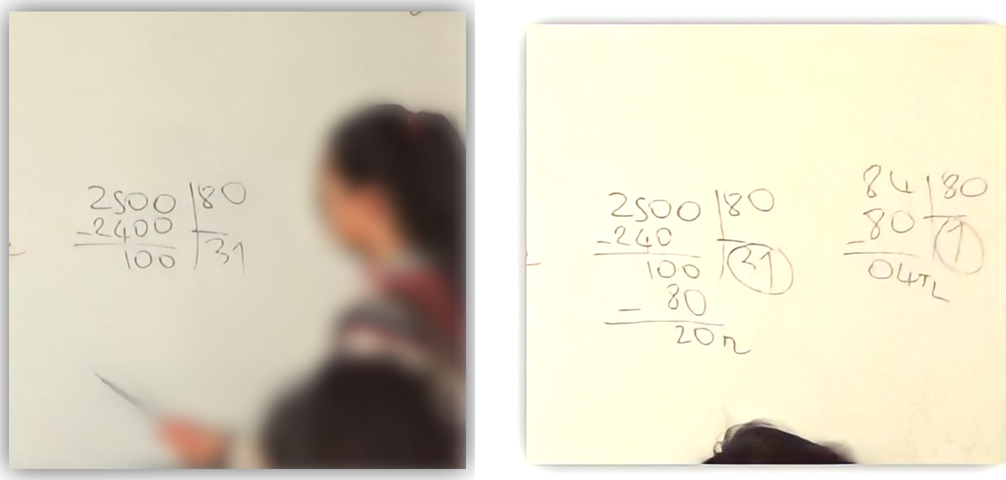


Figure 4. 71. Nazan’s solution for A15Q3

4.3.3. TAS 3 Reasoning About Covariance and Invariance Relationship within Ratio Table

There was a journey from a long ratio table with build-up strategies to short ratio tables with an abbreviated strategy for most students in class 7/X. During this journey, a horizontally extended ratio table was generally used by the students, and this became one of the practices of the students. This tendency can be explained by drawing a long ratio table, the whiteboard (rectangular area) size, and the ratio table visuals on the activities. Thus, this selection influenced their other practices as well.

Starting from Activity 6, the class explored and used the scale factors one by one. Teacher Merve had not named them as vertical and horizontal till then. Big Mathematical Idea of Activity 8 was about the meaning of a scale factor. Still, Nazan showed her way of solution and represented both vertical and horizontal scale factors altogether in A8Q2 without naming them. The researcher used the words *vertical* and *horizontal* to describe the students’ answers. The focus was on the way of the direction: Was it from upwards to downwards or vice versa? Was it from left to right or vice versa? Teacher Merve and the researcher explicitly introduced scale factors on Nazan’s table. After the introduction, the students started categorizing their solutions according to vertical and horizontal and asked whether they solved it horizontally or vertically. After a short explanation, the students’ solution methods were categorized based on this explanation.

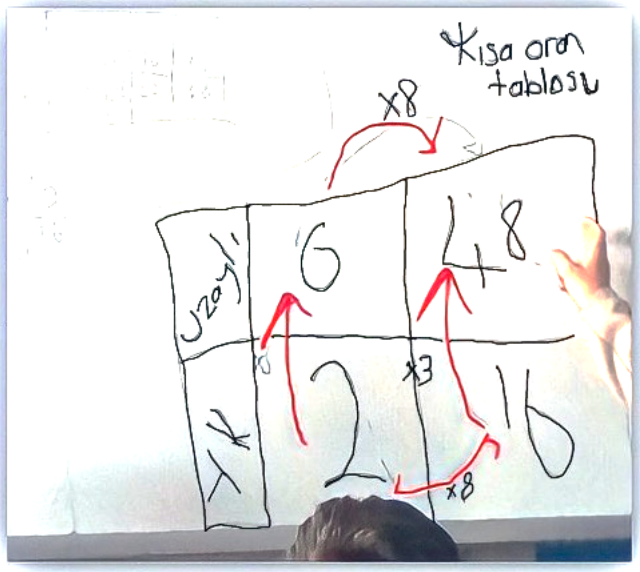


Figure 4. 72. Arda attending a discussion on Nazan’s representation

Researcher: Do you understand this question?

Berk: Ma’am, two directly multiplied by eight, which is sixteen. Six times eight is forty-eight. (Describing horizontal scale factor)

Researcher: That's what you did; you looked from here (To Berk).

Ahu: That's what I did. (Describing horizontal scale factor)

Researcher: So, Berk did it this way. I guess Ahu, you did it this way too. We can call it horizontal. It's done horizontally.

Unnamed student: So, Ma’am, I understood it more easily horizontally.

Researcher: Who did it horizontally?

Arda: So, Ma’am, it can be done both ways (Datum)

Teacher Merve: Does anyone do this? (Representing vertical)

Researcher: Well, who did vertical?

Zeynep: I did both, Ma’am. (Datum)

Berk described which scale he used and preferred using the horizontal scale factor. Ahu also agreed that she used the same scale factor as Berk. According to the student's competency, their scale factor selection changed. One unnamed student said they felt comfortable with the horizontal scale factor. The researcher claimed that the result does not change even the scale factor changes. This claim was supported by Fatma’s drawing attention to the results of two scale factors. The rest of the students in the discussion provided warrants to help this claim.

Upon this, Teacher Merve added that in practice, it would be better if Teacher Merve could use the scale factor to reach the result faster.

While we were progressing in the instructional sequence, the tendency to use a vertical or horizontal scale factor appeared. Activity 9 encouraged the students to use a horizontal scale factor with its number values as it was conjectured. The rule of this activity was "three food bars feed five aliens".

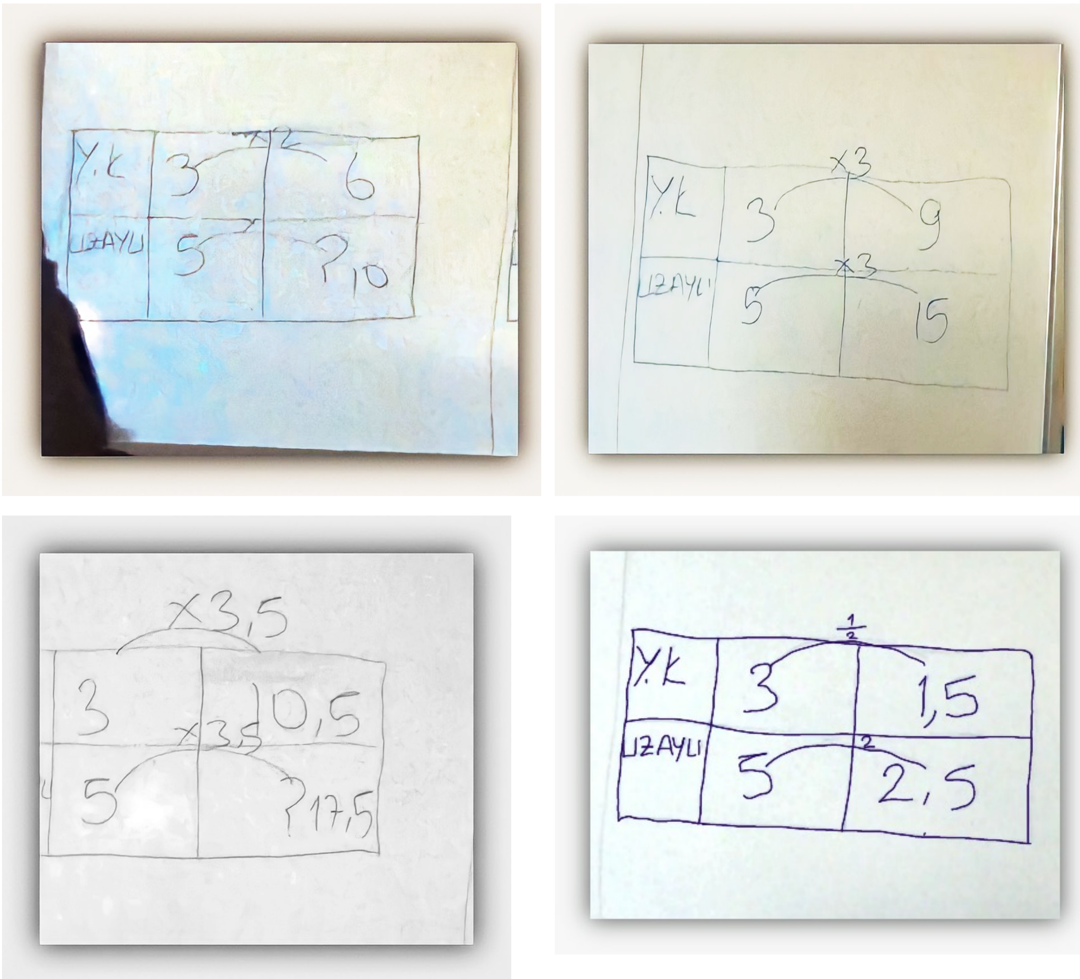


Figure 4. 73. Horizontal scale factor representation in ratio table as data (A9Q1)

In the first question, there are short ratio tables and one missing value problem in those tables. The first two ratio tables were easily filled. The last two ratio tables include decimal numbers in the cells. There are decimal scale factors in either way. Since vertical scale factor seemed difficult to calculate for most of the students in the class due to being decimal, they preferred horizontal scale factors. As it was previously described in CMP 2, Cansu struggled to explain her solution for scale factors which many of her friends did not understand at all. However, students were directly drawing the ratio table, the arrows on the table and the exact number of the scale factors. Additionally, they added "x" or " ÷" to describe the exact operation which also described the direction of the arrow. As an example, for verbal representation was "10,5 is 3,5 times as much of three" or "2,5 is half of five". The fourth question in Figure 4.73 had division symbol on the arrow, which meant that the direction was from left to right for horizontal scale factor. This representation became a datum, but backings and warrants were dropped off in their explanations while discussing decimal scale factors.

Knowledge about decimals was significant because there were students who could not deduce the exact decimal scale factor just by looking the ratio table given third picture in Figure 4.73. With these concerns, Teacher Merve encouraged Batu's explanation that he found scale factor "3,5" by dividing 10,5 into 3 although he dropped off warrants. Teacher Merve wanted to be sure about the students' conducting correct calculations with decimals and large numbers. While finding the missing values in the ratio tables, the students used division and multiplication operations to find the result.

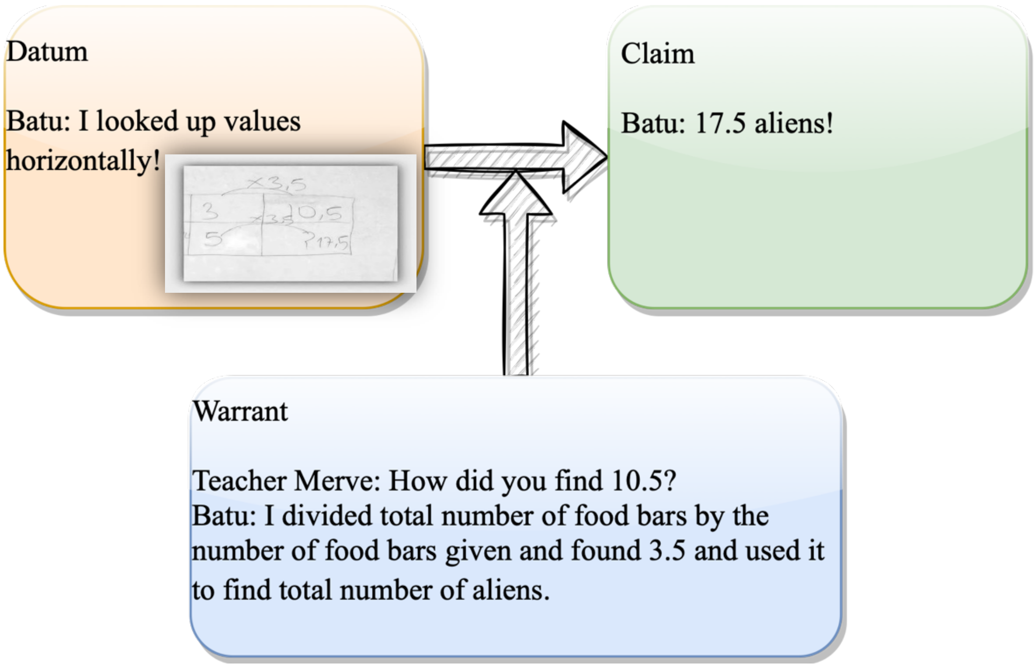


Figure 4. 74. A sample for a common scheme of HSF

Figure 4.74 shows a common scheme of HSF used by the students. The warrant of using arithmetic operations was dropped off, and drawing ratio tables and adding an arrow with scale factor became a norm for the further question. It could also be noted that the warrant of arithmetic operations was constructed as a division of the larger number by a smaller number related to the non-integer number issues.

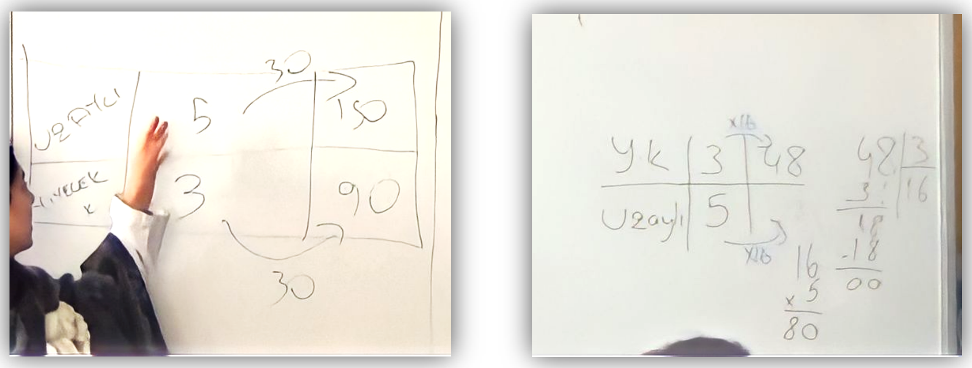


Figure 4. 75. Students' solutions for large numbers in the same activity

As it was conjectured, the students used HSFs dominantly during Activity 9, and this representation became a strategy within the ratio table as a horizontal "relationship". The students avoided the decimal scale factor among the variables while they selected the frequently used arithmetic operations in multiplication. They chose an "easy" way for the solution each time, even if it meant drawing all the variables or using a long ratio table (see Figure 4.75).

HSF was not previously used within the long ratio table, and it only emerged during the development of the short ratio table. On the contrary, the vertical scale factor (VSF) was used implicitly in the long ratio table to fill the cells. Furthermore, the vertical scale factor was another strategy the students preferred using in a short ratio table, provided that the appropriate conditions were delivered within the activity. Activity 7 included the tasks where the students focused on vertical scale factors (VSF) in general. Similar to Activity 8, the first question involves short ratio tables and missing values with the rule "two food bars feed six aliens". The other questions also ask how many food bars are needed to feed 9, 27, and 48 aliens, respectively. Akın started drawing the ratio table and filled the cells first. Afterward, he represented the vertical scale factor from upwards to downwards by drawing an arrow and wrote VSF as "×3" on the arrow (see Figure 4.76).

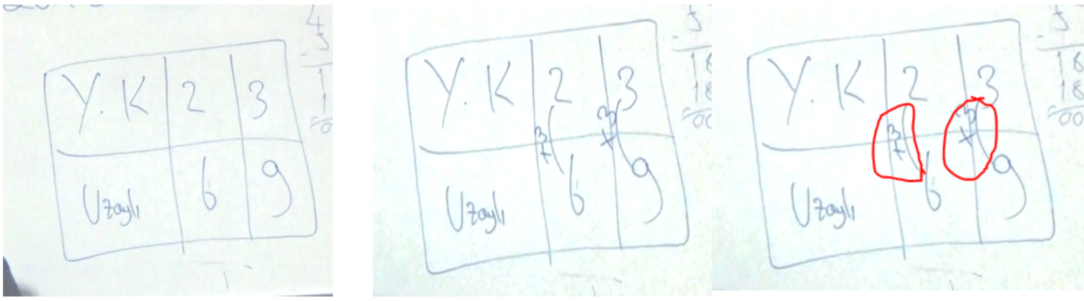


Figure 4. 76. Akın’s strategy for VSF

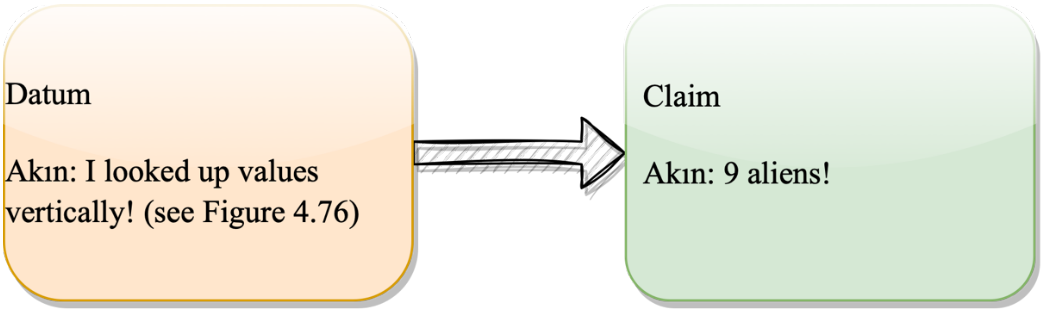


Figure 4. 77. A sample for a common scheme of HSF

VSF representation on the short ratio table became practice, as given in Figure 4.77 in missing value problems, and this representation as a datum was accepted by the students (see Figure 4.76). Although the frequency of the scale factor was changeable according to non-integer numbers, some volunteers attempted to use decimal scale factors without considering vertical or horizontal scale factors. Teacher Merve proactively guided the students to use different scale factors, even if it was decimal. Dane preferred using VSF for A9Q5, which asks, “use a table to find how many aliens to feed with 48 food bars.” with the rule “three food bars feed five aliens”.

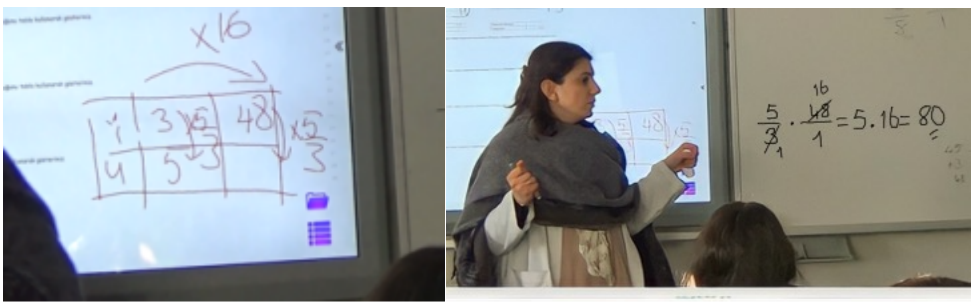


Figure 4. 78. A decimal VSF presented by Dane

Dane drew the short ratio table on the smartboard, and she first showed the VSF as a norm and calculated the missing value by multiplying VSF by 48 on the whiteboard (see Figure 4.3.3.8). In this question, Dane molded strategies of unit ratio and ratio table. Teacher Merve and the students supported this kind of relationship between strategies. Additionally, Teacher Merve added HSF to the ratio table to emphasize both scale factors were helpful.

Throughout the instructional sequence, the students provided their understanding of vertical or horizontal scale factors. The class concluded that the scale factors do not change the result but knowing these two factors may influence whether the students could solve the problem. If a student does not feel confident about the decimal scale factor, they sure choose the integer scale factor and the way of the solution. Activity 8 is also one of the activities that support both scale factors. As in the activity, they internalized the names of the scale factors. Implicitly, the vertical scale factor in a horizontal ratio table represents the constant multiplicative relationship between different variables. On the other hand, the horizontal scale factor represents the multiplicative relationship between the same variable. The difference between them is that VSF does not change between different variables, while the HSF can change from column to column. These issues did not take place in this lesson.

The students used "horizontal" and "vertical" for naming effectively within a horizontally extended ratio table. During introducing the scale factors, Teacher Merve started with the vocabulary meaning of horizontal and vertical that we interrelated this meaning with a mathematical context in a ratio table. In this perspective, Teacher Merve revoiced the students' solutions by emphasizing the relationship between or within variables. However, these attempts could have been implicit for several students during the instruction that Teacher Merve and the researcher came across a misconception. Teaching and naming these relationships were affected by the shape of the ratio table. In classroom 7/X, although the first formal ratio table of Deren emerged vertically, what became a practice was the horizontal ratio table. The two issues influenced the students. First, the whiteboard was a rectangle in height which was way shorter than its length. Second, the instructional sequence also horizontally represented the ratio table—those issues affected how to teach vertical and horizontal scale factors. In the beginning, Teacher Merve and the researcher focused on the direction of the relationship between the numbers. If it goes upwards to downwards or reverses, there is a vertical relationship in the horizontal ratio table. Still, there is a horizontal relationship if it goes from left to right or reverse. There were still a few students who sometimes used vertical ratio tables or other representations that Teacher Merve and the researcher neglected. Thus, the students felt confused about whether they were using horizontal or vertical scale factors.

Simge experienced a situation in A17Q3 asking, “Sue can walk 15 km in 5 hours; how far can she walk in 3 hours?”. Some students used only the unit ratio strategy and division/multiplication algorithm. Additionally, Buket and Arda used a short ratio table on the board using VS, and the researcher wondered about a solution strategy using HSF for this question. Simge claimed that she found the result by using HSF.

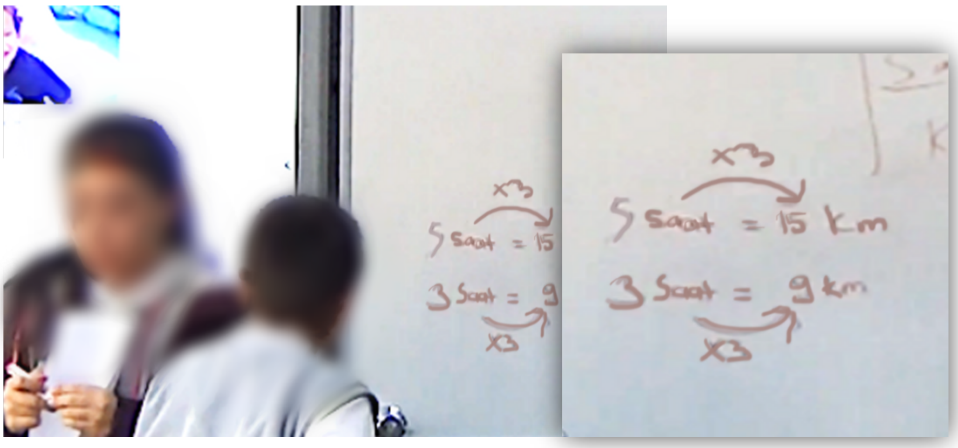


Figure 4. 79. Simge’s representation of HSF for A17Q5

The position of the arrows on Simge’s representation of solution misguided her as seen in Figure 4.79, and she said that she used HSF. Nevertheless, she did not get any rejection or reaction from the class. The researcher drew the attention of the classroom to the ratio table on the board drawn by Buket previously. She matched the numbers Simge wrote in equations on the board and the numbers in the ratio table. The inquiry focused on transferring the numbers into the cells of the ratio table. Simge shared the taken-as-shared ideas for the horizontal relationship within the horizontally extended short ratio table, but it was wrong for this solution. The numbers that Simge wrote in equations on the board were not aligned with the numbers in the horizontal ratio table. The inquiry focused on transferring the numbers into the cells of the ratio table (see Figure 4.80).

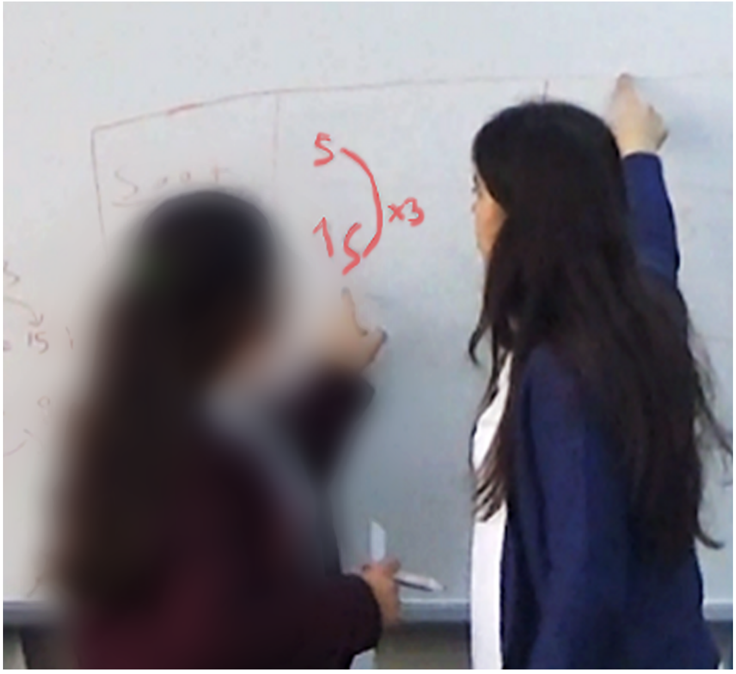


Figure 4. 80. The researcher explaining VSF and HSF based on within state ratios

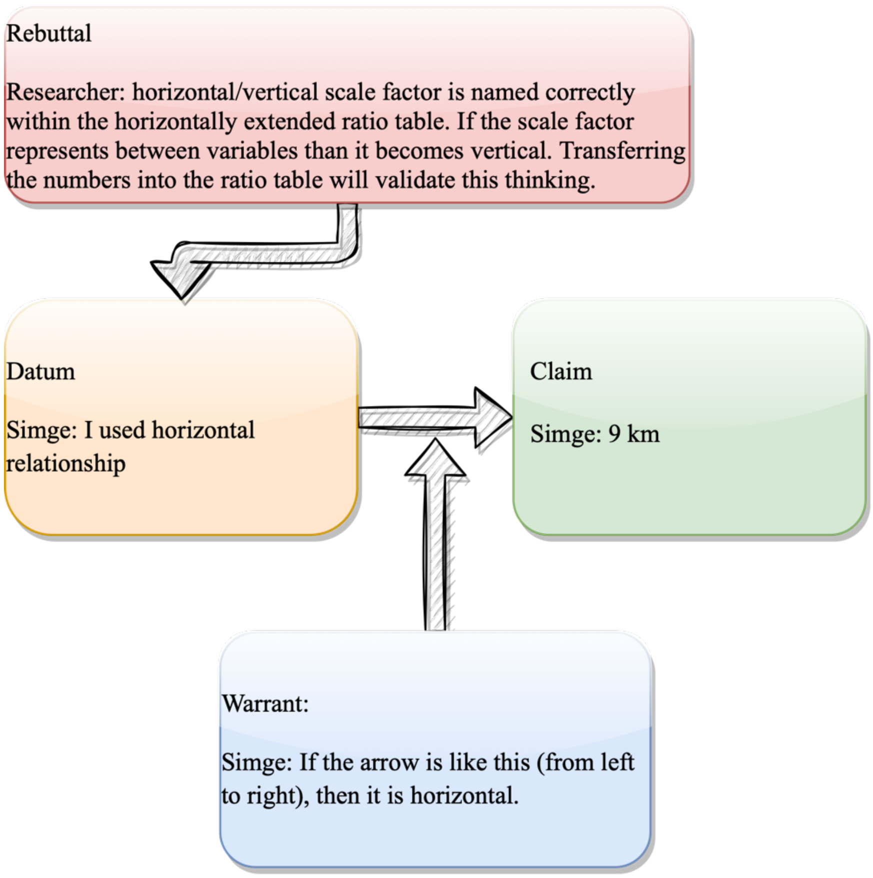


Figure 4. 81. Simge's misconception and the researcher's rebuttal for naming HSF

The teacher's and the researcher's guidance for VSF and HSF started from the direction of the arrows, and naming was also related with those arrows. However, the focus was on the relationship with variables: within or between. Simge used the relationship between the variables: hour and km, which described the vertical scale factor as Simge remembered on the short ratio table. On the other hand, horizontal scale factor described the relationship between the same variable. The researcher recommended the imagery of ratio table to remember the scale factors correctly, and Teacher Merve and the researcher talked about within and between variables concept more frequently in their revoicing and teaching during the instructional sequence instead of vertical and horizontal terms.

4.3.4. TAS 4 Reasoning about Cross-Product Algorithm and Inverse Proportions

Throughout the instructional sequence, Teacher Merve and the researcher encouraged the students to present their distinct strategies. They tested whether it was an efficient and mathematically meaningful solution for the question in the classroom discussions. In this kind of discussion moment, Ayten came to the whiteboard and used division and multiplication operations differently, as shown in Figure 4.82. She said that she found a novel strategy. Unlike the division algorithm explained in CMP 2, she reversed the numbers and calculated accordingly. Based on the previous practices, she was supposed to group the given number of aliens and multiply the quotient with the number of food bars shown in the rule to find the required number for 30 aliens. Instead, she multiplied the number of aliens by the number of food bars in the rule, finding 60. Next, she divided this finding into the number of aliens given in the rule. Each equation involved "alien × food bar" as a label, and she implicitly equated those two multiplication operations. At first glance, her use of cross multiplication was not evident that she was using cross multiplication. She said that you got the result of the question when you cross-multiplied the numbers in the table.

Figure 4.82 represents Ayten's utility of the cross-product algorithm. Ayten was trying to develop a novel solution method from the beginning, as was her groupmate, Cansu. Ayten reported that she deduced those arithmetic operations from the short ratio table. Based on her claim and datum, Teacher Merve asked her to show it on the ratio table as a warrant.

Teacher Merve had been cautious about the student-invented strategies from the beginning and questioned them with from the perspective of sociomathematical norm. That is, a strategy should be correct in every similar condition and be also defined by the owner of the strategy to check its correctness. She wanted to define a qualifier for the utility of cross-product algorithm since it was an overused strategy without considering its conceptual meaning. She asked to the class whether this strategy was true in every situation and gave it as homework. Ayten added that her strategy was true for all the questions in Activity 8.

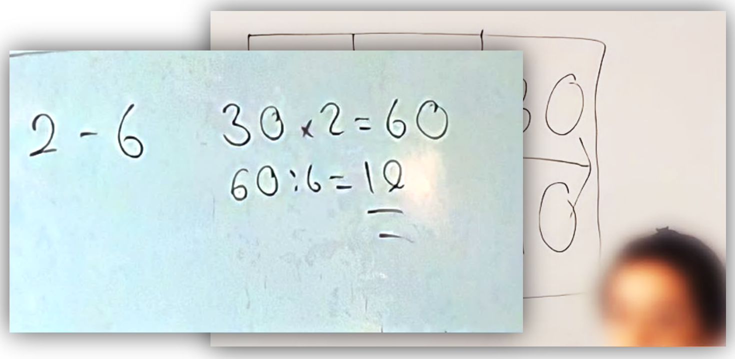
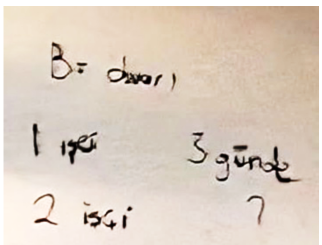


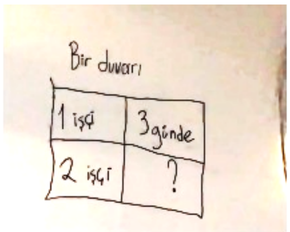
Figure 4. 82. Ayten's using cross-product algorithm

Before starting to Activity 9, Teacher Merve reminded each student about Ayten's new idea, cross-product algorithm. Focusing on the limitations of this strategy, Teacher Merve gave homework about finding counter arguments for Ayten's cross-product algorithm in the last lesson. Cross-product algorithm was common procedural knowledge used in Turkish middle school textbooks, but Teacher Merve had never mentioned this strategy within the context of instructional sequence. Nevertheless, Teacher Merve and the researcher planned to talk about this strategy later but improvised it through molding inversely proportional situations after Ayten's emergent strategy. A few students searched about the conditions that did not support cross-product relationship. Zeynep proposed one, asked, "If it takes a worker three days to build a wall, how long will it take two workers to build the same wall?". Then, she organized the given values on the white board. There were two claims: "six days (*Claim 1*)" and "one and half days (*Claim 2*)", which was already accepted due to the students' previous practices with scale factors and cross-multiplication.

Zeynep: I found one, I guess. (Zeynep shows what she did)



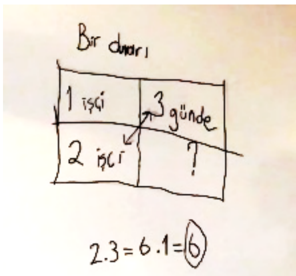
Teacher Merve: Look, we have a wall, okay? Let's make a table. Let's create a ratio table according to the worker this time, not the alien. In how many days will they complete the wall?



Unnamed student: Ma'am! Six days. (Claim 1)

Several students: One and a half days. (Claim 2)

Teacher Merve: In a day and a half. Well, I will say something. If only I had done as Ayten did. I'd multiply that [3] by that [2] and divide by that [1]. Two times three is six. I divided six by one. Usually, it should take six days, but being logical, what did you find? (Datum 1 for Claim 1)



Most students: One and a half.

Teacher Merve: One and a half, right? Well, what Ayten found is valid in every question? (Datum 2-Inversely Proportional Situations)

Unnamed student: Nooo…

Nur: Inversely proportional!

Teacher Merve: Inversely proportional? Ok! Do research about it and bring some examples if there are any. Normally, what if we used the scale factors? But in reality, what would be the number of days worked if the number of workers doubled?

Several students together: It is divided into two.

Teacher Merve: It decreases to half as soon as it doubles. Think of it logically. What happens if the number of workers has increased logically?

Several students together: Days decrease.

Teacher Merve: It decreases. What will happen this time will be reversed. So, if it's doubled here, it will be halving here. So, it will be a day and a half. Isn't it?

Teacher Merve: Now I'm deleting this place. Everyone should think about that cross-product algorithm. Look, you don't multiply it by two just because there is two as the scale factor in between. If the desired state is a situation that will increase, you multiply. Like how? As the number of aliens increases, the number of food bars will increase, right? So, we can multiply and double and find, but for example, as the number of workers increases, what will be the number of days working?

Class: It decreases.

Teacher Merve: It decreases as logic. Then instead of multiplying, I can do the opposite, half. What do I do if I get three times as much? I'll take one-third when I find the other, right? Let's do whatever is logical, okay? So, let's understand the problem first.

Teacher Merve explained this situation step by step and tried to convince the students about the new situation. Teacher Merve drew a ratio table on Zeynep's writing and offered the students to use Ayten's cross-multiplication strategy as a Datum for Claim 1. Nevertheless, the students chose Claim 2 as a result. The main question of Teacher Merve was, "Is it possible to use cross-multiplication in each situation?". Teacher Merve guided the students to be reasonable and did not conduct any strategy without questioning. Previously, the students worked on the ratio table and the same scale factor between each column or row. Since the students did not select cross-multiplication as a datum for this question, using cross-multiplication did not become a common strategy for the whole class.

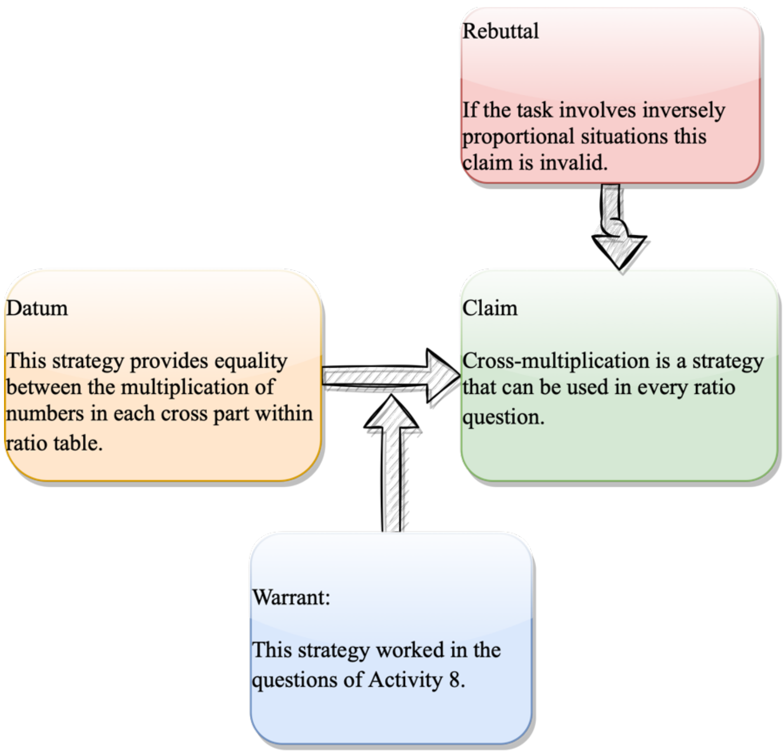


Figure 4. 83. A rebuttal condition for cross-product multiplication

Figure 4.83 summarizes the whole-class discussion about the condition for using the cross-multiplication algorithm. The discussion reflected the procedural details about the strategy, and Ayten put forward the correctness of her strategy within Activity 8 as a warrant. Ayten's contribution enabled the students to become familiar with the inversely proportional situations. Arda also brought an example related to inversely proportional situations. He used a vertically extended ratio table to show his understanding of inversely proportional situations and demonstrated his knowledge of increasing and decreasing relationships in this question. He used a ratio table and found the scale factors within it (horizontal in horizontally extended ratio table) the ratio table. However, Teacher Merve had not discussed the scale factors' construction in an inversely proportional question. Arda provided the first scale factor between the number of workers: 12:9 and 4:3. Since there was a decreasing relationship between the numbers in the table, as given in Figure 4.84, he assumed that there should be an increasing relationship according to his previous learning in the class. Nevertheless, he could not relate the scale factor within the numbers 27:*?*.

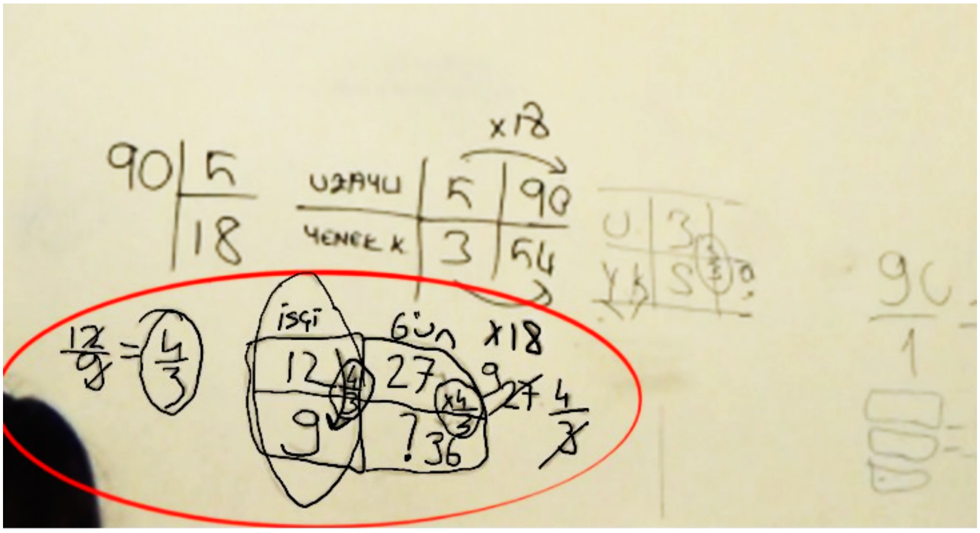


Figure 4. 84. Arda's question and solution for inversely proportional situation

Teacher Merve revoiced Arda's difficulty about the task and explained the inverse relationship between increasing and decreasing situations by relating inverse relationship between multiplication and division. As seen in Figure 4.84, the arrow was from upwards to downwards for each column, but there was division notation in the first column and multiplication notation in the second. To illustrate, while the number of workers were decreasing, the number of days to finish a job would increase. Arda conducted the calculation, and he found the result on the whiteboard after Teacher Merve's explanation. He multiplied 27 (total number of days to finish a job) by $\frac{4}{3}$. Although it was the only case about this situation, this discussion was already transferred to several students' solutions. To illustrate, Nur distinguished direct and inversely proportional situations, solved A9Q2 via the cross-product algorithm and transferred this solution strategy (see Figure 4.85). The solution on the board as the datum, warrant and claim.

Nur gained a practice for short ratio table by using cross-product algorithm for the direct proportional situation. The warrant involves using "doing calculation" strategy but with a new path. Short ratio tables were open to doing calculations as observed.

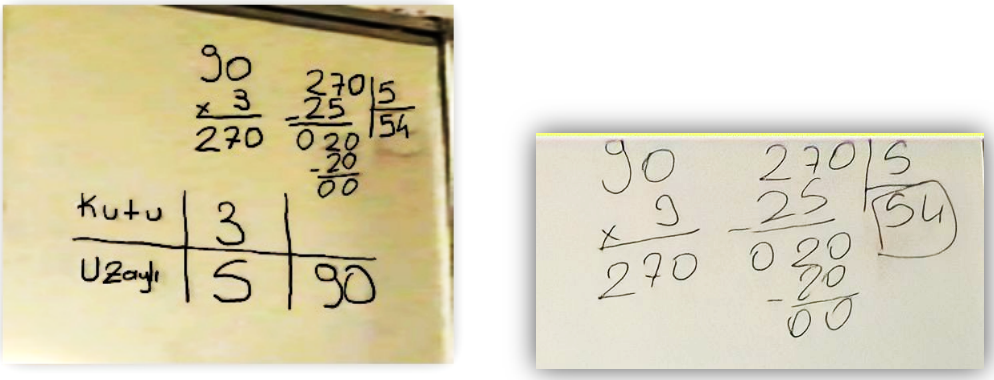


Figure 4. 85. Nur's cross-product algorithm representation (A9Q2)

As Nur's representation on the board, the ratio table became a tool to organize the numbers for the cross-product algorithm. She found the total number of food bars by multiplying three by 90 and dividing 270 by five (see Figure 4.85). Ayten came to the board again with an explanation of Nur's solution. She related Arda's solution with her previous solutions about using unit ratio.

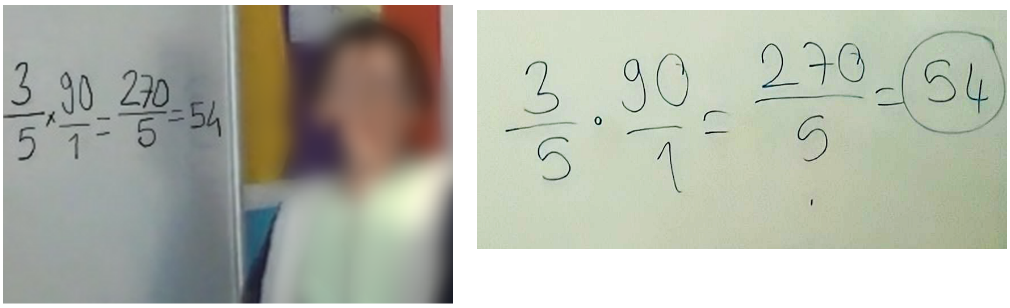


Figure 4. 86. Ayten's different representation of cross-multiplication

In this example, Ayten came to the board to show how to solve this question by using "fractions" as a datum. She improved her strategy during the instructional sequence and explained in a better way that her friends responded affirmatively to her strategy. She transformed the rule into a ratio so that it became $\frac{3}{5}$ and she multiplied it by 90 aliens. In the end, she found 54, similar to her friends. In Nur's cross-product algorithm, Ayten said, "she multiplied 90 with $\frac{3}{5}$, instead of multiplying 3". She explained the mechanism behind Nur's cross-product algorithm. In her previous examples, she could not achieve to reach her friends about her strategy, which was also not clear to her friends. She performed better than the other students in 7/X, as she already internalized this ratio understanding. Teacher Merve transferred "three-fifth food bar pieces for one alien" as a backing for the unit ratio strategy. After questioning ended, Zenan wondered about the scale factors and cross-product algorithm. Following this discussion, she already noticed that the cross-product algorithm did not use any scale factor. Teacher Merve reassured that scale factors and cross-product algorithm were distinct strategies and they were acceptable to solve this kind of problems.

Unit ratio strategy could be another explanation for cross-product algorithm, but this finding became explicit during the retrospective analysis. Indeed, Ayten emphasized this issue in her explanation, but Teacher Merve and the researcher handled it as a numerical similarity. As a further suggestion cross-product algorithm could be inserted into the instructional sequence as another relationship within the ratio table other than VSF and HSF.

4.3.5. TAS 5 Creating Third Linked Composite in A Ratio Table

Using ratio tables became a normative way of reasoning during the instructional sequence; however, there were still issues to explore within ratio tables. Activity 15 was the first that displayed an unusual frame for the ratio table (see Figure 4.87). This activity starts with filling a ratio table in which the columns, representing the number of the materials, were not progressing one by one and the rows have three variables. Additionally, each row has a different rule, in other words, ratio.

Calculator Costs

Below is a ratio table that illustrates the price for a certain type of calculator.

Number purchased	1	2	3	4	5	10	15	20
FCAT								$120
Scientific							$240	
Graphing						$800		

Figure 4. 87. Activity 15 and the ratio table

Several students adopted the number of relationships successfully. Fatma used multiplication to fill the table. First, she calculated the scale factor between units and multiplied the scale factor by the number of materials to find the total cost of these materials. She filled the cells without considering the previous column (independent from the column). She only multiplied and placed them into the appropriate cells, as given in Figure 4.88.

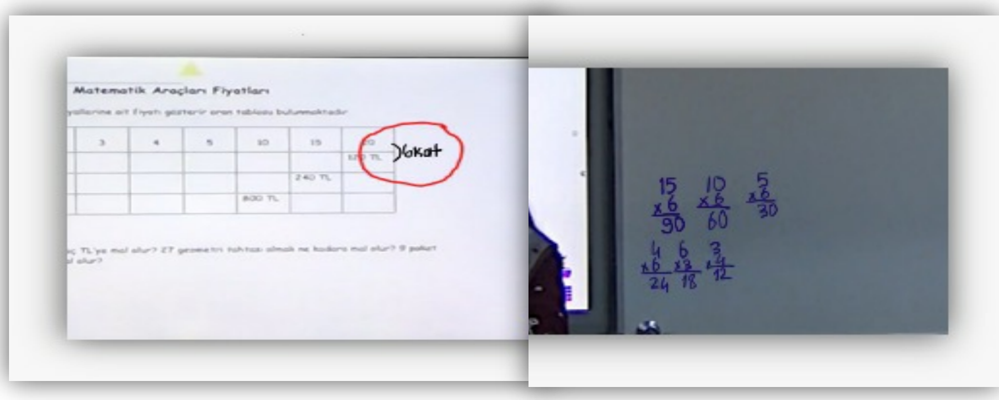


Figure 4. 88. Fatma's transfer previous learning into new ratio table

After Fatma, Eymen came to the board to solve the problem. He also found unit ratio at first. After that, he used build-up strategy to fill the consequent cells, as provided in Figure 4.89. He advanced one by one, starting from the cost of one material, then two, three, four, five, and six. He added 16 in each time since the number of materials increased one by one till five. Upon reaching 10 and 15, he multiplied the cost of five material by two and then by three respectively. His strategy was based on addition, and he was successful with his strategy with which he linked composite units correctly.

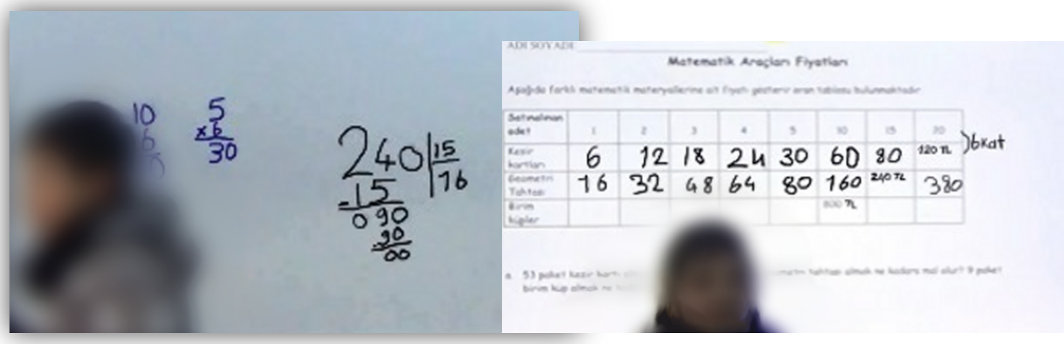


Figure 4. 89. Eymen's build-up strategy and linking composite units correctly

Dividing the given cost of materials by the number of columns provided an incorrect result for the table. Using this number and progressing additively could fall into a mistake. However, the number of materials given at the beginning of the table was paid attention to. To illustrate, Can divided the given cost of materials into the number of columns. Consequently, he found 240/7, a non-integer result, yet. His friends objected to his solution.

Activity 15 provided students with a different perspective on the ratio table in terms of flexibility between the number of columns, the number of rows, and the number of rules. Building on Activity 15, Activity 16 fostered the students to create third linked composite in the ratio table. Question 4 is “there are 3 boys for every 4 girls in Mrs. Smith’s class. If there are 28 students in the class, how many girls and how many boys are there?”. In this question, the students needed to create a third linked composite. Surprisingly, Adem rarely coming to the board was a volunteer to share his solution.

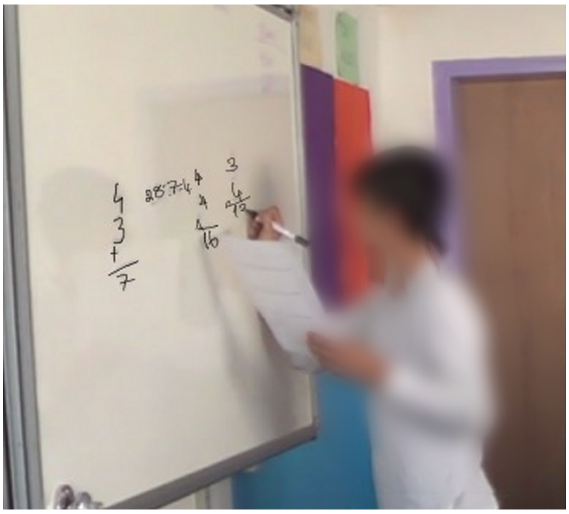


Figure 4. 90. Adem’s solution on the board

Adem created a third variable by adding the number of boys and the number of girls. He also found the scale factor between this number and divided by the total number of the students in the classroom. He applied the scale factor on the number of boys and the number of girls respectively. However, he was able to explain only the arithmetical operations, the procedures, which did not satisfy the class. Teacher Merve was in search of another person to elicit a warrant for this question. Adem’s claim, and the datum were accepted by Teacher Merve but the classroom needed a warrant to accept it.

Teacher Merve: Simge can you explain what did Adem try to do?

Simge: Teacher, when we add four to three, there are seven people. So, when we divide twenty-eight by four, it will be a multiple of four. (Repeating Adam)

Teacher Merve: Why do you think he added four to three? (Referring Adam)

Simge: Ma'am, to find the number of people, that is, according to that ratio. When we divide it, it comes out four times. (Repeating)

Teacher Merve: So when we divide twenty-eight by four, what do we actually find?

Simge: Scale factor relationship. (*Warrant*)

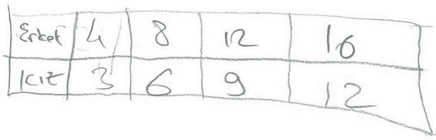
Teacher Merve: Are we finding the scale factor? In other words, if it is four times in total, the girls will be four times as much as the boys will be four times as much. (*Warrant*)

Deren: Ma'am, I don't understand. (Need a new warrant)

Evren: Can I do it? (Providing a new warrant)

Teacher Merve: Did you divide twenty-eight into four? Evren will you come? Everyone look at the board How did Evren do it?

Evren: Ma'am, it says there are three girls against four boys. I went four by four, Ma'am. In girls, Ma'am, it says there are three girls, I went three at a time. (It has progressed by building the sum, not the multiplicative relationship)



Teacher Merve: You say you went until I found twenty-eight total? Their total is twenty-eight. I'll say something now, children, Evren got it right, but some of you have tried this. Until you find the total of twenty-eight, you went four for the boys, how much for the girls? You went three by three. Ok, you found it easy as there are twenty-eight people here. What if the number value was something larger if it talked about the population of a school?

Teacher Merve: Well, you divide twenty-eight by seven and find four, right? This is actually the equivalent of a scale factor in this class. So yes, Özkan's is also correct, and Simge's is also correct, but wouldn't we combine both and get a more general solution with Cemil's? Look what we did out of three solutions. We've come up with an easier, simpler solution, right? (A backing)

Simge emphasized the scale factor between the numbers in Adem's solution. Teacher Merve needed to provide more detail for this question as a new warrant because several students did not understand Adem's explanation. Evren provided a ratio table solution and successfully placed the composite units. He frequently preferred the drawing strategy for his representation, and his voluntary utilization of the ratio table was a success for the instructional sequence because he insisted on using the drawing strategy previously. In his explanation, he filled the cells by adding up the numbers in the rows; next, he added the numbers in each column till the total result was 28. This strategy was appreciated because of its time-efficient aspect. Teacher Merve was in search of finding a strategy to use for both larger and smaller numbers. She gathered Adem, Simge, and Evren's strategies together to find a shared solution. While studying Adem's answer, she used Evren's build-up strategy to show the increase for the rows and concluded that the increase was seven in total for each step. Teacher Merve also used Simge's scale factor within variables and identified four as the scale factor to find the number of girls and boys. This new warrant and previous warrants molded, and they together created a backing for Adem's solution.

The construction of a third linked composite via other variables was established inexplicitly. Activity 17 was exploited to reinforce this understanding. The researcher created an opportunity to discuss a similar issue with different numbers in A17Q4. The drill question is, "There are four girls in every six boys. If there are 250 students, how many boys and girls are there in this class?". In the explore phase, Teacher Merve and the researcher observed the students' solutions in their activity sheets and initiated the activity by accepting solutions with the ratio table strategy. As mentioned above, Alp has done the question by using multiplicative relationship, which means he successfully transferred.

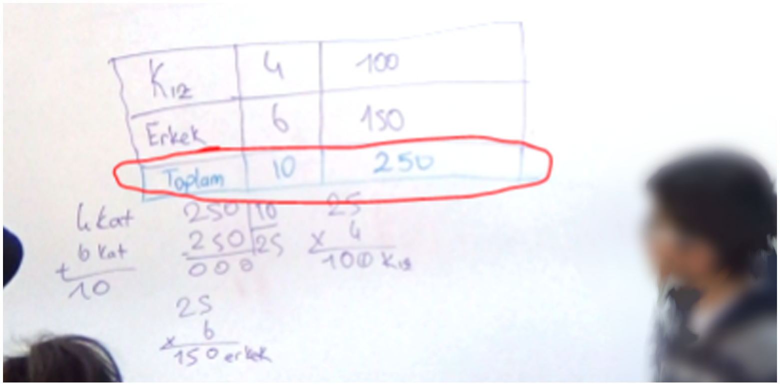


Figure 4. 91. Alp's representation of new variable

Alp used horizontal scale factors for the third linked composite and several students admitted of using it. Based on what Batu said, the teacher establishes a relationship to show the multiplicative relationship within the variable. The students did not establish vertical relationship among variables which was also expected due to non-integer scale factor.

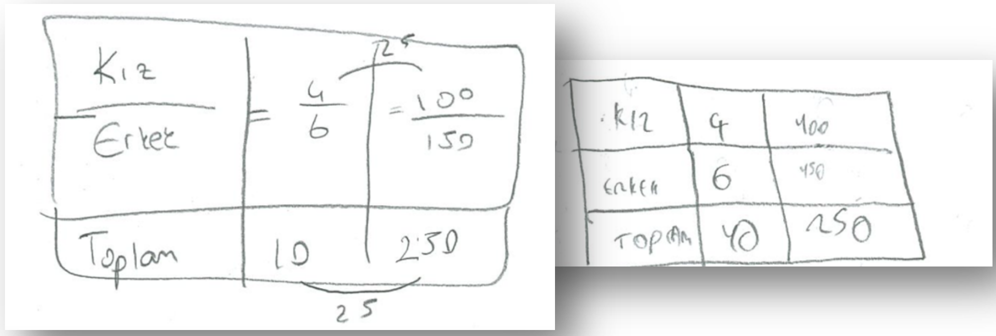


Figure 4. 92. Examples from students creating third variables

4.3.6. Summary of Classroom Mathematical Practice 3

The class explored conjectured and unconjectured relationships within long and short ratio tables among organized numbers in the preceding five ideas. The students did not follow a standard path. Some students who mainly used drawing transferred their knowledge into a long ratio table and explored the multiplicative relationship between the variables: vertical scale factors within the horizontally extended ratio table. On the other hand, some students who were already using scale factors brought cross-product algorithm strategy to solve direct ratio problems. Classroom Mathematical Practice 3 (CMP 3) consisted of five ideas from seventh-grade students who explored number relationships and mainly covariation within the ratio table. These five normative ways of reasoning are (1) Ratio tables can be filled out through covariation among composite units; (2) An alien can be half-fed, and a scale factor or the numbers in the cells can be decimal within the context of the problem; (3) The used strategy is not the difference between numerator and denominator in the equivalent ratios (additive thinking) but scale factors; (4) Cross-product algorithm can also be used in missing value problems provided that the problem is not the inversely proportional situation, (5) Third linked composite variable can be created in the ratio table and same horizontal scale factor can be used in this variable.

Table 4. 3

A Brief Summary of the Criteria for CMP 3

CMP 1	CMP Criteria
TAS 1	C3: Transfer of CMP 2 TAS 4 and using skip counting to fill out long ratio table.
TAS 2	C4: Reconstruction of CMP 1 TAS 1 C4 Rebuttal Not slicing an alien but half-full stomach
	C3: Transfer of units' attributes from CMP 1 to both discontinuous variables
	C3: Transfer of decimal scale factor into symbolic representation and real-life situations: One, decimal, or none.
TAS 3	C1: Dropping of warrants for CMP 2 TAS 4 for integer ratios.
	C1: Dropping of warrants for VSF and HSF
	C4: Rebuttal of "horizontal" misconception given as datum
	C4: Rebuttal of cross-product algorithm for inverse proportional situations
TAS 4	C2: Shifting "doing calculations" as a warrant
	C1: Using cross-product algorithm for direct proportional situations and dropping of warrants
TAS 5	C3: Transfer of HSF for third linked composite

As it was conjectured, the planned Big Idea developed how the students understand the nature of the scale factors and the procedures they used within ratio tables. While digging into the ideas with Teacher Merve, their perspectives on the problems were observable. As in the previous classroom mathematical practices, each idea was formed by datum, claim, warrants, backings, and rebuttals. Distinctively, the students' drawings formed this mathematical practice, and the students explored the number relationships in the table. This process reduced the discussion time spent with backing because their experience of overcoming the issues in the previous practices enabled them to build a collective mathematical language for each activity.

4.4. CMP 4: Analyzing Ratio and Proportion in Symbolic Representation

Classroom mathematical practice 4 focuses on raising awareness of the formal concepts about "ratio", "proportion", and "equivalent ratios" besides linking other representations to the symbolic representation. Previously, the students developed strategies, recognized the attribute of the units, and used decimals and larger numbers without explicitly saying "ratio". They started to reason by using symbolic expressions and terms.

4.4.1. TAS 1 Using Verbal and Symbolic Representations for Composite Units

After alien feeding episodes, new contexts were provided to assess the class's knowledge and the ability to transfer their skills, which was evidence of their social learning and normative reasoning. Activity 11 is about a pre-kindergarten "Tiny Tots" [9] designed based on the real-life context of rules to manage a pre-kindergarten. In this activity, the attributes of the units are as follows. The discontinuous variables are teachers and young children in lieu of aliens; on the other hand, food bars are continuous variables. The required number of teachers per child will be calculated for the new task. Both units are unbreakable to challenge the students, who are expected to explore the ratio concept, proportion, and their symbolic representations. Another challenge is that there are several different rules that students need to identify correctly. The way the students developed their ideas was formed around this aspect of ratio and proportion questions focusing on verbal and symbolic representations.

As a norm, Teacher Merve oriented students to the new activity and launched them by talking about the brochure that was distributed at the lesson's beginning, as shown in Figure 4.93. This knowledge was adapted to the Turkish context with the knowledge provided by the Ministry of Family and Social Services. The students aim to reason proportionally in this new context and learn the terminology and conventions of ratios, including how they are written and read.

In the launch phase, Berk identified "1:5" representation for the first time within the instructional sequence in A11 and asked what it meant. As given in the Launch part, Teacher Merve wondered about students' ideas about this representation, and the students were confused about the meaning. The estimates were "one or five", "one and five", and "numbers between one and five", which were not accepted by the other students or Teacher Merve, or the researcher. Ahmet started reading the verbal representation of division, "one divided by five," without the problem context. Ahsen contributed to his explanation of "one person from five-person". The rest of the class could not understand her explanation in the beginning. They missed the parts written in the brochure, which Teacher Merve wrote on the board. The booklet was so small that the writing was not readable for most students.

[9] As other activities, this activity was also adopted to the Turkish context as mentioned in the method part, but English as common language are used for writing.

Bu bilgiler http://cocukhizmetleri.aile.gov.tr/mevzuat/yonetmelikler adresinden 7 Kasım 2016 tarihinde edinilmiştir.

T.C. Aile ve Sosyal Politikalar Bakanlığı

Bu broşür "Oran-orantı " dersi kapsamında öğrenciler için hazırlanmıştır.

Özel kreş ve Gündüz Bakımevleri ile Özel Çocuk Kulüplerinin Kuruluş ve İşleyiş Esasları Hakkında Yönetmelik

T.C. Aile ve Sosyal Politikalar Bakanlığı

Çocuğunuzun Bakım Merkezini yakından tanıyın!

Bu kısa broşür, özel kreş ve gündüz bakımevleri ile özel çocuk kulüplerinin açılması ve çalışabilmesi için gereken minimum koşullar hakkında bilgilendirmek için hazırlanmıştır.

Gruplardaki Çocuk ve Personel sayısı

Gruplardaki personel-çocuk oranı:

a) 0-36 aylık çocuk grubunda 1:5

b) 37-66 aylık çocuk grubunda 1:10

c) 6-12 yaş grubunda 1:20

Çocuklara Yönelik Eğitim Programı

⇒ Kuruluşlar, Milli Eğitim Bakanlığının 0-36 aylık çocuklara yönelik Eğitim Programı ile 37-66 aylık çocuklar için Okul Öncesi Eğitim Programını uygular.

⇒ Uygulanan programlar ile ilgili formlar, her çocuk için düzenlenerek kayıt altına alınır ve denetimde gösterilmek üzere saklanır.

YÖNETMELİKLER

Çocukların Beslenmesi

⇒ Kuruluşlar, çocukların yaş ve gelişim özelliklerine göre gıda rasyonlarına uygun olarak, çocuklara öğlen yemeği ile sabah ve ikindi kahvaltısı, öğleden sonra gelen çocuklara öğlen yemeği ve ikindi kahvaltısı verir.

⇒ Kuruluşun 0-36 aylık çocuklara yönelik bir beslenme programı bulunmalıdır. Kuruluşta verilen yemek numuneleri uygun saklama ortamında yetmiş iki saat süre ile saklanır.

Kuruluş Binasında Aranacak Özellikler

⇒ Kuruluşta, bir idare odası veya bölümü oluşturulmalıdır.

⇒ Oyun, etkinlik, uyku, çalışma ve dinlenme odaları bol ışık almalıdır.

⇒ Çocukların kullandığı, oyun, uyku, çalışma odalarının kapıları içerden dışarıya açılır şekilde olmalıdır.

⇒ Çocukların oyun oynadığı ve etkinlik yaptığı odaların zemini mutlaka yumuşak malzemeden yapılmış olmalıdır.

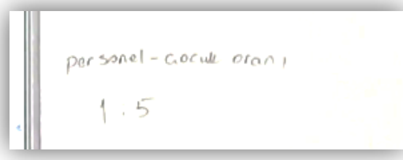
⇒ Çocukların bulunduğu oda ve salonlarda, taban çocukların sağlığına zarar vermeyecek, kolaylıkla silinip temizlenebilen bir madde ile döşenmeli ve zemin anti bakteriyel malzemeyle kaplanmalıdır. Çocukların oturarak oynamaları ve etkinlikler için, bazı bölümlere minder veya antialerjik halı konulabilir.

⇒ Çocukların bulunduğu oda ve salonların duvarları kolaylıkla silinip temizlenebilen bir madde ile boyanmalı veya kaplanmalıdır.

⇒ Grup odaları çocukların yaş gruplarına uygun ve psikolojik ve sosyal gelişimlerine yardımcı nitelikteki, Milli Eğitim Bakanlığı eğitim programlarında belirtilen eşyalar ve eğitim araç ve gereçleri ile donatılmalıdır. rahatça ulaşabileceği bir seviyede olmalıdır.

Figure 4. 93. Launch part for Activity 11

Teacher Merve: It was written like this. Staff in groups...



Berk: One man with five children.

Teacher Merve: Okay. The staff in the groups made a statement above as the ratio of children and said: What does one-five mean for a zero to thirty-six-month-old? That's one-five?

Ali: Teacher, I found it.

Teacher Merve: Tell me.

Akın: Ma'am, there are five people. For example, from the staff. One of the groups of children from zero to thirty-six months. Each of those groups means one by one.

Arda: Ma'am, I found it; I found it for sure.

Teacher Merve: Who else? Arda, tell me.

Arda: Ma'am, there are five children per staff, right? Ma'am, what was the exact rule of this anyway? One staff member, Ma'am, informs five children of thirty-six months old, Ma'am. *(Datum and Claim)*

Teacher Merve: Okay, look, Arda said. He said that there is one staff member for five of the children in this group.

Researcher: Do you agree?

Several students together: Yes.

Teacher Merve: Who attended Arda? As Arda said, it means one staff member for every five children. Does your friend Arda find it correct?

A few students: Yes, Ma'am.

Teacher Merve: Okay. Well, does anyone disagree with what Arda said? Who says I'll make a different statement?

This discussion process was a little bit different from the previous ones. The class tried to reach a valid statement for "what does 1:5 mean?". During the last discussion, Teacher Merve used Ahmet's "one divided by five" explanation to read "1:5" because ":" symbol was frequently used to represent division operation in their previous lessons. Berk said, "five children for one man". Teacher Merve tried to elicit similar explanations from the other students and asked the question again. Akın tried elaborating on Berk's explanation, but he could not express his ideas appropriately. As the instructional sequence suggested, Teacher Merve intervened in the discussion and gave a hint about the numerator and denominator. She asked, "which one represented the teacher, and which one represented the children? In the end, Arda concluded, "for each teacher, there are five children," different from Berk's conclusion.

Then, he started reading from the numerator to denominator. Teacher Merve also read “for five children there is one teacher”. The rest of the class did not object to this statement which was also constructed in the further questions. They connected the given numbers with the right units even if their places changed within the verbal representation. The focus was on the numbers and units’ relationship with ratio representation.

> Emergent Claim: It’s the rule of the problem. For five children, there is one teacher (molded with the problem context) because one represents the number of teacher (s), and five represents the number of child (ren).

The students already used similar claims in data organization strategies within the frame of the rule of the problem. When they saw a rule for the alien-food bar (see Figure 4.1 as an example), they naturally used “there are two food bars for five aliens” kind of statements to support their claim, data, or warrant. Shifting to the symbolic representation, the students just started to think about this representation. Nevertheless, they felt pretty complicated. After discussing about 1:5 in the brochure, Teacher Merve drew attention to the other rules: 1:10 and 1:20. She repeated Arda’s statement: For each teacher, there are 10 young children (37–66-month-old), and for each teacher, there are 20 children (6–12-year-olds). Teacher Merve asked, “what if we changed 1:5 to 5:1?”. Does the meaning change if the places of the numbers change?

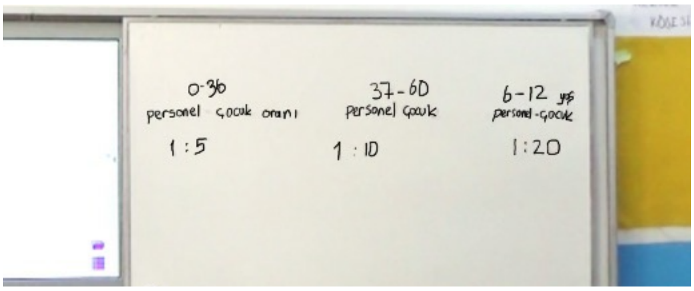


Figure 4. 94. Three ratios on the whiteboard (A11)

> Akın: It would change, Ma’am.
>
> Teacher Merve: He wrote a half-five over there; what would it be?
>
> Zenan: There would be five staff per child, Ma’am.
>
> Berk: One staff member for five children. No, no, no, no...
>
> Teacher Merve: How would it be written? Five over one. So you're saying if it was like this. If he had told me the child-staff ratio first, I would have written it as five to one

logically. I'm not saying let's break the rule. I'm just asking if the place will change if its location changes.

Arda: No, my teacher will not change.

Teacher Merve: You say it doesn't change. Fatma?

Fatma: I say it doesn't change either. Looking at it logically, five personnel cannot care for a child.

The students were confused about 1:5 and 5:1. Zenan and Akın claimed, “it does change”. Zenan read, “for one child, there are five teachers.” Unlike Zenan, Berk read “for five children; there is one teacher” by changing the order of the variables and concluded that “the ratio does not change”. Teacher Merve emphasized the ratio of children to teachers instead of teachers to children. This situation was not clear for Teacher Merve and the researcher, but Fatma enlightened the class about their thinking by stating that “5:1 cannot mean five teachers care about one child”. She concluded that this statement was not logical. As explained in Figure 4.94 and 4.95, Teacher Merve wrote variables *teacher* and *children* in the order above the symbolic representation 1:5. Still, this could not prevent the students from thinking that they would change places of the labels when reversing the numbers: $\frac{1\ teacher}{5\ children}$ turns into $\frac{5\ teachers}{1\ children}$. Teacher Merve and the researcher intervened in this process.

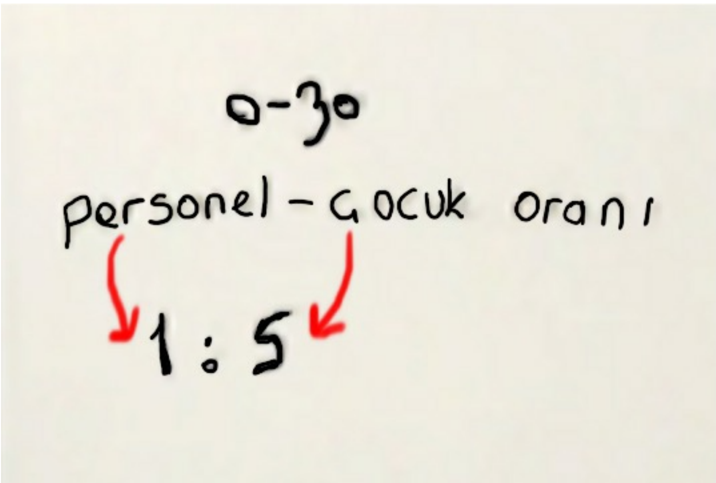


Figure 4. 95. Students’ reasoning about linking units on symbolic representation

In this discussion above, they implicitly presented an understanding of 1:5 and its relation to teachers to students. However, they used distinct imageries to explain the difference between 1:5 and 5:1. Does the ratio change? Does the rule change? Does the verbal representation change? This discussion continued within a new context to assess their understanding of these issues.

Researcher: So far, how can we write the ratio of girls to boys in this class? Can you calculate and write it right away?

Berk: Teacher, twenty boys, and eighteen girls.

Arda: The number of male students is twenty, and the number of female students is eighteen.

Teacher Merve: Okay, the ratio of girls to boys.

Akın: Ma'am, I think it's nine over ten.

Researcher: Nine over ten?

Akın: I divided eighteen by nine and twenty by ten.

In the new context, the researcher used the ratio of the number of girls to boys in 7/X. They started to identify the number of boys and girls first. Teacher Merve repeated the researcher's question. Akın said the ratio was "nine divided by ten," as Teacher Merve used previously, and the linkage between the variables and the numbers was correctly connected. Additionally, he simplified the ratio by two. Still, the impact of the representation change on the ratio, the rule, and the verbal representation was not apparent from the discussion till now.

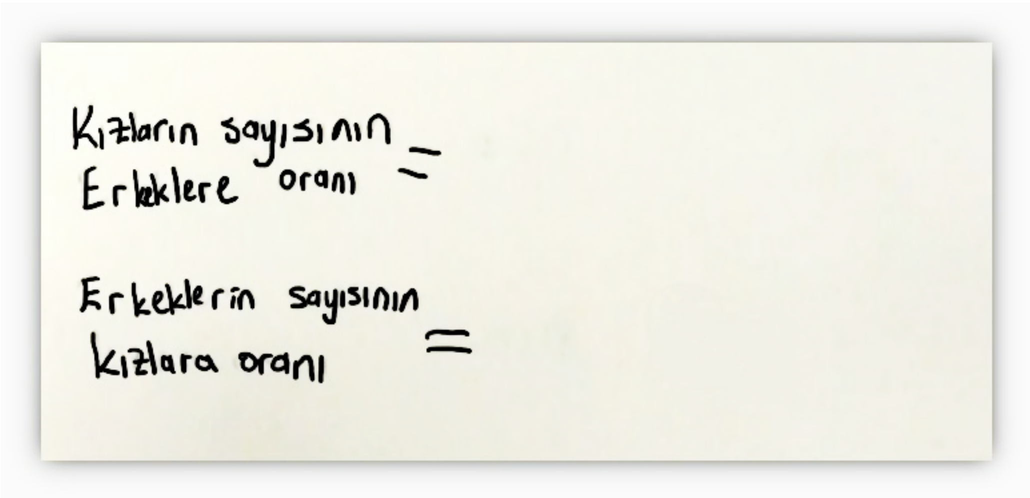


Figure 4. 96. Teachers' changing the label and Simge's response

Teacher Merve wrote the units on the board as "the ratio of the number of girls to the boys" first and then "the ratio of the number of boys to the girls" (see Figure 4.96). She asked the students to display these two ratios with ratio symbols. Simge wrote the ratio of the number of girls to boys as 18 to 20 and the ratio of the number of boys to girls as 20 to 18 without any reduction. Reducing a ratio was accepted by several students as a must for symbolic representation. In exchange, Teacher Merve emphasized that reduction was a choice but not a must, and that both symbolic representations were accurate in two situations. All in all, they concluded that Simge put the numbers in the correct places, which means they transferred the previous emergent claim to this context. Teacher Merve and the researcher continued questioning so that these verbal and symbolic representations would match.

> Teacher Merve: Did he just write it correctly or incorrectly?
>
> Most of the students: Correct!
>
> Teacher Merve: Then, can we come up with an idea? In other words, what do we mean to write first when we say the ratio of one thing to another? By looking at the reading, can we decide which one is written first?
>
> Several students together: Yes.
>
> Teacher Merve: Yes, we can. Otherwise, if there is no such rule, we can't know which one to write first [numerator] and which one to write later [denominator], right?
>
> Zeynep: Whatever we compare to what is written first. For example, if boys are compared to girls here, boys are written first. If I say girls, it's the opposite. (Warrant)
>
> Simge: But, Ma'am, we are rating two of them.
>
> Teacher Merve: So here we said the ratio of the number of girls to boys, we wrote the girls first, okay. Here, too, the number of men... OK, good. (Warrant)

Teacher Merve asked again about the order of the symbolic representation to the class, the place of the numerator and denominator. They repeated that it was right. Furthermore, Teacher Merve tried to create a warrant of matching symbolic and verbal representations. It seemed they were doing it right, but the class needed to clarify how they related both representations. As a conclusion to Teacher Merve's question (what if we changed the 1:5 to 5:1? Does the meaning change if the places of the numbers change?) the class found that the order of the units in verbal representation changes the ratio symbol. This question was constructed on Arda's emergent claim for 1:5. Simge's datum was also transferred from Arda's linking composite units. Throughout this discussion, the below argumentation layout emerged.

This argumentation was developed with a lengthy discussion in the classroom and contained unconjectured ideas. Students created a reading pattern, of which the language was also significant. Despite not being introduced previously, the students managed to use correct verbal representations without using "ratio" (a to b, a per b, a for b, a for each b, for every b, there are a). Now they learned the effective use of "the ratio of a to b". The concept of a ratio and its symbolic representation was embedded in this idea.

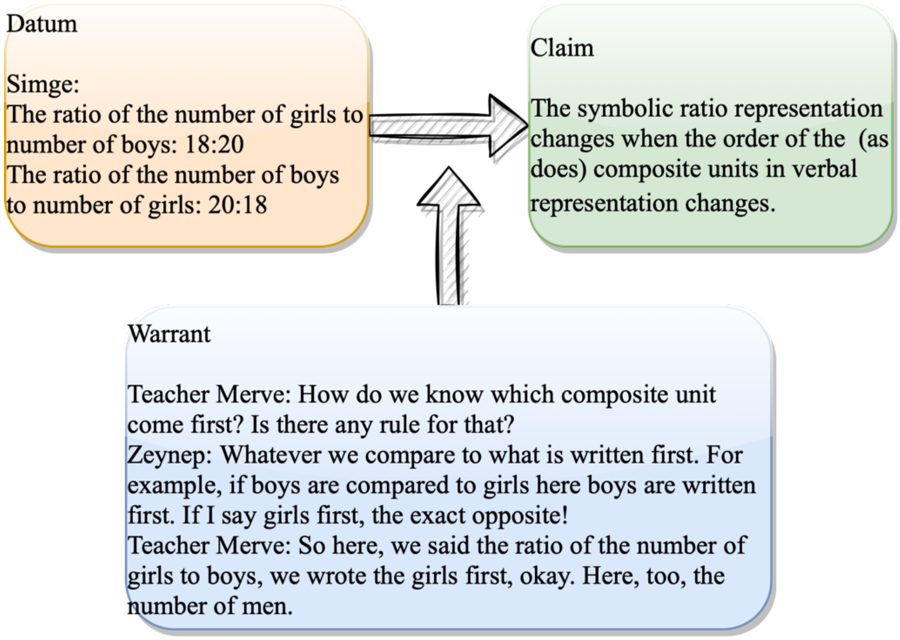


Figure 4. 97. Argumentation layout for the match between symbolic and verbal representation of ratio

4.4.2. TAS 2 Transferring Knowledge and Skills to Proportional Situations

The students already learned how to reason proportionally to find the missing value in the tasks because they were proportioning two ratios under the condition of equality. While developing Idea 1, reducing the ratio by two was discussed in short, and Teacher Merve concluded that reduction was not a must, but a choice. Nevertheless, Arda transferred this issue and questioned writing the ratio in a reduction mode, such as 9:10 instead 18:20. Therefore, he wanted to understand why it was written on the board as 9:10 in the end, as given in Figure 4.98

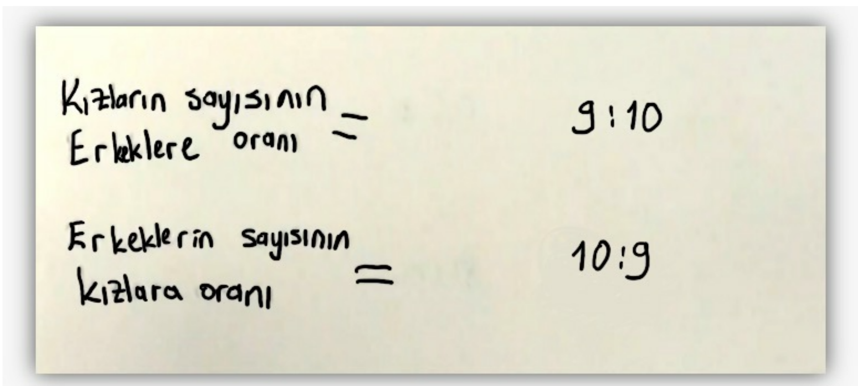


Figure 4. 98. Reduced ratio representation

Teacher Merve erased Simge's unreduced ratio representation on the board and wrote reduced ones despite Simge's unreduced choice. Arda noticed and assumed that there was an issue about reducing ratios.

Researcher: Can those who say both are correct raise their hand?

Arda: No, Ma'am. (*Claim1: Unreduced ratio is true*)

Researcher: What about Arda?

Arda: Ma'am, why are we only simplifying the process in half? We divide by the real number without halving (20 why do we divide 18 by 10 and 9?) (*Datum1*)

Teacher Merve: Why are we simplifying, who wants to explain? OK, Berk.

Berk: Teacher, to find the ratio, to simplify and find smaller numbers. (*Implicit Claim2: Both of them are true and Datum2*)

Teacher Merve: To express it with a smaller number.

Arda: But teacher, the most common method is your own. (*Warrant 1*)

Teacher Merve: Okay, this is the most common method, it's not a number, it's a ratio. In fact, we can write this ratio as follows. So eighteen over twenty. You know a fraction has a meaning of division, right? (Warrant 2)

Several students together: Yes.

Teacher Merve: Yes. Can't I write it like this?

Several students together: Yes.

Teacher Merve: Okay, haven't I been simplifying fractions until now? (*Warrant 2*)

Arda: We were doing. (*Accepting Warrant 2*)

Teacher Merve: So what do you think is abnormal now?

Student: Teacher, Arda says no to everything. (*Accepting Claim 2*)

Researcher: Okay, guys, let's put this in the hypothetical column. think about it!

Figure 4. 99. Hypothesis wall used for equivalent ratio

Arda was perplexed about the reduced (10:9) and unreduced (20:18) ratio form. He wondered why the unreduced ratio was divided by two (see Figure 4.98). Teacher Merve and the researcher asked the classroom why the ratios were reduced. According to him, 18:20 was the "normal" mode, which should be written as such. Teacher Merve and several students offered rebuttals as a new argument (*Claim2, Datum2, Warrant2*) to Arda's argument (*Claim1, Datum1, Warrant1*). Using small numbers and fraction imagery was the focus of this discussion. In the end, Arda's claim was directly related to the equivalent ratios and proportions. In what followed, Big Idea of Activity 11 (A11) was about discussing equivalent ratios, and this discussion provided a base for the upcoming discussions about equivalent ratios.

After the launch phase of Activity 11, explore phase started, and students solved the questions in A11 as given in Figure 4.100 based on the information provided in Figure 4.93. The students solved the first two questions by using all the strategies described in CMP 2 and CMP 3. Drawing all the units was a strategy used by the slow learners; although it was time-consuming, it supported slow learners' participation and expression of their understanding. VSF and HSF were commonly used in those examples. The students used VSF and HSF interchangeably at the same time. In the second question, Teacher Merve tried to draw attention to a new discussion about equivalent ratios by using a symbolic representation of the ratio.

Teacher Merve used fraction imagery and the division meaning of fractions for the students to recall their previous learning. She kept reading the ratio "1:5" as "one divided by five". She was looking for the answers who used "$\frac{1}{5}$" fraction bar and accepted the ratio table solution strategy for this question. The students continued offering cross-product algorithms, VSF, and HSF in returning to the teacher's question.

Tiny Tots

Tiny Tots Daycare has a teacher to infant ratio of 1:5.

Teachers	1	
Infants	5	

1. How many teachers must be in the room if there are 25 infants? Explain.

2. What is the maximum number of infants that can be in the room if there are 6 teachers? Explain.

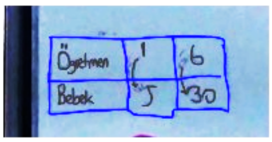
3. Tiny Tots Daycare has a toddler to teacher ratio of 8 to 1. Use a ratio table to determine how many teachers can be in the room if there are 24 toddlers. Show your work and explain.

4. Decide which of the ratios below belongs in the Teacher/Infant table and which belong in the Teacher/Toddler table.

2/16	4/20	80/400	2:10
8/64	6 to 48	7:35	7/56

Figure 4. 100. Activity 11 Tiny Tots

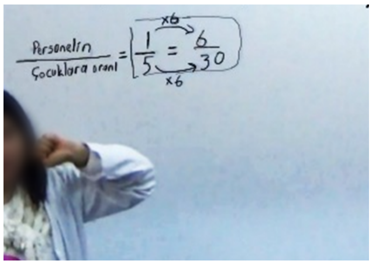
Teacher Merve: Now the ratio of personnel to children is one to five. The question tells us how many children six personnel take care of. So, the following table is actually coming from the ratio table. Can anyone write it as a symbolic representation? You know, we said in the last lesson that the ratio of personnel to children is one-fifth.



Ahmet: Ma'am, we can do a cross-product algorithm like this.

Teacher Merve: Apart from the cross-product algorithm, can we also do it with a multiplicative relationship? How many times does one go into six?

Ahmet: We can find six times one, Ma'am. It becomes six. We multiply six by five, and we get thirty, Ma'am.



Teacher Merve: That of... (Draws on the board). That's six times that. Ahmet said that Ma'am, one has been here six times. It's in the share section right here. Then he said that for this expression not to change, there must be six multiples here. He said thirty children are needed, right? Well, isn't this actually an example of equivalent fractions at the same time?

All of the students: Yes.

As given in the discussions, the students drew a ratio table on the board and filled in the cells with the correct numbers. One of the students showed a VSF relationship on the board. Teacher Merve tried connecting the ratio table with ratio representation and then planned to bring the topic to the proportion concept. She transferred the numbers into the equivalent ratio by representing them above the whiteboard. She introduced the conceptual knowledge about proportions. In the beginning, she wrote the units, the rule, and the required number of teachers and children, respectively. She explicitly compared equivalent ratio representation with equivalent fractions and emphasized their similarity. The students nodded positively at this similarity. In other words, Teacher Merve highlighted that the students have tried to think proportionally and find the missing part of the equivalent ratio. She also suggested HSF on the table via drawing arrows as in the ratio table to convince students. She added that if the students were to expand or reduce a rational number, the exact places of the number values in the denominator and numerator were necessary, which was about how to keep equality intact. She reminded again of Idea 1 and said, "If the ratio of teachers to children is said, we will write the personnel up and the children down in this process". At this point, the researcher intervened in this process.

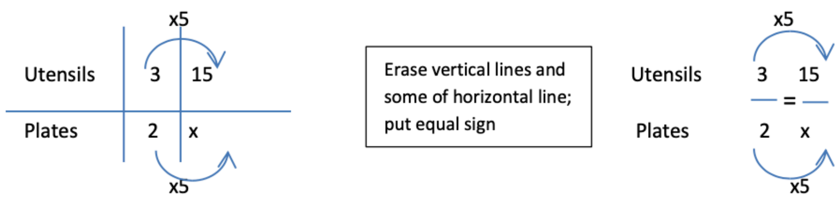


Figure 4. 101. A strategy for proportion concept on ratio table (Stephan et al., 2015)

Figure 4.101 is a strategy taken from the instructional sequence to represent the equality between ratios. It is recommended to support students' utility of symbolic representation efficiently. Therefore, the researcher erased borders of the ratio table as in Figure 4.102.

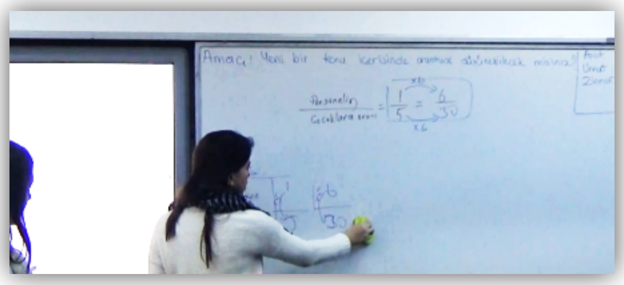


Figure 4. 102. Researcher erasing the vertical lines on the table

The researcher explained that the erased figure represented the mathematical statement of the problem, and it was a symbolic representation of equivalent ratios and proportions. Teacher Merve put an order of this strategy*: draw a table, erase the lines,* and *add an equal sign*, which would create equivalent ratios and proportions. Teacher Merve asked Simge to solve another problem to assess whether this strategy worked with the students. This time the ratio of personnel to children is one to eight and asking the required personnel for 24 children. Simge transferred previous demonstration of Teacher Merve and researcher and used HSF to find the missing value, reporting three as the multiplicative relationship.

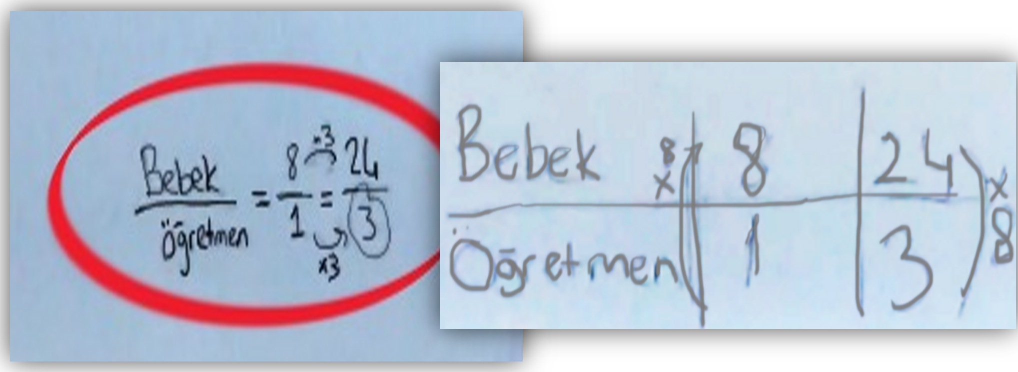


Figure 4. 103. VSF and HSF representation of Simge (left) and Batu (right)

In response to Simge's answer, Batu used short ratio table and her datum was about using VSF, eight as the multiplicative relationship. He showed all the process on the ratio table as before. He just copied Simge's representation of equivalent ratios. Normally, it was expected them to write one in the first row and eight in the second row of the short ratio table based on the collection of practices. However, Batu did not see any concern about filling the numbers.

After he was done with the solution, Teacher Merve reminded three steps to show in an equivalent relationship.

The transfer from VSF and HSF to equivalent ratio was not easy for them. It was observed that Simge preferred using the ratio of children to personnel, unlike other solutions, which included the smaller value in the numerator of the ratio representation. Students did not mind this situation, which was evidence of the transfer of Idea 1. It was also another evidence that Simge solved the question with the new strategy and did not need to explain her solution since there was no further question about it. Simge used within the variable connection between two equivalent ratios (HSF), but no need a warrant. This strategy consists of the same multiplicative relationships with the ratio table. VSF, HSF, and cross-multiplication algorithm in the horizontally extended ratio table can be transferred into this strategy. Batu said, "I solved this question with VSF and directly drew a ratio table". He did not transfer it into proportion. Teacher Merve again erased the frame of the ratio table and reminded the steps of proportion and equivalent ratios. VSF and HSF utility for proportion could not be achieved till now.

Following activity (A12) includes missing-value questions from different contexts, which requires decision-making about selection of VSF and HSF. The first one contains two discontinuous variables (human), and the second has two continuous ones (sugar and flour). The questions involved at least one integer scale factor in two cases, which encouraged students to use vertical and horizontal scale factors. As given in Figure 4.104, they put the numbers in the correct places while using equivalent ratios thanks to the ratio table, but unlike ratio table the units were not written in equivalent ratios representation. Teacher Merve especially asked their units and they verbally mentioned them on the board. In the discussion, there were several shy students trying to solve the problems, which was an attractive issue for the rest of the class, and they even expressed "I haven't seen my friend talking before" loudly.

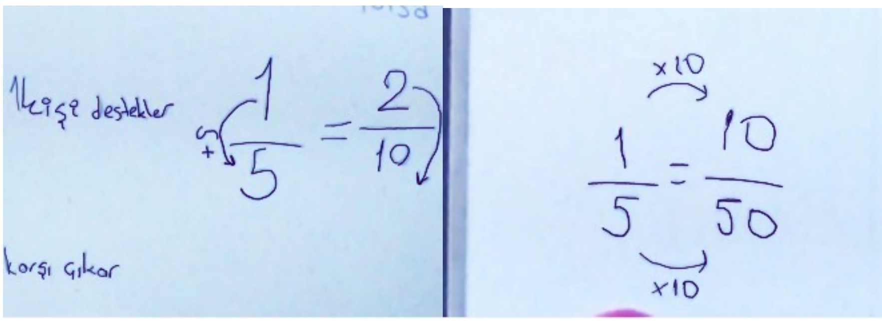
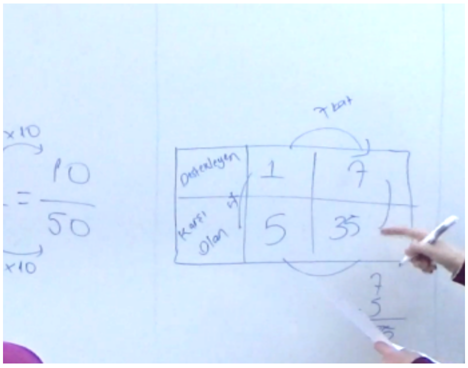


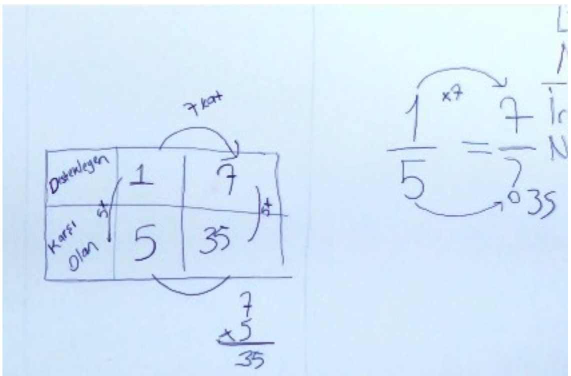
Figure 4. 104. Students' using VSF and HSF on the equivalent ratios

While Fatma was solving the same question in Activity 12, she used a ratio table unlike her peers (see Figure 4.104) and showed two scale factors on the table. Nazan offered that this problem could be solved through "fractions" as well. The students mentioned "fractions" as a number representation not as the meaning for "part-whole relationship". The Teacher also supported this word and repeated it. Nonetheless, there was an aim to learn the correct language "equivalent ratios" or "proportion. Teacher Merve and the researcher decided to emphasize formal words more for the other lessons even if the students preferred saying fractions.

Fatma: There are five multiples between this and that (by showing the difference between the number of people who are advocates and against the war), there will be multiples of five between that and that (by pointing to the second column), she states that she shows the 7×5 below and writes thirty-five. (She also added the horizontal relationship in the table) (CMP 3 Idea 3)



Teacher Merve: Yes, Fatma has solved it in two ways; everyone should look at the board. I see. If there are multiples of seven here, then there will be multiples of seven here, thirty-five. Or, as a second strategy, I can look vertically. Five to one is thirty-five to seven.



Nazan: Ma'am, we can do the same with fractions by deleting the lines. (CMP 4 Idea 2 construction continues)

Teacher Merve: Yes, you can do it with fractions. Let's do one with a fraction, then. Efe, come! Try to do it with fractions. Whatever Fatma does, do it with fractions.

Efe: While five are against, one person is an advocate, so it's five to one. I did it with a multiplicative relationship, Ma'am.

After Nazan's offer, Teacher Merve asked Efe to show as "fractions" on the board as other students did for the other questions (Figure 4.104) since Efe also used scale factor representation. Students were getting used to transfer VSF and HSF into proportion. Another example, Teacher Merve helped Burcu to use equivalent ratios in response that she did not understand how to transfer the data from table to equivalent ratio. Teacher Merve helped Burcu to use equivalent ratios in response that she did not understand how to transfer the data from table to equivalent ratio. She repeated the rule: think/draw a ratio table, erase the lines, and put the equal sign. In the representation, VSF was used, Ahmet and Fatma added that HSF also could be used even if it was decimal, as in Figure 4.105.

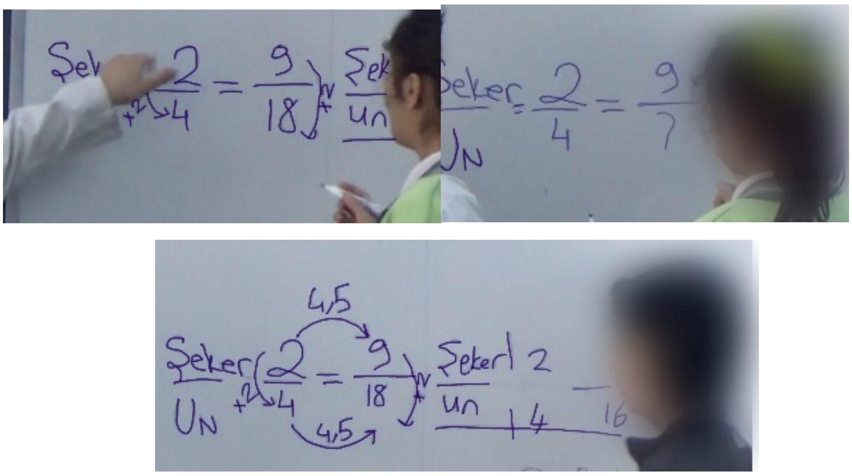


Figure 4. 105. Students' using VSF and HSF within proportion (A12Q2)

In the further activities, this strategy was used for the missing value problems and students used VSF and HSF to find the missing value. They did not question the reasoning behind scale factor utility since it was already discussed in CMP 3 in detail. Erasing the lines from ratio table was also eliminated in the further activities. They directly used equivalent ratios.

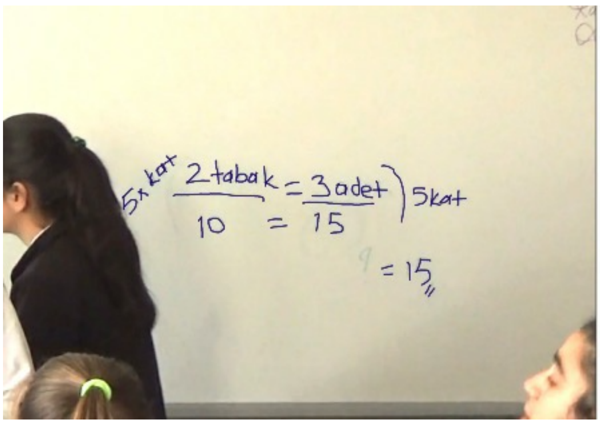


Figure 4.4.2.6 An example of proportion representation from Dane (A16)

In this classroom, the students and the teacher did not object to the claim and did not ask any further questions about equivalent ratios representation. The discussion continued with datum and claim within the frame of argumentation layout, which meant the warrant and backing were dropped off. In this respect, using equivalent ratios became a normative way of reasoning.

Starting with Activity 13, the students created equivalent ratios which involved larger numbers. In this activity, a ratio is provided without picture imagery that explains the rule in the alien feeding episodes. It asks students to find the missing value in the second ratio. The class already studied larger numbers within ratio tables, but additive thinking emerged with the contribution of equivalent ratio representation. Several students were confused about the meaning of equality, and they objected to scaling between two ratios.

Each new representation opened the similar challenge cycle which also involved decimals. The question asks, "How many 7^{th} graders prefer going to action movies if 120 8^{th} graders prefer going them (the rule is 75 7^{th} graders: 90 8^{th} graders)?" Reducing a ratio seemed harder than finding the difference between the numerator and denominator for Berk and Alp. Alp found the difference between numbers and added it to the third variable to find the missing value, but his peers rebutted it. The difference between number values had not been used until now, and Alp admitted that they had been searching for the multiplicative relationship. The emphasis on the meaning of equivalent ratios was still unclear for Berk.

> Berk: Ma'am, I am confused now. He said seventy-five divided by ninety. There are fifteen numbers in between. He said twenty-five divided by thirty; there are five numbers. I couldn't do anything there. (his additive thinking emerges)
>
> Teacher Merve: But we don't look at the difference. We look at whether these fractions are equivalent to each other. Now look; do three-quarters and one-half mean the same thing? (Fraction imagery)
>
> Beck: No.
>
> Teacher Merve: It won't tell. But the difference between them is one.
>
> Beck: Yes.
>
> Teacher Merve: The difference does not tell the same piece. Look. Three over four is as follows. Let that be three over four. That's three-quarters, that's one-half-two. Does it say the same thing?
>
> Several students together: No.
>
> Teacher Merve: Look, this is a little bit more (three quarters). If we look at the difference between the fractions and tell the same amount or not, wouldn't we be wrong?
>
> Several students together: We will.
>
> Teacher Merve: Do the differences always explain the equivalent part, the equivalent amount? (Asking for a qualifier)
>
> Student: No!
>
> Teacher Merve: Let's say a loaf of bread is enough for two people - let's think it's a little big- if one loaf is enough for two people, two loaves are enough for four people, right? Are you looking at the difference? How many for three people?
>
> Berk: One and a half.

Teacher Merve: One and a half is enough. Or three loaves are enough for six people. Are you looking at the difference? While one pita is enough for two people, the difference is one, but three loaves are enough for six people. Did you understand? (representation by using real-life context)

Berk: Got it.

Cansu: We look at the multiplicative relationship.

Berk, like Alp, assumed that the equivalent ratio involved a constant difference between the numerator and denominator parts of two ratios. Teacher Merve again provided ratios with small numbers: 1:2 and 3:4. These ratios have the same difference between numerator and denominator. Berk already knew that these two ratios did not refer to the same fraction. Teacher Merve also changed the ratios into fraction imagery and drew an area model. Berk accepted that 3:4 covers more area in the model. After that, Teacher Merve reinforced Berk's thinking using 1:2 ratio in a real-life context. She gave examples of two people eating one loaf of bread and continued asking how much bread three/four people would eat. Berk did not have a problem with small numbers or with decimals. Teacher Merve helped Berk realize the solution was not related to the difference between numerator and denominator. This part of the discussion backed the argument, which started with Alp's solution. Fraction imagery was a helpful tool to eliminate additive thinking instead of directly showing the scale factor. Refutation worked with non-equivalent ratios, which had smaller numbers and difference of "one" (1:2 and 3:4) between numerator and denominator. Additive thinking did not emerge till the instructional sequence ended. Nonetheless, posttest demonstrated there were students having problems with larger numbers.

Zeynep performed well at procedural knowledge and came up with decimal scale factor between variables for the same question as Berk's. She solved the problem, "First, we find the ratio of seventh graders to eighth graders. 90 to 75. Accordingly, eighth graders ask if there are one hundred and twenty seventh graders. I divided ninety by seventy-five and found the ratio between them. I found it *1,2* too. If the ratio of this is *1,2*, the ratio of this must be *1,2*. That's why I divided 120 by 1,2. I found the answer as 100.". In brief, she used VSF to determine the missing value. In CMP 4 Idea 2, the students already conducted VSF and HSF operations between the numbers (see Figure 4.106).

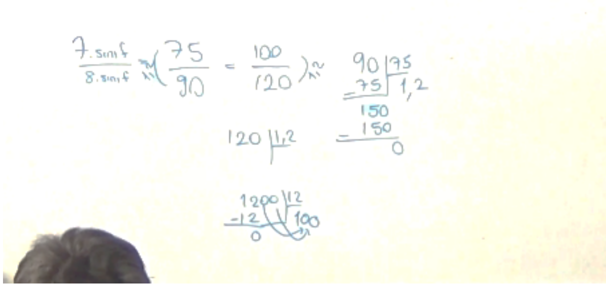


Figure 4. 106. Zeynep's explaining decimal scale factor

At first, Zeynep only showed equivalent ratios with the vertical scale factor representation. The students did not understand why the conclusion was 100. Zeynep wrote the arithmetical operations on the whiteboard to explain the result. Zeynep shared her claim and datum, but the other students said that they did not understand how she found the result. The question of the students was not about displaying equivalent ratios; nevertheless, conducting operations with decimals was still a challenge for several students in the class. As a result, equivalent ratio/ proportional understanding came along with scale factors within and between variables, not additive thinking. However, the situation lies under the students' procedural skills with large numbers. What became a practice in this idea was that students' knowing how to find the result but their incompetency of calculations made them use additive thinking. The instructional sequence focuses more on the smaller numbers, which makes everyone focuses on the conceptual understanding of the topic. On the other hand, procedural skills should have been developed with more drill and practice.

4.4.3. Summary of Classroom Mathematical Practice 4

In the preceding three ideas, the class engaged with the conceptual understanding of ratios and proportions by means of fraction imagery and ratio table tool. While exploring, the way of delivery through question context came to the fore, which influenced the students' ability to evaluate the symbolic representation of ratios and equivalent ratios. Classroom Mathematical Practice 4 (CMP 4) consisted of seventh-grade students' two ideas that emerged while trying to understand the symbolic representation of the ratio and proportion concept. These three normative ways of reasoning were (1) The symbolic ratio representation changes as does the order of the composite units in verbal representation; (2) Ratio table is used as a tool for proportion representation; therefore, VSF and HSF can also be adopted in proportion. The used strategy is not the difference between numerator and denominator in the equivalent ratios (additive thinking) but scale factors. Fraction imagery helped introduce symbolic ratio representation and equivalent ratios.

Table 4. 4

A Brief Summary of the Criteria for CMP 4

CMP 1	CMP Criteria
TAS 1	C3: Transfer of using verbal representation and symbolic representation interchangably
TAS 2	C2: Dropping of warrants for reducing ratio C3: Transfer of VSF and HSF in short ratio table to equivalent ratios C3: Transfer of VSF or HSF for larger numbers.

Moreover, transforming the ratio table tool into equivalent ratios recognizably enhanced the students' conceptualization. As conjectured, additive thinking was observed in larger numbers within the symbolic representation of equivalent ratios. Here is noteworthy, additive thinking, one of the anticipated answers in the previous questions, did not emerge through the ratio table and drawing strategy. The students' acceptance of the claim and data and the absence of clarification questions indicated the emergence of the datum and the claim type of argumentation layout. In this respect, the warrant and backing were dropped off, evidence of classroom mathematical practice.

4.5. CMP 5: Adapting Strategies for Comparing Non-Equivalent Ratios

Missing value problems are dominant in the instructional sequence until Activity 18 which also involves comparison of non-equivalent ratio questions. There is also rate concept embedded in these activities. When the students first met with rate concept, they entered a state of disequilibrium between missing value and comparison problems. They could not directly adopt strategies emerged for missing value problems. Although Teacher Merve and the researcher did not mention the type of questions, the students were aware of the distinct situation. They transferred bits and pieces from their former learning and normative way of reasonings. Classroom Mathematical Practice 5 encompassed the students' comparison of non-equivalent ratios and their normative ways of reasoning. In addition, ratio table was not an effective tool anymore.

4.5.1. TAS 1 Using Unit Ratio Concept to Compare Ratios

As the guideline for the instructional sequence recommends, teachers should encourage their students to explain how they find their results. This implies that students should know the associations of each number within their solution. Building on this understanding, Teacher Merve and the researcher tried to make the mathematical numbers and symbols experientially real from the beginning because the answer is directly related to its problem context. What does it mean? Previously, the units' attributes, such as continuous and discontinuous variables, were mentioned. Previously, the questions involved "is enough", and "how many", which was clear for the students to understand. There are also some problem contexts that ask "intensity, speed, darker, and the like" to determine a solution strategy. From now on, there is only ratio comparison problem type dealing with those concepts from Activity 18 to Activity 21. Activity 18[10] asks to determine the most intense grape molasses with the ratio given in Figure 4.107.

Figure 4. 107. Production of grape molasses and talking

The question in A18 involves grapes juice and a special powder to make an intense molasses. There are three recipes, involving those two continuous variables with units spoon and liter. Each recipe has different intensity. Before doing this question, there are several steps to achieve: What makes molasses intense? How to represent it symbolically as a ratio? How to

[10] This activity was adopted into the Turkish context.

compare three ratios? Till present activity, the students studied on two ratio comparison. Each step has a challenge for them. Acar was a student who shared his ideas rarely on the board and Teacher Merve encouraged all the students to share their ideas even if it was wrong or incomplete.

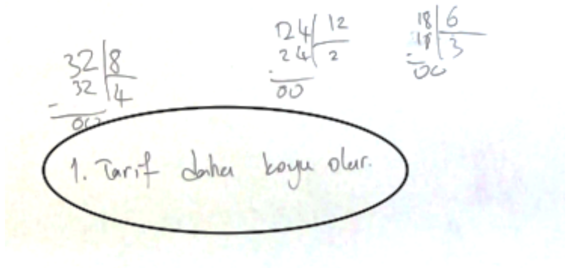


Figure 4. 108. Acar’s solution for A18

Acar came to the whiteboard and directly conducted division of the total number of grape juices by the total number of spoons. It looked like he selected division algorithm with a quotient of our since it was the largest number among three quotients. Teacher Merve expected an explanation for this datum to give meaning. Batu accepted datum and claim. He perceived division as the unit ratio and evaluated the quotient as the spoon per one liter. Batu completed the unfinished explanation of Acar through unit ratio strategy as given below:

Batu: Teacher, I'll explain.

Teacher Merve: Okay, let's listen to Batu.

Batu: Ma’am, this is what should be the intense here. It is because to a liter it takes four spoons. It gets more intense. (Showing 4 in the quotient he found in his division) (*Warrant*)

Teacher Merve: Look what Batu says? Do you listen to what Batu says? How many spoons per liter?

Several students together: Four.

Batu: Teacher, this is what should be the most intense here. Two spoons for that, three spoons for that. This is more intense because the number of spoons is more.

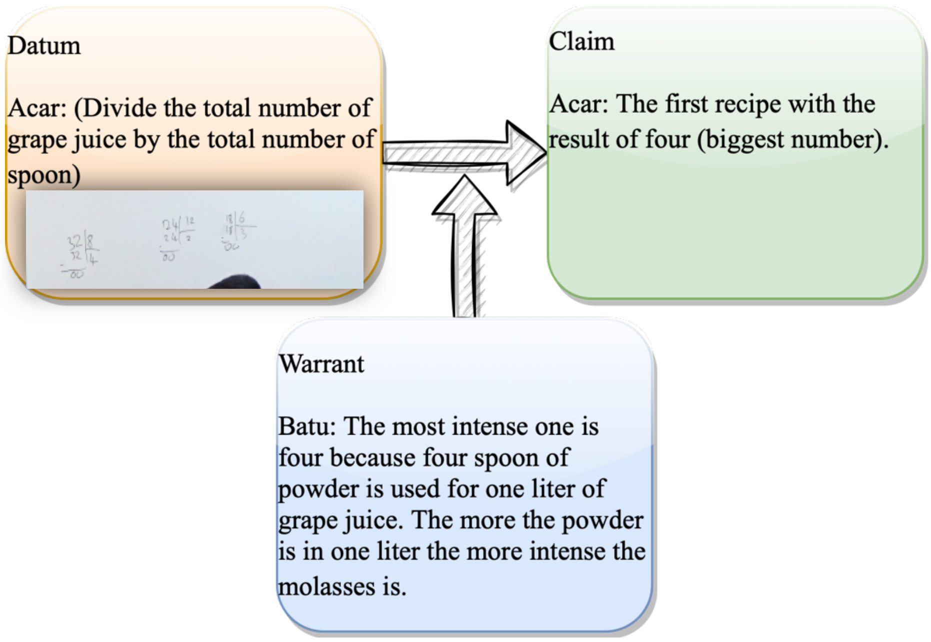


Figure 4. 109. Argumentation layout for the unit ratio strategy with comparison problem

Simge was good at scale factors lately during the instructional sequence and used ratio representation correctly with units. She ordered the recipes in ratio representation and revealed the multiplicative relationship between grape molasses and powder. She selected the largest vertical scale factor to claim the most intense recipe. It was accepted as correct by her peers, but it could also be a coincidental issue for Simge to reach this result because comparing numbers without giving contextual meaning could lead the students an incorrect result like Acar.

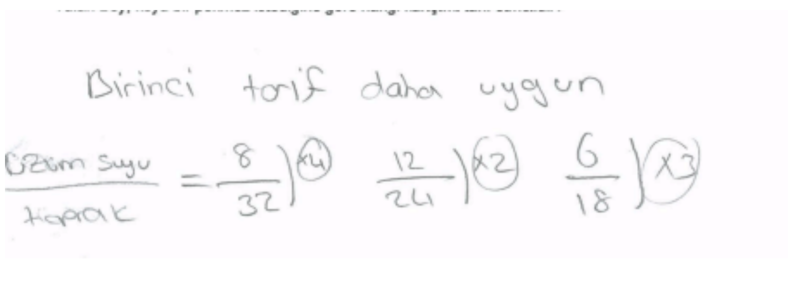


Figure 4. 110. Simge's VSF for comparison intensity

While Teacher Merve and the researcher were going around the class in the explore phase, several issues were observed as it was conjectured. For example, the most powder there is the most intense the molasses is or the least the grapes juice is the most intense the molasses is. Teacher Merve tried to eliminate the students' incorrect reasoning through drawing on students' previous learning. She asked students to compare 8:32 and 16:32 related to the amount of grapes juice. By making the amount of grapes juice constant helped students think reasonably. 4 spoons of powder for 1 liter and 2 spoon of powder for 1 liter comparison explained what intensity is.

Comparison problem emphasizes the contextual meaning of ratios, and students need to define first what makes a ratio the most or the least. Batu provided a warrant Teacher Merve used several times during the discussion. Teacher Merve highlighted that the amount of the total or one material could not determine a ratio to be the most or the least. Ayten repeated Teacher Merve's explanation in her activity sheet. She wrote that she selected the first recipe by considering the amount of the powder alone.

Molding the symbolic representation with the contextual representation is significant due to fraction imagery. While writing equivalent ratios, this imagery helped Teacher Merve talk about proportion. Previously, Simge showed three recipes one by one by comparing them with scale factors, and Batu's explanation completed this whole argumentation. The students would be confused if the connection between the context and symbolic representation was not constructed in harmony In Figure 4.111, Arda ordered the numbers by using simple fraction imagery of the ratios. Notably, there was a tendency to write the smaller number to the numerator while representing a ratio. Simple fraction imagery could lead the students to incorrect results without giving contextual meaning.

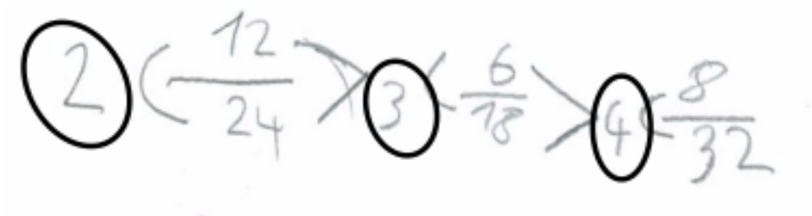


Figure 4. 111. The confusion of Arda about comparing the ratios

Arda first ordered the recipes as if they were fractions and found the second recipe the most intense molasses at first. He was not sure about this issue and used a "> (greater than)" sign first. Next, he added a "< (less than)" sign with their scale factors after the discussion. Fraction imagery led the students to order fractions. At this point, Batu's warrant and Simge's scale factor were helpful for Arda to change his way of ordering.

Activity 19[11] is similar to Activity 18 and encompasses "which pasta (packages) has the most sauce (tablespoon)?". The rates involve two packages to 10 tablespoons, one package to 4

[11] This activity was adopted into the Turkish context.

tablespoons, six packages to 18 tablespoons, and eight packages to 48 tablespoons. Zeynep transferred Simge's solution to the board at first. For datum, she adopted Batu's explanation and found the unit ratio for each recipe. In conclusion, she showed the largest number of teaspoons to determine the pasta with the most sauce. Most of the students agreed on this strategy as a solution.

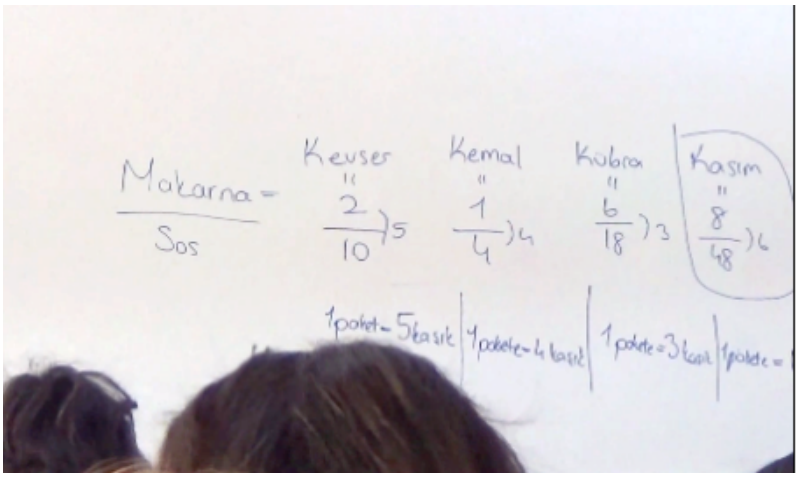


Figure 4. 112. Zeynep's solution for A19

To reach this claim and solve the problems, it was assured that the class developed a shared understanding of the unit ratio strategy and transferred this strategy successfully to the comparison problems. This strategy was repeated in other students' activity sheets. In the other lessons, Zeynep preferred writing the rate of paste to the sauce. There was a tendency in this class to write the ratios with a smaller number of variables as the numerator and a larger number of variables as the denominator. While comparing, she also used Batu's warrant and wrote on the board that a mathematical statement used five tablespoons of sauce for one package of paste.

4.5.2. TAS 2 Comparing Ratios with Least Common Multiple Strategy

Instructional sequence warns that students may have problems with decimal unit ratios and rates. Relatedly, the students were comfortable with using unit ratio with integers until Activity 21, which had a task incorporating several recipes with decimal unit ratios. This time, rates are 5 cups of water to 8 drops of potion (recipe 1), 3 cups to 6 drops (recipe 2), 6 cups to 9 drops (recipe 3), and 3 cups to 5 drops (recipe 4). The discussion started with a similar strategy that Simge used previously. She came to the whiteboard to explain her reasoning again.

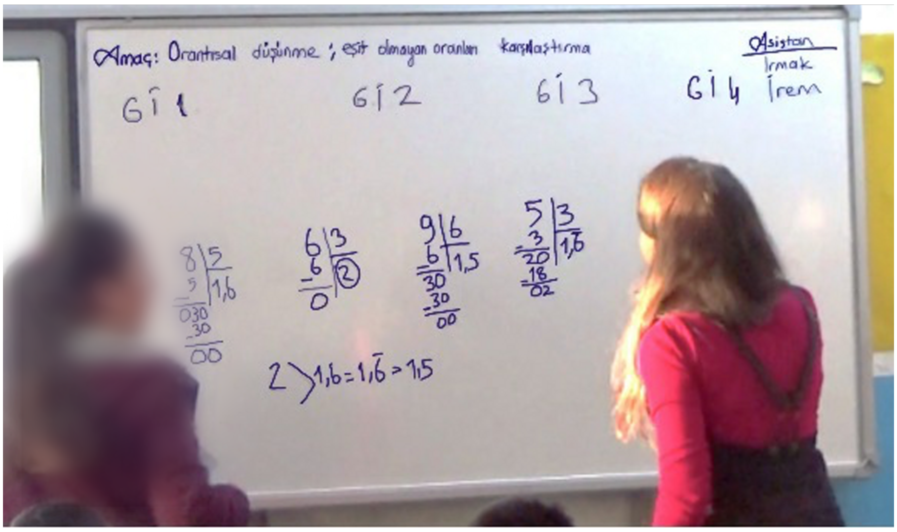


Figure 4. 113. Simge’s solution for Activity 21

Simge divided the larger number into the smaller number and repeated this procedure for all the division operations. She compared the values by using the *greater than* sign, as Arda did previously. She found 1,6–2–1,5–$1,\overline{6}$ respectively and calculated all decimals, including repeating ones, but she concluded that both recipes (1-4) were equal as written on the board (see Figure 4.113). According to her, the largest number was found to be the most effective potion, similar to the previous arguments claimed by the other students. The researcher attempted to reinforce this understanding by asking questions because the students had solution strategies which consisted of additive thinking. The following argumentation describes an example of additive thinking that emerged in the comparison problem.

> Buket: Ma’am, I did it like this. That is the upper range. I compared it like this, Ma’am. It says five cups, three cups, six cups, three cups. Here's the biggest of the six cups. Eight drops, six drops… Nine drops is the biggest one among them, so I said the third recipe [9:6] (*Datum, Claim*)
>
> Teacher Merve: Okay! Let's ask a friend. The children who did this said that since six and nine are both larger, this is the most effective recipe. She said that this is the greatest, since it is big in both the six and the nine. Is it correct? (*Warrant*)
>
> Several students together: No! (Rebuttal)
>
> Teacher Merve: Well, What should we look for then, instead not the largest? Ahsen, what should we be looking at?
>
> Ahsen: Teacher, the number of drops falling into a cup. (Rebuttal became a Datum)
>
> Teacher Merve: While these are different, we can't compare them, right? I mean, because you say six drops to six cups, but you say nine cups to six cups, this time it should be the same for you. So we need to fix one of the number, right? Let's extend one of the numbers so that we can make the right decision. That's why your friends say let's see how much

> they all drop in a cup of water.Let's reduce them into one cup. The most drops there is in a cup, the most efficient potion it is. Okay? (Warrant)

The above-mentioned, Buket went one step behind additive thinking. She first compared the largest number of potion drops to determine the most effective potion and reached the largest number of cups to determine "the biggest". Since she calculated the previous problems based on the ratio relationship, she went out of context and offered this datum. Some students did not object to this datum, and also, students firmly rejected her solution. Comparing the first and second units separately was not acceptable. She was using ordering fraction imagery from her previous years[12]. The students did not find any example to refute her immediately. Still, Teacher Merve asked a question to deny her datum and hinted about finding the least common multiple of the denominator or numerator of all the fractions. The class still needed much time to discuss this issue.

While diving into the students' strategies, Teacher Merve came across additive thinking as conjectured in the instructional sequence. Although Batu solved unit ratio problems smoothly, he claimed that adding the numerical values of the numerator and denominator could be used to find the most effective potion. Deren objected to this claim with a datum emphasizing "searching for a multiplicative relationship but not additive". This emphasis was done several times during the instructional sequence as described in the other classroom mathematical practices. The students concluded that they should not use addition and subtraction for comparison problems, but they needed more data and warrant not to switch to additive thinking.

> Zeynep: Ma'am, I have an idea. I compared two and four. (*Datum*)
>
> Teacher Merve: Did you compare two by two?
>
> Zeynep: For example, I said that the cups for two and four are the same. I eliminated the other one, not the big one. So I thought I should equate it with something like, for example. I tried the others. (*Datum*)
>
> Teacher Merve: Now, Zeynep, can you tell us how you did it? Can you guys listen?
>
> Zeynep: I did it by comparing the two. I compared the second potion with the fourth potion first. Since there are three cups for three cups, so the youth potion, the drop of youth, will rejuvenate more. That's why I said it's two. Then I compared one to three and

[12] There is a strategy to order fractions on the number line. The strategy involves making denominator/numerator equivalent through finding the least common multiple of denominator or numerator of all the fractions. Next, the extended fraction is rewritten since they are equivalent fractions. Lastly, students compare only numerator or denominator of different fractions.

tried to make the cups the same [LCM]. Therefore, thirty cups became forty-eight drops; thirty cups became forty-five drops. The drop of potion will make you more rejuvenated with the most. That's why I eliminated the third recipe. The last time I compared one and two, they were equal. I matched the number of cups and water. Twenty-four drops in fifteen cups became thirty drops in fifteen cups. Again, I chose the second option since the one with the most drops would rejuvenate more. (Revoicing Datum, Warrant, and Claim)

Deren: Well, it's not thirty, but there are forty-eight as well.

Zeynep: We will compare in pairs.

Teacher Merve: If thirty is forty-eight, you divide it in half, and fifteen becomes twenty-four.

Deren: I see, okay, okay.

Teacher Merve: Look, this time, your friend fixed the number of cups by not dividing but multiplying, right? So it is fixed. He said that whichever has more drops in an equal number of cups, the density of the youth potion will be higher. Ok…(Revoicing).

Teacher Merve mentioned the least common multiple (LCM) for all the ratios, and she tried to equate one of the variables to make students understand the ratio concept. From this perspective, one variable became the same for all the ratios, and the other variable changed with the appropriate scale factor. Zeynep compared recipes two by two, and this strategy helped her persuade her friends. She first selected the exact value of the units in the denominator and eliminated the smaller ones (3 cups: 6 drops and 3 cups: 5 drops). In Figure 4.114, Zeynep compared the second and fourth recipes and eliminated the fourth one. Then, she compared the first and third recipes.

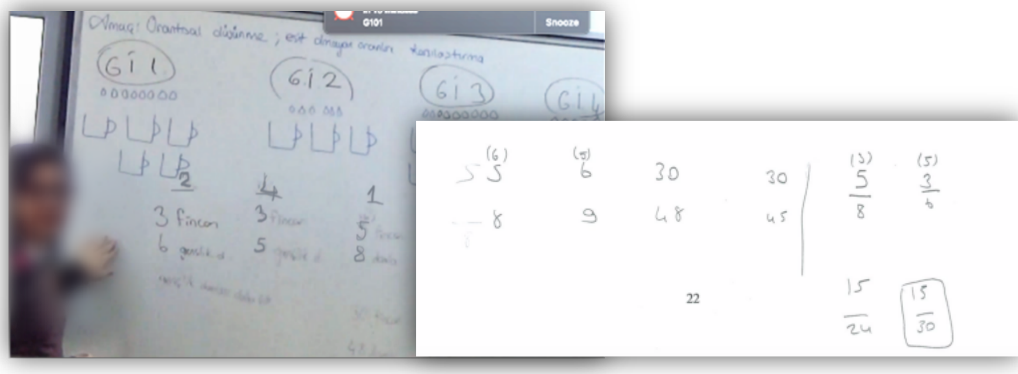
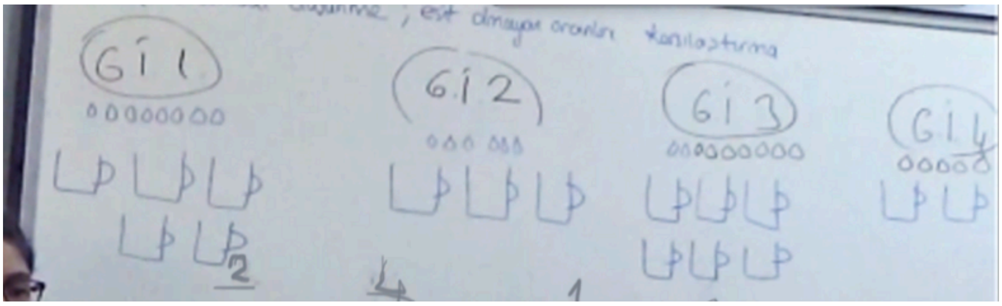


Figure 4. 114. Zeynep’s solution for Activity 21

Due to the lack of the same number value of cups or drops, she applied Teacher Merve’s offer of 30 as the least number that is divisible by both 5 and 6. The ratios of the first and third recipes became 30 cups: 48 drops and 30 cups: 45 drops. She eliminated the third recipe. Next, there were the first and second recipes to compare. She extended the second recipe to *15 cups* *to 30 drops* (this time, LCM became 15 for 3 and 5) and finally found the first recipe was the most effective potion. There was a point where she did not use fraction imagery, but she extended the numbers simultaneously with the same multiplier. She decided that the last potion was the most intense recipe among the others. Making the potion elements concrete by drawing strategy was helpful with the guidance of Teacher Merve. She supported the development of the ideas by starting by comparing smaller numbers. Teacher Merve saw two steps in Zeynep's understanding. The first was the comparison of ratios with the same number value, and the second was the comparison through LCM. She started checking the students' understanding of comparing ratios with the same numerical values. Nevertheless, the problem was in the second part.

As a teaching strategy, the researcher drew pictures to make students imagine the rates: drops and cups. In the first activity, Teacher Merve and the researcher observed its influences on students using unit ratio and other issues related to multiplicative thinking. Zeynep's solution did not get the expected reaction from the class, and Teacher Merve and the researcher were aware that comparison problems were a brand-new topic than missing value problems. The attribute of the units was already discussed, but this time knowing and evaluating the concept was much more involved for a valid claim. While wrapping up, Can could not keep up with the drawing strategy. He continued comparing the number of drops in each recipe to decide the most effective potion. The researcher reminded their previous learning of unit ratio for two continuous units via drawing.

> Researcher: For example, now I distribute these drops evenly. How many were here, eight of them? I started to distribute them one by one. (*Backing*)



> Teacher: Merve: Yes, you distributed the remaining three one by one.
>
> Researcher: Now, how can I distribute it (Showing Recipe 1)? I can usually divide it into five parts. For example, I distributed them one by one. Now let it stay like this. Look here. Two by two, the drops fell (Showing Recipe 2). First of all, will those here be more intense? Will they be over there? (*Backing*)
>
> Students: Those on the second side.

Researcher: In the second one, because there are two drops in a glass, but here there are not two drops in each cup. Isn't it? Here we can divide each drop and put it in, but I didn't do it that way to avoid confusing it further. (*Backing*)

Can: Ma'am, you found two. It will be bigger, okay.

Teacher Merve: Think of it as density. Here, since two drops do not fall into each cup, it is less dense than that.

Researcher: For example, let's pour it one by one here; how many are left again (Showing the third recipe)? Yes. For instance, we can quickly put half and half here. Is this now bigger?

Students: The seconddddd…

Researcher: Let's try this right here (Showing the fourth recipe). Let's distribute them one by one; there are two drops left. We will divide these two tiny drops into five, but will it be larger than half? Is it smaller than half?

Students: It would be smaller.

Researcher: OK. You thought well of that. At that time, was it the second or fourth recipes denser?

Students: Twoooo…..

Researcher: At that time, did two turn out to be bigger than all? In terms of density, the two gave the largest output.

Akın: Ma'am, I already thought that the higher the density, the more effective it would be.

Researcher: Of course. If we did it, it came out with a comma. Here, I drew a figure so that you can clearly see what Simge is doing.

Teacher Merve: Do you understand more clearly now?

Students: Yes.

Teacher Merve: So, we don't look directly at the numbers. What are we looking at? We look at the density. How much is dropped in a cup, or how many drops fall in an equal number of cups of water? It does not necessarily have to be one, but there can be an equal number of cups.

Teacher Merve and the researcher provided backing for Zeynep's procedural operations to make them meaningful. Handling the context and attribute of the units with drawing, Teacher Merve used the "intensity related to drops" concept for this task instead of "the most effective potion" since she drew on the common language of the class by using lucid words related to rate and ratio concepts. Both CMP 2 Strategy 1 and CMP 5 Idea 1 were used for this representation as they were already achieved by the students.

Activity 22 is similar to the previous comparison problems and provides recipes for a mixture of orange juice and mineral water with sprinkles,[13] as shown in Figure 4.115. It asks, "the

[13] This activity was adapted into the Turkish context.

orangey-est mixture." This time the measurement unit is the same: cups, which allowed the students to create a new variable (see CMP 3 Idea 5). Since the measurement unit and the liquid variable were studied previously, Teacher Merve and the researcher detected normative ways of reasoning for comparison problems in students' solutions.

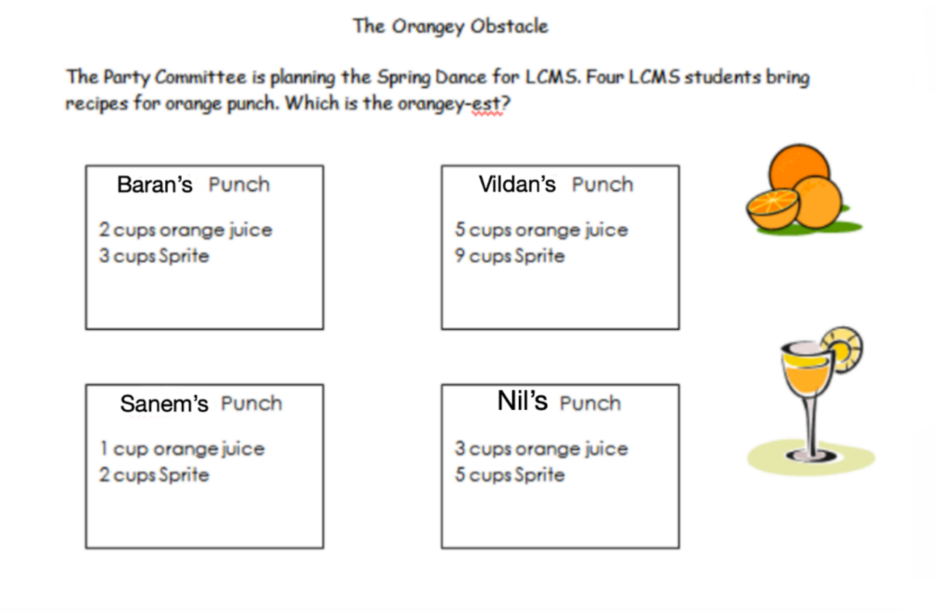


Figure 4. 115. Activity 22

Similar to Zeynep in Activity 21, Ayten came to the whiteboard and divided the amount of orange juice by the amount of mineral water. As Teacher Merve knew from the previous lessons, Ayten performed well with decimals and repeating decimal numbers. She conducted procedures successfully. At the same time, she used a unit ratio strategy to determine the amount of orange juice per cup of mineral water. Nevertheless, she selected the least amount of orange juice by selecting Sanem's punch. Although the procedures were conducted flawlessly by Ayten, she was unable to connect what it meant the quotient in the division algorithm (see Figure 4.116). Teacher Merve reversed the situation from the amount of mineral water to the amount of orange juice so that it was possible to select the least amount in the quotient.

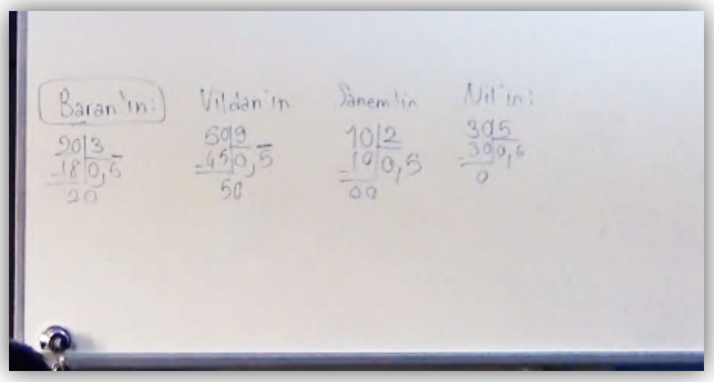


Figure 4. 116. Ayten’s last decision for the orangey-est punch

Teacher Merve noticed the unit ratio/rate strategy and the least common multiple strategies for comparison problems. She was aware that several students got confused about decimals in comparison and the contextual meaning of the rates. Simple fraction, mineral water divided by orange. Reversibly, Nur found the amount of mineral water per orange juice by dividing the amount of mineral water by the amount of orange juice, as provided in Figure 4.117. However, the students could not go further from comparing the value of the numbers separately.

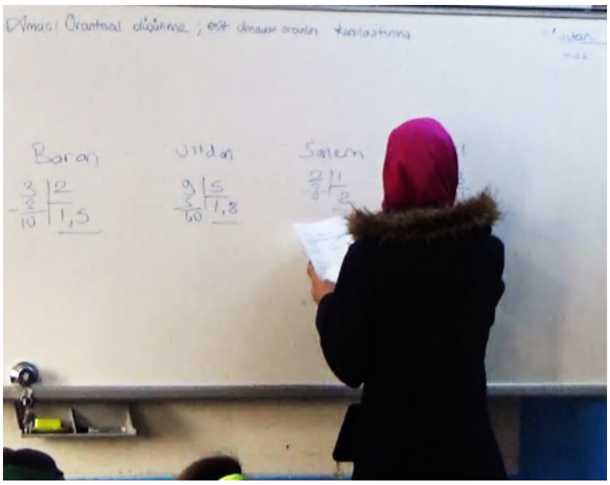


Figure 4. 117. Nur reversed the unit ratio

Based on this, Teacher Merve offered the least common multiple strategy as backing and compared the ratios two by two with Batu. Teacher Merve guided Batu to conduct the procedures. They compared Baran’s and Vildan’s punch, and then compared Sanem’s and Nil’s punch. As a result, Baran’s and Nil’s punches were compared and lastly Baran’s punch was selected as the most orangey-est juice. Akın admitted that he understood better in this way even if he used Nur’s strategy.

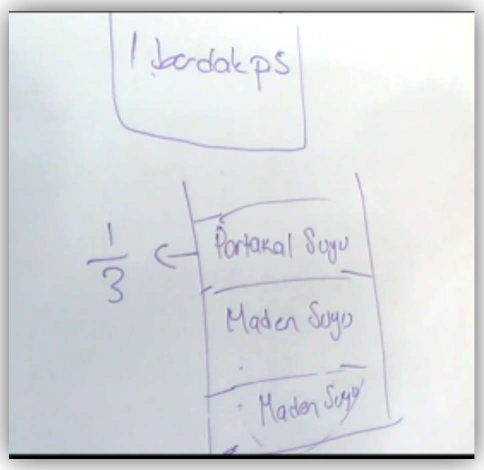


Figure 4. 118. Fatma’s representation of the ratio

Activity 22 lasted approximately three lesson hours. Equating the number of variables required the students to practice because procedural knowledge of LCM influenced their selected strategies. As an alternative, imagining the variables by drawing strategy was also presented as an opportunity to support their claim. Teacher Merve encouraged Fatma to show her drawing for the orange juice and mineral water mixture on the board. Fatma’s pictorial representation was also a helpful tool to make the punch experientially real to compare. The students struggled with comparison problems with non-integer unit ratios/rates. Therefore, the discussions were conducted with several students, and Teacher Merve and the researcher guided the class in proportional reasoning. Other than that, the solutions of the rest of the class helped define the limitations of some aspects and showed the class what not to do. It was also a beneficial learning technique that supported the students’ refutation and provided reasoning about the concept. Comparison problems are directly related to their context. Proportional reasoning cannot be reduced to just numbers.

4.5.3. Summary of Classroom Mathematical Practice 5

A comparison of ratios was discussed between Activity 18-22. In the preceding two ideas, the class engaged with the conceptual understanding of ratios by means of contextual issues of the task. The analysis evidenced the students' transfer of their previous learning during the instructional sequence to new tasks, and this evidence displayed the repetition of already developed classroom mathematical practices. The class basically studied how to deal with non-equivalent ratios without using usual fraction imagery.

Table 4. 5

A Brief Summary of the Criteria for CMP 5

CMP 1	CMP Criteria
Idea 1(I1)	C3: Transfer of unit ratio strategy to compare three or more ratios.
Idea 2(I2)	C2: Using Least Common Multiples to compare three or more ratios

Classroom Mathematical Practice 5 (CMP 5) consisted of two ideas for seventh-grade students' strategy adaptation processes for comparison problems. These two normative ways of reasoning are (1) The more material *a* is in one amount of material *b*, the more intense the mixture is; (2) The least common multiple for ordering ratios can be used for understanding the most or the least of an attribute in a mixture. Lastly, the students were provided a pictorial representation of materials to imagine the least or the most concept as a backing.

4.6. Pre-test and Post-test Results

Five classroom mathematical practices presented above reflected the formative assessment of the students' understanding related to ratio and proportion concepts in terms of collective construction of the knowledge. Besides, there is also summative assessment conducted with main group and pilot group to investigate the effect of the implementation of instructional sequence on students' achievement in the test.

The results of each class and school and the whole group were compared to determine whether the results were consistent or whether there were anomalies in the data. An independent-samples t-test was conducted to compare the students' scores for 7/X and 7/Y before the ratio and proportion instruction started. This is to understand the students' prior knowledge about the topic. There was no significant difference in scores for 7/X (M = 31.53, SD = 16.16) and 7/Y (M = 28.79, SD = 16.35; $t(67)$ = 1.62, p = .49, two-tailed). The magnitude of the differences in the means (mean difference = 2.74, 95% CI: –5.08 to 10.56) was very small (eta squared = .007). Following tables shows the statistical analysis of the pre-posttest results.

Table 4. 6
Descriptive Information Related to 7/X and 7/Y

	Group Statistics				
	Group	Mean	N	Std. Deviation	Std. Error Mean
XYPRETEST	7/X	31.53	36	16.16	2.69
	7/Y	28.79	33	16.35	2.85

Table 4. 7
Independent Samples T-Test for Pre-test of 7/X and 7/Y

	Levene's Test for Equality of Variances		t-test for Equality of Means						
	F	Sig.	t	df	Sig. (2-tailed)	Mean Difference	Std. Error Difference	95% Confidence Interval of the Difference Lower	Upper
E.V assumed	.005	.944	.700	67	.487	2.74	3.92	-5.08	10.56
E. V. not assumed			.699	66.34	.487	2.74	3.92	-5.08	10.56

For analysis of pre-and post-tests scores of 7/X students (Table 4.8), paired-samples t-test was applied to evaluate the difference. All assumptions for the paired samples t-test were checked based on independence of observation and normality. There was a statistically significant increase in the test scores of the students before (*M = 31.53*, *SD* =16.16) and after (*M* = 46.25, *SD* = 22.88), *t* (35) = -5.28, *p* <. 0005 (two-tailed). The mean decrease in scores was -14.72 with a 95% confidence interval ranging from 20.44 to 9.00 (Table 4.9). According to Cohen (1988, pp. 284-287), the eta squared statistic (.44) indicated a large effect size (.01=small, .06=moderate, .14=large effect). Following tables shows the statistical analysis of the pretest and posttest results.

Table 4. 8
Descriptive Information Related to Pretest and Posttest Results of 7/X

	Paired Samples Statistics			
	Mean	N	Std. Deviation	Std. Error Mean
XPRETEST	31.53	36	16.161	2.693
XPOSTTEST	46.25	36	22.878	3.813

Table 4. 9

Paired-Samples T-Test for Pre-test and Post-test of 7/X

Paired Samples Test								
	Mean	Std. Deviation	Std. Error Mean	95% Confidence Interval of the Difference		t	df	Sig. (2-tailed)
				Lower	Upper			
XPRETEST – XPOSTTEST	-14.72	16.90	2.82	-20.44	-9.00	-5.23	35	.000

The tests were also conducted with 7/Y classroom, which was the pilot group. For analysis of pre-and post-tests scores of students, paired-samples t-test was applied to evaluate the difference. All assumptions for the paired samples t-test were checked based on independence of observation and normality. There was a statistically significant increase in the instrument scores of the students before ($M = 28.79$, SD =16.35) and after (M = 39.85, SD = 22.02), t (32) =-4.32, p <. 0005 (two-tailed). The mean decrease in scores was -11.06 with a 95% confidence interval ranging from -16.28 to -5.84. The eta squared statistic (.37) indicated large effect size by the commonly used guidelines proposed by Cohen (1988, pp. 284–287): .01=small effect, .06=moderate effect, .14=large effect. Following tables shows the statistical analysis of the pretest and posttest results.

Table 4. 10
Descriptive Information Related to Pretest and Posttest Results of 7/Y

Paired Samples Statistics				
	Mean	N	Std. Deviation	Std. Error Mean
YPRETEST	28.7879	33	16.34732	2.84570
YPOSTTEST	39.8485	33	22.02573	3.83419

Table 4. 11
Paired-Samples T-Test for Pre-test and Post-test of 7/Y

Paired Samples Test								
				95% Confidence Interval of the Difference				
	Mean	Std. Deviation	Std. Error Mean	Lower	Upper	t	df	Sig. (2-tailed)
YPRETEST – YPOSTTEST	-11.06	14.72	2.56	-16.28	-5.84	-4.32	32	.000

CHAPTER 5

DISCUSSION

The current study aimed to investigate how middle school students develop an understanding of ratio and proportion concepts in a classroom setting, focusing on how these concepts are shared and developed within the classroom community. The study aimed to identify the progression of these mathematical concepts as they emerged and spread through classroom discourse and to understand the conditions under which this development occurred. The findings of the study offer insights into how students can effectively learn and comprehend ratio and proportion concepts through an instructional sequence focused on these topics. In this chapter, the results of the study were presented in the context of various mathematical concepts, and the implications of these results for future research on this topic were also explored.

5.1. Discussion about mathematical practices

The current study was conducted in two classrooms. Although classroom mathematical practices of one classroom (7/X) were presented, two classrooms performed similarly before and after the implementation of the sequence. Conducted pre-test and post-test results revealed that seventh graders' performance in proportional reasoning significantly changed due to instructional sequence implementation. However, this quantitative change was expected before the study. As a complementary issue, qualitative findings (thick description of CMPs) provided how students' communal ways of thinking changed in real classrooms. Some changes were reported in the relevant literature, but they needed to be viewed from a holistic perspective.

The hypothetical learning trajectory of the ratio and proportion instructional sequence was planned around the current study's Big Ideas, demonstrated in Appendix I, starting from linking composite units to the vertical/horizontal scale factors. Aligned with the design research process requirements, the HLT was constructed on some conjectures of the researcher and the participating teacher. The story of the instructional sequence is based on a Halloween theme, and several themes were not found experientially real by the design team for the cultural context. They were changed, maintaining the semiotic and contextual structure of the tasks. 23 Activities were employed in this classroom teaching experiment. In contrast, the implementation of the instructional sequence and retrospective analysis led to some minor revisions and instructional suggestions for further research, although most of the activities related to ratio and proportion were applied. Based on the taken-as-shared ways of reasoning of the participations, the conjectures were aligned with the anticipated hypothetical learning trajectory in the part of *Positive Topics of Discourse and Emerging Ideas*.

Table 5. 1

Initial Hypothetical Learning Trajectory (Shortened)

HLT Phase	Big idea	Tools/imagery	Activity Pages
Phase 1	Linking composite units	Connecting pictures of aliens to food bars	Page 1
Phase 2	Iterating linked composites	Informal symbolizing (e.g., tables, two columns of numbers, pictures of aliens and food bars)	Pages 2-4
Phase 3	Build up strategies	Ratio table	Pages 3-4
Phase 4	Additive versus multiplicative reasoning	Fold back to pictures; Shortened ratio table	Page 5
Phase 5	Structuring ratios multiplicatively	Shortened ratio tables through multiplication and division with scale factors	Pages 3-7
Phase 6	Creating equivalent ratios	Ratio tables with missing values; Traditional proportion representation (two ratios separated with equal sign)	Pages 8-17
Phase 7	Analyzing equivalent ratios	“Fraction” imagery	Page 11 (bottom) and Page 14 (top)
Phase 8	Comparing ratios	Ratio table No ratio tables, but can use arrow notations	Pages 18-23

The ratio and proportion instructional sequence was developed to support a hypothetical learning trajectory consisting of several phases and to help teachers perform a more systematic decision-making process for the teaching and learning process (Stephan et al., 2015). The sequence includes both vertical and horizontal mathematization processes as described in Realistic Mathematics Education. The authors of the study used the frame proposed by Battista and van Auken Borrow (1995) to construct horizontal mathematization in the instructional sequence. The implementation of the instructional sequence is described, including what worked well, and suggestions are presented based on CMPs to prepare for the next implementation of the instructional study. The phases covered several activities together. This study provided an opportunity to discuss the strengths and areas for improvement related to the taken-as-shared ideas and the anticipated and emergent instructional enrichments for the ratio and proportion instructional sequence.

Table 5. 2

Revised Hypothetical Learning Trajectory for Ratio and Proportion

HLT Phase	Big idea	Tools/imagery	Activity Pages
Phase 1	Reasoning about invariant structure of ratios through discrete/continuous units	Illustrations of the task variables	Page 1
Phase 2	Linking and iterating linked composites	Informal symbolizing (e.g., drawing, operations, two columns of numbers, pictures of aliens and food bars, unit ratio, ratio table)	Pages 2-4
Phase 3	Covariation among composite units	Build-up and multiplicative relations within ratio table; Shortened ratio table; Scale factors	Pages 3-7
Phase 4	Analyzing ratio and proportion in symbolic representation	Ratio tables with missing values; Traditional proportion representation (two ratios separated with equal sign); "Fraction" imagery	Pages 8-17
Phase 5	Comparing non-equivalent ratios	No ratio tables, but illustrations, drawing are encouraged	Pages 18-23

The ratio and proportion instructional sequence is designed to be compatible with the Launch-Explore-Discuss teaching model in terms of its content. During the launch phase, information was given about the topic of the day, including the teacher's introduction, warm-up activities, connecting with the previous topic, and drawing the students' attention to the current topic. In the explore phase, students were directed to discuss among themselves or work individually, and the teacher explained the importance of avoiding behaviors such as telling or showing the solution to students while working together. After everyone finished their solutions, the students presented their suggestions on the board. Questions were asked about these solutions, and students attempted to answer them. The "Anticipated Student Answers" section in the ratio-proportion teaching sequence was used to guide the researcher and teacher during the discussion phase to achieve the lesson's purpose. However, the student solutions diversified over time and differed from the "Anticipated Student Solutions" section. When this situation occurred, the teacher and researcher arranged these different student solutions to align with the lesson's purpose and ensured that the discussion continued.

It has been observed that the previous teaching model is mainly based on a communication style where the teacher tells, and the student listens (Bowers & Nickerson, 2001). The idea that the teacher is the only correct source of information in the previous learning environment has shaped some teacher and student behaviors in the classroom in the new learning environment. The teacher's role was to involve students in mathematical situations and uncover ways of mathematical thinking throughout the LED cycle. In the launch phase, the teacher started the lesson by summarizing what was done in the previous lesson, and this part was kept very short. In the Explore phase, where activity papers were distributed to the students, the students were left alone with the problems. At this stage, in the beginning, many students waited for the teacher's and the researcher's approval, asking questions such as "Teacher, is this right?" or "I did it like this. Is it wrong?" In the beginning, students were getting used to the sequence. In order to encourage students' participation, the researcher and the teacher browsed the students' papers in the explore phase to select and sort the students' solutions and requests for approval. Then, they supported them while solving the questions with appropriate questions. The instructional sequence provided insight into student solutions, helping to overcome potential difficulties in the explore phase. The teacher and the researcher did not comment on the correctness of what the students did in the Explore phase, encouraging them to decide for themselves whether their solutions were correct or not. One of the incentives can be considered getting support from a friend. However, it has also been observed that asking friends in the explore phase resulted in accepting the same solution as their friends did or continuing with their own wrong solution. The questions asked by the students at this stage were similar to "I found 24; is it correct?" They evaluated the answers to their questions for approval according to their own needs at the discussion stage. They were provided with solutions to their ongoing questions. Considering the process, there was no need to explain the concepts in the ratio-proportion issue directly. To encourage students to discuss and allow different ideas to emerge, the teacher often asked, "Does anyone answer this question differently?" during the discussion phase or "What is the difference between what you did and what Leyla did?" It was noted that he asked questions that would prompt students to discuss.

Table 5. 3

Changes in Classroom Philosophy Table Adapted from Brown (1992)

Role	Previous Classroom Setting	Intentional Learning Environment
Students	Find correct answer Listen to the teacher's explanations Respond to teacher directives Drill pre-given solution procedures	Learn other ways of reasoning Produce their reasoning Explain their ideas Challenge the thinking of the others
Teacher	The only source of knowledge Didactic teaching	Guided discovery/responsive guides to students' needs
Discourse	Initiation-Response-Evaluation (Student-Teacher)	Launch-Explore-Discussion (Student-Student, Student-Teacher)
Content	Ratio-Proportion Content from the Z-book	Ratio-Proportion Content based on HLT
Computer	Z book on the smartboard	Activity representation on the smartboard
Assessment	Summative assessment	Formative assessment Summative assessment
Guest participants	None	The researcher, as a participant-observer

From the perspectives of social and sociomathematical, it was observed that active participation increased, and the teacher expressed it. Comparing his friend's solution to "this is it!", "too long", "I didn't understand it", "mine (my solution) is easier", expressing his thoughts, showing them to his friends, and being evaluated by his friends is not a single solution method, but more than one solution method. Furthermore, it has been observed that some slow learner students have increased participation compared to their previous performances. In this sense, the students were encouraged to discuss in order to solve the questions meaningfully. Students tried to make sense of the answers to the questions in a classroom environment where only expressions such as "I do not agree" or "I was wrong" were not accepted without explanation. It was observed that the students' listening problems increased as the solutions on the board progressed from drawing to processing. After sharing their own explanations in the classroom environment, the teacher asked, “How did you find the solution?”, “Is it similar to yours?”, “Why did you divide it here?”, “What does this result tell us?” Involving questions such as students were asked. In the discussion stage, it was noted that as the student's solutions progressed from simple to complex (from drawing to processing), their contribution decreased, or their interpretations were problematic regarding mathematical reasoning. At these points, the teacher intervened and tried to make the students listen in simpler pieces that would enable the students to ask questions. Some of the question patterns asked by the teacher are as follows: "For example, if he had done this, how would you do it?", "What is the reason for doing this operation here?", "Can you show your explanation on the board?". In this way, it was noted that during the Discussion stage, it was tried to be organized by the teacher and the student to create a productive discussion environment.

Another significant issue was that most students had difficulty listening to their classmates since this was the first time they had experienced this environment. "Listening to each other" to learn has not become a social norm in this practice. Social norms are behavioral patterns that support learning, accepted by teachers and students, that can have interdisciplinary validity in learning environments where the social learning dimension is emphasized (Cobb & Yackel, 1996). It can be said that they learned to listen to each other and produce different solutions through the solutions of others during the discussion. At the same time, allowing students to work on their solutions during the discussion stage, asking them to share their ideas, and encouraging them to learn from each other also support the formation of socio-mathematical norms (Cobb & Yackel, 1996), which are also a part of the social dimension. Social and socio-mathematical norms help realize mathematics applications and, thus, learning (see Stephan & Akyuz, 2012).

5.1.1. Discussion about CMP 1

Classroom Mathematical Practice 1 (CMP 1) consists of three ideas of seventh-grade students foregrounding the discrete/continuous attributes of units and the presentation of the problem context through the linkage between composite units that emerged through the whole class discussion of Activity 1. These three normative ways of reasoning are around (1) Discrete units in composite units cannot be reduced for covariation due to problem context; (2) The invariant structure of ratio is independent of the random grouping of objects; (3) Maintaining the invariant structure of the composite units, continuous units in composite units can be reduced. This CMP magnifies issues related to the problem's context and the variables' attributes. Ideas involved both anticipated and unanticipated student contributions. Anticipated ones were dominantly using algorithms, drawing lines between aliens and food bars while connecting the visuals of two different measures.

TAS 1 and TAS 3 were directly related to the context variables as the context of the problem in terms of whether their referential content is continuous or discrete. Aliens and food bars represented different meanings within the problems. While linking the units in TAS 1, the students disobeyed the rule, halved a circle representing an alien, and demonstrated an unawareness of the invariant structure of a ratio. What was emphasized in the classroom discussion was that "slicing an alien made the context invalid" instead of the invariant structure of a ratio. This referential content as discrete variables made a few students confused. However, discussing the invariant structure of the ratio, the rule of the problem, was anticipated to be discussed under the linking composite units. In TAS 3, the students worked on slicing a food bar considering the invariant structure of the ratio, the rule, and the unit ratio strategy. Like TAS 1 and TAS 3, the relevant literature conveyed that the structural variations influenced students' understanding of proportion in the problems (Fernandez et al., 2012; Jeong et al., 2007; Lamon, 1993; Tournaire & Pulos, 1985). Attributes of the units become relevant to the problem context. Tournaire and Pulos (1985) suggested studying more on the continuous and discrete quantities in proportional reasoning. Besides, they added that students demonstrated better performance for proportional tasks involving discrete quantities (Tournaire & Pulos, 1985). Rather than continuous quantities, as Tournaire and Pulos (1985) reported, discrete quantities were hard to give meaning to solve the problems in non-integer ratios. On the other hand, the nature of counting carried on the discrete units may lead primary graders to think additively. Additionally, using continuous units supports multiplicative thinking (Jeong et al., 2007). Based on this, Fernandez et al. (2012) claimed that being discrete or continuous might affect students' using incorrect strategies as well; they recommended using missing-value problems for both discrete and continuous units instead of comparison problems and integrating a visual representation of the problem to overcome the complexities of the attribute of the units. Lamon also (1993) agrees about the referential context of easily represented units. Rapp et al. generated various examples of discrete units such as collective nouns (people, class of students), slices of a mass (pizza, pies, apples), discrete sets (marbles, balloons, grapes, crayons) (2015, p. 52). There occurred at least three situations for two linked units: discrete-continuous, continuous-continuous, discrete-discrete, and relevant literature may not directly suggest which referential content is to be selected at first during instruction. Moreover, context variables seem essential, but to what extent they affect students' learning of ratio and proportion is still not clear enough. Although this study considering its limitation, used the advantage of discussing the referential content of the context variable during instruction, it cannot say where to start considering the referential context. However, it may say that both discrete and continuous variables should be involved in the instructional materials. This is because three types of problems were available in this study, and it is argued that the referential context of the units did not lead to the overuse of proportionality or additive thinking. On the contrary, the students extended or limited their capabilities with the units, and event they started to understand unit ratio strategy by slicing the food bar. The point was that the units' attribute related to "being sliceable" was helpful. Moreover, these taken-as-share ideas (TAS 1 and TAS 3) showed that the solution was formed by the meaning given to the non-integer ratios with discrete units. The context "ratio of personnel to a child" or "ratio of alien to food bar" created a field to discuss when the comparison problem was in stage. When the given number of children is not the exact multiple of the ratio, what should a child say? Or is it possible to buy half when your money is more than buying a calculator? This kind of reasoning coming from real life should be developed, and the students should be encouraged to talk about these daily-life issues in the classrooms related to the phases of the HLT.

Another point is about partitioning units. With the transfer of the knowledge gained by TAS 1, TAS 3 explained students' progress in partitioning a unit considering the invariant structure of the ratio. This partitioning process of the continuous variables was defined as the unitizing process, which developed over time. Lamon asserted that using verbal and visual representation for partitioning signified conceptual development from additive to multiplicative thinking (Lamon, 1996). Lamon (1996, 2020) also referred to this partitioning process as significant for developing unitizing, which is one of the crucial skills for proportional reasoning. In this study, the students explored a unit ratio strategy based on partitioning the food bar through collective reasoning. Additionally, the drawing strategy was already acknowledged. However, students built on other strategies based on using this strategy, and the occurrence of the discussion for continuous and discrete characteristics of the units may be combined effectively under the guidance of provided didactic strategies. For this respect, Lamon (1996) targeted children's unitizing processes under partitioning strategies to develop didactic strategies. Apart from the simultaneous emergence of the partition of food bars in this study, the instructional sequence can be enriched in terms of partitioning, starting with the involvement of the various task types described by Lamon (1996). Since these tasks were preserved as part of the fraction, they can be added as warm-up tasks at the beginning of the instructional sequence to make sure that children are ready for the instructional sequence.

As a particular focus on TAS 3, the food bar illustration played the role of area model, which are diagrams that use shapes, usually circles or rectangles, to represent fractions. The shapes are divided into equal sections, and the fraction is shown by shading a certain number of those sections. The denominator of the fraction is the total number of sections in shape, while the numerator is the number of shaded sections. In this study, students divided the food bar into the required number of aliens given in the rule and linked them as composites. The selection of the figure "rectangle" as a food bar representation made students relate it to the fraction imagery and helped them to slice it equally. As an area model representation, the rectangular area was considered easier to partition (van de Walle et al., 2016), which led the students to think of unit ratio with the help of a drawing strategy. Within the limitation of this study, it may be deduced that the area model supported students' exploration of "a per b" understanding other than the part-whole relationship in fractions.

In CMP 1, unanticipated ideas enriched the lens used for the instructional sequence. Specifically, these ideas connected students to discover the attributes of the units (TAS 1 and TAS 3) and the attributes of the illustrations/visuals representing the units (TAS 2), which were not among the anticipated student thinking in the instructional sequence. Several students replaced the invariant structure of the rule/ratio with the number of the row because each row was illustrated as the iteration of the rule. The influence of the visuals could not be avoided on students' learning, although only two students demonstrated this misconception emerged in TAS 2 while they were in search of finding a distinct way of solution. In the dissertation of Lamon (1989), she proposed a design for the ratio tasks, and this task was used in other studies (see Figure 5.1).

In Figure 5.1, aliens were not distributed evenly or, as such, "three aliens in one row," in contrast to the ratio and proportion instructional sequence. Students need to find their own grouping and matching. As a notice, instead of using dark circles for food pellets, using rectangular area representation was more helpful due to the partition of the area, so the understanding of unit rate/ratio. In total, design can be used to engage students in the grouping, underline the composite units or rules of the task and facilitate discussion. Misailidou and Williams (2003, 2004) used a similar design to Lamon (1989) for the grouping, and students used different grouping strategies. They even predicted that the emphasized cultural device was using a shared context and a visual diagram to facilitate grouping as a precursor to a multiplicative strategy (Misailidou & Williams, 2003). Each idea influenced the emergence of other ideas. While linking the composite units, these ideas were dominantly formed by the relationship between the objects. This issue is considered under the problem context and affects students' informal strategies (Heller et al., 1990). Facilitation of this intuitive understanding is predicted to transform into relevant algorithms (Lawton, 1993).

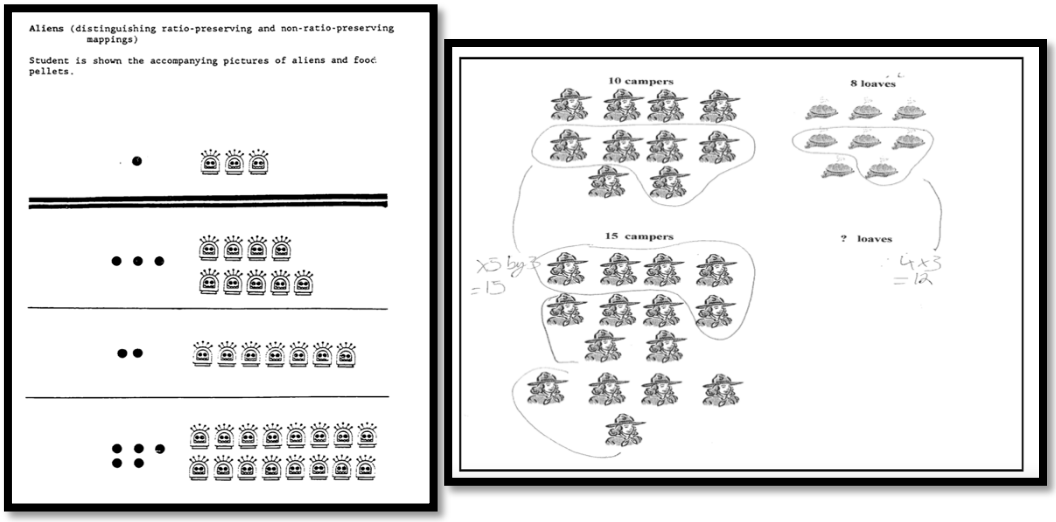


Figure 5. 1. Unitizing (Lamon, 1989, p. 62; Misailidou & Williams, 2004, p. 2)

All in all, CMP 1 originated from the visual design of Activity 1. Through the visual images included in the book, importance of the referential context for the learning of ratio and proportion has been supported by several studies (Fernandez et al., 2012; Jeong et al., 2007; Lamon, 1989, 1993, 1996; Misailidou & Williams, 2003, 2004; Tournaire & Pulos, 1985). These studies also signified that illustrations could also be the focal points of mathematical interaction of middle graders when interacting with the task similar with young children (Anderson et al., 2005; van Oers, 1996). The influence of visuals in the tasks may still infer middle graders' understanding of proportional reasoning situations based on the findings in CMP 1. By combining illustrations with text, tasks can contribute to the initial stages of interpreting and using representations, supporting the development of mathematical understanding for ratio and proportion as this study also advocated. Further studies can be conducted with middle graders for focusing on the tasks. This can be done through activities that help them practice forming, exchanging, and negotiating meaning in their everyday lives.

5.1.2. Discussion about CMP 2

CMP 2 consisted of four ideas about linking composite units and iterating them in various situations for missing-value and comparison problems. These four normative ways of reasoning are (1) Composite units can be represented and iterated in their pictorial or symbolic forms; (2) Division and multiplication algorithms require linking correct units for missing-value and comparison problems; (3) The unit ratio can be created to reconstruct composite units (unitizing); (4) Ratio table is composed of iteration of composite units. The HLT started with *linking composite units* to deal with the ratio and proportion problems, which was critical to developing higher-order thinking (Battista & van Auken Borrow, 1995; Stephan et al., 2015). They used different linking representations for the rule "one food bar (on the left) feeds three aliens". The first task was suggested for the students to develop those links between the units, and the pictures supported them to develop this skill by drawing the units and the link between composites.

CMP 2 TAS 1 signifies the multiple representation of mathematical entities. In the literature, children's drawings and enhancement of the learning can be related to future studies. Andreasen et al. (2005) also emphasized the relationship between pictures and mathematical interaction. Showing the link between units that make them composites through drawing was preferred by the students after they started the activities with visuals given in the instructional sequence. The rule given as drawn shapes raised the action of repeated iteration of the visuals. In other words, the ratio of 1:3 turned into 5:15 which involves the same proportional relation requiring concurrent iteration of the composite units one and three, five times. This informal way of reasoning was presented repeatedly in several studies (Ayan-Civak, 2020; Kaput & West, 1994; Lamon, 1993; van Auken Borrow, 1995). While students interacted with the iteration process, they intuitively gained an understanding of covariation for each unit (2–4, 4–8, 6–12…, see Figure 4.33), and they used ratio as a unit (Lamon, 1993). At the same time, they started a foundation for the build-up strategies (Kaput & West, 1994).

Understand the concepts of multiplication and division thoroughly enough to recognize how they play a part in the iteration process and abstractly reflect on the iteration process before applying multiplication and division in CMP 2 TAS 2. In other words, it is important to have a strong foundation in multiplication and division and to be able to think critically about the iteration process, before using these mathematical operations as part of the process (Battista and van Auken Borrow, 1995). While using algorithms, the students confronted several issues. The students had to link composite units in division and multiplication operations because of their relatedness. The students used various algorithms to solve the problems and either selected path 1 or path 2 (see Figure 4.35) according to the selection of the total number of food bars or the total number of aliens as reported in CMP 2 TAS 2. Path 1 is a solution in which the students used the given total number of aliens and the number of aliens in the rule.

They found the number of groups and for each group, there need to be some required food bar. With the iteration of that number of food bars (multiplication), the students found the total number of food bars required. On the other hand, path 2 is a solution that the students used the total number of food bars to multiply by the number of aliens in the rule. They multiplied them to find the exact number of aliens to be fed. Batu's explanation for path 2 was accepted as transparent, and no more questions were asked. Path 1 and path 2 were a simulation of partitive and quotative division problems. Partitive division problems require the number of objects in one of the partitioned small groups within the total number of the whole group; whilst the quotative division is called subtractive division in that the number of objects is repeatedly subtracted to reach the target number (Lamon, 2020). In other words, quotative division reflects the measurement each group gets; on the other hand, partitive division reflects the equal sharing issue and is related to the rate concept. (Lamon, 2020). In the teaching experiment study of Lo and Watanabe (1997), they demonstrated the capabilities of a fifth-grade student, Bruce, for proportional reasoning tasks and revealed that Bruce had difficulty transit from quotative to partitive division strategy. Based on this, it may be concluded that students' strategies for the calculation may differ, and it created difficulty in understanding. Bruce's ratio and proportion were influenced by his understanding of such topics as multiplication and division operations. However, his experiences with ratio and proportion tasks also provided contexts within which he was able to develop a more complex understanding of those operations. Unlikely, this study provided a snapshot from the classroom that describes a smooth transition between path 1 and path 2, an anticipated result from seventh graders. The students used two paths to solve missing-value problems. Within the limitation of this study, additional investigations, longitudinal in nature, are needed to answer many questions raised unanswered by this preliminary research (Lo & Watanabe, 1997). Frequently, children's strategies in solving multiplication and division problems neglect the actual context of the problem (Fosnot & Dolk, 2001; Russell, 2000). A few examples of multiplication and division strategies should provide a better grasp of children's approaches to solving problems.

Another point for the students' using paths 1 and 2 were similar to use internal and external ratios. For example, path 1 is a solution in which several students used the given total number of aliens (9) and the number of aliens in the rule (3). By using the same measure space, they used internal ratios (Kaput & West, 1994). On the other hand, path 2 is a solution that the students used the total number of food bars (2) to multiply by the number of aliens in the rule (3). They multiplied them to find the exact number of aliens to be fed (6), which represented external ratios (Kaput & West, 1994). In either perspective, the emphasis is put on the factor of change and its constancy. This factor can be called vertical or horizontal (Stephan et al., 2015), but the point is whether it is within the same measure space or between distinct measure spaces. Without describing the units, the students needed to understand measure spaces through the teacher's questioning. However, integer and non-integer results affected students thinking differently. What remainder means in path 1 confused students understanding for discrete units.

The students were not sure of the meaning of the components of four operations especially division and did not use labels. While conducting cross-product algorithms and conducting multiplication they did not also use labels. The middle-grade mathematics curricula do not involve labels or naming of the numerical values in ratio and proportion while conducting algorithms (CCSSI, 2010; Lamon, 2020; MoNE, 2013, 2018). Distinctively, Teacher Merve decided to ask what each component represented in the algorithm during the ratio and proportion instructional sequence. According to Cramer and Post (1993b), not using labels while solving the problem in the proportion quartet is called a fraction strategy. In this study, the students learned the composite units, but they might not develop the distinction between ratio and fraction because issues related to labeling in division and multiplication operation was not anticipated before the study started; however, labeling issue should be considered deeply while teaching ratio and proportion for all strategies students used. Cramer and Post (1993) highlighted the implicit expression of the labels, but these expressions should be questioned by the teacher during instruction. Clark et al. (2003) also criticized this issue in terms of textbooks in which there are decontextualized ratios for part-part-whole meaning, represented with the same notation. They provided an example of "4 girls and 5 boys" as 4/5 and "4 out of 5 people" as 4/5. They questioned, "returning to the problem of the ice and rock-salt, we wonder how any student could make sense of 1/7 and 1/8 as stand-alone fractions. How do students adjust from the emphasis on fractions as part–whole representations throughout elementary school to the introduction of fractional representations of part–part and associated-sets relationships with little or no explanation?" (Clark et al., 2003, p.312). As a suggestion they also questioned students' decontextualized written, verbal expressions to check out the labels and emphasized, "The ratio of (Unit 1) to (Unit 2) is (Number)," or, "There is (Ratio) of (Unit 1) for every one of (Unit 2)." (Clark et al., 2003, p. 312). Moreover, Lamon (2012, p.227) concluded labeling the quantities to reveal the operational differences between ratios and part-whole fractions while using fraction notation. Similarly, this study revealed that labeling the quantities are also significant while students conducted multiplication/division algorithm to solve the problems. They may start to form incorrect operations or may lose the label of the result. Their practices were a tendency not to use the labels, but teachers should insist on asking students what each number represents and write their labels on the board to make it all visible.

The present study has suggested several possible roots of students' difficulties with ratio and proportion tasks. These findings provide additional support for the view that the development of ratio and proportion concepts is embedded within the development of multiplicative structures. In this study, additional tasks related to procedural knowledge related to fraction was integrated as a reminder of their previous knowledge. The common claim of Nabor (2003) and Davis (2003) was that by dealing with fractional tasks it was possible to develop proportional reasoning. In her comprehensive study, Lamon revisited this idea and added to her hypothesis that "… However, understanding the larger concept of proportionality comes about later, through interaction with mathematical and scientific systems that involve the invariance of a ratio or a product." (2007, p. 640). It helps to differentiate between the different meanings of the fractions, especially with part-to-whole and ratio (van de Walle et al., 2016).

Unit ratio or rate was discovered by the students while they were discussing the continuous aspect of a unit which was accepted as another informal strategy for ratio by Kaput and West, (1994). Drawing and the referential context of the problem (rectangular visual representing the homogeneity and divisibility of the food bar) supported students' thinking in this instructional sequence. Similarly, Fernandez et al. (2012) described the possibility of using a unit ratio strategy with the emergence of non-integer ratios in proportional problems. Non-integer ratios and students' utility of the visual were not directly related to students' unit ratio strategy. There are studies claiming the importance of problem context (Cramer & Post, 1993b). However, the significant point is that using experientially real and divisible visuals compliant with their previous learning can activate transferring fractional imagery to the ratio and proportion context. Although fraction imagery may lead to misconceptions as well, the cases provided in the unit ratio strategy were found helpful to naturally come out with the understanding of "a per b".

They conducted iteration by drawing from pictorial to symbolic representations. Their drawings, if rich in geometrical shapes (see Lo & Watanabe, 1997), aided them to initiate the partition of the continuous units in which the potential of geometrical shapes emerges. Lamon referred to this partitioning process as significant for the development of unitizing, which is one of the crucial skills for proportional reasoning (Lamon, 2020). For this respect, Lamon (1996) targeted children's unitizing processes under partitioning strategies to develop didactic strategies. In this study, the drawing strategy was already acknowledged but students built on other strategies based on using this strategy and the occurrence of the discussion for continuous and discrete characteristics of the units may be combined in an effective way under the guidance of provided didactic strategies. Starting with the involvement of the various task types as described by Lamon (1996) the instructional sequence can be enriched in terms of partitioning. Since these tasks were preserved as part of the fraction, they can be added as warm-up tasks at the beginning of the instructional sequence to make sure that children are ready for the instructional sequence, or they can be given in previous years to be prepared for the formal teaching of ratio and proportion. In the current study, being continuous or discrete of a unit influenced the students' reasoning. A sliceable food bar in a rectangular form led students to share it equally among the aliens. Thanks to equal-sharing imagery, they explored the unit ratio strategy simultaneously and they studied on re-unitizing the given rule. For both partitioning and grouping experience of the children may seem irrelevant to the ratio and proportion concept. Nevertheless, this kind of paper and pencil task can be a tool to discover the physical acts of sharing as a facilitator of composite units (Lamon, 1996). Cutting food bars into pieces demonstrates the children's understanding of equivalence class as well (Lamon, 1996).

Each time the students faced different challenges (problems involving operations of whole numbers and non-integer numbers, missing value problems, comparison problems, and the number value of the units in the rule different than one), they updated their informal tools. As the formal tool, the ratio table was introduced with the help of the instructional sequence, and the student's interaction with the tool was observed and it took time to deal with the ratio table. The students tried to organize the numbers, and the iteration was embedded in their solutions explicitly or implicitly. Since the number value in the questions increased, the students moved to other iteration models, such as the ratio table strategy rather than the drawing they frequently used in the first segment of HLT. The ratio table was not evolved smoothly, but its utility was advanced by the students during the classroom discussions.

The ratio table is composed of iterations of composite units. It took time to deal with the ratio table. Nonetheless, a smooth adaptation of the students to the ratio table tool was ensured by Teacher Merve. After the introduction of the ratio table, the students first iterated and calculated additively the number values by considering their increase and decrease number.

Their representation of the iteration changed from writing the same digit (5 5 5 5 5 5 5, each five also representing 1 food bar) to an abbreviated ratio table. Sozen-Ozdogan et al. (2019) described the evolution of the ratio table in their study as given in Figure 5.4.

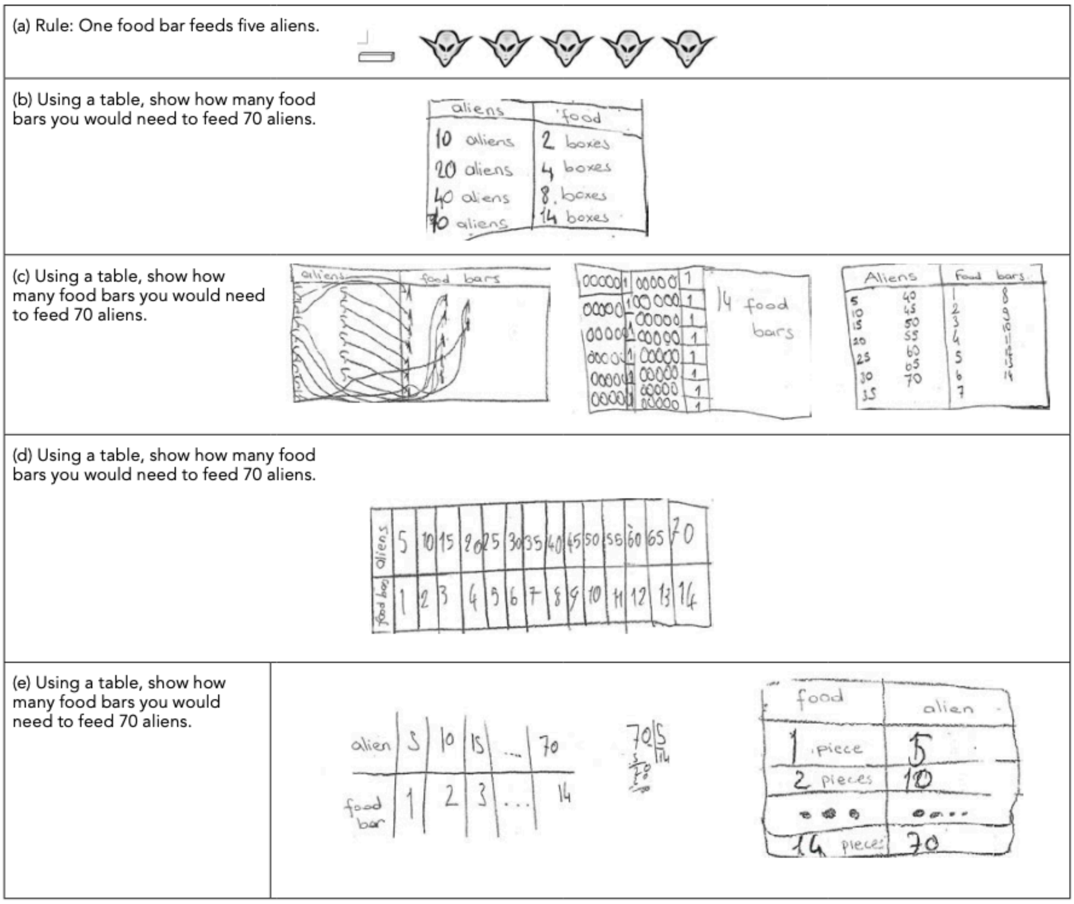


Figure 5.4. Evolution of abbreviated ratio table (Sozen Ozdogan et al., 2019, p. 19)

From this aspect, the ratio table is accepted as a conceptual and computational tool that employed both conceptual and procedural learning of ratio and proportion (Brinker, 1998; Ercole et al., 2011; Lamon, 2012; Middleton & van den Heuvel-Panhuizen 1995). Tables of values have long been acknowledged for their contribution to students' mathematical understanding (Warren, 1996), and constructing a table helps to identify the numerical relationship between the two quantities (Cramer & Post, 1993a). In this study, the LED teaching cycle included discussion phases that were meant to challenge students and help them address any difficulties or misunderstandings they had about ratio and proportion. The instructional sequence was organized to gradually introduce students to more complex concepts related to ratios, starting with linking composites and eventually working up to multiplicative reasoning. This progression allowed students to develop a deeper understanding of ratios and be able to solve a range of proportion problems, with little emphasis on cross-multiplication. The structure of the ratio table, which emphasizes learning outcomes such as distinguishing units, recognizing the binary number relationship in the ratio, and comprehending the multiplicative relationship between the units by completing the spaces in the cells, supports the understanding of the ratio-proportion issue (Abrahamson & Cigan, 2003). A ratio table is a tool that builds these connections in a way that allows students to develop an understanding of rational numbers - and as such, it is a good alternative to cross multiplication (Middleton & van den Heuvel-Panhuizen, 1995). It enables to develop of an understanding of the successive manipulation of the numbers to maintain the relationship between two quantities in the ratio and it is a conceptual tool for understanding equivalent ratios therefore it supports proportional thinking (Middleton & Van den Heuvel-Panhuizen, 1995; Abrahamson & Cigan, 2003; Ercole et al., 2011). As a challenge, it further seems to be the case that one can create the equation without this understanding.

Develop students' understanding of proportional relations before teaching computational procedures that are conceptually difficult to understand (e.g., cross-multiplication). Building on students' developing strategies for solving ratio, rate, and proportion problems, CMP 2 reflected on the various issues related to students' reasoning about their informal strategies to solve proportional tasks. Siegler et al. (2010) suggested conceptual development before procedures such as a cross-product algorithm for solving ratio, rate, and proportion problems. Using visual representations and alternative strategies (Siegler et al., 2010), the students achieved reasoning about composite units and iteration of at least two composite units. This was suggested for the successful transition to multiplicative reasoning (Battista & van Auken Borrow, 1995).

As Steffe (1994) encouraged, students were not told the correct ways of solving the tasks. They were guided to find their own ways of reaching the big ideas through their reasoning. This created "lots of iteration ideas" according to the students but as sociomathematical norms were shaping, they learned what makes a solution distinct from others. When the investigation focuses on the children's taken-as-shared ideas qualitative variations in the children's explanations and the solutions may vary (Yackel, 1995). The significance of the students' interpretations suggests that the context is what distinguishes the qualitative variances in the explanations of the students. Students made calculation mistakes due to their incompetencies with operations. Therefore, they found many results but, in the end, they agreed with one result. In this respect, students' reasoning demonstrates slow steps for a transition from informal tools to formal tools with collective and consistent sharing of ideas. These TASs also described students' informal and formal tools they used during instructional sequence implementation similar to the literature (Avcu & Avcu, 2010; Ayan-Civak, 2020; Artut & Pelen, 2015; Chapin & Anderson, 2003; Kaput & West, 1994; Lamon, 2020; Piskin Tunc, 2020).

5.1.3. Discussion about CMP 3

As a formal tool, ratio tables gradually took part in the students' activity sheets and their solutions on the board. There were arithmetical attempts for several students to internalize ratio tables and explore the multiplicative relationship in tiny steps. On the other hand, several students studied scale factors among the numbers and have used them effectively since then. They molded previous taken-as-shared ideas with the ratio table tool to better understand the relationship between the already organized composite numbers. In CMP 3, the students explored how to use vertical and horizontal scale factors conveniently, and their misconceptions during instructional sequence emerged and were eliminated and explored conjectured/unconjectured relationships within long and short ratio tables among organized numbers in the preceding five ideas. The students did not follow a standard path. Some students who mainly used drawing transferred their knowledge into a long ratio table and explored the multiplicative relationship between the variables: vertical scale factors within the horizontally extended ratio table. On the other hand, some students who were already using scale factors brought cross-product algorithms to solve direct ratio problems. CMP 3 consisted of five ideas from seventh-grade students who explored number relationships within the ratio table. These five normative ways of reasoning are (1) Ratio tables can be filled out through covariation among composite units; (2) An alien can be half-fed, and a scale factor or the numbers in the cells can be decimal within the context of the problem; (3) The used strategy is not the difference between numerator and denominator in the equivalent ratios (additive thinking) but one of the scale factors; (4) Cross-product algorithm can also be used in missing value problems provided that the problem is not the inversely proportional situation, (5) Third linked composite variable can be created in the ratio table and same horizontal scale factor can be used in this variable.

Understanding the structure of and development of strategies about how to use the ratio table tool requires reasoning about the covariation of the composites. Fractions, decimals, multiplication, and division are the basic components of ratio and proportion concepts and any combination of those components in instruction can be helpful for permanent teaching.

Decimal numbers created an issue to discuss the design of ratio and proportion learning environment. During the study, extra sources were provided to the students and the teacher focused more on the procedural algorithms related to decimals. When the components of the division operation did not provide an integer solution, students thought that they found a wrong result. This was because they had been familiar with the integer concepts in the division operation, or they did not capable of conducting procedural knowledge for decimals. The place of decimals in terms of non-integer problems cannot be avoided in ratio and proportion teaching (Lamon, 2020; van Dooren et al. 2009). In this situation, novel approaches can be adopted for the differentiation of teaching and learning in that based on the performance of the class several connections between mathematical topics can be done explicitly. Stemming from decimals as instances of ratio in terms of the invariant structure of ratios, Confrey and Lachance (2002) considered ratio as a broad construct and decimals as its subconstruct. Their instructional model puts the ratio at the center and teaches decimals on the ratio construct. According to their findings, fourth-grade children used ratios to understand and compare decimals (Confrey & Lachance, 2002). Depending on the needs of the students, decimal number situations emerged in this study, and a reminder of how to conduct four operations with decimals was made. Similar to the results of Confrey and Lachance's (2002) *nontraditional* way of instruction, this add-in was helpful for the students in this study.

Another issue related to TAS 2 is about connecting symbols with meanings. For ratio and proportion concepts, the students felt a need to understand what is “half of an alien?”. Does it mean a dead alien or a half-fed alien? This kind of reasoning can make learning more meaningful by reflecting the academic language of mathematics into daily-life language. Lachance and Confrey (2002) also suggested a reorganization of the curriculum to assist students to understand that different mathematical symbols have deeper meanings.

As it was conjectured, the planned Big Idea developed how the students understand the nature of the scale factors and the procedures they used within ratio tables. While digging into the ideas with Teacher Merve, their perspectives on the problems were observable. As in the previous classroom mathematical practices, each idea was formed by datum, claim, warrants, backings, and rebuttals. Distinctively, the students’ drawings formed this mathematical practice, and the students explored the number relationships in the table. This process reduced the discussion time spent with backing because their experience of overcoming the issues in the previous practices enabled them to build a collective mathematical language for each activity.

Besides, the cross-product algorithm strategy was carried to the classroom discussion, and so were the inversely proportional situations, which were not conjectured. Computing cross product algorithm provides a simple test for determining whether two ratios are equal. The fourth of the terms can be found by using cross-product relationships. It is suggested that most of the proportion problems in the intermediate grades should be able to be solved without using cross products. Cross multiplication can also be used to variate the multiplicative relationship within the ratio table. The relationship between the numbers does not have to be only from left and right or upwards to downwards or vice versa. It can be in a cross-direction. This can be used to criticize the direction of the arrow and the similar problem context given in a proportional situation. Cross product algorithm can be used if there is no inversely proportional situations. Cross-product algorithm emerged within the context of instructional situations without any guidance from teachers. Lamon (2020) evaluates it as a normal situation for seventh and eighth grades provided that they achieved multiplicative relations. After dealing with the ratio table in terms of multiplicative relationships and scale factors, integrating a cross-product algorithm can be effective to show a procedural operation.

Additionally, the conditions for selecting the scale factors and ratio table were influenced by the students' previous experience with decimals. As dealing with ratio and proportion context was in line with other mathematical concepts, the instructional sequence also conjectured difficulty in using decimals. Contributions of the taken-as-shared ideas to covariation among composite units are described in this classroom mathematical practice in detail.

5.1.4. Discussion about CMP 4

CMP 4 focuses on raising awareness of the formal concepts of "ratio", "proportion", and "equivalent ratios" besides linking other representations to the symbolic representation. Previously, the students developed strategies, recognized the attribute of the units, and used decimals and larger numbers without explicitly saying "ratio". They started to reason by using symbolic expressions and terms. consisted of seventh-grade students' two ideas that emerged while trying to understand the symbolic representation of the ratio and proportion concept. These three normative ways of reasoning were (1) The symbolic ratio representation changes as does the order of the composite units in verbal representation; (2) Ratio table is used as a tool for proportion representation; therefore, VSF and HSF can also be adopted in proportion. The used strategy is not the difference between the numerator and denominator in the equivalent ratios (additive thinking) but scale factors. Fraction imagery helped introduce symbolic ratio representation and equivalent ratios. The used strategy is not the difference between the numerator and denominator in the equivalent ratios (additive thinking) but scale factors. Fraction imagery helped introduce symbolic ratio representation and equivalent ratios. The class engaged with the conceptual understanding of ratios and proportions employing fraction imagery and ratio table tool. While exploring, the way of delivery through question context came to the fore, which influenced the students' ability to evaluate the symbolic representation of ratios and equivalent ratios.

Variations in context and modes of presentation may affect students' responses (Lamon, 1993). Symbolic representation of ratio can be presented frequently in three ways: for one food bar to three aliens, 1 to 3, or 1:3, or 1/3. These symbolic representations look like they represent the same context. However, without knowing, decontextualized context conveys distinct learning experiences. As Lamon (1993) described:

1. "3 to 5" describes the situation verbally, without any mathematical implication.
2. "3:5" describes the pattern using the concept of ratio.
3. "3/5" is a fraction and thus implies that the given relationship can be defined.

The first concept of ratio will be called the equivalent fraction concept. Students who perceive a ratio as a fraction tended to reduce the ratio to lower terms and then use an additive strategy, adding (or subtracting) the numerator of the reduced ratio to the original numerator of the ratio and the reduced denominator to the original denominator to find other equivalent fractions. These students consistently described the ratios as fractions, using words like "It is twenty-four fortieths" and "It is three-fifths." (Middleton & Van den Heuvel-Panhuizen, 1995). This situation leads students to the wrong imagery and the part-whole relationship between the numbers. (TAS 1).

Transforming the ratio table tool into equivalent ratios recognizably enhanced the students' conceptualization. As conjectured, additive thinking was observed in larger numbers within the symbolic representation of equivalent ratios. Here is noteworthy, additive thinking, one of the anticipated answers in the previous questions, did not emerge through the ratio table and drawing strategy. The students' acceptance of the claim and data and the absence of clarification questions indicated the emergence of the datum and the claim type of argumentation layout. In this respect, the warrant and backing were dropped off, evidence of classroom mathematical practice. The ratios 1 to 4 and 2 to 8 are equal. Understanding equal ratios are strictly encouraged. Multiplying the two quantities vs adding to the two quantities. Prior knowledge: writing equivalent fractions.

5.1.5. Discussion about CMP 5

Missing value problems are dominant in the instructional sequence until Activity 18 which also involves the comparison of non-equivalent ratio questions. A comparison of ratios was discussed between Activity 18-22. When the students first met with the rate concept, they entered a state of disequilibrium between missing values and comparison problems. They could not directly adopt strategies that emerged for missing value problems. Although Teacher Merve and the researcher did not mention the type of questions, the students were aware of the distinct situation. They transferred bits and pieces from their former learning and normative way of reasoning. CMP 5 encompassed the students' comparison of non-equivalent ratios and their normative ways of reasoning. In addition, the ratio table was not an effective tool anymore. CMP 5 consisted of two ideas for seventh-grade students' strategy adaptation processes for comparison problems: (1) The more material a is in one amount of material b, the more intense the mixture is; (2) The least common multiple for ordering ratios can be used for understanding the most or the least of an attribute in a mixture. Lastly, the students were provided a pictorial representation of materials to imagine the least or the most concept as a backing. The students were provided a pictorial representation of materials to imagine the least or the most concept as a backing. In the preceding two ideas, the class engaged with the conceptual understanding of ratios utilizing contextual issues of the task. The analysis evidenced the students' transfer of their previous learning during the instructional sequence to new tasks, and this evidence displayed the repetition of already developed classroom mathematical practices. The class basically studied how to deal with non-equivalent ratios without using usual fraction imagery.

> "In the middle grades, the concept of proportion might be introduced through an investigation in which students are given recipes for punch that call for different amounts of water and juice and are asked to determine which is "fruitier." Since no two recipes yield the same amount of juice, this problem is difficult for students who do not have an understanding of proportion. As various ideas are tried, with good questioning and guidance by a teacher, students eventually converge on using proportions" (NCTM, 2000, p. 52).

While students internalized ratio tables, it was observed that they used drawing, algorithm, or unit ratio strategies. Although they rightly conducted multiplicative operations, for a warrant or a backing, they provide these strategies to support their claims in the ratio table. This issue was defined by Tournaire and Pulos (1985) as a "fall-back" strategy as the utility of more elementary strategies by the same students in a more difficult problem.

In the current study, the students were more engaged with the missing value problems. Comparison problems were not easy to adapt to at first. Apart from missing value problems, comparison problems are also significant in helping students develop proportional thinking. The contexts involve research-based features such as recipes (Brinker, 1998) or mixture (Tournaire & Pulos, 1985). The students investigated the meaning of the ratio context while representing (1:5 or 5:1). Therefore, fraction imagery (especially simple fraction imagery) became a challenge without considering the meaning of the ratio (the ratio of boys to girls or the ratio of girls to boys) while comparing the tasks. In this respect, the students compared the ratios using a common numerator and denominator, considering their meanings. It is widespread to use continuous variables in comparison problems (Boyer et al. 2008) as in this instructional sequence. As it was reported (Tournaire & Pulos, 1985), students' understanding of the relationship of the ratio as a single entity became a challenge for this study as well. The students did not and comparing those single entities or mixture problems provide a challenge to determine the most, the best, the least of a new element that they created after a mixture of an intensive entity (orangeyest).

In the second TAS, the students devised the least common divisor to compare the mixture in terms of their intensive attributes. While unitizing, unit strategy with long division is not preferable in real-life to find the cost of one unit, or other intensive quantities (Lamon, 2002). From this respect, Lamon (2002) recommends other strategies related to reconceptualizing the units for fractions. In addition to fractions, creating a new chunk of units by using the least common divisor or building on distinct units are among the strategies to support flexibility in unitizing ratios as well.

5.2. Implications and Suggestions for the Further Research

The findings and conclusions of this study provide information about teaching and learning of ratio and proportion reflectively, which may guide all the stakeholders such as in-service mathematics teachers, preservice mathematics teachers, teacher educators, curriculum developers, policy makers, and researchers.

First, educational design research provided an environment in which various interrelated theoretical perspectives molded for the local instructional implementation, they were tested and revised in terms of learning the content. Moreover, their actualization in the classroom environment was presented in the study. What the researchers planned and how it was enacted in the classroom were the question that the design research methodology answers (Cobb et al., 2001), and in this case the tenets coming to the fore were represented in a bounded system. The involvement of stakeholders other than the researcher provided a valuable resource to develop an efficient way of teaching-learning environment for proportional reasoning. Plomp (2013) also highlighted the involvement of practitioners as a quite significant part of educational design research. Throughout this process, the discourse generated from whole-class discussion gave prominence to students' ways of thinking, and students built mathematical practices relating to ratio and proportion. This means that they internalized, organized, and reinvented a model for this mathematical content and for the learning of that content in a social environment (Gravemeijer et al., 2003). These kinds of characteristics were described in other design research studies as well (see Ayan-Civak, 2020; Bowers & Nickerson, 2001; Stephan & Akyuz, 2012; Stephan, 2015; Stephan et al., 2003). The dissemination of EDR, in which not only a single principle but also many principles are considered holistically, may constitute the infrastructure to support the teaching of specific content. In the further studies, EDR will unlock the doors of the classroom and make transparent of what is going on for the teaching and learning.

Second, the content was enriched by the instructional sequence from the first activity to the last activity by considering the stages of the LED model. It prepared the teacher for the lesson, provided flexibility to the teacher considering the progress of the class, and limited the teacher's going out of the subject by adhering to the purpose of the lesson. Apart from the researcher, the teacher guided the classroom learning with her questions, and together with the researcher, the teacher actively participated in the daily planning (Cobb & Yackel, 1996). Discussions were conducted by accepting that each student has his own mathematical reality. These facts are based on what students say and what students do when performing a mathematical activity. In other words, students had the opportunity to shape their understanding of mathematics individually by interacting with the social environment. The focus is on students' reasoning skills, and interactive mathematical communication has been at the center of this method. Considering these situations, it has contributed to revealing the skills expected from the student such as producing, discussing, and reasoning about distinct ideas mentioned in the literature (Lappan, 2014; MVP, 2017, 2019; Stein et al., 2008). It has been observed that bringing the solutions together in this way causes students to share their ideas with their group and classmates, change the strategies they use according to their own needs, develop or refute them, and in this sense, it has a positive effect on learning. By examining the different roles in the findings and considering their difficulties and contributions, instructional plans can be designed with the LED model, which combines individual learning and social learning. Such teaching models can be investigated more about its limitations and strengths. On the other hand, implications of the recent literature for practice (Bowers, & Nickerson, 2001; Lappan, 2014; MVP, 2017, 2019; Stein et al., 2008) may be employed by mathematics teachers in their teaching.

In this study, the learning environment, students', and teachers' roles were tried to be defined in the center of the LED model, and the aspects that needed to be developed were expressed. In this respect, it has been supported by other studies in the literature that the features that may arise when a teaching model such as LED, which is compatible with the teaching process, can be similar to the features described in this study when applied to other classes (Forman, 1996; Stein et al., 2008). On the other hand, Akar and Yıldırım (2005) expressed the difficulties that teacher candidates may experience in transitioning from a traditional classroom environment to another classroom environment that adopts the social constructivist approach, and it was emphasized that new research could be important in facilitating this transition. Findings that might pose a challenge in transition, such as the crowded classroom environment and taking responsibility for learning, expressed by Akar and Yıldırım (2005), did not emerge as a challenge as the LED model was applied in this study. In this sense, since the proposed teaching model is structured, it can be accepted as a method that can facilitate the transition from a learning environment based on the relationship between telling and listening to a discussion environment. Finally, since this study is limited to the ratio-proportion teaching sequence, it is recommended for further studies re-examine the roles that will emerge in the LED plans prepared based on other mathematics topics to measure their effects on student learning.

With the support of EDR, LED and aforementioned learning environment, this study investigated the development process of ratio and proportion concepts of seventh grades in terms of extraction of their classroom mathematical practices through the use of ratio and proportion instructional sequence. After documenting five classroom practices and presenting them with their argumentation process, the analysis demonstrated that the students' construction of knowledge was dependent to each other, and their ideas emerged within the web of knowledge related to proportional reasoning such as unitizing, covariant, and invariant structure of the ratio and proportionality. Their consideration of the visuals, organization of the data given in the tasks, smooth transitions from build-up strategies to multiplicative relations, from pictorial to symbolic representation, from small to large numbers, from integer to non-integer tasks, from one composites to many composites, from discrete to continuous task variables, from missing-value to comparison problems were significant evidences of to what extent this instructional sequence and the HLT worked in this classroom environment with the contribution of daily micro cycles which were conducted by the teacher and the researcher to connect argumentations, tasks, big ideas (learning outcome), and children's needs. Besides children's correct thinking, their misconceptions and errors were also emerged and used as a tool for effective discussion. Furthermore, incorrect, and correct strategies of the students may help teachers and teacher educators to have an idea about the reactions and normative ways of reasoning of their students about ratio and proportion. This sequence can be integrated in any seventh-grade mathematics course while teaching ratio and proportion concepts. Still, the analysis revealed that several points still needed to be improved by the updates to the instructional sequence for the rest of the concepts, knowledge and skills related to the proportional reasoning. Staying focused on the HLT is suggested but students' needs should be considered during teaching. Ratio and proportion concepts are fed by many other topics at the same time they are feeding the other topics.

Ratio table is an effective formal tool for teaching and learning of ratio and proportion (Abrahamson & Cigan, 2003; Ayan-Civak, 2020; Brinker, 1998; Cramer & Post, 1993a; Dole, 2008; Ercole et al., 2011; Karagöz Akar, 2014; Lamon, 2012; Middleton & van den Heuvel-Panhuizen, 1995; Sozen-Ozdogan et al., 2019; Warren, 1996). In the current study, the students also used ratio table to organize the data in the problems, to understand the build-up and multiplicative relationship between the number values in the cells through hands-on activity sheets. The students realized the multiplicative relationship between and within variables but the invariant structure within ratio (between measure spaces, vertical scale factor) and covariant structure among composites (within measure spaces, horizontal scale factors) in the ratio table were not emerged to discuss comprehensively. For the further studies, it is strictly recommended for teachers and teacher educators to study more on the ratio tables in terms of this aspect and even to intervene in the discourse to create a discussion topic. For a smooth development of this discussion, more drill and practices may be provided to study on number relationships for ratio and proportion within ratio table. The instructional sequence can be supported with other tools, technologies, and mathematical topics when needed

(Confrey & Lachance, 2002). Therefore, it is suggested that tasks may be organized as Web 2.0 or digital tools to make students study on distinct examples for discrete/continuous variables, integer/noninteger tasks, creating many linked composites, and proportion.

The current study signified ratio and proportion concepts dominantly and unitizing, linking and iterating composites representing, data organizing, conducting operations, comparing the ratios, applying build-up and multiplicative relations, creating, and analyzing equivalent ratios were realized in the classroom effectively. With this form it can be applied in sixth and seventh grade students. Though suggested improvements were mentioned in the Discussion of Classroom Mathematical Practices part. In summary, the improvements related to discrete/continuous variables and integration of the worksheets about decimals, inversely proportional situations, cross-product algorithm, linear relationship in direct proportions, digital tools for the visualization of the comparison problems are suggested to enhance the ratio and proportion instructional sequence in terms of other concepts, skills, and knowledge under the umbrella term proportional reasoning. This study can be conducted with seventh graders without considering the number of students in the class with these suggested improvements to extract students' reasoning and difficulties about these topics. Ayan-Civak (2020) conducted one of the studies with revised HLT for proportional reasoning considering this issue and actualized most of the topics but cross-product algorithm, discrete/continuous task variable were not emergent in the argumentations unlike from current study. Additionally, Stephan et al. (2015) designed instructional sequence considering rate and percent concepts as well. Due to time limitation, it was not employed in the current study but in addition to these improvements, the rest of the activities can be employed in the further studies to see the vertical and horizontal mathematization to see the transfer of both ratio table tool and formal strategies.

An analysis of the National Turkish Middle School Mathematics Curriculum showed that it is lacking in certain understandings, such as iterating linked composites, absolute and relative thinking, and qualitative reasoning (see Avcu & Avcu, 2010; Ayan-Civak, 2020; Artut & Pelen, 2015; Karagöz Akar, 2014; Piskin Tunc, 2020). As a result, it is suggested for the curriculum developers and policy makers that additional objectives related to the topic of ratio and proportion be added to the curriculum for sixth and seventh grades. Additionally, it is suggested that proportional reasoning can be emphasized as a fundamental skill throughout all grade levels (see Boyer et al., 2008), as it is connected to various other concepts and topics, including fractions, rational numbers, and multiplication and division (Lachance & Confrey, 2002). This could be done by seeing proportional reasoning as an overarching skill to be addressed in a variety of subjects, rather than as an isolated topic. Additionally, incorporating proportional reasoning into problem-solving activities in a variety of contexts may also be effective in promoting its development.

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APPENDICES

A. APPROVAL OF THE METU HUMAN SUBJECTS ETHICS COMMITTEE

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

12 EKİM 2016

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Sayı: 28620816 / 354

Konu: Değerlendirme Sonucu

Gönderilen: Doç.Dr. Didem AKYÜZ

İlköğretim Bölümü

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın : Doç.Dr. Didem AKYÜZ;

Danışmanlığını yaptığınız doktora öğrencisi Sinem SÖZEN ÖZDOĞAN'ın "Yedinci Sınıf Öğrencilerinin Oran Orantı Konusu ile ilgili Teknoloji Destekli Sınıf Ortamında Matematik Uygulamaları" başlıklı araştırması İnsan Araştırmaları Kurulu tarafından uygun görülerek gerekli onay **2016-EGT-139** protokol numarası ve **12.10.2016-30.06.2017** tarihleri arasında geçerli olmak üzere verilmiştir

Bilgilerinize saygılarımızla sunarız.

Prof. Dr. Canan SÜMER

İnsan Araştırmaları Etik Kurulu Başkanı

Prof. Dr. Meliha ALTUNIŞIK
İAEK Üyesi

Prof. Dr. Ayhan SOL
İAEK Üyesi

Prof. Dr. Mehmet UTKU
İAEK Üyesi

Prof. Dr. Ayhan Gürbüz DEMİR
İAEK Üyesi

Yrd .Doç .Dr. Pınar KAYGAN
İAEK Üyesi

Yrd. Doç. Dr. Emre SELÇUK
İAEK Üyesi

BU BÖLÜM, İLGİLİ BÖLÜMLERİ TEMSİL EDEN İNSAN ARAŞTIRMALARI ETİK ALT KURULU TARAFINDAN DOLDURULACAKTIR.

Protokol No: 2016-EGT-139

İAEK DEĞERLENDİRME SONUCU

Sayın Hakem,

Aşağıda yer alan üç seçenekten birini işaretleyerek değerlendirmenizi tamamlayınız. Lütfen "<u>Revizyon Gereklidir</u>" ve "<u>Ret</u>" değerlendirmeleri için gerekli açıklamaları yapınız.

Değerlendirme Tarihi: 12-10-2016

Ad Soyad:

☒ **Herhangi bir değişikliğe gerek yoktur. Veri toplama/uygulama başlatılabilir.**

☐ **Revizyon gereklidir**

- ☐ Gönüllü Katılım Formu yoktur.
- ☐ Gönüllü Katılım Formu eksiktir.
  Gerekçenizi ayrıntılı olarak açıklayınız:
- ☐ Katılım Sonrası Bilgilendirme Formu yoktur.
- ☐ Katılım Sonrası Bilgilendirme Formu eksiktir.
  Gerekçenizi ayrıntılı olarak açıklayınız:
- ☐ Rahatsızlık kaynağı olabilecek sorular/maddeler ya da prosedürler içerilmektedir.
  Gerekçenizi ayrıntılı olarak açıklayınız:
- ☐ Diğer.
  Gerekçenizi ayrıntılı olarak açıklayınız:

☐ **Ret**

Ret gerekçenizi ayrıntılı olarak açıklayınız:

ÖİDB

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

Sayı : 14588481-605.99-E.13159321 22.11.2016
Konu : Araştırma İzni

ORTA DOĞU TEKNİK ÜNİVERSİTESİNE
(Öğrenci İşleri Daire Başkanlığı)

İlgi: a) MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2012/13 nolu Genelgesi.
b) 11/11/2016 tarihli ve 4938 sayılı yazınız.

Enstitünüz İlköğretim Anabilim Dalı doktora öğrencisi Sinem Sözen Özdoğan'ın **"Yedinci Sınıf Öğrencilerinin Oran Orantı Konusu İle İlgili Teknoloji Destekli Sınıf Ortamında Matematik Uygulamaları"** konulu tez kapsamında uygulama talebi Müdürlüğümüzce uygun görülmüş ve uygulamanın yapılacağı İlçe Milli Eğitim Müdürlüğüne bilgi verilmiştir.

Görüşme formunun (20 sayfa) araştırmacı tarafından uygulama yapılacak sayıda çoğaltılması ve çalışmanın bitiminde bir örneğinin (cd ortamında) Müdürlüğümüz Strateji Geliştirme (1) Şubesine gönderilmesini rica ederim.

Vefa BARDAKCI
Vali a.
Milli Eğitim Müdürü

28-11-2016-18000

...nli Elektronik İmzalı
Aslı İle Aynıdır.
22.11.2016

...ÖZDEMİR

Konya yolu Başkent Öğretmen Evi arkası Beşevler ANKARA
e-posta: istatistik06@meb.gov.tr

Ayrıntılı bilgi için

B. CURRICULUM VITAE

1. Name Surname: Sinem SÖZEN ÖZDOĞAN
2. Organization: TEDU Faculty of Education, Department of Elementary Education, Primary Education Program
Duration of Employment: 10 years (by 2022)

Degree	Field	University	Year
Bachelor's Degree*	Elementary Mathematics Education	Middle East Technical University	2011
Master's Degree	Elementary Science and Mathematics Education	Middle East Technical University	2013

* The abovementioned student took courses at University of Inholland (Inholland Hogeschool) in Netherlands in the spring semester of the 2009-2010 academic year within the scope of the Erasmus Program.

3. Languages:

English: Advanced   Dutch: Beginner

German: Beginner   Japanese: Beginner

4. Publications:

Articles published international and national refereed journals:

- Karslı-Çalamak, K., Olkun, S., & **Sözen Özdoğan, S.** (2022). Çokkültürlü sınıflarda matematik eğitimi: Öğretmen uygulamaları üzerine bir inceleme [Teaching mathematics in culturally and linguistically diverse classrooms: An examination of teacher practices]. *Anadolu Journal of Educational Sciences International, 12* (1), 123-155. doi: 10.18039/ajesi.926493. (TR Dizin)
- Akın, S. & **Sozen-Ozdogan, S**. (2021). Öğretmen eğitiminde yapıtaşı: Türkiye, Singapur ve Hong Kong'da öğretmenlik mesleği genel yeterlikleri [Generic Teacher Competencies: The Building Blocks in Turkey, Singapore, and Hong Kong]. *Ankara University Journal of Faculty of Educational Sciences, 54*(1), 269-310.doi: 10.30964/auebfd.642519. (TR Dizin)
- **Sozen-Ozdogan**, S., Akyuz, D., & Stephan, M. (2019). Developing ratio tables to explore ratios. *The Australian Mathematics Educational Journal (AMEJ*), *1*(1), 16-21. https://hdl.handle.net/11511/75258. (ERIC)
- **Sozen-Ozdogan**, Çakır, B., & Orhan, B. (2019). A case of teacher-students mathematical problem-solving behaviors from the perspective of cognitive-

metacognitive framework [Special Issue]. *Studia Paedagogica*, 24(4), 221-223. doi:10.5817/SP2019-4-10 (Scopus)

- **Sözen, S.,** & Çabuk, A. (2013). Türkiye, Avusturya ve Almanya öğretmen yetiştirme sistemlerinin incelenmesi [An investigation of teacher education system of Turkey, Austria and Germany] [Special Issue]. *Uşak Üniversitesi Sosyal Bilimler Dergisi*, 213-230. (TR Dizin)

Books or chapters published by national/international publishers:

- **Sozen-Ozdogan,** S., Akyuz, D., & Stephan, M. **(**2022). Chapter 6 Patterns and relationships within ratio contexts: Students' emerging ideas through ratio tables. In P. Jenlink (Ed.), *Mathematics as the science of patterns: Making the invisible visible through teaching* (pp. 99-125). Information Age Publishing.
- **Sozen-Ozdogan, S.,** Akyuz, D., & Çakıroğlu, E. (2020). A phenomenological study: Incorporating the history of mathematics from the perspectives of the teachers. In P. Jenlink (Ed.), *The language of mathematics: How the teachers' knowledge of mathematics affects instruction* (pp. 113-141). Rowman & Littlefield Publishers.

Paper presentations presented at international conferences:

- **Sozen-Ozdogan**, S. & Şahin-Doğruer, Ş. (2022, July). *Collaborative learning practices of students through synchronous online learning environment*. Paper presented at Congress 45th Conference of the International Group for the Psychology of Mathematics Education (PME), Alicante, Spain.
- **Sozen-Ozdogan**, S., & Güzeller, G. (2019, December). Matematik ders planlarında çokkültürlülük vurgusu: GÖÇ-MAT projesi örneği (Multiculturality in Mathematics Instructional Plans). Paper presented at *International Teacher Education and Accreditation Congress* (ITEAC'2019), Ankara, Turkey.
- Akin, S., & **Sozen-Ozdogan**, S. (2019, September). Teacher Competences as building blocks for improving teacher education: Turkey, Singapore, and Hong Kong in perspective. Paper presented at *ECER 2019 Education in an Era of Rise*, Hamburg, Germany.
- **Sozen-Ozdogan,** S., & Akyuz, D. (2019, June). Educational design research in mathematics education: Developing classroom-centered learning environment. Paper presented at *VIth International Eurasian Educational Research Congress*, Ankara, Turkey.
- **Sozen-Ozdogan**, S., & Akyuz, D. (2019, April). Launch-Explore-Discuss cycle as teaching and learning mathematics. Paper presented at *28th International Conference on Educational Sciences*, Ankara, Turkey.
- Olkun, S., Karsli-Calamak, E., **Sozen-Ozdogan, S.,** Solmaz, G., & Haslaman, T. (2018, December). Subitizing and Beyond: Perception of Set Cardinality from Different Spatial Representations. Presented at *4. Cyprus International Congress of Education Research*, Kyrenia, North Cyprus.
- **Sozen-Ozdogan, S.,** and Akyuz, D. (2018, September). Drawing as a fresh start for learning ratio concept: The case of Maya. Presented at *EARLI SIG 1 Special Educational Needs*, Potsdam, Germany.

- **Sözen, S.,** Çakır, B. and Orhan, B. (2015, September). Metacognitive similarities and differences between teachers and students on mathematical problem solving skills. Presented *at ECER 2015 Education and Transition,* Budapest, Hungary.

Paper presentations presented at national conferences.

- Karsli, E., Olkun S., and **Sozen-Ozdogan, S.** (2016, September). GÖÇ-MAT: Mülteci öğrencilere yönelik çok temsilli matematik materyalleri geliştirilmesi (MIGRA-MATH: Developing mathematics lesson plans with multiple representations for refugee children). Presented at UFBMEK-12 *National Congress on Science and Mathematics Education,* Trabzon, Turkey.
- **Sözen, S.,** Aydemir, D., Ayan, R., and Çabuk, A. (2014, September). Türkiye'de gerçekleşen birebir özel ders ile ilgili ebeveynlerin görüşleri [Parents' perspectives for tutoring in Turkey]. Presented at UFBMEK-11 *National Congress on Science and Mathematics Education*, Adana, Turkey.
- **Sözen, S.,** & Çabuk, A. (2012, September). Türkiye, Avusturya ve Almanya öğretmen yetiştirme sistemlerinin incelenmesi [Investigating Turkey, Austria, and Germany teacher training systems]. Presented at *I. Uluslararası Katılımlı Öğretmen Yetiştirme ve Geliştirme Sempozyumu,* Uşak, Turkey.
- Aras, S., & **Sözen, S.** (2012, June). Türkiye, Finlandiya ve Güney Kore'de öğretmen yetiştirme programlarının incelenmesi [Investigating the teacher training programs of Turkey, Finland and South Korea]. Presented at *UFBMEK-10 National Congress on Science and Mathematics Education*, Niğde, Turkey.

5. Projects:

Project Code	**Project Title - Role**
TÜBİTAK 1003 (215K478)	MIG-MATH: Supporting Teachers of Immigrant Students with Respect to their Mathematics Education Professional Practices-Research Assistant
TEDU Institutional Research Fund (0BAP16B0015)	Subitizing and Beyond: Perception of Set Cardinality from Different Spatial Representations - Research Assistant
TÜBİTAK 1001 (111K545)	6-11 yaş Türk çocukları örnekleminde diskalkuliye yatkınlığı ayırt etmede kullanılacak bir ölçme aracı geliştirme çalışması (Developing a diagnosis instrument for 6-11 year-old students with dyscalculia) - Research Assistant for Collecting Data
TÜBİTAK 4006 (417B557-07)	The effects of instructional sequences of coordinate plane through Geogebra on seventh grade students' academic achievement-Assistant Advisor Teacher
TÜBİTAK 2237 June 19-21, 2014	Eğitim Bilimlerinde TÜBİTAK Proje Yazım Eğitimi Programı (A Training Program on Project Writing for National Science Foundation[TUBITAK]), Participant and Technical Assistant

TÜBİTAK 2237 April 23-27, 2021	Kuramdan Uygulamaya Nitel Araştırmalar 2 Kursu (A Training Program for Qualitative Research), Technical Assistant
TÜBİTAK 2237 June 17-21, 2021	Eğitim Alanındaki Lisansüstü Bilimsel Araştırma Becerilerinin Geliştirilmesi-2 Eğitimi (A Training Program for the Development of Scientific Research Skills), Technical Assistant.
TEDU Institutional Research Fund (T-21-B2010-90072)	Investigation of pre-service teachers' self-regulated learning skills and professional development using e-portfolio. Research Assistant
UNICEF-MEB	

6. Administrative Duties /Organizational and Professional Services:

- TEDU Pet Friendly Society 2020-Ongoing (Academic Advisor)
- Web Commission 2017-Ongoing (Member of the Commission)
- Accreditation Commission 2017-2021 (Member of the Commission)
- Erasmus Commission 2012-2015 (Member of the Commission)

7. Scholarships:

2013-2023 TÜBİTAK Scholarship for Graduate Students
2006-2011 Başbakanlık Scholarship: A scholarship given by the Turkish Government for successful undergraduate students who selected to high-achieved Teacher Training Programs.

8. The courses attended as a research assistant at TEDU

Academic Year- Semester	Courses Offered
• 2012-2013 Fall	• CMPE 101 Introduction to Information Technologies
• 2013-2014 Spring	• EDU 511 Assessment and Evaluation in Education • EDU 101 Introduction to Education
• 2014-2015 Fall	• EDU 201 School, Family, and Society • EGE 221 Primary School Mathematics
• 2014-2015 Spring	• EGE 222 Teaching Primary School Mathematics
• 2015-2016 Fall	• EDU 201 School, Family, and Society • EGE 221 Primary School Mathematics
• 2015-2016 Spring	• EGE 222 Teaching Primary School Mathematics
• 2016-2017 Fall	• EGE 221 Primary School Mathematics
• 2016-2017 Spring	• EGE 222 Teaching Primary School Mathematics
• 2018-2019 Spring	• EDUC 140 Instructional Technologies • EDUC 110 Educational Sociology

• 2019-2020 Spring	• LIT 201 Children's Literature
• 2020-2021 Fall	• ECE 207/EGEP 201 Drama in Primary Education • LIT 201/EGEP 251 Children's Literature • EGE 222/EGEP 303 Mathematics Teaching 1
• 2020-2021 Spring	• EGEP 304 Mathematics Education 2 • EGEP 351 Children at Risk
• 2021-2022 Fall	• CMPE 101 Introduction to Information Technologies • EGEP 305 Science Education

9. Community and Professional Development Services:

Sibel Akın Sabuncu & **Sinem Sözen Özdoğan** – TED University Faculty of Education-June 25, 2021- Başkent Workshops for Teachers VII (Başkent Öğretmen Atölyeleri), Öğretmenlik Mesleği Genel Yeterlikleri: Türkiye, Hong Kong ve Singapur Örnekleri (Generic Teacher Competences of Turkey, Hong Kong and Singapore).

Sinem Sozen Ozdogan & Sule Sahin Dogruer - Middle East Technical University- Faculty of Education- February 8, 2020- 7. Matematik Öğretmenleri Paylaşım Zirvesi (A national mathematics teachers' meeting platform)-Doğrusal Denklemler ve Geogebra: Gerçek Yaşamdan Örnekler (Linear Equations and Geogebra: Examples from Real-Life Experience)

Selda Aras & **Sinem Sozen Ozdogan** - TED University- Faculty of Education - April 20-21, 2019- Eğitimi Demokratikleştirmek Konferansı (Conference for Democratization in Education)- Erken çocukluk döneminde oyun temelli Matematik öğretimi: Zihnin araçları (Play-based mathematics teaching in early grades: Tools of the Mind)

10. Other Work Experience (Education, Industry etc.):

2011-2012	Research Assistant, Amasya Üniversitesi, Faculty of Education, Elementary Education Department

11. Other Research Interests and Personal Development Tracks

Village Institutes, Women Mathematicians, Neuroscience and Learning, Cognitive Science and Its Reflections on Education are the topics that I am interested in.

C. PROBLEM TYPES AND EXAMPLES (Hilton et al., 2013; Lamon, 1993; van Dooren, 2005)

Problem types	Semantic Categorization	Description	Example
Comparison Problems	Part-part-whole	ratio problems in which two complementary parts are compared with each other or the whole	If my recipe requires 10 cups of flour for 4 cups of sugar, will it be enough to use 20 cups of flour with 20 cups of sugar for two cakes?
	Well-known measures	relationships between two measures that result in a rate, which is itself a commonly used entity, e.g., distance/time=speed	One of the athletes runs 1000 m in 11 min. another athlete runs 900 m in 10 min. Which athlete is the most
	Associated sets	rate situations in which the relationship between quantities is defined within the question, e.g., birthday cake pieces and children at a party	Given data about the number of children who choose an activity and the total children in each of two classes, identify which activity was relatively the most popular?
	Stretchers and Shrinkers or Growth problems	situations that involve scaling up or down	If I double the length and width of a diagram, what will happen to the area of the diagram?
Missing-value problems	Part-part-whole	In the problem, three values are provided, and fourth value is asked.	There are 25 students in the class. The ratio of the number of girls to the number of boys is 2/3. How many girls are there in the class?
	Well-known measures	relationships between two measures that result in a rate, which is itself a commonly used entity, e.g., distance/time=speed	If a child runs at a certain pace and it takes a given amount of time, what will happen to the time if he runs at twice the pace?
	Associated sets	rate situations in which the relationship between quantities is defined within the question, e.g., birthday cake pieces and children at a party	If my recipe requires 10 cups of flour for 4 cups of sugar, how much flour will I need if I use 6 cups of sugar?

	Stretchers and Shrinkers or Growth Problems	situations that involve scaling up or down	I have a photo with the size of 6 cm in width and 9 cm in length. If I scale up, what will be the size of length, if the width becomes 12 cm?
Non-proportional situations	Additive	Situations that are related to constant difference between two variables.	Today, Bert becomes 2 years old and Lies becomes 6 years old. When Bert is 12 years old, how old will Lies be?
	Constant	Situations that require realistic consideration of the situation and no need for the calculations	A group of children sings a song. If we double the number of children, how long will it take to sing the song?
	Linear	Situations that involve a linear pattern of the form $f(x) = ax+b$ with $b \neq 0$.	In the hallway of our school, 2 tables stand in a line. 10 chairs fit around them. Now the teacher puts 6 tables in a line. How many chairs fit around these tables?

D. TIME SCHEDULE

WHEN	WHAT	WHERE	WHO
01.04.2016	First meeting with the school administrators and asking approval for the entrance to the field as a researcher	Middle Grade Public School in Yenimahalle	The school administrators
21.04.2016	Observation of the classroom and teaching-learning interaction	Classroom 6/Y	Students at 6/Y and Teacher Zehra
17.08.2016	First meeting with the Teacher Merve to get an appointment	On the phone	Teacher Merve
05.01.2016	Start approval process of ODTU Ethics Committee	ODTU Graduate School of Social Sciences	UEAM
05.10.2016	First appointment; Introduction to the study; Giving volunteer consent form	Teacher's room	Teacher Merve
14.10.2016	Introduction of the Geogebra	Teacher's room	Teacher Merve
17.10.2016	Approval of ODTU Ethics Committee	ODTU Graduate School of Social Science	UEAM
19.10.2016	Start approval process for Ministry of National Education	ODTU Elementary Education Department	Elementary Education Secretarial Staff
21.10.2016	Getting to know Geogebra	Teacher's room	Teacher Merve
24.10.2016	Study with Geogebra + Classroom Observation	Teacher's room and 7/X classroom observation	Teacher Merve and 7/X students
04.11.2016	2 hours of 7/X observation + 1 hour of 7/Y observation; Getting to know about about instructional sequence	7/X and 7/Y classroom observations	Teacher Merve and students of 7/X and 7/Y
22.11.2016	Approval of MoNE	MoNE	MoNE Statistic Department Secretarial Staff
24.11.2016	Talking about instructional sequence	Teacher's room	Teacher Merve
02.12.2016	Pretest application	Classroom 7/X and 7/Y	Teacher Merve and 7/X and 7/Y
05.12.2016	Start instructional sequence with 7/Y	Classroom 7/Y	Teacher Merve and 7/Y
08.12.2016	Pretest with 7/Z (classroom with no treatment)	Classroom 7/Z	Students of 7/Z
09.12.2016	Start instructional sequence with 7/X	Classroom 7/X	Teacher Merve and Students of 7/X

13.12.2016	Posttest application 7/X and 7/Y	Classroom 7/X and 7/Y	Students of 7/X and 7/Y
17.01.2016	End of the instructional sequence with 7/Y	Classroom 7/Y	Students of 7/Y
18.01.2016	End of the instructional sequence with 7/X	Classroom 7/X	Students of 7/X

E. INSTRUCTIONAL SEQUENCE IMPLEMENTATION (7/X)

Date	Activity	Time Period
2016.12.09	Feed the Aliens Activity 1 Feed the Aliens Activity 2	2 lesson hours
2016.12.12	Feed the Aliens Activity 2 (Continued)	1 lesson hour
2016.12.14	Feed the Aliens Activity 3 Feed the Aliens Activity 4	2 lesson hours
2016.12.15	Feed the Aliens Activity 4 (Continued) Feed the Aliens Activity 5	2 lesson hours
2016.12.16	Feed the Aliens Activity 5 (Continued) Feed the Aliens Activity 6	2 lesson hours
2016.12.19	Feed the Aliens Activity 8	1 lesson hour
2016.12.21	Feed the Aliens Activity 8 (Continued) Feed Honk and Ponk (Activity 7) given as homework	2 lesson hours
2016.12.22	Feed the Aliens Activity 9 Feed Honk and Ponk Activity 10	2 lesson hours
2016.12.23	Tiny Tots Activity 11	2 lesson hours
2016.12.26	Discussion with a remaining group (10 students) Summing up	City tour with the volunteer group of 7/X (27 students) 1 lesson hour
2016.12.28	Tiny Tots Activity 11 (Continued) Mixed Problems Activity 12	2 lesson hours
2016.12.29	Snow holiday	No lesson
2016.12.30	Snow holiday	No lesson
2017.01.02	Mixed Problems Activity 13	1 hour -time
2017.01.04	Maths Second Exam No Activity	No lessom
2017.01.05	Calculator Costs Activity 15 (1 hour) Schoolwise examination	2 lesson hours
2017.01.06	Calculator Costs Activity 15 Mixed Problems Activity 16	2 lesson hours
2017.01.09	Mixed Problems Activity 17	1 lesson hour
2017.01.11	Mixed Problems Activity 17 (continued) Making Blood Activity 18 Fright Might Activity 19 Pumpkin Pie Activity 20 (Launch)	2 lesson hours
2017.01.12	Pumpkin Pie Activity 20 (Continued) Crazy Couldron Activity 21 The Oranges Obstacle Activity 22	2 lesson hours
2017.01.13	The Oranges Obstacle Activity 22 (Continued)	2 hours'time
2017.01.16	The Oranges Obstacle Activity 22 (Continued)	1 lesson hour
2017.01.17	Rates and Ratios Activity 23 Posttest	2 lesson hours

F. INSTRUCTIONAL SEQUENCE IMPLEMENTATION (7/Y)

Date	Activity	Time Period
2016.12.06	Feed the Aliens Activity 1 Feed the Aliens Activity 2	2 lesson hours
2016.12.08	Feed the Aliens Activity 2 (Continued) Feed the Aliens Activity 3	1 lesson hour
2016.12.09	Feed the Aliens Activity 4	2 lesson hours
2016.12.12	Feed the Aliens Activity 4 (Continued)	2 lesson hours
2016.12.13	Feed the Aliens Activity 4 (Continued) Feed the Aliens Activity 5	2 lesson hours
2016.12.15	Feed the Aliens Activity 5	1 lesson hours
2016.12.16	Feed the Aliens Activity 6	2 lesson hours
2016.12.19	Feed the Aliens Activity 6 (Continued) Feed Honk and Ponk (Page 7 done in the classroom)	2 lesson hours
2016.12.20	Feed the Aliens Activity 8	2 lesson hours
2016.12.22	Feed the Aliens Activity 8 (Continued)	1 lesson hour
2016.12.23	Feed the Aliens Activity 8 (Continued) Feed the Aliens Activity 9	2 lesson hours
2016.12.27	Feed the Aliens Activity 9 (Continued) Extra worksheet prepared for decimal representation of fractions	2 lesson hours (time schedule changed, no Monday lesson)
2016.12.28	Feed Honk and Ponk Activity 10 Tiny Tots Activity 11	2 lesson hours
2016.12.29	Snow holiday	No lesson
2016.12.30	Snow holiday	No lesson
2017.01.03	Tiny Tots Activity 11 (Continued) Mixed Problems Activity 12	2 hour -time
2017.01.04	Maths Second Exam No Activity	1 lesson hour
2017.01.05	Mixed Problems Activity 12 (Continued) School wise examination	2 lesson hours
2017.01.06	Activity Mixed Problems (Page 14)	2 lesson hours
2017.01.10	Mixed Problems Activity 13 Mixed Problems Activity 14 Mixed Problems Activity 15	2 lesson hours
2017.01.11	Mixed Problems Activity 15 (Continued)	1 lesson hour
2017.01.12	Mixed Problems Activity 15 (Continued)	2 lesson hours
2017.01.13	Mixed Problems Activity 16	2 lesson hours
2017.01.17	Mixed Problems Activity 16 (Continued) Posttest	2 lesson hours

G. RATIO AND PROPORTION INSTRUCTIONAL SEQUENCE HYPOTHETICAL LEARNING TRAJECTORY (ORIGINAL)

Big idea	***Tools/imagery***	***Possible Topics of Discourse***	***Activity Pages***
Linking composite units	Connecting pictures of aliens to food bars	If the rule is 1 food bar feeds 3 aliens, the rule can't be broken if we add more food bars	Page 1
Iterating linked composites	Informal symbolizing (e.g., tables, two columns of numbers, pictures of aliens and food bars)	How students keep track of two quantities while making them bigger	Pages 2-4
Build up strategies	Ratio table	Keeping track of two linked quantities while they grow additively	Pages 3-4
Additive versus multiplicative reasoning	Fold back to pictures; Shortened ratio table	Adding or multiplying to build up	Page 5
Structuring ratios multiplicatively	Shortened ratio tables through multiplication and division with scale factors	Efficient ways of curtailing long ratio tables What does the horizontal scale factor represent? What does the vertical scale factor represent?	Pages 3-7
Creating equivalent ratios	Ratio tables with missing values; Traditional proportion representation (two ratios separated with equal sign)	Adding versus multiplying; Meanings of scale factors; What do decimal scale factors mean?	Pages 8-10
Creating equivalent ratios	Ratio tables	Adding versus multiplying; horizontal and vertical scale factors	Pages 11-12 and Pages 15-17
Analyzing equivalent ratios	"Fraction" imagery	Reducing ratios; Vertical and horizontal scale factors	Page 11 (bottom) and Page 14 (top)
Comparing ratios	Ratio table No ratio tables, but can use arrow notations	Finding common numerators or denominators; size of the scale factors; unit ratios	Pages 18-23
Comparing rates	Ratio table; arrow notation; standard proportion notation	Difference between a ratio and rate; unit rates; common denominator and numerators	Pages 24-27

H. A SAMPLE ARGUMENTATION LOG

<table>
<tr><td>Activity-Question Code</td><td colspan="3">A1Q1_A
Rule 1 food bar for 3 aliens
9 aliens 2 food bars?</td><td>A1Q1_B1</td></tr>
<tr><td>Discussion Topic</td><td>Linking the units by showing</td><td>Objects are given as drawn</td><td>Each row has the number of objects required for one food bar.</td><td>“Slicing the aliens” Idea</td></tr>
<tr><td>Question</td><td colspan="3">Birinci soruda yeterince yiyecek kutusu var mı yok mu?</td><td>Bundan farklı yaptım diyen var mı?</td></tr>
<tr><td>Claim</td><td colspan="3">Yetmez.</td><td>Tam gelir.</td></tr>
<tr><td>Data</td><td colspan="3">Yiyecek kutuları ile üçer grup uzaylıyı elips içerisine alacak şekilde kalem ucuyla havada çizim yapar “işte böyle yaptım”</td><td>Paylaşımla..</td></tr>
<tr><td>Warrant</td><td colspan="3">Şimdi şurada 9 tane uzaylı var (tahtadaki uzaylı çizimlerini işaret ederek) bir tanesi 9 taneyi besleyebiliyorsa bunlar şunları besleyebilir hocam. (yiyecek kutuları ile 3er grup uzaylıyı elips içerisine alacak şekilde kalem ucuyla havada çizim yapar (TRANSCRIPT_C1, Pos. 118)</td><td>Üçer kişi ya hocam. Sonra üç kişi kalıyor hocam. Bir kişiye bir kutu veririm öbürüne diğer kutuyu öbürünü de bölerim ortadan. (TRANSCRIPT_C1, Pos. 123)</td></tr>
<tr><td>Backing</td><td colspan="3"></td><td></td></tr>
<tr><td>Rebuttal</td><td colspan="3"></td><td>Uzaylıyı kesip bölersen nasıl yemek yiyebilir ölüyor nasıl olsa. (TRANSCRIPT_C1, Pos. 137)</td></tr>
<tr><td>Notes</td><td colspan="2">This is kind of big bang for the need of explanation that will make sense for the children in the class.</td><td>Showing on the board was ended but need for explanation age started.</td><td>This episode put an emphasis on the attribute of the units. Which object can be sliced or “shared”? Which objects can not be shared? This topic did not end in here.</td></tr>
</table>

I. EXTENSION AND REVISION OF THE INSTRUCTION MAP

Big idea	*Tools/imagery*	*Possible Topics of Discourse*	*Planned Big Idea*	*Possible Topics of Discourse (Expanded)*	*Emerging Ideas (CMP, Strategies, Errors, Misconceptions)*	*Activity Pages*
Linking composite units	Connecting pictures of aliens to food bars	If the rule is 1 food bar feeds 3 aliens, the rule can't be broken if we add more food bars	The idea of this page is to encourage students to link two composites together.	The rule can not be broken what about food bar and the aliens? (Let's discuss attribute of the units) How many food bar was enough for one alien? How to share one food bar among three aliens? How to use this ratio in the operations?	The number of rows: Is that important. Meaning of division and multiplication operation in terms of problem context.	Page 1
Iterating linked composites	Informal symbolizing (e.g., tables, two columns of numbers, pictures of aliens and food bars)	How students keep track of two quantities while making them bigger	*The idea of this page is to encourage students to link two composites together and to begin to organize these links when there are large quantities involved (2) *Begin to reason with short or long ratio table; constructing a build up strategy versus an abbreviated strategy (3) *using non-unit rates to create equivalent ratios; what does a unit rate mean; usefulness of unit rate (4)	Not drawing aliens, drawing food bars still makes drawing easy. What are the meaning of divisor, dividend, quotient and the remainder in the division algorithm within the problem context? Where to use unit ratio? How do students develop ratio table? Horizontal or vertical? Does that matter?	Drawing pictures, Division/Multiplication algorithm, Unit Ratio, Ratio Table are the four strategies emerged as mathematical practices during these activities. Dividing the number of aliens into the number of food bars does not give the number of aliens or number of food bars but a ratio. Dividing the number of aliens into the number of aliens does not give the number of food bars. Dividing the number of food bars into the number of food bars does not give the number of aliens.	Pages 2-4
Build up strategies	Ratio table	Keeping track of two linked quantities while they grow additively	*Begin to reason with short or long ratio table; constructing a build up strategy versus an abbreviated strategy *using non-unit rates to create equivalent ratios; what does a unit rate mean; usefulness of unit rate	What is a ratio table? What makes a table a ratio table? A ratio table can be expanded as horizontal or vertical. Does it change the properties?	A long ratio table could be more efficent than drawing, on the other hand, division/multiplication could be more efficient than long ratio table.	Pages 3-4
Additive versus multiplicative reasoning	Fold back to pictures;	Adding or multiplying to build up	Additive versus multiplicative reasoning/proportional reasoning	Can decimal rate guide the students to use additive thinking?	Difference between the given numbers do not help us to find the result.	Page 5
	Shortened ratio table					
Structuring ratios multiplicatively	Shortened ratio tables through multiplication and division with scale factors	Efficient ways of curtailing long ratio tables	*Begin to reason with short or long ratio table; constructing a build up strategy versus an abbreviated strategy	Drawing v. long ratio table. Which one is time efficient?	Using algorithm might lead students to wrong decisions such as multiplication with incorrect number combinations or vice versa.	Page 3-7
		What does the horizontal scale factor represent?	*using non-unit rates to create equivalent ratios; what does a unit rate mean; usefulness of unit rate	What does half of an alien represent		
		What does the vertical scale factor represent?	* Additive versus multiplicative reasoning/proportional reasoning * Using non-unit rates to create equivalent ratios; what does a unit rate mean; usefulness of unit rate * Linking 3 composite units	Students use the same data organization strategies for three composite units.		
Creating equivalent ratios	Ratio tables with missing values;	Adding versus multiplying;	explore the meaning of a decimal scale factor; explore additive versus multiplicative reasoning again	A new cross-product algorithm and its relationship with ratio representation	An alien can be half-fed and a scale factor can be decimal within the context of the problem.	Pages 8-10
	Traditional proportion representation (two ratios separated with equal sign)	Meanings of scale factors;	determining the most useful scale factor (horizontal vs. vertical); additive reasoning	Inverse ratio situations, can you use cross-product algorithm?		
		What do decimal scale factors mean?	Assessment of how students understand the nature of the scale factors and the procedures they have used with ratio tables	Horizontal or vertical ratio table development determines the naming of vertical and horizontal scale factor.	VSF and HSF gives the same result and they can be selected based on its usefulness which is basically determined by being non-integer or integer.	
Creating equivalent ratios	Ratio tables	Adding versus multiplying; horizontal and vertical scale factors	*reasoning about equivalent ratios in a new context			Pages 11-12 and Pages 15-17
			*Determining equivalent ratios; multiplicative versus additive reasoning with decimal scale factors			
			*Finding missing values in a "proportion". Although not introduced yet, students are reasoning proportional every time to find a missing value in a table because reasoning proportionally means to set two ratios equivalent.		Ratio table is used as a tool for proportion representation; therefore, VSF and HSF can also be adopted in proportion	
			* reason proportionally, introduce the word and symbols of proportionality	1:5 or 5:1? Does the verbal representation changes?Does the ratio changes? Is the rule broken?	There is a symbolic representation of ratio and verbal representation of the ratio influence the way of symbolic representation of the ratio.	
Analyzing equivalent ratios	"fraction" imagery	Reducing ratios;	Reasoning about equivalent ratios in a new context		Ratio table is used as a tool for proportion representation; therefore, VSF and HSF can also be adopted in proportion;	Page 11 (bottom) and Page 14 (top)
		Vertical and horizontal scale factors	creating equivalent ratios from beginning ratios that involve large numbers	Ratio table and equivalent ratios. What is the relationship between them?	Horizontal scale factors and vertical scale factors can be used interchangeably and conveniently in a horizontally extended ratio table.	
Comparing ratios	Ratio table	Finding common numerators or denominators; size of the scale factors; unit ratios	solving missing proportions	How to compare the ratios looking like "fractions". How to read the ratio matters for comparison of the ratios?	Mixture and understanding its related terms "intense", "darker", "sweet"	Pages 18-23
	No ratio tables, but can use arrow notations		comparing non-equivalent ratios		The more material a is in one amount of material b, the more intense the mixture is; LCM for ordering ratios can be used for understanding the most or the least of an attribute in a mixture.	

J. ORIGINAL INSTRUCTIONAL SEQUENCE (Stephan et al., 2015)

1

Ratio and Rates

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Contributors: This unit could not have been written without the support of our Principal Robin Dehlinger, Assistant Principal Tonya Fennell and the parents and students at Lawton Chiles Middle School.

LAUNCH

NOTES:

Alien Dream

Begin the unit by telling a story about a scary dream you had last night. The story can resemble the one below or be completely different.

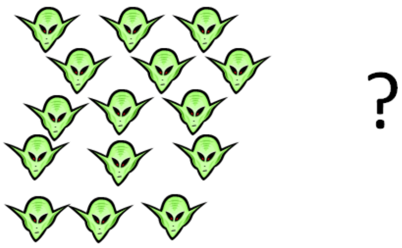
In your dream, a noise awoke you from your sleep and you ventured into the kitchen to investigate the source of that noise. You saw a swarm of aliens eating your food. They saw you and started chanting, “more bread, more bread.” You only had one bread loaf which wasn’t enough to feed the aliens, so they attacked you. You woke up with a new math lesson in mind!

RATIO AND RATES

1. Is there enough food? Explain.

2. Is there enough food? Explain.

3. How many food bars are needed to feed these aliens? Explain.



Anticipated Student Thinking:

NUMBER ONE
- Some students will say not enough food. Draw line from one food bar to 3 aliens but the last 3 do not have a food bar.
- Some will circle the entire collection of 3 aliens and 1 food bar

NUMBER TWO
- There is more than enough food because they draw a line from 3 aliens to 1 food bar until all aliens are used up
- Some will circle the aliens/food bar collection
- Some might say that you could feed 3 more aliens if you had another food bar

NUMBER THREE
- Many students will say that there are 5 food bars needed because they circle groups of 3 aliens,
- They draw a food bar and link it to 3 aliens as many times as they can (5 times)

Big Mathematical Idea(s): The idea of this page is to encourage students to link two composites together.

Rationale: It is an easy page for students so it should only take a couple of minutes as a beginning page. Make sure to highlight the links that students form between a food bar and a composite of aliens, either verbally or with symbols.

Teacher Notes:

LAUNCH: Let students know that that one food bar can feed 3 aliens comfortably. However, if the aliens are not fed adequately, there may be an intergalactic war. You need to ensure that this does not happen. Write some proof on your paper to show that there is enough food or there is more needed.

EXPLORE: About 2-5 minutes

DISCUSSION: The discussion should focus on the link between 3 and 1. Call on students who have drawn some type of symbol that illustrates the link. Ask students explicitly what that line stands for and hopefully they will explain that for every 3 aliens, there is one food bar.

As a follow up question to number 3, you might ask the class what would happen if one more alien were added. Some might say that you would need another food bar, some might say you would only need part of a food bar, some might say 1/3 food bar. Encourage that 3-1 link.

1. Will 12 food bars be enough to feed 36 aliens? Explain.

2. Will 24 food bars be enough to feed 72 aliens? Explain.

3. Will 6 food bars be enough to feed 18 aliens? Explain.

4. Will 8 food bars be enough to feed 20 aliens? Explain.

5. How many food bars are needed to feed 39 aliens? Explain.

Anticipated Student Thinking:

NUMBER ONE
- 36 divided by 12=3
- Draw 12 food bars with 3 aliens next to each
- Makes a table 1, 3; 2, 6; 3, 9; 4, 12; 5, 15 etc.

NUMBER TWO
- Some might relate this one to number one and double 12, so double 36.
- 72 divided by 24 is 3
- Some might add 12 to 12 to get 24 and add 12 to 36 to get 48
- Some might draw 24 food bars and 72 aliens.
- Make a table again with every value in it
- Continue the picture or table from number one

NUMBER 3
- 18 divided by 6 is 3
- Divide 24 in number 2 by 4, so divide 72 by 4 to get 18
- Draw or make a new table
- Look at their table from #1 or #2

NUMBER 4
- 39 divided by 3 is 13
- Draws 39 aliens and circles 3 at a time
- Makes a table and stops at 13

Big Mathematical Idea(s): The idea of this page is to encourage students to link two composites together and to begin to organize these links when there are large quantities involved.

Rationale: Students need to find a way to organize the links as they increase in size. A ratio table should be introduced from students' work on this page.

LAUNCH: As students to write something down to show how they solved each problem. They need to make sure they feed the aliens appropriately so the intergalactic war does not start.

EXPLORE: 10 minutes

DISCUSSION: Sequence the solutions by having a student who did the division strategy first. The student likely will not be able to explain why they divided. Move to the students who drew the picture, table second, additive reasoning (incorrect) next. Ask students what is common and different about the picture and table strategy. They will likely say that they both take a long time to create but the table is quicker. In fact, name the table method as a ratio table and acknowledge that it is quicker than drawing pictures but that it represents the picture in a more organized way. Do not play up the division strategy today. Let students know that they can use ratio tables or pictures to justify their thinking on future problems since those are the ones that seem to make sense to most students.

1. How many food bars are needed to feed 30 aliens? Explain

2. How many aliens can you feed with 5 food bars? Explain

3. Using a table show how many food bars you would need to feed 70 aliens?

4. How many food bars do we need to feed 35 aliens? Explain.

Anticipated Student Thinking:

NUMBER ONE:

- Students might construct a ratio table and multiply 5 aliens by 6 to get 30 aliens and 1 by 6 to get 6 foodbars
- Some might create a long ratio table 1, 5; 2, 10; 3, 15 etc.
- Some might say 30 divided by 5 is 6

NUMBER TWO:

- Some might look in their long ratio table from number 1 to get 25.
- Others might create an entirely new long table
- Some might create a short table multiplying 1 x 5 and 5 x 5
- Some might look at their short table from above and go back 1 food bar and 5 aliens

NUMBER THREE:

- Long table versus a short table for 14 food bars
- Some might use answer for number one 6 food bars for 30 aliens, so 12 food bars for 60 aliens, then add up to 70 aliens

NUMBER FOUR:

- Might start with number one and add one more food bar
- Short or long table

Big Mathematical Idea(s): Begin to reason with short or long ratio table; constructing a build up strategy versus an abbreviated strategy

Rationale: Students should begin to use a table to organize their thinking

Teacher Notes:

LAUNCH: Begin with a warm up…show a long table with the rule 1 food bar feeds 4 aliens, how many aliens to feed 44 aliens? Show a ratio table with 1 and 4 in it and ask students if anyone can go straight to the 44 aliens without having to write all the stops in between. They don't have to do it, but see if anyone can. Have a discussion about what the x11 means. It is more than the number you multiply by…it is 11 sets of 1 food bar and 11 sets of 4 aliens. Have students use a picture to explain it.

Launch this page and suggest that students use either a short or long ratio table but they should be prepared to explain their thinking.

EXPLORE: about 10 minutes

DISCUSSION: Begin with number one and ask students who drew a picture, drew a long ratio table, and a short ratio table to show their thinking at the board. Ask students to compare the ways. Which one is shorter? What does the x# mean in the short ratio table? Teacher should facilitate a horizontal scale factor as *groups of* food bars and *groups of* aliens. If anyone constructs an additive strategy, have them present it as a contrast.

Two food bars feed 4 aliens

1. Will 10 food bars be enough to feed 20 aliens? Explain.

2. Will 12 food bars be enough to feed 22 aliens? Explain.

3. How many aliens can 14 food bars feed? Explain.

4. How many aliens will 98 food bars feed? Explain.

5. How many food bars are needed to feed 16 aliens? Explain.

Anticipated Student Thinking:

NUMBER ONE:

- Some students will circle one food bar and 2 aliens; so 10 food bars will feed 20 aliens
- Some will put it in a ratio table and make a long one to find that it is true
- Others will make a short table
- Some will still be making a drawing

NUMBER TWO:

- Students will put 12 food bars in a ratio table and try to get as close to 22 as possible. They will see they can't.
- Others will find the "unit rate" as 1 for 2 and say that you'd need 11 food bars only

NUMBER THREE:

- Short or long ratio table; either going to unit rate 28 or long table
- Some will say 14 divided by 2 times 14

NUMBER FOUR:

- Short or long table; some unit rates, some not

NUMBER FIVE:

- Short or long tables with unit rate or not

Big Mathematical Idea(s): using non-unit rates to create equivalent ratios; what does a unit rate mean; usefulness of unit rate

Rationale: students might begin to construct unit rate as a mathematical object that can be useful for determining a number of equivalent ratios; students get more facile with short ratio tables

Teacher Notes:

LAUNCH: Bellwork: show a long table that starts at 1/2; 2/4; 3/6; 4/8; 5/10; 6/12. Ask students to predict how many aliens will be fed by 10 food bars; students might continue the long table; some might notice a vertical relationship between the food bars and aliens (there's always twice as many aliens as food bars), so

EXPLORE: about ten minutes

DISCUSSION: pictures, long tables, short tables, tables that go to the unit rate (in that order), have a discussion about unit rate and why it was a good one to go to for these problems. Highlight the unit rate strategy as a useful one and if it feels right, name it "Stephanie's" unit rate or whoever introduced it. Naming it after a person gives ownership and will be remembered more.

How many aliens will 26 food bars feed?

SARAH:

x 13

aliens	3	39
FB	2	26

x 13

SAMPSON:

aliens	3	6	9	12	15	18	21	24	27	30	33	36	39
FB	2	4	6	8	10	12	14	16	18	20	22	24	26

SALLY:

+ 24

aliens	3	27
FB	2	26

+ 24

STEVE:

aliens	3	39
FB	2 x1.5	26 x1.5

WHO DO YOU AGREE WITH AND WHY? It is okay to agree with as many people as you want.

Anticipated Student Thinking:

Almost all students will choose Sarah, Sampson and Steve because each of them gets 36 as the answer. However, they may not know why Sally's does not work and why Steve's DOES work.

Big Mathematical Idea(s): Additive versus multiplicative reasoning/proportional reasoning

Rationale: Students need to analyze the strategies that are correct and incorrect. Also, this is a chance to explore what the scale factors in Steve and Sarah's stands for (aliens always 1.5 times greater than foodbars; unit rate versus 13 groups of aliens). Why can't you add by the same amount?

Teacher Notes:

LAUNCH: Let students know that one of your classes last year solved the problem in four different ways. Which one or ones do they agree with and be prepared to defend the choice.

EXPLORE: 5 minutes

DISCUSSION: Who agrees with Sarah, Sally, Steve, Sampson? What are your rationales? What does Sarah's x13 stand for? Where does it show up in Sampson's long table? Why can't you just add 24 and 24? Didn't your teacher tell you last year that whatever you do to the top you do to the bottom? What does the x1.5 stand for in Steve's? Where does it show up in Sampson's long table? Where does it show up in the picture rule?

Two food bars will feed 5 aliens

1. Will 6 Bars feed 15 aliens? Explain.

2. How many aliens will 1 Bar feed? Explain

3. How many aliens will 20 Bars feed? Explain.

4. How many Bars will you need to feed 35 aliens?

Anticipated Student Thinking:

NUMBER ONE:

- Some students will make a short table starting with 2/5 and multiply both by 3 to see that 6 will feed 15.
- Some students will still draw out a long table

NUMBER TWO:

- Some students will argue that it cannot be done because you cannot feed half an alien
- Some students will say 1 food bar will feed 2.5 aliens.
- Some will be confused

NUMBER THREE:

- Some will make a short table either starting with 2/5 or 1/2.5 and scale up to 20 food bars

NUMBER FOUR:

- Some will make a long rate table and go up to 35 aliens
- Some might make a short table and scale up to 35 aliens, either starting with 2/5 or 1/2.5 to get 14 food bars.
- Some might accidentally go up to 35 food bars in a short table

Big Mathematical Idea(s): Using non-unit rates to create equivalent ratios; what does a unit rate mean; usefulness of unit rate

Rationale: In this case, the unit rate is a decimal and students will need to think about what that means.

Teacher Notes:

LAUNCH: same as usual

EXPLORE: 10 minutes

DISCUSSION: Ask if anyone had trouble answering question one and make sure they all agree with the answer. Begin discussion with number 2. Some students will argue that you cannot answer this one because it doesn't make sense to feed a half alien. Other students might argue that you can have an alien "half full". Praise this type of thinking and say that we will allow aliens to be partially full from now on.

Finish the page by discussing students' solutions that use the unit rate or not. If someone mixes up 35 food bars with aliens, reiterate that labels in the ratio table are very helpful for organizing work.

One Food Pack feeds 3 Wumps and 2 Krumps

1. How many Wumps and Krumps will 5 food packs feed? Explain.

2. How many Wumps and Krumps will 10 food packs feed? Explain.

3. How many Wumps and Krumps will 15 food packs feed? Explain.

4. There are 21 Wumps and 14 Krumps. How many food packs are needed? Explain.

5. There are 24 Wumps and 16 Krumps. How many food packs are needed? Explain.

Anticipated Student Thinking:

NUMBER ONE:
- Students might draw a picture of 5 food packs and 5 sets of 3 wumps/2 krumps to get 15 wumps and 10 krumps
- Some might draw a ratio table with three rows and either do a long or short table

NUMBER TWO:
- Same as above
- Some students may notice that there are twice as many food packs in this question so the wumps and krumps will be doubled as well

NUMBER THREE:
- Same as above
- Some might notice that the food packs keep going up by 5s, so the wumps will go up by another 15 and the krumps will go up by another 10

NUMBER FOUR:
- Short or long tables, pictures
- Some might notice that the wumps are 7 times bigger and the krumps are 7 times bigger, so the food packs must be 7 times bigger

NUMBER FIVE:
- Same as number four

Big Mathematical Idea(s): Linking 3 composite units

Rationale: This page is not necessary, but could be interesting if there is time. This page requires students to coordinate three units simultaneously which could expand their notion of a ratio/rate. Could be used as a challenge page or as homework.

Teacher Notes:

LAUNCH: A different galaxy introduces us to wumps and krumps aliens. In this case one food pack can feed 3 wumps and 2 krumps. Can you figure out the questions below? If you'd like to draw a picture or use a ratio table to show your thinking, please feel free.

EXPLORE: 10-15 minutes

DISCUSSION: Depending on how this paper is used, there may or may not be a discussion. If there is a discussion, focus on the multiplicative relationships and what the scale factors stand for in each problem.

1. Fill in the missing values in each table below.

Food Bars	2	1
aliens	6	?

Food Bars	2	7
aliens	6	?

Food Bars	2	?
aliens	6	7

Food Bars	2	?
aliens	6	1

2. Use a table to find how many food bars are needed to feed 48 aliens.

3. Use a table to find how many food bars will feed 9 aliens.

4. Use a table to find how many food bars will feed 27 aliens.

5. To find out how many food bars will feed 30 aliens, Carol says, "In problem 4, there were 27 aliens and now I have to feed 30 aliens. They just added 3 more aliens so I'll just add three more food bars!"

Is Carol right or wrong? Explain.

Anticipated Student Thinking:

NUMBER ONE:

- Some might multiply x3 vertically or divide by an appropriate scale factor horizontally
- Some might give up because either the scale factor is a decimal or the resulting food bar or alien is a decimal
- Some might use additive reasoning here

NUMBER TWO:

- Some might start from unit rate and scale up horizontally or vertically
- Some might scale up from 2/6 horizontally or vertically
- Some might get 48 aliens reversed with 48 food bars

NUMBER THREE:

- Some might start with the unit ratio to get 3 food bars
- Some might start from 2/6 and multiply either horizontal or vertically to get 3
- Some might start with 2/6 and add 3 to each to get 5 food bars

NUMBER FOUR:

- Same as above
- Some might take their answer from number 3 and triple it

Big Mathematical Idea(s): explore the meaning of a decimal scale factor; explore additive versus multiplicative reasoning again

Rationale: students should be fairly comfortable with solving alien problems. Introducing decimal scale factors might bring some of them back to additive reasoning

Teacher Notes:

LAUNCH: I decided to give you problems in table form now. See if you can figure out some of these tough ones.

EXPLORE: 10-15 minutes

DISCUSSION: Begin with problem one and discuss vertical and horizontal methods. Many students will suggest that the "rule" is missing but then others will counter that the rule can be found in the table…1 food bar can feed 3 aliens. Really dig in on the tables that have decimal scale factors. What do those mean? Which way is easier, horizontal or vertical? Sometimes one is easier to calculate than the other. Make sure you discuss number 5 to bring out the meaning of additive versus multiplicative again. Be careful…some students may have formed a very procedural understanding of this…"whatever you do to the top you do to the bottom, unless it is addition and subtraction". If needed, go back to a simple ratio situation to explore this in a more conceptual way.

1. Fill in the missing values in each table below.

Food Bars	3	6
aliens	5	?

Food Bars	3	9
aliens	5	?

Food Bars	3	10.5
aliens	5	?

Food Bars	3	?
aliens	5	2.5

2. Use a table to find how many food bars are needed to feed 90 aliens.

3. Use a table to find how many food bars will feed 150 aliens.

4. Use a table to find how many food bars will feed 75 aliens.

5. Use a table to find how many aliens can be feed with 48 food bars.

Anticipated Student Thinking:

NUMBER ONE:
- Horizontal scale factors
- Vertical scale factors
- Additive reasoning

NUMBER TWO:
- Use 3/5 to scale up horizontally to get 90 aliens 54 food bars
- Use unit rate

NUMBER THREE:
- Most popular will be horizontal scale factors

NUMBER FOUR:
- Same as above

NUMBER FIVE:
- Same as above
- Some might confuse 48 food bars with aliens

Big Mathematical Idea(s): determining the most useful scale factor (horizontal vs. vertical); additive reasoning

Rationale: students need more practice with decimal scale factors and determining which one will be easier (vertical or horizontal); help them become more flexible with choosing

Teacher Notes:

LAUNCH: same as before

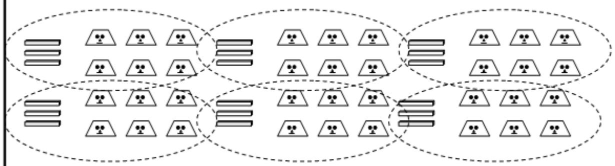
EXPLORE: 10-15 minutes

DISCUSSION: highlight solutions that used easy scale factors rather than decimals

Anticipated Student Thinking:

How many Wumps can be fed with 18 food bars?

JANE'S SOLUTION:



Jane says, "I drew a picture. Every time I added 3 foodbars, I added 6 Wumps. So, 3+3+3+3+3+3=18 food bars because there are three in each circle. Then, 6+6+6+6+6+6=36 Wumps since there are 6 Wumps in each of the 6 circles."

JACK'S SOLUTION:

+3 +3 +3 +3 +3

Foodbars	3	6	9	12	15	18
Wumps	6	12	18	24	30	36

+6 +6 +6 +6 +6

Jack says, "I started with the 3 foodbars and 6 Wumps. Every time I added 3 foodbars, I added 6 Wumps. I stopped when I got to 18 food bars and saw that I had 36 Wumps."

JILL'S SOLUTION:

Foodbars	3	18
Wumps	6	36

x6

Jill says, "I started with 3 foodbars and 6 Wumps. I took a shortcut and instead of adding 3 six times and adding 6 six times, I multiplied both times 6 to get my answer."

Sally says that all three of these give the same answer. She also says that Jack's is the same as Jane's but faster, and that Jill's is the same as both of them, just faster than both. Do you agree with Sally? Explain.

Big Mathematical Idea(s): Assessment of how students understand the nature of the scale factors and the procedures they have used with ratio tables

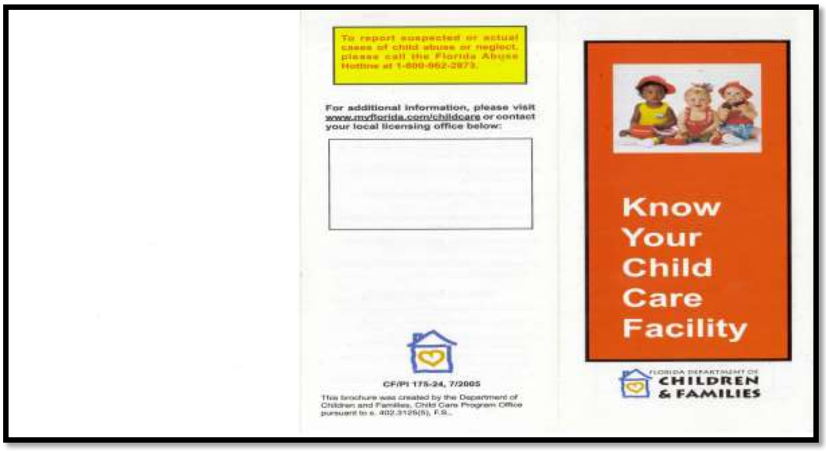
Rationale: what sense do students have of their actions so far

Teacher Notes:

LAUNCH: 2 minutes…quiz or homework

EXPLORE: 5-10 minutes

DISCUSSION: No discussion. This is an assessment. You might come back to discuss this page if students show you that the discussion is needed.



To report suspected or actual cases of child abuse or neglect, please call the Florida Abuse Hotline at 1-800-962-2873.

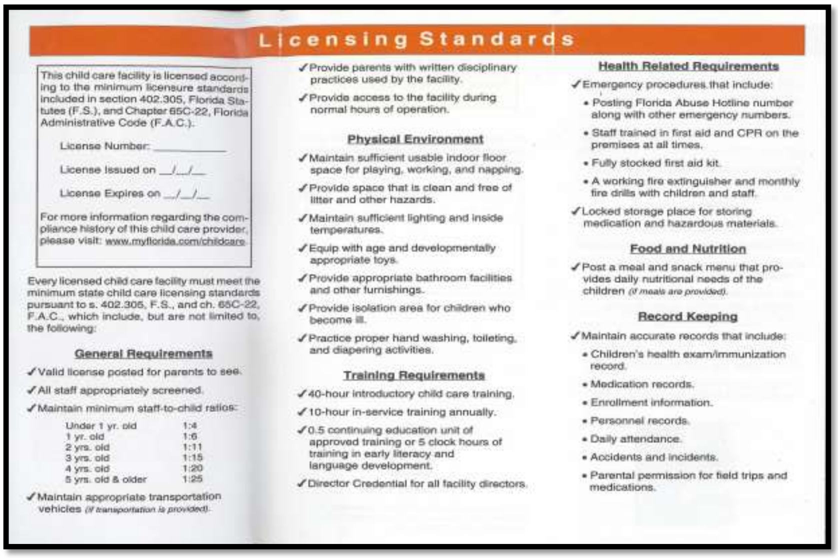
For additional information, please visit www.myflorida.com/childcare or contact your local licensing office below:

CF/PI 175-24, 7/2005

This brochure was created by the Department of Children and Families, Child Care Program Office pursuant to s. 402.3125(5), F.S..

Know Your Child Care Facility

FLORIDA DEPARTMENT OF CHILDREN & FAMILIES



Licensing Standards

This child care facility is licensed according to the minimum licensure standards included in section 402.305, Florida Statutes (F.S.), and Chapter 65C-22, Florida Administrative Code (F.A.C.).

License Number: ____________

License Issued on __/__/__

License Expires on __/__/__

For more information regarding the compliance history of this child care provider, please visit: www.myflorida.com/childcare

Every licensed child care facility must meet the minimum state child care licensing standards pursuant to s. 402.305, F.S., and ch. 65C-22, F.A.C., which include, but are not limited to, the following:

General Requirements

✓ Valid license posted for parents to see.

✓ All staff appropriately screened.

✓ Maintain minimum staff-to-child ratios:

Under 1 yr. old	1:4
1 yr. old	1:6
2 yrs. old	1:11
3 yrs. old	1:15
4 yrs. old	1:20
5 yrs. old & older	1:25

✓ Maintain appropriate transportation vehicles *(if transportation is provided)*.

✓ Provide parents with written disciplinary practices used by the facility.

✓ Provide access to the facility during normal hours of operation.

Physical Environment

✓ Maintain sufficient usable indoor floor space for playing, working, and napping.

✓ Provide space that is clean and free of litter and other hazards.

✓ Maintain sufficient lighting and inside temperatures.

✓ Equip with age and developmentally appropriate toys.

✓ Provide appropriate bathroom facilities and other furnishings.

✓ Provide isolation area for children who become ill.

✓ Practice proper hand washing, toileting, and diapering activities.

Training Requirements

✓ 40-hour introductory child care training.

✓ 10-hour in-service training annually.

✓ 0.5 continuing education unit of approved training or 5 clock hours of training in early literacy and language development.

✓ Director Credential for all facility directors.

Health Related Requirements

✓ Emergency procedures that include:

- Posting Florida Abuse Hotline number along with other emergency numbers.
- Staff trained in first aid and CPR on the premises at all times.
- Fully stocked first aid kit.
- A working fire extinguisher and monthly fire drills with children and staff.

✓ Locked storage place for storing medication and hazardous materials.

Food and Nutrition

✓ Post a meal and snack menu that provides daily nutritional needs of the children *(if meals are provided)*.

Record Keeping

✓ Maintain accurate records that include:

- Children's health exam/immunization record.
- Medication records.
- Enrollment information.
- Personnel records.
- Daily attendance.
- Accidents and incidents.
- Parental permission for field trips and medications.

Tiny Tots

Tiny Tots Daycare has a teacher to infant ratio of 1:5.

Teachers	1	
Infants	5	

1. How many teachers must be in the room if there are 25 infants? Explain.

2. What is the maximum number of infants that can be in the room if there are 6 teachers? Explain.

3. Tiny Tots Daycare has a toddler to teacher ratio of 8 to 1. Use a ratio table to determine how many teachers can be in the room if there are 24 toddlers. Show your work and explain.

4. Decide which of the ratios below belongs in the Teacher/Infant table and which belong in the Teacher/Toddler table.

2/16	4/20	80/400	2:10
8/64	6 to 48	7:35	7/56

Anticipated Student Thinking:

NUMBER ONE:
- Not much diversity

NUMBER TWO:
- Either 5 x 6 because of 1 x 6 or 25 divided by 5.

NUMBER THREE:
- Many students will use horizontal or vertical scale factors
- Some students will get confused with the 8 coming first

NUMBER FOUR:
- Many students will "reduce" each ratio to the unit ratio and compare
- Other students will see if the denominator is divisible by 8 or 5. Those that are divisible by both are problematic
- Other students will compare the numerator to the denominator and find a vertical scale factor (e.g., 2x8 is 16)

Big Mathematical Idea(s): reasoning about equivalent ratios in a new context

Rationale: can student reason proportionally in a new context? Compare equivalent ratios. Learn the terminology and conventions of ratios, the way they are worded as well as the way they are written.

Teacher Notes:

LAUNCH: Give each student a day care brochure provided. Ask them to look closely at General Requirements section and discuss what those numbers mean. We have been studying ratios in this class…any time we compare two amounts like food bars to aliens or infants to teachers, that is called a ratio. Add to word wall. Ask students what the ratio of teacher to student in your classroom is and write it on the board. Tell students that you want this next page will ask them questions about a fictional daycare called Tiny Tots.

EXPLORE: 10-15 minutes

DISCUSSION: Not much discussion on numbers 1 and 2. Some discussion might occur on 3 in terms of how to set up a ratio table and the way a ratio is written. Number four is the most interesting one, so have a variety of student thinking presented. No order necessary.

1. Bill conjectured that for every person that is in favor of the Iraq War, 5 people are against it.

For	1	2
Against	5	?

a. According to Bill, if 2 people are in favor of the war, how many people are against it?

b. If there are 10 people for the war, how many people are against it?

c. If there are 7 people for the war, how many people are against it?

d. If there are 45 people AGAINST the war, how many are for it?

2. A recipe for chocolate chip cookies calls for 2 cups of sugar and 4 cups of flour.

Sugar		2
Flour		4

a. How many cups of flour are needed if we use only one cup of sugar?

b. How many cups of flour are needed for a recipe that uses 8 cups of sugar?

c. How many cups of flour are needed for a recipe that uses 9 cups of sugar?

d. How many cups of flour are needed for a recipe that uses 11 cups of sugar?

e. How many cups of SUGAR are needed for a recipe that uses 12 cups of flour?

Anticipated Student Thinking:

NUMBER ONE:

- Horizontal or vertical scale factors
- Add one for and 5 against
- Add one for and 1 against

- Horizontal or vertical scale factors
- Correct/incorrect additive strategy

- Same as above

- Same as above
- Some people might not notice the AGAINST part and miscalculate

NUMBER TWO:

- Same as above for all questions
- From b to c, someone might add one more cup of sugar to get 9 and 2 more flour to get 18

Big Mathematical Idea(s): reasoning about equivalent ratios in a new context

Rationale: can student reason proportionally in a new context?

Teacher Notes:

LAUNCH: 1 minute

EXPLORE: 10 minutes

DISCUSSION: Have students present the each strategy listed above. Begin with additive strategy that is wrong. Move to the horizontal and vertical strategies last. Ensure that the last one in each section is discussed to make sure students notice the change in what is asked for.

At LCMS, 75 7th graders prefer action movies to 90 8th graders. Write a ratio table to figure out:

a. How many 8th graders prefer action movies if 25 7th graders prefer them?

b. How many 8th graders prefer action movies if 10 7th graders prefer them?

c. How many 7th graders prefer action movies if 120 8th graders prefer them?

At LCMS, 80 7th graders prefer Fergie to 64 8th graders. Write a ratio table to figure out:

a. How many 7th graders prefer Fergie if 120 8th graders prefer them?

b. How many 8th graders prefer Fergie if 10 7th graders prefer them?

c. How many 7th graders prefer Fergie if 16 8th graders prefer them?

Anticipated Student Thinking:

NUMBER ONE:

- Most likely use the horizontal scale factor
- Some try the vertical SF
- Some find the unit ratio of 1/1.2 and scale up

- Those who find the unit ratio scale up fairly easily
- Other try the HSF or VSF but with difficulty because of the decimals
- Others use an additive strategy incorrectly

- Might use answer from b to scale up quickly
- HSF and VSF or use of unit ratio

NUMBER TWO:

- Same as above

Big Mathematical Idea(s): creating equivalent ratios from beginning ratios that involve large numbers

Rationale: how will students deal with ratios that begin really large? Will they scale down appropriately with, sometimes, decimal scale factors or will they find the unit ratio first?

Teacher Notes:

LAUNCH: Survey was taken at LCMS (or whatever your school is named). Here are some of the results.

EXPLORE: 10-15 minutes

DISCUSSION: Discuss the first problem that has disagreement over the answers. Have students share various strategies and have them decide which strategies are both correct and easy.

1. Which ratios belong to the same ratio tables?

1/2	2/3
3/4	6/9
20/40	4/9
20/45	9/12

2. For every 3 shirts Marci sells, she earns $4. How much money will she earn if she sells 4 shirts?

Alex and Ann each solve this problem. Alex says the answer is $5.33 and Ann says it is $5. Their work is shown below. Who do you agree with and why?

ANN'S METHOD ALEX'S METHOD

+1 X 1 1/3

Shirts	3	4			3	4			
$$	4	5			4	5.33			

+1 X 1 1/3

Anticipated Student Thinking:

NUMBER ONE:

- Some might put ½ with 2/3 because the numerator and denominator are 1 apart in each ratio
- Some might reduce all ratios to lowest terms and see which ones match
- Create a unit rate in each one and match
- Find the VSF in each to see if they are the same
- Some might put 6/9 and 9/12 in same table because the numerators are 3 apart from each other and so are the denominators
- Some might make ratio tables to compare two at a time and find that either the SFs match or do not

NUMBER TWO:

- Some will pick Ann's and some Alex's

Big Mathematical Idea(s): Determining equivalent ratios; multiplicative versus additive reasoning with decimal scale factors

Rationale:

Teacher Notes:

LAUNCH: 1 minute

EXPLORE: 5-10 minutes

DISCUSSION: For number one, it will be important to discuss students' strategies. You might start the discussion with ½…which ones would show up in that table? Have students debate why 2/3 would not if anyone put that. If necessary, fall back on aliens and food bars picture to support students' arguments. Go through the others highlighting the various strategies above.

On number two, you might draw out T-shirts and dollars (3 shirts, 4 dollar bills) to support a discussion about what the SF versus additive number mean in that context. In Ann's, students should argue that it means 1 shirt costs $1 but Alex's means 1 shirt costs $1.33. Which is correct? A drawing might help with determining that solution.

Calculator Costs

Below is a ratio table that illustrates the price for a certain type of calculator.

Number purchased	1	2	3	4	5	10	15	20
FCAT								$120
Scientific							$240	
Graphing						$800		

1. How much does it cost to buy 53 FCAT calculators? How much to buy 27 scientific calculators? How much to buy 9 graphing calculators?

2. How many FCAT calculators can a school buy if it can spend $390? What if the school can spend only $84?

3. How many graphing calculators can a school buy if it can spend $2,500? What if the school can spend only $560?

Anticipated Student Thinking:

- Some students will find the unit rate for each calculator type and reason from there on all questions
- Some students might take half of 20 to get the price for 10, half for the price for 5 and then split that amount up by five to get each of 1-5. Same for graphing calculator.
-

Big Mathematical Idea(s): Finding missing values in a "proportion". Although not introduced yet, students are reasoning proportional every time the find a missing value in a table because reasoning proportionally means to set two ratios equivalent.**

Rationale: Reasoning proportionally in different contexts.

Teacher Notes:

LAUNCH: 1-2 minutes

EXPLORE: 15-20 minutes

DISCUSSION: This page tends to be pretty easy for students so it can be used as an assessment/quiz or homework

** This task was adapted from CMP2.

1. At a dining room table, there are 3 serving utensils for every 2 plates. If there are 10 plates, how many serving utensils are there?

2. Mitch paid $4.50 for 5 computer disks. How much did he pay for 12 disks?

3. If you can buy 2 packs of cookies for $3, how many packs of cookies can you buy for $21?

4. There are 3 boys for every 4 girls in Mrs. Smith's class. If there are 28 students in the class, how many girls and how many boys are there?

Anticipated Student Thinking:

NUMBER ONE:
- HSF or VSF
- Drawings

NUMBER TWO:
- HSF
- VSF easier
- Drawings

NUMBER THREE:
- Same as number one

NUMBER FOUR:
- Some students do not notice that the 28 refers to students not girls or boys
- Some students put ¾ in a table and then add up 3s and 4s until they get a column that totals 28
- Some students say that the ¾ boy/girl ratio adds to 7 people total and might even draw a third row in the ratio table for total. Then they scale the 7 up by a HSF of x7 and scale each individual girl, boy up by 7 each

Big Mathematical Idea(s): reason proportionally, introduce the word and symbols of proportionality

Rationale: Practice situations

Teacher Notes:

LAUNCH: 2 minutes, challenge them on number 4 but don't tell them what the challenge is

EXPLORE: 10-15 minutes

DISCUSSION: This is a good page to introduce typical proportional symbols and the definition. Problems 1-3 are very easy for students, so when discussing number 1, tell students that mathematicians like to make things easy, so instead of using a ratio table, (erase the line in between two ratios in the table) they put an equal sign in between [see example below]

For number four highlight the students who total the students. Distinguish between a ratio and fraction (part/part vs. part/whole)

** These tasks were adapted from a released FCAT test.

x5

Utensils	3	15
Plates	2	x

x5

Erase vertical lines and some of horizontal line; put equal sign

x5

Utensils $\frac{3}{2} = \frac{15}{x}$ Plates

x5

1. In hot chocolate, for every 2 cups of milk, one needs 4 teaspoons of cocoa. If a person has 5 cups of milk to make hot chocolate with, how many teaspoons of cocoa are needed?

2. These photos are the similar, but one is larger than the other. Explain how you would find the height of the larger photo.

6 cm

8 cm

? cm

16 cm

Length		
Width		

3. Sue can walk 15 miles in 5 hours. How far can she walk in 3 hours?

4. Suppose at UCF there are 4 girls for every 6 boys. There are 250 students altogether. How many boys and how many girls are in this class?

Anticipated Student Thinking:

NUMBER ONE

- HSF or VSF
- Might use a ratio table or a more traditional proportion set up

NUMBER TWO

- Some students draw an arrow from 8 cm to 16 cm and write x2; then draw arrow from 6 cm to ? and x2 to get 12 cm
- Some students use the ratio table and use HSF typically
- Some students might add 8 to the 6 cm.

NUMBER THREE

- Same as number 1

NUMBER FOUR

- Might make a ratio table with three rows, the last row containing the total of 10 students and scale up each by 25
- Some will scale up by trial and error until they find the total of 250

Big Mathematical Idea(s): solving missing proportions

Rationale:

Teacher Notes:

LAUNCH: Give a bellwork in purely symbolic form (ex. 2/5 = x/15)

EXPLORE: 10-15 minutes; challenge them with number 4 again

DISCUSSION: have students show their work, especially highlighting the symbolic solutions. Since number two is an area type problem, students might revert to additive reasoning. Make sure this one discussed if this happens. The pictures are drawn proportionally so that when students say, "I multiplied both by 2" you can ask what the x2 stands for in this context. It means that two of the short sides create the short side of the larger picture and two of the long sides create the long side of the big picture. Students can show that on the board for students.

Save #4 for last and compare strategies.

Making Blood

The Band wanted to make fake blood for their Halloween party. They found a recipe on the internet that uses water and food dye to make the blood.

Three Band students decided to experiment with the recipe to see who could make the darkest red color blood.

Michelle	Julianna	Kyle
8 oz of water 32 drops of red food coloring	12 oz of water 24 drops of red food coloring	6 oz of water 18 drops of red food coloring

Of the 3 students (Michelle, Julianna, and Kyle), who the darkest blood? Explain!

Anticipated Student Thinking:

- Some students will compare two recipes at a time changing Kyle's to 12/36 and choosing Kyle; they then compare Kyle to Michelle by changing each to a common numerator of 24 to choose Michelle
- Some students will create common numerators for all three at one time (24) and choose largest number of red drops
- Some students will create unit rates for each one (1/4, ½, 1/3) and choose Michelle. Others might choose ½ because it is the biggest number.
- Some students will just draw an arrow from numerator to denominator and say x4, x2, and x3, so Michelle's
- Some students will find the difference between the numerator and denominator (24, 12, and 12) and decide Michelle for the wrong reason
- Some might say that Michelle's is the most because it has the most red drops (32)

Big Mathematical Idea(s): comparing non-equivalent ratios

Rationale: how will students change their ratios in order to determine which amount to choose?

Teacher Notes:

LAUNCH: 1 minute

EXPLORE: 10-15 minutes

DISCUSSION: Have students debate their choices. Do not highlight the difference method since it coincidentally creates the correct solution. Have the x4 solutions go last and question the students about what that number means. Acceptable explanations either focus on there are four times as many red drops than water so that is the most. Or they can relate it to the already presented unit ratio strategy.

Fright Night

The LCMS Science Club is hosting Fright Night at Chiles Halloween night. Guests will pay $2 to walk through their Hall of Horrors. In this Hall of Horrors, the lights will be dimmed and guests will take turns feeling the contents of bowls that are filled with various foods that will feel like body parts. One of the displays in the Hall of Horrors is going to be a Bowl of Intestines. Four members of the Science Club looked on the internet to find recipes for the Bowl of Intestines and found the following:

Catherine's
2 pounds cooked noodles
10 teaspoons Karo syrup

Sean's
1 pounds cooked noodles
4 teaspoons Karo syrup

Caitlyn's
6 pounds cooked noodles
18 teaspoons Karo syrup

Kitty's
8 pounds cooked noodles
48 teaspoons Karo syrup

Anticipated Student Thinking:

Same strategies as last question.

Big Mathematical Idea(s): comparing non-equivalent ratios

Rationale: how will students change their ratios in order to determine which amount to choose?

Teacher Notes:

LAUNCH: 1 minute

EXPLORE: 10-15 minutes

DISCUSSION: Have students debate their choices. Do not highlight the difference method since it coincidentally creates the correct solution. Have the x6 solutions go last and question the students about what that number means. Acceptable explanations either focus on there are four times as many red drops than water so that is the most. Or they can relate it to the already presented unit ratio strategy.

Which recipe is the stickiest!?!

Pumpkin Pie

Below are the pumpkin and flour portions of four recipes for pumpkin pie. Which pumpkin pie recipe will be the most "pumpkiny" flavor?

Recipe A
2 cups flour
8 cups pumpkin mix

Recipe B
3 cups flour
9 cups pumpkin mix

Recipe C
8 cups flour
8 cups pumpkin mix

Recipe D
4 cups flour
2 cups pumpkin mix

Anticipated Student Thinking:

Same as before except some may say there is a tie between A and B if they reason about the difference

Some might say recipe B because it has the most pumpkin cups

Big Mathematical Idea(s): comparing non-equivalent ratios

Rationale: how will students change their ratios in order to determine which amount to choose?

Teacher Notes:

LAUNCH: 1 minute

EXPLORE: 10-15 minutes

DISCUSSION: Have students debate their choices. Highlight the difference method since recipe A and B have the same difference.

Anticipated Student Thinking:

Same as before but some might say a tie between 1 and 4 because they are both 1.6 scale factor

Crazy Cauldron Lady

Crazy Cauldron Lady wants to make the hottest guy in town, named Lars, fall in love with her this Halloween. However, she is an extremely ugly witch and needs the most help she can get. She has four different love potions to choose from, but needs the strongest love potion she can find to make Lars fall in love. Which of the following love potions will give her the best chance at love?

Love Potion 1	Love Potion 2	Love Potion 3	Love Potion 4
5 ounces water 8 love drops	3 ounces water 6 love drops	6 ounces water 9 love drops	3 ounces water 5 love drops

Big Mathematical Idea(s): comparing non-equivalent ratios

Rationale: how will students change their ratios in order to determine which amount to choose?

Teacher Notes:

LAUNCH: 1 minute

EXPLORE: 10-15 minutes

DISCUSSION: Have students debate their choices. Highlight the difference method since recipe 1, 2 and 3 have the same difference. Also, the one that has the most love power is the one with the smallest difference.

The Orangey Obstacle

The Party Committee is planning the Spring Dance for LCMS. Four LCMS students bring recipes for orange punch. Which is the orangey-est?

Buffy's Punch	Willow's Punch
2 cups orange juice 3 cups Sprite	5 cups orange juice 9 cups Sprite

Xander's Punch	Gile's Punch
1 cup orange juice 2 cups Sprite	3 cups orange juice 5 cups Sprite

Fraction Dilemma

Tara says, "1/2 of Xander's punch is Orange Juice."
Oz says, "1/3 of Xander's punch is Orange Juice."

1. Which students is correct and why? Explain.

2. Suppose 240 students bought tickets to the dance. How many cups of orange juice and sprite are needed for the dance if the committee uses Gile's Punch?

3. If the committee makes juice for 240 students, how many BATCHES of Buffy's Punch will they need to make if each student drinks 1/2 cup?

Anticipated Student Thinking:

- Some students will find the unit ratio and choose the biggest number when really it is the smaller number this time when the 1 is in the numerator
- Some will find the unit ratio
- Some might pick based upon the differences

Big Mathematical Idea(s):

Rationale:

Teacher Notes:

LAUNCH:

EXPLORE:

DISCUSSION:

Anticipated Student Thinking:

Rates and Ratios

We have learned that a ratio is a comparison between two numbers. What you may not know is that the two units compared in a ratio are always the same. For example, you compared **cups** of orange juice to **cups** of sprite. You compared **number** of 8^{th} graders to **number** of 7^{th} graders (both are numbers of people). Rates are ratios that compare two **different** units. For example, when you compared **number** of calculators with their **price**, price and number are two different units. This is called a rate. When your parent drives 65 miles per hour, her speed is called a rate because it compares **miles** to **hours**.

Suppose Sasha travels from Orlando to Tampa. If you have ever made this trip, you know that the traffic is bad on some parts of I-4 and not so bad in other spots. So, sometimes Sasha has to slow down and sometimes he can go fast. He stopped three tiems to record his time and distance:

Stop One: 5 miles in 20 minutes
Stop Two: 8 miles in 24 minutes
Stop Three: 15 miles in 40 minutes

You may use a rate table to answer the questions below.

On which part was Sasha traveling fastest? Slowest?

Big Mathematical Idea(s):

Rationale:

Teacher Notes:

LAUNCH:

EXPLORE:

DISCUSSION:

K. TURKISH SUMMARY / TÜRKÇE ÖZET

ORAN ve ORANTI KAVRAMLARI HAKKINDA YEDİNCİ SINIF ÖĞRENCİLERİNİN SINIF İÇİ MATEMATİKSEL UYGULAMALARI

1. Giriş

Piaget'nin yaklaşımına göre orantısal akıl yürütme resmi olarak formal düşüncenin gelişimiyle örtüşen ortaokul matematik programlarında vurgulanır (bkz. Adjiage & Pluvinage, 2007; Ben-Chaim vd., 2012; CCSSI, 2010; MoNE, 2013, 2018). Ancak varsayım olarak uygun biliş düzeyinde olsalar bile birçok nedenden dolayı öğrenciler (Cengiz & Rathouz, 2018; Çalışıcı, 2018; Fernández vd., 2012; Jensen, 2018; Jiang vd., 2017; Karplus vd., 1975; van Dooren vd., 2009) ve öğretmenler için (Ellis, 2013; Ledesma, 2011) orantısal düşünmenin gelişimi sorunlu olmaya devam etmektedir. Öğrencilerin perspektifinden, orantısal akıl yürütme, kaybolduklarını hissettikleri bir bilgi ve beceri ağı olarak tanımlanmıştır. Öğretim sürecinin sonunda, öğrencilerden orantısal akıl yürütme ile ilgili süreçleri yapmaları beklenmektedir. Ancak, orantısal akıl yürütmedeki süreç yeterliliklerini elde etmeden önce öğrencilerin kazanmaları gereken çeşitli örtük bilgi ve beceriler vardır. Başlangıçta, orantısal ve orantısal olmayan durumlar (de Bock vd., 2007; Modestou & Gagatsis, 2007; van Dooren vd., 2009), sürekli ve süreksiz değişkenler (Boyer vd., 2008; Fernández vd., 2012), ters orantı içeren problem durumları, problem türleri (Lamon, 1993), birimlerin ve "birimin birimi"nin anlaşılması (Battista & van Auken Borrow, 1995; Behr vd., 1994; Lamon, 1993) gibi konuları ayırt etmeleri gerekir.

Bu problemlerden yola çıkarsak, öğrencilerin orantısal düşünmenin gelişimi için basitten karmaşığa doğru tutarlı bir rehberliğe ihtiyaç duyduğu düşünülebilir. Son araştırmalar sayesinde öğretme-öğrenme sürecinin çeşitli kombinasyonları deneysel olarak test edilmeli ve sistemli bir şekilde gözden geçirilmelidir. Eğitim araştırmaları, uygulamadan izole edilmemelidir. Bunun için, araştırma bilgisi öğretim ve öğrenme için kullanışlı bilgiye dönüştürülmelidir. Mevcut çalışma gerçek bir sınıf ortamında yürütülmektedir. Eğitim tasarımı araştırmaları araştırmacılara, deneysel ve izole edilmiş sınıflardan orta ölçekli öğrenci ve öğretmenler tarafından işletilen sıradan sınıflara aktarılabilecek etkili müdahaleler yapmalarına olanak tanımaktadır (Brown, 1992). Çocukların gerçek yaşam ve sosyal olarak inşa edilmiş fikirlerini kullanarak, oran ve orantıyı anlamak için kendi öğrenme sistemlerini geliştirmeleri, temsil etmeleri, savunmaları ve çürütmeleri beklenmektedir. Sonuç olarak, alan yazındaki eğilim, iş birliği, etkin öğrenme, problem çözme ortamında oran ve orantının öğretimini ve öğrenimini etkili hale getirmek için bazı önceden belirlenmiş öğretim bölümlerinin kullanılması yönündedir.

Sınıf içi matematiksel uygulamalar, öğrencilerin matematik etkinliklerini yorumladığı bir öğretim dizisi içinde ortaya çıkan etkileşime dayalı dinamik bir akıl yürütme ile elde edilen özgün fikirler veya stratejilerdir (Bowers et al., 1999; Cobb et al., 2001, 2011). Bu uygulamalar, bir öğrencinin değil, *sınıfın akılcı düşünme şeklinin bir ürünüdür* (Bowers et al., 1999). Sınıf içi matematiksel uygulamaları oluşturmak için matematiksel anlayış ve akıl yürütme desteği sağlayan bir öğrenme ortamı yaratmak faydalı olacaktır ve bu sayede paylaşılan fikirlerin ortaya çıkması mümkün olacaktır (Cobb et al., 2001, 2011; Stephan et al., 2003). Bu uygulamalar, tartışma, işbirliği ve görüşme süreçlerine katılarak, işbirliği ortamı yaratan, toplu tartışma ortamı sağlayan akıl yürütme, açıklama ve diğerlerini ikna etme ile ilgili özgün bilgiler sağlarlar (Gravemeijer et al., 2000; Streefland, 1991). Bu çalışmanın odak noktası, Lamon' un (2007) önerdiği gibi orantısal düşünmeyi araştırmak için bir eğitim tasarımı araştırması yöntemi takip ederek gelişen fikirlerin ve öğrencilerin anlayışının genişliğini ve derinliğini dokümente etmektir. Lamon (2007), varsayımsal öğrenme rotasını destekleyen bir tasarımın gerekliliğini ortaya koymuştur. Stephan ve arkadaşlarının da (2015), alanın ihtiyaçlarını dikkate alarak oran ve orantı öğretimi için bir öğretim tasarımı geliştirdiler. Varsayımsal öğrenme rotasının amaçlarından biri, belirli bir matematiksel kavram için öğretim planı tasarlamak, bu konunun öğretimini ve öğrenimini teşvik etmektir (Simon ve Tzur, 2004). Bu çalışmanın önemi, Stephan ve arkadaşlarının (2015) geliştirdiği bu öğretim tasarımının ve öğrenme rotasının öğretmen ve öğrenci katılımı yoluyla gözden geçirilmesidir. Bu rotalar ve öğretim dizilerinin test edilmesi ve gözden geçirilmesi, oran ve orantı kavramlarıyla ilgili başarılı ve başarısız uygulamaları gösterebilir. Takip edilen tasarım tabanlı araştırma ise, "Nasıl, ne zaman ve neden" çalıştığını dikkate alarak "Ne işe yaradığını" gösteren retrospektif bir analiz ile bizi bu bilgiye ulaştırır (Cobb vd., 2003). Dolayısıyla, teori ve uygulama arasındaki bağlantıyı tasarıma ve test edilebilir sanılara yansımalarını ortaya çıkarabilir. Bu şekilde, öğrencilerin fikirlerinin ilerlemesinin açıklanması yoluyla alana özgü öğretim teorileri oluşturulur ve bu fikirler, alana özgü bu öğretim teorilerine katkıda bulunur (Cobb, 1999). Eğitimde tasarı tabanlı araştırma, bu araştırma için oran ve orantı kavramlarının alana özgü öğrenim ve öğretimini araştırmak için uygulandı ve orantılı düşünme gelişiminin bir sonucu olarak bir uygulama yapıldı. Özellikle, tasarım ekibi, görevlerin ve öğretim yaklaşımlarının çeşitliliği ile zengin bir öğrenme ortamı oluşturmuş ve öğrencilerin etkililiklerle etkileşimini aşamalı analizlerle (Confrey, 2006) ortaya koymuştur.

Özetle, kesir, rasyonel sayılar, nicel düşünme becerisi, olasılık, cebirsel düşünme gibi birçok matematiksel konu ve beceriyi içerisinde barındıran orantısal düşünme becerisi ile ilgili öğrencilerin yaşadığı sıkıntılar alan yazın çalışmalarında rapor edilmiştir. Bu sorunları çözmeye yönelik çeşitli öğrenme ve öğretmeye yönelik yaklaşımlar geliştirilmiştir. Bu yaklaşımlar göz önünde bulundurulduğunda sınıf ortamı ve araştırmalar arasında uygulama yönünden boşluk olduğu gözlenmiştir. Matematik eğitimi araştırmaları bu boşluğu doldurabilmek adına öğrenme ve öğretmenin sistematik olarak incelendiği tasarım tabanlı araştırmalara (Design-Based Research Collective, 2003) yönelmektedir. Bu araştırma yöntemi, deneysel çalışmalardan farklı olarak, öğretmen ve tasarım ekibi ile öğrenme ve öğretmeyi gözlemleme, değiştirme, yenilemeye teşvik etmektedir.

Bu çalışmanın amacı, eğitimde tasarı tabanlı araştırma yöntemi kapsamında belirli bir öğretim dizisi aracılığıyla yedinci sınıf öğrencilerinin oran ve orantı kavramlarına ilişkin *Sınıf İçi Matematiksel Uygulamalarını (SMU)* araştırmaktır. Çalışma boyunca oran ve orantı kavramları için varsayıma dayalı öğrenme rotası ve buna dayalı olarak geliştirilmiş öğretim dizisi uygulanmıştır. Bu öğretim dizisinde elde edilen sınıf içi matematiksel uygulamaların öğrencilerin bilişsel gelişimlerine paralel olarak değişir. Öğretmen, öğrenme sürecinde bir rehber ve öncüdür; öğrenciler ise kendi bireysel öğrenmelerinden sorumludur. Bu nedenle, bu çalışma oran ve orantı kavramları için önceden tasarlanmış olan varsayımsal öğrenme trotası kullandı. Araştırma sorusu şu şekilde belirlenmiştir: Eğitimde tasarı tabanlı araştırması ortamında, oran ve orantı öğretim dizisinin uygulanmasıyla yedinci sınıf öğrencilerinin hangi sınıf içi matematiksel uygulamaları ortaya çıkmıştır?

Bu çalışmanın teorik arka planı sosyal ve bireysel öğrenmeyi ele alan birleştirici yaklaşım (emergent perspective) tarafından yönlendirilse de, çalışmanın odağı sosyal boyuttaki sınıf içi matematiksel uygulamalarıdır. Sınıf içi matematiksel uygulamaların etkisi retrospektif analiz yoluyla görünür hale getirilir. Araştırma sorusunda belirtildiği gibi, bu çalışma yeni bir varsayımsal öğrenme rotası veya öğretim dizisi geliştirmeyi amaçlamamaktadır; bunun yerine oran ve orantı kavramları için zaten belirlenmiş olan önceden Stephan ve arkadaşları (2015) tarafından tasarlanmış varsayımsal öğrenme rotasını öğrencilerin sınıf içi matematiksel uygulamaları göz önünde bulundurularak öğretim süreci boyunca test etmek ve gözden geçirmek hedeflenmektedir.

2. Yöntem

2.1. Araştırma Deseni

Eğitim tasarımı araştırması, etkili bir öğretme ve öğrenme ortamında ortamın ortaya çıkan özelliklerine (Design-Based Research Collective, 2003) yanıt olarak bu öğretim materyallerinin, programların, müfredatın veya araçların tasarımını sistematik olarak araştırmak için bir araştırma perspektifini tanımlar. Stephan vd., 2015). Çalışmanın önemine uygun olarak oran ve orantı bağlamının gerçek paydaşları; yedinci sınıf öğrencileri, öğretmenleri ve matematik eğitiminde alan uzmanlarının ortak anlayış içinde öğretme-öğrenmeyi, içeriği, söylemi, materyalleri vb. birlikte araştırılmasına ihtiyaç duyulmuştur. Bu tür bir eğitim tasarımındaki hareketlilik, birçok yönden en iyi şekilde eğitim tasarımı araştırması çerçevesinde incelenebilir. Birincisi, tasarım araştırması paradigması kısmen, dinamik ve canlı bir sınıf ortamına duyulan ihtiyacın çoğunlukla göz ardı edildiği diğer araştırma metodolojilerinin eksikliklerinden ortaya çıkmıştır (Brown, 1992). Tasarım araştırmasının müdahaleci yapısı, çalışma yapılmadan önce, çalışma sırasında ve sonrasında araştırma planını ayarlayarak bu dezavantajın üstesinden gelmeyi amaçlar (Plomp, 2013). Öğretim materyalleri ve önerilen öğrenme süreci, gerçek sınıf ortamında gerçekleştirilir ve tasarım araştırmasında potansiyel öğrenme yolları ortaya çıkar (Cobb vd., 2003).

Tasarım, "öğrenmeyi teşvik etmek veya bir eğitim sorununu çözmek için tasarlanan nesne (modül, birim, araçlar, sınıf kültürü, örgütsel altyapı)" anlamına gelir (Bakker, 2018, s. 15). Brown'ın (1992) tanıtılmasıyla, tasarım araştırması, ortamın ortaya çıkan özelliklerine yanıt olarak bu öğretim materyallerinin, programların, müfredatın veya araçların tasarımını sistematik olarak araştırmak için bir paradigma haline geldi (Design-Based Research Collective, 2003). Bu çalışmanın tasarımı, aktif öğrenme ve öğretme için Stephan ve arkadaşları (2015) tarafından geliştirilen oran ve orantı öğretim dizisi ve paydaşların bu bağlamla etkileşimidir. Ayrıca, öğretmenlerin ve araştırmacıların eğitimde tasarı tabanlı araştırmalara paydaş olarak aktif katılımı, Plomp'un (2013) vurguladığı gibi, "uygulayıcıların katılımı" olarak bu araştırmanın süreç odaklı yönünü doğrudan etkiler. Öğrencilerin ihtiyaçlarına cevap veren daha uygun bir öğrenme ortamı elde etmek için araştırmacının ilk planını ve niyetlerini geliştirmeye öğretmenin katılımı gereklidir. Kısaca, mevcut araştırma tasarımı, paydaşların öğrenme ve öğretme etkileşimini anlamaları için önerilen oran ve orantı öğretim planının sistematik olarak irdelenmesi yoluyla araştırmaya dayalı bilgi toplamayı göz önünde bulundurarak öğretme ve öğrenme süreçlerini şekillendirir. Bu çalışmanın amacı ve araştırma sorusu ile tutarlı olan öğretme ve öğrenmenin merkezi olarak öğrencinin öğrenme etkinliklerini öğretimle ilişkilendirerek sınıf içi matematiksel uygulamalarını irdelemektir.

Eğitimde tasarı tabanlı araştırma yöntemi, tasarım ekibine uygulama süresince ortaya çıkan gelişmeleri doğal bağlamlarında değerlendirmeleri için fırsatlar sağlayan gerçek dünya ortamında bir müdahale geliştirmeyi amaçlar (van den Akker vd., 2006, s. 5). Tasarım araştırması, doğası gereği döngüsel olan sistematik eğitim müdahalelerini içerir (Plomp, 2013). Temel amaç, yalnızca ilk öğretim teorisini test etmek değil, her tasarım döngüsünde onu gözden geçirmek, geliştirmek ve değerlendirmektir. Bu yineleme, araştırmacının niyetini ve sınıftaki faydasını dengelemek içindir. Mikro döngüler ve makro döngüler, tasarım araştırmasında döngüsel sürecin temel bileşenleridir. Her bir mikro döngüde, araştırma ekibi öğretim dizisi üzerinde düşünce deneyleri gerçekleştirdi (Plomp, 2013).

2.2. Sınıf Tabanlı Tasarı Uygulaması (Classroom Design Study)

Tasarım araştırması, bire bir tasarım uygulaması (öğretmen-deneyci ve öğrenci), sınıf tabanlı tasarı uygulaması, hizmet öncesi öğretmen geliştirme uygulaması, hizmet içi öğretmen yetiştirme çalışmaları ve okul geliştirme çalışmaları gibi çeşitli ortamlarda yürütülebilir (Cobb vd., 2003). Bir sınıf ortamında öğrenme araçlarını bir tasarım ekibiyle birlikte araştıran bir sınıfsal tasarı deneyi, sınıf tasarımı çalışması (Cobb vd., 2016) olarak adlandırılmıştır (Cobb vd., 2003; Rasmussen ve Stephan, 2008). Bu araştırmada elde edilmesi planlanan sınıf içi matematiksel uygulamalar, sınıf tabanlı tasarı uygulamasından elde edilmesi öngörülmüştür. Bu noktada da sınıfsal tasarı uygulaması süreçleri araştırma yöntemini yönlendirmiştir. Stephan (2015) sınıf tabanlı tasarı uygulaması kapsamında üç aşama tanımlamıştır: Tasarım, Uygulama ve Analiz Aşamaları.

Bir sınıf tabanlı tasarı uygulamasında ilk aşama olan tasarım aşaması, Stephan ve diğerleri (2015) tarafından geliştirilen oran ve orantı öğretim dizisinin uygulamaya kadar hazırlanmasını ve düşünce deneyi sürecinden geçmesini açıklar (Bakker, 2018; Gravemeijer & Cobb, 2013). Bu dizi, oran ve orantı kavramlarının öğretimi ile ilgili yapılmış çalışmalardan başarılı olanların etkili öğrenmeyi hedefleyecek şekilde bir araya getirilerek oluşturulmuştur. Bu bölümde, *öğrenme ekolojisi* olarak adlandırılabilecek (Gravemeijer ve Cobb, 2013) öğrenme ortamının temel unsurlarını anlamak için hem bu tasarım için varsayılan yerel öğretim teorisi hem de sınıf tabanlı tasarı uygulaması için gerekli bağlamı ve hazırlığı sunmaktır. Sırasıyla bahsedilecek olursa

Öğretim dizisi, Ankara'nın Yenimahalle ilçesindeki bir devlet okulunda çalışan deneyimli ortaokul matematik öğretmeni tarafından tasarım ekibinin desteğiyle de gerçekleştirilmiştir. Veriler, 38 yedinci sınıf öğrencisinin sınıf oturumları, öğretmen görüşmeleri, alan notları ve öğrencilerin ve öğretmenlerinin belgeleri aracılığıyla toplanmıştır. Sınıf gözlemi ve uygulama

34 ders saati sürmüştür. Sınıf tartışmaları içinde yer alan argümantasyonlar içerisindeki SMU'ları oluşturan paylaşılmış fikirler, Toulmin tarafından oluşturulan argümantasyon modeli ve üç aşamalı sınıf içi matematiksel uygulamalar analizi ile çözümlenmiştir. Bu çalışmada öğrenci öğrenimini kolaylaştıran beş SMU şunlardır: ayrık/sürekli nesneler ve oran kuralı hakkında akıl yürütme; bileşik birimler arasında bağlantı ve yineleme; oran tablosu içinde bileşik birimler arasındaki ilişkili değişim (covariation); oran ve orantıyı simgelerle gösterilmesi; eş olmayan oranları karşılaştırmak için stratejiler geliştirme. Bu uygulamalar, orantısal akıl yürütme ile ilgili birçok boyutunu işaret etmekte ve öğretim dizisinin oran ve orantı kavramlarının öğretim kalitesini

3. Bulgular

Bu araştırma, yedinci sınıf öğrencilerinin oran ve orantı kavramlarına ilişkin öğrenmeyi ortaya koyan sınıf içi matematik uygulamalarını (SMU) ve öğretim dizisinin SMU'lara katkılarını incelemeyi amaçlamıştır. Öğrenme ortamı, tartışmacı bir sınıf içi söylemin teşvik edildiği Başlat-Keşfet-Tartıştır öğretim modeli üzerine kurgulanmış ve sınıf matematik uygulamalarının toplu olarak kurgulanması sağlanmıştır. Öğretim dizisi, basitten karmaşığa geçiş yoluyla öğrencilerin oran ve orantı anlayışını geliştirmek için varsayımsal bir öğrenme rotası ve büyük fikirlere dayalı olarak tasarlanmış etkinliklerden oluşuyordu. Her etkinlik deneyimsel olarak gerçek problemlerden oluşuyordu. Merve Öğretmen ve araştırmacı, öğrencilerin öğrenmelerini kolaylaştırmalarına yardımcı olmak ve rehberlik etmek için sınıfta bulundular. Aşağıda verilen araştırma soruları bu bölümün konusunu oluşturmuştur:

Bu araştırma sorularına dayalı olarak, bulgular niteliksel ve niceliksel olarak sunulmuştur. Sınıfın paylaşılan fikirlerini (TAS) yansıtan beş ana SMU ortaya çıkarıldı. Veriler ağırlıklı olarak sınıf tartışmalarından, alan notlarından ve sınıf tartışmalarından elde edilen görsellerden toplanmıştır. Ayrıca veriler öğrenci etkinlik kağıtları ve Merve Öğretmenin katkılarıyla zenginleştirilmiştir. Veriler, yorumlayıcı çerçevenin merceği aracılığıyla tamamlayıcı bir şekilde şekillendirildi. Dokümante edilmiş veriler, Toulmin'in (2003) sınıf matematik uygulamalarını araştırmak için Krummheuer (2007) tarafından aktarılan ve çürütme (rebuttal) durumunda Hitchcock ve Verheij (2006) tarafından zenginleştirilen argümantasyon düzeni kullanılarak analiz edilmiş ve sunulmuştur. Öğretim dizileri de büyük fikirlere ve varsayımlara dayalı olarak ayrıntılı olarak açıklanmıştır. Faaliyetlerin programı, bu faaliyetlerin ne zaman gerçekleştiğine dair zaman çizelgesini gösteren Ek E'de sunulmuştur. Bildirilen normatif akıl yürütme yollarının tekrarı nedeniyle bulgularda birkaç etkinlikten bahsedilmediğine dikkat edilmelidir. Uygulamaların nitel anlatımlarına ek olarak, öğrencilerin ön test ve son test puanları analiz edilerek nicel sonuçlar verilerek oran ve orantı dizisi uygulaması öncesi ve sonrasında öğrencilerin performansındaki değişim ortaya konulmuştur.

3.1. SMU 1: Aynı Oran İçerisinde Sürekli ve Süreksiz Miktarlar Üzerine Akıl Yürütme

Öğretim dizisinin başlatılması, yalnızca oran ve orantı içeriğinin geliştirilmesi için ortak bir temel oluşturması açısından değil, aynı zamanda etkili bir öğrenme ortamı oluşturması açısından da çok önemliydi. 22 etkinlik uygulandı ve bunların 10'u, bileşik birimleri birbirine bağlamaya dayalı olarak uzaylıları beslemek etrafında geliştirilen bölümlerdi. Bağlantı, sınıftaki etkinliğin kuralı olarak da adlandırıldı. İlk etkinlik, uzaylıları besleme bölümlerinin başlangıcıdır. Kural şudur: "Bir yiyecek kutusu üç uzaylıyı besler." Her kural, bileşik birimlerin (composite units) resimli bir temsili ile etkinlikte sunulur. Farklı bir şekilde, ilk bölüm, birimleri birbirine bağlayan ve diğer çözüm stratejilerini keşfettiren resimsel temsilleri içermektedir. Bu etkinlik, birimler arasındaki ilişkiye dair bir anlayış geliştirmeyi amaçladığı için öğrencilerin bir yiyecek kutusu ve üç uzaylı arasındaki kurdukları bağlantı türlerini ortaya çıkardı. Öğrencilerin bu bağlantıyı temsil ederken akıl yürütmeleri, sınıf tartışmasının odak noktasıydı. Bileşik birimleri işlerken öğrenciler adım adım oran kuralın korunumu bilincini geliştirdiler. Sınıf, yalnızca çözüm yöntemi üzerinde anlaşmaya varmakla kalmadı, aynı zamanda geçersiz argümanları çürütmeyi de öğrendiklerini gösteren düşünüş biçimleri ortaya koydular. Tartışarak kabul ettikleri fikirleri kullandılar ve çürütülmüş fikirleri kullanmayarak sınıf içi matematiksel uygulama haline geldiler. Öğrencilerin etkinliklerdeki değişkenlerin bağlamı ve değişken özellikleri hakkında akıl yürütmesi ile oran durumunun keşfedilmesinden önce şekillerle sunulan problem durumunun önemi ortaya çıkmış oldu.

Böylece, Sınıf İçi Matematiksel Uygulama 1 (SMU 1), yedinci sınıf öğrencilerinin ünitelerin sürekli /süreksiz niteliklere sahip nesnelerin ve problemin görsellerle desteklenmesi üç paylaşılan fikri (TAS) ortaya koydu: (1) Bileşik birimlerdeki süreksiz birimler, problem bağlamından dolayı kovaryasyon için indirgenemeyebilir; (2) Oranın değişmez yapısı, nesnelerin rastgele gruplanmasından bağımsızdır; (3) Bileşik birimlerin değişmez yapısı korunarak, bileşik birimlerdeki sürekli birimler azaltılabilir.

Merve Öğretmen ile fikirler sınıf ortamında irdelenirken öğrencilerin sorulara bakış açıları anlaşılır hale geldi. Ek olarak, her fikir veri (data), iddia (claim), gerekçe (warrant), destek (backing) ve çürütmelerle (rebuttal) oluşturulmuştur. Fikirler ayrıca hem beklenen hem de beklenmeyen öğrenci katkılarını içeriyordu. Beklenenler, ağırlıklı olarak algoritmalar kullanıyor, uzaylılar ve yemek kutuları arasına çizgiler çekiyordu. Öte yandan, beklenmedik fikirler öğretim dizisi için kullanılan kılavuzu zenginleştirdi. Spesifik olarak, bu fikirler öğrencileri, öğretim sürecinde beklenen öğrenci düşünmeleri arasında yer almayan ayrık ve sürekli (TAS 1 ve TAS 3) ve problem bağlamını temsil eden çizimleri (TAS 2) açısından ünitelerin özelliklerini keşfetmeye bağladı. Her bir fikir, diğer fikirlerin ortaya çıkışını etkilemiş ve ağırlıklı olarak nesneler arasındaki ilişkilerle şekillenmiştir. Bir oranın değişmez yapısı (problemin kuralı) hakkında muhakeme geliştirmeye, bileşik birimler içindeki ayrık/sürekli nesneler hakkında muhakeme yapmaya ve bir dizi bağlantılı nesneyi gruplandırmaya katkıda bulundular. Bu TAS'lar, diğer sınıf uygulamaları için sosyal ve sosyomatematiksel normlar oluşturmaya da yardımcı oldu çünkü öğrenciler matematiksel olarak anlamlı tartışma yollarını anlamaya ve fikirlerinin yanlış olup olmadığı konusunda diğerlerini uyarmaya başladılar.

3.2. SMU 2: Bileşik Birimleri İlişkilendirme ve Öteleme

Oran ve orantı öğretim dizisi, öğrencilerin ilk önce deneysel olarak gerçek bir bağlam (örneğin, uzaylılar ve yemek çubukları) içindeki orantılı durumları keşfetmelerine ve daha soyut orantılı akıl yürütmeye doğru aşamalı olarak ilerlemelerine izin verecek şekilde tasarlanmıştır. Etkinlik 1-4, öğrencilerin öğretim dizileri sırasında kullandıkları, birbirine bağlanan ve tekrar eden bileşik birimleri geliştirmekti. Verileri nasıl organize edeceklerini keşfettiler ve bu dört etkinlik, problemleri çözmek için matematiksel olarak doğru ve zamandan tasarruf eden akıl yürütmeye odaklanan tartışmalar yoluyla düşünüşlerini güçlendirdi. Bu etkinlikler sayesinde öğrenciler verilerin etkin bir şekilde düzenlenmesi ve net olmayan, açıklanamayan argümanların çürütülmesi için fikir üretmeye teşvik etti. Verilerin düzenlenmesi, bağlantıların şekiller/resimler üzerine çizilmesi yoluyla bileşik birimler hakkında akıl yürütmeleri ile başlamıştı. Bileşik birimler ve çeşitli veri düzenleme stratejileri, bileşik birimlerin tanımlanması ve ötelenmesi (iteration) hakkında soru işaretleri uyandırdı. Öğrenciler, diğer veri düzenleme stratejilerinden faydalanarak ötelemeyi açıklamak için akıl yürüttüler. Öğrenciler, öğretim dizisinin başından itibaren farklı çözümler ürettiler ve kullandılar. Her bir çözüm yöntemi etkinlik sayfalarında ve tahtada defalarca gözlemlendi. Sınıf tartışmalarında ve etkinlik sayfalarında gözlemlendiği gibi, öğrencilerin fikirleri değişkendi. Aynı öğrencinin tahtadaki çözümü ile etkinlik kağıdı arasında farklılaştığı bile gözlemlendi. Aşağıda temsil edildiği gibi, veri düzenleme stratejilerinin uygulamaları, sınıftaki entegre tartışmalardan çıkarıldı. Her soruda ya da problemde birden fazla çözüm yöntemini aynı anda veri, iddia, destek, çürütücü ya da destek olarak sunulduğu rapor edildi.

İkinci uygulama, öğrencilerin süreksiz/sürekli değişkenler hakkında akıl yürüttükten sonra farklı bağlama sürecine katılmasıyla oluşturulmuştur**.**

Sınıfta Matematiksel Uygulama 2, birleşik birimleri birbirine bağlamayı keşfeden ve bunları çeşitli durumlarda yineleyen yedinci sınıf öğrencilerinden gelen dört fikirden oluşuyordu. Bu dört normatif muhakeme yolu şunlardır: (1) Bileşik birimler resimsel veya sembolik formlarında temsil edilebilir ve yinelenebilir; (2) Bölme ve çarpma algoritmaları, eksik değer (missing-value) ve karşılaştırma (comparison) problemleri için doğru birimleri bağlamayı gerektirir; (3) Bileşik birimleri yeniden oluşturmak için birim oranı oluşturulabilir (birleştirme); (4) Oran tablosu, bileşik birimlerin yinelenmesinden oluşur. Oran tablosuyla uğraşmak zaman alsa da öğrencilerin oran tablosu aracına sorunsuz bir şekilde uyum sağlaması Etkinlik 3'ün odak noktasıydı. Bu nedenle Merve Öğretmen, BKT öğretim döngüsünün tartışma aşamasında oran tablosu dışındaki tüm çözüm stratejilerini dinledi. Her TAS, SMU'ların dört kriteri aracılığıyla normatif bir muhakeme yolu haline geldi. Paylaşılan her bir strateji, başka bir sınıf matematik uygulamasına aktarıldı.

Bu dört stratejide, sınıf, ders oturumları boyunca etkinlik sayfalarında verilen bileşik birimleri birbirine bağlamak ve yinelemek için verileri organize etmek için kendi muhakeme yollarını geliştirdi. Bu dört fikir, bilinmeyen değer problemlerinde veri organizasyonunun önemli bir bölümünü oluşturacak şekilde paylaşılan stratejiler olarak öne çıktı. Farklı zorluklarla karşılaştıkları her defasında (tam sayı ve tam sayı olmayan sayıların işlemlerini içeren problemler, eksik değer problemleri, karşılaştırma problemleri, kuraldaki birimlerin birden farklı sayı değeri), resmi olmayan araçlarının yeteneklerini test ettiler. Biçimsel araç olarak, öğretim dizisi yardımıyla oran tablosu tanıtılmış ve öğrencilerin araçla etkileşimi gözlemlenmiştir.

3.3. SMU 3 Oran Tablosunda Bileşik Birimler Arasında Kovaryasyon

Biçimsel bir araç olarak, oran tabloları yavaş yavaş öğrencilerin etkinlik kağıtlarındaki çözümlerinde ve tahtada yer aldı. Birkaç öğrencinin oran tablolarını içselleştirmesi ve çarpımsal ilişkiyi küçük adımlarla keşfetmesi için aritmetik girişimler ortaya çıktı. Öte yandan, birkaç öğrenci sayılar arasındaki kat ilişkisini inceledi ve o zamandan beri bunları etkili bir şekilde kullandı. Her durumda, öğrencilerin SMU 2'de belirtilen fikirleri kabul ettikleri argümanların transferiyle ortaya çıkmış oldu. Halihazırda düzenlenmiş bileşik birimler arasındaki ilişkiyi daha iyi anlamak için oran tablosu aracıyla da bu fikirleri şekillendirdiler. SMU 3'te ise öğrenciler, dikey ve yatay ilişkiyi ifade eden çarpanları uygun bir şekilde nasıl kullanacaklarını keşfettiler ve öğretim sırasındaki kavram yanılgıları ortaya çıktı ve öğretmenin müdahalesiyle ortadan kaldırıldı. Ayrıca, çapraz çarpım algoritması stratejisi sınıf tartışmasına taşındı ve varsayımsal olmayan ters orantılı durumlar da bu şekilde ele alındı. Ek olarak, kat ilişkisi ve oran tablosunda bunu seçme koşulları, öğrencilerin ondalık sayılarla ilgili önceki deneyimlerinden etkilendiği ortaya çıkmıştır. Oran ve orantı bağlamının ele alınması diğer matematiksel kavramlarla uyumlu olduğundan, öğretim dizisi ondalık sayıları kullanmanın zorluğunu da tahmin ediyordu. Paylaşılan fikirlerin bileşik birimler arasındaki kovaryasyona katkıları, bu sınıf matematik uygulamasında ayrıntılı olarak açıklanmaktadır.

Sınıf, bu uygulamadaki beş fikirde de sayılar arasındaki uzun ve kısa oran tablolarındaki ve varsayımsal olmayan ilişkileri araştırdı. Öğrencilerin standart bir yol izlemediği görülmüştür. Çoğunlukla çizim kullanan bazı öğrenciler, bilgilerini uzun bir oran tablosuna aktardılar ve değişkenler arasındaki çarpımsal ilişkiyi keşfettiler: yatay olarak genişletilmiş oran tablosundaki dikey kat ilişkisi ve yatay kat ilişkisi üzerinde de muhakeme yaptılar. Öte yandan, halihazırda kat ilişkisi kullanan bazı öğrenciler, doğrudan oran problemlerini çözmek için çapraz çarpım algoritması stratejisini getirdiler. Sınıf İçi Matematiksel Uygulama 3 (SMU 3), oran tablosunda sayı ilişkilerini ve esas olarak kovaryasyonu araştıran yedinci sınıf öğrencilerinden gelen beş fikirden oluşuyordu. Bu beş normatif muhakeme yolu şunlardır: (1) Oran tabloları, bileşik birimler arasındaki kovaryasyon yoluyla doldurulabilir; (2) Bir uzaylı yarı beslenebilir ve problem bağlamında kat ilişkisi veya hücrelerdeki sayılar ondalık olabilir; (3) Kullanılan strateji pay ve payda arasındaki eşdeğer oranlardaki fark değil (toplamsal düşünme) ölçek faktörleridir; (4) Ters orantı durumu olmaması kaydıyla bilinmeyen değer problemlerinde çapraz çarpım algoritması da kullanılabilir, (5) Oran tablosunda üçüncü bileşik birim değişkeni oluşturulabilir ve bunda aynı yatay kat ilişkisi kullanılabilir.

Tahmin edildiği gibi, planlanan aşamanın amacına uygun olarak, öğrencilerin kat ilişkisinin doğasını ve oran tablolarında nasıl kullandıklarını keşfettikleri gözlemlenmiştir. Merve Öğretmen paylaşılan fikirleri derinlemesine sorgularken öğrencilerin sorulara bakış açıları irdelendi. Önceki uygulamalarda olduğu gibi, her fikir veri, iddia, gerekçe, destek ve çürütücülerden oluşuyordu. Belirgin bir şekilde, öğrencilerin çizimleri bu matematiksel uygulamalarda önemli bir yer oluşturdu ve öğrenciler tablodaki sayı ilişkilerini bu şekilde keşfettiler. Bu süreçte, öğrencilerin yeni fikirler karşısında harcadıkları tartışma süresinin azaldığı çünkü önceki uygulamalardaki sorunların üstesinden gelme deneyimleri, her etkinlik için ortak bir matematik dili oluşturmalarını sağladı.

3.4. SMU 4: Sembolik Gösterimle Oran ve Orantı Kavramlarını İnceleme

Sınıf İçi Matematiksel Uygulama 4 diğer temsilleri sembolik temsile bağlamanın yanı sıra "oran", "orantı" ve "eşdeğer oranlar" hakkındaki resmi kavramlar hakkında farkındalık yaratmaya odaklanır. Önceden, öğrenciler stratejiler geliştiriyor, birimlerin özelliklerini tanıyor ve açıkça "oran" demeden ondalık sayıları ve daha büyük sayıları kullanıyorlardı. Sembolik ifadeler ve terimler kullanarak akıl yürütmeye başladılar.

Sınıf Matematik Uygulaması 4 (SMU 4), yedinci sınıf öğrencilerinin oran ve orantı kavramının sembolik temsilini anlamaya çalışırken ortaya çıkan iki fikrinden oluşmaktadır. Keşfederken, öğrencilerin oranların ve eşdeğer oranların sembolik temsilini değerlendirme becerilerini etkileyen soru bağlamının temsil edilmesi ön plana çıktı. Bu üç normatif muhakeme yolu şunlardı: (1) Sembolik oran temsili, sözel temsildeki bileşik birimlerin sırası gibi değişir; (2) Oran tablosu oran gösterimi için bir araç olarak kullanılır; bu nedenle, dikey ve yatay kat ilişkisi sembolik gösterimde de kullanılabilir. Kullanılan strateji pay ve payda arasındaki eşdeğer oranlardaki fark (toplam düşünme) değil, ölçek faktörleridir. Kesir görüntüleri, sembolik oran gösterimi ve eşdeğer oranların tanıtılmasına yardımcı oldu.

Ayrıca, oran tablosu aracının eşdeğer oranlara dönüştürülmesi, öğrencilerin kavramsallaştırmasını belirgin şekilde geliştirmiştir. Tahmin edildiği gibi, eşdeğer oranların sembolik temsili içinde daha büyük sayılarda toplamsal düşünme gözlemlendi. Burada dikkat çeken bir önceki sorularda beklenen cevaplardan biri olan toplamsal düşünme ortaya çıkmazken, sembolik temsilde bu durum tetiklendi. Öğrencilerin iddiayı ve verileri kabul etmesi ve açıklama sorularının olmaması, argümantasyon düzeninin veri ve iddiadan oluşan bir fikir paylaşımı olduğunu göstermektedir.

3.5. SMU 5: Denk Olmayan Oranların Karşılaştırılmasını İnceleme

Eşdeğer olmayan oran sorularının karşılaştırılmasını da içeren bilinmeyen değer problemleri, Etkinlik 18'e kadar öğretim dizisinde baskındır. Bu etkinlikten sonra karşılaştırmalı problemler ele alınmıştır. Bu faaliyetlere gömülü oran kavramı da vardır. Öğrenciler oran kavramı ile ilk tanıştıklarında bilinmeyen değer ve karşılaştırma problemleri arasında bir dengesizlik durumuna girmişlerdir. Bilinmeyen değer problemleri için ortaya çıkan stratejileri doğrudan benimseyemezler. Merve Öğretmen ve araştırmacı soru türlerinden bahsetmese de öğrenciler farklı problem durumlarını fark ettiler. Önceki öğrenmelerinden ve normatif muhakeme yollarından edindikleri deneyimleri aktardılar. Sınıf İçi Matematiksel Uygulama 5, öğrencilerin eşdeğer olmayan oranları ve normatif muhakeme yollarını karşılaştırmasını kapsıyordu. Ayrıca, oran tablosu karşılaştırma problemleri için artık etkili bir araç değildi.

Etkinlik 18-22 arasında oranların karşılaştırılması tartışıldı. Önceki iki fikirde, sınıf, problem bağlamı aracılığıyla oranların kavramsal olarak anlaşılmasıyla ilgilendi. Analiz, öğrencilerin öğretim dizisi sırasında önceki öğrenmelerini yeni görevlere aktardığını kanıtladı ve bu kanıt, halihazırda geliştirilmiş sınıf matematiksel uygulamalarının tekrarını gösterdi. Sınıf, temel olarak, olağan kesir imgelerini kullanmadan eşdeğer olmayan oranlarla nasıl başa çıkılacağını inceledi.

Sınıf Matematik Uygulaması 5 (SMU 5), yedinci sınıf öğrencilerinin karşılaştırma problemlerine yönelik strateji uyarlama süreçlerine yönelik iki fikirden oluşuyordu. Bu iki normatif muhakeme yolu şunlardır: (1) Bir miktardaki b maddesinde a maddesi ne kadar fazlaysa, karışım o kadar yoğun olur; (2) Sıralama oranlarının en küçük ortak katı, bir karışımdaki bir özelliğin en çok veya en azını anlamak için kullanılabilir. Son olarak, öğrencilere destek olarak en az veya en çok kavramı hayal etmeleri için materyallerin resimli bir temsili verildi.

4. Sonuç, Tartışma ve Öneriler

Bu çalışmanın bulguları ve sonuçları, oran ve orantı öğrenii ve öğretimi hakkında, matematik öğretmenleri, matematik öğretmen adayları, öğretmen eğitimcileri, müfredat geliştiriciler, politika yapıcılar ve araştırmacılar gibi tüm paydaşlara rehberlik edebilecek bilgiler sağlamaktadır.

İlk olarak, tasarı tabanlı eğitim araştırmaları, bölgesel öğretim uygulaması için birbiriyle ilişkili çeşitli teorik bakış açılarının şekillendirildiği, test edildiği ve içeriğin öğrenilmesi açısından revize edildiği bir ortam sağlamıştır. Ayrıca çalışmada gerçek sınıf ortamından kesitler sunulmuştur. Araştırmacıların planladıkları ve sınıfta nasıl hayata geçirdikleri, eğitimde tasarı tabanlı araştırma desenin ilgi alanını içermektedir (Cobb v.d., 2001) ve bu durumda öne çıkan ilkeler sınırlı bir sistemde temsil edildi. Araştırmacı dışındaki paydaşların katılımı, orantılı akıl yürütme için etkili bir öğretme-öğrenme ortamı geliştirmek için değerli bir kaynak sağlamıştır. Plomp (2013) ayrıca eğitim tasarımı araştırmasının oldukça önemli bir parçası olarak uygulayıcıların katılımını vurgulamıştır. Bu süreç boyunca, sınıf tartışmasından elde edilen söylemler, öğrencilerin düşünme biçimlerini ön plana çıkarmış ve öğrenciler, oran ve orantı ile ilgili sınıf içi matematiksel uygulamalar geliştirmiştir. Bu matematiksel içerik ve bu içeriğin sosyal bir ortamda öğrenilmesi için modeli içselleştirdikleri, organize ettikleri ve yeniden icat ettikleri anlamına gelir (Gravemeijer vd., 2003). Bu tür özellikler diğer tasarım araştırma çalışmalarında da tanımlanmıştır (bkz. Ayan-Civak, 2020; Bowers & Nickerson, 2001; Stephan & Akyuz, 2012; Stephan, 2015; Stephan vd., 2003). Tek bir ilkenin değil, birçok ilkenin bütüncül olarak ele alındığı eğitimde tasarı tabanlı araştırmaların yaygınlaştırılması, belirli bir içeriğin öğretimini destekleyecek altyapıyı oluşturabilir. İleriki çalışmalarda, bunun gibi eğitime özgü araştırma yöntemleri sınıfın kapılarını açacak ve öğretme ve öğrenme için neler olup bittiğini şeffaf hale getirecektir.

İkincisi, Başlat-Keşfet-Tartış modelinin aşamaları dikkate alınarak ilk etkinlikten son etkinliğe kadar uygulanan öğretim dizisi ile içerikten alınan verim arttırılmıştır. Öğretmeni derse hazırlamış, dersin gidişatını dikkate alarak öğretmene esneklik sağlamış, dersin amacına bağlı kalarak öğretmenin konunun dışına çıkmasını sınırlamıştır. Araştırmacının dışında, öğretmen sorularıyla sınıfın öğrenmesine rehberlik etmiş ve araştırmacıyla birlikte öğretmen etkin olarak günlük planlamaya katılmıştır (Cobb & Yackel, 1996). Her öğrencinin kendine ait bir matematiksel gerçekliği olduğu kabul edilerek tartışmalar yürütülmüştür. Bu gerçekler, öğrencilerin bir matematik etkinliği gerçekleştirirken ne söylediklerine ve ne yaptıklarına dayanmaktadır. Başka bir deyişle, öğrenciler sosyal çevre ile etkileşime girerek matematik hakkındaki bilgi ve becerilerini hem sosyal hem de bireysel olarak şekillendirme fırsatı bulmuştur. Öğrencilerin akıl yürütme becerilerine odaklanıldığından etkileşimli matematiksel iletişim bu yöntemin merkezinde yer alır. Bu durumlar göz önünde bulundurularak alan yazında bahsedilen farklı fikirler hakkında öğrenciden beklenen üretme, tartışma ve akıl yürütme gibi becerilerin ortaya çıkarılmasına katkı sağlamıştır (Lappan, 2014; MVP, 2017, 2019; Stein vd., 2008). Çözümlerin bu şekilde bir araya getirilmesi öğrencilerin fikirlerini grup ve sınıf arkadaşlarıyla paylaşmalarına, kullandıkları stratejileri kendi ihtiyaçlarına göre değiştirmelerine, geliştirmelerine veya çürütmelerine olumlu etkisi olduğu gözlemlenmiştir. Bulgulardaki farklı roller incelenerek, zorlukları ve katkıları dikkate alınarak, bireysel öğrenme ile sosyal öğrenmeyi birleştiren Başlat-Keşfet-Tartış modeli ile öğretim planları tasarlanabilir. Bu tür öğretim modelleri, sınırlamaları ve güçlü yönleri ile daha fazla araştırılabilir. Öte yandan, uygulamaya yönelik son çalışmaların (Bowers ve Nickerson, 2001; Lappan, 2014; MVP, 2017, 2019; Stein ve diğerleri, 2008) çıkarımları göz önünde bulundurularak matematik öğretmenleri tarafından öğretimlerinde kullanılabilir.

Bu çalışmada Başlat-Keşfet-Tartış modelinin merkezinde öğrenme ortamı, öğrenci ve öğretmen rolleri tanımlanmaya çalışılmış ve geliştirilmesi gereken yönler ifade edilmiştir. Bu açıdan LED gibi öğretim süreci ile uyumlu bir öğretim modelinin diğer modellere uygulandığında ortaya çıkabilecek özelliklerin bu çalışmada açıklanan özelliklere benzeyebileceği literatürdeki diğer çalışmalarla desteklenmiştir (Stein vd., 2008). Öte yandan Akar ve Yıldırım (2005) geleneksel sınıf ortamından sosyal yapılandırmacı yaklaşımı benimseyen başka bir sınıf ortamına geçişte öğretmen adaylarının karşılaşabilecekleri güçlükleri dile getirmiş ve yeni araştırmaların bunu kolaylaştırmada önemli olabileceği vurgulanmıştır. Akar ve Yıldırım (2005) tarafından ifade edilen kalabalık sınıf ortamı ve öğrenme sorumluluğu alma gibi geçişte zorluk teşkil edebilecek bulgular, bu çalışmada Başlat-Keşfet-Tartış modeli uygulandığı için bir zorluk olarak ortaya çıkmamıştır. Bu anlamda önerilen öğretim modeli yapılandırılmış olduğundan, bir tartışma ortamından anlatma ve dinleme ilişkisine dayalı bir öğrenme ortamından geçişi kolaylaştırabilecek bir yöntem olarak kabul edilebilir. Son olarak, bu çalışma oran-orantı öğretim dizisi ile sınırlı olduğundan, diğer matematik konularına dayalı olarak hazırlanan Başlat-Keşfet-Tartış planlarında ortaya çıkacak rollerin öğrenci öğrenmesi üzerindeki etkilerini ölçmek için daha sonraki araştırmalar için yeniden incelenmesi önerilir.

Eğitimde tasarı tabanlı araştırmalar, Başlat-Keşfet-Tartış ve öğrenme ortamının desteğiyle, bu çalışma yedinci sınıfların oran ve orantı kavramlarının gelişim sürecini, oran ve orantı öğretim dizisi kullanılarak sınıf matematik uygulamalarının çıkarılması açısından etkili olmuştur. Beş sınıf içi matematiksel uygulama raporlanmış ve yapılan analizler öğrencilerin bilgiyi yapılandırmalarının birbirine bağlı olduğunu ve fikirlerinin orantısal akıl yürütme ile ilgili ilişkilendirme (linking), oranın yapısı, orantı, oranın değişkenliği (covariance) ve değişmezliği (invariance) gibi bilgi ağı içinde ortaya çıktığını gösterdi. Görselleri dikkate almaları, problem durumlarında sunulan verileri düzenlemeleri, oluşturma stratejilerinden çarpımsal ilişkilere, resimden sembolik temsile, küçükten büyüğe, tam sayıdan tam sayı olmayan görevlere, bir bileşik birimden diğerine, süresiz değişkenlerden sürekli değişkenlere, bilinmeyen değer problemlerinden ve karşılaştırma problemlerine kadar pek çok konu öğretmen ve öğretmen tarafından yürütülen günlük mikro döngülerin katkısıyla öğretim dizisinin ve varsayılan öğretim rotasının bu sınıf ortamında ne ölçüde işe yaradığının önemli kanıtlarıydı. Çocukların doğru düşünmelerinin yanı sıra kavram yanılgıları ve hataları da ortaya çıkarılarak etkili bir tartışma aracı olarak kullanıldı. Ayrıca öğrencilerin yanlış ve doğru stratejileri, öğretmenlerin ve öğretmen eğitimcilerinin, öğrencilerin oran ve orantı konusundaki tepkileri ve normatif akıl yürütme biçimleri hakkında fikir sahibi olmalarına yardımcı olabilir. Bu öğretim dizisi, oran ve orantı kavramları öğretilirken herhangi bir yedinci sınıf matematik dersine entegre edilebilir. Ayrıca analiz öğretim dizisinin olumlu yönlerinin yanı sıra, orantısal akıl yürütmeyle ilgili diğer kavram, bilgi ve beceriler için bazı noktaların hala iyileştirilmesi gerektiğini de ortaya çıkardı ve süreçte yapılan müdahalelerle birkaç durumun nasıl ele alındığı da detaylı olarak verildi.

Oran tablosu oran ve oranın öğretilmesi ve öğrenilmesi için etkili bir araçtır (Abrahamson & Cigan, 2003; Ayan-Civak, 2020; Brinker, 1998; Cramer & Post, 1993a; Dole, 2008; Ercole vd., 2011; Karagöz Akar, 2014; Lamon, 2012; Middleton & van den Heuvel-Panhuizen, 1995; Sözen-Özdoğan ve diğerleri, 2019; Warren, 1996). Bu çalışmada, öğrenciler uygulamalı etkinlik sayfaları aracılığıyla hücrelerdeki sayı değerleri arasındaki oluşum ve çarpma ilişkisini anlamak için problemlerdeki verileri düzenlemek için oran tablosunu da kullanmışlardır. Öğrenciler değişkenler arasındaki ve içindeki çarpımsal ilişkiyi fark ettiler, ancak oran tablosundaki oran içindeki değişmez yapı (ölçü uzayları arasındaki, dikey ölçek faktörü) ve bileşikler arasındaki kovaryant yapı (ölçü uzayları içindeki, yatay ölçek faktörleri) kapsamlı bir şekilde tartışmak için ortaya çıkmadı. Bundan sonraki çalışmalarda, öğretmenlerin ve öğretmen eğitimcilerinin bu yönüyle oran tabloları üzerinde daha fazla çalışmaları ve hatta tartışma konusu oluşturmak için söyleme müdahale etmeleri kesinlikle önerilmektedir. Bu tartışmanın sorunsuz bir şekilde ilerlemesi için, oran tablosunda oran ve orantı için sayı ilişkileri üzerinde çalışmak üzere daha fazla alıştırma ve uygulama sağlanabilir. Öğretim dizisi, gerektiğinde diğer araçlar, teknolojiler ve matematiksel konularla desteklenebilir (Confrey ve Lachance, 2002). Bu nedenle, öğrencilerin süreksiz/sürekli değişkenler, tamsayı/tamsayı olmayan görevler, birçok bağlantılı bileşik oluşturma ve orantı için farklı örnekler üzerinde çalışmalarını sağlamak için görevlerin Web 2.0 veya dijital araçlar olarak düzenlenmesi önerilir.

Oran ve orantı kavramlarını ağırlıklı olarak ifade eden bu çalışmada, ilişkilendirme, öteleme, bileşik birimleri temsil etme, verileri düzenleme, işlemleri yürütme, oranları karşılaştırma, oluşturma ve çarpma ilişkilerini uygulama, eşdeğer oranları oluşturma ve analiz etme sınıf ortamında etkili bir şekilde gerçekleştirilmiştir. Bu şekliyle altıncı ve yedinci sınıf öğrencilerine uygulanabileceği öngörülmektedir. Özet olarak, orantısal akıl yürütme şemsiye terimi altında diğer kavram, beceri ve bilgilerin eklenmesiyle orantı öğretim dizisinde yapılacak olan kesikli/sürekli değişkenlerle ilgili iyileştirmeler, ondalık sayılar, ters orantılı durumlar, çapraz çarpım algoritması, doğru orantılarda doğrusal ilişki, karşılaştırma problemlerinin görselleştirilmesine yönelik ilaveler, dijital araçlarla ilgili çalışma yaprakları kullanılması oran ve orantı konularının öğretiminin iyileştirilmesi açısından önerilmektedir. Bu çalışma, öğrencilerin bu konulardaki muhakemelerini ve zorluklarını ortaya çıkarmak için önerilen bu iyileştirmelerle sınıftaki öğrenci sayısı dikkate alınmadan yedinci sınıf öğrencileri ile yapılabilir. Ayan-Civak (2020) orantısal muhakeme için revize edilmiş varsayımsal öğrenme rotası ile bu konuyu dikkate alan çalışmalardan birini gerçekleştirmiş ve konuların çoğunu öğretim sürecinin içerisinde ele almıştır ancak çapraz çarpım algoritması, kesikli/sürekli görev değişkeni mevcut çalışmadan farklı olarak argümantasyonlarda ortaya çıkmamıştır. Ek olarak, Stephan ve ark. (2015), oran ve yüzde kavramlarını da dikkate alarak öğretim dizisi tasarlamıştır. Zaman kısıtlaması nedeniyle bu çalışmada yüzdeler konusu ele alınmadı ancak bu iyileştirmelere ek olarak hem oran tablosu aracının hem de öğretim dizisinin tamamının sunduğu dikey ve yatay matematikleştirmeyi görmek için geri kalan etkinlikler sonraki çalışmalarda kullanılabilir.

Ulusal İlköğretim Matematik Programı üzerinde yapılan bir analiz, bağlantılı bileşikleri tekrarlama, mutlak ve göreli düşünme ve nitel muhakeme gibi belirli konularda yeterli vurguların yapılmadığı söylenebilir (bkz. Avcu ve Avcu, 2010; Ayan-Civak, 2020; Artut ve Pelen, 2015; Karagöz Akar, 2014; Pişkin Tunç, 2020). Sonuç olarak, program geliştiricilere ve politika yapıcılara altıncı ve yedinci sınıflar için oran orantı konusuyla ilgili ek kazanımların müfredata eklenmesi önerilmektedir. Ayrıca orantısal akıl yürütmenin kesirler, rasyonel sayılar, çarpma ve bölme gibi diğer çeşitli kavram ve konularla bağlantılı olması nedeniyle tüm sınıf seviyelerinde temel bir beceri olarak vurgulanabileceği önerilmektedir (bkz. Boyer vd., 2008; Lachance ve Confrey, 2002). Bu, ancak orantısal düşünmeyi bağımsız bir konu olarak değil, çeşitli konularla iç içe ele alınması gereken kapsayıcı bir beceri olarak ele alınmasıyla mümkün olabilir. Sonuç olarak, program geliştiricilerin ve politika yapıcıların orantısal düşünmeyi çeşitli bağlamlarda problem çözme faaliyetlerine dahil etmek de gelişimini teşvik etmede etkili olabilir.

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