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Killing-Yano charges of asymptotically maximally symmetric black holes

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ABSTRACT

We construct an asymptotic conserved charge for a current that has been defined using Killing-Yano tensors. We then calculate the corresponding conserved charges of the Kerr and AdS-Kerr black holes, and their higher-dimensional generalizations, Myers-Perry and Gibbons-Lü-Page-Pope black holes. The new charges all turn out to be proportional to the angular momenta of their parent black holes.

Contents

1. Introduction

A solution to the field equations of General Relativity may or may not support Killing or Killing-Yano tensors. However when it does, that is generally a sign for the existence of "hidden symmetries" of the particular geometry, typically helps in the separation of variables in the relevant Hamilton-Jacobi equations, and usually implies the existence of conserved currents and charges. While it is not always easy to give a clear physical interpretation of such conserved currents, conserved charges constructed out of these typically have an interpretation in terms of the parameters of the particular solution.

One such enigmatic current, which seems to carry important information, is that of Kastor-Trachen (KT) [1]. Finding conserved charges constructed out of this current for various interesting geometries would certainly help in a better understanding of the physical relevance of this current. The present paper is a work which, we hope, provides a positive step in this direction.

In this paper we derive asymptotic conserved charges for the Kerr [2], the AdS-Kerr [3] and the Gibbons-Lü-Page-Pope (GLPP) [4] black holes, that are higher dimensional generalizations of the AdS-Kerr metric. The charges for the Myers-Perry (MP) [5] black holes easily

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follow from those of the GLPP ones by taking the cosmological constant to zero. Our derivation uses ideas from [6] and techniques drawn from discussions of the KT-current for Killing-Yano tensors (KYTs) as developed in [7] and [8], and are further extended here.

Briefly, the procedure we follow is to split the metric into a background plus a deviation, and consider cases where the background has a KYT. We then further restrict to the cases when the KT-current linearized with respect to the background is conserved and it is possible to determine the corresponding potential for the linearized current. These then make it feasible to construct (asymptotically) conserved charges as integrals over sub-manifolds. By "saturating" the current potential with a certain set of vectors, we can always take a flux-integral over a $(D-2)$-dimensional subspace. This approach is used for explicitly calculating the charges for the GLPP and MP black holes in $4 \le D \le 8$ dimensions. Our systematic results are tabulated in sec. 6.

The organization of the paper is as follows: In sec. 2 we introduce KYTs and the KT-current. Sec. 3 contains a review of asymptotic charges in this context, expounds the discussion given in the previous paragraph and introduces the charge definition employed in the rest of the paper. A treatment of the Kerr metric along these lines is contained in sec. 4 and of the AdS-Kerr metric in sec. 5. Sec. 6 then presents our derivation of the charges for the GLPP and MP black holes. We end by a brief discussion of our results in sec. 7.

The setting of our discussion is General Relativity in $D \ge 4$ dimensions; i.e. the geometry is pseudo-Riemannian with metric g, Levi-Civita connection Γ, curvature tensor R, etc. We also implicitly assume that the geometry supports KYTs, as described in sec. 2 below, and indices of KYTs are raised and lowered with the metric g that supports them in the first place.

2. The KT-current

Let us start by recalling some basics on KYTs. They can be thought of as generalizations of Killing vectors to rank-n antisymmetric tensor fields. So they can be viewed as the components of n-forms $f_{a_1\ldots a_n} = f_{[a_1\ldots a_n]}$ satisfying

$$\nabla_{(a} f_{a_1)a_2\ldots a_n} = 0\,, \qquad 1 \le n \le D\,. \tag{2.1}$$

For a spacetime that admits a rank-n KYT f, one can also show that the current[2] [1]

$$\begin{aligned} j^{a_1\ldots a_n} &= N_n\, \delta^{a_1\ldots a_n d_1 d_2}_{b_1\ldots b_n c_1 c_2}\, f^{b_1\ldots b_n}\, R_{d_1 d_2}{}^{c_1 c_2} \\ &= -\frac{(n-1)}{4}\, R^{[a_1 a_2}{}_{bc}\, f^{a_3\ldots a_n]bc} + (-1)^{n+1}\, R_c{}^{[a_1}\, f^{a_2\ldots a_n]c} - \frac{1}{2n}\, R\, f^{a_1\ldots a_n} \end{aligned} \tag{2.2}$$

is covariantly conserved. Here $\delta^{a_1\ldots a_m}_{b_1\ldots b_m} = \delta^{[a_1}_{b_1}\cdots\delta^{a_m]}_{b_m}$ is totally antisymmetric in all up and down indices, and

$$N_n = -\frac{(n+1)(n+2)}{4n}\,. \tag{2.3}$$

The covariant conservation of (2.2), i.e., $\nabla_{a_1} j^{a_1\ldots a_n} = 0$ follows from the Bianchi identities:

$$\nabla_{[a} R_{bc]}{}^{de} = 0\,, \quad \nabla_a R_{bcd}{}^a + 2\nabla_{[b} R_{c]d} = 0\,, \quad \nabla_a R^a{}_b - \tfrac{1}{2}\nabla_b R = 0\,, \tag{2.4}$$

and the properties of f.

Since a covariantly conserved antisymmetric rank-n tensor field is equivalent to a co-closed n-form, one can use the extension of the Poincaré lemma to the exterior co-derivative and express the original rank-n tensor field as the co-derivative of an $(n+1)$-form in a suitably chosen (simply-connected) open set. Thus, one should be able to write, at least locally,

$$j^{a_1\ldots a_n} = \nabla_c\, \ell^{ca_1\ldots a_n} \tag{2.5}$$

for some $(n+1)$-form potential $\ell^{ca_1\ldots a_n} = \ell^{[ca_1\ldots a_n]}$. The problem of determining the potential ℓ for the KT-current j (2.2) is still open. However, this is not what we are interested in here and there is more to the story: Suppose that one has somehow found a totally antisymmetric potential ℓ for the KT-current j (2.2) satisfying

$$\nabla_{a_1} j^{a_1\ldots a_n} = \nabla_{a_1} \nabla_c\, \ell^{ca_1\ldots a_n} = 0\,. \tag{2.6}$$

Consider a set of linearly independent arbitrary vectors $x^{(i)}$, $(i = 1,\ldots,n-1)$ in the spacetime that admits the rank-n KYT f that goes into the KT-current j (2.2). Then it follows that

$$\nabla_d\, \nabla_c \left(\ell^{cda_1\ldots a_{n-1}}\, x^{(1)}_{a_1} \ldots x^{(n-1)}_{a_{n-1}}\right) = 0\,. \tag{2.7}$$

This is easily seen if we observe that the covariant derivatives turn into a curvature tensor acting on a second rank antisymmetric tensor. Alternatively, to be explicit, we expand the derivatives on the left hand side,

$$\begin{aligned} &\left(\nabla_d \nabla_c\, \ell^{cda_1\ldots a_{n-1}}\right) x^{(1)}_{a_1} \ldots x^{(n-1)}_{a_{n-1}} \\ &+ \left(\nabla_d \ell^{cda_1\ldots a_{n-1}}\right) \sum_{i=1}^{n-1} x^{(1)}_{a_1} \ldots (\nabla_c x^{(i)}_{a_i}) \ldots x^{(n-1)}_{a_{n-1}} + (d \longleftrightarrow c) \\ &+ \ell^{cda_1\ldots a_{n-1}} \sum_{i=1}^{n-1} \sum_{j\neq i}^{n-1} x^{(1)}_{a_1} \ldots (\nabla_c x^{(i)}_{a_i}) \ldots (\nabla_d x^{(j)}_{a_j}) \ldots x^{(n-1)}_{a_{n-1}} \end{aligned}$$

[2] We refer to (2.2) as the *KT-current* henceforth.

$$+\ell^{cda_1\dots a_{n-1}}\sum_{i=1}^{n-1}x^{(1)}_{a_1}\dots\left(\nabla_d\nabla_c x^{(i)}_{a_i}\right)\dots x^{(n-1)}_{a_{n-1}}\,. \tag{2.8}$$

The first term in (2.8) vanishes by (2.6). The terms on the second and third lines are symmetric on the indices c and d, whereas ℓ itself is totally antisymmetric; so they vanish. The last term can be rewritten in terms of the Riemann tensor by using the antisymmetry of ℓ again:

$$\ell^{cda_1\dots a_{n-1}}\sum_{i=1}^{n-1}x^{(1)}_{a_1}\dots\left(\nabla_d\nabla_c x^{(i)}_{a_i}\right)\dots x^{(n-1)}_{a_{n-1}}=2\ell^{cda_1\dots a_{n-1}}\sum_{i=1}^{n-1}x^{(1)}_{a_1}\dots\left(R_{dca_i}{}^{b}\,x^{(i)}_{b}\right)\dots x^{(n-1)}_{a_{n-1}}\,, \tag{2.9}$$

which vanishes by the Bianchi identity $R_{[dca_i]}{}^{b}=0$. Hence (2.7) does hold whenever a spacetime admits a rank-n KYT f and there is a proper set of $(n-1)$ vectors $x^{(i)}$.

In the next section we will discuss how these can be used for defining conserved charges.

3. Linearized currents and asymptotic charges

The existence of asymptotic charges based on the KT-current (2.2) was shown in [1,7] for asymptotically flat and asymptotically AdS geometries. The method is a generalization of employing asymptotic Killing vectors [6] to define the corresponding conserved charges. Let us quickly recapitulate this construction for convenience. The starting point is a D-dimensional spacetime with a metric g_{ab} whose asymptotic Killing-Yano charge(s) are to be computed. The metric g_{ab} does not necessarily have to admit exact KYTs, but the assumption is that it can be split into a background $\bar{g}_{ab}$ plus a deviation as

$$g_{ab}\equiv\bar{g}_{ab}+h_{ab}\quad\text{so that}\quad g^{ab}=\bar{g}^{ab}-h^{ab}+\mathcal{O}(h^2)\,, \tag{3.1}$$

where $h^{ab}=\bar{g}^{ac}h_{cd}\,\bar{g}^{db}$. It is also assumed that h_{ab} vanishes sufficiently fast at the relevant (spatial) boundary, and that $\bar{g}_{ab}$ admits a completely antisymmetric rank-n KYT $\bar{f}_{a_1\dots a_n}$.

With the understanding that all indices are raised and lowered with the generic background metric $\bar{g}_{ab}$ from now on, e.g. $h\equiv\bar{g}^{ab}h_{bc}$ and $\bar{\Box}\equiv\bar{\nabla}^c\,\bar{\nabla}_c$, one finds the following linearized curvature, Ricci tensor and curvature scalar (to $\mathcal{O}(h)$)

$$(R_{ab}{}^{cd})_L=\bar{R}_{abe}{}^{[c}h^{d]e}+2\,\bar{\nabla}_{[a}\bar{\nabla}^{[d}h_{b]}{}^{c]}\,, \tag{3.2}$$

$$(R^a{}_b)_L=\frac{1}{2}\left(\bar{\nabla}^c\bar{\nabla}^a h_{bc}+\bar{\nabla}_c\bar{\nabla}_b h^{ac}-\bar{\nabla}^a\bar{\nabla}_b h-\bar{\Box}h^a{}_b\right)-h^{ac}\bar{R}_{bc}\,, \tag{3.3}$$

$$R_L=\bar{\nabla}_a\bar{\nabla}_b h^{ab}-\bar{\Box}h-h^{ab}\bar{R}_{ab}\,. \tag{3.4}$$

For maximally symmetric backgrounds that we will be concerned with, the linearized Bianchi identities are identical to (2.4) with all curvature terms replaced by their linearized counterparts. This, in turn, guarantees the background conservation of the linearized current (see [8] for technical details), i.e. $\bar{\nabla}_{a_1}(j^{a_1\dots a_n})_L=0$. Since the current is antisymmetric, the covariant conservation gives rise to an ordinary conservation law via

$$\bar{\nabla}_{a_1}(j^{a_1\dots a_n})_L=\frac{1}{\sqrt{|\bar{g}|}}\,\partial_a\big(\sqrt{|\bar{g}|}\,(j^{a_1\dots a_n})_L\big)=0\,. \tag{3.5}$$

This is used in constructing conserved charges for linearized currents in [1,7], as flux integrals over a suitably chosen $(D-1-n)$-dimensional subspace by further invoking Stokes' theorem: The crucial step is the determination of a potential $\bar{\ell}$ for the current j_L, as further elucidated below.

The asymptotic charges for the KT-current were given in [1] for an arbitrary rank-n KYT in an asymptotically flat background and in [7] for an arbitrary rank-n KYT in an asymptotically AdS background. As mentioned, their existence rests on the linearized KT-current being expressible as the covariant divergence of an $(n+1)$-form potential. The construction of such a potential is nontrivial, a condition that the background has to satisfy for this procedure to work was derived in [8].

The relevant linearized part of (2.2) is

$$(j^{a_1\dots a_n})_L=N_n\,\delta^{a_1\dots a_n d_1 d_2}_{b_1\dots b_n c_1 c_2}\,\bar{f}^{b_1\dots b_n}\,(R_{d_1d_2}{}^{c_1c_2})_L\,. \tag{3.6}$$

In terms of the linearized Riemann tensor in (3.2), the current may be written as

$$(j^{a_1\dots a_n})_L=N_n\,\delta^{a_1\dots a_n d_1 d_2}_{b_1\dots b_n c_1 c_2}\,\bar{f}^{b_1\dots b_n}\left(\bar{R}_{d_1d_2e}{}^{[c_1}\,h^{c_2]e}+2\,\bar{\nabla}_{d_1}\bar{\nabla}^{c_2}h_{d_2}{}^{c_1}\right). \tag{3.7}$$

For a flat background, this may be written as [1]

$$(j^{a_1\dots a_n})_L=\bar{\nabla}_c\,\bar{\ell}^{ca_1\dots a_n}=\frac{1}{\sqrt{|\bar{g}|}}\partial_c\left(\sqrt{|\bar{g}|}\,\bar{\ell}^{ca_1\dots a_n}\right) \tag{3.8}$$

where the $(n+1)$-form $\bar{\ell}^{ea_1\dots a_n}=\bar{\ell}^{[ea_1\dots a_n]}$ is

$$\bar{\ell}^{ea_1\dots a_n}=2N_n\,\delta^{a_1\dots a_n e d_2}_{b_1\dots b_n c_1 c_2}\,\bar{f}^{b_1\dots b_n}\,\bar{\nabla}^{c_2}\,h_{d_2}{}^{c_1}-\frac{1}{2n}\left(h\,\bar{\nabla}^e\,\bar{f}^{a_1\dots a_n}-(n+1)\,h^{d_2[e}\,\bar{\nabla}_{d_2}\,\bar{f}^{a_1\dots a_n]}\right). \tag{3.9}$$

Similar manipulations as in [1] give the following result for the general case[3] [8]

$$(j^{a_1\ldots a_n})_L = \bar{\nabla}_e\,\bar{\ell}^{ea_1\ldots a_n} + N_n\Big(\bar{f}^{[a_1\ldots a_n}\,\bar{R}_{c_1c_2e}{}^{c_1}\,h^{c_2]e} + 2\,h_{c_2}{}^{[c_1}\,\bar{\nabla}^{c_2}\,\bar{\nabla}_{c_1}\,\bar{f}^{a_1\ldots a_n]}\Big) \tag{3.10}$$

with $\bar{\ell}$ as in (3.9). It was shown in [8] that the vanishing of the terms in parentheses (3.10) can be expressed in terms of the background curvature as[4]

$$\bar{f}^{[a_1\ldots a_n}\,\bar{R}_{c_1c_2e}{}^{c_1}\,h^{c_2]e} + 2(-1)^n h_{c_2}{}^{[c_2}\,\bar{R}_e{}^{c_1}{}_{c_1}{}^{a_1}\,\bar{f}^{a_2\ldots a_n]e} = 0\,. \tag{3.11}$$

It is clearly fulfilled for the flat case which leads to the results in [1]. For a maximally symmetric background

$$\bar{R}_{abcd} = \lambda\,(\bar{g}_{ac}\,\bar{g}_{bd} - \bar{g}_{ad}\,\bar{g}_{bc})\,, \quad \bar{R}_{ab} = (D-1)\lambda\,\bar{g}_{ab}\,, \quad \bar{R} = D(D-1)\lambda\,,$$

(3.11) is also fulfilled and leads to the results in [7]. This agrees with the known cases where the linearized Bianchi identities ensure conservation of the KT-current.

To give a concrete example, let us explicitly write the potential for the rank-2 case, i.e. $(j^{ab})_L = \bar{\nabla}_c\,\bar{\ell}^{cab}$ for a maximally symmetric background [1,7]:

$$\bar{\ell}^{abc} = -\frac{3}{2}\,\bar{f}^{d[a}\,\bar{\nabla}^b\,h^{c]}{}_d + \frac{3}{4}\,\bar{f}^{[ab}\,\bar{\nabla}^{c]}\,h + \frac{3}{4}\,h^{d[c}\,\bar{\nabla}_d\,\bar{f}^{ab]} - \frac{3}{4}\,\bar{f}^{[ab}\,\bar{\nabla}_d\,h^{c]d} - \frac{1}{4}\,h\,\bar{\nabla}^{[a}\,\bar{f}^{bc]}\,. \tag{3.12}$$

Furthermore, the GLPP [4] or the MP [5] black holes we will consider in this work clearly have maximally symmetric backgrounds, and all can be cast into the form [1]

$$\bar{g}_{ab} = (n^c n_c) n_a n_b + r_a r_b + q_{ab}\,, \tag{3.13}$$

where q_{ab} is the metric on the $(D-2)$-dimensional space Σ, n^a and r^a are mutually orthogonal unit vectors to Σ, with n^a a non-null vector, which we will choose to be timelike so that $n^c n_c = -1$. The conserved charge can be written for the $n = 2$ case using the 3-form potential (3.12) as

$$Q = \int_\Sigma d^{D-2}x\, n_{[a}\, r_{b]}\,\big(\sqrt{|q|}\,(j^{ab})_L\big) = \int_\Sigma d^{D-2}x\, n_{[a}\, r_{b]}\,\partial_c\big(\sqrt{|\bar{g}|}\,\bar{\ell}^{cab}\big). \tag{3.14}$$

Near the spatial boundary, one can further write $q_{ab} = y_a y_b + \gamma_{ab}$, where at spatial infinity y^a is the unit normal to the $(D-3)$-dimensional boundary $\partial\Sigma$ and γ_{ab} is the metric on $\partial\Sigma$. Finally, Stokes' theorem lets one write

$$Q = \int_{\partial\Sigma} d^{D-3}x\, n_{[a}\, r_b\, y_{c]}\,\big(\sqrt{|\gamma|}\,\bar{\ell}^{abc}\big) \tag{3.15}$$

at the spatial boundary.

However, one can define a conserved charge in an alternative way. In view of (3.5) and (3.8), it is easy to see that the potential $\bar{\ell}$ satisfies

$$\bar{\nabla}_d\bar{\nabla}_c\,\bar{\ell}^{cda_1\ldots a_{n-1}} = 0\,. \tag{3.16}$$

This means that the linearization process can also be applied to (2.7) resulting in its linearized analog

$$\bar{\nabla}_d\bar{\nabla}_c\,\Big(\bar{\ell}^{cda_1\ldots a_{n-1}} x^{(1)}_{a_1}\ldots x^{(n-1)}_{a_{n-1}}\Big) = 0\,, \tag{3.17}$$

for a suitably chosen set of vectors $x^{(i)}$, $(i = 1, \ldots, n-1)$ on the background geometry. The term inside the parentheses in (3.17) can be thought of as a 2-form "potential" $\bar{L}^{cd}$, for a 1-form "conserved current" $\bar{J}^d$, with $\bar{J}^d = \bar{\nabla}_c\,\bar{L}^{cd}$, which can be employed in a consistent definition for a "conserved charge". What about a simple but reasonable choice for the background vectors $x^{(i)}$ though? Suppose that the background space admits the foliation[5]

$$\bar{g}_{ab} = -n_a n_b + r_a r_b + q_{ab} \quad \text{with} \quad q_{ab} = \sum_{i=1}^{n-1} x^{(i)}_a x^{(i)}_b + \gamma_{ab}\,, \tag{3.18}$$

where for consistency it must be that $D - 2 = (n-1) + \dim\gamma$, so that $n + 1 \leq D$. Here it is implicitly assumed that the induced metric γ on the $(D-1-n)$-dimensional subspace is non-degenerate. Now let $x^{(i)}$ be mutually orthogonal spacelike normal vectors of this subspace, i.e. $x^{(i)}{}_a x^{(j)a} = 0$ when $i \neq j$. In fact, we also demand that they are hypersurface orthogonal, i.e. $x^{(i)}{}_{[a}\bar{\nabla}_b x^{(i)}{}_{c]} = 0$ for all $i = 1, \ldots, n-1$. With these, one can now integrate over the $(D-2)$-dimensional spatial boundary[6] to get a conserved charge, and write

[3] Note that there are no additional curvature terms generated in the process.
[4] When $n = 1$, (3.11) simply reads $h\bar{R}^{ab}\,\bar{f}_b - h^{bc}\,\bar{R}_{bc}\,\bar{f}^a = 0$.
[5] This is indeed the case for the maximally symmetric backgrounds of the MP [5] and the GLPP [4] black holes we will consider.
[6] instead of integrating over the $(D-1-n)$-dimensional subspace as in (3.15).

$$\mathcal{Q} = \int_{\Sigma} d^{D-2}x\, n_{[a}\, r_b\, x^{(1)}_{c_1} \ldots x^{(n-1)}_{c_{n-1}]}\, \bar{\ell}^{abc_1\ldots c_{n-1}} \sqrt{|q|}\,. \tag{3.19}$$

In what follows we will calculate this charge $\mathcal{Q}$ for Kerr, AdS-Kerr, and the GLPP [4] black holes, which are higher dimensional generalizations of the AdS-Kerr metric. The charges for the MP [5] black holes will follow from those of the GLPP ones by taking the cosmological constant to zero, of course.

4. The Kerr metric

Let us examine all of these ideas first on the celebrated Kerr metric [2] cast in Boyer-Lindquist coordinates [3]:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4aMr\sin^2\theta}{\Sigma}\, dt\, d\phi + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 \\ &\quad + \left(r^2 + a^2 + \frac{2a^2 Mr\sin^2\theta}{\Sigma}\right) \sin^2\theta\, d\phi^2\,, \end{aligned} \tag{4.1}$$

where

$$\Sigma := r^2 + a^2\cos^2\theta \qquad \text{and} \qquad \Delta := r^2 + a^2 - 2Mr\,. \tag{4.2}$$

The rank-2 KYT of the Kerr metric is [9,10]

$$f = a\cos\theta\, dr \wedge \left(dt - a\sin^2\theta\, d\phi\right) + r\sin\theta\, d\theta \wedge \left((r^2 + a^2) d\phi - a dt\right). \tag{4.3}$$

The background is found by setting $M = 0, a = 0$ in (4.1):

$$\begin{aligned} &\bar{g}_{ab} = \text{diag}(-1, 1, r^2, r^2\sin^2\theta)\,, \quad n^a = (-\partial_t)^a\,, r^a = (\partial_r)^a \text{ with } n^a n_a = -1 \text{ and } r^a r_a = 1\,, \\ &q_{ab} = \text{diag}(0, 0, r^2, r^2\sin^2\theta)\,. \end{aligned} \tag{4.4}$$

Clearly q_{ab} is the metric on a sphere S^2 of radius r. The spatial boundary is defined by $r \to \infty$. From (4.3), the background KYT reads $\bar{f} = r^3 \sin\theta\, d\theta \wedge d\phi$, $\sqrt{|q|} = r^2 \sin\theta$ and after some calculation

$$\bar{\ell}^{tr\theta} = \frac{aM\sin\theta\left(a^2\cos^2\theta + 3r^2\right)}{2r\Sigma^2}\,. \tag{4.5}$$

With the choice $x^a = (\partial_\theta / r)^a$ from (3.18), it follows that

$$\begin{aligned} \mathcal{Q} &= \lim_{r\to\infty} \int_{S^2} d^2x\, n_{[a}\, r_b\, x_{c]} \bar{\ell}^{abc} \sqrt{|q|} = \lim_{r\to\infty} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, r^3 \sin\theta\, \bar{\ell}^{tr\theta} \\ &= \lim_{r\to\infty} \left(\frac{\pi^2 aMr\left(2a^6 + 16r^5\left(r - \sqrt{a^2+r^2}\right) - 4a^2r^3\left(7\sqrt{a^2+r^2} - 9r\right) + 3a^4 r\left(7r - 3\sqrt{a^2+r^2}\right)\right)}{\sqrt{a^2+r^2}\left(a^3 + 2ar\left(r - \sqrt{a^2+r^2}\right)\right)^2} \right) \\ &= \frac{3\pi^2}{2} aM\,. \end{aligned} \tag{4.6}$$

$\mathcal{Q}$ (4.6) is proportional to aM, i.e. the angular momentum of the Kerr black hole.

5. The AdS-Kerr metric

In Boyer-Lindquist coordinates the AdS-Kerr metric [11] is

$$\begin{aligned} ds^2 &= -\frac{\Delta_\theta}{\Xi}(1 - \lambda r^2)\, dt^2 + \frac{\Sigma}{\Delta}\, dr^2 + \frac{\Sigma}{\Delta_\theta}\, d\theta^2 \\ &\quad + \frac{(r^2 + a^2)}{\Xi} \sin^2\theta\, d\phi^2 + \frac{2Mr}{\Sigma} \left(\frac{\Delta_\theta}{\Xi}\, dt - \frac{a\sin^2\theta}{\Xi}\, d\phi \right)^2, \end{aligned} \tag{5.1}$$

where

$$\begin{aligned} &\Delta := (r^2 + a^2)(1 - \lambda r^2) - 2Mr\,, \qquad \Xi := 1 + \lambda a^2\,, \\ &\Delta_\theta := 1 + \lambda a^2\cos^2\theta\,, \qquad \Sigma := r^2 + a^2\cos^2\theta\,. \end{aligned} \tag{5.2}$$

The rank-2 KYT of the AdS-Kerr metric (5.1) is

$$f = \frac{a\cos\theta}{\Xi}\, dr \wedge \left(\Delta_\theta\, dt - a\sin^2\theta\, d\phi\right) + \frac{r\sin\theta}{\Xi}\, d\theta \wedge \left((r^2 + a^2) d\phi - a(1 - \lambda r^2) dt\right). \tag{5.3}$$

The AdS background

$$\bar{g}_{ab} = \text{diag}\left(-(1-\lambda r^2), \frac{1}{1-\lambda r^2}, r^2, r^2\sin^2\theta\right) \tag{5.4}$$

is obtained by setting $M = 0$, $a = 0$ in (5.1). This background can be foliated as in (3.18), where the unit normal vectors n^a, r^a and x^a are explicitly

$$n^a = \left(-\frac{1}{\sqrt{1-\lambda r^2}}\partial_t\right)^a, \quad r^a = \left(\sqrt{1-\lambda r^2}\,\partial_r\right)^a, \quad x^a = \left(\frac{\partial_\theta}{r}\right)^a, \tag{5.5}$$

with $\sqrt{|q|} = r^2\sin\theta$. The background KYT reads $\bar{f} = r^3\sin\theta d\theta \wedge d\phi$ as for the Kerr case and leads to the potential with a nontrivial component

$$\bar{\ell}^{tr\theta} = \frac{aM\sin\theta\,\Delta_\theta\left(a^2\cos^2\theta + 3r^2\right)}{2r\,\Xi^2\Sigma^2}. \tag{5.6}$$

The conserved charge can be obtained by an integration on the $(D-2)$-dimensional boundary S^2 at the spatial infinity $r \to \infty$:

$$\mathcal{Q} = \lim_{r\to\infty}\int_{S^2} d^2x\, n_{[a}\, r_b\, x_{c]}\bar{\ell}^{abc}\sqrt{|q|} = \frac{3\pi^2\,(3+\Xi)}{8\,\Xi^2}aM, \tag{5.7}$$

which correctly reduces to (4.6) when the cosmological constant $\lambda \to 0$. Note also that $\mathcal{Q}$ (5.7) is again proportional to aM, the angular momentum of the AdS-Kerr black hole.

6. Higher-dimensional metrics

The charge $\mathcal{Q}$ (3.19) can be computed for the GLPP metrics [4] in higher dimensions. To that end, we first recall their form (as they are cast in Appendix E of [4] in Boyer-Lindquist coordinates):

$$\begin{aligned} ds^2 = &-W(1-\lambda r^2)\,dt^2 + \frac{2M}{VF}\left(W\,dt - \sum_{i=1}^{N}\frac{a_i\mu_i^2}{\Xi_i}\,d\phi_i\right)^2 + \sum_{i=1}^{N}\frac{r^2+a_i^2}{\Xi_i}\mu_i^2\,d\phi_i^2 \\ &+\frac{VF}{V-2M}\,dr^2 + \sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}\,d\mu_i^2 + \frac{\lambda}{W(1-\lambda r^2)}\left(\sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{\Xi_i}\mu_i\,d\mu_i\right)^2, \end{aligned} \tag{6.1}$$

where

$$\begin{aligned} &W := \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{\Xi_i}, \qquad V := r^{\epsilon-2}(1-\lambda r^2)\prod_{i=1}^{N}(r^2+a_i^2), \\ &F := \frac{r^2}{1-\lambda r^2}\sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}, \qquad \Xi_i := 1+\lambda a_i^2. \end{aligned} \tag{6.2}$$

The "evenness" integer[7] ϵ and the number of azimuthal angular coordinates N is defined as [4]

$$\epsilon := (D-1)\bmod 2, \qquad N := [(D-1)/2],$$

so that the number of latitudinal coordinates μ_i is $N+\epsilon$, and there are N azimuthal angular coordinates ϕ_j. Thus $D = 2N+\epsilon+1$, where the finger counting goes as follows: There are the time coordinate t, the radial coordinate r, $N+\epsilon-1$ independent latitudinal coordinates[8] μ_i, and N azimuthal angular coordinates ϕ_j. The latitudinal coordinates μ_i range over $[0,1]$ except for the last one in *even* dimensions, μ_{N+1}, which ranges over $[-1,1]$. As usual, the azimuthal angular coordinates ϕ_j are periodic with period 2π. The GLPP metric (6.1) satisfies the cosmological Einstein equations [4]

$$R_{ab} = (D-1)\lambda g_{ab},$$

and reduces to the MP metric [5] when $\lambda \to 0$. Moreover, the rank-2 closed conformal KYT of the GLPP metric (6.1) is [12][9]

$$k = \sum_{i=1}^{N}a_i\,\mu_i\,d\mu_i \wedge \left[a_i\,dt - \frac{(r^2+a_i^2)}{\Xi_i}\,(d\phi_i - \lambda a_i\,dt)\right] + rdr \wedge \left[dt - \sum_{i=1}^{N}\frac{a_i\mu_i^2}{\Xi_i}\,(d\phi_i + \lambda a_i\,dt)\right]. \tag{6.3}$$

[7] $\epsilon = 1$ for even D and $\epsilon = 0$ for odd D.

[8] There is a constraint on latitudinal coordinates: $\sum_{i=1}^{N+\epsilon}\mu_i^2 = 1$.

[9] Initially we thought our work was the first where this was reported. However we became aware of the thesis [12] (see formula (B.15) therein) after posting the first version of our paper. In passing, it should be mentioned that the associated KYTs for the MP black holes were first reported in [13], which paved the way for applications of KYTs to higher dimensional stationary black holes.

The background can be found by setting $M = 0$ and $a_i = 0$ in (6.1):

$$d\bar{s}^2 = -(1-\lambda r^2)\,dt^2 + \frac{dr^2}{1-\lambda r^2} + r^2\left(\sum_{i=1}^{N+\epsilon} d\mu_i^2 + \sum_{i=1}^{N} \mu_i^2\,d\phi_i^2\right). \tag{6.4}$$

However, this is a bit misleading since there is a constraint on the latitudinal coordinates μ_i whose elimination resurrects the $\bar{g}_{\mu_i\mu_j}$ components. This considerably complicates the charge $\mathcal{Q}$ integration (3.19), since with a non-diagonal q_{ab} (3.18), it is difficult to identify a proper set of vectors $x^{(i)}$ that heeds the requirements mentioned in the penultimate paragraph of sec. 3. The remedy is a transformation from μ_i to the quasi-spheroidal coordinates θ_i, which diagonalize the $(D-2)$-dimensional q_{ab} in the foliation of the background (3.18), and are given by

$$\mu_i(\theta) := \left(\prod_{j=1}^{N+\epsilon-i} \sin\theta_j\right)\cos\theta_{N+\epsilon-i+1}\,, \tag{6.5}$$

where the last coordinate $\theta_{N+\epsilon}$ and $d\theta_{N+\epsilon}$ are set to 0 in order to write the transformation compactly. All θ_i range over $[0, \pi/2]$, except for θ_N in *even* dimensions which ranges over $[0, \pi]$.

In the new coordinates (t, r, θ_i, ϕ_j), the background metric (6.4) is

$$\bar{g}_{ab} = \mathrm{diag}\left(-(1-\lambda r^2)\,, \frac{1}{1-\lambda r^2}\,, r^2\left(\prod_{k=1}^{N+\epsilon-1-i} \sin^2\theta_{N+\epsilon-k}\right), r^2\mu_j^2(\theta)\right), \tag{6.6}$$

where i runs from 1 to $N+\epsilon-1$, j runs from 1 to N and we have kept $\mu_j = \mu_j(\theta)$ (6.5) in the last entry for economy of notation. Now, $\bar{g}_{ab}$ (6.6) can be put into the form (3.18) with the timelike and spacelike unit normal vectors

$$n^a = \left(-\frac{1}{\sqrt{1-\lambda r^2}}\,\partial_t\right)^a\,, \quad r^a = \left(\sqrt{1-\lambda r^2}\,\partial_r\right)^a\,, \tag{6.7}$$

and q becomes the metric on S^{D-2}, the $(D-2)$-dimensional sphere, of radius r, where the spatial boundary is defined by $r \to \infty$. For this background, the rank-2 closed conformal KYT (6.3) simplifies to

$$\bar{k} = r dr \wedge dt\,. \tag{6.8}$$

The rank-$(D-2)$ background KYT $\bar{f}$ that will go into the rank-$(D-1)$ potential $\bar{\ell}$ (3.9) can be found by taking the Hodge dual of $\bar{k}$ (6.8) [14] with respect to the background (6.6)

$$\bar{f} = \star\bar{k} := r\sqrt{|\bar{g}|}\,d\theta_1 \wedge \cdots \wedge d\theta_{N+\epsilon-1} \wedge d\phi_1 \wedge \cdots \wedge d\phi_N\,. \tag{6.9}$$

Finding the potential $\bar{\ell}$ (3.9) is not an easy feat now since the coordinate transformation we have introduced earlier (6.5) changes the form of (6.1) drastically, which in turn changes the deviations h_{ab} (3.1) that go into $\bar{\ell}$ (3.9). In retrospect, a choice had to be made in the trade-off between "being able to integrate with simpler set of vectors $x^{(i)}$" and "working with more straightforwardly calculable (and perhaps less complicated) deviations h_{ab}, and hence potential $\bar{\ell}$". We have opted for the first and paid a price in the determination of the deviations and later the potentials. The calculations involved are hardly illuminating, so we prefer not to show the gory details, and briefly summarize the steps taken instead: First, we have cast (6.1) in the (t, r, θ_i, ϕ_j) coordinates for dimensions $4 \le D \le 8$, then determined h_{ab} (3.1) in each case using the background $\bar{g}_{ab}$ (6.6) and carefully calculated the rank-$(D-1)$ potential $\bar{\ell}$ (3.9) in each case. We have found that there are N independent components of $\bar{\ell}$ that can be used in the charge $\mathcal{Q}$ integration (3.19):

$$\bar{\ell}^{tr\theta_1\ldots\theta_{N+\epsilon-1}\phi_1\ldots\widehat{\phi_j}\ldots\phi_N}\,, \quad (2N+\epsilon-1 = D-2)\,, \tag{6.10}$$

where a wide hat on a ϕ_j-component indicates that it is to be omitted and $j = 1, \ldots, N$. So we have chosen the $N+\epsilon-1$ vectors $x^{(i)}$ and the N vectors $y^{(i)}$ that will saturate the $(D-2)$ components of the rank-$(D-1)$ potential $\bar{\ell}$ as

$$x^{(i)} = r\left(\prod_{k=1}^{N+\epsilon-1-i} \sin\theta_{N+\epsilon-k}\right) d\theta_i \quad \text{and} \quad y^{(j)} = r\,\mu_j(\theta)\,d\phi_j\,. \tag{6.11}$$

Thus we have been able to calculate N charges (3.19) in D dimensions as

$$\mathcal{Q}_D^{(j)} = \int_{S^{D-2}} d^{D-2}x\, n_{[a}\, r_b\, x^{(1)}_{c_1} \ldots x^{(N+\epsilon-1)}_{c_{N+\epsilon-1}}\, y^{(1)}_{c_{N+\epsilon}} \ldots \widehat{y^{(j)}_{c_{N+\epsilon+i-1}}} \ldots y^{(N)}_{c_{D-3}]}\, \bar{\ell}^{abc_1\ldots c_{D-3}} \sqrt{|q|} \tag{6.12}$$

in the limit $|r| \to \infty$. Our findings are tabulated in Table 1.

We find that the generic expression for $\mathcal{Q}_D^{(j)}$ can be written as

$$\text{GLLP:} \quad \mathcal{Q}_D^{(j)} = \frac{(D-1)\,\Omega(D-1)}{2D(D-2)}\,\frac{p_D^{(j)}(\lambda)}{\Xi_j\left(\prod_{k=1}^{N}\Xi_k\right)}\,a_j M\,, \qquad \Omega(D-1) := \frac{2\pi^{D/2}}{\Gamma(D/2)}\,. \tag{6.13}$$

Table 1
Charges $\mathcal{Q}$ for GLLP and MP black holes in dimensions $4 \le D \le 8$.

D	Charge	Potential	GLLP	MP
4	$Q_4^{(1)}$	$\ell^{abc_1} x_{c_1}^{(1)}$	$\frac{3aM\pi^2}{8\Xi^2}(3+\Xi)$	$\frac{3aM\pi^2}{2}$
5	$Q_5^{(1)}$	$\ell^{abc_1c_2} x_{c_1}^{(1)} y_{c_2}^{(2)}$	$-\frac{16a_1M\pi^2}{45\Xi_1^2\Xi_2}(2\Xi_1+3\Xi_2)$	$-\frac{16a_1M\pi^2}{9}$
	$Q_5^{(2)}$	$\ell^{abc_1c_2} x_{c_1}^{(1)} y_{c_2}^{(1)}$	$\frac{16a_2M\pi^2}{45\Xi_1\Xi_2^2}(2\Xi_2+3\Xi_1)$	$\frac{16a_2M\pi^2}{9}$
6	$Q_6^{(1)}$	$\ell^{abc_1c_2c_3} x_{c_1}^{(1)} x_{c_2}^{(2)} y_{c_3}^{(2)}$	$-\frac{5a_1M\pi^3}{48\Xi_1\Xi_2^2}(2\Xi_1+3\Xi_2+\Xi_1\Xi_2)$	$-\frac{5a_1M\pi^3}{8}$
	$Q_6^{(2)}$	$\ell^{abc_1c_2c_3} x_{c_1}^{(1)} x_{c_2}^{(2)} y_{c_3}^{(1)}$	$\frac{5a_2M\pi^3}{48\Xi_1^2\Xi_2}(2\Xi_2+3\Xi_1+\Xi_1\Xi_2)$	$\frac{5a_2M\pi^3}{8}$
7	$Q_7^{(1)}$	$\ell^{abc_1c_2c_3c_4} x_{c_1}^{(1)} x_{c_2}^{(2)} y_{c_3}^{(2)} y_{c_4}^{(3)}$	$\frac{16a_1M\pi^3}{175\Xi_1^2\Xi_2\Xi_3}(2\Xi_1(\Xi_2+\Xi_3)+3\Xi_2\Xi_3)$	$\frac{16a_1M\pi^3}{25}$
	$Q_7^{(2)}$	$\ell^{abc_1c_2c_3c_4} x_{c_1}^{(1)} x_{c_2}^{(2)} y_{c_3}^{(1)} y_{c_4}^{(3)}$	$-\frac{16a_2M\pi^3}{175\Xi_1\Xi_2^2\Xi_3}(2\Xi_2(\Xi_3+\Xi_1)+3\Xi_3\Xi_1)$	$-\frac{16a_2M\pi^3}{25}$
	$Q_7^{(3)}$	$\ell^{abc_1c_2c_3c_4} x_{c_1}^{(1)} x_{c_2}^{(2)} y_{c_3}^{(1)} y_{c_4}^{(2)}$	$\frac{16a_3M\pi^3}{175\Xi_1\Xi_2\Xi_3^2}(2\Xi_3(\Xi_1+\Xi_2)+3\Xi_1\Xi_2)$	$\frac{16a_3M\pi^3}{25}$
8	$Q_8^{(1)}$	$\ell^{abc_1c_2c_3c_4c_5} x_{c_1}^{(1)} x_{c_2}^{(2)} x_{c_3}^{(3)} y_{c_4}^{(2)} y_{c_5}^{(3)}$	$\frac{7a_1M\pi^4}{288\Xi_1^2\Xi_2\Xi_3}(2\Xi_1(\Xi_2+\Xi_3)+3\Xi_2\Xi_3+\Xi_1\Xi_2\Xi_3)$	$\frac{7a_1M\pi^4}{36}$
	$Q_8^{(2)}$	$\ell^{abc_1c_2c_3c_4c_5} x_{c_1}^{(1)} x_{c_2}^{(2)} x_{c_3}^{(3)} y_{c_4}^{(1)} y_{c_5}^{(3)}$	$-\frac{7a_2M\pi^4}{288\Xi_1\Xi_2^2\Xi_3}(2\Xi_2(\Xi_1+\Xi_3)+3\Xi_1\Xi_3+\Xi_1\Xi_2\Xi_3)$	$-\frac{7a_2M\pi^4}{36}$
	$Q_8^{(3)}$	$\ell^{abc_1c_2c_3c_4c_5} x_{c_1}^{(1)} x_{c_2}^{(2)} x_{c_3}^{(3)} y_{c_4}^{(1)} y_{c_5}^{(2)}$	$\frac{7a_3M\pi^4}{288\Xi_1\Xi_2\Xi_3^2}(2\Xi_3(\Xi_1+\Xi_2)+3\Xi_1\Xi_2+\Xi_1\Xi_2\Xi_3)$	$\frac{7a_3M\pi^4}{36}$

$\Omega(D-1)$ is the surface area of the unit sphere S^{D-1}. Here $p_D^{(j)}(\lambda)$ is a polynomial that contains Ξ terms, which can be written iteratively starting from the first nontrivial one $p_4^{(1)}$. For $D \ge 5$, $p_D^{(1)}(\lambda)$ explicitly reads

$$p_D^{(1)}(\lambda) = p_{2N}^{(1)}(\lambda)\,\Xi_N + 2\prod_{k=1}^{N-1}\Xi_k - (1-\epsilon)\prod_{k=1}^{N}\Xi_k \quad \text{with} \quad p_4^{(1)} := 3 + \Xi_1\,. \tag{6.14}$$

The remaining $p_D^{(j)}(\lambda)$, for $j = 2, \dots, N$, follows by cycling the N indices that j runs over. For the MP metrics for which $\lambda = 0$, $\Xi_j = 1$ and $p_D^{(j)} = D$ for all j, so that (6.13) simplifies to

$$\text{MP:} \quad \mathcal{Q}_D^{(j)} = \frac{(D-1)\,\Omega(D-1)}{2D(D-2)}\,a_j M\,. \tag{6.15}$$

The charges $\mathcal{Q}_D^{(j)}$ (6.13) are proportional to a_jM and are clearly related to the angular momenta of the black holes. The correct angular momenta that fulfills the first law of black hole thermodynamics have been calculated through the Komar integral and reads[10]

$$J_j = \frac{\Omega(D-2)}{4\pi\ \Xi_j\left(\prod_{k=1}^{N}\Xi_k\right)} M a_j \tag{6.16}$$

in our conventions. The main difference stems from the polynomial $p_D^{(j)}(\lambda)$ that our $\mathcal{Q}_D^{(j)}$ (6.13) has. However, recall that our main objective is to come up with some conserved charge out of the KT-current. In that sense, we never expected $\mathcal{Q}_D^{(j)}$ (6.13) to have a clear-cut or a definite physical meaning in the first place.

7. Discussion

In this paper we first reviewed and amended the construction of asymptotic charges starting from the KT-current for KYTs. We then applied these ideas to the GLPP and MP black holes to arrive at the charge formulae (6.13) and (6.15), respectively.

The construction we have presented heavily relies on a set of properly chosen vectors $x^{(i)}$, which in itself implicitly depends on the foliation of the background. This naturally brings in the question of whether the charge $\mathcal{Q}$ (3.19) is background gauge invariant, i.e. how does the charge $\mathcal{Q}$ (3.19) change as the deviation h_{ab} (3.1) transforms as

$$\delta_{\bar{\zeta}} h_{ab} = \bar{\nabla}_a \bar{\zeta}_b + \bar{\nabla}_b \bar{\zeta}_a \tag{7.1}$$

under an infinitesimal diffeomorphism generated by a vector $\bar{\zeta}^a$? The answer to this question, foremost, depends on how the linearized current $(j)_L$ (3.6) itself transforms under (7.1). Using (3.2), we find for an arbitrary background that

$$\delta_{\bar{\zeta}} (R_{ab}{}^{cd})_L = 2\bar{R}_{ab}{}^{e[c}\bar{\nabla}_e\bar{\zeta}^{d]} - 2\bar{R}^{cd}{}_{e[a}\bar{\nabla}_{b]}\bar{\zeta}^e\,. \tag{7.2}$$

[10] See Section 4.1 of [15] for details.

However, for maximally symmetric backgrounds we have been working with the right hand side vanishes so that $\delta_{\bar{\xi}}(R_{ab}{}^{cd})_L = 0$, which means, via (3.6), that the linearized current $(j)_L$ and hence its potential $\bar{\ell}$, via (3.10), are both background gauge invariant.[11] Since the infinitesimal diffeomorphisms in question do not change the foliation of the background, the vectors $x^{(i)}$ are also left invariant. Thus we conclude that the charge $\mathcal{Q}$ (3.19) is background gauge invariant for maximally symmetric backgrounds.

Our main aim has been to associate a conserved charge to the enigmatic KT-current. In that sense, we feel our work gets to first base.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests:

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Data availability

No data was used for the research described in the article.

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[11] The potential is left invariant up to an exact term.
[12] However the entire responsibility for the publication is ours. The financial support received from TÜBİTAK does not mean that the content of the publication is approved in a scientific sense by TÜBİTAK.