Synchronization of oscillators not sharing a common ground

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Abstract

Networks of coupled LC oscillators that do not share a common ground node are studied. Both resistive coupling and inductive coupling are considered. For networks under resistive coupling, it is shown that the oscillator-coupler interconnection has to be bilayer if the oscillator voltages are to asymptotically synchronize. Also, for bilayer architecture (when both resistive and inductive couplers are present) a method is proposed to compute a complex-valued effective Laplacian matrix that represents the overall coupling. It is proved that the oscillators display synchronous behavior if and only if the effective Laplacian has a single eigenvalue on the imaginary axis.

1 Introduction

A network of two-terminal electrical oscillators coupled via two-terminal components gives rise to a pair of graphs. One of them is the coupler graph, whose edges represent the couplers. The other is the oscillator graph, where the edges stand for the oscillators. As an illustration, let us reproduce in Fig. 1a the example array of six Chua's oscillators coupled via linear resistors presented in [13]. The corresponding coupler graph and oscillator graph are given in Fig. 1b and Fig. 1c, respectively. Note that, since one terminal of each oscillator rests on the so called ground node (g), the oscillator graph in this example is a star.[1] Such networks with a star oscillator graph have been considered, for instance, in [12, 5, 7, 4]. Another type of topology appears in the works [8, 1, 6] which study synchronization in a series-connected array of oscillators. Those networks enjoy an oscillator graph that is a path; see Fig. 2.

The majority of research effort on electrical networks focuses on those with a star oscillator graph. There is a good reason for that. Man-made systems, be it a large power network or a tiny microchip, are usually designed so that there is a common ground node in the overall circuit to which the units (generators, oscillators, etc.) are directly connected. Still, this real-world significance of networks with a common ground, we believe, is no justification for the general neglect in the literature of less restrictive architectures. Let us briefly speculate why. Clearly, not having to be confined to star topology means more flexibility both in design and in analysis. Flexibility in design can be important, for instance, if the system to be built is part of a microchip, where the design constraints are already very tight. And, flexibility in analysis may prove useful, e.g., in understanding natural phenomena: Certain biological systems are long known to be able to be modeled by interconnected electrical oscillators and it is very unlikely that nature should have a certain preference for networks where the oscillators share a common ground node. In addition to these practical benefits, understanding the general case has value in its own right. Motivated by these, we study in this paper the collective behavior of coupled LC circuits without assuming that the oscillator graph is a star; two simple examples are shown in Fig. 3.

We begin our investigation by studying the linear time invariant (LTI) network of identical LC oscillators coupled by resistors only. We show that such a network displays synchronous behavior if and only if the oscillators communicate through a bilayer coupling structure (see Fig. 4) and the graphs representing the layers are both connected. In the second half of the paper we focus our attention on a more general situation, where not only resistors but also inductors are allowed as couplers. In this case the overall

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[1] I.e., a tree with diameter no larger than two.

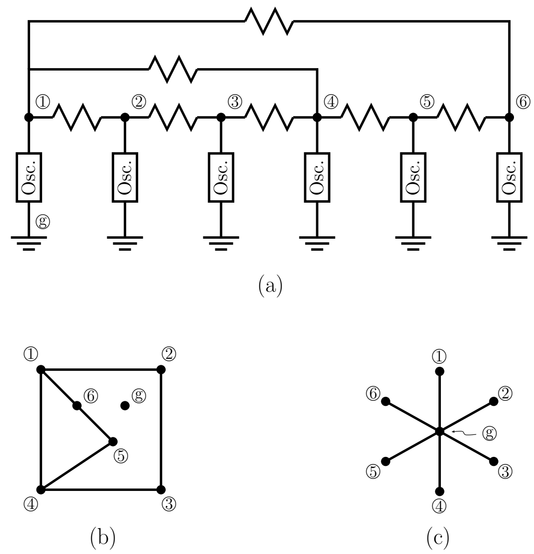

Figure 1: (a) An array of coupled oscillators. (b) The coupler graph. (c) The oscillator graph.

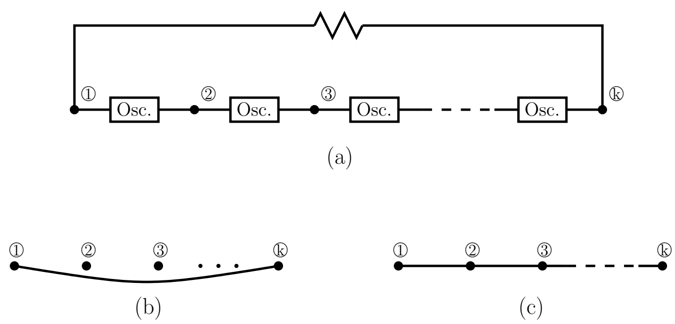

Figure 2: (a) An array of series-connected oscillators. (b) The coupler graph. (c) The oscillator graph.

coupling gives rise to four Laplacian matrices G_1, G_2, B_1, B_2. The matrices G_1 and G_2 represent the resistors in the first and second layers, respectively; B_1 and B_2 represent the inductors in the first and second layers, respectively. We propose a method to generate a single complex matrix (which we call the effective Laplacian) out of these four real matrices[2] and study some of its properties. Then we show that the oscillators asymptotically synchronize if and only if this effective Laplacian has a single eigenvalue on the imaginary axis.

Possible contributions of this paper are intended to be in two places. First. Through this preliminary work, we hope to draw attention to nonstar networks and the possible riches they may contain. At first, their apparent lack of structure might suggest that nothing interesting will come out of them. To this view, however, we present a counterevidence in this paper. As mentioned earlier, when one attempts to extract synchronous behavior from a nonstar network, an elegant structure (bilayer coupling) emerges as a necessary condition (Theorem 1). Possibly, other different phenomena flourish on other interesting structures; and considering networks where oscillators do not share a common ground node would allow one to discover those interconnection forms. This is somewhat in contrast to the general style in the literature, where the analysis usually rests on an initially assumed architecture, e.g., networks with star oscillator graph. Second. We make a first step toward a systematic approach (which involves studying the spectral properties of the effective Laplacian) for understanding the joint tendencies of electrical

[2] The situation is a bit subtler. There are indeed six (not four) matrices to be taken into consideration. The details are given in Section 6.

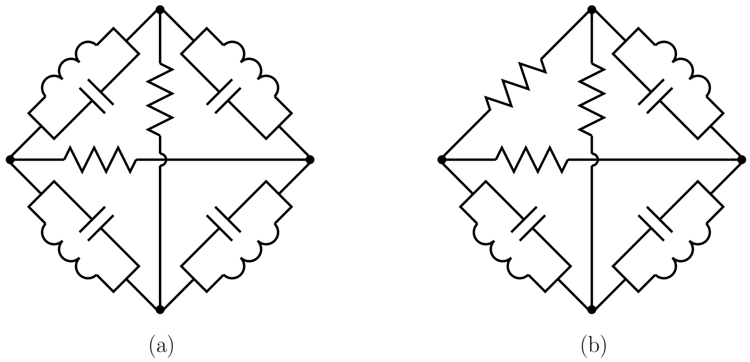

Figure 3: Networks of coupled LC tanks with nonstar oscillator graph.

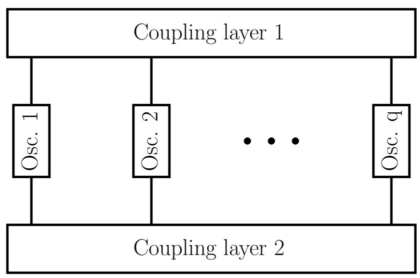

Figure 4: A network of oscillators with bilayer coupling structure.

oscillators under bilayer coupling. To the best of our knowledge, this is a novelty, for the mature literature on synchronization of harmonic oscillators (see, e.g., [9, 10, 14, 11]) does not seem to provide one with off-the-shelf tools to determine the asymptotic behavior of coupled LC tanks in the absence of a common ground node.

2 Notation

Let $I \in \mathbb{R}^{n\times n}$ denote the identity matrix and $I(:, r) \in \mathbb{R}^n$ be its r^{th} column. The vector of all ones is denoted by $\mathbf{1}_q \in \mathbb{R}^q$. The set of eigenvalues of a square matrix P is denoted by $\text{eig}(P)$ and its pseudoinverse by P^+. A matrix $F \in \mathbb{R}^{r\times \ell}$ is said to be class $\mathcal{F}$ if it satisfies the following properties: (i) each entry of F is either 0 or 1, (ii) each column of F contains a single nonzero entry, and (iii) F has no zero rows. Note that $F^T\mathbf{1}_r = \mathbf{1}_\ell$ when F is class $\mathcal{F}$. In this paper we assume that the reader is familiar with some basic graph theoretic terms such as *tree*, *path*, and *cycle*.

3 Problem statement

Recall that the voltage $v_k \in \mathbb{R}$ of an uncoupled LC oscillator satisfies $c\ddot{v}_k + l^{-1}v_k = 0$, where $l, c > 0$ are the associated inductance and capacitance, respectively. Consider now a network of q identical LC oscillators coupled by LTI resistors and inductors. This network has n nodes (denoted by $\nu_1, \nu_2, \ldots, \nu_n$) and each node is assumed to be incident to at least one oscillator. The node voltages (defined with respect to some arbitrary reference point) are denoted by $e_1, e_2, \ldots, e_n \in \mathbb{R}$. Let $A \in \mathbb{R}^{n\times q}$ be the (oriented) incidence matrix associated to the oscillators. This matrix is constructed as follows. Suppose the k^{th} oscillator extends between the nodes ν_r and ν_s[3]; and the positive terminal of the oscillator rests on ν_r while the negative terminal on ν_s. (Note that this implies: the voltage v_k of the k^{th} oscillator reads

[3]We allow at most one oscillator extending between any pair of nodes. In other words, no two oscillators can be parallel.

$v_k = e_r - e_s$.) Then the $k^{\rm th}$ column of A satisfies $A(:, k) = I(:, r) - I(:, s)$. Since each node is incident to at least one oscillator, A has no zero rows. We denote the $sk^{\rm th}$ entry of A by α_{sk}. Let $g_{sr} = g_{rs} \geq 0$ be the conductance of the resistor connecting the nodes ν_r and ν_s. ($g_{rs} = 0$ means there is no resistor between ν_r and ν_s.) Likewise, let $b_{sr} = b_{rs} \geq 0$ denote the reciprocal ($b_{rs} = l_{rs}^{-1}$) of the inductance l_{rs} of the inductor connecting ν_r and ν_s.[4] ($b_{rs} = 0$ means there is no inductor between ν_r and ν_s.) The set of equations describing the evolution of this network then reads

$$\sum_{k=1}^{q} \alpha_{rk}(c\ddot{v}_k + l^{-1}v_k) + \sum_{s=1}^{n} g_{rs}(\dot{e}_r - \dot{e}_s) + \sum_{s=1}^{n} b_{rs}(e_r - e_s) = 0\,, \qquad r = 1, 2, \ldots, n \tag{1a}$$

$$v_k = \sum_{r=1}^{n} \alpha_{rk} e_r\,, \qquad k = 1, 2, \ldots, q \tag{1b}$$

Definition 1 *The network of oscillators* (1) *is said to be* (nontrivially) synchronous *if* $|v_k(t)| - |v_\ell(t)| \to 0$ *as* $t \to \infty$ *for all pairs* (k, ℓ) *and all initial conditions; and* $v_k(t) \not\to 0$ *for some* k *and some initial conditions.*

Remark 1 *If the network* (1) *is synchronous, then the new network obtained by reversing the polarities of some oscillator voltages (i.e., letting* $v_k^{\rm new} = -v_k$ *for some* k*) is still synchronous. In other words, whether a network is synchronous or not is independent of how the oscillator voltage polarities are chosen.*

A simple example for a synchronous network is given in Fig. 3a, where the oscillator voltages (nontrivially) satisfy $|v_k(t)| - |v_\ell(t)| \to 0$. The network in Fig. 3b, however, is not synchronous because $v_k(t) \to 0$ for all k.

To simplify analysis let us first construct the vectors $v = [v_1\ v_2\ \cdots\ v_q]^T$ and $e = [e_1\ e_2\ \cdots\ e_n]^T$. These two vectors are related to one another through the identity $v = A^T e$. Secondly, let us introduce the $n \times n$ Laplacian matrices

$$G = \begin{bmatrix} \sum_s g_{1s} & -g_{12} & \cdots & -g_{1n} \\ -g_{21} & \sum_s g_{2s} & \cdots & -g_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -g_{n1} & -g_{n2} & \cdots & \sum_s g_{ns} \end{bmatrix}, \qquad B = \begin{bmatrix} \sum_s b_{1s} & -b_{12} & \cdots & -b_{1n} \\ -b_{21} & \sum_s b_{2s} & \cdots & -b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{n1} & -b_{n2} & \cdots & \sum_s b_{ns} \end{bmatrix}.$$

Observe that G and B are symmetric and positive semidefinite. Thirdly, we let (without loss of generality) $c = 1$ and define $\omega_0 = 1/\sqrt{lc}$. We can now rewrite (1) as

$$\begin{aligned} AA^T(\ddot{e} + \omega_0^2 e) + G\dot{e} + Be &= 0 & (2a) \\ v &= A^T e\,. & (2b) \end{aligned}$$

The problem we intend to solve in this paper is this.

Find conditions on the triple (A, G, B) *under which the network* (1) *is synchronous.*

Note that the network we study comprises LTI passive components. Therefore it is obvious physically that the solutions have to be bounded. The following remark formalizes this simple observation and will prove useful for later analysis.

Remark 2 *Let* $v(t) = A^T e(t)$ *be an arbitrary solution of the network* (2)*. Construct the nonnegative function* $W(t) = \frac{1}{2}e(t)^T Be(t) + \frac{1}{2}\omega_0^2 v(t)^T v(t) + \frac{1}{2}\dot{v}(t)^T\dot{v}(t) = \frac{1}{2}e(t)^T Be(t) + \frac{1}{2}\omega_0^2 e(t)^T AA^T e(t) + \frac{1}{2}\dot{e}(t)^T AA^T\dot{e}(t)$*. Computing the time derivative of* W *along the solutions of* (2) *we obtain*

$$\begin{aligned} \dot{W} &= \dot{e}^T Be + \omega_0^2 \dot{e}^T AA^T e + \dot{e}^T AA^T \ddot{e} \\ &= \dot{e}^T(AA^T(\ddot{e} + \omega_0^2 e) + Be) \\ &= -\dot{e}^T G\dot{e} = -\sum_{r<s} g_{rs}(\dot{e}_r - \dot{e}_s)^2\,. \end{aligned} \tag{3}$$

[4]Recall that the susceptance of an LTI inductor with inductance l_{rs} at some frequency ω equals $-(\omega l_{rs})^{-1}$. Hence the parameter b_{rs} is the magnitude of the susceptance of the inductor at the frequency $\omega = 1$ rad/s.

Hence $\dot{W} \leq 0$, meaning W is nonincreasing. As a result, the solution $v(t)$ has to be bounded. Since produced by an LTI system, $v(t)$ can be written as a sum (of finitely many terms)

$$v(t) = \sum_k \mathrm{Re}\left(\pi_k(t)e^{\lambda_k t}\right) \tag{4}$$

where $\pi_k(t)$ are polynomials with vector coefficients and $\lambda_k \in \mathbb{C}$ are distinct. That $v(t)$ is bounded therefore implies $\mathrm{Re}(\lambda_k) \leq 0$ for all k; and when $\mathrm{Re}(\lambda_k) = 0$ for some k the corresponding polynomial $\pi_k(t)$ must be of degree zero, i.e., a constant vector.

4 Linkage

Some of the conditions (for synchronization) we present are of structural nature and require the introduction of a graph-like object associated to the network (1). It is defined as follows. Let $\mathcal{V} = \{\nu_1, \nu_2, \ldots, \nu_n\}$ be the set of nodes of our network. Over this node set we construct two (undirected) graphs. The first one is the oscillator graph $(\mathcal{V}, \mathcal{E}_{\mathrm{o}})$, where the edge set $\mathcal{E}_{\mathrm{o}}$ contains $q \geq 2$ unordered distinct pairs $\{\nu_r, \nu_s\} \subset \mathcal{V}$ of nodes between which there is an oscillator in the network. In other words, $\{\nu_r, \nu_s\} \in \mathcal{E}_{\mathrm{o}}$ if the incidence matrix A has a column that reads either $A(:, k) = I(:, r) - I(:, s)$ or $A(:, k) = I(:, s) - I(:, r)$. Since each node in the network is incident to at least one oscillator, the graph $(\mathcal{V}, \mathcal{E}_{\mathrm{o}})$ has no isolated nodes. The second graph is the coupler graph $(\mathcal{V}, \mathcal{E}_{\mathrm{c}})$, where $\{\nu_r, \nu_s\} \in \mathcal{E}_{\mathrm{c}}$ when $g_{rs} + b_{rs} > 0$. We now combine these two graphs into a single object $(\mathcal{V}, \mathcal{E}_{\mathrm{o}}, \mathcal{E}_{\mathrm{c}})$, which we call the *linkage* of the network (2).

Recall that a graph $(\mathcal{V}, \mathcal{E})$ is *bipartite* if we can find two disjoint sets of nodes $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$ such that $\mathcal{V}_1 \cup \mathcal{V}_2 = \mathcal{V}$ and each edge $\epsilon \in \mathcal{E}$ extends between $\mathcal{V}_1$ and $\mathcal{V}_2$, i.e., we can write $\epsilon = \{\nu_r, \nu_s\}$ for some $\nu_r \in \mathcal{V}_1$ and $\nu_s \in \mathcal{V}_2$. Such pair $(\mathcal{V}_1, \mathcal{V}_2)$ is called a *bipartition* of $(\mathcal{V}, \mathcal{E})$, which is not necessarily unique. An equivalent definition would have been the following: A graph is bipartite if it has no odd cycles [2]. Below we introduce a generalization of this.

Definition 2 *The linkage $(\mathcal{V}, \mathcal{E}_{\mathrm{o}}, \mathcal{E}_{\mathrm{c}})$ is said to be* bipartite *if $\mathcal{E}_{\mathrm{o}} \cap \mathcal{E}_{\mathrm{c}} = \emptyset$ and no cycle of the graph $(\mathcal{V}, \mathcal{E}_{\mathrm{o}} \cup \mathcal{E}_{\mathrm{c}})$ contains an odd number of edges from $\mathcal{E}_{\mathrm{o}}$.*

For instance, the linkage of the network in Fig. 3a is bipartite, whereas that of the network in Fig. 3b is not. The sister definition is given next.

Definition 3 *The linkage $(\mathcal{V}, \mathcal{E}_{\mathrm{o}}, \mathcal{E}_{\mathrm{c}})$ is said to be* bilayer *if the graph $(\mathcal{V}, \mathcal{E}_{\mathrm{o}})$ has a bipartition $(\mathcal{V}_1, \mathcal{V}_2)$ such that no edge in $\mathcal{E}_{\mathrm{c}}$ extends between $\mathcal{V}_1$ and $\mathcal{V}_2$. When this holds there exist two disjoint subgraphs $(\mathcal{V}_1, \mathcal{E}_1)$ and $(\mathcal{V}_2, \mathcal{E}_2)$, called the* layers*, satisfying $(\mathcal{V}_1, \mathcal{E}_1) \cup (\mathcal{V}_2, \mathcal{E}_2) = (\mathcal{V}, \mathcal{E}_{\mathrm{c}})$.*

We now establish that bipartite and bilayer mean the same thing.

Lemma 1 *The linkage $(\mathcal{V}, \mathcal{E}_{\mathrm{o}}, \mathcal{E}_{\mathrm{c}})$ is bipartite if and only if it is bilayer.*

Proof. Given the linkage $\mathrm{L} = (\mathcal{V}, \mathcal{E}_{\mathrm{o}}, \mathcal{E}_{\mathrm{c}})$, let the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}_{\mathrm{o}} \cup \mathcal{E}_{\mathrm{c}})$ has κ components. This means we can find disjoint node sets $\mathcal{V}^1, \mathcal{V}^2, \ldots, \mathcal{V}^\kappa \subset \mathcal{V}$ as well as edge sets $\mathcal{E}_{\mathrm{o}}^1, \mathcal{E}_{\mathrm{o}}^2, \ldots, \mathcal{E}_{\mathrm{o}}^\kappa \subset \mathcal{E}_{\mathrm{o}}$ and $\mathcal{E}_{\mathrm{c}}^1, \mathcal{E}_{\mathrm{c}}^2, \ldots, \mathcal{E}_{\mathrm{c}}^\kappa \subset \mathcal{E}_{\mathrm{c}}$ such that $\mathcal{G} = \bigcup_{i=1}^{\kappa}(\mathcal{V}^i, \mathcal{E}_{\mathrm{o}}^i \cup \mathcal{E}_{\mathrm{c}}^i)$. It is not difficult to see that L is bipartite (bilayer) if and only if all the triples $(\mathcal{V}^i, \mathcal{E}_{\mathrm{o}}^i, \mathcal{E}_{\mathrm{c}}^i)$ are separately bipartite (bilayer). Hence, without loss of generality, we henceforth assume $\kappa = 1$, i.e., $\mathcal{G}$ is connected.

Part I: BP $\Longrightarrow$ BL. Suppose L is bipartite. Let $\mathcal{T}$ be a tree of $\mathcal{G}$. Now, using $\mathcal{T}$ we partition the node set $\mathcal{V} = \{\nu_1, \nu_2, \ldots, \nu_n\}$ into two disjoint sets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$ as follows. We begin with letting $\nu_1 \in \mathcal{V}_1$. Then for $r = 2, 3, \ldots, n$ the node ν_r is made belong to $\mathcal{V}_1$ if the (unique) path (in $\mathcal{T}$) connecting ν_1 to ν_r (denoted $\mathcal{P}_{\nu_1 \to \nu_r}$) contains an even number of edges from $\mathcal{E}_{\mathrm{o}}$ (called *o-edges*). Once such construction of $\mathcal{V}_1$ is complete, we let $\mathcal{V}_2 = \mathcal{V} \setminus \mathcal{V}_1$. Now, our claim is that the pair $(\mathcal{V}_1, \mathcal{V}_2)$ has to be a bipartition of the graph $(\mathcal{V}, \mathcal{E}_{\mathrm{o}})$. To see that, suppose otherwise. That is, there exists an edge $\{\nu_r, \nu_s\} \in \mathcal{E}_{\mathrm{o}}$ such that either $\nu_r, \nu_s \in \mathcal{V}_1$ or $\nu_r, \nu_s \in \mathcal{V}_2$. Without loss of generality let both ν_r and ν_s belong to $\mathcal{V}_1$, which implies both paths $\mathcal{P}_{\nu_1 \to \nu_r}$ and $\mathcal{P}_{\nu_1 \to \nu_s}$ contain an even number of o-edges. Let us consider the following two cases separately. *Case 1: The edge $\{\nu_r, \nu_s\}$ belongs to the tree $\mathcal{T}$.* Note that we either have $\mathcal{P}_{\nu_1 \to \nu_s} = \mathcal{P}_{\nu_1 \to \nu_r} \cup \mathcal{P}_{\nu_r \to \nu_s}$ or $\mathcal{P}_{\nu_1 \to \nu_r} = \mathcal{P}_{\nu_1 \to \nu_s} \cup \mathcal{P}_{\nu_s \to \nu_r}$. Without loss of generality let the former be the case. Then, since $\{\nu_r,\, \nu_s\} \in \mathcal{E}_{\rm o}$, if the number of o-edges in the path $\mathcal{P}_{\nu_1\to\nu_r}$ is p then the path $\mathcal{P}_{\nu_1\to\nu_s}$ must contain $p+1$ o-edges. Since p and $p+1$ have different parities, it is not possible that both are even. That is, ν_r and ν_s cannot both belong to $\mathcal{V}_1$. Having ruled out this case let us now consider the other scenario. *Case 2: The edge $\{\nu_r,\, \nu_s\}$ does not belong to the tree $\mathcal{T}$.* Then there is a unique cycle $\mathcal{C}$ of $\mathcal{G}$ with the following property. The cycle $\mathcal{C}$ contains the edge $\{\nu_r,\, \nu_s\}$ and all its other edges belong to the tree $\mathcal{T}$. Let ν_u be the node of $\mathcal{C}$ such that the path $\mathcal{P}_{\nu_1\to\nu_u}$ is shorter than $\mathcal{P}_{\nu_1\to\nu}$ for any other node ν in $\mathcal{C}$. Note that both paths $\mathcal{P}_{\nu_1\to\nu_r}$ and $\mathcal{P}_{\nu_1\to\nu_s}$ should pass from the vertex ν_u, i.e., we have

$$\mathcal{P}_{\nu_1\to\nu_r} = \mathcal{P}_{\nu_1\to\nu_u} \cup \mathcal{P}_{\nu_u\to\nu_r} \tag{5a}$$
$$\mathcal{P}_{\nu_1\to\nu_s} = \mathcal{P}_{\nu_1\to\nu_u} \cup \mathcal{P}_{\nu_u\to\nu_s}\,. \tag{5b}$$

Let now p_1 and p_2 be the numbers of o-edges that $\mathcal{P}_{\nu_u\to\nu_r}$ and $\mathcal{P}_{\nu_u\to\nu_s}$ contain, respectively. Since both ν_r and ν_s belong to $\mathcal{V}_1$, the numbers p_1 and p_2 must have the same parity by (5). Hence p_1+p_2 must be even. Note that $\mathcal{C} = \mathcal{P}_{\nu_u\to\nu_r} \cup \mathcal{P}_{\nu_r\to\nu_s} \cup \mathcal{P}_{\nu_s\to\nu_u}$. Therefore (thanks to $\{\nu_r,\, \nu_s\} \in \mathcal{E}_{\rm o}$) the number of o-edges in the cycle $\mathcal{C}$ must be p_1+p_2+1, which is an odd number. But this contradicts the fact that Ł is bipartite.

Having shown that $(\mathcal{V}_1,\, \mathcal{V}_2)$ is a bipartition of the graph $(\mathcal{V},\, \mathcal{E}_{\rm o})$, we now establish the second condition for Ł to be bilayer. That is, no edge in $\mathcal{E}_{\rm c}$ (called *c-edge*) extends between $\mathcal{V}_1$ and $\mathcal{V}_2$. The demonstration is very similar to that of the first condition. Once again we employ contradiction. Suppose there is an edge $\{\nu_r,\, \nu_s\} \in \mathcal{E}_{\rm c}$ such that $\nu_r \in \mathcal{V}_1$ and $\nu_s \in \mathcal{V}_2$. Let us first study *Case 1: The edge $\{\nu_r,\, \nu_s\}$ belongs to the tree $\mathcal{T}$.* Then both paths $\mathcal{P}_{\nu_1\to\nu_r}$ and $\mathcal{P}_{\nu_1\to\nu_s}$ must contain the same number of o-edges, which implies either both ν_r and ν_s belong to $\mathcal{V}_1$ or neither do. But this cannot coexist with our initial assumption that $\nu_r \in \mathcal{V}_1$ and $\nu_s \in \mathcal{V}_2$. Now we consider *Case 2: The edge $\{\nu_r,\, \nu_s\}$ does not belong to the tree $\mathcal{T}$.* Then there is a unique cycle $\mathcal{C}$ of $\mathcal{G}$ with the following property. The cycle $\mathcal{C}$ contains the edge $\{\nu_r,\, \nu_s\}$ and all its other edges belong to the tree $\mathcal{T}$. Let ν_u be the node of $\mathcal{C}$ such that the path $\mathcal{P}_{\nu_1\to\nu_u}$ is shorter than $\mathcal{P}_{\nu_1\to\nu}$ for any other node ν in $\mathcal{C}$. Note that both paths $\mathcal{P}_{\nu_1\to\nu_r}$ and $\mathcal{P}_{\nu_1\to\nu_s}$ should pass from the vertex ν_u, i.e., we have (5). Let now p_1 and p_2 be the numbers of o-edges that $\mathcal{P}_{\nu_u\to\nu_r}$ and $\mathcal{P}_{\nu_u\to\nu_s}$ contain, respectively. Since $\nu_r \in \mathcal{V}_1$ and $\nu_s \in \mathcal{V}_2$, the numbers p_1 and p_2 must have different parities by (5). Hence p_1+p_2 must be odd. Note that $\mathcal{C} = \mathcal{P}_{\nu_u\to\nu_r} \cup \mathcal{P}_{\nu_r\to\nu_s} \cup \mathcal{P}_{\nu_s\to\nu_u}$. Therefore the number of o-edges in the cycle $\mathcal{C}$ must be p_1+p_2, which is an odd number. But this contradicts the fact that Ł is bipartite. To sum up, we have established that the pair $(\mathcal{V}_1,\, \mathcal{V}_2)$ makes a bipartition of the graph $(\mathcal{V},\, \mathcal{E}_{\rm o})$ and that no c-edge extends between $\mathcal{V}_1$ and $\mathcal{V}_2$. Hence the linkage Ł $= (\mathcal{V},\, \mathcal{E}_{\rm o},\, \mathcal{E}_{\rm c})$ is bilayer. In the second part of the proof we show the other direction.

Part II: BL $\Longrightarrow$ BP. Suppose Ł is bilayer. Then the graph $(\mathcal{V},\, \mathcal{E}_{\rm o})$ has a bipartition $(\mathcal{V}_1,\, \mathcal{V}_2)$ such that no c-edge extends between $\mathcal{V}_1$ and $\mathcal{V}_2$. Let $\mathcal{C}$ be a cycle of the graph $\mathcal{G}$ with the node sequence $(\nu_{r_1},\, \nu_{r_2},\, \nu_{r_3},\, \ldots,\, \nu_{r_\alpha},\, \nu_{r_1})$ where each consecutive pair of nodes $\{\nu_{r_\beta},\, \nu_{r_{\beta+1}}\}$ (letting $r_{\alpha+1}=r_1$) is an edge of $\mathcal{G}$. Note that $\{\nu_{r_\beta},\, \nu_{r_{\beta+1}}\}$ is an o-edge if the constituent nodes belong to different node subsets: $\nu_{r_\beta} \in \mathcal{V}_1$ and $\nu_{r_{\beta+1}} \in \mathcal{V}_2$ or vice versa. Otherwise, i.e., when $\nu_{r_\beta},\, \nu_{r_{\beta+1}} \in \mathcal{V}_1$ or $\nu_{r_\beta},\, \nu_{r_{\beta+1}} \in \mathcal{V}_2$, the edge $\{\nu_{r_\beta},\, \nu_{r_{\beta+1}}\}$ is a c-edge. That the sequence $(\nu_{r_1},\, \nu_{r_2},\, \nu_{r_3},\, \ldots,\, \nu_{r_\alpha},\, \nu_{r_1})$ begins and ends with the same node has one obvious implication: the number of transitions between $\mathcal{V}_1$ and $\mathcal{V}_2$ while one traverses this sequence must always be even. Since the number of transitions equals the number of o-edges, we conclude that the number of o-edges in the cycle $\mathcal{C}$ is even. This is true for any cycle because $\mathcal{C}$ was arbitrary. Therefore Ł is bipartite. ■

5 Purely resistive coupling

In this section we focus on the special case where the LC oscillators are coupled via resistors only. Namely, we study the network (2) under $B=0$. Consider therefore

$$AA^T(\ddot{e}+\omega_0^2 e) + G\dot{e} = 0\,, \qquad v = A^T e\,. \tag{6}$$

We will show that whether the network (6) is synchronous or not depends only on the structure of the coupling. This would imply that two separate (resistively coupled) networks are both synchronous if they share the same linkage. In Section 8 we will see that this is not the case for the general situation where inductive coupling is also present. The following lemma is the first step toward the main result (Theorem 1) of this section.

Lemma 2 *The network* (6) *is synchronous only when its linkage is bipartite.*

Proof. Suppose the network (6) is synchronous. Let $v(t) = A^T e(t)$ be an arbitrary solution satisfying $v(t) \not\to 0$. Recall that $v(t)$ is bounded and satisfies (4). Hence $v(t) \not\to 0$ implies that the sum (4) must include a term of the form $\mathrm{Re}(\xi e^{j\omega t})$ for some $\omega \in \mathbb{R}$ and nonzero $\xi \in \mathbb{C}^q$. The network being LTI, the mapping $t \mapsto \mathrm{Re}(\xi e^{j\omega t})$ must itself be a solution. Hence, it might have been that $v(t) = \mathrm{Re}(\xi e^{j\omega t})$. We henceforth focus on this case. Observe that the solution $v(t) = \mathrm{Re}(\xi e^{j\omega t})$ is periodic; consequently, so is the corresponding $W(t) = \frac{1}{2}\omega_0^2 v(t)^T v(t) + \frac{1}{2}\dot{v}(t)^T \dot{v}(t)$. Recall that $W(t)$ is also nonincreasing; see Remark 2. Hence $W(t)$ must be constant, which yields $0 = \dot{W} = -\dot{e}^T G \dot{e}$ by (3). Since G is symmetric positive semidefinite, $\dot{e}^T G \dot{e} = 0$ implies $G\dot{e} = 0$. Under this condition the equation (6) simplifies into $AA^T(\ddot{e} + \omega_0^2 e) = 0$. Multiplying this from left with $(\ddot{e} + \omega_0^2 e)^T$ allows us to write $0 = (\ddot{e} + \omega_0^2 e)^T AA^T(\ddot{e} + \omega_0^2 e) = \|A^T(\ddot{e} + \omega_0^2 e)\|^2 = \|\ddot{v} + \omega_0^2 v\|^2$. That is, our choice $v(t) = \mathrm{Re}(\xi e^{j\omega t})$ satisfies $\ddot{v} + \omega_0^2 v = 0$. This at once brings $\omega = \omega_0$. Furthermore, since the network (6) is synchronous, we should have $v(t) = [v_1(t)\ v_2(t)\ \cdots\ v_q(t)]^T$ with $v_k(t) = a_k \mu \sin(\omega_0 t + \phi)$, where $\mu > 0$, $\phi \in [0, 2\pi)$, and $a_k \in \{-1, 1\}$. Without loss of generality we can let $\mu = 1$ and $\phi = 0$. Hence we have established the following. The synchronous network (6) admits a pair $(v(t), e(t))$ such that

$$v(t) = [a_1\ a_2\ \cdots\ a_q]^T \times \sin(\omega_0 t)\,, \qquad G\dot{e}(t) = 0 \tag{7}$$

where $a_k \in \{-1, 1\}$ for $k = 1, 2, \ldots, q$.

Using (7) we now show that the linkage $(\mathcal{V}, \mathcal{E}_o, \mathcal{E}_c)$ is bipartite. To this end, let us first establish $\mathcal{E}_o \cap \mathcal{E}_c = \emptyset$. Suppose otherwise. That is, there exists an edge $\{\nu_r, \nu_s\} \in \mathcal{E}_o \cap \mathcal{E}_c$. This means there are an oscillator (say the kth oscillator) and a resistor (with conductance $g_{rs} > 0$) in the network both extending between the nodes ν_r and ν_s. Then we either have $v_k = e_r - e_s$ or $v_k = e_s - e_r$. Without loss of generality let $v_k = e_r - e_s$. Since $G\dot{e} = 0$ we have $\dot{e}_r(t) - \dot{e}_s(t) \equiv 0$ by (3). This however contradicts

$$\dot{e}_r(t) - \dot{e}_s(t) = \dot{v}_k(t) = a_k \omega_0 \cos(\omega_0 t) \not\equiv 0\,.$$

The second thing we have to show is that no cycle of the graph $(\mathcal{V}, \mathcal{E}_o \cup \mathcal{E}_c)$ contains an odd number of edges from $\mathcal{E}_o$. Again suppose otherwise. Then there is a cycle with the set of edges $\mathcal{S} = \{\{\nu_{r_1}, \nu_{r_2}\}, \{\nu_{r_2}, \nu_{r_3}\}, \ldots, \{\nu_{r_{p-1}}, \nu_{r_p}\}, \{\nu_{r_p}, \nu_{r_1}\}\} \subset \mathcal{E}_o \cup \mathcal{E}_c$, comprising p edges, an odd number $2m + 1 \leq p$ of which belongs to $\mathcal{E}_o$. Without loss of generality let $\{\nu_{r_p}, \nu_{r_1}\} \in \mathcal{E}_o$. Consider the identity

$$(\dot{e}_{r_1} - \dot{e}_{r_p}) = (\dot{e}_{r_1} - \dot{e}_{r_2}) + (\dot{e}_{r_2} - \dot{e}_{r_3}) + \cdots + (\dot{e}_{r_{p-1}} - \dot{e}_{r_p})\,.$$

Let us remove from the righthand side the terms $(\dot{e}_{r_\ell} - \dot{e}_{r_{\ell+1}})$ for which $\{\nu_{r_\ell}, \nu_{r_{\ell+1}}\} \in \mathcal{E}_c$. We can do this because such terms satisfy $(\dot{e}_{r_\ell} - \dot{e}_{r_{\ell+1}}) = 0$ thanks to $G\dot{e} = 0$ and (3). Then the above equality simplifies into

$$(\dot{e}_{r_1} - \dot{e}_{r_p}) = \sum_{\{\nu_{r_\ell}, \nu_{r_{\ell+1}}\} \in \mathcal{S} \cap \mathcal{E}_o} (\dot{e}_{r_\ell} - \dot{e}_{r_{\ell+1}})\,. \tag{8}$$

Since each term in (8) corresponds to some edge in $\mathcal{E}_o$, we can find indices $k_0, k_1, \ldots, k_{2m} \in \{1, 2, \ldots, q\}$ and coefficients (determined by the polarities of the oscillator voltages) $c_0, c_1, \ldots, c_{2m} \in \{-1, 1\}$ that allow us to rewrite (8) as

$$c_{k_0}\dot{v}_{k_0} = c_{k_1}\dot{v}_{k_1} + c_{k_2}\dot{v}_{k_2} + \cdots + c_{k_{2m}}\dot{v}_{k_{2m}}\,. \tag{9}$$

Combining (9) and (7) then implies $a_{k_0}c_{k_0} = a_{k_1}c_{k_1} + a_{k_2}c_{k_2} + \cdots + a_{k_{2m}}c_{k_{2m}}$ which never admits a solution because the lefthand side is always odd, while the righthand side is even. The result hence follows by contradiction. ∎

Theorem 1 *The network* (6) *is synchronous if and only if the linkage* $(\mathcal{V}, \mathcal{E}_o, \mathcal{E}_c)$ *is bilayer and both its layers are connected.*

Proof. Let the network (6) be synchronous. Then, by Lemma 1 and Lemma 2, its linkage $(\mathcal{V},\,\mathcal{E}_{\rm o},\,\mathcal{E}_{\rm c})$ is bilayer. Let the graphs $(\mathcal{V}_1,\,\mathcal{E}_1)$ and $(\mathcal{V}_2,\,\mathcal{E}_2)$ be a pair of associated layers. Recall that the pair $(\mathcal{V}_1,\,\mathcal{V}_2)$ is then a bipartition of the graph $(\mathcal{V},\,\mathcal{E}_{\rm o})$ and we have $(\mathcal{V}_1,\,\mathcal{E}_1)\cup(\mathcal{V}_2,\,\mathcal{E}_2)=(\mathcal{V},\,\mathcal{E}_{\rm c})$. Let n_1 and n_2 denote the number of nodes in $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. Suppose now one of the layers, say $(\mathcal{V}_1,\,\mathcal{E}_1)$, is not connected. Then we can find disjoint subgraphs $(\mathcal{V}_1',\,\mathcal{E}_1')$ and $(\mathcal{V}_1'',\,\mathcal{E}_1'')$ satisfying $(\mathcal{V}_1',\,\mathcal{E}_1')\cup(\mathcal{V}_1'',\,\mathcal{E}_1'')=(\mathcal{V}_1,\,\mathcal{E}_1)$. Let n_1' and n_1'' denote the number of nodes in $\mathcal{V}_1'$ and $\mathcal{V}_1''$, respectively. Without loss of generality (see Remark 1) we can suppose that the Laplacian matrix G and the incidence matrix A of the network (6) have the following structures

$$G=\left[\begin{array}{ccc} G_1' & 0 & 0\\ 0 & G_1'' & 0\\ 0 & 0 & G_2\end{array}\right]\,,\qquad A=\left[\begin{array}{cc} F_1' & 0\\ 0 & F_1''\\ -F_2' & -F_2''\end{array}\right]$$

where $G_1'\in\mathbb{R}^{n_1'\times n_1'}$, $G_1''\in\mathbb{R}^{n_1''\times n_1''}$, and $G_2\in\mathbb{R}^{n_2\times n_2}$ are all separately Laplacian matrices; and $F_1'\in\mathbb{R}^{n_1'\times q_1}$, $F_1''\in\mathbb{R}^{n_1''\times q_2}$, and $[F_2'\ F_2'']=F_2\in\mathbb{R}^{n_2\times q}$ are class $\mathcal{F}$ matrices. Consider now the case where the first n_1' node voltages in the network read $\sin(\omega_0 t)$ while the remaining are fixed at zero. The corresponding node voltage vector is $e(t)=[\mathbf{1}_{n_1'}^T\ \ 0_{1\times n_1''}\ \ 0_{1\times n_2}]^T\times\sin(\omega_0 t)$. Note that

$$G\dot{e}(t)=\left[\begin{array}{ccc} G_1' & 0 & 0\\ 0 & G_1'' & 0\\ 0 & 0 & G_2\end{array}\right]\left[\begin{array}{c}\mathbf{1}_{n_1'}\\ 0\\ 0\end{array}\right]\omega_0\cos(\omega_0 t)=\left[\begin{array}{c} G_1'\mathbf{1}_{n_1'}\\ 0\\ 0\end{array}\right]\omega_0\cos(\omega_0 t)=0$$

because $G_1'\mathbf{1}_{n_1'}=0$. Note also that $\ddot{e}(t)+\omega_0^2 e(t)=0$. Therefore $e(t)$ satisfies (6), meaning the resulting oscillator voltage vector $A^Te(t)=v(t)=[v_1(t)\ v_2(t)\ \cdots\ v_q(t)]^T$ is a possible solution of the network. We can write

$$v(t)=A^Te(t)=\left[\begin{array}{ccc} F_1'^T & 0 & -F_2'^T\\ 0 & F_1''^T & -F_2''^T\end{array}\right]\left[\begin{array}{c}\mathbf{1}_{n_1'}\\ 0\\ 0\end{array}\right]\sin(\omega_0 t)=\left[\begin{array}{c} F_1'^T\mathbf{1}_{n_1'}\\ 0\end{array}\right]\sin(\omega_0 t)=\left[\begin{array}{c}\mathbf{1}_{q_1}\\ 0\end{array}\right]\sin(\omega_0 t)$$

from which we obtain

$$v_k(t)=\left\{\begin{array}{cl}\sin(\omega_0 t) & \text{for } k\in\{1,\,\ldots,\,q_1\}\,,\\ 0 & \text{for } k\in\{q_1+1,\,\ldots,\,q\}\,.\end{array}\right. \tag{10}$$

Clearly, (10) contradicts that the network (6) is synchronous. Hence both layers $(\mathcal{V}_1,\,\mathcal{E}_1)$ and $(\mathcal{V}_2,\,\mathcal{E}_2)$ must be connected.

Now we show the other direction. This time we start with assuming the linkage $(\mathcal{V},\,\mathcal{E}_{\rm o},\,\mathcal{E}_{\rm c})$ is bilayer and both its layers are connected. Therefore, without loss of generality (see Remark 1), we can let

$$G=\left[\begin{array}{cc} G_1 & 0\\ 0 & G_2\end{array}\right]\,,\qquad A=\left[\begin{array}{c} F_1\\ -F_2\end{array}\right]$$

where each of the Laplacian matrices $G_1\in\mathbb{R}^{n_1\times n_1}$ and $G_2\in\mathbb{R}^{n_2\times n_2}$ represents one of the layers and $F_1\in\mathbb{R}^{n_1\times q}$ and $F_2\in\mathbb{R}^{n_2\times q}$ are class-$\mathcal{F}$ matrices. Since the layers are connected the ranks of the matrices G_1 and G_2 are n_1-1 and n_2-1, respectively. In particular, we have $\operatorname{null} G_1=\operatorname{span}\{\mathbf{1}_{n_1}\}$ and $\operatorname{null} G_2=\operatorname{span}\{\mathbf{1}_{n_2}\}$. Suppose now the network (6) is not synchronous. This means (see the proof of Lemma 2) either of the following two cases must take place.

Case 1: There exists a solution of the form $v(t)=\operatorname{Re}(\xi e^{j\omega_0 t})$ *with nonzero* $\xi=[a_1\ a_2\ \cdots\ a_q]^T$ *satisfying* $a_k/a_\ell\notin\{-1,\,1\}$ *for some* $(k,\,\ell)$. Let $e(t)=\operatorname{Re}(\zeta e^{j\omega_0 t})$ be the corresponding node voltage vector, i.e., $A^Te(t)=v(t)$. Then let $\zeta_1\in\mathbb{C}^{n_1}$ and $\zeta_2\in\mathbb{C}^{n_2}$ satisfy $[\zeta_1^T\ \zeta_2^T]^T=\zeta$. Since $G\dot{e}(t)=0$ (see the proof of Lemma 2) we have to have $G_1\zeta_1=0$ and $G_2\zeta_2=0$. Also, $A^Te(t)=v(t)$ implies $F_1^T\zeta_1-F_2^T\zeta_2=\xi$. Observe that since $\zeta_1\in\operatorname{span}\{\mathbf{1}_{n_1}\}$ and F_1 is a class-$\mathcal{F}$ matrix, we have $F_1^T\zeta_1\in\operatorname{span}\{\mathbf{1}_q\}$. Similarly, $F_2^T\zeta_2\in\operatorname{span}\{\mathbf{1}_q\}$. As a result $\xi\in\operatorname{span}\{\mathbf{1}_q\}$. But this violates our initial assumption on ξ.

Case 2: All solutions satisfy $v(t)\to 0$. To show that this case, too, is impossible we construct a counterexample solution which does not vanish as $t\to\infty$. Let the first n_1 node voltages in the network read $\sin(\omega_0 t)$ while the remaining are fixed at zero. That is, $e(t) = [\mathbf{1}_{n_1}^T \ 0_{1\times n_2}]^T \times \sin(\omega_0 t)$. This allows us to write

$$G\dot{e}(t) = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} \mathbf{1}_{n_1} \\ 0 \end{bmatrix} \omega_0 \cos(\omega_0 t) = \begin{bmatrix} G_1 \mathbf{1}_{n_1} \\ 0 \end{bmatrix} \omega_0 \cos(\omega_0 t) = 0$$

because $G_1 \mathbf{1}_{n_1} = 0$. Note also that $\ddot{e}(t) + \omega_0^2 e(t) = 0$. Therefore $e(t)$ satisfies (6), meaning the resulting oscillator voltage vector $v(t) = A^T e(t)$ is a possible solution of the network. Since $F_1^T \mathbf{1}_{n_1} = \mathbf{1}_q$ we can proceed as

$$v(t) = A^T e(t) = [F_1^T \ \ -F_2^T] \begin{bmatrix} \mathbf{1}_{n_1} \\ 0 \end{bmatrix} \sin(\omega_0 t) = F_1^T \mathbf{1}_{n_1} \sin(\omega_0 t) = \mathbf{1}_q \sin(\omega_0 t)\,.$$

Clearly, $v(t) \not\to 0$. Both cases are now ruled out. Hence the network (6) has to be synchronous. ■

6 Generalized eigenvalues

In the previous section we have discovered it is necessary that the linkage of the network (6) is bilayer for synchronous behavior. In the remainder of the paper we will assume this condition for the general framework (2). Also, for simplicity of analysis, we will consider the type of networks where the subspace that contains the oscillator voltage vector $v = A^T e$ is the entire space, i.e., $\mathrm{range}(A^T) = \mathbb{R}^q$. To sum up, we henceforth make

Assumption 1 *The following hold.*

- *The linkage of the network* (2) *is bilayer.*
- $\mathrm{rank}(A) = q$.

Let $P, Q \in \mathbb{C}^{n\times n}$. Recall that a generalized eigenvalue $\lambda \in \mathbb{C}$ of the pair $(P,\, Q)$ satisfies $(P - \lambda Q)x = 0$ for some $x \neq 0$, where $x \in \mathbb{C}^n$. Observe that if $(P,\, Q)$ is a Laplacian pair, which implies $P\mathbf{1}_n = Q\mathbf{1}_n = 0$, then the corresponding set of eigenvalues according to this definition is the entire complex plane $\mathbb{C}$. To avoid this kind of outrage, we introduce a slightly modified version.

Definition 4 *Let $P, Q \in \mathbb{C}^{n\times n}$. A* restricted generalized eigenvalue *$\lambda \in \mathbb{C}$ of the pair $(P,\, Q)$ satisfies*

$$(P - \lambda Q)x = 0 \quad \textit{for some} \quad Qx \neq 0$$

where $x \in \mathbb{C}^n$. The set of all restricted generalized eigenvalues is denoted by $\mathrm{reig}(P,\, Q)$.

In this section we study the problem of computing $\mathrm{reig}(G + jB,\, AA^T)$ for the network (2). To this end, we propose a method to reduce this generalized eigenvalue problem to a standard eigenvalue problem. Then, in the next section, we uncover the link between these generalized eigenvalues and network synchronization.

Consider the network (2) under Assumption 1. Since the linkage of the network is bilayer we will henceforth let, without loss of generality (see Remark 1), the matrices $G,\, B \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times q}$ have the following structures

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \qquad A = \begin{bmatrix} F_1 \\ -F_2 \end{bmatrix} \tag{11}$$

where the submatrices $G_1,\, B_1 \in \mathbb{R}^{n_1\times n_1}$ and $G_2,\, B_2 \in \mathbb{R}^{n_2\times n_2}$ are all Laplacians and $F_1 \in \mathbb{R}^{n_1\times q}$ and $F_2 \in \mathbb{R}^{n_2\times q}$ are class $\mathcal{F}$ matrices. We now investigate the properties of the solution $(E,\, Y)$ of the following equation

$$\underbrace{\begin{bmatrix} G + jB & -A \\ A^T & 0 \end{bmatrix}}_{M} \begin{bmatrix} E \\ Y \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ I \end{bmatrix}}_{N} \tag{12}$$

where $E \in \mathbb{C}^{n\times q}$ and $Y \in \mathbb{C}^{q\times q}$.

Theorem 2 *The equation (12) admits a solution* $(E,\,Y)$ *with a unique* Y*. Moreover, the following hold.*

1. $Y = Y^T$.
2. *Each eigenvalue* λ *of* Y *satisfies* $\mathrm{Re}(\lambda) \geq 0$ *and* $\mathrm{Im}(\lambda) \geq 0$.
3. $Y\mathbf{1}_q = 0$
4. *If* $B = 0$ *then* Y *is real and* $Y \geq 0$.

Proof. *Existence.* A solution $(E,\,Y)$ exists for (12) if and only if $\mathrm{range}(M) \supset \mathrm{range}(N)$, which is equivalent to

$$\mathrm{null}(M^*) \subset \mathrm{null}(N^*)\,. \tag{13}$$

To establish (13) suppose otherwise, that is, $\mathrm{null}(M^*) \not\subset \mathrm{null}(N^*)$. This implies that we can find a nonzero vector η satisfying $M^*\eta = 0$ and $N^*\eta \neq 0$. Let us partition this vector as $\eta = [\eta_1^T\ \eta_2^T]^T$ where $\eta_1 \in \mathbb{C}^n$ and $\eta_2 \in \mathbb{C}^q$. Expanding $M^*\eta = 0$ gives us

$$\begin{aligned} (G - jB)\eta_1 + A\eta_2 &= 0 \\ -A^T\eta_1 &= 0\,. \end{aligned} \tag{14}$$

which allows us to write

$$\eta_1^* G\eta_1 - j\eta_1^* B\eta_1 = \eta_1^*(G - jB)\eta_1 = -\eta_1^* A\eta_2 = (-A^T\eta_1)^*\eta_2 = 0\,.$$

Since the Laplacians G and B are symmetric positive semidefinite we have $\eta_1^* G\eta_1 = 0$ and $\eta_1^* B\eta_1 = 0$, which implies $G\eta_1 = 0$ and $B\eta_1 = 0$. Revisiting (14) with this information lets us see $A\eta_2 = 0$, whence we infer $\eta_2 = 0$ because A is full column rank by Assumption 1. But this contradicts $N^*\eta \neq 0$ because $N^*\eta = \eta_2$.

Uniqueness & symmetry. Let the pairs $(E_1,\,Y_1)$ and $(E_2,\,Y_2)$ both satisfy (12). We can write

$$\begin{aligned} Y_1 &= [0\ \ I]\begin{bmatrix} -E_1 \\ Y_1 \end{bmatrix} \\ &= [E_2^T\ Y_2^T]\begin{bmatrix} G + jB & A \\ -A^T & 0 \end{bmatrix}\begin{bmatrix} -E_1 \\ Y_1 \end{bmatrix} \\ &= [E_2^T\ Y_2^T]\begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= Y_2^T\,. \end{aligned} \tag{15}$$

The choice $(E_1,\,Y_1) = (E_2,\,Y_2)$ gives us at once the symmetry $Y_1 = Y_1^T$. Then, thanks to this symmetry, (15) implies the uniqueness $Y_1 = Y_2$.

Eigenvalues. Let $\lambda \in \mathbb{C}$ be an eigenvalue of Y and $v \in \mathbb{C}^q$ be the corresponding unit eigenvector, i.e., $Yv = \lambda v$ and $v^*v = 1$. Multiplying both sides of (12) by v from right and letting $e = Ev$ we obtain

$$\begin{aligned} (G + jB)e - \lambda Av &= 0 \\ A^T e &= v\,. \end{aligned}$$

Using these identities we can write $\lambda = \lambda v^*v = \lambda(e^*A)v = e^*(\lambda Av) = e^*Ge + je^*Be$. Since G and B are symmetric positive semidefinite, it follows that $\mathrm{Re}(\lambda) \geq 0$ and $\mathrm{Im}(\lambda) \geq 0$.

Null space. Let $u \in \mathbb{R}^n$ be $u = [0_{1\times n_1}\ \mathbf{1}_{n_2}^T]^T$. Using (11) and the identities $G_2\mathbf{1}_{n_2} = 0$, $B_2\mathbf{1}_{n_2} = 0$, $F_2^T\mathbf{1}_{n_2} = \mathbf{1}_q$ we can write

$$\begin{bmatrix} 0 \\ \mathbf{1}_q \end{bmatrix} = \begin{bmatrix} 0_{n_1\times 1} \\ 0_{n_2\times 1} \\ \mathbf{1}_q \end{bmatrix} = \begin{bmatrix} G_1 + jB_1 & 0 & F_1 \\ 0 & G_2 + jB_2 & -F_2 \\ -F_1^T & F_2^T & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{1}_{n_2} \\ 0 \end{bmatrix} = \begin{bmatrix} G + jB & A \\ -A^T & 0 \end{bmatrix}\begin{bmatrix} u \\ 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned} Y\mathbf{1}_q &= [E^T\ Y]\begin{bmatrix} 0 \\ \mathbf{1}_q \end{bmatrix} \\ &= [E^T\ Y]\begin{bmatrix} G+jB & A \\ -A^T & 0 \end{bmatrix}\begin{bmatrix} u \\ 0 \end{bmatrix} \\ &= [0\ \ I]\begin{bmatrix} u \\ 0 \end{bmatrix} \\ &= 0\,. \end{aligned}$$

thanks to (12) and the symmetry of Y.

Positive semidefiniteness. Let $B=0$. Now that the matrix M is real, a real solution (E, Y) exists for (12). Then $Y \in \mathbb{R}^{q\times q}$ by uniqueness. To show that $Y \geq 0$ let us write (for real E)

$$\begin{aligned} Y &= [E^T\ Y^T]\begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= [E^T\ Y^T]\begin{bmatrix} G & -A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} E \\ Y \end{bmatrix} \\ &= E^T G E \end{aligned}$$

whence the positive semidefiniteness follows by $G \geq 0$. ■

Theorem 2 motivates the following definition.

Definition 5 *The* effective Laplacian *of the network* (2) *is defined as*

$$Y = [0\ \ I]\begin{bmatrix} G+jB & -A \\ A^T & 0 \end{bmatrix}^{+}\begin{bmatrix} 0 \\ I \end{bmatrix}$$

which satisfies the properties 1-4 listed in Theorem 2.

Remark 3 *When $n_1 = n_2$ and $F_1 = F_2 = I$, the effective Laplacian Y equals the* parallel sum *of the matrices $Y_1 = G_1 + jB_1$ and $Y_2 = G_2 + jB_2$, i.e., $Y = Y_1(Y_1+Y_2)^+Y_2$; see [3].*

Remark 4 *The solution $v(t)$ of the resistively coupled network* (6) *satisfies $\ddot{v} + \omega_0^2 v + Y\dot{v} = 0$.*

We end this section with the observation that the generalized eigenvalues of the pair $(G+jB,\, AA^T)$ coincide with the eigenvalues of the effective Laplacian of the network. This will allow us to work with Y instead of the pair $(G+jB,\, AA^T)$ in the next section.

Theorem 3 $\mathrm{reig}(G+jB,\, AA^T) = \mathrm{eig}(Y)$.

Proof. Let $\lambda \in \mathrm{eig}(Y)$ and $v \in \mathbb{C}^q$ be the corresponding eigenvector, i.e., $Yv = \lambda v$. Let $e = Ev$ where E satisfies (12). Now, multiplying both sides of (12) from right by v and using $Yv = \lambda v$ and $e = Ev$ we obtain

$$\begin{bmatrix} G+jB & -A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} e \\ \lambda v \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix}.$$

Hence we can write $(G+jB)e - \lambda Av = 0$ and $v = A^T e$. Substituting v in the first equation by $A^T e$ yields $(G+jB-\lambda AA^T)e = 0$. Being an eigenvector, $v \neq 0$. As a result $A^T e \neq 0$ which implies $AA^T e \neq 0$. This means $\lambda \in \mathrm{reig}(G+jB,\, AA^T)$. Therefore $\mathrm{reig}(G+jB,\, AA^T) \supset \mathrm{eig}(Y)$.

We now show the other direction. Let $\lambda \in \mathrm{reig}(G+jB,\, AA^T)$. Then we can find $e \in \mathbb{C}^n$ satisfying $(G+jB-\lambda AA^T)e = 0$ and $AA^T e \neq 0$. The latter implies $A^T e \neq 0$. Letting the nonzero vector $v = A^T e$ we can cast $(G+jB-\lambda AA^T)e = 0$ into

$$\begin{bmatrix} G+jB & A \\ -A^T & 0 \end{bmatrix}\begin{bmatrix} -e \\ \lambda v \end{bmatrix} = \begin{bmatrix} 0 \\ v \end{bmatrix}. \tag{16}$$

Using (16), (12), and the symmetry of Y we can write

$$\begin{aligned} Yv &= [E^T \ Y]\begin{bmatrix} 0 \\ v \end{bmatrix} \\ &= [E^T \ Y]\begin{bmatrix} G+jB & A \\ -A^T & 0 \end{bmatrix}\begin{bmatrix} -e \\ \lambda v \end{bmatrix} \\ &= \left(\begin{bmatrix} G+jB & -A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} E \\ Y \end{bmatrix}\right)^T \begin{bmatrix} -e \\ \lambda v \end{bmatrix} \\ &= [0 \ \ I]\begin{bmatrix} -e \\ \lambda v \end{bmatrix} \\ &= \lambda v\,. \end{aligned}$$

Therefore $\mathrm{reig}(G + jB,\, AA^T) \subset \mathrm{eig}(Y)$. The result then follows. ■

7 RL coupling

In Section 6 we considered an eigenvalue problem focusing on the generalized eigenvalues of the pair $(G + jB,\, AA^T)$ associated to the network (2) that has q oscillators. We introduced a $q \times q$ matrix Y called the effective Laplacian of the network and established that the above mentioned generalized eigenvalues coincide with the eigenvalues of Y. Our analysis there was a preparation necessary for our present investigation of the collective behavior of the coupled oscillators of the network (2). Now we will show that whether the oscillators asymptotically synchronize or not can be determined through the spectrum of the effective Laplacian. As before, here, too, we posit Assumption 1 holds and the matrices G, B, A satisfy (11).

Remark 5 *It is not difficult to see that* $e(t) = [\mathbf{1}_{n_1}^T \ 0_{n_2\times 1}]^T \times \sin(\omega_0 t)$ *and* $v(t) = \mathbf{1}_q \sin(\omega_0 t)$ *satisfy* (2).

Below is our main result.

Theorem 4 *The network* (2) *is synchronous if and only if the effective Laplacian* Y *has a single eigenvalue on the imaginary axis.*

We prove this theorem in two steps.

Lemma 3 *The network* (2) *is not synchronous if and only if there exist* $\omega \in \mathbb{R}$, $\bar{e} \in \mathbb{C}^n$, *and* $\bar{v} \in \mathbb{C}^q \setminus \mathrm{span}\{\mathbf{1}_q\}$ *satisfying*

$$((\omega_0^2 - \omega^2)AA^T + B)\bar{e} = 0 \qquad (17a)$$

$$G\bar{e} = 0 \qquad (17b)$$

$$A^T\bar{e} = \bar{v}\,. \qquad (17c)$$

Proof. Suppose the network (2) is not synchronous. Then, by Remark 2 and Remark 5, there must exist a solution $v(t) = \mathrm{Re}(\bar{v}e^{j\omega t})$ satisfying (2) with some $e(t) = \mathrm{Re}(\bar{e}e^{j\omega t})$, where $\omega \in \mathbb{R}$ and $\bar{v} \notin \mathrm{span}(\mathbf{1}_q)$. Substituting this particular pair $(v(t),\, e(t))$ into (2) yields

$$(\omega_0^2 - \omega^2)AA^T\bar{e} + j\omega G\bar{e} + B\bar{e} = 0 \qquad (18a)$$

$$\bar{v} = A^T\bar{e}\,. \qquad (18b)$$

Without loss of generality let $\bar{v}$ be a unit vector, i.e., $\bar{v}^*\bar{v} = 1$. Now, multiplying (18a) from left with $\bar{e}^*$ and rearranging the terms allow us to write

$$\omega^2 - \omega_0^2 = \bar{e}^*B\bar{e} + j\omega\bar{e}^*G\bar{e}\,. \qquad (19)$$

Since G and B are symmetric positive semidefinite, we have $\bar{e}^*B\bar{e} \geq 0$ and $\bar{e}^*G\bar{e} \geq 0$. Then (19) lets us see that $\omega \geq \omega_0$, $\bar{e}^*G\bar{e} = 0$, and consequently $G\bar{e} = 0$. Finally, combining $G\bar{e} = 0$ with (18) yields (17).

To show the other direction suppose (17) is satisfied by some choice of parameters ω, $\bar{e}$, and $\bar{v} \notin \mathrm{span}\{\mathbf{1}_q\}$. Construct the signals $v(t) = \mathrm{Re}(\bar{v}e^{j\omega t})$ and $e(t) = \mathrm{Re}(\bar{e}e^{j\omega t})$, which clearly satisfy (2). Hence $v(t) = \mathrm{Re}(\bar{v}e^{j\omega t})$ is a possible solution. Then (since the network is LTI) by Remark 5 the signal $\hat{v}(t) = \mathrm{Re}(\bar{v}e^{j\omega t}) + \mathbf{1}_q \sin(\omega_0 t)$ is also a possible solution, through which we see that the network (2) cannot be synchronous, because $\bar{v} \notin \mathrm{span}\{\mathbf{1}_q\}$. ■

Lemma 4 *There exist $\omega \in \mathbb{R}$, $\bar{e} \in \mathbb{C}^n$, and $\bar{v} \in \mathbb{C}^q \setminus \mathrm{span}\{\mathbf{1}_q\}$ satisfying* (17) *if and only if Y has two or more eigenvalues on the imaginary axis.*

Proof. By Theorem 2 we have $Y\mathbf{1}_q = 0$. Therefore $\lambda_1 = 0$ is an eigenvalue of Y with the eigenvector $\mathbf{1}_q$. Suppose now this eigenvalue at the origin is not the only eigenvalue on the imaginary axis. That is, there exists a second eigenvalue $\lambda_2 = j\mu$ with $\mu \in \mathbb{R}$. (We note that $\mu \geq 0$ by Theorem 2.) This implies there exists a unit eigenvector $\bar{v} \notin \mathrm{span}\,\{\mathbf{1}_q\}$ satisfying $Y\bar{v} = j\mu\bar{v}$. This is obvious if $\lambda_2 \neq 0$. To see that it is still true even if the eigenvalue at the origin is repeated (i.e., $\lambda_2 = 0$) suppose otherwise. That is, $\mathbf{1}_q$ is the only eigenvector for the repeated eigenvalue at the origin. This requires that there exists a generalized eigenvector w satisfying $Yw = \mathbf{1}_q$. But then the symmetry $Y = Y^T$ produces the contradiction $0 = (Y\mathbf{1}_q)^T w = \mathbf{1}_q^T(Yw) = \mathbf{1}_q^T\mathbf{1}_q = q$. Consider now (12), which is satisfied with some $E \in \mathbb{C}^{n\times q}$. Let $\bar{e} = E\bar{v}$. Multiplying both sides of (12) from right by $\bar{v}$ yields

$$\begin{bmatrix} G + jB & -A \\ A^T & 0 \end{bmatrix}\begin{bmatrix} \bar{e} \\ j\mu\bar{v} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{v} \end{bmatrix}$$

whence we extract

$$\begin{aligned} G\bar{e} + j(B\bar{e} - \mu A\bar{v}) &= 0 \qquad (20a) \\ A^T\bar{e} &= \bar{v} \qquad (20b) \end{aligned}$$

Using these identities and $\bar{v}^*\bar{v} = 1$ we can write

$$\begin{aligned} 0 &= \bar{e}^*G\bar{e} + j\bar{e}^*(B\bar{e} - \mu A\bar{v}) \\ &= \bar{e}^*G\bar{e} + j(\bar{e}^*B\bar{e} - \mu(A^T\bar{e})^*\bar{v}) \\ &= \bar{e}^*G\bar{e} + j(\bar{e}^*B\bar{e} - \mu)\,. \end{aligned}$$

Since G and B are symmetric positive semidefinite matrices, we have to have $\bar{e}^*G\bar{e} = 0$ yielding

$$G\bar{e} = 0 \qquad (21)$$

Using (21), (20), and letting $\omega = \sqrt{\omega_0^2 + \mu}$ we can write

$$((\omega_0^2 - \omega^2)AA^T + B)\bar{e} = 0 \qquad (22)$$

Combining (20b), (21), and (22) then yields (17).

Now we show the other direction. Suppose (17) holds for some $\omega \in \mathbb{R}$, $\bar{e} \in \mathbb{C}^n$, and $\bar{v} \in \mathbb{C}^q \setminus \mathrm{span}\,\{\mathbf{1}_q\}$. Defining the real number $\mu = \omega^2 - \omega_0^2$ we can mold (17) into

$$\begin{bmatrix} G + jB & A \\ -A^T & 0 \end{bmatrix}\begin{bmatrix} -\bar{e} \\ j\mu\bar{v} \end{bmatrix} = \begin{bmatrix} 0 \\ I \end{bmatrix}\bar{v}\,.$$

Choose some $E \in \mathbb{C}^{n\times q}$ satisfying (12). Using the symmetries of G, B, Y we can write

$$\begin{aligned} Y\bar{v} &= [E^T\ Y]\begin{bmatrix} 0 \\ I \end{bmatrix}\bar{v} \\ &= [E^T\ Y]\begin{bmatrix} G + jB & A \\ -A^T & 0 \end{bmatrix}\begin{bmatrix} -\bar{e} \\ j\mu\bar{v} \end{bmatrix} \\ &= [0\ \ I]\begin{bmatrix} -\bar{e} \\ j\mu\bar{v} \end{bmatrix} \\ &= j\mu\bar{v}\,. \end{aligned}$$

Recall $Y\mathbf{1}_q = 0$. Then $Y\bar{v} = j\mu\bar{v}$ implies Y has at least two eigenvalues on the imaginary axis because $\bar{v} \notin \mathrm{span}\,\{\mathbf{1}_q\}$ and μ is real. ■

Proof of Theorem 4. Combine Lemma 3 and Lemma 4. ■

8 An example

In this section we provide an illustration of Theorem 4, where an example network of four harmonic oscillators under bilayer RL coupling is studied.

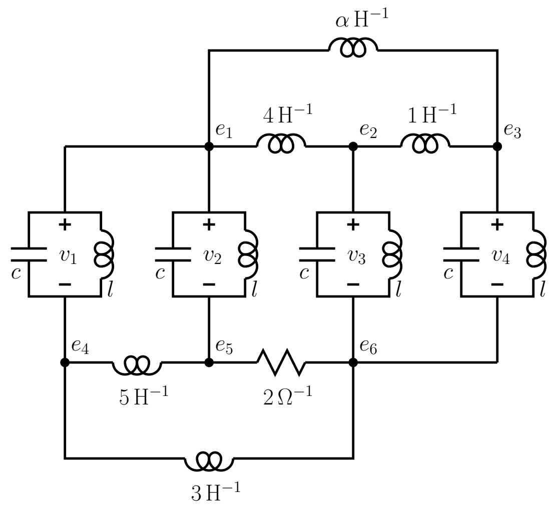

Figure 5: A network of identical LC tanks under bilayer RL coupling.

Consider the network of $q = 4$ coupled LC tanks shown in Fig. 5. The network has $n = 6$ nodes. For the labeling shown in the figure the matrices $G \in \mathbb{R}^{6\times 6}$, $B \in \mathbb{R}^{6\times 6}$, and $A \in \mathbb{R}^{6\times 4}$ enjoy the structures given in (11) where the submatrices read

$$G_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} \alpha+4 & -4 & -\alpha \\ -4 & 5 & -1 \\ -\alpha & -1 & \alpha+1 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$G_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 8 & -5 & -3 \\ -5 & 5 & 0 \\ -3 & 0 & 3 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

For the parameter choice $\alpha = 1$ the eigenvalues of the effective Laplacian $Y \in \mathbb{C}^{4\times 4}$ can be computed to be $\{\lambda_1 = 0,\ \lambda_2 = 0.5795 + j1.8886,\ \lambda_3 = 0.6283 + j4.1990,\ \lambda_4 = 1.4393 + j11.3242$. Since $\lambda_1 = 0$ is the only eigenvalue on the imaginary axis, by Theorem 4 we can say that the oscillators will asymptotically synchronize when $\alpha = 1$. For $\alpha = 4$, however, the eigenvalues read $\lambda_1 = 0$, $\lambda_2 = j6$, $\lambda_3 = 1.1989 + j11.3818$, $\lambda_4 = 1.3931 + j2.3622$. This time there are two eigenvalues on the imaginary axis, namely, $\lambda_1 = 0$ and $\lambda_2 = j6$. Therefore the oscillators are not guaranteed to synchronize for this case. This example tells us that synchronization cannot be determined merely by the structure of the coupling. In other words, without the actual parameter values, knowing only which oscillator is connected to which and by what type of connector is in general not sufficient to make definite conclusions about the collective behavior of the oscillators.

9 Conclusion

In this paper we studied networks of coupled LC tanks with nonstar oscillator graph. We first considered the special case, where the coupling is purely resistive. For such networks we showed that the interconnection has to be bilayer in order for the oscillator voltages to asymptotically synchronize. Then we moved on to analyze the general case (where both resistive coupling and inductive coupling are simultaneously active) under bilayer coupling structure. The layered architecture generates six real matrices (three for each layer) which do not readily tell whether the oscillators achieve synchronization or not. To overcome this difficulty we proposed a method to construct a single complex matrix (called the effective Laplacian) out of those six matrices and presented a simple test to study synchronization. The test is this. The oscillators synchronize if and only if the effective Laplacian has a single eigenvalue on the imaginary axis.

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