CLASSIFICATION OF CERTAIN SIX MANIFOLDS

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ABSTRACT

CLASSIFICATION OF CERTAIN SIX MANIFOLDS

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Classification of manifolds is a central question in the field of topology. A complete classification for orientable manifolds of dimension greater than or equal to 4 is not possible because of surgery theoretical reasons: any finitely generated group can be realized as the fundamental group of such a manifold and classification of finitely generated groups is theoretically impossible. As a first invariant, one usually fixes the fundamental group.

In this thesis, we are interested in classifying certain 6-dimensional orientable manifolds. Namely, we study the problem of classifying 6-manifolds with fundamental group isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \) and trivial second homotopy group. Following the work of Kreck and Lueck, we show that such manifolds are stably diffeomorphic if and only if they are bordant.

Keywords: surgery, homotopy groups, 6-manifolds, stable diffeomorphism, bordism
ÖZ

BELİRLİ ALTI MANİFOLDLARIN SINIFLANDIRILMASI

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Manifoldların sınıflandırılması topoloji alanındaki merkezi bir sorudur. Ameliyat teorisine bağlı sebeplerden ötürü 4 ve üstü boyutlu yönlendirilebilir manifoldların tam bir sınıflandırması mümkün değildir: her sonlu sayıda elemanla üretilmiş grup böyle bir manifoldın temel grubu olarak görülebilir ve sonlu sayıda elemanla üretilmiş grupların sınıflandırılması teorik olarak imkansızdır.

Bu tezde, belli başlı 6 boyutlu yönlendirilebilir manifoldların sınıflandırılması ile ilgileniyoruz. Spesifik olarak temel grubu \( \mathbb{Z} \oplus \mathbb{Z} \) ye izomorfik olan ve ikinci homotopi grubu açıkgrup olan 6-manifoldları inceliyoruz. Kreck ve Lueck’in çalışmalarını takip ederek bu tarz manifoldların ancak ve ancak bordant olduklarında stabil difféomorfik olduklarını gösteriyoruz.

Anahtar Kelimeler: ameliyat, homotopi grupları, 6-manifoldlar, stabil difféomorfizm, bordizm
To my family
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CHAPTER 1

INTRODUCTION

Classification of manifolds is a central question in the study of topology. Ever since John Milnor’s discovery in 1956 (see [13]) of a 7-dimensional manifold which is homeomorphic to $S^7$ but not diffeomorphic to it, mathematicians have been trying to come up with invariants and techniques that tell us which smooth manifolds are diffeomorphic and which are not. Although homeomorphism and diffeomorphism are interchangeable for manifolds of dimension $\leq 3$, for dimensions $\geq 4$ there is a wild world out there. Classifying manifolds up to diffeomorphism is not an easy task which would require constructing a complete set of invariants with properties such that:

(i) the invariants that we attach to a manifold can be computed,

(ii) two manifolds attain the same invariants if and only if they are diffeomorphic,

(iii) every possible set of invariants is realized through a list of non-diffeomorphic manifolds.

Unfortunately a complete classification of manifolds of dimension $\geq 4$ up to diffeomorphism is not possible since any group can be seen as the fundamental group of a manifold of dimension $\geq 4$ and it is impossible to classify groups based on their representation up to isomorphism which is known as the word problem in algebra. However through relatively new developments such as surgery, handle attaching and cobordism there have been significant advancement on the classification of manifolds of dimension $\geq 4$. To see the difficulty of classifying manifolds of high dimensions we divert our attention to closed manifolds. For closed manifolds we only have $S^1$
In dimension 1, all closed manifolds are determined by their orientability and their genus. In dimension 3, the classification problem has recently been resolved by which is now known as Poincare Theorem. In dimensions $\geq 4$ a classification up to diffeomorphism is not possible due to the word problem as we said so as our first invariant we should fix the fundamental group.

Even with a fixed fundamental group it is still hard to talk about diffeomorphism so instead we look at manifolds that are stably diffeomorphic to one another. Stable diffeomorphism means that two $(2n)$-dimensional manifolds $M$ and $N$ become diffeomorphic after taking connected sum with $S^n \times S^n$ i.e.,

$$M \# a(S^n \times S^n) \cong N \# b(S^n \times S^n)$$

for $a, b \geq 0$.

In 1964 Wall [18] introduced the notion of stable diffeomorphism which is easier to work with compared to diffeomorphism on many occasions and stable diffeomorphism is what we are focusing on in this thesis.

In this thesis, we deal with closed connected orientable manifolds of dimension 6 with fundamental group $\mathbb{Z} \oplus \mathbb{Z}$. We also want the second homotopy group of our manifolds to be trivial. For 6 dimensional manifolds we will see that information about the homotopy groups up to middle dimensional is enough for our classification. In fact we do not even need the third homotopy group. Let us first try to clarify the importance of dimension 6 for such manifolds.

1.0.1 Dimension $\leq 4$

In dimension 1, all closed connected orientable manifolds are diffeomorphic to $S^1$ so there is nothing interesting to see. In dimension 2 all manifolds of the type we are looking for are diffeomorphic to $T^2 = S^1 \times S^1$. In dimension 3, there is no manifold with fundamental group $\mathbb{Z} \oplus \mathbb{Z}$. To see this let $f: M \to T^2$ be the 3-connected classifying map (see Definition 19) of the universal covering of such a manifold which actually comes from the Postnikov tower for $M$ where $T^2 = K(\pi_1(M), 1)$ and $f$ is a fibration as explained in the next section i.e.,

$$f: M \to T^2$$
satisfy the homotopy lifting property for every topological space $X$. In other words for every homotopy

$$H : X \times [0, 1] \rightarrow T^2$$

and for every lift

$$\tilde{H}_0 : X \rightarrow M$$

lifting $H|_{X \times \{0\}} = H_0$ there exists a not necessarily unique homotopy

$$\tilde{H} : X \times [0, 1] \rightarrow M$$

lifting $H$ with $\tilde{H}_0 = \tilde{H}|_{X \times \{0\}}$. This can be seen more clearly in the following commutative diagram:

\[
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\tilde{H}_0} & M \\
\downarrow & & \downarrow f \\
X \times [0, 1] & \xrightarrow{H} & T^2
\end{array}
\]

Here 3-connected means the induced map between $n$-th homotopy groups of $M$ and $T^2$ is an isomorphism for $n = 1, 2$ and an epimorphism for $n = 3$. By Hurewicz Theorem

$$H_p(f) : H_p(M) \rightarrow H_p(T^2)$$

is an isomorphism for $p \leq 2$. Then by Universal Coefficient Theorem (UCT)

$$H^p(f) : H^p(T^2) \rightarrow H^p(M)$$

is also an isomorphism. Poincare Duality (PD) implies that $H^1(M) \cong H_2(M)$ which is a contradiction since $H_2(T^2) \cong \mathbb{Z}$ while $H_1(T^2) \cong \mathbb{Z}^2$.

In dimension 4, again we have no manifold of the type we are looking for by the following argument. Like in the case for 3-manifolds we conclude that $H_2(M) \cong \mathbb{Z}$ and by Universal Coefficient Theorem $H^2(M) \cong \mathbb{Z}$ which implies $\chi(M) = 1$. Now we note that finite coverings of $M$ are also manifolds of the type we are looking for. The fundamental group of a finite covering is a subgroup of finite index in $\mathbb{Z} \oplus \mathbb{Z}$ and the only such example is $\mathbb{Z} \oplus \mathbb{Z}$ itself. The higher homotopy groups of a covering remain the same. Now we consider a subgroup of index $k > 1$ in $\pi_1(M)$ and let $N$ be the corresponding covering. Then $\chi(N) = k \cdot \chi(M)$ but since $N$ is also a manifold of the type we are looking for $\chi(N) = 1$ which is a contradiction.
1.0.2 Dimension 5

In dimension 5, we have at least two manifolds with the specific homotopy groups we want, namely $T^2 \times S^3$ and $T^2 \times S^3$, the twisted sphere bundle over $T^2$. They are not homotopy equivalent and by using techniques from homotopy theory it is possible to show that there are exactly two homotopy types of manifolds under consideration, namely those given by $T^2 \times S^3$ and $T^2 \times S^3$. Since we look for a stricter classification the next logical question is the determination of the homeomorphism and diffeomorphism type of these manifolds. Both of these are determined by the first Pontryagin class and in our case both $T^2 \times S^3$ and $T^2 \times S^3$ have trivial Pontryagin classes. Hence in dimension 5 homotopy types of our manifolds actually give their diffeomorphism and homeomorphism types. For more information on this topic, the reader should refer to [11].

1.0.3 Dimension 6

In dimension 6, one obvious example of such a manifold described as above is $T^2 \times S^4$. Other such manifolds can be constructed through the process of surgery (see Definition [22]). For this purpose, we consider $M' = T^2 \times X$ where $X$ is a simply connected smooth 4-manifold with trivial second Stiefel–Whitney class i.e., $w_2(\tau_X) = 0$. Then for a basis of $\pi_2(T^2 \times X) \cong \pi_2(X)$, we choose disjoint embeddings of $S^2$ into $T^2 \times X$, one for each basis element. By Whitney Embedding Theorem (see Definition [2.1.12]) we can choose these maps as disjoint smooth embeddings. The normal bundle of these embeddings is trivial since $w_2(T^2 \times X) = 0$ so we use a tubular neighbourhood (see Definition [2.1.13]) to construct the desired embeddings:

$$S^2 \times D^4 \to T^2 \times X.$$ 

We can now form a new manifold by deleting the interiors of these embeddings and gluing in $D^3 \times S^3$ to each deleted component. The resulting manifold is denoted by $M$. Since the embedded $S^1$’s are not touched during this process $\pi_1(M)$ is still $\mathbb{Z} \oplus \mathbb{Z}$. Also note that after enough surgeries on the generators of $\pi_2(T^2 \times X)$, we have $\pi_2(M) = 0$ and $w_2(\tau_M) = 0$ for our new manifold $M$. 

4
For construction of non-spin examples, one can consider a suitable non-spin thickening of \( T^2 = K(\mathbb{Z} \oplus \mathbb{Z}, 1) \) as in [5, Lemma 3.3] (see also [19]): We start with a single \( D^6 \). Then we take two 1-handles and attach them to \( D^6 \) to get the boundary connected sum, \( (S^1 \times D^5)\#(S^1 \times D^5) \). Next, we attach a 2-handle \( D^2 \times D^4 \) such that \( D^2 \) is attached on top of the loop represented by \( aba^{-1}b^{-1} \) where \( a \) and \( b \) are the generators of the fundamental group of

\[
\partial \left( (S^1 \times D^5)\#(S^1 \times D^5) \right) = (S^1 \times S^4) \#(S^1 \times S^4)
\]

which is \( \mathbb{Z} \ast \mathbb{Z} \). This attaching is possible since by the Whitney Embedding Theorem we can embed \( S^1 \) into \( (S^1 \times S^4) \#(S^1 \times S^4) \) and by the Tubular Neighbourhood Theorem we can extend this embedding into an embedding of the total space of the normal bundle \( S^1 \times D^4 \). Hence the resulting manifold has the fundamental group \( \mathbb{Z} \oplus \mathbb{Z} \). This manifold deformation retracts onto \( T^2 \) and it is called a 6-dimensional thickening of \( T^2 \). We can also attach more 2-handles nullhomotopically (without affecting the fundamental group). Now we attach 3-handles to kill the elements of the second homotopy group. This is possible because the manifold we constructed is spin so the normal bundle over \( S^2 \) embedded into \( (S^1 \times S^4) \#(S^1 \times S^4) \) is trivial. Let this manifold be denoted by \( W(\mathcal{P}) \) where

\[
\mathcal{P} = \{a, b : aba^{-1}b^{-1}\}
\]

is the finite presentation of the group \( \mathbb{Z} \oplus \mathbb{Z} \). The framing on our 2-handle can be modified in the following way which in turn makes our manifold non-spin:

Take the non-trivial \( w \in H^2(W(\mathcal{P}); \mathbb{Z}_2) \cong H^2(T^2; \mathbb{Z}_2) \). A cocycle representative for the class \( w \) gives a function on the 2-cells to \( \mathbb{Z}/2 \), which can be used to vary the given framings by an element of \( \pi_1(SO(4)) \cong \mathbb{Z}_2 \) for each 2-handle. For every point \( x \in S^1 \) we have a \( D^4 \) sitting on top of it. We get a different isomorphism \( f|_{\{x\} \times D^4} : D^4 \to D^4 \) for every point in \( S^1 \) when we restrict our framing to \( \{x\} \times D^4 \).

\[
\begin{array}{ccc}
D^4 & \xrightarrow{f|_{\{x\} \times D^4}} & D^4 \\
\downarrow & & \downarrow \\
x & & x
\end{array}
\]

Recall that \( \pi_1(SO(4)) \cong \mathbb{Z}_2 \) which means there are 2 distinct homotopy classes of maps \( S^1 \to SO(4) \). Take a non-contractible map \( \alpha : S^1 \to SO(4) \) corresponding
to the action of $w$. We have an isomorphism from $D^4$ to $D^4$ in the form of $\alpha(x) \in \pi_1(SO(4)) \cong \mathbb{Z}_2$. The composition

$$\alpha(x) \circ f|_{\{x\} \times D^4} : D^4 \to D^4$$

gives us a new framing when enlarged to the whole $S^1 \times D^4$.

$$E(\nu_{S^1}) \xrightarrow{f} S^1 \times D^4 \xrightarrow{id \times \alpha(x)} S^1 \times D^4$$

This new framing makes our manifold non-spin which we now call $M$.

For 6-manifolds $M$ under consideration like the one we constructed above, we have the following invariant: The second cohomology is isomorphic to $\mathbb{Z}$, in order to see this note that we can see $T^2$ as an embedding in $M$ by the Whitney Embedding Theorem and $\pi_2(M, T^2) = 0$ since $\pi_2(M) = \pi_2(T^2) = 0$. Likewise $\pi_1(M, T^2) = 0$ since $\pi_1(M) = \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$. So by the Relative Hurewicz Theorem

$$H_2(M, T^2; \mathbb{Z}) = 0$$

so

$$H_2(M; \mathbb{Z}) \cong H_2(T^2; \mathbb{Z}) \cong \mathbb{Z}.$$  

By the Universal Coefficient Theorem

$$H^2(M, \mathbb{Z}) \cong \text{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(M; \mathbb{Z}), \mathbb{Z})$$

and we know that $H_1(M; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ so $\text{Ext}(H_1(M; \mathbb{Z}), \mathbb{Z}) = 0$ hence

$$H^2(M, \mathbb{Z}) \cong \text{Hom}(H_2(M; \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}.$$  

Thus we can choose a generator $x \in H^2(M; \mathbb{Z})$. This generator is well defined up to sign. Let $[M]$ be the fundamental class in $H_6(M; \mathbb{Z}) \cong \mathbb{Z}$, i.e., its generator. Taking the cup product of $x$ with the Pontryagin class $p_1(\tau_M) \in H^4(M; \mathbb{Z}) \cong \mathbb{Z}$ and evaluating on $[M]$ gives us our invariant:

$$\pm \langle x \smile p_1(\tau_M), [M] \rangle \in \mathbb{Z}$$

which is unique up to a sign $\pm$. This invariant agrees with the Pontryagin class of the manifold $X$, which we used in the construction of $M$, evaluated at $[X]$ up to sign:

$$\pm \langle x \smile p_1(\tau_M), [M] \rangle = \pm \langle p_1(\tau_X), [X] \rangle.$$
This invariant is important because it does not change if we take the connected sum of $M$ with $S^3 \times S^3$ which means it is an invariant of the stable diffeomorphism type. Two closed manifolds $M_1$ and $M_2$ of dimension $2k$ are said to be stably diffeomorphic, if there exist integers $a$ and $b$, such that $M_1 \# a(S^k \times S^k)$ is diffeomorphic to $M_2 \# b(S^k \times S^k)$. The next theorem tells us how our new invariant is crucial to classifying manifolds of the type we are looking for up to stable diffeomorphism:

**Theorem 1.0.1.** ([10] Stable Classification of Certain Six-Dimensional Manifolds) Two smooth 6-dimensional closed orientable manifolds $M_1$ and $M_2$ with $\pi_1(M_1) \cong \pi_1(M_2) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $\pi_2(M_1) = \pi_2(M_2) = 0$ are stably diffeomorphic if and only if

(i) in both cases $w_2$ vanishes or it does not vanish,

(ii) $\pm \langle x_1 \smile p_1(\tau_{M_1}), [M_1] \rangle = \pm \langle x_2 \smile p_1(\tau_{M_2}), [M_2] \rangle$.

Throughout this thesis our main objective is providing necessary background and details for the proof of this theorem.

1.0.4 The Outline Of The Thesis

We first define fibrations and in particular vector bundles over manifolds which allow us to talk about Postnikov Towers. Then we go over certain spectral sequences that later help us to determine specific cobordism groups. After that we briefly overview Stiefel-Whitney classes which appear as obstructions to manifolds admitting certain structures on them. Then we study Grassmann manifolds which are closely related to how we classify vector bundles and all this lead us into surgery and cobordism which give us the final piece of background to tackle our main theorem regarding the stable diffeomorphism between certain closed, oriented 6-dimensional manifolds.
CHAPTER 2

BACKGROUND

2.1 Basic Definitions

In this thesis, our primary object of study is closed connected orientable 6-manifolds with \( \pi_1 \cong \mathbb{Z} \oplus \mathbb{Z} \) and trivial \( \pi_2 \). In the next section we show that such manifolds exist but in order to do that we first need to establish some well known and fundamental results in topology. Let us start with vector bundles.

2.1.1 Vector Bundles

Roughly speaking vector bundles are special type of fiber bundles where each fiber is endowed with a vector space structure and the structure group is the orthogonal group. Under some mild assumptions on the topological spaces we are working with we can consider fibrations as examples of fiber bundles. For more details on this subject we refer the reader to [14]. Let us start by defining fibrations and from there we work our way up to vector bundles and their properties.

**Definition 1.** [1] A fibration is a map between topological spaces \( f : X \to B \) such that for every topological space \( Y \) we have the following commutative diagram

\[
\begin{array}{ccc}
Y \times \{0\} & \xrightarrow{\bar{R}_0} & X \\
\downarrow & & \downarrow f \\
Y \times [0,1] & \xrightarrow{H} & B \\
\end{array}
\]

where we can lift the homotopy \( H : Y \times [0,1] \to B \) to a not necessarily unique homotopy \( \bar{H} : Y \times [0,1] \to X \). If for every \( Y \) there exists such a homotopy \( \bar{H} \) then it
is said that the fibration \( f : X \to B \) satisfies the homotopy lifting property for every \( Y \). The space \( f^{-1}(b) \) is called the fiber over \( b \in B \).

Although we do not need it for this thesis, for the sake of completeness let us define cofibrations which can be seen as the dual of fibrations.

**Definition 2.** \([1]\)** A cofibration is a map between topological spaces \( i : A \to X \) such that for any topological space \( Y \) and any map \( f : A \to Y \) which can be extended to \( f' : X \to Y \) it is possible to extend a homotopy of maps \( H : A \times I \to Y \) to a homotopy of maps \( H' : X \times I \to Y \) such that \( H(a, 0) = f(a) \) and \( H'(x, 0) = f'(x) \).

All this can be more easily seen in the following commutative diagram.

\[
\begin{array}{ccc}
A & \xrightarrow{H} & Y^I \\
\downarrow{i} & & \downarrow{q_0} \\
X & \xrightarrow{f'} & Y
\end{array}
\]

where \( Y^I = \text{Hom}_{\text{Top}}(I, Y) \) is the path space of \( Y \) i.e., the space of paths in \( Y \), \( H' : X \to Y^I \) is the extension of \( H : A \to Y^I \) and \( q_0(\sigma) = \sigma(0) \) for every \( \sigma \in Y^I \). If for every \( Y \) there exists such a homotopy \( H' \) then it is said that the fibration \( i : A \to X \) satisfies the homotopy extension property for every \( Y \). The space \( X / i(A) \) is called the fiber of \( i \).

If we take \( B \) as a paracompact space in the definition of a fibration then our fibration is also a fiber bundle which differs from a vector bundle as with fiber bundles there is no need of fibers having a vector space structure. So let us directly define vector bundles instead.

**Definition 3.** \([14]\)** Let \( B \) be a fixed topological space which is called the base space. A real \( k \)-dimensional vector bundle (or \( k \)-plane bundle) \( \xi \) over \( B \) (denoted by \((B, \xi)\)) consists of the following:

(i) a topological space \( E = E(\xi) \) called the total space

(ii) a (continuous) map \( p : E \to B \) called the projection map, and

(iii) for each \( b \in B \), the fiber \( p^{-1}(b) \) over \( b \), is a real \( k \)-dimensional vector space.
Additionally, a vector bundle must satisfy the local triviality condition: For each point \( b \in B \) there should exist a neighbourhood \( U \subset B \) and a homeomorphism

\[ h: U \times \mathbb{R}^k \to p^{-1}(U) \]

so that, for each \( b \in U \), the correspondence \( x \mapsto h(b, x) \) defines an isomorphism between the vector space \( \mathbb{R}^k \) and the vector space \( p^{-1}(b) \).

We call such a pair \((U, h)\) a local coordinate system for \( \xi \) about \( b \). If \( U \) can be taken as the entire total space, then the \( k \)-plane bundle is said to be trivial and it is isomorphic to the \( k \)-plane bundle \( e^k \) which we define later. The vector space \( p^{-1}(b) \) is called the fiber over \( b \), denoted by \( F_b \) or \( F_b(\xi) \). For topological spaces there is the similar notion of charts. A chart for a topological space \( X \) is a homeomorphism \( \varphi \) from an open subset \( U \) of \( X \) to an open subset of a Euclidean space. It is also denoted by \((U, \varphi)\).

An indexed family \( \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I} \) of charts is called an atlas for \( X \), denoted by \( \mathfrak{U} \), if it covers \( X \) i.e., \( \bigcup_{\alpha \in I} U_\alpha = X \).

For a smooth vector bundle, we require that \( B \) and \( E \) be smooth manifolds, that \( p \) be a smooth map, and that, for each \( b \in B \) there exist a local coordinate system \((U, h)\) with \( b \in U \) such that \( h \) is a diffeomorphism. In order to understand the topology of vector bundles, we need to look at the topology of the general linear and orthogonal groups:

**Definition 4.** [15]  

(i) The \( k \)-dimensional general linear group

\[ \text{GL}(k) = \text{Aut}_{\mathbb{R}}(\mathbb{R}^k) \]

is the automorphism group of the standard \( k \)-dimensional real vector space \( \mathbb{R}^k \), consisting of invertible \( k \times k \) real matrices.

(ii) The \( k \)-dimensional orthogonal group

\[ \text{O}(k) = \text{Aut}_{\mathbb{R}}(\mathbb{R}^k, 1) \]

is the automorphism group of the standard \( k \)-dimensional symmetric bilinear form \((\mathbb{R}^1, 1)\), consisting of orthogonal \( k \times k \) real matrices.
We give \( O(k) \subset \text{GL}(k) \subset M_{k,k}(\mathbb{R}) \) the subspace topology where the topology on \( M_{k,k}(\mathbb{R}) \) is given by the standard topology on \( k^2 \)-dimensional Euclidean space since the set of all \( k \times k \) matrices can be identified with the \( k^2 \)-dimensional Euclidean space \( M_{k,k}(\mathbb{R}) = \mathbb{R}^{k^2} \).

**Proposition 2.1.1.** [15] With the topology mentioned above

(i) the \( k \)-dimensional general linear group \( \text{GL}(k) \) is an open manifold of dimension \( k^2 \),

(ii) the \( k \)-dimensional orthogonal group \( O(k) \) is a compact manifold of dimension \( \frac{1}{2}k(k-1) \),

(iii) the inclusion \( O(k) \hookrightarrow \text{GL}(k) \) is a homotopy equivalence with the homotopy inverse \( \text{GL}(k) \to O(k) \) given by the Gram-Schmidt orthonormalisation process.

We can see how the \( k \)-dimensional orthogonal group \( O(k) \) relates to vector bundles in the following: Let \( (U, h), (U', h') \) be two local coordinate systems for a \( k \)-plane bundle \( \xi \) with non-empty intersection. The transition functions

\[
h'^{-1}h : (U \cap U') \times \mathbb{R}^k \to p^{-1}(U \cap U') \to (U \cap U') \times \mathbb{R}^k
\]

are of the form \( (x, v) \mapsto (x, l(x)(v)) \) for some continuous function

\[
l = l_{U'}^U : U \cap U' \to \text{GL}(k)
\]
satisfying the usual compatibility conditions

\[
l_{U'}^{U''}(x) = l_{U'}^{U''}(x)l_{U'}^{U'}(x) : \mathbb{R}^k \to \mathbb{R}^k \quad (x \in U \cap U' \cap U'').
\]

By Proposition 2.1.1 (iii) we can always deform the transition functions of a vector bundle to be of the form

\[
(U \cap U') \times \mathbb{R}^k \to (U \cap U') \times \mathbb{R}^k ; (x, v) \mapsto (x, l(x)(v))
\]

with \( l : U \cap U' \to O(k) \). In this thesis we are interested in bundles of this type. Since the standard inner products on the fibres are preserved by the transition functions, each vector \( v \in E(\xi) \) has a length \( ||v|| \geq 0 \).
Since each fiber of a vector bundle has a vector space structure on it, we should be able to talk about orientation. More generally:

**Definition 5.** Let $V$ be a $k$-dimensional real vector space.

(i) An orientation for $V$ is an equivalence class of (ordered) bases such that if two bases differ by an element of $\text{GL}^+(k)$, which is the group of invertible matrices with determinant greater than 0, then they are equivalent. There are $[\text{GL}(k) : \text{GL}^+(k)] = 2$ possible orientations for $V$.

(ii) A $k$-plane bundle $\xi$ is orientable if the transition functions $l: U \cap U' \to \text{GL}(k)$ are orientation-preserving i.e., $l(U \cap U') \subseteq \text{GL}^+(k)$.

(iii) A $k$-plane bundle is nonorientable if it is not orientable.

(iv) An orientation for an orientable $k$-plane bundle $\xi$ is a compatible choice of orientation for each of the $k$-dimensional vector spaces (fibres) $F_b(\xi)$ where $b$ is an element of the base space $B$.

(v) The $k$-dimensional special orthogonal group

$$\text{SO}(k) = \{a \in \text{O}(k) | \det(a) = 1\}$$

is the index 2 subgroup of $\text{O}(k)$ consisting of the orientation-preserving elements. The inclusion $\text{SO}(k) \hookrightarrow \text{GL}^+(k)$ is a homotopy equivalence with the homotopy inverse given by Gram-Schmidt orthonormalisation process.

There are structures associated with vector bundles other than orientation which help their classification, namely:

**Definition 6.** Let $B$ be a base space.

(i) A $k$-plane bundle over a space $B$ is trivial if it is isomorphic to the bundle $\epsilon^k$ with projection

$$p: E(\epsilon^k) = B \times \mathbb{R}^k \to B; \ (b, y) \mapsto b.$$  

(ii) A framing (or trivialisation) of a $k$-plane bundle $\eta$ is an isomorphism to the trivial $k$-plane bundle

$$b: \eta \cong \epsilon^k.$$
(iii) Two framings $b_0, b_1$ of a $k$-plane bundle $\eta$ over $B$ are isomorphic if there exists a continuous family of framings

$$b_t : \eta \cong e^k \quad (0 \leq t \leq 1)$$

or equivalently if $b_0, b_1$ extend to a framing of the $k$-plane bundle $\eta \times I$ over $B \times I$ with total space $E(\eta \times I) = E(\eta) \times I$.

(iv) A section of a $k$-plane bundle $\eta$ over a space $B$ with projection $p: E(\eta) \rightarrow B$ is a map $s: B \rightarrow E(\eta)$ such that

$$ps = 1: B \xrightarrow{s} E(\eta) \xrightarrow{p} B$$

Hence for each $b \in B$ there is given a continuous choice of element $s(b) \in F_b(\eta)$.

(v) A section $s$ of $\eta$ is non-zero if $s(b) \neq 0(b) \in F_b(\eta)$ for every $b \in B$.

(vi) The zero section of $\eta$ is the section

$$z: B \rightarrow E(\eta); \; b \mapsto 0(b).$$

**Definition 7.** [14] Given two vector bundles $\xi$ and $\eta$ over the same base space $B$ with $E(\xi) \subset E(\eta)$. We say that $\xi$ is a sub-bundle of $\eta$ (written as $\xi \subset \eta$) if each fiber $F_b(\xi)$ is a sub-vector-space of the corresponding fiber $F_b(\eta)$.

Given two vector bundles $\xi_1, \xi_2$ with projection maps $p_i: E_i \rightarrow B_i$, for $i = 1, 2$. It is possible to consider their Cartesian product $\xi_1 \times \xi_2$ which is defined to be the vector bundle with projection map

$$p_1 \times p_2: E_1 \times E_2 \rightarrow B_1 \times B_2$$

where each fiber

$$(p_1 \times p_2)^{-1}(b_1, b_2) = F_{b_1}(\xi_1) \times F_{b_2}(\xi_2)$$

is given the obvious vector space structure. Note that $\xi_1 \times \xi_2$ is locally trivial.

In this thesis, we are particularly interested in two special types of vector bundles, namely tangent bundles and normal bundles. We also need to define the canonical line bundle $\gamma_1^n$ over real projective space $\mathbb{P}^n$ which we need in order to introduce the axioms of Stiefel-Whitney classes.
• The tangent bundle $\tau_M$ of a smooth manifold $M$ is defined as follows: The manifold $DM$ is the total space of $\tau_M$ consisting of all pairs $(x, v)$ with $x \in M$ and $v$ is tangent to $M$ at $x$. The projection map $p: DM \to M$ is defined as $p(x, v) = x$, while the vector space structure in $p^{-1}(x)$ is given by

$$t_1(x, v_1) + t_2(x, v_2) = (x, t_1v_1 + t_2v_2).$$

It satisfies the local triviality condition hence tangent bundle is a vector bundle. The total space $DM$ for an $n$-dimensional base space $M$ is an $2n$-dimensional manifold. Also recall that $M$ is called parallelizable if $\tau_M$ is a trivial bundle.

• The normal bundle $\nu$ of a smooth manifold $M \subset \mathbb{R}^n$ is defined as follows: The total space $E \subset M \times \mathbb{R}^n$ is the set of all pairs $(x, v)$ such that $v$ is orthogonal to the tangent space $DM_x$. Like the case with the tangent bundle, the projection map $p: E \to M$ and the vector space structure in $p^{-1}(x)$ is given by the formulas $p(x, v) = x$ and

$$t_1(x, v_1) + t_2(x, v_2) = (x, t_1v_1 + t_2v_2),$$

respectively. It satisfies the local triviality condition hence normal bundle is a vector bundle.

• Let $\mathbb{P}^n = S^n/(x \sim -x)$, where $S^n \subset \mathbb{R}^{n+1}$ is the $n$-dimensional unit sphere and $x, -x \in S^n$, be the $n$-dimensional real projective space which is topologized as a quotient space of $S^n$. Let $E(\gamma^1_n)$ be the subset of $\mathbb{P}^n \times \mathbb{R}^{n+1}$ consisting of all pairs $(\{\pm x\}, v)$ such that the vector $v$ is a multiple of $x$. Define $p: E(\gamma^1_n) \to \mathbb{P}^n$ by

$$\pi(\{\pm x\}, v) = \{\pm x\}$$

so that each fiber $p^{-1}(\{\pm x\})$ can be identified with the line through $x$ and $-x$ in $\mathbb{R}^{n+1}$. With each such line with its usual vector space structure we have the resulting vector bundle $E(\gamma^1_n)$ which is called the canonical line bundle over $\mathbb{P}^n$.

**Definition 8.**[14] Two vector bundles $\xi$ and $\eta$ over the same base space $B$ are said to be isomorphic, written $\xi \cong \eta$, if there exists a homeomorphism

$$f: E(\xi) \to E(\eta)$$

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between the total spaces which maps each vector space $F_b(\xi)$ isomorphically onto the corresponding vector space $F_b(\eta)$.

Now let $\xi$ be a vector bundle with projection $p: E \to B$ and let $B_1$ be an arbitrary topological space. Given any map $f: B_1 \to B$ one can construct the pullback (or induced) bundle $f^*\xi$ over $B_1$. The total space $E_1$ of $f^*\xi$ is the subset $E_1 \subset B_1 \times E$ consisting of all pairs $(b, e)$ with $f(b) = p(e)$. The projection map $p_1: E_1 \to B_1$ is defined by $p_1(b, e) = b$. So one has the following commutative diagram

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\bar{f}} & E \\
p_1 \downarrow & & \downarrow p \\
B_1 & \xrightarrow{f} & B \\
\end{array}
$$

where $\bar{f}(b, e) = e$. The vector space structure in $p_1^{-1}(b)$ is defined by

$$
t_1(b, e_1) + t_2(b, e_2) = (b, t_1e_1 + t_2e_2).
$$

The bundle $f^*\xi$ satisfies the local triviality condition. There is also the universality of pullback which means that if we have maps from a topological space $X$ to both $B_1$ and $E$ then there is a unique map from $X$ to $E_1$. The diagram above is an example of a bundle map, in fact it is a pullback bundle map both of which we define now:

**Definition 9.** Let $\eta$ and $\xi$ be vector bundles. A bundle map

$$(f, \bar{f}): (Y, E(\eta)) \to (X, E(\xi))$$

is a commutative diagram of maps

$$
\begin{array}{ccc}
E(\eta) & \xrightarrow{\bar{f}} & E(\xi) \\
p_2 \downarrow & & \downarrow p_1 \\
Y & \xrightarrow{f} & X \\
\end{array}
$$

such that the restriction of $\bar{f}$

$$
\bar{f}|_{F_y(\eta)} : F_y(\eta) \to F_{f(y)}(\xi); \ v \mapsto \bar{f}(v)
$$

is a linear map of vector spaces for each $y \in Y$.  

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Definition 10. [15] A pullback bundle map is a bundle map \((f, \bar{f}): (X', E(\eta')) \to (X, E(\eta))\) such that each of the linear maps
\[
\bar{f}|_{F_{x'(\eta')}}: F_{x'(\eta')} \to F_{f(x')(\eta)} \quad (x' \in X')
\]
is an isomorphism of vector spaces, i.e., such that the function
\[
E(\eta') \to E(f^*\eta); \; y \mapsto (p'(y), \bar{f}(y))
\]
is a homeomorphism.

Note that an isomorphism \(\bar{f}: E(\eta') \to E(\eta)\) of bundles over the same space \(X\) is a pullback bundle map of the type \((1, \bar{f}): (X, E(\eta')) \to (X, E(\eta))\).

Proposition 2.1.2. [15] Let \(X\) and \(X'\) be topological spaces.

(i) There is a pullback bundle map
\[
(f, \bar{f}): (X', f^*\eta) \to (X, \eta)
\]
for any \(k\)-plane bundle \(\eta\) over \(X\) and any map \(f: X' \to X\).

(ii) A pullback bundle map \((f, \bar{f}): (X', \eta') \to (X, \eta)\) is a bundle map such that
\[
(1, \bar{f}): (X', \eta') \to (X', f^*\eta)
\]
is an isomorphism of bundles.

A way of constructing a new vector bundle from two vector bundles over the same base space \(B\) is by considering their Whitney sum: Let \(\xi_1\) and \(\xi_2\) be two such vector bundles and let \(d: B \to B \times B\) denote the diagonal embedding i.e., \(d(b) = (b, b)\) for every \(b \in B\). The bundle \(d^*(\xi_1 \times \xi_2)\) over \(B\) is called the Whitney sum of \(\xi_1\) and \(\xi_2\), and it is denoted by \(\xi_1 \oplus \xi_2\). Note that each fiber \(F_b(\xi_1 \oplus \xi_2)\) is canonically isomorphic to the direct sum \(F_b(\xi_1) \oplus F_b(\xi_2)\).

The trivial \(k\)-plane bundle \(\epsilon^k\) over a base space \(B\) can be written as the \(k\)-fold Whitney sum of the trivial line bundle
\[
\epsilon^k = \epsilon \oplus \epsilon \oplus \cdots \oplus \epsilon,
\]
with \( k \) linearly independent sections. Note also that, for any map \( f: Y \to X \) the pullback \( f^*\epsilon^k \) is isomorphic to the trivial \( k \)-plane bundle \( \epsilon^k \) over \( Y \).

The next proposition establishes a connection between Whitney sums and non-zero sections of a vector bundle.

**Proposition 2.1.3.** [15] A \( k \)-plane bundle \( \eta \) admits a non-zero section if and only if it is isomorphic to \( \eta' \oplus \epsilon \) for a \((k - 1)\)-plane bundle \( \eta' \).

A direct consequence of the previous proposition is that a \( k \)-plane bundle admits \( k \)-many linearly independent non-zero sections if and only if it is isomorphic to \( \epsilon^k \).

**Definition 11.** [15] Let \( \eta \) be a \( k \)-plane bundle and \( \eta' \) be a \( k' \)-plane bundle over the same base space \( B \).

(i) A stable isomorphism between \( \eta \) and \( \eta' \) is a bundle isomorphism

\[
c: \eta \oplus \epsilon^j \cong \eta' \oplus \epsilon^{j'}
\]

where \( j, j' \geq 0 \) with \( j + k = j' + k' \).

(ii) A stable bundle over \( B \) is an equivalence class \([\eta]\) of bundles where \( \eta \sim \eta' \) if there exists a stable isomorphism \( \eta \oplus \epsilon^j \cong \eta' \oplus \epsilon^{j'} \) for some \( j, j' \geq 0 \).

(iii) The bundle \( \eta \) is stably trivial if \( \eta \oplus \epsilon^j \) is trivial for some \( j \geq 0 \).

**Lemma 2.1.4.** [14] Let \( \xi_1, \xi_2 \subset \eta \) be vector bundles such that each vector space \( F_b(\eta) \) is equal to the direct sum of the sub-spaces \( F_b(\xi_1) \) and \( F_b(\xi_2) \). Then \( \eta \) is isomorphic to the Whitney sum \( \xi_1 \oplus \xi_2 \).

The following question arises from the previous lemma; is there a complementary sub-bundle \( \xi_2 \subset \eta \) such that \( \eta = \xi_1 \oplus \xi_2 \) for a given sub-bundle \( \xi_1 \subset \eta \)? If \( \eta \) is provided with a Euclidean metric then it is possible to construct such a \( \xi_2 \) which is called the orthogonal complement of \( \xi_1 \) in \( \eta \) and it is denoted by \( \xi_1^\perp \).

### 2.1.2 Postnikov Towers

Postnikov towers are important constructions which allow us to view \( CW \)-complexes as sort of twisted products of Eilenberg-MacLane spaces, up to homotopy equiva-
lence. In order to construct the Postnikov tower of a CW-complex we first need to understand what an Eilenberg-MacLane space is.

**Definition 12.** A topological space $X$ with only one non-trivial homotopy group $\pi_n(X) \cong G$ is called an Eilenberg-MacLane space and it is denoted by $K(G, n)$.

The following theorem establishes an important connection between Eilenberg-Maclane spaces and singular cohomology.

**Theorem 2.1.5.** Let $\langle X, K(G, n) \rangle$ be the set of basepoint-preserving homotopy classes of maps from CW-complex $X$ to $K(G, n)$. There are natural bijections

$$T: \langle X, K(G, n) \rangle \rightarrow H^n(X; G)$$

for all $n > 0$, with $G$ an abelian group. Such a bijection has the form $T([f]) = f^*(\alpha)$ for a certain distinguished class $\alpha \in H^n(K(G, n); G)$.

For connected CW-complexes $X$ we can replace $\langle X, K(G, n) \rangle$ in the theorem with $[X, K(G, n)]$, the nonbasepointed homotopy classes. This theorem allows us to represent each element in a cohomology ring with a map between CW-complexes. Now with that out of the way we can define Postnikov towers.

**Definition 13.** Let $X$ be a path-connected space. A Postnikov tower for $X$ is a commutative diagram

$$
\begin{array}{cc}
\vdots & \\
\downarrow & \\
X_3 & \\
\downarrow & \\
X_2 & \\
\downarrow & \\
X_1 & \\
\end{array}
$$

which satisfies the following conditions:

1. The map $X \rightarrow X_n$ induces an isomorphism on $\pi_i$ for $i \leq n$.
2. $\pi_i(X_n) = 0$ for $i > n$. 

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We can take the map $X_n \to X_{n-1}$ as a fibration and call its fiber $F_n$, so we have the following homotopy long exact sequence:

$$\ldots \to \pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1}) \to \pi_i(F_n) \to \pi_i(X_n) \to \pi_i(X_{n-1}) \to \ldots$$

Since the Postnikov tower is commutative and

$$\pi_i(X) \cong \pi_i(X_{n-1})$$

for $i \leq n - 1$, we have

$$\pi_i(X_n) \cong \pi_i(X_{n-1})$$

so $\pi_i(F_n) = 0$. For $i > n$, we have

$$\pi_i(X_n) \cong \pi_i(X_{n-1}) = 0$$

so again $\pi_i(F_n) = 0$. For $i = n$, both $\pi_{i+1}(X_{n-1})$ and $\pi_i(X_{n-1})$ are trivial so

$$\pi_i(F_n) \cong \pi_i(X_n) \cong \pi_i(X).$$

Hence we can conclude that $F_n \cong K(\pi_n(X), n)$. Generally the fibration

$$F_n \leftarrow X_n \longrightarrow X_{n-1}$$

does not need to be trivial hence $X_n$ is the twisted product of Eilenberg-MacLane spaces.

### 2.1.3 Spectral Sequences

Before we delve more into vector bundles and how they relate to manifolds, let us briefly discuss spectral sequences, more specifically the Leray-Serre spectral sequence. Although there are many versions of spectral sequences, in this thesis we only use the homology Leray-Serre and the Atiyah-Hirzebruch spectral sequences so there is no need to define spectral sequences in a general sense. Let us begin with the homology Leray-Serre spectral sequence which relates the homology groups of base, fiber and total space of a fibration.
**Definition 14.** Let $F \to X \to B$ be a fibration where $B$ is path connected and $\pi_1(B)$ acts trivially on $H_*(F; G)$. The Leray-Serre spectral sequence of this fibration is a 2-dimensional array of abelian groups $E^r_{p,q}$ and maps, called differentials, between them

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that

$$d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$$

where $r$ denotes the page number and $p$ and $q$ denote the positioning of this group on the page. To begin with, on the second page we have

$$E^2_{p,q} = H_p(B; H_q(F; G))$$

and on each consecutive page $r \geq 3$

$$E^r_{p,q} = \frac{\ker(d^r_{p,q}^{-1})}{\text{im}(d^r_{p+r,q-r+1})}.$$

The terms on the second page are as follows:

<p>| | | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>2</td>
<td>$H_0(B; H_2(F; G))$</td>
<td>$H_1(B; H_2(F; G))$</td>
</tr>
<tr>
<td>1</td>
<td>$H_0(B; H_1(F; G))$</td>
<td>$H_1(B; H_1(F; G))$</td>
</tr>
<tr>
<td>0</td>
<td>$H_0(B; H_0(F; G))$</td>
<td>$H_1(B; H_0(F; G))$</td>
</tr>
</tbody>
</table>

The pages of this spectral sequence stabilize, that is there exists an $r_0$ such that for $r \geq r_0$ and for any $p, q$ we have

$$E^r_{p,q} \cong E^{r+1}_{p,q} \cong \cdots \cong E^\infty_{p,q}$$

at which point we say the term $E^{r_0}_{p,q}$ is stabilized. Each stable term $E^{r_0}_{p,n-p} \cong E^\infty_{p,n-p}$ for any fixed $n$ is isomorphic to the successive quotient $\frac{E^p_{p,n}}{E^{p+1}_{p,n}}$ in a filtration

$$0 \subset F^0_n \subset \cdots \subset F^n_n = H_n(X; G)$$

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of $H_n(X; G)$.

A generalization of Leray-Serre spectral sequence can be made in the form of Atiyah-Hirzebruch spectral sequence which we define now.

**Definition 15.** Let

$$ F \hookrightarrow E \to B $$

be a fibration where $B$ is a path-connected CW-complex. Let $G_*$ be an additive homology theory. Here homology theory is a consistent way of attaching unique sequences of groups and homomorphisms between two such sequences to topological spaces and continuous mappings between them, respectively. Additive on the other hand means if $\{X_\alpha\}_{\alpha \in I}$ are disjoint open sets of a topological space $X$ such that

$$ \bigcup_{\alpha \in I} X_\alpha = X $$

and

$$ i_\alpha : X_\alpha \hookrightarrow X $$

is the inclusion map then

$$ G_*(X) \cong \bigoplus_{\alpha \in I} i_{\alpha*}(G_*(X_\alpha)) $$

where $i_{\alpha*}$ is the map between homologies induced by $i_\alpha$. There exists a spectral sequence

$$ E_2^{p,q} \cong H_p(B; G_q F) \Longrightarrow G_{p+q}(E). $$

More specifically this says that there exists

1. a filtration

$$ 0 \subset F_n^0 \subset \ldots \subset F_n^p \subset \ldots \subset G_n(E) $$

of $G_n(E) = \bigcup_p F_n^p$,

2. a bigraded spectral sequence $(E_\ast^{r,s}, d^r)$ such that $d^r$ has bidegree $(-r, r-1)$, so

$$ d^r(E_r^{r,s}) \subset E_r^{r-r,s+r-1}. $$

Hence we have

$$ E_{p,q}^{r+1} = \frac{\ker(d_{p,q}^r)}{\text{im}(d_{p+r,q-r+1}^r)}. $$
(3) isomorphisms \( E^2_{p,q} \cong H_p(B; G_q F) \).

Here \( G_q F \) represents a local coefficient system. For more information on the subject, the reader should see [3].

This spectral sequence is especially useful for computation of various bordism groups that we define later which are additive homology theories, namely it allows us to compute \( \Omega^\text{Spin}_q(M) \) where we have the following approximation

\[
H_p(M; \Omega^\text{Spin}_q(\text{pt})) \xrightarrow{\cong} \Omega^\text{Spin}_{p+q}(M)
\]

for the fibration

\[
\text{pt} \hookrightarrow M \xrightarrow{id} M.
\]

### 2.1.4 Stiefel-Whitney Classes

In this section we study the Stiefel-Whitney cohomology classes of a vector bundle. These will be important obstructions once we get to universal bundles. For a more detailed reading on this subject, the reader is referred to [14]. We define the Stiefel-Whitney classes by introducing four axioms which characterize them. For the existence and uniqueness of such cohomology classes satisfying these axioms, the reader again should see [14]. While we define the Stiefel-Whitney class for an arbitrary vector bundle for now, later when we say the Stiefel-Whitney class of a manifold \( M \) we mean the Stiefel-Whitney class of the tangent bundle \( \tau_M \) over \( M \).

By the expression \( H^i(B; G) \) we mean the \( i \)-th singular cohomology ring of \( B \) with coefficients in \( G \). For extensive details on singular cohomology, the reader should see [6]. Now we can define the four axioms.

1. There is a sequence of cohomology classes

\[
w_i(\xi) \in H^i(B(\xi); \mathbb{Z}_2) \quad i = 0, 1, 2, \ldots,
\]

   corresponding to each vector bundle \( \xi \) called the Stiefel-Whitney classes of \( \xi \). The class \( w_0(\xi) \) is equal to the unit element

\[
1 \in H^0(B(\xi); \mathbb{Z}_2),
\]
and \( w_i(\xi) = 0 \) for \( i > n \) if \( \xi \) is an \( n \)-plane bundle.

2) Naturality. If \( f: B(\xi) \to B(\eta) \) is covered by a bundle map

\[
(f, \tilde{f}): (B(\xi), E(\xi)) \to (B(\eta), E(\eta)), \text{ then } w_i(\xi) = f^*w_i(\eta)
\]

3) The Whitney Product Theorem. For vector bundles \( \xi \) and \( \eta \) over the same base space

\[
w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta)
\]

where \( \cup \) denotes cup product in cohomology rings.

For example, we have

\[
w_1(\xi \oplus \eta) = w_1(\xi) + w_1(\eta)
\]

\[
w_2(\xi \oplus \eta) = w_2(\xi) + w_1(\xi)w_1(\eta) + w_2(\eta)
\]

4) The Stiefel-Whitney class \( w_1(\gamma_1) \) is non-zero for the line bundle \( \gamma_1 \) over the circle \( \mathbb{P}^1 \).

Proposition 2.1.6. We have the following as immediate consequences of (2).

1) If \( \xi \cong \eta \), then \( w_i(\xi) = w_i(\eta) \).

2) If \( \epsilon \) is a trivial vector bundle, then \( w_i(\epsilon) = 0 \) for \( i > 0 \).

For a trivial vector bundle \( \epsilon \) there exists a bundle map from \( \epsilon \) to a vector bundle over a point. When we combine this information with the Whitney product theorem, we have:

3) If \( \epsilon \) is trivial, then \( w_i(\epsilon \oplus \eta) = w_i(\eta) \).

4) If \( \xi \) is an \( \mathbb{R}^n \)-bundle with a Euclidean metric which possesses a nowhere zero cross-section, then \( w_n(\xi) = 0 \). If \( \xi \) possesses \( k \) cross-sections which are nowhere linearly dependent, then

\[
w_{n-k+1}(\xi) = w_{n-k+2}(\xi) = \ldots = w_n(\xi) = 0.
\]
We can view the trivial vector bundle $\epsilon$ over a manifold $M$ as the Whitney sum of the tangent bundle $\tau_M$ and stable normal bundle $\nu_M$ so by the second proposition above we have $w_i(\epsilon) = w_i(\tau_M \oplus \nu_M) = 0$ for $i > 0$. Combining this with the Whitney Product Theorem we can see that

$$w_1(\tau_M \oplus \nu_M) = 0 = w_1(\tau_M) + w_1(\nu_M)$$

so we have

$$w_1(\tau_M) = -w_1(\nu_M).$$

Since both $w_1(\tau_M)$ and $w_1(\nu_M)$ are elements of the cohomology ring $H^1(M; \mathbb{Z}_2)$ it is easy to see that $w_1(\tau_M) = w_1(\nu_M)$. In addition to this if we assume $M$ is orientable then we have

$$w_2(\tau_M \oplus \nu_M) = 0 = w_2(\tau_M) + w_1(\tau_M)w_1(\nu_M) + w_2(\nu_M).$$

By orientability $w_1(\tau_M)w_1(\nu_M) = 0$ so $w_2(\tau_M) + w_2(\nu_M) = 0$ and by a similar argument we have $w_2(\tau_M) = w_2(\nu_M)$. We say $M$ is an orientable manifold if $w_1(\tau_M) = 0$. Likewise we say $M$ admits a spin structure if $w_1(\tau_M) = w_2(\tau_M) = 0$. Our manifold $M$ becomes an oriented (or spin) manifold by fixing an orientation (or spin structure) over $M$.

### 2.1.5 Grassmann Manifolds and Universal Bundles

Next, we define Grassmann manifolds and universal bundles. For more details on this subject we refer the reader to [14].

The $(nk)$-dimensional Grassmann manifold $G_n(\mathbb{R}^{n+k})$ is defined as the set of all $n$-dimensional planes through the origin of the Euclidean space $\mathbb{R}^{n+k}$. By assigning each $n$-plane its orthogonal complement in $\mathbb{R}^{n+k}$, which is a $k$-plane, we can define a homeomorphism between $G_n(\mathbb{R}^{n+k})$ and $G_k(\mathbb{R}^{n+k})$. An $n$-frame in $\mathbb{R}^{n+k}$ is an $n$-tuple of linearly independent vectors of $\mathbb{R}^{n+k}$. The collection of all $n$-frames in $\mathbb{R}^{n+k}$ forms an open subset of the $n$-fold Cartesian product $\mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$, called the Stiefel manifold $V_n(\mathbb{R}^{n+k})$.

$$q: V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$$
is a canonical map that sends each $n$-frame to the $n$-plane which it spans. The topology on $G_n(\mathbb{R}^{n+k})$ is given by the quotient topology: a subset $U \subset G_n(\mathbb{R}^{n+k})$ is open if and only if $q^{-1}(U) \subset V_n(\mathbb{R}^{n+k})$ is open. A canonical vector bundle $\gamma^n(\mathbb{R}^{n+k})$ over $G_n(\mathbb{R}^{n+k})$ can be constructed in the following way. Let

$$E = E(\gamma^n(\mathbb{R}^{n+k}))$$

be the set of all pairs 

$$(\text{n-plane in } \mathbb{R}^{n+k}, \text{vector in that n-plane}).$$

The projection map $p: E \to G_n(\mathbb{R}^{n+k})$ is defined by $p(X, x) = X$, and the vector space structure on the fiber over $X$ is defined by $t_1(X, x_1) + t_2(X, x_2) = (X, t_1x_1 + t_2x_2)$. This bundle that we constructed satisfies the local triviality condition.

Given an $n$-dimensional smooth manifold $M^n$ embedded in $\mathbb{R}^{n+k}$ the generalized Gauss map

$$g: M \to G_n(\mathbb{R}^{n+k})$$

is the function which sends each $x \in M$ to its tangent space $DM_x \in G_n(\mathbb{R}^{n+k})$. This is covered by a bundle map

$$\bar{g}: E(\tau M) \to E(\gamma^n(\mathbb{R}^{n+k})),
$$

where $\bar{g}(x, v) = (DM_x, v)$, i.e., we have the following commutative diagram

$$
\begin{array}{ccc}
E(\tau M) & \xrightarrow{\bar{g}} & E(\gamma^n(\mathbb{R}^{n+k})) \\
p \downarrow & & \downarrow p \\
M & \xrightarrow{g} & G_n(\mathbb{R}^{n+k})
\end{array}
$$

Note that both $g$ and $\bar{g}$ are continuous.

The vector bundle $\gamma^n(\mathbb{R}^{n+k})$ is called a "universal bundle" because for $k$ sufficiently large enough not only tangent bundles, but most other $\mathbb{R}^n$ bundles can be mapped into $\gamma^n(\mathbb{R}^{n+k})$.

If for $G_n(\mathbb{R}^{n+k})$, we let $k$ go to infinity we have the infinite Grassmann manifold $G_n(\mathbb{R}^\infty)$ (also denoted by $BO(n)$ or simply $G_n$). It is the set of all $n$-dimensional linear subspaces of $\mathbb{R}^\infty$, topologized as the direct limit of the sequence

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+2}) \subset \cdots.$$
A subset of $G_n$ is open (or closed) if and only if its intersection with $G_n(\mathbb{R}^{n+k})$ is open (or closed) as a subset of $G_n(\mathbb{R}^{n+k})$ for each $k$. Whitney sum on infinite Grassmann manifolds works as follows:

$$\oplus: \text{BO}(j) \times \text{BO}(k) \xrightarrow{\cong} \text{BO}(j+k).$$

The canonical bundle $\gamma^n$ over $\text{BO}(n)$ can be constructed just as in the finite dimensional case. The following two theorems imply that this bundle $\gamma^n$ over $\text{BO}(n)$ is a "universal" $\mathbb{R}^n$-bundle.

**Theorem 2.1.7.** Any $\mathbb{R}^n$-bundle $\xi$ over a paracompact base space $B$ admits a bundle map $(g, \bar{g}): (B(\xi), E(\xi)) \to (\text{BO}(n), E(\gamma^n))$.

In this theorem paracompact space stands for a topological space $X$ such that every open cover of $X$ has a refinement by a locally finite cover. An open cover $\{U_\alpha\}_{\alpha \in I}$ is a refinement of another open cover $\{V_\beta\}_{\beta \in J}$ if each open set in the first cover is contained in an open set in the second cover. In particular, note that compact spaces are paracompact.

This theorem says that for any such $\mathbb{R}^n$-bundle $\xi$ we have a map $g$ just like in the commutative diagram above.

Two bundle maps $(g_1, \bar{g}_1), (g_2, \bar{g}_2): (B(\xi), E(\xi)) \to (\text{BO}(n), E(\gamma^n))$ are said to be bundle-homotopic if there exists a one-parameter family of bundle maps

$$h_t: E(\xi) \to E(\gamma^n) \quad 0 \leq t \leq 1$$

with $h_0 = \bar{g}_1, h_1 = \bar{g}_2$, such that $h$ is continuous as a function of both variables, i.e. the associated function

$$h: E(\xi) \times [0, 1] \to E(\gamma^n)$$

must be continuous.

**Theorem 2.1.8.** Any two bundle maps

$$(g_1, \bar{g}_1), (g_2, \bar{g}_2): (B(\xi), E(\xi)) \to (\text{BO}(n), E(\gamma^n))$$

are bundle-homotopic.
This theorem says that bundle maps are unique up to bundle-homotopy. So these two theorems establish that the bundle $\gamma_n$ over $G_n$ is a universal $\mathbb{R}^n$-bundle. For the rest of this thesis we will be using $g: B \to \text{BO}(n)$ to denote the $n$-plane bundle over base space $B$ where the total space is that of the bundle $g^*\gamma^n$.

We can also consider for any $n$, all $n$-dimensional planes in $\mathbb{R}^\infty$, in which case the corresponding Grassmann manifold is denoted by BO: consider the natural inclusion $\text{BO}(n) \hookrightarrow \text{BO}(n + 1)$ corresponding to adding a one-dimensional trivial bundle to $E(\gamma^n)$. We can see this map as an embedding so that it makes sense to consider the union

$$\text{BO} = \bigcup_{n=1}^{\infty} \text{BO}(n)$$

in the inductive limit topology i.e., $U \in \text{BO}$ is open if it is open in each $\text{BO}(n)$ for every $n$. With the knowledge of Grassmann manifolds, we can rewrite the definitions of tangent bundle and normal bundle as follows:

**Definition 16.** [15] The tangent bundle of an $m$-dimensional manifold $M$ with atlas $\mathcal{U}$ is the $m$-plane bundle $\tau_M: M \to \text{BO}(m)$ with total space the open $2m$-dimensional manifold

$$E(\tau_m) = \left( \prod_{(U, \varphi) \in \mathcal{U}} U \times \mathbb{R}^m \right) / \sim$$

where the identification is given by

$$(x \in U, h \in \mathbb{R}^m) \sim (x' \in U', h' \in \mathbb{R}^m)$$

for

$$x = x' \in U \cap U' \subseteq M \text{ and } d(\varphi'^{-1}\varphi)(\varphi^{-1}(x))(h) = h' \in \mathbb{R}^m$$

with the projection map

$$p: E(\tau_m) \to M ; (x, h) \mapsto x.$$ 

The tangent space to $x \in M$ is the $m$-dimensional vector space

$$\tau_m(x) = \left( \prod_{(U, \varphi) \in \mathcal{U}, x \in U} \{x\} \times \mathbb{R}^m \right) / \sim$$

such that

$$E(\tau_M) = \bigcup_{x \in M} \tau_M(x).$$
In order to understand the definition of normal bundle in this new setting we need the definition of an isotopy between two embeddings of manifolds first.

**Definition 17.** [15] An isotopy between two embeddings of manifolds $f_0, f_1 : N^n \hookrightarrow M^m$ is a homotopy $f : N \times I \to M ; (x,t) \mapsto f_t(x)$ which is an embedding $f_t : N \hookrightarrow M$ at each level $t \in I$.

Now we are ready to define the stable normal bundle of an $m$-dimensional manifold $M$.

**Definition 18.** [15] The stable normal bundle of an $m$-dimensional manifold $M$ is the bundle with the classifying map

$$\nu_M : M \to BO$$

represented by the normal $k$-plane bundle $\nu : M \to BO(k)$ of any embedding $f : M^m \hookrightarrow \mathbb{R}^r$, with $r$ suitably large so that each embedding of $M^m$ into $\mathbb{R}^r$ is isotopic to one another, such that

$$\tau_M \oplus \nu_M = \epsilon^\infty : M \to BO.$$ 

Here the word stable means that our normal bundle does not depend on any particular choice of an embedding. If we embed our manifold into a Euclidean space with a suitably large dimension compared to the dimension of our manifold, then the normal bundle is independent of our choice of an embedding.

Until now we did not restrict our vector bundles to orientable ones. If we do that, this gives rise to the notion of oriented Grassmann manifold. Let $\tilde{G}_n(\mathbb{R}^{n+k})$ be the Grassmann manifold of all oriented $n$-planes in $\mathbb{R}^{n+k}$. It has the quotient topology just as in the case of $G_n(\mathbb{R}^{n+k})$. It is clear that $\tilde{G}_n(\mathbb{R}^{n+k})$ is a 2-fold covering space of the unoriented Grassmann manifold $G_n(\mathbb{R}^{n+k})$. Similar to the case with unoriented $n$-plane, if we let $k$ go to infinity we have $\tilde{G}_n(\mathbb{R}^\infty)$ (or $BSO(n)$) and then if we let $n$ go to infinity we have BSO.
For a smooth manifold $M$, the generalized Gauss map
\[ g: M \to BO \]
can be extended to a map
\[ \tilde{g}: M \to BSO \]
if and only if the first Stiefel-Whitney $w_1(\tau_M)$ class is trivial.

The first Stiefel-Whitney class for vector bundles over $S^1$ allows us to classify these bundles as follows. The first Stiefel-Whitney class defines an isomorphism
\[ w_1: \pi_1(BO(k)) \to H^1(S^1; \mathbb{Z}_2) = \mathbb{Z}_2; \quad \omega \mapsto w_1(\omega). \]

The first Stiefel-Whitney class $w_1$ in this case is an element of $H^1(S^1; \mathbb{Z}_2)$ so it defines a map from $H_1(S^1; \mathbb{Z}_2)$ but in our case $\pi_1(BO(k))$ and $H_1(S^1; \mathbb{Z}_2)$ are isomorphic. The following theorems allow us to establish this isomorphism.

**Theorem 2.1.9.** [14] The number of $r$-cells in $G_n(\mathbb{R}^m)$ is equal to the number of partitions of $r$ into at most $n$ integers each of which is $\leq m - n$.

**Theorem 2.1.10.** [6] Every map $f: X \to Y$ of CW complexes is homotopic to a cellular map. If $f$ is already cellular on a subcomplex $A \subset X$, the homotopy may be taken to be stationary on $A$.

As an immediate result of the first theorem we can see that $BO(k) = G_k(\mathbb{R}^\infty)$ has one 0-cell and one 1-cell in its 1-skeleton. By the second theorem we can view the map $S^1 \to BO(k)$ as a cellular map i.e., it takes the 1-skeleton of $S^1$, which is $S^1$ itself, to the 1-skeleton of $BO(k)$, which is again $S^1$. Hence we have a clear isomorphism
\[ \pi_1(BO(k)) \cong H_1(S^1; \mathbb{Z}_2). \]

so instead of chains in $H_1(S^1; \mathbb{Z}_2)$, $H^1(S^1; \mathbb{Z}_2)$ can also eat the elements of $\pi_1(BO(k))$.

An orientable $k$-plane bundle $\omega: S^1 \to BO(k)$ is isomorphic to the trivial bundle $\epsilon^k$.

A nonorientable $k$-plane bundle $\omega: S^1 \to BO(k)$ is isomorphic to the Whitney sum $\mu \oplus \epsilon^{k-1}$ where $\mu: S^1 \to BO(1) = \mathbb{P}^\infty$ is the nonorientable line bundle with total space the open Möbius band
\[ E(\mu) = \mathbb{R} \times I/\{(x, 0) \sim (-x, 1)\}. \]
The 1-disk bundle (each fiber is diffeomorphic to a 1-disk) is the closed Möbius band.

Before we move onto the next section we should look at the homotopy groups of \( \text{BO}, \text{BSO} \) and a new manifold \( \text{BSpin} \) which we talk about in detail later. These will become handy once we get to the normal \( k \)-type of manifolds.

The homotopy groups of the infinite dimensional orthogonal group \( O = \lim_{n \to \infty} (O(n)) \) are given by the Bott periodicity theorem which in turn gives us the homotopy groups of infinite Grassmann manifold \( \text{BO} \).

**Theorem 2.1.11.** [2] *There is an isomorphism \( \pi_i(O) \cong \pi_{i+8}(O) \) for every \( i \geq 0 \).*

The table below gives the homotopy groups of the orthogonal group \( O \).

<table>
<thead>
<tr>
<th>( i )</th>
<th>( \pi_i(O) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>1</td>
<td>( \mathbb{Z}_2 )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>( \mathbb{Z} )</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>( \mathbb{Z} )</td>
</tr>
</tbody>
</table>

Since \( O \) consists of two connected components both of which are diffeomorphic to \( \text{SO} \) we have

\[ \pi_0(\text{SO}) \text{ is trivial and } \pi_i(O) = \pi_i(\text{SO}) \text{ for } i \geq 1. \]

As for \( \text{Spin} = \lim_{n \to \infty} (\text{Spin}(n)) \), since \( \text{Spin}(n) \) is the universal cover of \( \text{SO}(n) \) for \( n \geq 3 \) we have

\[ \pi_1(\text{Spin}) = 0 \text{ and } \pi_i(\text{Spin}) = \pi_i(\text{SO}) \text{ for } i \geq 2. \]

Homotopy groups of \( \text{O}, \text{SO}, \text{Spin} \) give us the homotopy groups of \( \text{BO}, \text{BSO}, \text{BSpin} \) as follows: For any discrete group \( G \), in our case \( \text{O}, \text{SO} \) and \( \text{Spin} \), the classifying space \( BG \) can be obtained as the quotient \( BG = EG/G \) where \( G \) acts on \( EG \) which is weakly contractible i.e., all of its homotopy groups are trivial. We have the following fibration

\[ G \longrightarrow EG \]
\[ \downarrow \]
\[ BG \]

which gives the homotopy long exact sequence

\[ \cdots \longrightarrow \pi_{i+1}(EG) \longrightarrow \pi_{i+1}(BG) \longrightarrow \pi_i(G) \longrightarrow \pi_i(EG) \longrightarrow \cdots \]
Since $\pi_i(EG) = 0$ for any $i \geq 0$ we can say that
\[
\pi_{i+1}(BG) \cong \pi_i(G) \text{ for any } i \geq 0,
\]
hence $\pi_i(O), \pi_i(SO), \pi_i(\text{Spin})$ directly give us $\pi_{i+1}(BO), \pi_{i+1}(BSO), \pi_{i+1}(B\text{Spin})$ for any $i \geq 0$.

### 2.1.6 Surgery and Bordism

We are almost ready to discuss surgery but before that let us start with the definition of connectedness.

**Definition 19.** [15] Let $n \geq 1$

(i) A space is $n$-connected if it is connected and
\[
\pi_i(X) = 0 \text{ for } i \leq n.
\]

(ii) A map $f: X \to Y$ of connected spaces is $n$-connected if $f_*: \pi_i(X) \to \pi_i(Y)$ is onto for $i = n$ and an isomorphism for $i < n$, or equivalently if
\[
\pi_i(f) = 0 \ (i \leq n).
\]

(iii) A pair of connected spaces $(Y, X)$ is $n$-connected if the inclusion map $f: X \to Y$ is $n$-connected, or equivalently if
\[
\pi_i(Y, X) = 0 \ (i \leq n).
\]

The definition of an $n$-connected map is useful for us to show that there is no manifold in dimensions 3 and 4 with the desired homotopy groups. The next two theorems will be important when we discuss surgery on a manifold shortly after.

**Theorem 2.1.12.** [15] *Whitney Embedding Theorem*\] Let $f: N^n \to M^m$ be a map of manifolds such that

either $2n + 1 \leq m$

or $m = 2n \geq 6$ and $\pi_1(M) = 1$.

Then $f$ is homotopic to an embedding $N^n \hookrightarrow M^m$. 

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Theorem 2.1.13. [15] Tubular Neighbourhood Theorem] An embedding \( f : N^n \hookrightarrow M^m \) can be extended to a codimension 0 embedding \( E(\nu_f) \hookrightarrow M^m \) of the total space of the normal \((m - n)\)-plane bundle \( \nu_f : N \to BO(m - n) \). As for an immersion \( f : N \looparrowright M \), we have a similar situation with \( E(\nu_f) \looparrowright M \) which is again an extension to a codimension 0 embedding. Recall that \( BO(n) \) represents the classifying space for the orthogonal group \( O(n) \).

We combine these theorems to define a framed \( n \)-embedding.

**Definition 20.** [8] Let \( M^m \) be a manifold of dimension \( m \).

(i) A framed embedding of a manifold \( N^n \) in \( M \) is an embedding \( f : N^n \times D^{m-n} \hookrightarrow M \). Equivalently, (by the tubular neighbourhood theorem), it is an embedding of \( N \) in \( M \) with a fixed trivialization of the normal bundle.

(ii) An \( n \)-embedding in \( M \) is an embedding \( f : S^n \hookrightarrow M \).

(iii) A framed \( n \)-embedding is a framed embedding \( f : S^n \times D^{m-n} \hookrightarrow M \).

Before we get to surgery we need to define handle attachment.

**Definition 21.** [15] Let \( (W, \partial W) \) be an \((m+1)\)-dimensional manifold with boundary. Consider the \((m+1)\)-disk \( D^{m+1} \) as the product \( D^k \times D^{m-k+1} \) and suppose we have a framed \((k-1)\) embedding \( f : \partial D^k \times D^{m-k+1} \hookrightarrow \partial W \). Now let

\[
W' = D^k \times D^{m-k+1} \cup_f W.
\]

We say that the manifold \( W' \) is obtained by attaching an \((m+1)\)-dimensional \( k \)-handle \( D^k \times D^{m-k-1} \) to \( W \), and that \( D^k \times D^{m-k+1} \subseteq W' \) is a \( k \) handle on \( W' \).

Now we are ready to define surgery.

**Definition 22.** [8] Let \( M^m \) be a manifold of dimension \( m \) (possibly with boundary), let \( f : S^n \times D^{m-n} \hookrightarrow M \) be a framed \( n \)-embedding, and let \( B \) be such that

\[
\partial(cl(M^m \setminus f(S^n \times D^{m-n}))) = \partial M \amalg f(S^n \times S^{m-n-1}) = \partial M \amalg B.
\]

The operation of cutting out \( S^n \times D^{m-n} \) and gluing in \( D^{n+1} \times S^{m-n-1} \) along \( B \) is called \( n \)-surgery (or just surgery when no ambiguity occurs) on \( f(S^n \times D^{m-n}) \subseteq M \).
The resulting manifold
\[(D^{n+1} \times S^{m-n-1}) \cup S^n \times S^{m-n-1} \text{ cl}(M \setminus f(S^n \times D^m))\]
is called the effect of the surgery, and \(\tilde{f} = f|_{S^n \times \{0\}} \in \pi_n(M)\) is said to be killed by the surgery. Note that the effect of the surgery has the same boundary as \(M\).

Surgery on an \(m\)-dimensional manifold \(M\) gives us a new \(m\)-dimensional manifold \(M'\) which we called the effect of surgery. A nice application of surgery is that it allows us to construct manifolds of dimension \(\geq 4\) with fundamental group \(G\) where \(G\) can be any finitely presented group. Let 
\[G = \langle g_1, \ldots, g_k | r_1, \ldots, r_l \rangle\]
be our group where \(g_i\) represent the generators whereas \(r_j\) represent the relations. Let 
\[M = (S^1 \times S^{m-1}) \# \ldots \# (S^1 \times S^{m-1})\]
so 
\[\pi_1(M) = \text{free group on } k \text{ generators.}\]

Each \(r_i\) represents a loop in \(M\) which we can take as an embedding of \(S^1\) to \(M\) by Whitney Embedding Theorem. Both \(M\) and \(S^1\) are orientable so the normal bundle over \(S^1\) is also orientable hence trivial so by Tubular Neighbourhood Theorem we have an embedding \(S^1 \times D^{n-1} \hookrightarrow M\) and through surgery we kill each \(r_i\) and hence \(\pi_1(M') \cong G\). Now let us define bordism.

**Definition 23.** Bordism is a triple \((W_0, M_1, M_2)\) where \(W_0\) is an \((m+1)\)-dimensional compact differentiable manifold with boundary, \(M_1\) and \(M_2\) are closed \(m\)-manifolds, and \(i: M_1 \hookrightarrow \partial W_0\) and \(j: M_2 \hookrightarrow \partial W_0\) are embeddings such that \(\partial W_0 = i(M_1) \amalg j(M_2)\). If this is the case, then \(M_1\) and \(M_2\) are said to be bordant to each other.

We say that \((W_0, M_1, M_2)\) is an oriented bordism, if each manifold in the above triple is oriented and \(\partial W_0 = i(M_1) \amalg \overline{j(M_2)}\) where \(\overline{j(M_2)}\) denotes \(j(M_2)\) with the reversed orientation.

We can view the effect of an \(n\)-surgery on \(M\) as a manifold \(M'\) such that the elementary (meaning there is no additional structure on the manifolds such as orientation
etc.) bordism \((W^{m+1}, M, M')\) is obtained by attaching an \((n + 1)\)-handle to the trivial bordism \((M^m \times I, M, M)\). We can see this more clearly in
\[
W = (D^{n+1} \times D^{m-n}) \cup_{(S^n \times D^{m-n}) \times \{1\}} (M \times I)
\]
so that
\[
\partial W = (cl(M \setminus S^n \times D^{m-n}) \cup_{S^n \times S^{m-n-1}} (D^{n+1} \times S^{m-n-1})) \amalg M
\]
i.e. \(M'\) is the effect of an \(n\)-surgery on \(M\), and \(M\) is bordant to \(M'\).

Bordism is an equivalence relation on manifolds. This allows us to separate manifolds into bordism classes. Two \(m\)-dimensional manifolds \(M\) and \(M'\) are in the same bordism class if they are bordant to each other. The bordism class of \(M\) is denoted by \([M]\).

Similarly, we can define the oriented bordism class of \(M\), assuming \(M\) is oriented. An \(m\)-dimensional oriented manifold \(M'\) is in the same oriented bordism class with \(M\), if there exists an \((m + 1)\)-dimensional oriented manifold \(W\) with \(\partial W = M \amalg M'\).

We will use \([M]\) to denote the bordism class of \(M\) and whether it is the oriented bordism class or not will be specified in the text.

**Definition 24.** [16] The \(m\)-dimensional oriented bordism group is denoted by \(\Omega^\text{SO}_m\). Each element of \(\Omega^\text{SO}_m\) is the oriented bordism class \([M]\) of an \(m\)-dimensional oriented manifold \(M\) for each such \(M\).

Addition in \(\Omega^\text{SO}_m\) is given by \([M] + [M'] = [M \amalg M']\). Since \(M \amalg M'\) is always bordant to \(M \# M'\) we can also define the sum as \([M] + [M'] = [M \# M']\). The identity element in \(\Omega^\text{SO}_m\) is given by 0 = \([\emptyset]\). We can represent 0 by any bounding \(m\)-manifold so in particular the identity can also be represented by \(S^n\). Thus we have \([S^n] = [\emptyset]\). Reversing the orientations gives us the inverse in \(\Omega^\text{SO}_m\): we have \(-[M] = [M]\).

The oriented bordism ring \(\Omega^\text{SO}_*\) can be constructed by putting the bordism groups \(\Omega^\text{SO}_m\) together for all \(m \in \mathbb{N}\). We define the multiplication by \([M] \cdot [N] = [M \times N]\). The unit element of this ring is \([\text{pt}] \in \Omega^\text{SO}_0\). Like the oriented bordism group for oriented manifolds we can define other bordism groups for manifolds by using additional structures on them.
One such bordism group is the spin bordism group $\Omega_{m}^{\text{Spin}}$ for $m$-dimensional spin manifolds. It is the group of $m$-dimensional manifolds endowed with spin structures where two $m$-dimensional manifolds $M_1$ and $M_2$ are in the same spin bordism class if their disjoint union is the boundary of an $(m + 1)$-dimensional spin manifold $W^{m+1}$ and the spin structures on $M_1$ and $M_2$ are induced by the spin structure on $W$. The tables below give us the oriented bordism groups $\Omega_{m}^{\text{SO}}$ for $m \leq 11$ and spin bordism groups $\Omega_{m}^{\text{Spin}}$ for $m \leq 8$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_{m}^{\text{SO}}$</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
</tbody>
</table>

Likewise the table below shows the spin cobordism groups $\Omega_{m}^{\text{Spin}}$ for $m \leq 8$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Omega_{m}^{\text{Spin}}$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>0</td>
<td>$\mathbb{Z}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}$</td>
</tr>
</tbody>
</table>

We can also define bordism groups by considering manifolds with certain structures and with a reference map to a predetermined topological space. For example, we can define $\Omega_{m}^{\text{Spin}}(X)$, which is the singular bordism group of spin $m$-manifolds with a reference map to $X$, i.e., an element $(M, f)$ of $\Omega_{m}^{\text{Spin}}(X)$ is represented by an $m$-manifold $M^m$ endowed with a spin structure and a continuous map $f: M \to X$. Two elements $(M_1, f_1), (M_2, f_2)$ of $\Omega_{m}^{\text{Spin}}(X)$ are considered to be equivalent if and only if the disjoint union of $M_1$ and $M_2$ is the boundary of an $(m + 1)$-dimensional spin manifold $W^{m+1}$ with a reference map $F: W \to X$ such that $F|_{M_1} = f_1$ and $F|_{M_2} = f_2$ with the spin structures on $M_1$ and $M_2$ being induced by the spin structure on $W$.

Here by a spin manifold we mean an orientable manifold which admits a spin structure. Equivalently, a spin manifold $M$ of dimension $m$ is a manifold such that the tangent bundle over $M$ can be constructed as a pullback bundle $g^*\xi$ where $g: M \to B\text{Spin}(m)$ is a map from $M$ to the classifying space of the spin group $\text{Spin}(m)$ which is a Lie group i.e., it is a manifold which also has a group structure on it. In fact $B\text{O}(m), B\text{O}, \text{SO}(m), \text{SO}$ are all Lie groups. The space $\text{Spin}(m)$ is the double cover
of $\SO(m)$ such that there exists a short exact sequence

$$1 \to \mathbb{Z}_2 \to \Spin(m) \to \SO(m) \to 1$$

of Lie groups for $m > 2$.

For $m \geq 3$, $\Spin(m)$ is the universal cover of $\SO(m)$. The second Stiefel-Whitney class $w_2(\tau_M) \in H(M, \mathbb{Z}_2)$ is the obstruction of lifting the classifying map $g: M \to B\SO(m)$ to $\tilde{g}: M \to B\Spin(m)$.

As we discussed before, we can assign the first Stiefel-Whitney class $w_1$ to the tangent bundle $\tau_M$ or to the stable normal bundle $\nu_M$ over the same manifold $M$ since they basically give the same element $w_1(\tau_M) = w_1(\nu_M)$ of $H^1(M; \mathbb{Z}_2)$. Furthermore, if our manifold is orientable, then we can also say $w_2(\tau_M) = w_2(\nu_M)$.

One bordism group which has great importance for this thesis is the normal bordism group $\Omega_m(B; \xi)$ where $B$ is a topological space and $\xi$ is a stable oriented vector bundle over $B$. Shortly, we will see that if this $B$ sits atop BO as a fibration $p: B \to BO$ such that the generalized Gauss map $g: M \to BO$ can be lifted to $\tilde{g}: M \to B$ and $p \circ \tilde{g} = g$ then $B$ is actually what we call a $B$-structure on a manifold $M$. Elements of $\Omega_m(B; \xi)$ are represented by triples $(M, f, \alpha)$ where $M$ is a closed $m$-dimensional smooth manifold, $f: M \to B$ is a continuous map and $\alpha$ is an isomorphism between $f^*\xi$ and stable normal bundle $\nu_M$ over $M$. Such a triple is called a normal map in $(B, \xi)$. Two such triples $(M_1, f_1, \alpha_1), (M_2, f_2, \alpha_2)$ are in the same bordism class if there exists an $(m + 1)$-dimensional smooth manifold $W$ such that $\partial W = M_1 \amalg M_2$,

$$f_1 \amalg f_2: M_1 \amalg M_2 \to B$$
can be extended to a map
\[ F: W \to B \]
and
\[ \alpha_1 \amalg \alpha_2: \nu_{M_1} \amalg \nu_{M_2} \to f_1^* \xi_1 \amalg f_2^* \xi_2 \]
can be extended to an isomorphism
\[ \beta: \nu_W \to F^* \xi_0 \]
where \( \xi_0 \) is the stable vector bundle over \( W \) which induces \( \xi_1 \) and \( \xi_2 \) over \( M_1 \) and \( M_2 \) respectively.

In general, structures such as an orientation or a spin structure on a manifold \( M \) can be understood as a \( B \)-structure on \( M \). In order to see what constitutes a \( B \)-structure on a manifold \( M \) we require some key definitions.

**Definition 25.** Let \( p_n: B_n \to \text{BO}(n) \) be a fixed fibration and let \( f: M \to \text{BO}(n) \) be the classifying map of an \( n \)-plane vector bundle \( \xi \). A \( B_n \)-structure on \( \xi \) is a homotopy class of maps \( \tilde{f}: M \to B_n \) such that \( p_n \circ \tilde{f} = f \). Thus we have the following commutative diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & \text{BO}(n) \\
\downarrow{p_n} & & \\
B_n & \xleftarrow{\tilde{f}} & \\
\end{array}
\]

The following lemma gives us a correspondence between the \( B_n \)-structures on two normal bundles over the same base space.

**Lemma 2.1.14.** [17] For \( n \) sufficiently large depending on \( m \), there is a \( 1 - 1 \) correspondence between the set of \( B_n \)-structures of the normal bundles of any two embeddings \( i_0, i_1: M^m \hookrightarrow \mathbb{R}^{m+n} \).

Since a \( B_n \)-structure is a homotopy class of maps, this lemma basically says that for \( n \) sufficiently large and letting \( i_0, i_1: M^m \hookrightarrow \mathbb{R}^{m+n} \) be two embeddings and \( A_0, A_1 \) be the sets of \( B_n \)-structures of the normal bundles of \( i_0, i_1 \) respectively, there exists a homotopy \( H_j: M \times [0, 1] \to B_n \) with \( H_j(x, 0) = f_j(x) \) and \( H_j(x, 1) = g_j(x) \) where \( [f_j] \in A_0, [g_j] \in A_1 \) and \( j \in \{ \text{number of } B_n \text{ structures on } M \} \). Here \( A_0 \) and \( A_1 \) are ordered in such a way that the homotopic classes of maps such as \( [f_j] \) and \( [g_j] \) have
the same index. Hence for $n$ sufficiently large the $B_n$-structure does not depend on any particular choice of an embedding. This lemma coupled with the fibered stable vector bundle which we define now allows us to define stable vector bundle as in [10].

**Definition 26.** [17] A fibered stable vector bundle is a triple $B = (B_n, p_n, \tilde{f}_n)$ consisting of the following data: a sequence of fibrations $p_n: B_n \to \text{BO}(n)$ together with a sequence of maps $\tilde{f}_n: B_n \to B_{n+1}$ such that we have the following commutative diagram

\[
\begin{array}{ccccccc}
B_n & \xrightarrow{\tilde{f}_n} & B_{n+1} & \xrightarrow{\tilde{f}_{n+1}} & \cdots & \to & B \\
p_n & & \downarrow{p_{n+1}} & & \downarrow{p} & & \\
\text{BO}(n) & \xrightarrow{j_n} & \text{BO}(n+1) & \xrightarrow{j_{n+1}} & \cdots & \to & \text{BO}
\end{array}
\]

where $j_n$ is the standard inclusion. We let $B = \lim_{n \to \infty} (B_n)$.

Note that, this diagram actually gives rise to a sequence of inclusions

\[
B_n \xrightarrow{\tilde{f}_n} B_{n+1} \xrightarrow{\tilde{f}_{n+1}} B_{n+2} \xrightarrow{\tilde{f}_{n+2}} \cdots
\]

with bundle maps $(\tilde{f}_n, j_n): (B_n, E_n) \to (B_{n+1}, E_n \oplus \epsilon)$ where $\epsilon$ is the trivial line bundle over $B_{n+1}$, $E_n$ is the total space of the vector bundle classified by the map $p_n: B_n \to \text{BO}(n)$ and each vector bundle $\xi_n$ is of dimension $n$. Such a system

\[
B_n \to \text{BO}(n) \to \text{BO}(n+1) \to \cdots \to \text{BO}
\]

is called a stable vector bundle over $B_n$.

Now, a $B_n$-structure on the normal bundle of an embedding $i: M \to \mathbb{R}^{m+n}$ defines a unique $B_{n+1}$-structure on the normal bundle of the composition of $i$ with the standard inclusion $\mathbb{R}^{m+n} \hookrightarrow \mathbb{R}^{m+n+1}$.

\[
\begin{array}{ccccccc}
B_n & \xrightarrow{\tilde{f}_n} & B_{n+1} & \xrightarrow{\tilde{f}_{n+1}} & \cdots & \to & B \\
p_n & & \downarrow{p_{n+1}} & & \downarrow{p} & & \\
M & \xrightarrow{j_n} & \text{BO}(n) & \xrightarrow{j_{n+1}} & \text{BO}(n+1) & \xrightarrow{j_{n+1}} & \cdots & \to & \text{BO}
\end{array}
\]

Hence we have the following definition:

**Definition 27.** [17] Let $B$ be a fibered stable vector bundle. A $B$-structure on a manifold $M$ is an equivalence class of sequences of $B_n$-structures on the normal
bundle of $M$ where the sequence is given in the diagram above and two such structures are equivalent if they become equivalent for $n$ sufficiently large. A $B$-manifold is a pair $(M, \tilde{g})$ with $M$ a compact manifold and $\tilde{g}$ a $B$-structure on $M$.

In summary, a normal $B$-structure on a manifold $M$ can be defined as the generalized Gauss map $g: M \to BO$, the map where the pullback bundle $g^*\gamma$ classifies the stable normal bundle over $M$, having a lift $\tilde{g}: M \to B$. This gives us a connection between a $B$-structure on a manifold $M$ and a concept that we introduce now that is called the normal $k$-smoothing. So, as our next step let us define the normal $k$-smoothing and the normal $k$-type.

**Definition 28.** Let $M$ be a smooth manifold and $B \to BO$ be a fibration.

(i) A normal $k$-smoothing of a manifold $M$ is a lift $\tilde{g}: M \to B$ of the generalized Gauss map $g: M \to BO$ such that $\pi_i(\tilde{g}) = 0$ for $i \leq k + 1$, that is $\tilde{g}$ induces isomorphisms between $i$-th homotopy groups of $M$ and $B$ for $i \leq k$ and an epimorphism for $i = k + 1$. Note that a normal $k$-smoothing of a manifold $M$ is $(k + 1)$-connected.

(ii) The normal $k$-type of a manifold $M$ is the fiber homotopy type of the fibration $p: B \to BO$ such that $\pi_i(p) = 0$ for $i \geq k + 2$, admitting a normal $k$-smoothing of $\tilde{g}: M \to B$ that is a lift of $g: M \to BO$.

In this definition by fiber homotopy type we mean the class of fibrations over $BO$ such that between the total spaces of these fibrations there are maps which we call fiber homotopy equivalences. To understand this more thoroughly, consider the following commutative diagram of two fibrations over the same base space $BO$.

```
B^1 \xrightarrow{f} B^2
\downarrow{p} \quad \downarrow{q}
BO
```

This diagram is called a map of spaces over $BO$. If $f$ is a fiber homotopy equivalence, then for any $b \in BO$ the restriction $f: p^{-1}(b) \to q^{-1}(b)$ is a homotopy equivalence. This gives us a notion of a homotopy equivalence over $BO$ which is called a fiber homotopy equivalence. The naming comes from the following proposition.
Let \( p : B^1 \to BO \) and \( q : B^2 \to BO \) be fibrations and let \( f : B^1 \to B^2 \) be a map such that \( q \circ f = p \). If \( f \) is a homotopy equivalence then \( f \) is also a fiber homotopy equivalence.

As to why \( \pi_i(\tilde{g}) = 0 \) for \( i \leq k + 1 \) implies \( \pi_i(M) \cong \pi_i(B) \) for \( i < k + 1 \) and an epimorphism \( \tilde{g}_* : \pi_{k+1}(M) \to \pi_{k+1}(B) \), see the diagram below.

\[
\cdots \longrightarrow \pi_{k+2}(\tilde{g}) \longrightarrow \pi_{k+1}(M) \longrightarrow \pi_{k+1}(B) \longrightarrow \pi_{k+1}(\tilde{g}) \\
\cdots \longleftarrow \pi_k(\tilde{g}) \longleftarrow \pi_k(B) \longleftarrow \pi_k(M)
\]

Since this sequence is exact and \( \pi_i(\tilde{g}) = 0 \) for \( i \leq k + 1 \) it implies \( \pi_i(M) \cong \pi_i(B) \) for \( i < k + 1 \) and

\[\tilde{g}_* : \pi_{k+1}(M) \to \pi_{k+1}(B)\]

is an epimorphism where \( \tilde{g}_* \) is the map between homotopy groups of \( M \) and \( B \) induced by \( \tilde{g} \).

Likewise for the second part of the definition we have the following long exact sequence

\[
\cdots \longrightarrow \pi_{k+3}(p) \longrightarrow \pi_{k+2}(B) \longrightarrow \pi_{k+2}(BO) \longrightarrow \pi_{k+2}(p) \\
\cdots \longleftarrow \pi_{k+1}(p) \longleftarrow \pi_{k+1}(B) \longleftarrow \pi_{k+1}(BO)
\]

where \( \pi_i(p) = 0 \) for \( i \geq k + 2 \) which implies \( \pi_i(B) \cong \pi_i(BO) \) for \( i \geq k + 2 \) and

\[p_* : \pi_{k+1}(B) \to \pi_{k+1}(BO)\]

is a monomorphism.

The following theorem allows us to classify 6-dimensional manifolds, whose fundamental groups are isomorphic to \( \mathbb{Z} \oplus \mathbb{Z} \) and second homotopy groups are trivial, based on their 2-type which will be useful once we get to the proof our main theorem.

**Theorem 2.1.16.** \([10]\) Suppose that \( M_1 \) and \( M_2 \) are two closed \( 2k \)-dimensional manifolds with the same normal \((k-1)\)-type admitting bordant normal \((k-1)\)-smoothings. Then \( M_1 \) and \( M_2 \) are stably diffeomorphic.

Now, let us define what it means for a fibration \( p : B \to BO \) to be \( k \)-universal.
Definition 29. [9] Let \( p: B \to \text{BO} \) be a fibration. We say that this fibration is \( k \)-universal if the fibre \( F \) of the map \( p \) is connected and its homotopy groups vanish in dimension \( \geq k + 1 \).

The fibration in this definition gives us the following homotopy long exact sequence

\[
\cdots \to \pi_{k+2}(F) \to \pi_{k+2}(B) \to \pi_{k+2}(\text{BO}) \to \pi_{k+1}(F) \to \cdots
\]

where \( \pi_i(F) = 0 \) for \( i \geq k + 1 \) which implies \( \pi_i(B) \cong \pi_i(\text{BO}) \) for \( i \geq k + 2 \) and \( p_*: \pi_{k+1}(B) \to \pi_{k+1}(\text{BO}) \) is a monomorphism.

At this point, it is clear that the fibration \( p: B \to \text{BO} \) under consideration for the normal \( k \)-type of a manifold \( M \) must be \( k \)-universal in addition to admitting a normal \( k \)-smoothing of \( g: M \to \text{BO} \).

It is possible to talk about the normal \( k \)-type of any manifold \( M \) since there exists a \( k \)-universal fibration \( B^k \to \text{BO} \) admitting a normal \( k \)-smoothing of \( M \) by the theory of Moore-Postnikov decompositions which can be seen in detail in [11]. By obstruction theory if \( B \) and \( B' \) are both \( k \)-universal and admit a normal \( k \)-smoothing of the same manifold \( M \), then the two fibrations are fiber homotopy equivalent i.e., they are of the same fiber homotopy type which ensures the uniqueness of the normal \( k \)-type of \( M \). So the fibre homotopy type of the fibration \( B^k \to \text{BO} \) is an invariant of the manifold \( M \) and we call it the normal \( k \)-type of \( M \) denoted \( B^k(M) \).

As an example which will be useful later on, let us consider the normal 2-type of a manifold \( M \): The generalized Gauss map \( g: M \to \text{BO} \) has a lift \( \tilde{g}: M \to B \) such that we have the following commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\tilde{g}} & B \\
\downarrow{g} & & \downarrow{p} \\
& \text{BO} &
\end{array}
\]

where \( \pi_i(p) = 0 \) for \( i \geq 4 \) and \( \pi_i(\tilde{g}) = 0 \) for \( i \leq 3 \). So \( p_*: \pi_i(B) \cong \pi_i(\text{BO}) \) for \( i \geq 4 \),

\[p_*: \pi_3(B) \to \pi_3(\text{BO})\]
is a monomorphism, $\tilde{g}_*: \pi_i(M) \cong \pi_i(B)$ for $i < 3$ and
\[
\tilde{g}_*: \pi_3(M) \to \pi_3(B)
\]
is an epimorphism.

If $W$ is a compact manifold with boundary $\partial W$ then we can restrict a $B$-structure on $W$ to a $B$-structure on $\partial W$ by choosing the inward pointing normal vector along $\partial W$.

In particular, if $(M, \tilde{g})$ is a closed $B$-manifold where $\tilde{g}: M \to B$ is the map which characterizes the normal bundle over $M$, then there is a canonical $B$-structure $\tilde{g}$ on $W = M \times [0, 1]$ which restricts to $(M, \tilde{g})$ on $M \times \{0\}$. The restriction of this $B$-structure on $W$ to $M \times \{1\}$ is denoted by $-\tilde{g}$. Hence by construction $(M \amalg M, \tilde{g} \amalg -\tilde{g})$ is the boundary of $(M \times [0, 1], \tilde{g})$.

The following definition is a generalization of oriented cobordism and spin cobordism.

**Definition 30.** [17] Two closed $B$-manifolds $(M_0, \tilde{g}_0)$ and $(M_1, \tilde{g}_1)$ are said to be $B$-bordant if there exists a compact $B$-manifold $(W, \tilde{g})$ such that $\partial(W, \tilde{g}) = (M_0 \amalg M_1, \tilde{g}_0 \amalg -\tilde{g}_1)$.

As said before, we denote the class of $B$-bordant manifolds by $[M]$ while referring to the $B$-structure in the text except a few instances where the notation is important in a specific context in which case we use $[M, \tilde{g}]$.

The following proposition generalizes the group operation, disjoint union, of the usual bordism group to all $B$-bordism groups $\Omega^B_m$.

**Proposition 2.1.17.** [17] The set of $B$-bordism classes of closed $m$-manifolds with $B$-structure,
\[
\Omega^B_m = \{[M, \tilde{g}]\}
\]
forms an abelian group under the operation of disjoint union with the inverse given by $-[M, \tilde{g}] = [M, -\tilde{g}]$ where $\tilde{g}$ represents the $B$-structure on $M$.

Another important characteristic class of a real vector bundle over a closed, connected, orientable 4-manifold $M$ is its Pontryagin class.
\[
p_1(\tau_M) \in H^4(M; \mathbb{Z}) \cong \mathbb{Z}.
\]
The isomorphism above follows from Poincare Duality (PD) and the connectedness of \( M \)

\[
H^4(M; \mathbb{Z}) \cong H_0(M; \mathbb{Z}) \cong \mathbb{Z}.
\]

In order to understand the Pontryagin class thoroughly, we first need to look at the intersection form and the signature of a manifold.

**Definition 31.** [16] Given any closed oriented 4-manifold \( M \), its intersection form is the following composition

\[
Q_M: H^2(M; \mathbb{Z}) \times H^2(M; \mathbb{Z}) \xrightarrow{\sim} H^4(M; \mathbb{Z}) \xrightarrow{PD} H_0(M; \mathbb{Z}) \cong \mathbb{Z}
\]

\[
Q_M(\alpha, \beta) = (\alpha \smile \beta)[M] .
\]

This form is bilinear and it is represented by a matrix of determinant \( \pm 1 \). If \( Q_M(\alpha, \alpha) \) is even for all classes \( \alpha \), then the form is said to be even, otherwise it is odd.

Signature is an invariant of an intersection form and it can be obtained as follows: We first diagonalize \( Q_M \) as a matrix over \( \mathbb{R} \) (or \( \mathbb{Q} \)), separate the resulting positive and negative eigenvalues, then subtract their counts; that is

\[
\text{sign } Q_M = \text{number of positive eigenvalues} - \text{number of negative eigenvalues}.
\]

An oriented 4-manifold \( M^4 \) has zero signature if and only if it is the boundary of an oriented 5-manifold \( W^5 \). Because signature is additive over disjoint union of orientable 4-manifolds and reversing the orientation of a manifold reverses the sign of its signature ([16, p. 120]), we can say that two oriented 4-manifolds \( M_1 \) and \( M_2 \) are in the same cobordism class in \( \Omega_4^{SO} \) if and only if they have the same signature. This relation is more clear below:

\[
\text{sign } Q_{M_1} = \text{sign } Q_{M_2} = b \iff \text{sign } Q_{M_1 \sqcup M_2} = \text{sign } Q_{M_1} + \text{sign } Q_{M_2} = b - b = 0.
\]

\[
\iff M_1 \sqcup M_2 = \partial W \text{ for some oriented 5-manifold } W .
\]

\[
\iff M_1, M_2 \in [M_1] \in \Omega_4^{SO} .
\]

From here we can easily see that if we have \( \partial W = M \sqcup \emptyset \) for an oriented 4-manifold \( M \) and an oriented 5-manifold \( W \) then \( \text{sign } Q_M = 0 \) since \( \text{sign } Q_{\emptyset} = 0 \). So \( \partial W = M \sqcup \emptyset = M \) if and only if \( \text{sign } Q_M = 0 \).
It follows that

$$\Omega_{SO}^4 \cong \mathbb{Z},$$

with the generator complex projective plane $\mathbb{CP}^2$ and the isomorphism given by

$$[M] \mapsto \text{sign } Q_M.$$

Another special case of a $B$-bordism group is

$$\Omega_{\text{Spin}}^4 \cong \mathbb{Z}$$

with the $K3$ surface as the generator and the isomorphism given by

$$[M] \mapsto \frac{1}{16} \cdot \text{sign } Q_M$$

which is always an integer by Rokhlin’s Theorem which we state now:

**Theorem 2.1.18.** [16] If $M^4$ is smooth, orientable manifold and it has $w_2(\tau_M) = 0$, then its intersection form must have

$$\text{sign } Q_M = 0 \pmod{16}.$$

There is also a relation between Pontryagin class of a tangent bundle over a closed 4-manifold $M$ and the signature of its intersection form which is given by Hirzebruch’s Signature Theorem:

**Theorem 2.1.19.** [16] For every closed, oriented 4-manifold $M$ we have

$$p_1(\tau_M) \cong 3 \cdot \text{sign } Q_M$$

where the isomorphism is given by the Pontryagin class evaluated on the fundamental class $[M] \in H_4(M; \mathbb{Z})$, i.e.,

$$\langle p_1(\tau_M), [M] \rangle = 3 \cdot \text{sign } Q_M.$$

This theorem actually gives the relation between signature and the Pontryagin number which we define now.

**Definition 32.** Let $M^{4n}$ be a smooth, compact, oriented manifold of dimension $4n$ with the fundamental class $[M]$ and let $I = i_1, i_2, \ldots, i_r$ be a partition of $n$. Then the $I$th Pontryagin number is an integer defined as

$$p_I[M^{4n}] = \langle p_{i_1}(\tau_M) \smile p_{i_2}(\tau_M) \smile \cdots \smile p_{i_r}(\tau_M), [M] \rangle.$$
Since there is only one possible partition of the integer 1, for a smooth, closed, oriented 4-manifold \( M^4 \) there is only one Pontryagin number, namely \( \langle p_1(\tau_M), [M] \rangle \), which by Hirzebruch Signature Theorem is equal to \( 3 \cdot \text{sign} Q_M \). Since signature for such manifolds is an oriented cobordism invariant, Pontryagin number is also an oriented cobordism invariant and an oriented 4-manifold \( M^4 \) has zero Pontryagin number if and only if it is the boundary of an oriented 5-manifold \( W^5 \).

Now we are finally ready to tackle the proof of our main theorem.
Let us start this section by restating the main theorem which we will be proving.

**Theorem 3.0.1.** \([10]\) *Stable Classification of Certain Six-Dimensional Manifolds*

Two smooth 6-dimensional closed orientable manifolds \(M_1\) and \(M_2\) with \(\pi_1(M_1) \cong \pi_1(M_2) \cong \mathbb{Z} \oplus \mathbb{Z}\) and \(\pi_2(M_1) = \pi_2(M_2) = 0\) are stably diffeomorphic if and only if

1. in both cases \(w_2\) vanishes or it does no vanish,
2. \(\pm \langle x_1 \circ p_1(\tau_{M_1}), [M_1] \rangle = \pm \langle x_2 \circ p_1(\tau_{M_2}), [M_2] \rangle\).

Before we tackle the proof of this theorem let us look at an example of the Leray-Serre spectral sequence which will be useful for us in the proof of the main theorem. Consider the fibration

\[
\begin{array}{ccc}
\tilde{M} & \longrightarrow & M \\
\downarrow^{f} & & \downarrow \\
K(\pi_1(M), 1) & & 
\end{array}
\]

and let us compute \(H_2(M; \mathbb{Z})\) using the corresponding Leray-Serre spectral sequence. Let \(M\) be a closed, orientable manifold of dimension 6 such that \(\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}\), \(\pi_2(M)\) is trivial, and let \(\tilde{M}\) be the universal covering of \(M\). Note that for such a manifold \(M\), we have \(K(\pi_1(M), 1) = T^2\). The terms on the second page are

\[
E^2_{p,q} = H_p(T^2; H_q(\tilde{M}; \mathbb{Z})).
\]

We know that \(H_1(\tilde{M}; \mathbb{Z}) = H_2(\tilde{M}; \mathbb{Z}) = 0\) by Hurewicz Theorem since \(\pi_1(\tilde{M}) = \pi_2(\tilde{M}) = 0\) so the terms on the second page should be seen as follows:
Observe that, this spectral sequence stabilizes on the second page since each differential
\[ d^2_{p,q} : H_p(T^2; H_q(\overline{M}; \mathbb{Z})) \to H_{p-2}(T^2; H_{q+1}(\overline{M}; \mathbb{Z})) \]
is 0, so we have
\[ E_{p,q}^\infty = \ldots = E_{p,q}^4 = E_{p,q}^3 = \frac{\ker(d^2_{p,q})}{\text{im}(d^2_{p+2,q-1})} = E_{p,q}^2 \]

hence we have the filtration
\[ 0 \subset \overline{M}_2^0 \subset \overline{M}_2^1 \subset \overline{M}_2^2 = H_2(M) \]
where
\[ \frac{\overline{M}_2^p}{\overline{M}_2^{p-1}} \cong E_{p,2-p}^\infty \]
so
\[ E_{0,2}^\infty \cong 0 \cong \frac{\overline{M}_2^0}{\overline{M}_2^{-1}} \implies \overline{M}_2^0 \cong 0. \]

Likewise
\[ E_{1,1}^\infty \cong 0 \cong \frac{\overline{M}_2^1}{\overline{M}_2^0} \cong \frac{\overline{M}_2^1}{0} \implies \overline{M}_2^1 \cong 0 \]
and
\[ E_{2,0}^\infty \cong \mathbb{Z} \cong \frac{\overline{M}_2^2}{\overline{M}_2^1} \cong \frac{\overline{M}_2^2}{0} \implies \overline{M}_2^2 \cong \mathbb{Z} \]

hence
\[ H_2(M; \mathbb{Z}) \cong \overline{M}_2^2 \cong \mathbb{Z} \]
By using Universal Coefficient Theorem
\[ H^2(M; \mathbb{Z}) = \text{Hom}_\mathbb{Z}(H_2(M; \mathbb{Z}); \mathbb{Z}) \oplus \text{Ext}_\mathbb{Z}(H_1(M; \mathbb{Z}); \mathbb{Z}) \]
where \( H_1(M; \mathbb{Z}) = \pi_1(M) = \mathbb{Z} \oplus \mathbb{Z} \) so there is no torsion element hence
\[ H^2(M; \mathbb{Z}) = \text{Hom}_\mathbb{Z}(H_2(M; \mathbb{Z}); \mathbb{Z}) = \mathbb{Z}. \]

We also know that the fourth homology and cohomology groups of \( M \) are
\[ H^2(M; \mathbb{Z}) \xrightarrow{\cong} H_4(M; \mathbb{Z}) \cong \mathbb{Z} \]
\[ \mathbb{Z} \cong H^4(M; \mathbb{Z}) \xrightarrow{\cong} H_2(M; \mathbb{Z}) \]
by Poincare Duality.

Now that we know the second and fourth homology and cohomology of such a manifold we are good to go. Let us start our proof.

**Proof.** Let \( M \) be a smooth 6-dimensional orientable manifold with \( \pi_1(M) = \mathbb{Z} \oplus \mathbb{Z} \) and trivial \( \pi_2(M) \). Our first step is to determine the normal 2-type of \( M \). Since \( M \) is orientable we have the first Stiefel-Whitney class \( w_1(\nu_M) = w_1(\tau_M) = 0 \) so we can lift the generalized Gauss map \( g: M \to BO \) to a map \( \tilde{g}: M \to BSO \) which now classifies the orientable stable normal bundle \( \nu_M \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
M & \xrightarrow{g} & BO \\
\downarrow \tilde{g} & & \downarrow p_{BSO} \\
BSO
\end{array}
\]

The normal 2-type of \( M \) is the fiber homotopy type of the fibration \( p_B: B \to BSO \) in the following commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\tilde{g}} & B \\
\downarrow g & & \downarrow p_B \\
BSO
\end{array}
\]

such that
\[ p_{B*}: \pi_i(B) \xrightarrow{\cong} \pi_i(BSO) \]
for \( i \geq 4, p_{B*}: \pi_3(B) \to \pi_3(BO) \) is a monomorphism,
\[ \tilde{g}_*: \pi_i(M) \xrightarrow{\cong} \pi_i(B) \]
for \( i < 3 \) and \( \bar{g}_*: \pi_3(M) \to \pi_3(B) \) is an epimorphism.

The normal 2-type of \( M \) depends on the second Stiefel-Whitney class of \( \nu_M \), which determines whether or not \( \tilde{g} \) has a lift to BSpin. For our first case let \( M \) be a spin manifold, that is \( w_2(\nu_M) = w_2(\tau_M) = 0 \). Then we can lift the map \( \tilde{g}: M \to BSO \) to a map \( \bar{g}: M \to BSpin \).

Let \( B = T^2 \times BSpin \). Here \( T^2 = S^1 \times S^1 \) comes from the second stage of the Postnikov tower of \( M \), that is \( \pi_1(T^2) \) and \( \pi_2(T^2) \) are isomorphic to \( \pi_1(M) \) and \( \pi_2(M) \), and higher homotopy groups of \( T^2 \) are trivial (for more on Postnikov towers see \cite{6} page 410).

Also, recall that BSpin is the classifying space of the group Spin, which is the universal cover of SO. The fibration

\[
p_B: T^2 \times BSpin \to BSO
\]

is simply the following composition

\[
p_B: T^2 \times BSpin \xrightarrow{pr_2} BSpin \xrightarrow{p} BSO
\]

where \( pr_2 \) is the projection to the second component and the fibration

\[
p: BSpin \to BSO
\]

is induced by the homomorphism Spin \( \to \) SO. Now, we have the following commutative diagram

\[
\begin{array}{ccc}
T^2 \times BSpin & \xrightarrow{\bar{g}} & BSO \\
\downarrow{p_B} & & \downarrow{p} \\
M & \xrightarrow{\bar{g}} & BSO
\end{array}
\]

where \( \bar{g} = f \times \tilde{g} \) with \( f: M \to T^2 \) being the classifying map of the universal covering which induces an isomorphism on fundamental groups.

Observe that,

\[
p_{B*}: \pi_i(T^2 \times BSpin) \cong \pi_i(BSO)
\]

for \( i \geq 4 \) since BSpin is the universal cover of BSO and \( \pi_i(T^2) \) is trivial for \( i \geq 3 \). In addition to that since

\[
\pi_3(BSpin) \cong \pi_3(BSO) \cong \pi_2(SO) = 0
\]
and $\pi_3(T^2) = 0$ we have $\pi_3(T^2 \times \text{BSpin}) = 0$ so the map

$$p_{B*}: \pi_3(T^2 \times \text{BSpin}) \to \pi_3(\text{BSO})$$

is a monomorphism. On the other hand we have

$$\bar{g}_*: \pi_i(M) \xrightarrow{\cong} \pi_i(T^2 \times \text{BSpin})$$

for $i < 3$ and since $\pi_3(T^2 \times \text{BSpin}) = 0$ the map

$$\bar{g}_*: \pi_3(M) \to \pi_3(T^2 \times \text{BSpin})$$

is an epimorphism. Therefore, if $M$ is a spin manifold the fibration $B \to \text{BSO}$ is the normal 2-type of $M$. More than that, $M$ actually represents a bordism class $[M] \in \Omega^\text{Spin}_6(T^2)$. Note that, since we have

$$\bar{g} = f \times \bar{g}: M \to T^2 \times \text{BSpin}$$

we can view the map $f: M \to T^2$ as the reference map from $M$ to $T^2$ so

$$\Omega^{T^2 \times \text{BSpin}}_p \cong \Omega^\text{Spin}_p(T^2).$$

Now, we can use the Atiyah-Hirzebruch spectral sequence to compute $\Omega^\text{Spin}_6(T^2)$. We have the following fibration

$$\text{pt} \hookrightarrow T^2 \to T^2$$

which allows us to make the approximation

$$H_p(T^2; \Omega^\text{Spin}_q(\text{pt})) \Longrightarrow \Omega^\text{Spin}_p(T^2).$$

For $p + q = 6$, the terms $E^2_{p,q} = H_p(T^2; \Omega^\text{Spin}_q)$ on the second page are as follows:

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
6 & H_0(T^2; \Omega^\text{Spin}_6(\text{pt})) & H_1(T^2; \Omega^\text{Spin}_6(\text{pt})) & H_2(T^2; \Omega^\text{Spin}_6(\text{pt})) & \cdots \\
5 & H_0(T^2; \Omega^\text{Spin}_5(\text{pt})) & H_1(T^2; \Omega^\text{Spin}_5(\text{pt})) & H_2(T^2; \Omega^\text{Spin}_5(\text{pt})) & \cdots \\
\vdots & \vdots & \vdots & \vdots \\
0 & 1 & 2 & \cdots \\
\end{array}
\]
The differentials on the second page are the duals of Steenrod squares which are denoted by $Sq^2$ for $q = 1$ or this composed with reduction mod 2 for $q = 0$. For more information on Steenrod powers the reader should see [6, p. 487]. This sequence stabilizes on the second page and the only nonzero entry on the line $p + q = 6$ is $H_2(T^2; \Omega_4^{Spin}(pt)) \cong \mathbb{Z}$ so we have

$0 \subset F_6^0 \subset F_6^1 \subset F_6^2 \subset \ldots \subset F_6^6 = \Omega_6^{Spin}(T^2)$

hence $\Omega_6^{Spin}(T^2) \cong \mathbb{Z}$. This group is generated by $[(K3 \times T^2, \text{pr}_2)]$ where our reference map $\text{pr}_2$ is the projection onto the second component. Recall that, $\Omega_4^{Spin}(pt) \cong \mathbb{Z}$ and $K3$ surface is the generator of this group. The isomorphism

$$\Omega_6^{Spin}(T^2) \xrightarrow{\cong} H_2(T^2; \Omega_4^{Spin}) \cong \mathbb{Z}$$

is given by

$$[(M, f)] \mapsto \langle x \prec p_1(\tau_M), [M] \rangle$$

where $x$ is chosen as a generator of $H^2(M; \mathbb{Z}) \cong \mathbb{Z}$. When we take the cup product of the image of a generator $e$ of $H^2(T^2; \mathbb{Z}) \cong \mathbb{Z}$ under the induced map

$$f^*: H^2(T^2; \mathbb{Z}) \xrightarrow{\cong} H^2(M; \mathbb{Z})$$

and the first Pontryagin class $p_1(\tau_M) \in H^4(M; \mathbb{Z})$ of $M$ we get an element of $H^6(M; \mathbb{Z})$. This element then eats the fundamental class $[M] \in H_6(M; \mathbb{Z})$ of $M$ and gives us an integer

$$\langle x \prec p_1(\tau_M), [M] \rangle = k.$$ 

For each bordism class in $\Omega_6^{Spin}(T^2)$ we get a unique integer $k$ thanks to the isomorphism

$$\Omega_6^{Spin}(T^2) \xrightarrow{\cong} H_2(T^2; \Omega_4^{Spin}) \cong \mathbb{Z}$$

so the number $k$ is a bordism invariant. If we have

$$[(N, h)] \mapsto k$$

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for an arbitrary class \([(N, h)] \in \Omega^\text{Spin}_6(T^2)\) then this means that \((N, h)\) is in the same bordism class with

\[(K3 \times T^2 \# \ldots \# K3 \times T^2, p_2 \# \ldots \# p_2)\]

where we take the connected sum of \(k\)-many \(K3 \times T^2\) and \(k\)-many projection maps \(pr_2\) respectively.

Now let \(M_1\) and \(M_2\) be two manifolds in \(\Omega^\text{Spin}_6(T^2)\) with

\[
\pi_1(M_1) = \pi_1(M_2) = \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad \pi_2(M_1) = \pi_2(M_2) = \{1\}.
\]

We know for each manifold we can construct the following commutative diagrams

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\bar{g}_1} & T^2 \times \text{BSpin} \\
\downarrow & & \downarrow \text{pr}_B \\
\text{BSO} & \xrightarrow{g_1} & \text{BSO} \\
\end{array}
\quad \quad
\begin{array}{ccc}
M_2 & \xrightarrow{\bar{g}_2} & T^2 \times \text{BSpin} \\
\downarrow & & \downarrow \text{pr}_B \\
\text{BSO} & \xrightarrow{g_2} & \text{BSO} \\
\end{array}
\]

where we have

\[
\bar{g}_1 = f_1 \times \bar{g}_1 : M \rightarrow T^2 \times \text{BSpin} \quad \text{and} \quad \bar{g}_2 = f_2 \times \bar{g}_2 : M \rightarrow T^2 \times \text{BSpin}
\]

such that \(f_i\) is the classifying map of the universal covering of \(M_i\) and \(\bar{g}_i\) is the lift of the map \(\bar{g}_i : M_i \rightarrow \text{BSO}\).

So, \(M_1\) and \(M_2\) have the same normal 2-type admitting bordant normal 2-smoothings.

We also know that

\[
\pm \langle x_1 \sim p_1(\tau_{M_1}), [M_1] \rangle = \pm \langle x_2 \sim p_1(\tau_{M_2}), [M_2] \rangle
\]

by assumption so \(M_1\) and \(M_2\) are bordant since this is a bordism invariant. Hence by (2.1.16) our manifolds are stably diffeomorphic.

Now, let us consider the second case in which \(w_2(\nu_M)\) is not trivial. Because of this we cannot lift the classifying map \(\bar{g} : M \rightarrow \text{BSO}\) to a map \(M \rightarrow \text{BSpin}\). So for this case, a modification is needed.

First, we choose an oriented vector bundle \(\xi\) over \(T^2\) such that \(f^* w_2(\xi) = w_2(\nu_M)\) where the map

\[
f^* : H^2(T^2; \mathbb{Z}_2) \rightarrow H^2(M; \mathbb{Z}_2)
\]
is induced by $f : M \rightarrow T^2$. Here the significance of $T^2$ is due to the fact that it sits on the second stage of the Postnikov tower of $M$ so

$$f_* : \pi_i(M) \rightarrow \pi_i(T^2)$$

is an isomorphism for $i < 3$ and it is an epimorphism for $i = 3$ since $\pi_3(T^2)$ is trivial. We can represent $w_2(\bar{\gamma}) \in H^2(\text{BSO}; \mathbb{Z}_2)$ as a map

$$\text{BSO} \xrightarrow{w_2(\bar{\gamma})} K(\mathbb{Z}/2, 2)$$

as in [6, Theorem 4.57] where $\bar{\gamma}$ is the universal oriented vector bundle over $\text{BSO}$ and likewise we use

$$T^2 \xrightarrow{w_2} K(\mathbb{Z}/2, 2)$$

to represent $w_2 = w_2(\xi)$. We have the following commutative diagram

$$
\begin{array}{ccc}
\text{BSpin} & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) \\
\text{id} & & \\
\text{BSpin} & \xrightarrow{j} & T^2 \\
\end{array}
$$

as in [4, p. 157] where the map $w_2(\bar{\gamma})$ pulls back the second Stiefel-Whitney class for the universal oriented vector bundle $\bar{\gamma}$ over $\text{BSO}$.

Here we claim that $T^2\langle w_2 \rangle$ is the normal $2$-type of $M$. To see this consider

$$
\begin{array}{ccc}
\text{BSpin} & \xrightarrow{w_2} & K(\mathbb{Z}/2, 2) \\
\text{id} & & \\
\text{BSpin} & \xrightarrow{j} & T^2 \\
\end{array}
$$

which gives us the fibration

$$
\text{BSpin} \xrightarrow{i} T^2\langle w_2 \rangle \xrightarrow{j} T^2.
$$

As a result, we have the following homotopy long exact sequence

$$\ldots \rightarrow \pi_{i+1}(T^2) \rightarrow \pi_i(\text{BSpin}) \rightarrow \pi_i(T^2\langle w_2 \rangle) \rightarrow \pi_i(T^2) \rightarrow \ldots .$$

For $i \geq 2$ we have trivial $\pi_i(T^2)$ so

$$i_* : \pi_i(\text{BSpin}) \cong \pi_i(T^2\langle w_2 \rangle).$$
For $i = 1$ we have

$$0 \rightarrow \pi_1(\text{BSpin}) \rightarrow \pi_1(T^2\langle w_2 \rangle) \rightarrow \pi_1(T^2) \rightarrow 0$$

where $\pi_1(\text{BSpin}) = \pi_0(\text{Spin})$ which is trivial hence

$$\pi_1(T^2\langle w_2 \rangle) \cong \pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}.$$ 

We already have maps

$$\tilde{g}: M \to \text{BSO} \quad \text{and} \quad f: M \to T^2$$

so by using the universality of pullback we have a map

$$\begin{array}{ccc}
M & \xrightarrow{g} & T^2\langle w_2 \rangle \\
\downarrow f & & \downarrow j \\
T^2\langle w_2 \rangle & \xrightarrow{j} & T^2 \\
\downarrow \eta & & \downarrow w_2 \\
\text{BSO} & \xrightarrow{\gamma} & K(\mathbb{Z}/2, 2)
\end{array}$$

which is a normal 2-smoothing of $M$. We also know that the map

$$\eta: T^2\langle w_2 \rangle \to \text{BSO}$$

induces an isomorphism

$$\eta_*: \pi_i(T^2\langle w_2 \rangle) \xrightarrow{\cong} \pi_i(\text{BSO})$$

for $i \geq 2$ since

$$\eta \circ i: \text{BSpin} \to \text{BSO}$$

is the covering map where $\text{BSpin}$ is the universal cover of $\text{BSO}$ and $i_*$ gives an isomorphism between homotopy groups for $i \geq 2$ as we showed. Hence $T^2\langle w_2 \rangle$ is the normal 2-type of $M$.

The Atiyah-Hirzebruch spectral sequence used to compute $\Omega_6(T^2\langle w_2 \rangle)$ has the same $E_2$-terms as the one we used for trivial $w_2(\nu_M)$ with the difference being that the differentials are twisted by $w_2$ however it still stabilizes on the second page. This time the approximation comes in the form of

$$H_p(T^2; \Omega^\text{Spin}_q(\text{pt})) \Longrightarrow \Omega_{p+q}(T^2\langle w_2 \rangle).$$
We still get $\Omega_6(T^2\langle w_2 \rangle) \cong \mathbb{Z}$. Now just like the previous case we have

$$\pm \langle x_1 \sim p_1(\tau_{M_1}), [M_1] \rangle = \pm \langle x_2 \sim p_1(\tau_{M_2}), [M_2] \rangle$$

where $M_1$ and $M_2$ are our non-spin manifolds with the same normal 2-type and $x_1, x_2$ are generators of $H^2(M; \mathbb{Z}) \cong \mathbb{Z}$. This equality tells us that $M_1$ and $M_2$ are bordant. Hence by (2.1.16) our manifolds are stably diffeomorphic.
REFERENCES


