DYNAMICS OF RATIONAL FUNCTIONS AND WANDERING DOMAINS

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DYNAMICS OF RATIONAL FUNCTIONS AND WANDERING DOMAINS

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As Sullivan proved in 1985, there is no rational map whose Fatou components possess a wandering domain in one dimension, and in the proof quasi-conformal mappings were used. After that, it is expected that there should be a map, but not in one dimension, and that not every component of the Fatou set of this rational map is eventually periodic. This thesis will be a review of which maps have been built over the years to see the dynamics of wandering components. Chronologically, polynomial skew-product mappings in two dimensions were constructed using parabolic implosion techniques, and it’s seen that they have wandering domains [3]. In 2004, Lilov proved that, near an invariant super-attracting fiber the wandering Fatou components of polynomial skew-products cannot exist [2]. Then, the examples of mappings with wandering domains have been given explicitly in [3] and [12]. The concept of strongly-attracting fiber is defined in the paper of Ji(2018) [8] concluding that there are no wandering Fatou components near strongly-attracting fiber with the given conditions.
Keywords: iteration, Fatou-Julia sets, wandering domain, polynomial skew-product
ÖZ

RASYONEL FONKSİYONLARIN DİNAMİKLERİ VE PERİYODİK OLMAYAN FATOU KÜMELERİ

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Anahtar Kelimeler: iterasyon, Fatou-Julia kümeleri, periyodik olmayan Fatou kümesi, polinomiyal ayrık çarpım
To every being I have, for holding my hand in the way of being...
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Dynamics of rational functions is mainly divided into two sets: Julia and Fatou. Julia set is the unstable one which is the boundary of Fatou set in the one-dimension. Fatou set is stable relative to Julia set. As we iterate rational functions, we can obtain eye-catching fractals; the most popular one is Mandelbrot Set. In particular, the dynamics of the Mandelbrot set, which can be the subject of intellectual conversations from time to time, was a topic of discussion for a long time. After No Wandering Domain Theorem that states there is no rational function possessing a wandering domain which holds only in dimension one, Fatou Theory was dormant. Still, Sullivan has woken up by proving this theorem using quasi-conformal mappings.

While the question of whether the No Wandering Domain Theorem holds in higher dimensions or not is around and Lilov and Astorg et al. bring the idea of polynomial-skew product mappings to take advantage of its resemblance to dynamics in one-dimension. Contributions were made to this, and some examples of mappings having wandering Fatou components were presented by different authors over the years.

The content of the thesis is organized as follows. In the second chapter, the preliminaries in Complex Analysis and Complex Dynamics are given in dimension one. The iteration of a function will be defined, and adding on that, definitions of some well-known concepts in Fatou-Julia Theory will also be stated. The No Wandering Domain Theorem (Theorem 2.0.8), proved by Sullivan in 1985, will be given, and in order to prove it, we will need a new notion which brings us to the next chapter. Also, the Lemma used in the proof of Sullivan will be given at the end of the second
In the third chapter, the quasi-conformal mappings are handled in detail. Necessary theorems and lemmata will be given in order to prove the No Wandering Domain Theorem. All of these concepts work in the one-dimensional theory.

In the fourth chapter, we increase the dimension by one and work on the two-dimensional Fatou Theory. The published works in the area of the existence or nonexistence of Fatou components will be stated chronologically, one by one. Also, polynomial-skew products and bulging Fatou components will be defined. Using them, the No Wandering Fatou Components Theorem will be proved with the help of some illustrations.

In the last chapter, we will focus on the maps with wandering Fatou components. The projective spaces and Lavaurs maps will be defined to grasp the idea of the Fundamental Theorem, which is Theorem 5.1.1. The structure of the Fundamental Theorem is significant as it possesses a wandering Fatou component. Closing the chapter, and also the thesis, will be made by the proof of the Fundamental Theorem.
CHAPTER 2

PRELIMINARIES

In this chapter, we’ll give a survey of the dynamics of rational maps in $\mathbb{C}$. For the notations, we’ll follow the book [1].

**Definition 2.0.1.** A rational function of $z \in \mathbb{C}$ can be considered as follows:

\[
R(z) = \frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m}.
\]

**Definition 2.0.2.** The composition $R^n(z) = R \circ R \circ \ldots \circ R(z)$ is called the "nth iteration" of the rational function $R$.

In turn, we obtain a sequence

\[ z_0, z_1 = R(z_0), z_2 = R(z_1), z_3 = R(z_2), \ldots \]

Many questions arise:

- Does the sequence of points $z_n$ converge?
- If so, for which values of $z_0$ do we obtain the convergent sequence $z_n$?
- Can we say anything about the behavior and dynamics of the iteration if $(z_n)_{n \in \mathbb{N}}$ doesn’t converge?
- Under small changes in $z_0$, how vigorous are the answers to the above questions?

We can look at the sequence $\{z_0, z_{-1}, z_{-2}, z_{-3}, \ldots\}$ as well as looking at the sequence, $(z_n)_{n \in \mathbb{N}}$. Again, our notation is $z_{n+1} = R(z_n)$. Assume we are given $z_0$. There may be cases where $z_{-1}$ is not unique. Even, there may be more possibilities for $z_{-2}$, and so on.

By observing fixed points of $R$, we can get an insight since fixed points have an important role in the Iteration of Rational Functions Theory.
Definition 2.0.3. A point $\zeta \in \mathbb{C}$ is called a fixed point of $R$ if $R(\zeta) = \zeta$.

Theorem 2.0.1. Let $R$ be a rational map of degree $d$. When $d \geq 1$, $R$ has precisely $d + 1$ fixed points in the extended complex plane (i.e., $\mathbb{C} \cup \{\infty\}$).

Assume the sequence $(z_n)_{n \in \mathbb{N}} \to w \in \text{Dom}(R)$ for an arbitrary choice of $z_0$. Notice that $R$ is continuous at $w$. Then, we have

$$w = \lim_{n \to \infty} z_{n+1} = \lim_{n \to \infty} R(z_n) = R(\lim_{n \to \infty} z_n) = R(w)$$

Finally, we obtain $R(w) = w$, thus we can conclude that $w$ is a fixed point of $R$. So, if $z_n \to w$, then $R(w) = w$.

Here is an example. Consider the rational map $R(z) = z^2 - 4z + 6$. Choose some initial point $z_0$; it will eventually converge to 2, 3 or $\infty$.

Definition 2.0.4. We can classify fixed points into three types. Let $\zeta \in \mathbb{C}$ be the fixed point. Then, it’s:

(i) an attracting fixed point if $|R'(\zeta)| < 1$

(ii) a repelling fixed point if $|R'(\zeta)| > 1$

(iii) an indifferent fixed point if $|R'(\zeta)| = 1$.

If $z$ is close to $\zeta$, then, approximately,

$$|R(z) - \zeta| = |R(z) - R(\zeta)| \approx |R'(\zeta)||z - \zeta|$$

by Mean Value Theorem

Roughly speaking, when we apply $R$ repeatedly, the points close to an attracting fixed point $\zeta$ get closer to $\zeta$. On the other hand, those close to a repelling fixed point tend to draw away from it. In the previous example, $R$ has two attracting fixed points which are 2 and $\infty$; and 3 is its repelling fixed point.

To describe the idea of the theory, construct two sequences $z_n$ and $w_n$ by choosing some initial points $z_0$ and $w_0$. When $w_0$ is sufficiently close to $z_0$ and as $n \to \infty$, can we say that the sequences’ dynamics are roughly the same? The answer obviously depends on the initial point $z_0$ we chose. To be able to say yes or no to the previous question, $\mathbb{C}$ (the complex plane) is divided into two sets called ”(F) Fatou” and ”(J) Julia” sets in this theory.

$$\mathbb{C} = F \cup J$$
“ (\(\mathcal{F}\)) Fatou” represents the space where the answer is ‘yes’ and
“(\(\mathcal{J}\)) Julia” represents the space where the answer is ‘no’ to the previous question.

Later, we’ll define these sets precisely. Now, we can consider the given example regarding Fatou and Julia sets.

Observe that the fixed points of \(R(z) = z^2\) are 0, 1, and \(\infty\). The unit circle \(C = \{z : |z| = 1\}\) plays an important role on the dynamics of \(R\). When \(|z_0| < 1\), \(z_n\)’s converge to 0; while they converge to infinity if \(|z_0| > 1\).

In our choice of \(R(z) = z^2\),

\[\mathcal{J} = C \text{ and } \mathcal{F} = \{z : |z| < 1\} \cup \{z : |z| > 1\}.\]

Notice that \(\mathcal{J}\) is always the boundary of the Fatou set \(\mathcal{F}\).

In the above representation, the blue and orange regions represent Fatou sets. The black circle (unit circle) is the Julia set which is the boundary of the Fatou set \(\mathcal{F}\).

\[\text{Figure 2.1}\]

Definition 2.0.5. We say that \(\zeta\) is a periodic point of \(R\) if \(R^n(\zeta) = \zeta\) for some iterate \(n\). For such periodic point \(\zeta\), the elements of sequence, for some \(n \in \mathbb{Z}^+\),

\[\{*\} \zeta, R(\zeta), R^2(\zeta), R^3(\zeta), ..., R^{(n-1)}(\zeta)\]
are not the same but we have \( R^n(\zeta) = \zeta \). The cycle of \( \zeta \) is the finite set of points above \([^*]\), and \( n \) is called the period of \( \zeta \).

Notice that fixed points are of period 1.

\( \zeta \) is of period \( n \) \( \iff \) \( \zeta \) is a fixed point of \( R^n \) but not of any \( R^m \) for \( m < n \).

(\textbf{the Mandelbrot Set}) We’ll identify the set of values \( c \) for which \( z^2 + c \) has an attracting fixed point in \( \mathbb{C} \). We will use the previous Theorem 2.0.1. As \( R \) is of degree 2, we have precisely \( 3 \) \((2 + 1)\) fixed points, and one of them is \( \infty \). Say \( \alpha \) and \( \beta \) are the fixed points of \( R \) in \( \mathbb{C} \). So, they are solutions to the following equation

\[
\alpha^2 - \alpha + c = 0 \quad \text{as} \quad R(z) = z^2 + c = z
\]

which is

\[
\alpha^2 - \alpha + c = 0 \quad \text{and} \quad \beta^2 - \beta + c = 0.
\]

Clearly, \( \alpha + \beta = 1 \) and \( \alpha \beta = c \).

Then, \( R'(\alpha) + R'(\beta) = 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \) as we showed \( \alpha + \beta = 1 \). So, we can conclude that both \( \alpha \) and \( \beta \) can not be an attracting point (as we cannot get 2 when we add two numbers whose norm is less than 1. If they were both attracting, then \( |R'(\alpha)| < 1 \) and \( |R'(\beta)| < 1 \). Hence, we can have at most one attracting fixed point. Without loss of generality, say it is \( \alpha \). We know that

\[
|2\alpha| = 2|\alpha| = |R'(\alpha)| < 1
\]

which implies that \( |\alpha| < 1/2 \).

Yet, from the previous equation \( 22 \) we have;

\[
c = \alpha - \alpha^2.
\]

So, the set of \( c \)'s we are looking for is the image of the disc \( \{ \alpha : |\alpha| < 1/2 \} \) under \( z \to z - z^2 \). Notice that we can obtain this map by some conjugations.

Choosing \( h(z) = z - \frac{1}{2}, g(z) = z^2 \) and \( f(z) = \frac{1}{4} - z \),

\[
f(g(h(z))) = f(g(z - \frac{1}{2})) = f((z - \frac{1}{2})^2) = f(z^2 - z + \frac{1}{4}) = z - z^2.
\]
The set of $c$ can be represented as a cardioid, and the illustration is the following:

![Cardioid Illustration](image)

Choosing $c = -1$, we see that the rational map $R(z) = z^2 - 1$ induce the sequence

$$0, -1, 0, -1, 0, ...$$

Then, 0 and 1 will be the attracting fixed points of the second iterate of $z^2 - 1$ which is

$$R(R(z)) = (z^2 - 1)^2 - 1 = z^2(z) = z \implies R^2(z) - z = 0.$$ 

If $\alpha$ and $\beta$ are the fixed points of $R$, they are also fixed points of $R^2(z)$. So $(R^2(z) - z)$ should be divisible by $(R(z) - z)$.

Therefore; we are able to write

$$R^2(z) - z = (z^2 - z + c)(z^2 + z + 1 + c) = (z - \alpha)(z - \beta)(z - u)(z - v).$$

Now, we look for conditions on $c$ that imply $\{u, v\}$ is an attracting two-cycle.
(ie. $R(u) = v$, $R(v) = u$ (comes from 2-cycle) and $|(R^2)'(u)| < 1$, $|(R^2)'(v)| < 1$ (comes from attractiveness)). Clearly, $u$ and $v$ are attracting fixed points of $R^2$. Apply Chain Rule, we obtain

$$(R^2)'(u) = R'(R(u))R'(u) = R'(v)R'(u) = 2v.2u = 4uv = 4(1 + c).$$

We obtain that the set of $c$ we look for is $\{c : |1 + c| < \frac{1}{4}\}$. Recall the cardioid illustrated in the previous pages. Now, combine the disk $\{c : |1 + c| < \frac{1}{4}\}$ and the cardioid, we’ll obtain

![Figure 2.3](image)

The Figure 2.4 is the well-known **Mandelbrot set**.

**Definition 2.0.6** (Forward Invariance). Let $E \subset X$. If $g : X \to X$ is a map of $X$ into itself and $g(E) = E$, we say that $g$ is **forward invariant**.

**Definition 2.0.7** (Backward Invariance). Let $E \subset X$. If $g : X \to X$ is a map of $X$ into itself and $g^{-1}(E) = E$, we say that $g$ is **backward invariant**.
**Definition 2.0.8** (Complete Invariance). Let \( E \subset X \). If \( g : X \to X \) is a map of \( X \) into itself and \( g(E) = E = g^{-1}(E) \), we say that \( g \) is **completely invariant** (i.e., both forward and backward invariant).

**Theorem 2.0.2.** [7] The Fatou set \( F \), and the Julia set \( J \) are completely invariant under \( R \), where \( R \) is any rational map.

The definitions of Fatou and Julia sets are on the next page. If \( g \) is onto, \( g(X) = X \), then the backward and complete invariance overlap.

**Definition 2.0.9.** A point \( z \in \mathbb{C} \) is called a **critical point** of \( R \) if \( R \) cannot be injective in any neighborhood of \( z \). If \( w = R(z) \) (i.e., \( w \) is an image of some critical point \( z \)), then \( w \) is called a **critical value** for \( R \).

In the case where \( w \) is not a critical value, we can make some observations. As it is not a critical value and \( R(z) = w \) where \( R \) is a rational map of degree \( d \), we can see that \( R \) is one-to-one in every neighborhood of \( z \). So, the preimage of \( w \) consists exactly \( d \) distinct points, named \( z_1, z_2, ..., z_d \). None of \( z'_i \)'s are critical, then there are neighbourhoods \( N \) of \( w \), \( N_1 \) of \( z_1 \), \( N_2 \) of \( z_2 \) and so on till we obtain \( N_d \) of \( z_d \) from each \( N_j \) to \( N \). When we restrict \( R \) to these neighbourhoods \( N_j \)'s and call this map \( R_j \), the \( R_j \)'s has an inverse

\[
R^{-1}_j : N \to N_j.
\]
These $R_j^{-1}$’s are called the branches of $R^{-1}$ at $w$.

**Definition 2.0.10.** A family $F$ of maps of $(X, d)$ (where the domain of functions of the family $F$ is $X$ and the induced metric on $X$ is $d$) into $(X_1, d_1)$ is called equicontinuous at $x_0$ if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in X \text{ and } \forall f \in F, d(x, x_0) < \delta \implies d_1(f(x), f(x_0)) < \epsilon.$$  

If this holds for all $x_0 \in X$ for some subset $X_0 \subset X$, then we say that $F$ is equicontinuous on $X_0 \subset X$.

**Definition 2.0.11.** Let $R$ be a non-constant rational function. The largest open subset of the extended complex plane where $\{R^n\}$ is equicontinuous is the Fatou set (F) of $R$ while the complement of this largest open set is Julia set (J) of $R$.

**Definition 2.0.12.** If every infinite sequence of functions $(f_n) \in F$ has a subsequence that converges locally uniformly on $X$ where our functions are from $(X, d)$ to $(Y, d')$, then $F$ is called a normal family.

($f_n$ converges locally uniformly to $f : X \to Y$ if $\forall x \in X, \exists$ a neighbourhood $V$ of $x$ s.t. $f_n \xrightarrow{\text{uniformly}} f$ on $V$)

**Theorem 2.0.3** (Arzela-Ascoli). Let $D \subset \mathbb{C}_\infty$ and $F$ be a family of continuous functions of $D$ into $S^2$. Then

$F$ is equicontinuous in $D \iff F$ is normal family in $D$.

**Definition 2.0.13.** The smallest completely invariant set that is containing $z$ construct an equivalence class and we denote it by $[z]$. We call $z$ exceptional point when $[z]$ is finite.

Now, we take our consideration that under which special conditions this equivalence class can be finite. Let’s denote the set of points belonging to that equivalence class by $E(R)$.

**Theorem 2.0.4.** The number of exceptional points can be at most 2 when the map $R$ is a rational map with degree $d \geq 2$. When there is one such point, i.e. $E(R) = \{\eta\}$, $R$ is conjugate to a polynomial with $\eta$ corresponding to infinity. In the case where the
number of exceptional points is two and they are distinct, call them $\eta_1, \eta_2$, we say that $R$ is conjugate to some map which maps $z$ to $z^d$ and $\eta_1$ and $\eta_2$ correspond to 0 and infinity.

**Corollary 2.0.1.** The exceptional points of the rational map $R$ where $\deg(R) \geq 2$ lie in the Fatou set of $R$.

**Theorem 2.0.5** (Montel Theorem). [5] Let $D$ be a subset of $\mathbb{C}_\infty$ and $E$ be the domain in $\mathbb{C}_\infty \setminus \{0, 1, \infty\}$. Then $\mathcal{F}$ of all analytic maps from $D$ to $E$ is normal in $D$. (i.e., if it misses three points, then it’s normal. In the statement, $\{0, 1, \infty\}$ is used, but by using appropriate Möbius transformations, we can obtain desired 3 points.)

**Theorem 2.0.6.** [1] Either the Julia set $J$ has an empty interior, or it is equal to the extended complex plane $\mathbb{C}_\infty$.

**Theorem 2.0.7.** [1] The closure of the set of periodic points of $R$ where $\deg(R) \geq 2$ contains the Julia set $J$.

**Definition 2.0.14.** A component $\Omega$ of $F(R)$ is

- **periodic** when $R^n(\Omega) = \Omega$ for some $n \in \mathbb{Z}^+$

- **eventually periodic** when $R^m(\Omega)$ is periodic for some $m \in \mathbb{Z}^+$

- **wandering** when the sets $R^n(\Omega)$ are pairwise disjoint, $n \geq 0$.

**Definition 2.0.15.** A domain $\Omega \subseteq \mathbb{C}_\infty$ is called **simply connected** if its complement is connected.

**Definition 2.0.16.** A periodic Fatou component $\Omega$ of period $m$ is called

- **a super-attracting basin** when $\Omega$ contains a super-attracting fixed point,

- **an attracting basin** when $\Omega$ contains an attracting fixed point,

- **a parabolic basin** when there exists a fixed point $p \in \partial \Omega$ with $|f'(p)| = 1$, and the sequence of the iterations of $f$ converges normally to $p$,

- **a Siegel disc** when it’s simply connected and some sub-sequence of the iterations of $f^n|_{\Omega}$ converges to identity $|\Omega'$. 
• **a Herman ring** when it’s doubly connected and some sub-sequence of the iterations of $f_{|\Omega}^{mnk} \rightarrow \text{identity}_{|\Omega}$.

**Theorem 2.0.8.** \[7\] Every component of the Fatou set of $R$ is eventually periodic.

The theorem above holds in one dimension, and it was proved by Dennis Sullivan in 1985 using quasi-conformal mappings. In higher dimensions, the techniques to prove it cannot be used as they are true only in dimension one. The proof of this theorem will be done in the following chapters. Firstly, we need to define some concepts.

**Definition 2.0.17.** A complex function $f$ is said to be **analytic (holomorphic)** on the domain $\Omega$ if its complex derivative exists at every point on $\Omega$.

**Definition 2.0.18.** When a function $f$ is both analytic and one-to-one on a domain $\Omega$, we say that $f$ is a **univalent** function on $\Omega$.

**Definition 2.0.19.** The **Möbius maps** are the rational maps whose degree is one and of the form

$$z \mapsto \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$. These maps are crucial as they establish a group of analytic homeomorphisms of the complex sphere to itself.

**Theorem 2.0.9** (Riemann Mapping Theorem). \[5\] Let $\Omega$ be a simply connected, proper subset of $\mathbb{C}$ and $p$ be a point in $\Omega$. Then there exists a holomorphic function $f(z)$ of $\Omega$ onto the unit disk $\mathbb{D}$ in a one-to-one fashion and $f(p)=0$. Moreover, if we normalize it so that $f'(p) > 0$, then $f$ is unique.

**[Example of a Riemann Mapping]** Notice that $\mathbb{H}$ (i.e., upper half-plane) is simply connected, and it is a proper subset of $\mathbb{C}$.

**Definition 2.0.20.** A **homeomorphic** function $f$ is a function where $f$ is a bijection and $f$ and $f^{-1}$ is continuous.

**Definition 2.0.21.** Given two rational maps $R$ and $S$. $R$ and $S$ are called **conjugate** if there is some Möbius map $g$ with $S = gRg^{-1}$.

**Theorem 2.0.10.** \[7\] Say $R$ is a rational map that is not constant, and $g$ is a Möbius map with $S = gRg^{-1}$. Then $F(S)=g(F(R))$ and $J(S)=g(J(R))$. 

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Lemma 2.0.1. Let $R$ has a wandering domain. Then for some component $W$ of the Fatou set of $R$, the components of the sequence

$$W, R(W), R^2(W), ..., R^n(W), ...$$

of $F(R)$ are simply connected, pairwise disjoint and does not contain any critical points of $R$. Moreover, $R$ maps each component onto the next component homeomorphically.

Notice that, as $R$ is continuous, when $W$ is simply connected, $R(W)$ is simply connected too. (From the fact that image of a simply connected domain under a continuous map is simply connected.)

The Lemma shows that the existence of wandering domain implies the existence of simply connected wandering domain. Thus; it reduces the work of Sullivan to the simply connected case.
CHAPTER 3

CONFORMAL STRUCTURES

3.1 QUASI-CONFORMAL MAPPINGS

Recall that when \( f \) is analytic,
\[
\frac{\partial f}{\partial \bar{z}} = 0,
\]
\[
\frac{\partial f}{\partial z} = f'(z)
\]
where these operations are defined as
\[
\frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right),
\]
\[
\frac{\partial f}{\partial z} = \frac{1}{2i} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right).
\]

Our generalization of these equations is the Beltrami equation
\[
\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}
\]
where \( \mu \) is a suitable complex-valued function on our domain \( D \).

Notice that, in our Beltrami equation
\[
\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}
\]
when \( \mu = 0 \) throughout \( D \), any sufficiently smooth solution \( f \) is analytic.

In the general solution, \( \mu \) is taken as a measure of the deviation of a solution \( f \) from conformality.
Recall that an invertible mapping is called **conformal** if it preserves angles. Formally, if $U$ is an open subset of $\mathbb{C}$, then a function $f$ from $U$ to $\mathbb{C}$ is conformal if it is holomorphic (analytic) and its derivative is everywhere nonzero on $U$.

When $f$ is given, we can find $\mu$ by solving the equation above; in that case, $\mu$ is called **complex dilatation of** $f$ and it is denoted by $\mu_f$.

**Theorem 3.1.1.** [1] Let $\mu$ be a Beltrami coefficient on $D$ on $\mathbb{C}_\infty$. Then we can find a quasi-conformal map with Beltrami coefficient $\mu$ on $D$. Moreover, if there are two such quasi-conformal maps, call $\phi$ and $\psi$, then $\phi\psi^{-1}$ is analytic.

**Lemma 3.1.1.** [1] Let $\phi$ be a homemorphism on $D$. Then TFAE:

(i) $\phi$ is $\mu$-conformal on $D$,
(ii) $\phi$ is a chart for $D[\mu]$,
(iii) $\mu : D[\mu] \to \mathbb{C}_\infty[0]$ is analytic.

Each $\mu$ forms a Riemann surface on $D$, that is the surface $D[\mu]$ is determined by the Beltrami-coefficient $\mu$. So, $D[\mu]$ stands for $\mu$-conformal structure on $D$ and it is a Riemann surface. When $\mu \equiv 0$, it’s simply $D$.

**Lemma 3.1.2.** [1] Let $\mu$ and $\nu$ be Beltrami coefficients on $U$ and $V$ respectively, and $h$ be an analytic map from $U$ onto $V$. Then:

$$h : U[\mu] \to V[\nu] \text{ is analytic if and only if } \nu(hz) = \frac{|h'(z)|}{|h'(\overline{z})|} \mu(z) \text{ a.e. in } U.$$  

**Lemma 3.1.3.** [1] Let $R$ be a rational map with degree $d$ and $\psi$ be a $\nu$-conformal map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. Then,

$$\psi R \psi^{-1} \text{ is rational if and only if } \nu(Rz) = \frac{|R'(z)|}{|\overline{R'(z)}|} \nu(z) \text{ a.e. on } \mathbb{C}_\infty.$$  

Moreover; when this holds, $\deg(\psi R \psi^{-1}) = d = \deg(R)$.

Now, using the necessary lemmata above and the concepts stated before, we are able to write the proof of Theorem 2.0.8, which is known as No Wandering Domain Theorem.
Assume on the contrary that $R$ has a wandering domain (as our statement in the theorem was every Fatou component is eventually periodic). Then, by Lemma 2.0.1, there exists a component $W$ of $F(R)$ which is simply connected, pairwise disjoint. Since $W$ is simply connected, we are able to use Riemann Mapping Theorem. Thanks to that mapping, there is a conformal equivalence $g$ of the unit disk $\mathbb{D}$ onto our wandering domain $W$. With the help of Lemma 3.1.2, using $g$, we can transfer $\mu$ to $\nu$ on $W$. Then, we can extend $\nu$ to $\mathbb{C}_\infty$. By solving the Beltrami equation with Beltrami coefficient $\nu$ throughout $\mathbb{C}_\infty$ and using Lemma 3.1.3, we get a $\nu$-conformal map $\phi$ from $\mathbb{C}_\infty$ onto itself where $\phi R \phi^{-1}$ is rational and its degree is equal to degree of $R$. Assume that $\phi$ fixes three points $\{0, 1, \infty\}$, by combining it with a Möbius map. Observe that, we’ve had the below composition map from the unit disk $\mathbb{D}$ to itself:

$$
\mu \mapsto \nu \mapsto \phi \mapsto \phi R \phi^{-1}[\ast]
$$

It begins from the space of Beltrami coefficients on the unit disk $\mathbb{D}$ and ends at the space of rational maps of degree $d$. The space of Beltrami coefficients on $\mathbb{D}$ is infinite-dimensional while the space of rational functions of degree $d$ is finite-dimensional. So, this composite map cannot be 1-1. Thus, we have that the map $\mu \mapsto \phi R \phi^{-1}$ must map a large subspace of the space of Beltrami differentials onto a single function $S$. To justify that, we need a last Lemma and the proof of this lemma ends the proof of The No Wandering Domain Theorem.

**Lemma 3.2.1.** Let $\eta_0 > 0$. In the unit disk $\mathbb{D}$, for each $t$ where $t \in [0, 1]$, we can construct a Beltrami coefficient $\mu_t$ satisfying the following:

i) $||\mu_t||_\infty < \eta_0$

ii) The composite map $[\ast]$ above maps each $\mu_t$ to the same rational function $S$.

Before the proof of the above Lemma 3.2.1, we need to observe some consequences of it.

First of all, for all $t$ belonging to $[0, 1]$, using the part (ii), we have

$$
\phi_t R \phi_t^{-1} = S = \phi_0 R \phi_0^{-1}. \quad [\ast]
$$

Put $\Phi_t = \phi_t \phi_t^{-1}$. As $\Phi_0 = \phi_0 \phi_0^{-1} = I$, we can conclude that $\Phi_0(z) = z$ on the complex sphere $\mathbb{C}_\infty$. 

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Also, notice that
\[ R(z)\phi^{-1}_0 = \phi_0^{-1} \phi_0 R(z) \phi_0^{-1} = R\phi_0^{-1} \phi_t \phi_t^{-1}, \]
\[ \phi_0^{-1} \phi_t R(z) \phi_t^{-1} = R\phi_0^{-1} \phi_t \phi_t^{-1}, \]
\[ \Phi_t R(z) = \phi_0^{-1} \phi_t R(z) = R\Phi_t. \]

In the second line, we used the [*] above stating \( \phi_t R\phi_t^{-1} = S = \phi_0 R\phi_0^{-1} \). So,
\[ \Phi_t R(z) = R\Phi_t. \]

In other words, the map \( \Phi_t \) commutes with the map \( R \) for any \( t \in [0, 1] \).

Additionally, the map \( \Phi_t \) is continuous, 1-1, onto, and its inverse is also continuous on the complex sphere to itself. This yields that \( z \mapsto \Phi_t(z) \) is a homeomorphism of the complex sphere onto itself, for any \( t \in [0, 1] \). Furthermore, \( t \mapsto \Phi_t(z) \) is a continuous map on the interval \([0,1]\) for any \( z \). We’re done with the observations. Now, we’ll state another lemma,

**Lemma 3.2.2.** On the Julia set of \( R \), for any \( t \in [0, 1] \), the map \( \Phi_t \) is equal to the identity map \( I \). Besides, it maps each Fatou component of \( R \) onto itself.

**Proof.** Let \( p \) be the iteration number of \( R \) and denote the set of fixed points of \( R^p \) by \( A_p \). Recall that, the map \( \Phi_t \) commutes with the map \( R \) for any \( t \in [0, 1] \). Fix a point \( z \) in \( A_p \).

\[ z \in A_p \implies R^p(z) = z \implies \Phi_t(z) = \Phi_t(R^p(z)) = \Phi_t(R(R^{p-1}(z))). \]

By commutation of \( R \) and \( \Phi_t \)
\[ = R(\Phi_t(R^{p-1}(z))) = R(\Phi_t(R(R^{p-2}(z)))) \]

If we apply this procedure \( p - 2 \) times, we get
\[ \Phi_t(z) = \Phi_t(R^p(z)) = \Phi_t(R(R^{p-1}(z))) = R(\Phi_t(R(R^{p-2}(z)))) = ... = R^p(\Phi_t(z)). \]

In that case, we are able to say that if \( z \in A_p \), then \( \Phi_t(z) \in A_p \). Thence, each finite set \( A_p \) is mapped into \( A_p \) by the map \( \Phi_t \).

Recall also that \( t \mapsto \Phi_t(z) \) is a continuous map on the interval \([0,1]\) for any \( z \). So, any \( z \) in \( A_p \) is mapped to the discrete set \( A_t \). Therefore, the map \( \Phi_t(z) \) is
independent of $t$. Keeping in mind that $\Phi_0(z) = z$ on the complex sphere $C_\infty$, we say that $\Phi_t(z) = z$ for any $t \in [0,1]$ and $z$ in $A_p$, thanks to the independency from $t$. So, we induce that $\Phi_t$ is the identity map on the union of $A_p$ for any $t$. And, as $z \mapsto \Phi_t(z)$ is a homeomorphism of the complex sphere onto itself, for any $t \in [0,1]$, it also holds on the closure of that union.

Remember that the closure of the set of periodic points of $R$ where $\deg(R) \geq 2$ contains the Julia set $J$. So, when we restrict the map $\Phi_t$ to the Julia set, it is also the identity map. Notice that, as we are looking at the set of fixed points of $R^p$, we looked at its periodic points since fixed points are of period 1. Thus, $\Phi_t = I$ on $J(R)$.

Based on this and that $\Phi_t$ is a homeomorphism from $C_\infty$ to $C_\infty$, we say that the components of $F(R)$ must be permuted by each $\Phi_t$.

Begin by taking an arbitrary $z$ in $F_0$ where $F_0$ is an arbitrary Fatou component of $F(R)$. As $t \mapsto \Phi_t(z)$ is a continuous map on the interval $[0,1]$ for any $z$, the map $t \mapsto \Phi_t(z)$ maps the interval $[0,1]$ to a curve whose initial point is $z$ since $\Phi_0(z) = z$ on $C_\infty$. This follows from the fact that $[0,1]$ is connected and its image should be connected as the map is continuous. Since the image (curve) is connected, it lies completely in $F_0$ by the characteristics of the Fatou set and connectivity. Therefore, the arbitrary component $F_0$ should be mapped to itself by $\Phi_t$, and this holds for each $t$. This finishes the proof.

Observe that;

\[ \Phi_t(F_0) = F_0 = \phi_t^{-1}(\phi_t(F_0)), \]
\[ \phi_0(F_0) = \phi_t(F_0), \]
\[ \Phi_t(W) = \phi_t^{-1}(\phi_t(W)) = W, \]
\[ \phi_t(W) = \phi_0(W) = W_0. \]

First equality above comes from the conclusion that $\Phi_t$ maps each $F_0$ onto itself. Following that, the $\phi_t$’s, homeomorphisms, should map $W$ onto $W_0$ where $W_0$ is a simply connected domain and is independent of $t$. So, we can write $\phi_t$ instead of $\phi_0$ thanks to independency.
Thus,

In the Figure 3.1, the triangle symbol represents the unit circle $\mathbb{D}$. Notice that $h \phi_t g : \mathbb{D} \mapsto \mathbb{D}$.

As represented above, we can conclude that there exists a map $h$ from $W_0$ to $\mathbb{D}$ which is an conformal equivalence by Riemann Mapping.

Now, it’s better to recall the lemmata we already stated:

**Lemma 3.2.3.** [1] Let $\mu$ and $\nu$ be Beltrami coefficients on $U$ and $V$ respectively, and $h$ be an analytic map from $U$ onto $V$. Then;

$$h : U[\mu] \rightarrow V[\nu] \text{ is analytic } \iff \nu(hz) = \left[ \frac{h'(z)}{h'(z)} \right] \mu(z) \text{ a.e. in } U.$$

**Lemma 3.2.4.** [1] Let $R$ be a rational map with degree $d$ and $\psi$ be a $\nu$-conformal map from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$. Then,

$$\psi R \psi^{-1} \text{ is rational } \iff \nu(Rz) = \left[ \frac{R'(z)}{R'(z)} \right] \nu(z) \text{ a.e. on } \mathbb{C}_\infty.$$

Moreover, when this holds, $\text{deg}(\psi R \psi^{-1}) = d = \text{deg}(R)$.

In the construction of the previous composition map [*], we had $\nu$ on the complex sphere $\mathbb{C}_\infty$ starting with $\mu$ which is on the unit disk $\mathbb{D}$. Using the above lemmatas, we
see that;

\[ \| \nu \|_\infty = \| \mu \|_\infty. \]

Then, owing to Lemma 3.2.1.’s first part, we figure out that \( \| \nu_t \|_\infty \leq \eta_0 \) and this \( \nu_t \) is obtained by beginning composing the map \([^\star]\) by \( \mu_t \).

**Lemma 3.2.5.** \([1]\) ∀ε > 0, ∃δ > 0 s.t. when the norm of the Beltrami coefficient \( \nu \) on the complex sphere is less than \( \delta \), we have \( \sigma(F(z), z) < \epsilon \) in the complex sphere where \( \sigma \) is the hyperbolic metric and \( F \) is a function that fixes the three points \( \{0, 1, \infty\} \) and also a \( \nu \)-conformal map of complex sphere onto itself.

Notice that, when we choose \( \epsilon = \frac{1}{8} \) in the above lemma and say \( \delta_0 \) is the corresponding value for our \( \epsilon \), we get the following lemma:

**Lemma 3.2.6.** \([1]\) Let \( W \) on \( \mathbb{C}_\infty \) be a simply connected domain and \( \rho \) be the induced hyperbolic metric defined on \( W \). If \( \nu \) on the complex sphere has the property that its norm is less than \( \delta_0 \), then \( \rho(\Phi(z), z) < \log 2 \) in our simply connected domain \( W \) for all \( \Phi \) that is \( \nu \)-conformal map of \( \mathbb{C}_\infty \) onto itself and \( \Phi(W) = W \) and also the map \( \Phi \) is the identity map \( I \) on the boundary of \( W \).

By this lemma, we conclude that the composition map \( g^{-1}\Phi_t g \) of \( \mathbb{D} \) onto itself has bounded displacement function where displacement function of a homeomorphism \( \Phi \) from a simply connected domain \( \Omega \) onto \( \Omega \) is defined by

\[ z \mapsto \varrho(\Phi z, z) \]

where \( \varrho \) is the hyperbolic metric of our simply connected domain \( \Omega \) which is not conformally equivalent to \( \mathbb{C}_\infty \) or \( \mathbb{C} \). Then, it extends to \( I \) on the boundary of \( \mathbb{D} \).

Now, we’ll make some observations. Since the composition of analytic maps is analytic, the below map is analytic:

\[ \mathbb{D}[\mu_t] \xrightarrow{g} W[\mu_t] \xrightarrow{\phi_t} W[0] \xrightarrow{h} \mathbb{D}[0] \]

So, the composition \( h\phi_t g \) is a \( \mu_t \)-conformal map of the unit disk \( \mathbb{D} \) onto \( \mathbb{D} \). Call this composition map \( \psi_t := h\phi_t g \). It extends to a homeomorphism of \( \overline{\mathbb{D}} \) onto \( \overline{\mathbb{D}} \). Besides,
we have the following equality in the open disc:

\[ \psi_0^{-1} \psi_t = g^{-1} \Phi_t g, \]

\[ (g^{-1} \phi_0^{-1} h^{-1} b \phi_t g = g^{-1} \phi_0^{-1} \phi_t g = g^{-1} \Phi_t g). \]

So, on the boundary of \( \mathbb{D} \); as we earlier said that \( g^{-1} \Phi_t g \) extends to identity \( I \) on \( \partial \mathbb{D} \), we have \( \psi_t = \psi_0 \).

Recall Theorem 3.1.1. and say \( \Psi_t \) is a \( \mu_t \)-conformal map of \( \mathbb{D} \) onto \( \mathbb{D} \). Regarding the theorem, we say that \( \psi_t \Psi_t^{-1} \) is analytic. Then its inverse \( \psi_t^{-1} \Psi_t^{-1} \) is analytic, hence it’s conformal too. Since conformal maps of \( \mathbb{D} \) can be expressed as Möbius automorphisms, we can write \( \psi_t = M_t \Psi_t \) for some Möbius automorphism \( M_t \) of the unit disk \( \mathbb{D} \). As \( \psi_t = \psi_0 \) on \( \partial \mathbb{D} \), we deduce:

\[ \psi_t = M_t \Psi_t = M_0 \Psi_0 = \psi_0 \]
on the boundary of \( \mathbb{D} \).

Allow me to explain the path we took to prove our Main Theorem which Sullivan proved: Firstly, we will obtain some contradiction to our Lemma 3.2.1., then following it, this contradiction will complete the proof. For the desired contradiction, we should construct some maps like above \( \Psi_t \) in order to obtain \( \Psi_t = \Psi_0 \) on some open arc of the boundary of the unit disk. Then, we can conclude that if they are equal on three distinct points, then they’re equal. So, we need to check the open arc, which will give us that they’re equal everywhere. In this way; we can get that \( M_t = M_0 \), and this will mean that \( \Psi_t = \Psi_0 \). But, our construction will clarify that this is not the case. So, let’s begin to construct the maps \( \Psi_t \) with the desired properties and complete the remaining part of the proofs.

Recall that a rational map of degree \( d \) is of the form

\[ R(z) = \frac{a_0 + a_1 z + \ldots + a_d z^d}{b_0 + b_1 z + \ldots + b_d z^d}. \]

Considering it as a vector where it is formed by the coefficients of the above polynomials,

\[ (a_0, a_1, \ldots, a_d, b_0, b_1, \ldots, b_d) \]

and noticing that these are complex values, we see that this defines a \( 2d + 2 \) dimensional complex vector space. Making some coefficient 1 by dividing all the other
terms, we can have dimension $2d + 1$. Then, considering them as reals, we have dimension $4d + 2$ (which is $2(2d + 1)$).

Now, take an integer $N$ bigger than $4d + 2$ and construct the cube $C := [0, \epsilon_1]^N$, where $\epsilon_1$ is sufficiently small. For each vector $T \in C$, we should construct a Beltrami coefficient $\mu_T$ on $\mathbb{D}$ and a $\mu_T$-conformal map $\Psi_T$ from $\mathbb{D}$ to $\mathbb{D}$.

Firstly, we will construct the maps $\Psi_T$. We need a function $\omega_j$ for this construction. For the definition of this function, divide $[0, 2\pi]$ into $2N$ equal arcs. So, the lengths of our arcs will be $\frac{2\pi}{N}$. Name that consecutive arcs $\sigma_1, \tau_1, \sigma_2, \tau_2, \sigma_3, \tau_3, \ldots, \tau_N, \sigma_N$. We construct an infinitely many times differentiable function $\omega_j$ for each of these arcs $\sigma_j$ on $[0, 2\pi]$. This function $\omega_j$ will satisfy the following:

(i) $\begin{cases} \omega_j(x) > 0 & x \in \text{int}(\sigma_j) \\ \omega_j = 0 & \text{otherwise} \end{cases}$

(ii) $|\omega_j'(x)| < \frac{1}{2} \quad \forall x, j$

Now, for $z = re^{i\theta} \in \mathbb{D}$, define the map $\Psi_T(z)$ as

$$\Psi_T(z) = z e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)}.$$

Using the $\Psi_T$, we’ll compute the complex dilatation of it, or in other words, the Beltrami coefficient $\mu_T$.

Recall that

$$\frac{\partial \Psi_T}{\partial z} = \mu_T \frac{\partial \Psi_T}{\partial z} \implies \mu_T = \frac{\partial \Psi_T}{\partial z} \frac{\partial \Psi_T}{\partial \overline{z}}.$$

$$\frac{\partial}{\partial z} = \frac{1}{2} \frac{\partial}{\partial r} e^{i\theta} - \frac{1}{2ir} \frac{\partial}{\partial \theta} e^{i\theta},$$

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2} \frac{\partial}{\partial r} e^{-i\theta} + \frac{1}{2ir} \frac{\partial}{\partial \theta} e^{-i\theta}.$$

Then, $\mu_T$ has become;

$$\mu_T = \frac{1}{2} \frac{\partial \Psi_T}{\partial r} e^{i\theta} - \frac{1}{2ir} \frac{\partial \Psi_T}{\partial \theta} e^{i\theta} = e^{2i\theta} \frac{\partial \Psi_T}{\partial r} - \frac{1}{ir} \frac{\partial \Psi_T}{\partial \theta}.$$

where

$$\Psi_T(z) = z e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} = re^{i\theta} e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)}.$$

So, by plugging $\Psi_T(z)$ into the above $\mu_T$, we get;
\[ \mu_T = e^{2i\theta}(e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j'(\theta)}) - \frac{1}{i\rho}(ir e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} + r e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} - i \sum_{j=1}^{N} t_j \omega_j'(\theta)) \]

\[ = e^{2i\theta} \frac{e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j'(\theta)} + e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} + e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} - e^{i \sum_{j=1}^{N} t_j \omega_j'(\theta))}{2e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} + e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} + e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} - e^{i \sum_{j=1}^{N} t_j \omega_j'(\theta))}. \]

Taking in \( e^{i\theta}e^{i \sum_{j=1}^{N} t_j \omega_j(\theta)} \) parantheses;

\[ \mu_T = \frac{e^{2i\theta} \sum_{j=1}^{N} t_j \omega_j'(\theta)}{2 + \sum_{j=1}^{N} t_j \omega_j'(\theta)}, \]

therefore for all \( T \) we have;

\[ |\mu_T| = \left| \frac{e^{2i\theta} \sum_{j=1}^{N} t_j \omega_j'(\theta)}{2 + \sum_{j=1}^{N} t_j \omega_j'(\theta)} \right| \leq \frac{\sum_{j=1}^{N} |t_j \omega_j'(\theta)|}{2 - \sum_{j=1}^{N} |t_j \omega_j'(\theta)|} \leq \frac{N\varepsilon_1}{2 - N\varepsilon_1}. \]

Notice that, the inequalities above are obtained by triangle inequality and the properties of \( \omega_j \) function. Also, as \( \omega_j = 0 \) on \( \tau_j \), when argument of \( z \) lies on any \( \tau_j \), \( \Psi_T(z) = ze^{i \sum t_j \omega_j} = z.1 = z \). Since \( T \) is a vector in the cube \( C \), it’s dimension is \( N \) and distinct values of \( T \) are distinct vectors, so \( \Psi_T \)’s varies. These are required for \( \Psi_T \) in the proof of No Wandering Domain Theorem. Further; the second inequality in (32) implies that as \( \varepsilon_1 \to 0 \), \( ||\mu_T||_\infty < \eta \) for any given \( \eta \). The proof of Lemma 3.2.1. will be finished once we fix \( \varepsilon_1 \). When we prove it, our main theorem’s proof will be done, too. It’s time to state the proof of Lemma 3.2.1.

**Proof.** Let \( \mu_T \) be any Beltrami coefficient, then [*] in the beginning of this section becomes;

\[ T \mapsto \mu_T \mapsto \nu_T \mapsto \phi_T \mapsto R_T = \phi_T R_{\phi_T}^{-1} \]

To mention about the idea of the proof, we first need to factor the map \( T \to R_T \) through the intermediate stage of a vector that the zeros and the poles of \( R_T \) constructs it. Then, we aim to show that, on some curve, the first factor of that vector, so
the composition, is constant. If we have the same $T$ on same curve, then we obtain the same result $R_T$. Replace $R$ by conjugate by a Möbius map to satisfy the following: $R(0) = 1$, $R$ has $d$ distinct poles $p_1, p_2, \ldots, p_d$ and $d$ distinct zeros $z_1, z_2, \ldots, z_d$ in $\mathbb{C}$. Recall that, $\phi_T$ fixes 0, 1 and $\infty$ if we combine it with a Möbius map. Owing to these fixings, we see that $R_T$ is the rational function whose zeros are $\phi_T(z_j)$’s and poles are $\phi_T(p_j)$’s, and this $R_T$ is unique. Also, $R_T(0) = 1$.

So, it is enough to show that, on some curve,

$$\prod : \mathcal{C} = (0, \varepsilon_1)^N \to \mathbb{C}^{2d}$$

$$T \mapsto (\phi_T(z_1), \phi_T(z_2), \ldots, \phi_T(z_d), \phi_T(p_1), \phi_T(p_2), \ldots, \phi_T(p_d))$$

$$\prod(T) = (\phi_T(z_1), \phi_T(z_2), \ldots, \phi_T(z_d), \phi_T(p_1), \phi_T(p_2), \ldots, \phi_T(p_d))$$

from $\mathcal{C}$ into $\mathbb{C}^{2d}$ is constant.

Ahlfors and Bers concluded in their original work [9] that, for all $z$, $t \mapsto \phi_t(z)$ changes as smoothly as $t \mapsto \mu_t(z)$. Thanks to the smoothness of the map $T \mapsto \mu_T(z)$, we have that $T \mapsto \nu_T(z)$ is smooth. Therefore, we get that the maps $t \mapsto \phi_T(z_i)$’s and $t \mapsto \phi_T(p_i)$’s are all smooth, which yields that $\prod$ is also a smooth map from $\mathcal{C}$ into $\mathbb{C}^{2d}$.

Now, define a new function $f$ where

$$f : \text{Rank}(\prod) : \mathcal{C} \to \mathbb{C}^{2d},$$

$$\max(f) = k \leq 2d$$

The inequality above follows from the fact that maximum rank can be the dimension. Since $f$ is upper semi-continuous, $f^{-1}(k)$ is open in $\mathcal{C}$. Call this open subset of $\mathcal{C}$, $\mathcal{C}_0$. Then,

$$\prod|_{\mathcal{C}_0} : \mathcal{C}_0 \to \mathbb{C}^{2d}$$

has constant rank derivative. It is time to state a theorem followed by the Preimage Theorem which follows from Implicit Function Theorem (which also follows from Inverse Function Theorem):
**Theorem 3.2.1.** ([1]) Say $M, N$ are smooth manifolds of dimension $m$ and $n$ respectively and $f$ is a smooth mapping s.t. $f : M \to N$. If $f$ has constant rank $k$ on $M$ and $p \in f(M)$, then $f^{-1}(p)$ is a regular, closed submanifold of $M$ whose dimension is equal to $M - k$.

Therefore, using the theorem above, we can conclude that

$$
\prod_{\mathcal{C}_0}^{-1}(p) \text{ is a } N - k \text{ dimensional submanifold}
$$

where $p$ is some point s.t. $p = \phi_{T_0} = (\phi_{T_0}(z_1), \phi_{T_0}(z_2), \ldots, \phi_{T_0}(z_d), \phi_{T_0}(p_1), \phi_{T_0}(p_2), \ldots, \phi_{T_0}(p_d))$.

\qed

Therefore, the proof of the No Wandering Domain theorem is also finished as we proved the lemma above.
In Chapter 3, we argued that the wandering domains cannot exist in one dimension, for rational maps of degree greater than or equal to two. Until the early 2000s, not much was known on the existence of Fatou components of rational maps in higher dimension. In chronological order:

- (1985) Sullivan proved that every component of the Fatou set of an rational map of degree greater than two is eventually periodic, and this was in dimension one (No Wandering Domain Theorem) [1].

- (2004) Lilov proved that the wandering Fatou components of polynomial skew-products cannot exist in the basin of an invariant super-attracting fiber. Also, he showed that, the orbit of any horizontal disk in the super-attracting basin must eventually intersect a fattened Fatou component. Based on that statement, he made the conclusion that there are no wandering Fatou components in that construction [2].

- (2016) Thanks to examples that are built in the paper of Astorg et. al., we can see that wandering Fatou components can exist in the special case where we work with polynomial skew-products with an invariant parabolic fiber [3]. They gave specific example of polynomial skew-products that have wandering Fatou components. This paper will be detailed in the next chapter.

- (2016) The mystery of wandering Fatou components near the attracting-fiber tried to be solved. Peters and Vivas showed that what Lilov was proved is false in the geometrically-attracting case. (ie. we cannot say that the orbit of any horizontal disk in the geometrically-attracting basin must eventually intersect
a fattened Fatou component \cite{5}). Notice that, in the geometrically attracting case, the existence of wandering Fatou components remains open as we cannot make the conclusion of intersection.

- (2018) In the published paper of Ji, the concept of strongly attracting fiber has come up. The map $P(t, z) = (\lambda t, f(t, z))$, is defined for $0 < |\lambda| < \lambda_0$ where $\lambda_0 > 0$ and depends on $f$. The conclusion has made stating that there are no wandering Fatou components near strongly attracting fiber.

- (2023) Buff and Raissy published a paper that briefly explains wandering Fatou components of polynomial skew-products with an invariant parabolic fiber, and gave specific examples. \cite{12}

4.1 POLYNOMIAL SKEW-PRODUCTS

In this section, we will introduce polynomial skew-products in order to get a grasp what Lilov mentioned in their main theorem. Here is the definition.

**Definition 4.1.1.** A two dimensional polynomial skew-product is of the form

$$F(t, z) = (f(t), g(t, z)).$$

The utility of this polynomial-skew product concept is that, it maps vertical lines to vertical lines. So, when $f(t_0) = t_0$, the fiber $\{t = t_0\}$ should be mapped to itself by this utility and in the case where $g_{t_0}(z)$ is a polynomial, we can use the theorem which Sullivan proved and conclude that this fiber does not consist 1-dimensional wandering Fatou components.

**Definition 4.1.2.** A local skew product is an analytic map $F : \Omega \to \mathbb{C}^2$ defined in a neighbourhood of $(0,0)$, which preserves vertical lines and fixes origin.

( i.e. $F(t, z) = (f(t), g(t, z))$, $f(0) = 0$, $g(0, 0) = 0$ and $(0, 0) \in \Omega \subset \mathbb{C}^2$ )

4.2 BULGING FATOU COMPONENTS

$\{t = 0\}$ is called a super-attracting fiber of $F$ if $f'(0) = 0$. In the case of super-attracting fiber, by change of variables, we may assume that $f(t) = t^{d_1}, d_1 \geq 2$. 

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Definition 4.2.1. A bulging(fattened-up) Fatou component $U \subset \mathbb{C}_0$ is a Fatou component of $f_0 = F(0, z)$ where $U$ is contained in a Fatou component $\tilde{U} \subset \mathbb{C}^2$ of $F$.

Theorem 4.2.1. [NO WANDERING FATOU COMPONENTS] Say $F(t, z)$ is a skew product such that $F(t, z) = (t^{d_1}, f_t(z))$ where $d_1 \geq 2$ and $f_t(z)$ is of the form

$$f_t(z) = a_0(t)z^d + a_1(t)z^{d-1} + \ldots + a_d(t)$$

with $d \geq 2$ and $a_j(t)$ is analytic and defined for $|t| < \delta_0$, where $0 < \delta_0 < 1$ and $a_0(0) \neq 0$. Then, every Fatou component $U$ of $F$ such that $\text{proj}_t(U) \subset \{|t| < \delta_0\}$ eventually iterates to one of the fattened-up Fatou components of $f_0$.

In particular, there are no wandering Fatou components of $F$.

Proof. Assume on the contrary that there exists a Fatou component that does not iterate to one of the fattened-up Fatou components of $f_0$ (i.e. it is wandering) and take an open ball $U_0 \subset \mathbb{C}_{t_0}$ from a vertical slice of that wandering Fatou component. Take an arbitrary point $z_0$ from that vertical disc $U_0$.

We’ll use the notation

$$U_{n+1} := f_{t_n}(z_n)$$
$$t_{n+1} := t^{d_1} \quad \ldots = t^{d_1}$$
$$z_{n+1} := f_{t_n}(z_n)$$

The proof includes some constants and notation that were used in the Ph.D. Thesis of Lilov. One can review the third section of Lilov’s thesis to see it more clearly. Without going into details, let’s list them:

- The constants $M$, $\delta_{cr,1}$, $\delta_{cr,2}$, $\delta_d$, $r_{max}$ are the same as Lemma 3.2.1 in [2].
- The constant $R$ will be as in the 3.2. part of Lilov’s thesis.
- Let $\eta$ and $k$ use from Prop. 3.2.8 from the same thesis [2].
- Julia set of $f_0 := \mathcal{J}_0$
- $d_{\mathcal{J}_0(z)} := \text{dist}(z, \mathcal{J}_0) =$ distance between $z$ and $\mathcal{J}_0$
Figure 4.1

- $\varrho(z_n, U_n) := \sup \{r > 0 \mid B(z_n, r) \subset U_n\}$
  (a.k.a. the in-radius of $U_n$ centered at $z_n$)

Firstly, we’ll make some observations:

$$t = 0 \implies F(0, z) = (0, f_0(z))$$

In the second coordinate, we have $f_0(z) = p(z)$; so, we can behave it like an one-dimensional map. In dimension-one, we’ll be able to use No Wandering Domain Theorem. Also, in the figure, $|t| = 1$ line is drawn deliberately. Notice that, when $|t| > 1$, the forward orbit falls into the basin of attraction of $\infty$. Also, when $|t| < 1$, the points will converge to 0.

Now, in the first construction of our proof, we’ll show that the orbit $(t_n, z_n)$ cluster only on the Julia set of $f_0$, $J_0 \subset \mathbb{C}_0$. We’ve already said that $|t_n| \to 0$. If some subsequence of $z_n$, say $z_{n_k}$, has converged to a point $z_\infty$ in $U \subset \mathcal{F}_0$, then $(t_{n_k}, z_{n_k})$ would eventually falls into $U' \subset \mathbb{C}^2$. We’ve begun with a wandering domain $U_0$ and a point $z_0 \in U_0$, but it goes to a non-wandering domain. We’ve obtained a contradiction. Thus, it is impossible for $(t_0, z_0)$ to be contained in a wandering Fatou component of $F$. So; any limit $F^{n_j}(U)$ is contained in $J_0$. 

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Observe that $\mathcal{J}_0$ has no interior in $\mathbb{C}_0$ since, in the one-dimension, we have the fact that $\mathcal{J}_0$ either has no interior or equal to the whole $\mathbb{C}_\infty$. In the first observation located in the beginning of this page, we saw that when $t = 0$, we have a polynomial in the second coordinate. We know that polynomials are analytic. So, it takes open sets to open sets. However, as $\mathcal{J}_0$ has empty interior, we have no open set. Then it must be constant. Therefore, we came to the conclusion that $F^{n_j}|_U$ is a constant map. So, $U_0$ is mapped to a single point in $\mathcal{J}_0$. As any convergent sub-sequence has this property, we have $\rho_n \rightarrow 0$ (ie. $|U_n| \rightarrow 0$), and we can choose $n_0$ large enough so that $\rho_n < \frac{1}{3}, \forall n \geq n_0$.

Now, assume on the contrary that the vertical disc $B(z_{n_0}, \frac{\rho_{n_0}}{4})$ and $\mathcal{J}_0$ are separated (ie. $B(z_{n_0}, \frac{\rho_{n_0}}{4}) \cap \mathcal{J}_0 = \emptyset$ ). Then, $d_{\mathcal{J}_0}(z_{n_0}) \geq \frac{\rho_{n_0}}{4}$. Using Prop. 3.2.8. from Lilov’s thesis [2], we have

$$\tilde{r}(z_{n_0}) \geq \frac{k}{4^\eta} d_{\mathcal{J}_0}(z_{n_0})^\eta$$

$$\Rightarrow \frac{|t_{n_0}|}{\tilde{r}(z_{n_0})} \leq \frac{\rho_{n_0}^{\mu}}{k} \frac{4^\eta}{\rho_{n_0}^\eta} = \frac{4^\eta}{k} \frac{\rho_{n_0}^\eta}{\rho_{n_0}^\eta} < 1.$$  

As stated in the constants and notation part in the Chapter 3 from Lilov’s, by definition of $\tilde{r}$, there exists a function $\phi(t)$ for $|t| < \tilde{r}(z_{n_0})$. Also, notice that in the Figure 4.2, the graph of $\phi(t)$ is contained in $\mathcal{F}$ and $\phi(0) = z_{n_0}$.

![Figure 4.2](image-url)
Using the boundedness of $\phi(t) - \phi(0)$ by $R$, we have the following inequalities:

$$|\phi(t_{n_0}) - z_{n_0}| = |\phi(t_{n_0}) - \phi(0)| \leq R \frac{|t_{n_0}|}{\tilde{r}(z_{n_0})} \leq R \frac{\rho_{n_0}^{\nu}}{kd \tilde{r}(z_{n_0})^{\eta}} \leq R \frac{4^\eta}{k} (\rho_{n_0})^{\nu - \eta} < \frac{\rho_{n_0}}{4}$$

$$\Rightarrow |\phi(t_{n_0}) - z_{n_0}| < \frac{\rho_{n_0}}{4} \Rightarrow \phi(t_{n_0}) \in B(z_{n_0}, \frac{\rho_{n_0}}{4}) \subseteq U_{n_0}.$$

**Figure 4.3**

Observe in Figure 4.2 that $\phi(0) = z_{n_0} \in \mathcal{F}_0$, the graph of $\phi$ is connected and $\mathcal{F}$ contains it. So, there should be a fattened Fatou component that contains some neighbourhood of $\phi(t_{n_0})$. (We say fattened Fatou component when it contains a Fatou component of $f_0$. ) Denote it by $\tilde{\mathcal{F}}_{n_0}$. Also, in the Figure 4.3 above, $\tilde{\mathcal{F}}_0$ denotes the Fatou component of $\phi(0)$, and $\tilde{\mathcal{F}}_0 \subseteq \tilde{\mathcal{F}}_0$ (i.e. $\tilde{\mathcal{F}}_0$ is fattened). We see that $\mathcal{F}_{n_0} = \tilde{\mathcal{F}}_0$. By Sullivan, we know that $f_0$ is nonwandering. So, $U_{n_0}$ cannot be wandering, therefore $U_{n_0}$ is periodic or preperiodic but it contradicts with our initial assumption that $U_0$ is wandering. Hence, our conjecture stating that the vertical disc $B(z_{n_0}, \frac{\rho_{n_0}}{4})$ and $\mathcal{J}_0$ are separated fails down. For this reason, $B(z_{n_0}, \frac{\rho_{n_0}}{4}) \cap \mathcal{J}_0 \neq \emptyset$.

Now, since the intersection of $B(z_{n_0}, \frac{\rho_{n_0}}{4})$ and $\mathcal{J}_0$ is not empty, we can choose a point from the intersection, say $w_{n_0}$. Call $\rho'_{n_0} := \frac{\rho_{n_0}}{4}$ and $B := B(w_{n_0}, \rho'_{n_0})$. Observe that $B$ is included in $U_{n_0}$. Choose another point from $B$, for our arbitrary choice $z'_{n_0}$;
by triangle inequality,
\[ |z'_{n_0} - z_{n_0}| \leq |z'_{n_0} - w_{n_0}| + |w_{n_0} - z_{n_0}|. \]

Also, as points are chosen from \( B \),
\[ |z'_{n_0} - z_{n_0}| \leq 2\rho'_{n_0}. \]

Figure 4.4

Now, make the below observation using Figure 4.4:
\[ \varrho(z'_{n_0}, U_{n_0}) \geq \varrho(z_{n_0}, U_{n_0}) - |z'_{n_0} - z_{n_0}| \geq 4\rho_{n_0} - 2\rho'_{n_0} \geq \rho'_{n_0} \]

Therefore; we get that \( B(z'_{n_0}, \rho'_{n_0}) \) is included in \( U_{n_0} \).

Similarly, we will use the notation \( z'_{n+1} := f_0(z'_n) \) for the forward orbit of \( z'_{n_0} \in B \). Choose a point from \( B \), call \( z'_{n_0} \). If \( z'_{n} \) is included in \( U_{n} \), the forward iterates \( f^n_0(B) \subseteq U_{n} \) as the choice of \( z'_{n_0} \) was arbitrary. Recall that, in the beginning, the choice of \( U_0 \) has been made so that it is a relatively compact subset of the wandering Fatou component. Hence; diameter of \( U_{n} \) is uniformly bounded. So, by Montel Theorem, it follows that \( f^n_0 \) restricted to \( B \) is a normal family. But, we chose \( w_{n_0} \) in \( B \cup J_0 \), so this contradicts with being in the normal family. Therefore, it is enough to prove that \( z'_{n} \) is in \( U_{n} \). In order to proceed the proof, we first construct the sequence \( B(z'_{n_1}, \rho'_{n_1}) \) where \( B(z'_{n_1}, \rho'_{n_1}) \subseteq U_{n} \) and \( 0 < \rho'_{n_1} < \delta \). Additionally; we can find a subsequence \( \rho_{n_1} \) satisfying the following.
More specifically, we have:

\[ n_{l+1} - n_l \leq N + M \text{ such that } |t_n| \leq (\rho_n')^\mu \text{ where } n = n_l, l \geq 0. \]

We’ll use the induction technique:

\[ n = n_0 \implies (\rho_{n_0}')^\mu = \left( \frac{\rho_{n_0}}{4} \right)^\mu \geq \left( \frac{|t_n|}{4} \right)^\mu \geq |t_{n_0}|. \]

For \( n \geq n_0 \), we’ll make some observations; but initially, we need a lemma:

**Lemma 4.2.1.** [2] Let \( r_{max} \) be in the constants and notations part. For any \( \delta_{cr} > 0 \), there exist constants \( \delta_{d, \rho} (\delta_{cr}) := \delta_{d, \rho} \) where \( 0 < \delta_{d, \rho} < \frac{\delta_{cr}}{10} \), and \( c(\delta_{cr}) := c > 0 \) s.t. when \( |t| < r_{max} \), \( \text{dist}(z, C \cup C_t) > \delta_{cr} \) and \( 0 < \rho < \delta_{d, \rho} \), we have the following: \( f_t \) is 1-1 on \( B(z, \rho) \) and \( f_t(B(z, \rho)) \) contains a ball centered at \( f_t(z) \) of radius bigger than or equal to \( c \rho \).

Say \( n_m \) and \( \rho_p \) were constructed where \( l \geq 0, m \leq l \) and \( p \leq n_l \). For the rest of the proof, set \( n := n_l \). We have two cases: \( \text{dist}(z'_n, C \cup C_t) \geq \delta_{cr,2} \) and \( \text{dist}(z''_n, C \cup C_t) < \delta_{cr,2} \). For the first case, use the notation \( z''_{n+1} = f_{t_{n+1}}(z'_n) \). Lemma 4.2.1 implies that \( B(z''_{n+1}, c\rho'_n) = B(z''_{n+1}, \rho''_{n+1}) \subseteq U_{n+1} \). But,

\[ \implies B(z''_{n+1}, \rho) := B(z''_{n+1}, \rho''_{n+1} - |z'_n - z''_{n+1}|) \subseteq B(z''_{n+1}, \rho''_{n+1}) \]

\[ \implies \rho := \rho''_{n+1} - |z'_n - z''_{n+1}| \geq c\rho'_n - |f_0(z'_n) - f_{t_n}(z'_n)| \]

\[ \geq c\rho'_n - K|t_n|, \]

\[ \geq c\rho'_n - K(\rho'_n)^\mu \geq \frac{c}{2}\rho'_n =: \rho'_{n+1} > 0, \]

\[ \implies U_{n+1} \text{ contains } z'_{n+1}. \]

Moreover, we have:

\[ \frac{(\rho'_{n+1})^\mu}{|t_{n+1}|} = \left( \frac{c}{2} \right)^\mu (\rho_n')^\mu \geq \left( \frac{c}{2} \right)^\mu (\rho_n')^{\mu(1-d_1)} \geq \left( \frac{c}{2\rho_n'} \right)^\mu \geq 1 \]

as \( \delta \geq \rho'_n \).

Set \( n_{l+1} := n + 1 \).

For the second case; use the notation \( z''_{n+1} = f_{t_n}(z'_n) \) and \( z''_{n+k} = f_{t_{n+k-1}}(z''_{n+k-1}) \).
Since $2 \leq k \leq N + 1$, make the following observation:

$$|z''_{n+k} - z'_{n+k}| \leq |f_{n+k}(z''_{n+k-1}) - f_0(z''_{n+k-1})| + |f_0(z''_{n+k-1}) - f_0(z'_{n+k-1})|$$

$$\leq K|t_{n+k-1}| + K|z''_{n+k-1} - z'_{n+k-1}|$$

$$\leq K|t_n|d_{k-1} + K|z''_{n+k-1} - z'_{n+k-1}|.$$ 

Using the above inequality and $|z''_{n+1} - z'_{n+1}| = |f_{t_n}(z'_n) - f_0(z'_n)| \leq K|t_n|$ together, we have

$$|z''_{n+k} - z'_{n+k}| \leq K|t_n|d_{k-1} + K^2|t_n|d_{k-2} + K^3|t_n|d_{k-3} + \ldots + K^k|t_n|d_{k-k}$$

$$= K|t_n|d_{k-1} + K^2|t_n|d_{k-2} + \ldots + K^k|t_n|$$

$$\leq kK^k|t_n|\forall k, 1 \leq k \leq N + 1.$$

**Lemma 4.2.2.** We can find constants $\delta_1 \geq 0$ and $c \geq 0$ satisfying the following:

When $|t| \leq \delta_1$ and the arbitrary vertical ball $B(z, r)$ is included in $\mathbb{C}_t$, the ball $B(f_t(z), r') \subset \mathbb{C}_{r'}$, of radius $r' \geq crd$ is contained in $f_t(B(z, r))$.

Using Lemma 4.2.2. above, we get that the ball $B(z''_{n+k}, c_k(\rho'_n)^d) \subset U_{n+k}$. Call

$$\rho''_{n+k} := c_k(\rho'_n)^d.$$ 

calling $\rho := \rho''_{n+k} - |z'_{n+k} - z''_{n+k}|$ the ball $B(z'_{n+k}, )$ also contained in $U_{n+k}$. Also, notice that

$$\rho = \rho''_{n+k} - |z'_{n+k} - z''_{n+k}|,$$

$$\geq c_k(\rho'_n)^d - K^k|t_n|,$$

$$\geq c_k(\rho'_n)^d - kK^k\rho''_n,$$

$$\geq c_{N+1}(\rho'_n)^{d^{N+1}} - (N + 1)K^{N+1}\rho'_n,$$

$$\geq c'(\rho'_n)^{d^{N+1}}.$$ 

Say $\rho''_{n+k} := c'(\rho'_n)^{d^{N+1}}$. Fix $z''_{n+N+2} = z'_{n+N+2}$ and $z''_{n+N+j} = f_{t_n+N+j-1}(z''_{n+N+j-1})$ for $j \in \{3, 4, ..., M\}$. To conclude, we need two lemmata:

**Lemma 4.2.3.** We can find constants $\alpha, r_{\max}$ and $K$ where $0 < \alpha \leq 1$, $K$ is positive and $0 < r_{\max} < \min(1, \delta_0)$ so that the intersection of the $t=0$ line and an arbitrary
connected component of $C \cap \{|t| < r_{\text{max}}\}$ is non-empty, moreover, the intersection point is unique. Call the connected component $C_k$ and the unique point $c_0$. Then, the following holds for any arbitrary point $(t, z)$ in the connected component $C_k$;

$$|f_t(z) - f_0(c_0)| \leq K|t|^\alpha.$$  

**Lemma 4.2.4.** Consider the constant $r_{\text{max}}$ as in the above lemma. Choose a $\delta_{cr}$ positive. We can find constants $\delta_d, \delta_J$ and $c > 0$ where $0 < \delta_d, \delta_J < \delta_{cr} \leq \delta_{cr}/10$ satisfying that if $|t| < r_{\text{max}}$ and $0 < \rho < \delta_d < \text{dist}(z, C \cap C_t)$, then $f_t$ is 1-1 on the ball $B(z, \rho)$. Moreover, $f_t(B(z, \rho))$ contains a ball centered at $f_t(z)$ and of radius bigger than or equal to $c\rho$.

Applying Lemma 4.2.4. to $z''_{n+N+j}$ where $j \in [2, M]$ and using the equation 3.3.2 in Lilov's thesis, we can see that the ball $B(z''_{n+N+j}, \rho''_{n+N+j})$ is contained in $U_{n+N+j}$ and $\rho''_{n+N+j} \geq c^{j-1} \rho' n + N + j$.

Now, observe the following;

$$\rho''_{n+N+j} - |z'_{n+N+j} - z''_{n+N+j}| \geq c^{j-1} \rho' n + N + j \geq c^{j-1} \rho' n + (j-1)K^{j-1}|t_{n+N+1}|$$

$$= c^{j-1} \rho' n + (j-1)K^{j-1}|t_{n+1}|d_1^{N+1}$$

$$\geq c^{j-1} \rho' n + (j-1)K^{j-1}(\rho' n)^{d_1^{N+1}}$$

$$\geq \frac{c^{j-1} \rho' n}{2} d_1^{N+1}.$$  

Last inequality follows from the choices of constants we make in the beginning of the proof. Thanks to that choices; we see

$$c\rho - K\rho^\mu \geq \frac{c}{2} \rho \geq 1$$

for any $0 < \rho < \delta < \frac{1}{9}$.  

Particularly, we have that $z'_{n+N+M}$ is included in $U_{n+N+M}$.
Now; call $\rho'_{n+N+j} := \frac{c^{j-1}c'}{2}(\rho'_n)^{dN+1}$. Then;

$$
\frac{(\rho'_{n+N+M})^\mu}{|t_{n+N+M}|} = \left( \frac{c^M \rho'}{2} \right)^\mu (\rho'_n)^{dN+1} 
\geq \left( \frac{c^M \rho'}{2} \right)^\mu (\rho'_n)^{d^{N+1}-d^{N+M}} 
\geq \left( \frac{c^M \rho'}{2} \right)^\mu (\rho'_n)^{d^{N+1}-2^{N+M}} \geq 1.
$$

Plug $n_{l+1} = n + N + M$, then the proof of the No Wandering Fatou Components Theorem is complete. \qed
Definition 5.0.1 (Projective Spaces). Let $V$ be a vector space and $K$ be a field. The projective space over $V$, $\mathbb{P}(V)$ is the set of equivalence classes where the equivalence relation $\sim$ is defined by $a \sim b$ when $a = \lambda b$, where $\lambda \in K - \{0\}$. The elements of that projective space can be seen as points. They are notated as coordinates using brackets and :, i.e.

$$[x_0 : x_1 : x_2 : x_3 : \ldots : x_n]$$

and this emphasises the equivalence class. Notice that if $[x_0 : x_1 : x_2 : x_3 : \ldots : x_n]$ are projective coordinates of some point, then when $\lambda \in K - \{0\}$, the projective coordinate $[\lambda x_0 : \lambda x_1 : \lambda x_2 : \lambda x_3 : \ldots : \lambda x_n]$ corresponds to that point as well.

5.1 ATTRACTING, REPELLING FATOU COORDINATES AND LAVAURS MAP

To define Lavaurs Maps, first we need to define attracting Fatou coordinates and repelling Fatou coordinates as their composition gives us the Lavaurs map. In this section, $f$ will be a polynomial from $\mathbb{C}$ to $\mathbb{C}$, which is of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \mathcal{O}(z^4)$$

where $a_2$ is a complex number other than zero.

Changing coordinates by

$$Z = \frac{-1}{a_2 z},$$

we get the new map $F(Z)$ as

$$F(Z) = Z + 1 + \frac{1 - \frac{a_3}{a_2}}{Z} + \mathcal{O}(\frac{1}{Z^2}).$$
Call
\[ b := 1 - \frac{a_3}{a_2}, \]
we obtain
\[ F(Z) = Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right). \]

Recall that a holomorphic function on an open subset of the complex plane is called univalent if it is 1-1. Suppose \( R > 0 \) is large enough that \( F \) is univalent on \( \mathbb{C} \setminus \overline{D}(0, R) \), and the below inequalities hold
\[
\sup_{|Z| > R} |F(Z) - Z - 1| < 1/10, \\
\sup_{|Z| > R} |F'(Z) - 1| < 1/10.
\]

Also, we have some notations:
\[
\mathbb{H}_R := \{ Z \in \mathbb{C} | \text{the real part of } Z \text{ is bigger than } R \} \\
-\mathbb{H}_R := \{ Z \in \mathbb{C} | \text{the real part of } Z \text{ is smaller than } -R \} \\
\log z : \mathbb{C} \setminus \mathbb{R}^- \to \mathbb{C} := \text{the principal branch of logarithm}
\]

Notice that,
\[
F(Z) - b \log(F(Z)) = Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right) - b \log(Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right)), \\
= Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right) - b \log(Z(1 + \frac{1}{Z} + \frac{b}{Z^2} + O\left(\frac{1}{Z^3}\right))), \\
= Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right) - b(\log(Z) + \log(1 + \frac{1}{Z} + \frac{b}{Z^2} + O\left(\frac{1}{Z^3}\right))).
\]

Using the following fact
\[
\log(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots;
\]
we get
\[
F(Z) - b \log(F(Z)) = Z + 1 + \frac{b}{Z} + O\left(\frac{1}{Z^2}\right) - b - b(1 + \frac{1}{Z} + \frac{b}{Z^2} + O\left(\frac{1}{Z^3}\right)) + \ldots, \\
F'(Z) - b \log(F'(Z)) = Z - b \log(Z) + 1 + O\left(\frac{1}{Z^2}\right).
\]

As \( F \) is univalent on \( \mathbb{C} \setminus \overline{D}(0, R) \), it is univalent on \( \mathbb{H}_R \), too. So, when \( Z \in \mathbb{H}_R \), we
have

\[
F(F(Z)) - b \log(F(F(Z))) = F(Z) - b \log(F(Z)) + 1 + \mathcal{O}\left(\frac{1}{F(Z)^2}\right),
\]

\[
F^{\circ 2}(Z) - b \log(F^{\circ 2}(Z)) = Z + 1 + \frac{b}{Z} + \mathcal{O}\left(\frac{1}{Z^2}\right) - b \log(F(Z)) + \mathcal{O}\left(\frac{1}{F(Z)^2}\right),
\]

\[
= Z - b \log(Z) + 2 + \mathcal{O}\left(\frac{1}{F(Z)^2}\right).
\]

So; as \( m \) tends to \( \infty \),

\[
F^{\circ m}(Z) - b \log(F^{\circ m}(Z)) = Z - b \log(Z) + m + \mathcal{O}(1).
\]

Then, the sequence of univalent maps

\[
Z \mapsto F^{\circ m} - b \log(F^m(z)) - m
\]

is normal. Moreover, it is locally uniformly convergent to a univalent map \( \Phi_F : \mathbb{H}_R \to \mathbb{C} \) which satisfies the following

\[
\Phi_F \circ F = T_1 \circ \Phi_F
\]

where \( T_1 \) is translation by 1 (ie. \( T_1(Z) = Z + 1 \)).

Use the notation \( \Re(Z) \) for the real part of \( Z \). As \( \Re(Z) \) tends to infinity,

\[
\Phi_F(Z) = Z - b \log(Z) + o(1).
\]

Recall that, in the beginning of this section, we changed coordinates by

\[
Z = \frac{-1}{a_2 z}.
\]

When we return to the initial coordinate \( z \in \mathcal{B}_f \) where the parabolic basin of \( f \) denoted by \( \mathcal{B}_f \), we have that the sequence of mappings

\[
z \mapsto -\frac{1}{a_2 f^m(z)} - m - b \log(m) \in \mathbb{C}
\]

is locally uniformly convergent to an attracting Fatou coordinate \( \phi_f : \mathcal{B}_f \to \mathbb{C} \). This attracting Fatou coordinate semiconjugates \( f : \mathcal{B}_f \to \mathcal{B}_f \) to the translation map \( T_1 : \mathbb{C} \to \mathbb{C} \). To write semiconjugation more clearly;

\[
\phi_f \circ f = T_1 \circ \phi_f.
\]

Also; as \( \Re\left(\frac{-1}{z}\right) \to \infty \), the following will be satisfied;

\[
\phi_f(z) = -\frac{1}{a_2 z} - b \log\left(\frac{-1}{a_2 z}\right) + o(z).
\]
Define the attracting petal of \( f \) by
\[
P^\text{att}_f := \{ z \in \mathbb{C} \ ; \ \mathcal{R}( - \frac{1}{a_2 z} ) > R \}.
\]

Observe that, when we strict \( \phi_f \) to \( P^\text{att}_f \), we obtain the map
\[
z \mapsto \Phi_F( - \frac{1}{a_2 z} )
\]
where we can say that it is \( 1 - 1 \) and analytic, which means univalent. Moreover, in \[51\] the convergence is locally uniform with respect to the function \( f \) in \( B := \{(f, z); z \in B_f \} \) which is an open set.

Now, we have the following proposition;

**Proposition 5.1.1.** \([3]\) The map \( \phi_f \) has an analytic dependence on \( f \).

Now, we’ll move on repelling Fatou coordinate. As \( \mathcal{R}(Z) \to -\infty \), we have
\[
F(Z + b \log(-Z)) = Z + b \log(-Z) + 1 + \frac{b}{Z + b \log(-Z)} + \mathcal{O}(\frac{1}{Z^2}),
\]
\[
= (Z + 1) + b \log(-Z - 1) + \mathcal{O}(\frac{1}{Z^2}).
\]

Then, when \( \mathcal{R}(Z) < -R \) and \( R \) is sufficiently large, as \( m \) tends to infinity, we get;
\[
F^m(Z - m) + b \log(m - Z) = \mathcal{O}(1).
\]

In that case, we have that the sequence of univalent mappings
\[
Z \mapsto F^m(z - m + b)
\]
is locally uniformly convergent to the map \( \Psi_F : -\mathbb{H}_R \to \mathbb{C} \). This map semiconjugates \( F \) to the translation map \( T_1 : \mathbb{C} \to \mathbb{C} \) which is
\[
\Psi_F \circ T_1 = F \circ \Psi_F.
\]

Also, as \( \mathcal{R}(Z) \to -\infty; \)
\[
\Psi_F(Z) = Z + b \log(-Z) + o(1).
\]

When we return to the initial coordinate, we get the sequence of maps
\[
\mathbb{C} \ni Z \mapsto \frac{1}{a_2 \cdot (Z - m + b \log m)}
\]
\[
(52)
\]
is locally uniformly convergent to an repelling Fatou parametrization \( \psi_f : \mathbb{C} \rightarrow \mathbb{C} \) where it semiconjugates \( T_1 : \mathbb{C} \mapsto \mathbb{C} \) to \( f : \mathbb{C} \mapsto \mathbb{C} \). To write it more clearly, it satisfies

\[
\psi_f \circ T_1 = f \circ \psi_f,
\]

and moreover it satisfies the following equation as \( \mathcal{R}(Z) \rightarrow -\infty \),

\[
-\frac{1}{\psi_f(Z)} = Z + b \log(-Z) + o(1).
\]

**Proposition 5.1.2.** \([3]\) The map \( \psi_f \) has an analytic dependence on \( f \).

As we’ve described attracting Fatou coordinates and repelling Fatou coordinates, we are finally able to define the Lavaurs Maps. Although, we will be dealing with Lavaurs maps with phase zero, the generalized definition will be stated.

**Definition 5.1.1.** The Lavaurs map with phase \( \sigma \) is defined as follows:

\[
\mathcal{L}_{f,\sigma} := \Psi_f \circ T_{\sigma} \circ \Phi_f : B_f \rightarrow \mathbb{C}
\]

where \( T_{\sigma} \) is translation by \( \sigma \), that is \( T_{\sigma}(z) := z + \sigma \).

Notice that,

\[
\mathcal{L}_{f,0} := \mathcal{L}_f = \Psi_f \circ T_0 \circ \Phi_f = \Psi_f \circ \Phi_f.
\]

**Proposition 5.1.3.** \([12]\) Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a polynomial satisfying \( f(z) = z + z^2 + O(z^3) \) where \( f \) has that expansion at 0. Let \( (M_n)_{n \geq 0} \) be a sequence of integers, \( (\varepsilon_n)_{n \geq 0} \) be a sequence of real numbers close to 0 and satisfies the following,

\[
M_n - \frac{\pi}{\varepsilon_n} \rightarrow 0
\]

Now, we’ll define a new map \( \varepsilon_f \) such that

\[
\varepsilon_f := \phi_f \circ \psi_f : \mathcal{U}_f \rightarrow \mathbb{C}.
\]

The domain \( \mathcal{U}_f \) above is defined as \( \mathcal{U}_f := \psi_f^{-1}(B_f) \). Also, there is semiconjugation between \( \varepsilon_f : \mathcal{U}_f \rightarrow \mathbb{C} \) and \( \mathcal{L}_f : B_f \rightarrow \mathbb{C} \) by \( \psi_f \), ie. \( \psi_f \circ \varepsilon_f = \mathcal{L}_f \circ \psi_f \).

Proposition 5.1.1. and Proposition 5.1.2. together yield the following proposition:
Proposition 5.1.4. The map $\varepsilon_f$ defined above and the Lavaurs map $L_f$ have analytic dependence on $f$.

Below, the commutation between $\varepsilon_f$ and the translation map $T_1$ is observed. Begin with writing $\phi_f \circ \psi_f$ instead of $\varepsilon_f$:

$$\phi_f \circ \psi_f \circ T_1 = \phi_f \circ f \circ \psi_f.$$  

The equality above is followed by the fact that $\psi_f$ semi-conjugates $T_1$ and $f$. Then,

$$\phi_f \circ f \circ \psi_f = T_1 \circ \phi_f \circ \psi_f$$  

which is also followed by the semiconjugation between $T_1$ and $f$ via $\psi_f$. Combining the two equalities; we get:

$$\phi_f \circ \psi_f \circ T_1 = T_1 \circ \phi_f \circ \psi_f,$$

which is equal to

$$\varepsilon_f \circ T_1 = T_1 \circ \varepsilon_f.$$  

Thus, we are able to conclude that $\varepsilon_f$ commutes with the translation map $T_1$.

Define the universal cover map by

$$\exp : \mathbb{C} \ni Z \to e^{2\pi i Z} \in \mathbb{C} \setminus \{0\}.$$  

This universal map semi conjugates our map $\varepsilon_f$ defined in the previous page to a map $e_f$ which is the following map with $U_f := \exp(U_f)$,

$$e_f : U_f \to \mathbb{C} \setminus \{0\}.$$  

That is, $\exp \circ \varepsilon_f = e_f \circ \exp$. Recall that $U_f \subset \mathbb{C} \setminus \{0\}$.  

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5.2 RATIONAL MAPS WITH WANDERING FATOU COMPONENT

Now, it is time to state our main theorem, which can also be found in the paper of Astorg et al. To understand this main theorem, we needed to understand the Lavaurs map since the theorem uses the assumption related to it.

**Theorem 5.2.1 (FUNDAMENTAL THEOREM).** Say we have two polynomials \( f, g : \mathbb{C} \rightarrow \mathbb{C} \) that have the forms below:

\[
\begin{align*}
  f(z) &= z + z^2 + O(z^3), \\
  g(w) &= w - w^2 + O(w^3).
\end{align*}
\]

Define the polynomial-skew product map \( H : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) such that

\[
H(z, w) := \left( f(z) + \frac{\pi^2 w}{4}, g(w) \right).
\]

If \( L_f : B_f \rightarrow \mathbb{C} \) (Lavaurs map with phase 0) has an attracting fixed point, then the skew product map \( H \) defined above possesses a Fatou component which is wandering.

See the following proposition as an example in the complex case (coefficient is complex) whose Lavaurs map admits an attracting fixed point:

**Proposition 5.2.1.** Say \( f : \mathbb{C} \rightarrow \mathbb{C} \) is a polynomial of degree 3 which is defined as follows:

\[
f(z) := z + z^2 + cz^3
\]

where \( c \) is a complex number. When \( c \) is in a disk with center \((1-r)\) and radius \( r \) where \( r > 0 \) and close enough to 0, the Lavaurs map \( L_f : B_f \rightarrow \mathbb{C} \) has an attracting fixed point.

Also, we have another proposition which gives an example of a map in the real case (coefficient is real) whose Lavaurs map admits a super-attracting fixed point:

**Proposition 5.2.2.** Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a polynomial of degree 4 in the form

\[
f(z) := z + z^2 + az^4
\]

where \( a \) is a real number. Then, we can find a coefficient \( a \) which belongs to the interval \((-\frac{8}{27}, 0)\) satisfying that the Lavaurs map of \( f, L_f \), admits a fixed critical point.
Moreover, the fixed critical point is super-attracting.

In Proposition 5.1.6, set \( b = -0.2136 \in (-\frac{8}{27}, 0) \). The real wandering domain for the map \( P = (z + z^2 - 0.2136z^4 + \frac{z^2}{4}w, w - w^2) \) is illustrated in the paper Astrog et all.(2016). It is in Figure 5.1 below.

![Figure 5.1](image)

To prove Theorem 5.1.1. (our Fundamental Theorem), we need the following proposition as it will be deduced from Proposition 5.1.7. But first, we’ll let you know the idea behind it. It’s important that the initial point \((z_0, w_0)\) is chosen from
$B_f \times B_g$ to get the iterations of $P^n(z_0, w_0)$ returns infinitely many times close to the Lavaurs map $\mathcal{L}_f$’s attracting fixed point. The proof is constructed in such a way that for $n \geq n_0$, the return times are the integers $n^2$. We aim to analyze the orbit segment of length $(2n + 1)$, between $n^2$ and $(n + 1)^2 = n^2 + 2n + 1$.

Notice also that the aim of using polynomial skew products to construct examples is to build one-dimensional dynamics, as we can use some theorems in one dimension.

**Proposition 5.2.3.** The sequence of maps

$$\mathbb{C}^2 \ni (z, w) \rightarrow P^{(2n+1)}(z, g^{on^2}(w)) \in \mathbb{C}^2$$

is locally uniformly convergent to the map

$$B_f \times B_g \ni (z, w) \rightarrow (\mathcal{L}_f(z), 0) \in \mathbb{C} \times \{0\}$$

as $n$ tends to infinity.

Say the Lavaurs Map $\mathcal{L}_f$ has an attracting fixed point, $\xi \in B_f$. Take a disk $A$ around that attracting fixed point $\xi$, which satisfies that the image of $A$ under $\mathcal{L}_f$ is compactly contained in $A$. Observe that iterations of $\mathcal{L}_f^m(A)$ is convergent to the fixed point $\xi$ as $m$ tends to infinity. Also, take another arbitrary disk $B$ which is compactly contained in the parabolic basin of $g, B_g$. Using Proposition 5.2.3, we can find a positive integer $n_0$ such that the first projection of the map

$$P^{(2n+1)}(A \times g^{on^2}(B))$$

is compactly contained in $A$, and for all $n \geq n_0$,

$$\pi_1 \circ P^{(2n+1)}(A \times g^{on^2}(B)) \subseteq A.$$

Next is to state a lemma, but first define $U$ to be a connected component of the open set $P^{-n^2}(A \times g^{on^2}(B))$.

**Lemma 5.2.1.** The sequence of $(P^{on^2})_{n \geq 0}$ is locally uniformly convergent to $(\xi, 0)$ on $U$ where $\xi$ is an attracting fixed point of the Lavaurs map.

**Proof.** We’ll prove it by induction for every integer $n$ bigger than or equal to $n_0$. By definition of $U$, we have;

$$P^{on^2}(U) \subseteq A \times g^{on^2}(B).$$
When $n = n_0$, this holds. Assume that the inclusion above holds for some integer $n \geq n_0$. By induction, we must look at what happens in the $n + 1$ case. Observe that;

$$\pi_1 \circ P^{(n+1)^2}(U) = \pi_1 \circ \pi_1 \circ P^{(2n+1)}(n^2(U)) \subset \pi_1 \circ P^{(2n+1)}(A \times g^{n^2}(B)) \subset A.$$ 

Notice that first equality follows from the separation of $(n + 1)^2$ into $(2n + 1)$ and $n^2$. The first inclusion follows from the induction hypothesis. Lastly, the inclusion, in the end, follows from Prop. 5.1.7. Therefore, the equation 53 is obtained. Thus, $(P^{n^2})_{n \geq 0}$ is uniformly bounded. The conclusion that it is normal on $U$ yields by the following theorem that is stated in the paper of Narasimhan(1971), and also in the paper of Rudin(1986): 

\textbf{Theorem 5.2.2.} \cite{14,15} [Montel in Higher Dimension] Let $\Omega \subset \mathbb{C}^n$. A locally bounded family $\mathcal{F}$ of analytics functions on $\Omega$ is normal, i.e. any infinite sequence $f_k \in \mathcal{F}$ contains a subsequence that is uniformly convergent on every compact subset.

Therefore; $(P^{n^2})_{n \geq 0}$ in normal on $U$.

It is important to mention the following: say the sequence $P^{n^2}$ converges to some $\phi$ on $U$. As $P^{n^2}$ is holomorphic, $\phi$ is holomorphic. Then, $\phi(U)$ is open in $\mathbb{C}^2$ or $\phi$ is constant. But, $\phi(U)$ cannot be open in $\mathbb{C}^2$ as $\pi_2 \phi(U) = 0$. So, it is constant, and for some $z \in A$, any limit value is of the form $(z, 0)$.

Moreover; relating to the subsequence $n_k$, $(z, 0)$ is a cluster point if and only if $(L_f(z), 0)$ is a cluster point relating with the subsequence $(1 + n_k)$. Thus, under the Lavaurs map $L_f : A \to A$, the set of cluster limits is totally invariant. As it is totally invariant, this set should be diminished to $\xi$, the attracting fixed point of our Lavaurs map $L_f$. 

It follows from the definition of $P$ that the convergence of $P^{n^2}$ and $P^n$ are equivalent. Thus, we obtain the following Corollary of Lemma 5.1.1.

\textbf{Corollary 5.2.1.} $\mathcal{F}_P$ (the Fatou set of $P$) contains $U$.

Now that we have the competence to prove it, we can state the proof of Theorem 5.1.1, also called the Fundamental Theorem.

\textit{Fundamental Theorem’s Proof.} Say $\Omega$ is the component of $\mathcal{F}_P$ and $U \subset \Omega$ as in the Figure 5.2.
Using Lemma 5.1.1, we have that \( (P^{o(n^2+i)})_{n \geq 0} \) is locally uniformly convergent to \( P^{oi}(\xi, 0) \) on \( U \). Observe that, \( P^{oi}(\xi, 0) = (f^{oi}(\xi), 0) \) on \( U \). So, \( (P^{o(n^2+i)})_{n \geq 0} \) is locally uniformly convergent to \( P^{oi}(\xi, 0) = (f^{oi}(\xi), 0) \) on \( \Omega \) by Identity Principle.

![Figure 5.2](image)

To change the domain to \( P^{oi}(\Omega) \), notice that,

\[
P^{o(n^2+i)} = P^{o(n^2)}(P^{oi}) = P^{oi}(P^{on^2}) = P^{oi}(i-1)(f(\xi) + \frac{\pi^2}{4}, 0, g(0)) \rightarrow (f^{oi}(\xi), 0).
\]

Then, we can conclude that, the sequence \( (P^{o(n^2)})_{n \geq 0} \) is locally uniformly convergent to \( (f^{oi}(\xi), 0) \) on \( P^{oi}(\Omega) \).

Say \( i, j \in \mathbb{Z}^+ \cup \{0\} \) satisfying \( P^{oi}(\Omega) = P^{oj}(\Omega) \). Then, we see that

\[f^{oi}(\xi) = f^{oj}(\xi).\]

Recall that \( \xi \) was chosen from \( B_f \) (parabolic basin of \( f \)). Thus, \( \xi \) isn’t (pre)-periodic and \( i = j \). This shows that \( \Omega \) is not (pre)-periodic under the iteration of polynomial-skew product mapping \( P \). It fits with the definition of wandering Fatou component. Hence, for \( P, \Omega \) is a wandering Fatou component. □
REFERENCES


