ROBUST MAXIMAL COVERING LOCATION MODELS CONSIDERING PARTIAL COVERAGE

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ABSTRACT

ROBUST MAXIMAL COVERING LOCATION MODELS CONSIDERING PARTIAL COVERAGE

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Maximal Coverage Location Problem (MCLP) attempts to find a predetermined number of facilities to maximize the number of demand points that can be covered. In MCLP, while all demand points within a critical distance of a facility are completely covered, demand points exterior this region are not covered at all. In Partial MCLP (MCLP-P), another critical distance is introduced which allows coverage between two critical distances, monotonically decreasing with respect to demand points' distance from facilities. In this thesis, we study MCLP-P under coverage uncertainty. We utilize robust optimization framework and introduce two different strategies to hedge against uncertainty. We propose multiple solution approaches for both strategies. We show interpretation of the proposed robust optimization models from the perspective of game theory using payoff tables. We present the impact of the models and compare the performance of the proposed solution approaches on randomly generated datasets.

Keywords: Maximal coverage location problem, robust optimization, decision making under uncertainty, facility location, combinatorial optimization

ÖZ

KISMİ KAPSAMA ALTINDA GÜRBÜZ MAKSİMUM KAPSAMA MODELLERİ

Köksal, Burak
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Maksimum Kapsama Problemi (MCLP), kapsanacak talep miktarını maksimize edebilecek en iyi tesisleri bulmaya çalışır. MCLP'de tesisler, kritik mesafe adı verilen bir uzaklığa kadar yer alan tüm talep noktalarını tamamen kapsarken, bu uzaklığın dışındaki talep noktalarını hiç kapsamaz. Kısmi MCLP'de (MCLP-P) tanımlanan ikinci kritik mesafe, talep noktalarının tesislerden uzaklığına göre monoton olarak azalacak şekilde iki kritik mesafe arasında kapsamaya izin verir. Bu tezde, MCLP-P kapsama belirsizliği altında incelenmiştir. Gürbüz optimizasyon kullanılarak iki farklı çözüm yaklaşımı geliştirilmiştir. Önerilen çözüm stratejileri için farklı çözüm yöntemleri sunulmuştur. Sunulan gürbüz optimizasyon modelleri, sonuç tabloları kullanılarak oyun teorisi perspektifinden incelenmiştir. Önerilen çözüm yöntemlerinin etkisi ve modellerin performansı rassal olarak üretilmiş verisetleri üzerinde test edilmiştir.

Anahtar Kelimeler: Maksimum kapsama problemi, gürbüz optimizasyon, belirsizlik altında karar verme, tesis yerleştirme, kombinatoryal optimizasyon

To my family…

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LIST OF ABBREVIATIONS

ABBREVIATIONS

B&B	Branch-and-Bound
B&C	Branch-and-Cut
DRO	Distributionally Robust Optimization
DSM	Double Standard Model
FLP	Facility Location Problem
GNS	Greedy Neighborhood Search
HLP	Hub Location Problem
IP	Integer Program
LP	Linear Program
LSCM	Location Set Covering Model
MCLP	Maximal Coverage Location Problem
MCLP-P	Maximal Coverage Location Problem with Partial Coverage
MIP	Mixed Integer Program
MOILP	Multi-Objective Integer Linear Program
MOINP	Multi-Objective Integer Nonlinear Program
RNS	Random Neighborhood Search
RO	Robust Optimization
SO	Stochastic Optimization

CHAPTER 1

INTRODUCTION

Facility locations problems aim to choose the "best" possible sites for a set of facilities to meet a particular set of demand points. Maximal Coverage Location Problem (MCLP) deals with finding a predetermined number of facilities to maximize the number of demand points that can be covered. In the classical MCLP, it is assumed that all demand points within a critical distance from a facility are completely covered, while the demand points outside this critical distance are not covered at all (Church and ReVelle, 1974).

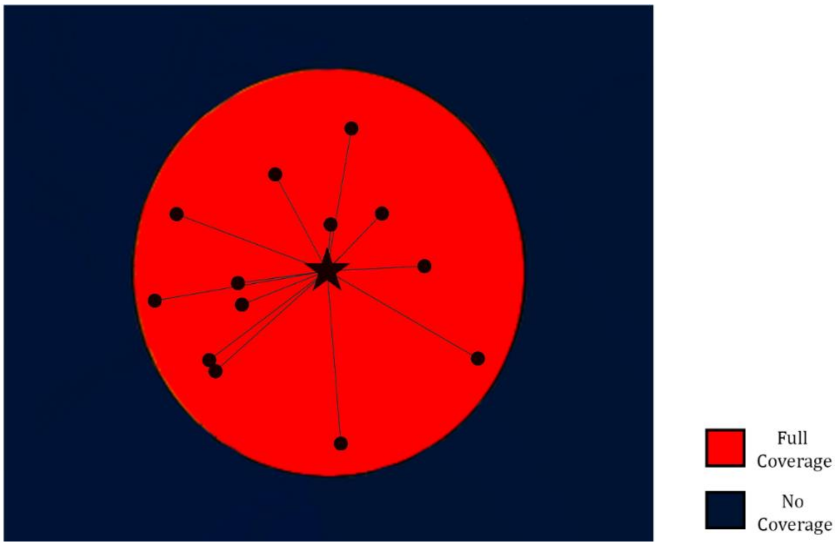


Figure 1.1. Coverage in the classical MCLP (★: Open facility, ●: Demand point)

However, in the classical MCLP problem, optimal solution is highly sensitive to the choice of the critical distance. Any demand point is fully covered until it reaches a certain critical distance from the center of the facility, but not covered at all exterior of this critical distance. Therefore, determining a single critical distance value may be problematic as it may lead to erroneous solutions. Based on this problem, Berman et al. (2003) and Karasakal and Karasakal (2004) define a second critical distance by introducing the idea of partial coverage.

In their study, while all demand points up to the minimum critical distance from the center of the facility are fully covered, demand points between the minimum and maximum critical distances to the facility are "partially" covered. This coverage decreases along with the distance from the facility. Full coverage is achieved up to the minimum critical distance. Between the minimum and the maximum critical distances, coverage gradually decreases as demand point approaches the maximum critical distance. Finally, facilities do not perform any coverage outside the maximum critical distance.

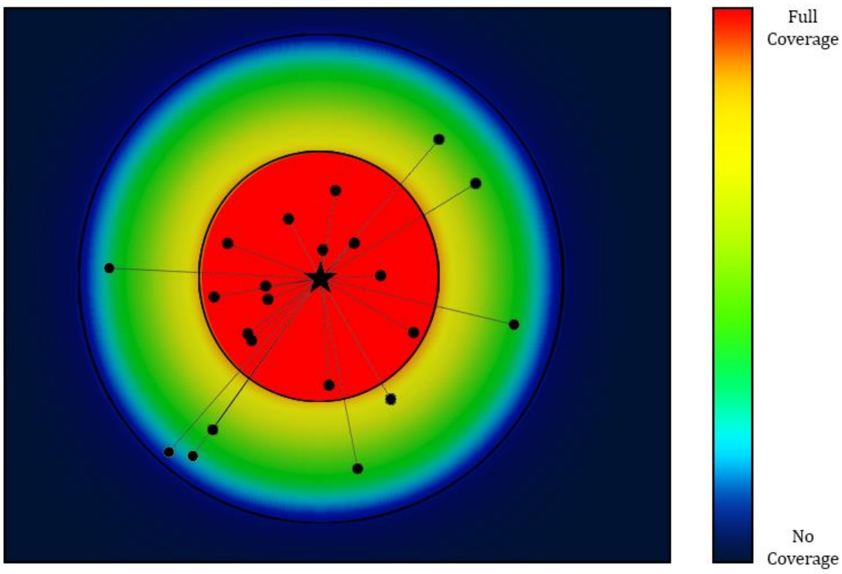


Figure 1.2. Coverage in the MCLP-P (★: Open facility, ●: Demand point)

This approach clearly enables us to find more realistic solutions than the classical MCLP approach. However, it is not easy to determine critical distances in real life. Critical distances of facilities are typically under the influence of many exogenous factors that are out of control of the decision-makers. Hence, disregarding their varying nature may result in finding solutions that are impractical in real-life. Therefore, the decision of selecting facilities that will cover the most demand points should be taken while carefully acknowledging this uncertainty.

We adopt the robust optimization methodology to address this uncertainty. In robust optimization, it is assumed that uncertain data belongs to an uncertainty set. Uncertainty set is a set that contains all possible values for uncertain parameters aimed to be dealt with in a robust optimization problem. Robust optimization is widely popular as it is computationally tractable for many different types of uncertainty sets and problems.

In this thesis, we introduce an approach to hedge against coverage uncertainty for MCLP-P based on the robust optimization framework presented in Bertsimas and Sim (2003). We propose two different models, namely robust and semi-robust models to deal with uncertainty. For the robust model, we propose a mixed integer linear programming formulation and a Benders decomposition algorithm to solve the problem. For the semi-robust model, we present two exact and two heuristic solution approaches. We propose a mixed integer linear program and a Benders decomposition algorithm as exact methods. For the Benders decomposition algorithm, we propose a method to obtain Benders cuts in linear time. Then, we present Greedy Neighborhood Search and Random Neighborhood Search as heuristics. We further investigate the process of finding robust solutions from the perspective of game theory using payoff tables, and present its interpretation.

Organization of this thesis is as follows. In Chapter 2, we present the literature review for the study. In Chapter 3, we introduce the robust model and the proposed solution approaches. In Chapter 4, we present the semi-robust model, and the developed solution methodologies. In Chapter 5, we present the computational experiments. In

Chapter 6, we present solution insights. In Chapter 7, we summarize our findings and conclude the thesis.

CHAPTER 2

LITERATURE REVIEW

2.1 Maximal Covering Location Problem

Berman and Krass (2002) investigate the MCLP case where partial coverage is allowed. In the study, degree of coverage for any demand point is a non-increasing step function of the distance to the closest facility. They show that the problem is equivalent to the uncapacitated facility location problem (UFLP).

Berman et al. (2003) study an extension of the generalized MCLP model analyzed in Berman and Krass (2002). They consider general forms of the coverage decay function in the paper. In addition to the UFLP-based formulation, authors develop an alternative formulation that yields significant computational improvements.

Drezner et al. (2004) minimizes the total weighted non-coverage of demand points instead of maximizing the coverage. They convert the formulation to the Weber problem by imposing a special structure on its cost function.

Karasakal and Karasakal (2004) formulates the problem based on the classical p-median formulation where they maximize the coverage of demand points instead of minimizing the total distance. In their study, they relax the restriction on the coverage function for their model by allowing the use of any coverage function if the coverage level decreases as distance increases. Authors develop a solution procedure based on Lagrangian relaxation.

Eydi and Mohebi (2018) examine the gradual MCLP with variable radius over multi-periods. Each facility has a fixed cost together with a variable cost which depends on the coverage radius of the facility. Authors attempt to maximize the coverage and minimize the relocation cost at the same time.

Berman et al. (2009) studies the combination of ordered median location problem and gradual coverage location problem. Ordered median location problem is a generalization of most of the classical location problems such as p-median or p-center which considers only relative customer-to-facility distance. By combining it with gradual coverage, authors are able to obtain a model that takes both relative and absolute customer-to-facility distances into account.

Drezner and Drezner (2014) discuss the gradual covering problem case where every facility is allowed to cover multiple demand points and each demand point is allowed to be covered by multiple facilities. Authors argue that partial coverage can be interpreted as the probability that the full coverage will occur. In the paper, authors aim to maximize the minimum coverage of each demand point.

Berman et al. (2019) discuss the multiple gradual cover location problem in the presence of joint coverage. Formulations in the study is based on the work of Drezner and Drezner (2014). However, different from the work of Drezner and Drezner (2014), authors aim to maximize the total joint cover of all demand points in this study.

Álvarez-Miranda and Sinnl (2019) propose an exact solution framework for the multiple gradual cover problem. They consider the case where joint coverage is allowed and build their formulations upon the work of Drezner and Drezner (2014) and Drezner et al. (2019). Authors present four different mixed integer programming formulations for the problem by exploiting the submodularity of the objective function.

Peker and Kara (2015) demonstrate the concept of gradual coverage to p-hub maximal covering problem. Authors present several mixed integer programming models that are applicable for both binary and partial coverages.

Karatas (2017) studies a multi-objective facility location problem which combines gradual coverage, cooperative coverage, and variable coverage concepts. Objectives in this study consists of covering demand points at a satisfactory level inexpensively and maintaining balanced workload among facilities.

Drezner et al. (2019) investigates the case where demand points are represented by circular discs instead of mathematical points. In the study, demand covered by a facility is obtained as the intersection area of the disc centered at the demand and the disc centered at the facility. Partial cover of the demand is the intersection area divided by the area of the demand's disc.

Drezner et al. (2020a) study an extension of the directional approach to gradual cover. Unlike the work of Drezner et al. (2019), they aim to maximize the minimum cover for all demand points.

Drezner et al. (2020b) discuss the gradual decline in attraction from 1 to 0 for the competitive facility location problems. Authors indicate that in competitive location problems, increasing function of market share is assumed as profit or revenue. Hence, maximizing market share is equivalent to maximizing profit or revenue. In this study, authors present formulations to maximize total market share for both single-facility and multi-facility cases.

Table 2.1 Relevant MCLP Studies

	Model Type		Factors Affecting Coverage		
Authors	**Deterministic**	**Stochastic**	**Distance**	**Facility Features**	**Solution Methods**
Drezner et al. 2020b	X			X	B&B, multi-start
Drezner et al. 2020a	X		X		Tabu search, simulated annealing
Álvarez-Miranda and Sinnl, 2019		X	X		MIP, B&C, greedy heuristic
Berman et al. 2019		X	X		Simulated annealing, ascent algorithm, tabu search
Drezner et al. 2019	X		X		Greedy heuristic, tangent line approximation, ascent algorithm, tabu search
Eydi and Mohebi, 2018		X	X		MIP, simulated annealing
Karatas, 2017		X	X	X	MOINP, MOILP
Peker and Kara, 2015	X				MIP
Drezner and Drezner, 2014		X	X		Tabu search, ascent algoritm
Berman et al., 2009	X		X	X	IP
Drezner et al., 2004	X		X		B&B
Karasakal and Karasakal, 2004	X		X		Lagrangian relaxation
Berman et al., 2003	X		X		Greedy heuristic
Berman and Krass, 2002	X		X		IP

2.2 Robust Optimization

In real life decision-making problems, it is not possible to know everything in advance in a deterministic sense. Hence, solving problems assuming that the data is precisely known and using exact parameters could yield impractical solutions. In order for output of a model to be more applicable in practice, such uncertainties should be taken into account and the goal should be to develop models that are immune to these uncertainties as much as possible. Knowing that, if we correctly define these uncertainties and try to hedge against them in the most appropriate way possible, we would obtain significantly better solutions.

There are various ways to deal with uncertainty in the decision-making literature. Two main approaches are stochastic and robust optimization. Stochastic optimization is based on the assumption that the probability distribution of uncertain data must be known. If this assumption holds for the data at hand, and if the stochastic reformulation is tractable, then the problem can be solved by stochastic optimization. However, it is not always the case. It may not be very straightforward to fit a probability distribution to data. In addition, chance constrained problems commonly are not computationally tractable. Yet, there are still many problems that can be solved using stochastic optimization, and it is one of the most popular approaches to handle uncertainty in optimization problems. The other common methodology is robust optimization. In robust optimization, the assumption is that the data belongs to an uncertainty set rather than a probability distribution. Robust optimization is popular as it is tractable for many types of uncertainty sets and problems. (Gorissen et al., 2015)

In this thesis, we address the coverage uncertainty in MCLP-P. In MCLP-P, coverage values are defined for each demand-facility pair and is a function of the distance between these pairs. Knowing that the MCLP-P is already NP-Hard, obtaining the underlying probability distribution of each coverage value for all demand-facility pairs and solving the resulting optimization problem would pose a significant challenge. Therefore, we adopt the robust optimization methodology as it allows to obtain rather tractable reformulations.

The first paper in robust optimizaton dates back to 1970s, however it has been mostly developed in the last 20 years (Gorissen et al., 2015). In Soyster (1973), author considers data uncertainty in columns. It aims protection at the highest level for each constraint, thus it is one of the most conservative approaches in the robust optimization literature. To overcome this over-conservatism, Ben-Tal and Nemirovski (2000) proposed ellipsoid uncertainty sets and developed algorithms to address convex optimization optimization problems with uncertain data. However, as it contains conic quadratic formulation, it cannot be applied to combinatorial optimization problems directly. Bertsimas and Sim (2004) introduce another approach which not only enables adjusting conservatism level quite flexibly by varying a single parameter, but also leads to computationally tractable reformulations. Based on their work published in that paper, in Bertsimas and Sim (2003), authors demonstrate that their approach can be adopted for discrete optimization and network flow problems. They address data uncertainty both in cost coefficients and constraints, and demonstrate that their approach retains the original nominal problem's complexity. In this thesis, we utilize the robust optimization framework introduced in Bertsimas and Sim (2003).

In this thesis, we additionally demonstrate the connection between discrete robust optimization and game theory for MCLP. In order to do that, we examine maximin and maximax concepts over payoff tables and explain their possible interpretations for discrete robust optimization along with illustrations.

2.3 Robust Optimization in Location Problems

Schmid and Doerner (2010) study ambulance location and relocation problem to cover potential future demand in a time-efficient manner. They prefer to approach this problem in a multi-period fashion since the travel times differ throughout the day and solving the static ambulance location problem may not be sufficient to address the problem effectively.

Dibene et al. (2016) extend this model by including multiple scenarios considering factors such as the time of day and the day of the week. With the use of real-world emergency data obtained from the Red Cross of Tijuana, the scenarios are generated. Authors attempt to solve all these scenarios in a single optimization problem. To solve this problem, authors propose static and robust versions of three different coverage models: the MCLP, the Location Set Covering Model (LSCM), and the Double Standard Model (DSM).

Vatsa and Jayaswal (2021) model the problem of assigning doctors to health centers as a robust capacitated multi-period MCLP with server uncertainty. Demand nodes can be covered fully or partially. Scenario dominance rules are presented to reduce the size of the formulation. Minimax regret approach is adopted.

In Lei et al. (2014), authors aim to maximize the expected demand coverage while considering possible facility failures. The suggested model considers geographically varying facility failure probabilities.

Álvarez-Miranda et al. (2015) examine the recoverable robust facility location problem. They define a location and allocation strategy in two stages such that the first stage solution they obtain should be robust to the data which can only be revealed in the second stage. Thereby, if required, it is possible to recover the solution in the second stage at low cost. Authors state that the proposed model is robust to any kind of provider-side, receiver-side, and in-between uncertainties.

Wang and Qin (2021) address partial coverage situation in the uncertain hub maximal covering location problem. Authors introduce the partial coverage parameter by considering travel times as uncertain variables. They present specific decay functions for the expected value of partial coverage parameter. Their objective is to maximize the service ability and economic effectiveness in a multi-objective model.

Coco et al. (2018) investigate the min-max regret MCLP. They address a generalization of the classical MCLP where they seek to find a set of columns with the maximum benefit sum in a matrix of benefits. Benefits of each column are uncertain and defined as interval data. They further define scenarios for all possible benefit realizations of these data intervals defined for each column. In their study, they enforce that each row is covered at least by a single column. Then they aim to minimize the maximum regret over all possible scenarios in their study. They propose exact and approximation algorithms. However, they indicate that even though the large running times, they obtain results with high optimality gaps.

In Chauhan et al. (2019), authors study MCLP with drones that have limited battery capacity. To acknowledge this fact, authors add a battery constraint which limits drones' travel distance. Chauhan et al. (2021) extends this work by considering uncertainties in battery capacity and consumption aiming to find robust solutions. Facilities act as launching sites for drones and have limited supply to cover the customer demand. Drones make single delivery trips until their battery is exhausted. They attempt to maximize the demand covered by drones while considering uncertainties in battery availability and consumption. The uncertainty is modeled by utilizing a penalty-based approach and gamma robustness.

Peng et al. (2017) formulate a two-stage robust facility location model that takes demand and transportation cost uncertainties, and facility disruptions into account. They address the uncertainty by introducing budget uncertainty set.

Saif and Delage (2021) discuss distributionally robust version of the capacitated facility location problem, where customer demand is the uncertain parameter.

Models for both single and two-stage problems are presented. In the single-stage problem, all decisions are made at the beginning, whilst in the two-stage problem location decisions are made under distributional ambiguity and demands are allocated to facilities once demands become known.

Santos et al. (2019) focus on the stochastic version of Equitable Sensor Location Problem which is a unique type of MCLP. The objective of this problem is to cover all locations equally given a limited number of sensors. Both ambiguous and resilient versions are considered. While the resilient version attempts to solve the problem under the assumption that the sensors are subject to partial or complete failure, the ambiguous version investigates the problem under uncertain surveying probabilities.

Baldomero-Naranjo et al. (2021) examine the single-facility MCLP on a network. They consider the case where the demand is uncertain with only a known interval estimation and distributed along the edges. To hedge against uncertainty in demand, authors propose a minmax regret model, where the facility can be located anywhere in the network.

Du and Zhou (2018) study p-center facility location problem under cost uncertainty. They adopt symmetric interval and multiple allocation strategy and utilize three uncertainty sets for the robust problem: box uncertainty, ellipsoidal uncertainty, and cardinality-constrained uncertainty. The objective is minimizing the maximum cost of covering a demand node.

Robustness takes significant place in the hub location problems (HLP) as inadequate applications most likely result in undesired outcomes such as high costs and discontented customers. Hence, robust hub location problems have been studied extensively in the literature.

Boukani et al. (2016) consider both single and multiple allocation HLP under fixed set-up cost and capacity uncertainties. Five different scenarios were defined for each uncertain parameter and minmax regret model is proposed.

Amin-Naseri et al. (2018) aim to minimize the overall transportation cost and maximum uncertainty in network by selecting location of the hubs and allocation of other nodes to the hubs. Authors employ a desirability function-based approach to consider both objectives.

Li et al. (2020) consider HLP under flow and set-up cost uncertainties. Budget uncertainty set is utilized to address the uncertainty. They offer formulations for both single and multiple allocation situations. The set-up cost of a hub depends on the total flow through the hub. They aim to obtain protected solutions against worst cases of different uncertain parameters.

Meraklı and Yaman (2016) investigate the robust multiple allocation p-hub median problem under polyhedral demand uncertainty. They address the uncertainty by utilizing hose and hybrid models. The hose model only imposes aggregate upper bounds on inbound and outbound traffic of each node, whereas the hybrid model additionally introduces lower and upper bounds on individual traffic demands. Their objective is to minimize the cost of the network under the worst-case scenario by employing a minmax criterion. In Meraklı and Yaman (2017), authors consider the capacitated hub location problem under hose demand uncertainty.

Ghaffarinasab (2022) approaches this problem with the same objective by using budget of uncertainty parameter which allows adjusting the level of conservatism. They utilize the budget of uncertainty approach to cope with uncertainty. Minimax criterion is used.

Table 2.2 Location Problems Under Uncertainty in the Literature

		Source of Uncertainty				Model Type		
Authors	**Problem Type**	**Cost**	**Demand**	**Travel Time**	**Other**	**Deterministic**	**Stochastic**	**Handling Uncertainty**
Köksal et al., 2023	MCLP		X			X		Γ-robustness
Wang and Qin, 2021	hub MCLP			X			X	Expected value model, chance constrained programming”
Ghaffarinasab, 2022	HLP		X			X		Minmax regret, Γ-robustness
Baldomero-Naranjo et al., 2021	MCLP		X					Minmax regret
Vatsa and Jayaswal, 2021	MCLP				X	X		Minmax regret
Saif and Delage, 2021	FLP		X				X	DRO
Chauhan et al., 2021	MCLP				X	X		Penalty based approach, Γ-robustness
Li et al., 2020	HLP	X			X	X		Uncertainty budget
Coco et al., 2018	MCLP				X	X		Minmax regret
Du and Zhou, 2018	FLP	X				X		Box, ellipsoidal, and cardinality-constrained uncertainty
Meraklı and Yaman, 2017	HLP		X			X		Hose uncertainty
Peng et al., 2017	Two-stage FLP	X	X		X	X		Budget uncertainty set
Boukani et al., 2016	HLP	X			X	X		Minmax regret
Dibene et al., 2016	MCLP, LSCM, DSM			X		X		Scenario-based optimization
Meraklı and Yaman, 2016	HLP		X			X		Hose uncertainty, hybrid uncertainty, minmax regret
Álvarez-Miranda et al., 2015	FLP	X	X	X	X	X		Two-stage RO
Schmid and Doerner, 2010	DSM			X		X		Multi-period modelling

2.4 Contributions to the Literature

Different from the studies presented in this chapter, we adopt robust optimization framework to address coverage uncertainty in MCLP-P. We propose two models to hedge against uncertainty, namely robust and semi-robust approach. For the robust approach, we propose a mixed integer linear optimization formulation and a Benders decomposition algorithm. For the semi-robust strategy, we present a mixed integer linear optimization formulation and a Benders decomposition algorithm as exact methods and two heuristic solution approaches. In Benders decomposition algorithm, we propose a method to obtain Benders cuts in linear time. Then, using game theory payoff tables, we demonstrate how the suggested robust optimization model may be interpreted from the game theory standpoint. Finally, we discuss the impact of our model and compare the proposed solution approaches.

CHAPTER 3

ROBUST MCLP-P

In this chapter, we consider MCLP-P under coverage uncertainty within the robust optimization framework. In MCLP-P, coverage is a function of the distance between each facility-demand point pair and the provided coverage level depends on the critical distance of each facility. Proposed robust MCLP-P model assumes that upper and lower bounds of critical distances are the only available information. Given this limited information, motivation of this study is to obtain the best possible coverage levels under worst-case scenarios. The level of conservatism could be simply adjusted with Γ parameter. In the following two sections we define the problem and provide a mathematical formulation of it. Then, we propose a mixed integer linear programming reformulation and a Benders decomposition algorithm to solve the problem.

3.1 Problem Definition

For the sake of completeness, we begin this chapter by presenting the deterministic MCLP-P formulation. Let I denote the set of demand points, J denote the set of all potential facility locations. S is the minimum critical distance, and T is the maximum critical distance for any potential facility j. Let c_{ij} represent the coverage level provided by potential facility j at demand point i and d_{ij} represent the distance from a demand point i to a potential facility j. M_i is the set of facilities that are eligible to cover demand point i. P is the number of facilities to be opened.

The MCLP-P modeled by Karasakal and Karasakal (2004) is as follows:

$$(\boldsymbol{MCLP-P}): \qquad \max_{x,y} \sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} \tag{1}$$

$$s.t. \sum_{j\in J} y_j = P \tag{2}$$

$$x_{ij} \le y_j \qquad \forall i \in I, j \in M_i \tag{3}$$

$$\sum_{j\in M_i} x_{ij} \le 1 \qquad \forall i \in I \tag{4}$$

$$x_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J \tag{5}$$

$$y_j \in \{0,1\} \qquad \forall j \in J \tag{6}$$

where

$$x_{ij} = \begin{cases} 1, if\ demand\ point\ i\ is\ either\ partially\ or\ fully\ covered\ by\ facility\ j \\ 0, \qquad otherwise \end{cases}$$

$$y_j = \begin{cases} 1, if\ a\ facility\ is\ sited\ at\ j \\ 0, \qquad otherwise \end{cases}$$

Coverage function under the MCLP-P is as follows:

$$c_{ij} = \begin{cases} 1, & if\ d_{ij} \le S \\ f(d_{ij}), & if\ S < d_{ij} \le T \\ 0, & otherwise \end{cases}$$

Partial coverage function is represented as $f(d_{ij}) \in \mathbb{R}^{(0,1]}$. Partial coverage value for a demand point depends on the employed partial coverage function. The selected function should be monotonic, and nonincreasing within the increasing distance from the facility. Berman et al. (2003) utilize linear decay function. In Karasakal and Karasakal (2004), authors compare four different possible coverage functions, sigmoid, classical, linear, and weighted linear partial coverage functions. For their computational study they use sigmoid function. In this study, we utilize a linear coverage function. However, as long as the partial coverage function is monotone decreasing with the increase in distance, any nonlinear function could also be utilized.

Objective function in $Eq. (1)$ maximizes the coverage level within the maximum critical distance. $Eq. (2)$ enforces that P facilities are opened. $Eq. (3)$ ensures that if facility j covers demand point i, facility j must be open. $Eq. (4)$ limits the number of facilities that can cover any demand point i simultaneously to 1. $Eq. (5)$ and $Eq. (6)$ enforces x_{ij} and y_j to take binary values.

MCLP-P extends the classical MCLP by introducing novel coverage levels. Let R denote the critical distance under the classical MCLP, its coverage function would be defined as follows:

$$c_{ij} = \begin{cases} 1, & if\ d_{ij} \leq R \\ 0, & otherwise \end{cases}$$

Since MCLP only takes coverage as a binary function, it does not permit facilities to cover demand points that are exterior to their critical distance, even if they are away by a very small margin. However, this is not quite practical in real life. Facilities would be willing to cover such demand points since they would be able to cover many more demand points only by slightly extending their critical distance. In order to address this problem, in MCLP-P authors propose another critical distance which enables facilities to operate under more realistic settings.

In MCLP-P, as in MCLP, the performance of the proposed solutions could deteriorate if critical distances are selected inaccurately. In such a case, if the critical distances are selected assuming the best-case scenario, we may try to cover demand points that could be impractical in real-life, at least under certain scenarios. On the other hand, if the critical distances are chosen assuming the worst-case scenario, we may end up being too pessimistic and disregard possible coverage opportunities. Hence, both of these cases could lead to suboptimal solutions.

However, selecting critical distances is not a simple task. They are under the influence of many exogenous factors and vary all the time. Hence, it is highly unlikely define the critical distances that reflect the real-life situations perfectly. In this study, we address this issue in MCLP-P and aim to find solutions that are robust to such changes in the critical distances. Since MCLP-P is a generalization of the classical MCLP, the proposed approach is applicable to the classical MCLP as well.

Characterization of uncertainty accurately is the first step to tackle this problem effectively. Since the coverage values are a function of distance between each demand point-facility pair, we aim to address uncertainty in the critical distances. To explain our proposition in a coherent way, we only focus on uncertainty in the maximum critical distance in this thesis. Yet, the proposed model is valid for both critical distances, and applying the proposed approach to the minimum critical distance is straightforward.

We utilize the robust optimization framework demonstrated in Bertsimas and Sim (2003), which is developed to address the uncertainties in discrete optimization problems. We aim to obtain solutions that are robust to exogenous factors. Any solution found by this model is expected be practical under various real-life situations. To achieve that, we define two scenarios for the maximum critical distance, i.e., the worst-case scenario and the average-case scenario. We represent these distances by T' and T, respectively. $f\left(d_{ij}\right)$ and $f'\left(d_{ij}\right)$ are the partial coverage functions under average-case and worst-case scenarios, respectively.

3.1.1 Coverage in the Average-Case Scenario

For the average-case scenario, the coverage occurs as follows:

$$c_{ij} = \begin{cases} 1, & d_{ij} \leq S \\ f(d_{ij}), & S < d_{ij} \leq T \\ 0, & T < d_{ij} \end{cases}$$

This is the equivalent coverage definition to MCLP-P. In average-case scenario we assume that facilities can serve demand points as they usually do and no significant exogenous factor can deterioate this coverage level. To give a practical example, we can think of average-case scenarios as hospitals covering demand points in a particular region while the traffic congestion is moderate or relatively low.

Under this scenario, demand point i can be completely covered by facility j within the minimum critical distance S, can be partially covered between the minimum and the maximum critical distances T, and no coverage can take place after the maximum critical distance.

The partial coverage function $f(d_{ij})$ should be monotonic and nonincreasing in the increasing distance from the facility. This would ensure that for given demand points i^1 and i^2 and a potential facility j, if $d_{i^1 j} \geq d_{i^2 j}$, then $c_{i^2 j} \geq c_{i^1 j}$.

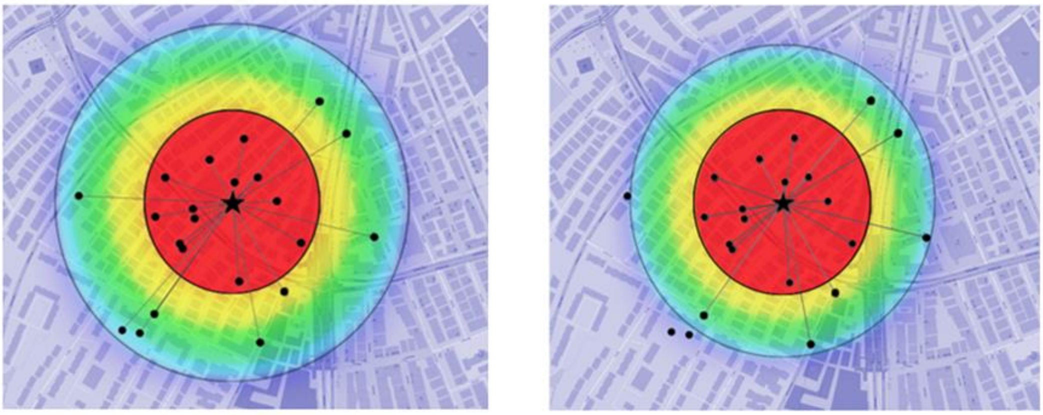
3.1.2 Coverage in the Worst-Case Scenario

For the worst-case scenario, the coverage function is as follows:

$$c'_{ij} = \begin{cases} 1, & d_{ij} \leq S \\ f'(d_{ij}), & S < d_{ij} \leq T' \\ 0, & T' < d_{ij} \end{cases}$$

In the worst-case scenario, we define a novel maximum critical distance T'. Different than the average-scenario, in this case we assume that the coverage level of facilities is affected by exterior elements. Thus, the facilities can only provide service to smaller regions. As a practical example, this would be the case where hospitals serving demand points while traffic congestion takes place. This would lead ambulances to be able to only reach shorter distances in a given time, resulting in the decreased level of coverage.

Preferred partial coverage function characteristics defined in section 3.1.1. is also valid for the worst-case scenario.



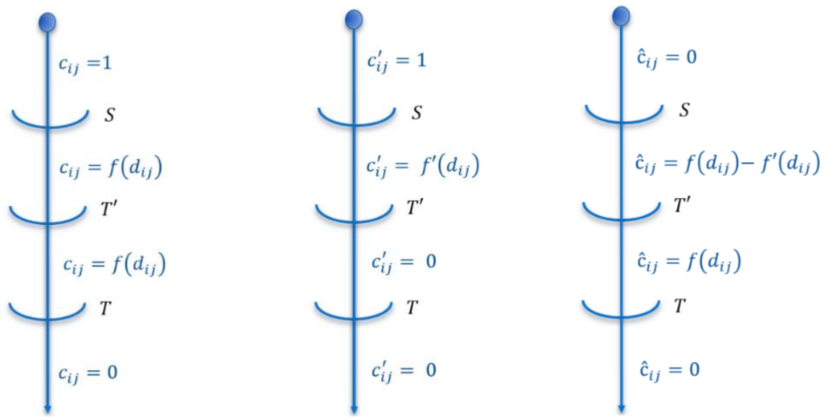
a) The average-case scenario   b) The worst-case scenario

Figure 3.1. Comparison of coverage under the average-case scenario (a) and the worst-case scenario (b) using heat map visualization. (★: Open facility, ●: Demand point)

3.1.3 Uncertainty Set for Each Demand Point-Potential Facility Pair

To accurately identify the uncertainty set, we can investigate loss in coverage based on incorrect choice of the critical distances. We can define the coverage loss value as variation and denote it as $\hat{c}_{ij}$. For each demand point-potential facility pair, variation becomes:

$$\hat{c}_{ij} = \begin{cases} 0, & d_{ij} \leq S \\ f(d_{ij}) - f'(d_{ij}), & S < d_{ij} \leq T' \\ f(d_{ij}), & T' < d_{ij} \leq T \\ 0, & T < d_{ij} \end{cases}$$



a) Average-case scenario b) Worst-case scenario c) Variation

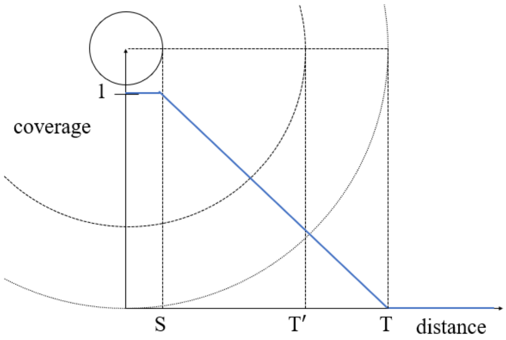
Figure 3.2. Visual demonstration of coverage under different scenarios and the variation

In Figure 3.2, let the blue dots on the top represent the location of facility j. Semicircles from top to bottom are the minimum critical distance, the maximum critical distance under the worst-case scenario and the maximum critical distance under the average-case scenario. For any demand point-facility pair, this figure visualizes the coverage values under the average-case scenario, the worst-case scenario, and the variation values respectively. Note that to accurately incorporate distance uncertainty the same coverage function should be preferred both for the average and the worst-case scenarios.

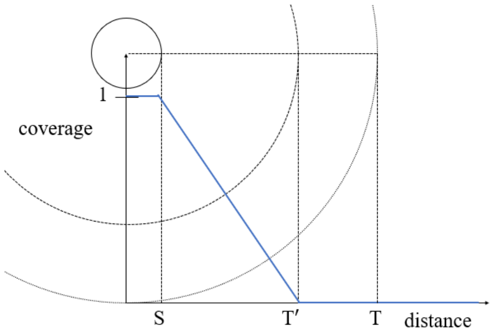
We can analyze the set of coverage values for each demand point and potential facility pair by utilizing the coverage values that have been specified so far. We know that coverage values should take a value between the worst-case and the average-case scenarios. So, if we denote $c_{ij} \in \mathbb{R}^{[0,1]}$ as the actual coverage value for any demand point-facility pair, we can investigate the four possible cases as follows:

- $d_{ij} \leq S$: $c_{ij} \in [1, 1]$. So, $c_{ij} = 1$.
- $S < d_{ij} \leq T'$: $c_{ij} \in [\mathrm{f}'(d_{ij}),\ \mathrm{f}(d_{ij})]$.

- $T' < d_{ij} \leq T$: $c_{ij} \in [0,\ \mathrm{f}(d_{ij})]$.
- $T < d_{ij}$: $c_{ij} \in [0, 0]$. $c_{ij} = 0$.



a) Coverage in the average-case scenario



b) Coverage in the worst-case scenario

Figure 3.3. Coverage values against distance

After defining the uncertainty set, the next step is to find an appropriate way to incorporate that knowledge in the model and then formulate the problem to obtain solutions that are robust to exogenous factors.

3.2 The Mathematical Model

We devise the mathematical model based on the robust optimization framework for discrete optimization problems presented in Bertsimas and Sim (2003). We formulate the model in a way that while the overall objective function aims to find solutions that will maximize the total coverage level, the lower level objective attempts to penalize this coverage given a conservatism level parameter, Γ. When the Γ parameter is equal to 0, we obtain the same result as MCLP-P, and as we increase Γ, the coverage level decreases.

Nomenclature

Sets:

I: Set of demand points; $i \in \{1, 2, ..., |I|\}$

J: Set of facility sites; $j \in \{1, 2, ..., |J|\}$

U: Set of facilities that are subject to uncertainty. U⊆J.

M_i: Set of facility that are eligible to cover demand point i fully or partially, $j \in \{1, 2, ..., |M_i|\}$. $M_i \subseteq J$.

Parameters:

c_{ij}: Average-case coverage level of demand point i provided by facility j.

$\hat{c}_{ij}$: Difference of coverage level between the average and the worst-case scenarios of demand point i provided by facility j.

c'_{ij}: Worst-case coverage level of demand point i provided by facility j.

d_{ij}: Distance between demand point i and facility j.

S : The minimum critical distance.

T: The maximum critical distance in the average-case scenario.

T': The maximum critical distance in the worst-case scenario.

Γ: Level of conservatism, i.e., the number of facilities that will function in the worst-case scenario.

P: Number of facilities to be opened.

Decision Variables:

$x_{ij}: \begin{cases} 1, if\ demand\ point\ i\ is\ either\ partially\ or\ fully\ covered\ by\ facility\ j \\ 0, otherwise \end{cases}$

$y_j: \begin{cases} 1, if\ facility\ j\ is\ opened \\ 0, otherwise \end{cases}$

$w_j: \begin{cases} 1, if\ facility\ j\ is\ in\ its\ worst-case\ scenario \\ 0, otherwise \end{cases}$

3.2.1 Formulation

The proposed formulation for the robust MCLP-P is as follows:

$$\max_{x,y} \left\{ \sum_{i\in I} \sum_{j\in J} c_{ij} x_{ij} + \min_{\{U|U\subseteq J, |U|\leq \Gamma\}} \sum_{i\in I} \sum_{j\in U} -\hat{c}_{ij} x_{ij} \right\} \qquad (7)$$

$$Constraints\ (2)-(6)$$

$Eq.\ (7)$ aims to find solutions that maximize the coverage while ensuring the impact of uncertainty is maximized. We allow variations up to Γ parameter.

To write the subproblem in the open form, a new variable, w_j, is introduced. w_j is the variable that allows robustness and takes the value of 1 if facilty j is in its worst-case scenario and takes the value of 0 if facility j is in the average-case scenario. This variable, coupled with the bilevel form of the problem, allows us to incorporate coverage variation that is caused by exogenous factors. Then, we can write the proposed model equivalently as follows:

$$(\boldsymbol{R-MCLP-P}): \quad \max_{x,y} \left\{ \sum_{i\in \mathrm{I}} \sum_{j\in J} c_{ij} x_{ij} + \min_{w} \left\{ \sum_{i\in \mathrm{I}} \sum_{j\in U} -\hat{c}_{ij} x_{ij}\, w_j \right\} \right\} \qquad (8)$$

$$s.t. \sum_{j \in U} w_j \leq \Gamma \tag{9}$$

$$0 \leq w_j \leq 1 \qquad \forall j \in U \tag{10}$$

$$Constraints\ (2) - (6)$$

In this formulation, $Eq. (8)$ maximizes the overall coverage level while minimizing the negative coverage variation. Minimizing the negative variation would lead maximizing the worst-case scenario which is the aim of the robust optimization framework. Hence, the objective is to maximize coverage under the worst-case scenario. $Eq. (9)$ limits the total number of facilities that are in their worst-case scenario to Γ. $Eq. (10)$ does not allow any facility to perform any worse than their worst-case scenario. The inner optimization problem ensures that we maximize the coverage variation while allowing us to adjust the level of conservatism. As we are maximizing the total coverage variation, this model is going to select facilities in a robust way assuming that the worst-case scenarios are likely to take place.

3.2.2 A Primal-Dual Solution Approach

The problem is in the form of bilevel optimization problem. Since the inner minimization problem is a linear optimization problem, we can convert it to a single level problem as follows.

Given that $x \in X$ is fixed, we apply strong duality and formulate the dual of the inner minimization problem. We obtain the following formulation:

$$(\boldsymbol{R-LP}): \qquad \max_{x,y,\theta,z} \left\{ \sum_{i\in I} \sum_{j\in J} c_{ij} x_{ij} + \max \left\{ \Gamma\theta + \sum_{j\in U} z_j \right\} \right\} \tag{11}$$

$$s.t.\, z_j + \theta \leq \sum_{i\in I} -\hat{c}_{ij} x_{ij} \qquad \forall j \in U \tag{12}$$

$$z_j \leq 0 \qquad \forall j \in U \tag{13}$$

$$\theta \leq 0 \tag{14}$$

$$Constraints\ (2) - (6)$$

Since the overall objective function is already maximization type, we can remove the inner maximization and write it directly as follows:

$$\max_{x,y,\theta,z} \left\{ \sum_{i\in I} \sum_{j\in J} c_{ij} x_{ij} + \Gamma\theta + \sum_{j\in U} z_j \right\} \tag{15}$$

$$Constraints\ (2) - (6), (12) - (14)$$

Now, we have a mixed integer linear programming problem. Next, we propose a Benders decomposition algorithm by keeping x_{ij} and y_j variables in the master problem, and taking the remaining variables θ and z_j to the subproblem.

3.3 Benders Decomposition Algorithm

Benders decomposition is an exact solution approach for large-scale combinatorial optimization problems based on row generation. The procedure involves solving two problems, namely master problem and subproblem, iteratively until a solution is found. As the computational difficulty increases with problem size, rather than solving a single large-scale problem, Benders decomposition algorithm iteratively solves smaller problems to become more efficient in terms of computational effort. In Benders decomposition algorithm, variables in the original formulation are divided into two sets and splitted among these two problems. Master problem is solved using one of these variable sets and subproblem is solved given the solution output of master problem. Depending on the solution of the subproblem, feasibility or optimality cuts are generated and added to the master problem. Each iteration provides a lower and upper bound for the optimal solution and the algorithm is repeated until either the gap between these bounds are sufficiently small or there is no optimal solution.

In the Benders decomposition algorithm, we keep the decision variables related to the coverage assignment and facility selection, x_{ij} and y_j, in the master problem whilst we take θ and z_j variables as complicating variables and project them out to the subproblem. The decomposition of the problem is equivalent to $(R - MCLP - P)$. Our master problem would be:

$$(\boldsymbol{R-MP}): \qquad \max_{x,y,q} \left\{ \sum_{i\in I} \sum_{j\in J} c_{ij}x_{ij} - q \right\} \tag{16}$$

$$s.t.\, q \geq \sum_{j\in U} \sum_{i\in I} \hat{c}_{ij}x_{ij}\bar{w}_j \tag{17}$$

$$Constraints\ (1), (6)$$

Here, we introduce a new continuous variable, q. For a given solution found in the master problem, this new variable tracks the solution of the subproblem and allow us to obtain the maximum possible coverage whilst incorporating the variation knowledge acquired from the subproblem. Hence, $Eq. (17)$ is an optimality cut that is generated at each iteration after solving the subproblem. As the subproblem cannot be unbounded for any combination of x_{ij} or y_j solutions, infeasibility cuts are not required.

After solving the master problem and obtaining a feasible solution of coverage assignments, which are shown by $\bar{x}$, the maximum variation in coverage for this given solution can be obtained using the following subproblem:

$$(\boldsymbol{R-SP}(\bar{x})): \quad \max_{\theta,z}\left\{\Gamma\theta+\sum_{j\in\mathrm{U}} z_j\right\} \tag{18}$$

$$s.t.\, z_j+\theta \leq \sum_{i\in\mathrm{I}} -\hat{c}_{ij}\bar{x}_{ij} \qquad \forall i\in I, j\in M_i \tag{19}$$

$$Constraints\ (13),(14)$$

This subproblem attempts to maximize the variation in coverage for the coverage assignment solution, $\bar{x}_{ij}$, obtained in the master problem. Note that this problem always yields feasible and bounded solutions. Therefore, its dual is always has to be feasible and bounded. Then, the dual formulation of this subproblem for a given $\bar{x}$ is as follows:

$$(\boldsymbol{R-DSP}(\bar{x})): \quad \min_{w}\left\{\sum_{j\in\mathrm{U}}\sum_{i\in\mathrm{I}} -\hat{c}_{ij}\bar{x}_{ij}\, w_j\right\} \tag{20}$$

$$Constraints\ (9),(10)$$

This subproblem is equivalent to the subproblem given in $(R-MCLP-P)$, where we explain the constraints and the objective function in detail. After solving the dual subproblem, we add optimality cuts to the master problem. We iteratively continue this process until a solution is found. Implemetation of the proposed Benders decomposition algorithm can be found in Algorithm 1.

ALGORITHM 1: BENDERS DECOMPOSITION ALGORITHM FOR THE ROBUST MCLP-P

Data:	$LB = -\infty,\ UB = \infty, \varepsilon = 0.05$		
While:	$UB - LB > \varepsilon$		
	Step 1:	*Solve the master problem, obtain* $x_{ij}{}^*$ *and* $y_j{}^*$.	
		Set $\bar{x}_{ij} \leftarrow x_{ij}{}^*$	
		Set $\bar{y}_j \leftarrow y_j{}^*$	
		Set $\bar{q} \leftarrow q^*$	
		Set $UB \leftarrow \sum_{j\in J}\sum_{i\in I} c_{ij}\bar{x}_{ij} - \bar{q}$	
	Step 2:	*Solve the subproblem with the updated master problem variables,* $\bar{x}_{ij}$.	
		Set $\bar{w}_j \leftarrow w_j{}^*$	
		Set $LB \leftarrow \sum_{j\in J}\sum_{i\in I} c_{ij}\bar{x}_{ij} - \sum_{j\in U}\sum_{i\in I} -\hat{c}_{ij}\bar{x}_{ij}\,\bar{w}_j$	
	Step 3:	*Add the following optimality cut to the master problem.*	
		$q \leq \sum_{j\in U}\sum_{i\in I} -\hat{c}_{ij}x_{ij}\bar{w}_j$	
End			

CHAPTER 4

SEMI-ROBUST MCLP-P

In this chapter, we consider MCLP-P under coverage uncertainty in an optimistic manner. As we have shown in Chapter 3, the robust optimization framework aims to find solutions while maximizing the variation caused by worst-case scenario. In this chapter, we present an approach that acknowledges worst-case scenario for each possible coverage and provide solutions that are affected by the worst-case scenarios at the least. Here the basic idea is to evade the effect of the worst-case scenarios as much as possible instead of mitigating the impact of the worst-case scenarios on the system when it happens. We utilize the uncertainty definition presented in Chapter 3. After we model the problem, we propose a nonlinear programming formulation and its linearization. Then, we propose a Benders decomposition algorithm and two heuristic methods to solve this problem. In Benders decomposition algorithm, we propose an approach to obtain cuts in linear time which provides significant improvements in computational performance for realistic large instances.

4.1 The Mathematical Model

Using the notation given in Chapter 3, the proposed model Semi-Robust MCLP-P, is given as follows:

$$(\boldsymbol{SR-MCLP-P}): \quad \max_{x,y} \left\{ \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} - \min_{\{U | U \subseteq J, |U| = \Gamma\}} \sum_{i \in I} \sum_{j \in U} \hat{c}_{ij} x_{ij} \right\} \quad (21)$$

$$s.t. \sum_{j \in J} y_j = p \quad (22)$$

$$x_{ij} \leq y_j \qquad \forall i \in I, j \in M_i \quad (23)$$

$$\sum_{j \in M_i} x_{ij} \leq 1 \qquad \forall i \in I \qquad (24)$$

$$x_{ij} \in \{0,1\} \qquad \forall i \in I, j \in J \qquad (25)$$

$$y_j \in \{0,1\} \qquad \forall j \in J \qquad (26)$$

Objective function given in $Eq.(21)$ aims to find the solution that maximizes coverage while ensuring the impact of uncertainty is minimized. We allow variations up to Γ facilities. $Eq.(22)$ limits the total number of facilities to be opened to P. $Eq.(23)$ ensures that facility j can only cover demand point i, if facility is j is opened. $Eq.(24)$ enforces that each demand point i is covered at most once.

The objective function is different than the objective function of the model proposed in Chapter 3. Thus, we need certain modifications in our model. First, we need to ensure that the worst-case can only occur for the facilities that are open. As we are minimizing the total variation, the model would be inclined to select potential facilities to be in their worst-case scenarios even though they are not open if we do not specify otherwise. Because of that, we need to add the following logical constraint to our subproblem,

$$w_j \leq y_j$$

Then, as the overall objective function is of maximization type and there is a minus in front of the subproblem, the model would not select any potential facility to be in their worst-case scenario since it deterioates the overall objective value. Hence, for the sum of number of selected worst-case facilities, rather than less than or equal to relationship we need equality type constraint. Therefore, we need to enforce that into the subproblem as follows:

$$\sum_{j \in U} w_j = \Gamma$$

Now, we apply a simple mathematical operation on $Eq. (21)$ and obtain an equivalent formulation. We convert the sign prior the subproblem to positive and multiply the subproblem objective coefficient by negative. We retain the constraints. The resulting model is as follows:

$$\max_{x,y} \left\{ \sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \max_{\{U | U \subseteq J, |U| = \Gamma\}} \sum_{i \in I} \sum_{j \in U} -\hat{c}_{ij} x_{ij} \right\} \tag{27}$$

$$Constraints\ (22) - (26)$$

If we write this model in the open form, we obtain the following single-level formulation.

$$\max_{x,y,w} \sum_{i \in I} \sum_{j \in J} \left(c_{ij} x_{ij} - \hat{c}_{ij} x_{ij} w_j \right) \tag{28}$$

$$\sum_{j \in U} w_j = \Gamma \tag{29}$$

$$w_j \leq y_j \qquad \forall j \in U \tag{30}$$

$$w_j \in \{0,1\} \qquad \forall j \in \mathrm{U} \tag{31}$$

$$Constraints\ (22) - (26)$$

Differently than the model presented in Chapter 3, we end up with a nonlinear model. Nonlinear problems can be challenging to solve due to their complex mathematical nature. In contrast, linear problems are simpler to solve and have well-established solution methods that can find the global optimum more easily when compared to nonlinear problems. Therefore, we next give the linearization of the proposed model to be able to tackle the problem more effectively.

4.1.1 Linearization

Nonlinearity in the model is caused by the multiplication of x_{ij} and w_j variables in $Eq.(28)$. To linearize the model, we introduce a new variable, z_{ij}, and eliminate this nonlinear term in the objective function. The following are the linearization constraints:

$$z_{ij} \leq w_j$$

$$z_{ij} \leq x_{ij}$$

$$z_{ij} \geq x_{ij} + w_j - 1$$

Among these constraints, since $z_{ij} \geq 0$ and it has a negative objective coefficient in a maximization problem as given in $Eq.(37)$, we can safely remove the first two constraints, since z_{ij} is already going to be minimized. Hence, the resulting optimization model is as follows:

$$(\boldsymbol{SR-LP}): \quad \max_{x,y,w,z} \sum_{i \in I} \sum_{j \in J} \left(c_{ij} x_{ij} - \hat{c}_{ij} z_{ij} \right) \tag{32}$$

$$z_{ij} \geq x_{ij} + w_j - 1 \quad \forall i \in I, \forall j \in \mathrm{U} \tag{33}$$

$$z_{ij} \geq 0 \quad \forall i \in I, j \in U \tag{34}$$

$$Constraints\ (22) - (26), (29) - (31)$$

Now, we have a mixed integer linear programming problem. In the sequel, we propose a Benders decomposition algorithm by keeping y_j and w_j variables in the master problem, and taking the remaining variables x_{ij} and z_{ij} to the subproblem. Then, for realistic instances where the number of demand points are very high and number of potential facilities are reasonably low, we propose an approach to obtain Benders cuts in linear time.

4.2 Benders Decomposition Algorithm

In the Benders decomposition algorithm, we keep only variables related to facility selection, y_j and w_j, in the master problem. We take x_{ij} and z_{ij} variables as complicating variables and transfer them to the subproblem. Our premiere focus is to select the average-case and the worst-case facilities in this algorithm. Our master problem is as follows:

$$(\boldsymbol{SR-MP}): \qquad \max_{q,y,w} q \tag{35}$$

$$s.t.\, q \le \sum_{j\in U}\sum_{i\in I}(1-w_j)\bar{\alpha}_{ij} + \sum_{j\in J}\sum_{i\in I} y_j\bar{\beta}_{ij} + \sum_{i\in I}\bar{\pi}_i \qquad \forall i \in I, \forall j \in \mathrm{U} \tag{36}$$

$$Constraints\ (22), (26), (29) - (31)$$

A new variable q is being introduced, which is used to monitor the subproblem solution and to update the master problem accordingly. Thus, after solving the subproblem, $Eq. (36)$ is introduced as an optimality cut in each iteration. There is no requirement for infeasibility cuts since the subproblem cannot be unbounded for any given value of the master problem variables, y_j and w_j.

The following subproblem is utilized to determine the best possible coverage values for a given $\bar{y}_j$ and $\bar{w}_j$ values obtained by the master problem:

$$(\boldsymbol{SR-SP}(\bar{y},\bar{w})): \qquad \max_{x,z}\left(\sum_{i\in I}\sum_{j\in J} c_{ij}x_{ij} - \sum_{i\in I}\sum_{j\in \mathrm{U}} \hat{c}_{ij}z_{ij}\right) \tag{37}$$

$$s.t.\ z_{ij} \ge x_{ij} + \bar{w}_j - 1 \qquad \forall i \in I, j \in M_i \tag{38}$$

$$x_{ij} \le \bar{y}_j \qquad \forall i \in I, j \in M_i \tag{39}$$

$$Constraints\ (24), (25), (34)$$

$(SR - SP(\bar{y}, \bar{w}))$ is an assignment problem for a given set of facility variables $\bar{y}_j$ and robustness variables $\bar{w}_j$. In this problem, the integrality constraints on x_{ij} can be relaxed and problem can be converted into a linear program. This is due to the fact that, for fixed $\bar{y}$ and $\bar{w}$, problem could be decomposed into seperate problems for $\forall i \in I$. If $\bar{w}_j = 1$, then $z_{ij} = x_{ij}$ due to $Eq.$ (38), also the objective function of $(SR - SP(\bar{y}, \bar{w}))$ ($Eq.$ (37)) for the given facility j becomes $\max_x(\sum_{i \in I} c_{ij} x_{ij} - \sum_{i \in I} \hat{c}_{ij} x_{ij})$. If $\bar{w}_j = 0$, then $z_{ij} = 0$ due to $Eq.$ (38), and the objective function $Eq.$ (37), for the given facility j becomes $\max_x(\sum_{i \in I} c_{ij} x_{ij})$. Hence, the problem becomes seperable for $\forall i \in I$ again due to $Eq.$ (24) and (39), and can be solved by inspection. Let us define $J^1 = \{j: \bar{y}_j = 1, \bar{w}_j = 0\}$ and $U^1 = \{j: \bar{y}_j = 1, \bar{w}_j = 1\}$. For a given demand point i, the maximum coverage is obtained by $\max\left\{\max_{j \in J^1}(c_{ij}), \max_{j \in U^1}(c'_{ij})\right\}$ and we decide coverage assignments depending on the highest coverage for the solution vectors $\bar{y}$ and $\bar{w}$. This proves two points: i) if we know solution vectors $\bar{y}$ and $\bar{w}$, we can compute the objective value of the $SR - SP$ in linear time, ii) relaxing the integrality constraints on x_{ij} yields exactly the same objective value as the binary problem. As a result, it would be safe to relax the integrality constraints $Eq.$ (25) for this subproblem as its linear relaxation would yield the same solution as its discrete form. We use the relaxation approach to solve the subproblem.

Note that for any given value of $\bar{y}$ and $\bar{w}$, $(SR - SP(\bar{y}, \bar{w}))$ is always both feasible and bounded. Therefore, the dual of this subproblem is always feasible and bounded. Let α_{ij}, β_{ij}, and π_i denote the dual variables for $Eq.$ (38), $Eq.$ (39), and $Eq.$ (24), respectively. Then, the dual formulation of $(SR - SP(\bar{y}, \bar{w}))$ is as follows:

$$(\boldsymbol{SR-DSP}(\bar{y},\bar{w})): \quad \min_{\alpha,\beta,\pi} \sum_{j\in U}\sum_{i\in I}(1-\bar{w}_j)\alpha_{ij} + \sum_{j\in J}\sum_{i\in I}\bar{y}_j\beta_{ij} + \sum_{i\in I}\pi_i \qquad (40)$$

$$s.t. \; -\alpha_{ij} \geq -\hat{c}_{ij} \qquad \forall i\in I, j\in U \qquad (41)$$

$$\alpha_{ij} + \beta_{ij} + \pi_i \geq c_{ij} \qquad \forall i\in I, j\in J \qquad (42)$$

$$\alpha_{ij} \geq 0 \qquad \forall i\in I, j\in U \qquad (43)$$

$$\beta_{ij} \geq 0 \qquad \forall i\in I, j\in J \qquad (44)$$

$$\pi_i \geq 0 \qquad \forall i\in I \qquad (45)$$

Next, we provide an approach to obtain Benders cuts in linear time similar to Cordeau et al. (2017). According to the complementary slackness theorem, the following equations must hold at optimality for the variables of the primal and the dual subproblem:

$$\alpha_{ij}^{*}\left(-z_{ij}^{*} + x_{ij}^{*} - 1 + \bar{w}_j\right) = 0 \qquad \forall i\in I, j\in U \qquad (46)$$

$$\beta_{ij}^{*}\left(x_{ij}^{*} - \bar{y}_j\right) = 0 \qquad \forall i\in I, j\in J \qquad (47)$$

$$\pi_i^{*}\left(\sum_{j\in J} x_{ij}^{*} - 1\right) = 0 \qquad \forall i\in I \qquad (48)$$

Both utilizing $Eq.(46)-(48)$ and $SR-DSP$, we can obtain the dual solution in linear time. Next, we show the computation of the $SR-DSP$ solution utilizing these information.

Recall that $J^1 = \{j: \bar{y}_j = 1,\ \bar{w}_j = 0\}$ and $U^1 = \{j: \bar{y}_j = 1,\ \bar{w}_j = 1\}$. If a given demand point i is covered, let us denote the index of the covering facility by k.

If $\sum_{j\in M_i}\overline{y}_j < 1$, meaning that if demand point i cannot be covered by any open facilities, then we are certain that all ${x_{ij}}^* = 0$, ${z_{ij}}^* = 0$ for demand point i. Then, we can see that:

- ${\alpha_{ij}}^* = 0$ for $\forall j \in J\backslash U^1$, and ${\alpha_{ij}}^* \geq 0$ for $\forall j \in U^1$ due to $Eq.(46)$.
- $\beta_{ij} = 0$ for $\forall j \in (J^1 \cup U^1)$, and $\beta_{ij} \geq 0$ for $\forall j \in J\backslash(J^1 \cup U^1)$ due to $Eq.(47)$.
- ${\pi_i}^* = 0$ due to $Eq.(48)$.

If $\sum_{j\in M_i}\overline{y}_j \geq 1$, then there are two possibilities. Facility k that covers demand point i could be in i) the average-case scenario ($k \in J^1$) or ii) the worst-case scenario ($k \in U^1$).

If $\sum_{j\in M_i}\overline{y}_j \geq 1$, and $k \in J^1$ we can deduce that:

- ${\alpha_{ij}}^* \geq 0$ for $\forall j \in (U^1 \cup \{k\})$, and ${\alpha_{ij}}^* = 0$ for $\forall j \in J\backslash(U^1 \cup \{k\})$ due to $Eq.(46)$.
- $\beta_{ij} \geq 0$ for $\forall j \in J\backslash(J^1 \cup U^1 - \{k\})$, and $\beta_{ij} = 0$ for $\forall j \in (J^1 \cup U^1 - \{k\})$ due to $Eq.(47)$.
- ${\pi_i}^* \geq 0$ due to $Eq.(48)$.

If $\sum_{j\in M_i}\overline{y}_j \geq 1$, and $k \in U^1$, we can infer that:

- ${\alpha_{ij}}^* \geq 0$ for $\forall j \in U^1$, and ${\alpha_{ij}}^* = 0$ for $\forall j \in J\backslash U^1$ due to $Eq.(46)$.
- $\beta_{ij} \geq 0$ for $\forall j \in J\backslash(J^1 \cup U^1 - \{k\})$, and $\beta_{ij} = 0$ for $\forall j \in (J^1 \cup U^1 - \{k\})$ due to $Eq.(47)$.
- ${\pi_i}^* \geq 0$ due to $Eq.(48)$.

Utilizing these information and considering $Eq.(40) - (45)$, we can generate optimal solutions and obtain the dual variable values as follows:

If $\sum_{j\in M_i}\overline{y}_j < 1$:

${\alpha_{ij}}^* = 0$ for $\forall j \in J\backslash U^1$, and ${\alpha_{ij}}^* \in [0, \hat{c}_{ij}]$ for $\forall j \in U^1$.

- $\beta_{ij} = 0$ for $\forall j \in (J^1 \cup U^1)$, and $\beta_{ij} = c_{ij}$ for $\forall j \in J \backslash (J^1 \cup U^1)$.
- $\pi_i{}^* = 0$.

If $\sum_{j \in M_i} \bar{y}_j \geq 1$, and $k \in J^1$:

- $\alpha_{ij}{}^* \in [0, c_{ik} - \beta_{ij}]$ for $\forall j \in (U^1 \cup \{k\})$, and $\alpha_{ij}{}^* = 0$ for $\forall j \in J \backslash (U^1 \cup \{k\})$.
- $\beta_{ij} \in [0, c_{ik} - \alpha_{ij}]$ for $\forall j \in J \backslash (J^1 \cup U^1 - \{k\})$, and $\beta_{ij} = 0$ for $\forall j \in (J^1 \cup U^1 - \{k\})$.
- $\pi_i{}^* = c_{ik}$.

If $\sum_{j \in M_i} \bar{y}_j \geq 1$, and $k \in U^1$:

- $\alpha_{ij}{}^* = \hat{c}_{ij}$ for $\forall j \in U^1$, and $\alpha_{ij}{}^* = 0$ for $\forall j \in J \backslash U^1$.
- $\beta_{ij} \in [0, c_{ik} - \pi_i]$ for $\forall j \in J \backslash (J^1 \cup U^1 - \{k\})$, and $\beta_{ij} = 0$ for $\forall j \in (J^1 \cup U^1 - \{k\})$.
- $\pi_i{}^* \in [0, c_{ik} - \beta_{ij}]$.

Note that dual variables may take any value in the specified intervals each yielding an alternative optimal solution. This approach is especially useful for realistic cases where the number of demand points is very high and the number of potential facilities are reasonably low. For these instances, computational experiment made for this approach is given in Chapter 5.

The proposed Benders decomposition algorithm first solves the master problem to optimality, obtains $\bar{y}_j$ and $\bar{w}_j$ values from the master solution. Then, these values are transferred to the dual subproblem. After solving this dual subproblem, we add optimality cuts to the master problem. We repeat this process until we find the optimal solution. The algorithm of the proposed approach is given in Algorithm 2.

ALGORITHM 2: BENDERS DECOMPOSITION ALGORITHM FOR THE SEMI-ROBUST MCLP-P

Data: $LB = -\infty,\ UB = \infty, \varepsilon = 0.05$

While: $UB - LB > \ \varepsilon$

Step 1: *Solve the master problem, obtain* $y_j{}^*$ *and* $w_j{}^*$.

Set $\bar{y}_j \leftarrow y_j{}^*$

Set $\bar{w}_j \leftarrow w_j{}^*$

Set $\bar{q} \leftarrow q^*$

Set $UB \leftarrow \bar{q}$

Step 2: *Solve the subproblem with the updated variables,* $\bar{y}_j$ *and* $\bar{w}_j$.

Set $\bar{\alpha}_{ij} \leftarrow \alpha_{ij}{}^*$

Set $\bar{\beta}_{ij} \leftarrow \beta_{ij}{}^*$

Set $\bar{\pi}_i \leftarrow \pi_i{}^*$

Set $LB \leftarrow \sum_{j\in U}\sum_{i\in I}(1-\bar{w}_j)\bar{\alpha}_{ij} + \sum_{j\in J}\sum_{i\in I}\bar{y}_j\bar{\beta}_{ij} + \sum_{i\in I}\bar{\pi}_i$

Step 3: *Add the following optimality cut to the master problem.*

$q \le \sum_{j\in U}\sum_{i\in I}(1-w_j)\bar{\alpha}_{ij} + \sum_{j\in J}\sum_{i\in I} y_j\bar{\beta}_{ij} + \sum_{i\in I}\bar{\pi}_i$

End

The linearized model and the Benders decomposition algorithm can be computationally expensive and may take a long time to solve problems for certain instances. On the other hand, heuristics are computationally less expensive and can find near-optimal solutions very quickly. Hence, to obtain good solutions in shorter time periods, we propose two heuristic algorithms. The proposed heuristics are not applicable for the Robust model since it has different objectives in its bilevel form.

4.3 Greedy Adding with Neighborhood Search (GNS)

In the GNS algorithm, we attempt to find optimal solution by iteratively selecting facilities that provide the greatest coverage and then seek for possible coverage improvements at each step. We begin this approach with the selection of the average-case facilities and continue with selecting the worst-case facilities until we reach the facility limit P.

For a given problem (assuming $P - \Gamma \geq 1$, meaning that there is at least one facility to be opened in the average-case scenario.), this algorithm starts with selecting the facility with the highest total coverage value for the average-case scenario. In each subsequent iteration, the algorithm first selects the facility that provides the maximum average-case coverage for the remaining uncovered demand points. Selected facilities are added to the set. After selecting these facilities according to their coverage, in each iteration, the GNS algorithm seeks if swapping selected facilities with non-selected facilities provides coverage improvements. If there exists any coverage improvements, swapping takes places and the selected facilities are updated. If there is no improvements, they are retained. This process is repeated until either number of selected facilities reaches to $P - \Gamma$ or all demand points are covered. Once the number of average-case facilities is equal to $P - \Gamma$, the GNS algorithm seeks to find Γ number of worst-case facilities in the same way with a single difference that is the selected average-case facilities are fixed and cannot be swapped or selected again.

After we select facilities utilizing this algorithm, we find the best solution by solving the problem only for the selected facilities. Solution can be obtained in linear time as shown in Section 4.2.

Flowcharts of the GNS and the Neighborhood Search Algorithms are given in Figure 4.1 and Figure 4.2 respectively.

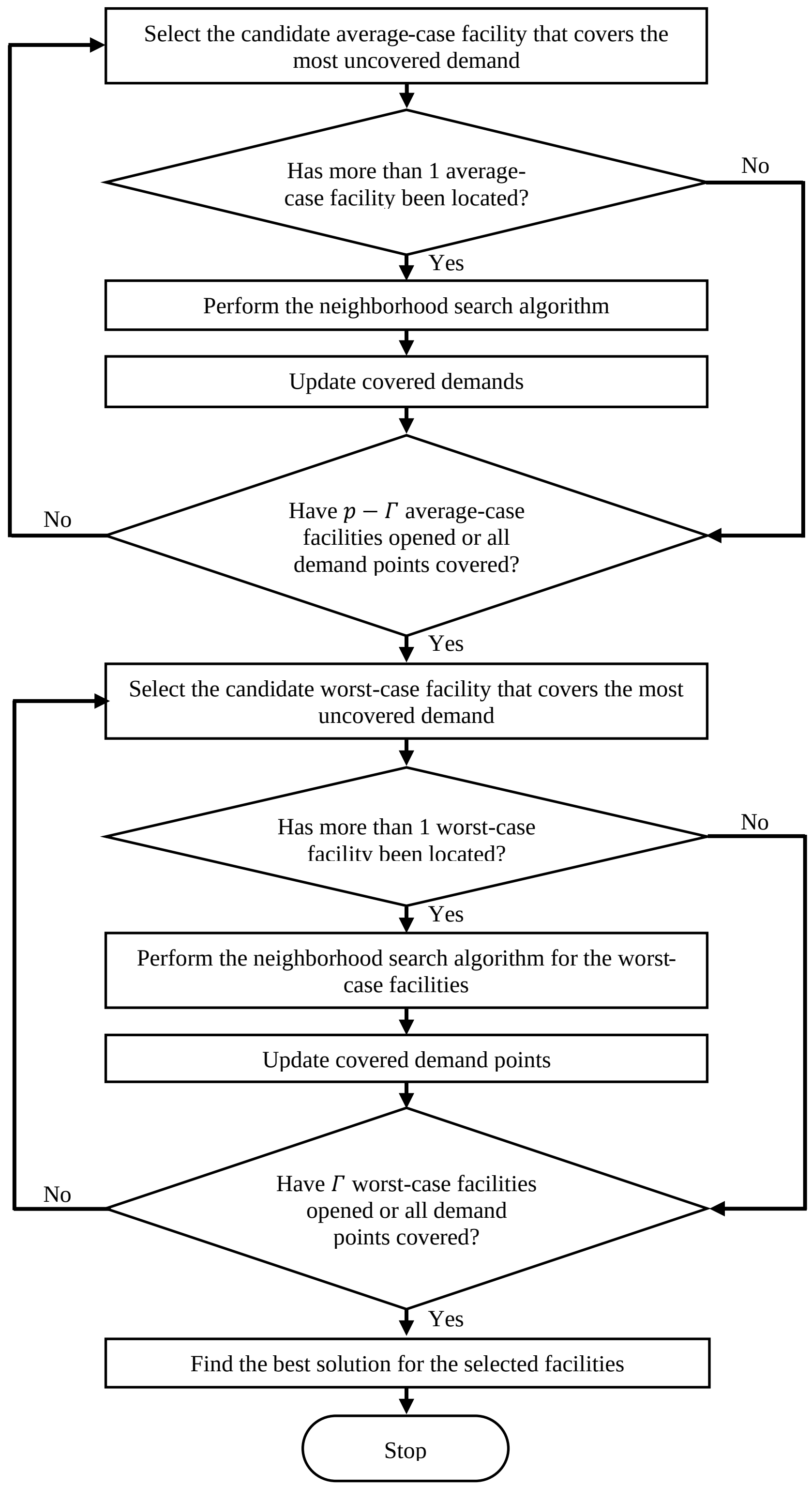


Figure 4.1. GNS Algorithm Flowchart

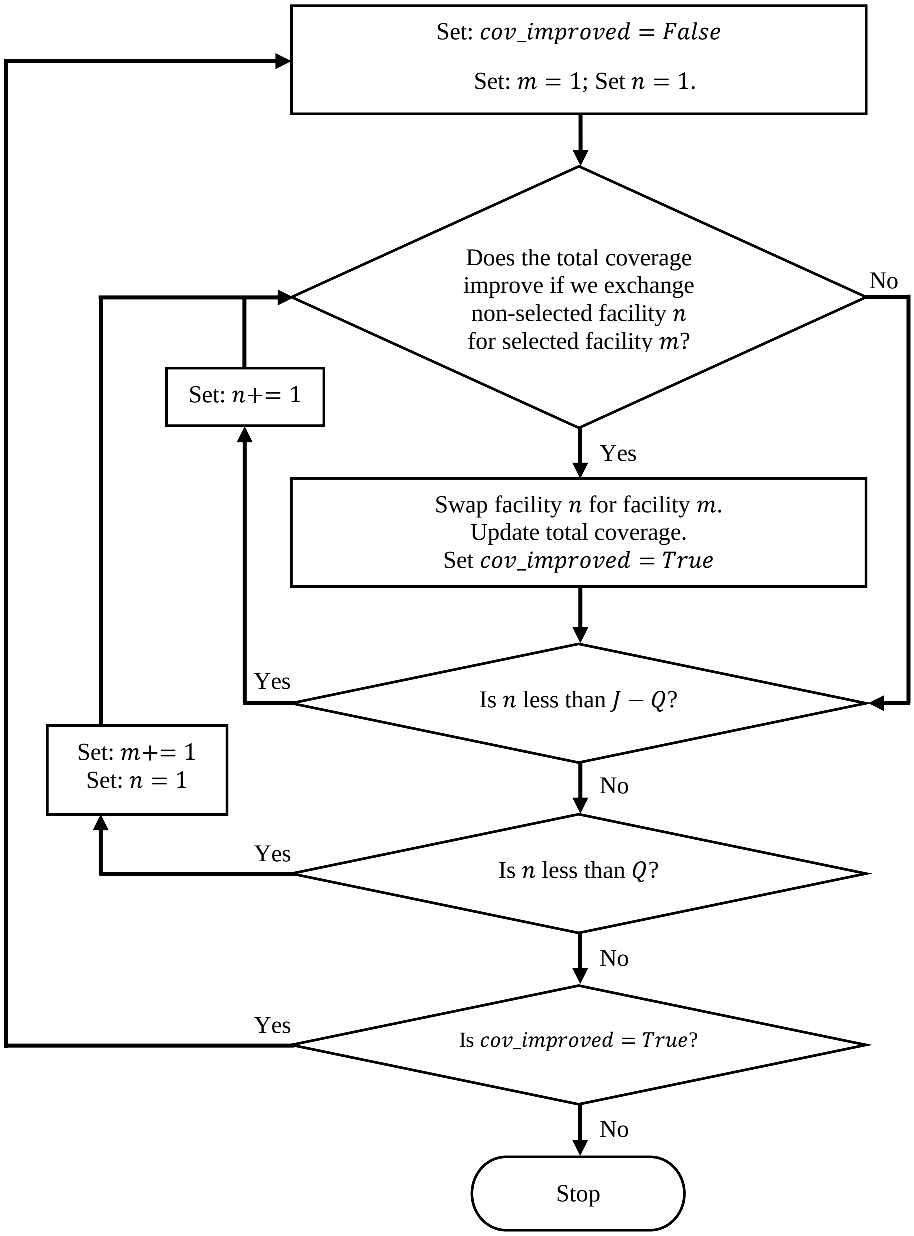


Figure 4.2. Neighborhood Search Algorithm Flowchart. (J: Number of candidate facilities, Q: Number of selected facilities)

4.4 Random Selection with Neighborhood Search (RNS)

In the RNS algorithm, we initially begin with arbitrarily selecting $P - \Gamma$ facilities for the average-case and Γ facilities for the worst-case scenario. For the given set of average-case facilities, we obtain $max(\sum_{i\in I}\sum_{j\in J^1} c_{ij}x_{ij})$. Next, for the given set of worst-case facilities and the remaining uncovered demand points, we calculate $max\,(\sum_{i\in I}\sum_{j\in U^1} c'_{ij}x_{ij})$. Summation of these two values becomes the initial solution. The RNS algorithm continues by seeking coverage improvements by performing the Neighborhood Search Algorithm given in Figure 4.2 for both average-case and worst-case facilities. Then, we find the incumbent solution by solving the problem for the selected facilities. Solution can be found in linear time as shown in Section 4.2. We start again by arbitrarily generating a set of facilities. We repeat this process until we reach the iteration limit. Flowchart of the RNS algorithm is given in Figure 4.3.

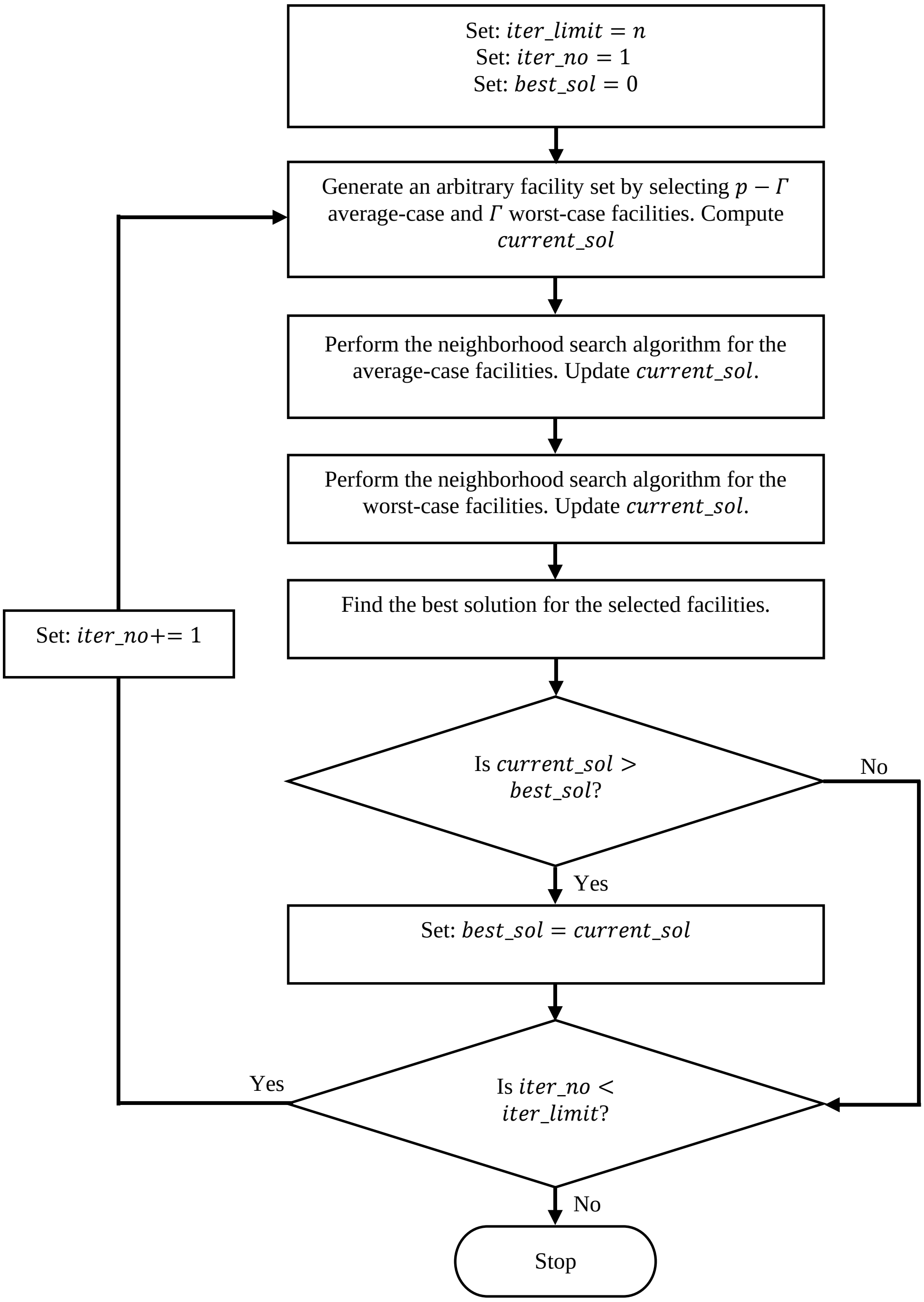


Figure 4.3. RNS Algorithm Flowchart

CHAPTER 5

COMPUTATIONAL EXPERIMENTS

In this chapter, we test the presented models on different instances and report their results. In each instance, along with the proposed models we also compare our solutions to the MCLP-P solutions to have a better idea about the effect of the proposed approaches. Time limit allowed for each solution approach was set to 3600 seconds. We executed all computational operations on Python 3.10 via CPLEX 22.1.0. The runs were conducted on a Windows 11 PC with a 2.60 GHz Intel Core i7 CPU and 8 GB RAM.

Some abbreviations used in the solution tables of the experiments. z corresponds to objective value, $Gap(\%)$ is the optimality gap, t_{sol} denotes the CPU time (in seconds) spent solving any given model, t_{total} denotes total CPU time spent while solving any model (including initialization processes such as creating a mathematical model), t_{sub} and t_{master} are CPU times for solving subproblems and master problems of Benders decomposition algorithms, respectively. $\bar{z}$ denotes the average solution value for all three models. $\bar{z}_{nom}$ represents the nominal coverage value for the robust models. $\bar{t}$, t_{min}, t_{max} and s denotes average, minimum, maximum and the standard deviation of the CPU time values.

5.1 Data Generation

We test the proposed approaches on randomly generated data sets. Let X, Y be the random variables for corresponding x, y coordinates of any point in datasets. Data are generated as $X_{demand} \sim U[0, 50]$, $Y_{demand} \sim U[0, 100]$, $X_{facility} \sim U[5, 45]$, and $Y_{facility} \sim U[10, 90]$. Euclidean distance is used to compute the distance between each demand-facility points and calculate the coverage values. Partial coverage functions used in this study are $f(d_{ij}) = \begin{cases} \frac{T-d_{ij}}{T-S}, & d_{ij} \in [S,T] \\ 0, & d_{ij} \notin [S,T] \end{cases}$ for the average-case scenario, and $f'(d_{ij}) = \begin{cases} \frac{T'-d_{ij}}{T'-S}, & d_{ij} \in [S,T'] \\ 0, & d_{ij} \notin [S,T'] \end{cases}$ for the worst-case scenario.

5.2 Experiment Design

Effects of five different factors are investigated, number of demand points $|I|$, number of potential facility points $|J|$, magnitude of the worst-case scenario $\frac{T-T\prime}{T-S}$ denoted as δ, number of facility points to be opened P, and robustness parameter Γ. All the results are compared to MCLP-P to investigate the effect of uncertainty on solutions.

Each factor has a certain number of levels, and we investigate all possible combinations of these factor levels. All factors and their corresponding factor levels are presented in Table 1.

Table 5.1 Factors and their levels of the experiment

$\|I\|$	$\|J\|$	P	Γ	δ
500	8	$0.25\|J\|$	$0.25P$	35%
2000	16	$0.5\|J\|$	$0.5P$	70%
8000	32	$0.75\|J\|$	$0.75P$	

In this experiment S and T were set to 15 and 25. For fractional Γ values, we round them to the higher nearest integer value. And if there exists any duplication instances caused due to rounding fractional values, we remove such instances from our experiment. Plus, the instances where $p = \Gamma$ which were the result of the rounding are also omitted.

5.3 Robust MCLP-P

From the experiments of the Robust MCLP-P we can observe that the primal-dual model performs well for almost all instances. Increase in the number of decision variables and constraints does not seem to have a strong impact on the solution time. The greatest effect on computation time for the primal dual model seems to be the number of demand points.

Benders decomposition algorithm seems to provide slightly higher computational times than the Primal Dual model for most instances. We note that in a few instances it was not able to find the optimal solution in the time limit. Robust MCLP-P experiment summary is given in Table 5.2.

Table 5.2. shows important experiment results against every factor level. Complete experiment can be found in Appendix A.

We employed t-test with 95% confidence level to compare the performance of the proposed solution approaches. For this given experiment, p-value of the t-test is found to be 0.007. Hence, we can say that the primal dual formulation performs better than the Benders decomposition algorithm in a statistically significant manner.

5.4 Semi-Robust MCLP-P

In the experiments of the Semi-Robust MCLP-P, we compared four different solution approaches. Summary of the findings are given in Table 5.3. This summary provides a bird-eye view for the proposed models and their performance. Complete experiment can be found in Appendix.

As can be seen from Table 5.3, proposed heuristic algorithms provides optimal or near-optimal solutions for most of the instances. Average optimality gap for all instances were found as 1.49% and 1.08% for the GNS and the RNS algorithms respectively.

Table 5.2 Robust MCLP-P experiment summary

	$MCLP$	$R-LP$						$Benders$			
	$\bar{z}$	$\bar{z}$	$\bar{z}_{nom}$	$\bar{t}$	t_{min}	t_{max}	s	$\bar{t}$	t_{min}	t_{max}	s
I=500	429.89	412.47	427.59	0.38	0.04	3.45	0.55	4.34	0.49	69.56	10.02
I=2000	1745.07	1677.15	1732.90	3.78	0.23	41.15	7.03	108.60	2.17	2252.32	390.20
I=8000	6997.10	6724.92	6944.83	98.35	1.65	2450.82	361.83	471.29	10.23	3600.00	1026.15
J=8	2607.86	2433.42	2579.61	2.56	0.04	23.80	2.87	17.55	0.49	205.95	22.91
J=16	3027.36	2890.28	3004.03	15.66	0.07	245.42	27.34	184.05	0.76	3600.00	368.15
J=32	3387.00	3322.59	3369.85	73.76	0.12	2450.82	207.21	323.57	1.39	3600.00	549.01
P=0.25|J|	3014.05	2878.90	2999.91	105.87	0.23	2450.82	203.27	513.99	1.54	3600.00	561.72
P=0.5|J|	3063.29	2943.17	3041.24	17.31	0.10	598.44	49.11	152.15	0.59	3600.00	347.49
P=0.75|J|	3080.29	2972.71	3052.44	3.23	0.04	20.70	3.53	24.50	0.49	249.71	29.40
Γ=0.25P	3140.91	3069.78	3129.71	12.62	0.08	251.32	21.45	91.91	0.75	2581.92	212.11
Γ=0.5P	3010.04	2891.19	2986.15	13.08	0.05	255.97	22.33	155.68	0.55	3383.37	306.90
Γ=0.75P	3127.88	2973.15	3101.49	89.29	0.07	2450.82	206.92	396.04	0.83	3600.00	548.48
δ=35%	3092.57	3024.60	3085.46	8.74	0.04	245.42	21.01	48.85	0.49	1534.01	129.81
δ=70%	3022.13	2851.77	2984.75	59.60	0.10	2450.82	207.73	340.64	0.76	3600.00	641.28

To analyze the proposed exact solution methods of the Semi-Robust MCLP-P, we conducted t-test to compare the solution times of the $\mathrm{SR-LP}$ and the Benders decomposition algorithm. p-value found as 10^{-7} with 95% confidence level, meaning that the linear model is computationally more efficient than the Benders decomposition algorithm.

Table 5.3 Semi-Robust MCLP-P Experiment Summary

	$MCLP$	$SR-LP$						$Benders$				$GNS\ Gap(\%)$				$RNS\ Gap(\%)$			
	$\bar{z}$	$\bar{z}$	$\bar{z}_{nom}$	$\bar{t}$	t_{min}	t_{max}	s	$\bar{t}$	t_{min}	t_{max}	s	avg	min	max	s	avg	min	max	s
I=500	430.15	423.83	427.68	0.23	0.06	1.08	0.17	75.59	1.73	1748.14	261.95	1.31	0.00	4.42	1.30	0.94	0.00	4.42	1.06
I=2000	1756.91	1731.66	1747.56	2.03	0.30	18.94	2.75	98.33	8.25	567.73	135.99	1.77	0.00	4.86	1.63	1.41	0.00	4.86	1.43
I=8000	7032.72	6931.88	6994.64	22.08	1.97	120.84	25.79	347.04	19.96	2126.86	547.10	1.41	0.00	4.17	1.37	0.94	0.00	3.99	1.09
J=8	2674.79	2577.25	2640.17	2.59	0.06	22.50	2.45	21.78	1.73	115.80	15.24	1.08	0.00	4.86	0.80	0.92	0.00	4.86	0.74
J=16	3006.38	2960.44	2987.06	9.27	0.08	120.84	13.73	227.34	2.42	2126.86	292.26	2.09	0.00	4.74	1.33	1.43	0.00	4.54	1.04
J=32	3405.78	3399.05	3403.83	10.65	0.17	100.42	12.24	212.05	4.14	1748.14	219.51	1.27	0.00	4.28	1.02	0.94	0.00	3.87	0.77
P=0.25|J|	3010.84	2948.56	2986.22	20.00	0.16	120.84	17.45	482.10	2.42	2126.86	337.07	2.27	0.00	4.54	1.16	1.67	0.00	4.54	0.94
P=0.5|J|	3057.54	3012.69	3041.00	5.18	0.09	28.89	5.31	144.93	1.74	1267.92	154.00	1.49	0.00	4.74	1.14	1.05	0.00	3.87	0.85
P=0.75|J|	3130.60	3099.26	3119.18	3.13	0.06	19.25	3.13	37.33	1.73	181.29	30.51	1.06	0.00	4.86	0.95	0.82	0.00	4.86	0.81
Γ=0.25P	3092.68	3079.50	3087.27	5.92	0.08	33.74	5.41	119.01	1.74	621.09	109.95	1.93	0.00	4.57	1.16	1.61	0.00	4.54	0.99
Γ=0.5P	3112.95	3079.55	3099.16	8.01	0.06	77.11	10.46	220.64	1.73	2126.86	283.81	1.03	0.00	4.08	0.82	0.55	0.00	3.05	0.53
Γ=0.75P	3092.87	3023.05	3068.09	12.16	0.14	120.84	14.48	201.52	4.14	2007.63	208.97	1.84	0.00	4.74	1.17	1.29	0.00	4.42	0.86
δ=35%	3091.42	3065.89	3086.98	6.35	0.06	74.51	9.48	116.66	1.91	1479.30	187.56	1.42	0.00	4.86	1.23	1.05	0.00	4.86	1.00
δ=70%	3055.10	2992.36	3026.27	9.88	0.06	120.84	15.77	216.11	1.73	2126.86	315.45	1.63	0.00	4.74	1.30	1.18	0.00	4.54	1.04

5.5 Impact of Robust Decision-Making on the Coverage Performance

In this section, we demonstrate the practical impact of the proposed models on a single randomly generated dataset. We perform two different experiments to understand i) how the classical MCLP-P perform under coverage uncertainty, ii) how the proposed models perform under completely deterministic settings.

To make the comparison clear, we utilize a single data set where certain parameters are fixed ($|I| = 500, |J| = 50,\ p = 15,\ S = 5, T = 10$), while other parameters (δ and Γ) are varying. δ parameter takes three values, $\%25, \%60,$ and $\%90,$ and Γ parameter is gradually increased from 2 to 14.

5.5.1 Robustness of the Classical MCLP-P

In this experiment, we seek to understand how the classical MCLP-P performs under coverage uncertainty and compare its performance against the proposed models, R-MCLP-P and SR-MCLP-P.

We solve the generated instance with all three models. For a given MCLP-P solution, let $\bar{x}_{ij}$ denote the coverage assignment solution. Then let z_R and z_{SR} represent the optimal solution of the robust models. To compare how the MCLP-P performs against the R-MCLP-P model, we first compute the R-MCLP-P objective with the MCLP-P solution, $z_1 = (\sum_{i\in I}\sum_{j\in J} c_{ij}\bar{x}_{ij} + \underset{\{U|U\subseteq J,|U|\leq\Gamma\}}{min} \sum_{i\in I}\sum_{j\in U} -\hat{c}_{ij}\bar{x}_{ij})$. Then to make the comparison, we compute $MCLP_R = \frac{z_R - z_1}{z_1}$ to find out how the R-MCLP-P model performs against the classical MCLP-P. Similarly, to compare the coverage performance of the MCLP-P against the SR-MCLP-P model, we compute $z_2 = (\sum_{i\in I}\sum_{j\in J} c_{ij}\bar{x}_{ij} - \underset{\{U|U\subseteq J,|U|=\Gamma\}}{max} \sum_{i\in I}\sum_{j\in U} \hat{c}_{ij}\bar{x}_{ij})$. To make the comparison, we calculate $MCLP_{SR} = \frac{z_{SR} - z_2}{z_2}$. Results of this experiment can be found in Table 5.4.

Table 5.4 Robustness of the Classical MCLP-P

	δ					
	25%		65%		90%	
Γ	$MCLP_R$	$MCLP_{SR}$	$MCLP_R$	$MCLP_{SR}$	$MCLP_R$	$MCLP_{SR}$
2	0.04%	0.03%	0.54%	0.70%	0.11%	0.57%
4	0.11%	0.16%	1.42%	1.08%	1.11%	1.27%
6	0.17%	0.25%	2.50%	2.10%	1.93%	2.78%
8	0.22%	0.34%	3.47%	2.85%	2.72%	3.85%
10	0.33%	0.43%	4.75%	3.75%	5.29%	5.15%
12	0.56%	0.48%	5.46%	4.65%	7.68%	7.60%
14	0.58%	0.49%	6.64%	5.22%	9.14%	18.73%

The impact of the proposed models could be seen in Table 5.4. As the value of δ or Γ increases, the proposed models provide significant coverage improvements when compared to the classical MCLP-P.

5.5.2 Nominal Coverage Performance of the Robust Models

In this experiment, we aim to understand how the robust models perform if there were no coverage uncertainty.

Similar to Chapter 5.5.1, we first solve the generated instance with all three models.

For a given R-MCLP-P and SR-MCLP-P solution, let ${\bar{x}_{ij}}^R$ and ${\bar{x}_{ij}}^{SR}$ denote the coverage assignment solutions. Then let z_{MCLP} represent the optimal solution of the MCLP-P model.

To find out how the R-MCLP-P performs if there were no coverage uncertainty, we first compute the MCLP-P objective using the R-MCLP-P solution, $z_3 = \sum_{i \in I} \sum_{j \in J} c_{ij} {\bar{x}_{ij}}^R$. Then to compare with the classical MCLP-P, we compute $R_{nom} = \frac{z_{MCLP} - z_3}{z_3}$. Then, to compare the coverage performance of the SR-MCLP-P against

MCLP-P, we compute $z_4 = \sum_{i \in I} \sum_{j \in J} c_{ij} \bar{x}_{ij}{}^{SR}$ To make the comparison, we calculate $SR_{nom} = \frac{z_{MCLP} - z_4}{z_4}$. Results of this experiment can be found in Table 5.5.

Table 5.5 Nominal coverage performance of the MCLP-P model against Robust and Semi-Robust models

	δ					
	25%		65%		90%	
Γ	R_{nom}	SR_{nom}	R_{nom}	SR_{nom}	R_{nom}	SR_{nom}
2	0.02%	0.05%	0.38%	0.75%	0.50%	2.93%
4	0.07%	0.45%	1.69%	3.20%	2.64%	6.38%
6	0.07%	0.52%	2.37%	5.16%	4.12%	11.17%
8	0.30%	1.00%	3.35%	7.73%	4.31%	17.00%
10	0.53%	1.68%	3.76%	10.26%	8.10%	22.87%
12	0.48%	1.76%	4.41%	13.02%	7.76%	30.00%
14	0.43%	2.40%	3.56%	18.78%	9.55%	31.37%

Table 5.5 demonstrates the nominal coverage performance of the classical MCLP-P against the robust models. Since the classical MCLP-P only aims to maximize the nominal coverage, it provides higher coverage performance when no facilities are under coverage uncertainty. However, in practice, this is usually not the case. Model parameters are not completely deterministic, hence assuming the model input are completely reliable may lead to undesirable solutions.

5.6 Impact of Critical Distance Selection

The factors introduced in Section 5.2. are the main factors that have influence on solution and computational complexity. One last factor that could worth analyzing is the values of the critical distances, S and T. As the experiment table already involves high number of instances, we found it appropriate to seperately present its influence over solutions and computation time. We demonstrate its impact on the linear models both for R-MCLP-P and SR-MCLP-P. To exclusively investigate the impact of critical distances, the following parameters are fixed $|I| = 500$, $|J| = 16$, $p = 4$, and $\delta = 0.5$. In addition to varying critical distances, we also set three different parameters for the Γ parameter. In Table 5.6, we present the impact of the critical distances on nominal coverage performance and the CPU time for varying $S, T,$ and Γ parameters.

As can be seen from the Table 5.6, greater critical distances leads to higher coverage and increased computation times as expected. Then, as the distance between S and T decreases, both the R-MCLP-P and the SR-MCLP-P provides closer nominal coverage to the classical MCLP-P model. This is also another anticipated outcome since if S and T would be equivalent, there would be no need for introducing robustness to the problem.

In Table 5.7, we present how critical distances affect robust coverage performance.

From Table 5.7, we can observe that, as the difference between S and T increases (especially when both parameters are reasonably high) the need for the robust models becomes clearer under our experiment setting.

Table 5.4 Impact of the critical distances on nominal coverage and time

		$S = 0.25T$					$S = 0.5T$					$S = 0.75T$				
		$MCLP$	R_{nom}	t_{sol}	SR_{nom}	t_{sol}	$MCLP$	R_{nom}	t_{sol}	SR_{nom}	t_{sol}	$MCLP$	R_{nom}	t_{sol}	SR_{nom}	t_{sol}
$\Gamma = 1$	T=4	15.81	15.81	0.04	14.77	0.06	16.01	15.14	0.03	15.23	0.06	20.46	20.46	0.02	20.26	0.04
	T=8	47.86	47.78	0.05	42.58	0.08	51.45	51.45	0.04	51.45	0.06	76.73	76.73	0.04	74.62	0.09
	T=16	141.20	141.05	0.23	127.21	0.43	201.06	200.94	0.15	190.09	0.26	256.07	255.67	0.17	248.59	0.27
	T=32	349.73	340.88	4.28	335.24	1.55	446.54	439.07	1.47	442.09	0.55	489.17	488.41	0.36	488.10	0.36
$\Gamma = 2$	T=4	14.34	13.38	0.04	12.04	0.05	17.53	17.53	0.04	16.33	0.05	25.48	24.39	0.03	24.49	0.04
	T=8	46.81	46.81	0.07	35.80	0.11	62.13	61.63	0.05	52.95	0.08	74.39	74.39	0.04	71.50	0.06
	T=16	146.76	142.61	0.35	112.37	0.63	191.62	191.14	0.26	165.01	0.35	236.65	235.24	0.25	223.58	0.27
	T=32	354.89	347.09	8.28	295.91	4.39	439.97	430.91	2.83	432.85	0.71	492.75	491.75	0.72	491.41	0.51
$\Gamma = 3$	T=4	15.70	15.40	0.07	11.99	0.07	22.61	21.77	0.05	19.28	0.06	24.46	24.32	0.03	22.71	0.05
	T=8	41.10	36.74	0.10	22.44	0.10	68.32	68.32	0.06	52.35	0.08	79.40	77.77	0.04	73.14	0.07
	T=16	154.05	150.73	0.97	93.38	0.61	190.79	186.99	0.49	144.84	0.51	237.45	230.68	0.19	213.29	0.30
	T=32	345.71	327.55	121.37	252.73	5.98	431.64	418.68	6.81	390.86	1.81	481.76	480.11	0.59	469.36	0.65

Table 5.5 Impact of the critical distances on the robust coverage performance

		$S = 0.25T$				$S = 0.5T$				$S = 0.75T$			
		$MCLP_R$	R	$MCLP_{SR}$	SR	$MCLP_R$	R	$MCLP_{SR}$	SR	$MCLP_R$	R	$MCLP_{SR}$	SR
$\Gamma = 1$	T=4	13.28	13.28	14.15	14.77	13.50	14.10	15.08	15.23	19.57	19.57	20.05	20.05
	T=8	39.05	39.23	40.97	41.20	44.68	44.68	49.18	49.18	71.89	71.89	74.14	74.62
	T=16	113.51	113.85	115.89	122.18	175.53	175.96	184.86	184.86	240.54	242.33	247.01	249.30
	T=32	280.81	290.28	311.58	329.31	408.63	417.83	435.70	440.87	479.72	480.98	485.37	487.58
$\Gamma = 2$	T=4	8.78	8.80	11.32	11.32	14.17	14.17	14.53	16.19	22.39	22.85	24.43	24.43
	T=8	29.21	29.21	32.87	33.32	48.12	48.42	51.79	51.86	66.66	66.66	68.69	68.69
	T=16	90.50	92.70	100.61	101.76	141.76	143.63	150.46	155.85	212.84	213.81	219.69	221.59
	T=32	232.17	244.09	261.61	287.62	365.77	385.71	422.83	429.39	482.89	484.36	487.92	488.68
$\Gamma = 3$	T=4	7.68	8.66	10.82	10.82	12.88	14.11	15.74	17.28	19.90	21.00	19.90	22.71
	T=8	16.92	18.45	19.97	21.96	46.41	46.41	49.36	49.40	68.68	69.07	70.77	70.77
	T=16	65.29	67.56	71.04	78.34	120.36	126.27	131.23	135.69	192.32	203.24	198.84	210.11
	T=32	166.51	186.56	203.89	223.36	342.79	353.48	352.31	381.60	460.59	462.10	465.53	467.33

5.7 Impact of Iteration Limit on the RNS

In the RNS algorithm, we first randomly select facilities and then implement the Neighborhood Search Algorithm repeatedly until we reach the iteration limit. Hence, the number of iterations affects the success of the solutions found. In this section, we investigate the impact of iteration limit by defining four different levels, 1, 5, 10, and 25. We work on 8 different cases.

Table 5.6 Iteration limit experiment instances

Case	$\lvert I \rvert$	$\lvert J \rvert$	P	Γ	δ
1	8000	8	4	2	35%
2	8000	8	4	2	70%
3	8000	16	4	1	35%
4	8000	16	4	1	70%
5	8000	16	8	2	35%
6	8000	16	8	2	70%
7	8000	16	8	4	35%
8	8000	16	8	4	70%

For each case, we solve the linear model, the GNS, and the RNS with different iteration limits and compare the solutions and present the results in Table 5.9.

As can be seen from the Table 5.9, the RNS algorithm was able to obtain better results when compared to the GNS algorithm for the given problems. The RNS found good starting solutions in first iterations for some instances, however it is clear that as the number of iterations increases, it provides solutions with smaller optimality gap. According to this experiment, using 10 iterations gives reasonably well results for most instances in short time periods. Since the computational efficiency of the RNS algorithm is going to be valid for larger problem sizes, it may be a reasonable choice to use it against the linear model for such instance. Yet it is quite clear that when the number of iterations increases the solutions get better.

Table 5.7 Iteration limit experiment

	$SR-LP$		GNS		RNS									
					Iter: 1		Iter: 10		Iter: 25		Iter: 50		Iter: 100	
Case	z	t_{total}	GAP (%)	t_{total}	GAP (%)	t_{total}	GAP (%)	t_{total}	GAP (%)	t_{total}	GAP (%)	t_{total}	GAP (%)	t_{total}
1	5604.8	11.56	1.59%	0.37	1.59	0.42	1.59	0.76	1.59	1.19	1.59	1.78	1.59	2.22
2	5694.99	36.06	4.48%	0.22	4.48	0.86	4.48	1.63	4.48	2.13	4.48	2.39	1.63	2.91
3	6187.94	74.65	0.00%	1.03	7.14	0.56	0.00	0.96	0.00	1.75	0.00	2.27	0.00	3.11
4	5896.50	139.72	1.97%	0.71	1.97	0.71	1.27	1.05	0.39	1.86	0.39	2.59	0.39	2.92
5	7523.01	13.61	2.05%	1.55	3.65	1.21	1.72	2.16	1.72	2.93	1.72	3.61	1.72	4.54
6	7819.66	19.18	2.57%	1.82	2.57	1.26	2.50	2.27	1.67	2.94	0.21	3.87	0.01	4.76
7	7689.92	14.47	0.20%	1.18	0.19	0.72	0.19	2.18	0.19	3.24	0.19	4.47	0.19	5.54
8	7436.17	23.14	3.04%	1.27	1.90	0.97	1.90	2.02	1.90	3.37	0.76	4.42	0.76	5.61

5.8 The Performance of SR-MCLP-P Models for Large Instances

In this experiment, for the SR-MCLP-P model, we test the impact of obtaining Benders cuts in linear time as proposed in Section 4.2. We compare its performance against the $SR - LP$ for realistic large instances. Large instances parameters are selected in a way that the number of customers are much larger than the number of potential facility locations.

Table 5.8 Large Instance Performance

						$SR-LP$	$Benders$			
Case	$\|I\|$	$\|J\|$	P	Γ	δ	t_{total}	$\#\ of\ cuts$	t_{sol}: t_{sub}	t_{sol}: t_{master}	t_{total}
1	50,000	8	4	1	0.70	735.71	30	6.03	0.41	48.29
2	50,000	8	4	1	0.35	756.96	24	5.22	0.30	43.09
3	50,000	8	4	2	0.70	1574.8	38	7.09	0.54	92.13
4	50,000	8	4	2	0.35	995.42	27	5.37	0.46	65.02
5	50,000	8	4	3	0.70	1305.2	31	5.08	0.57	96.30
6	50,000	8	4	3	0.35	805.40	28	5.13	0.45	88.83

From Table 5.8, we can clearly see that being able to add Benders cuts in linear time can provide significant advantages in terms of computational advantage, especially under the realistic cases where number of demand points are much higher than potential facilities.

CHAPTER 6

INSIGHTS

6.1 Maximin and Its Interpretation for Robust MCLP-P

Game theory is a field of mathematics that studies how individuals or groups make decisions in situations where the outcome depends on the choices of others. In game theory, strategy shapes the decisions players make to achieve a desired outcome, taking into account the potential choices of other players and the possible consequences of their actions. There exists different strategies, and maximin is one of the strategies players might adopt in game theory. In this section, we demonstrate the relation between the discrete robust optimization and maximin strategy in game theory through payoff tables.

In game theory, equilibrium refers to a state where each player's strategy is optimal given the strategies of all the other players. Assume there exists a unique, yet dangerous equilibrium point which player might find risky.

Table 6.1 Payoff Table

		Player II	
		B1	B2
Player I	A1	2, 1	2, -20
	A2	3, 0	-10, 1
	A3	-100, 2	3, 3

First values represent the payoff value for Player I, second values represent the payoff value for Player II.

In this case, unique equilibrium is (A3, B2) with a payoff of (3, 3). However, it may not be the most likely outcome. Player I has the greatest payoff in (A3) with (3), however there exists a risk that Player II selects (B1) to obtain his/her greatest payoff. Given that option (A3, B1) could be catastrophic for Player I, he/she could choose the option (A1). But under this scenario, if Player II is aware of that Player I might avoid option (A3) and will select (A1), he/she might not prefer to select option (B2) since then there is a risk of receiving a very low payoff value (-20). Therefore, Player II will likely select option (B1).

Under maximin strategy, each player attempts to maximize their minimum payoff in each scenario and choose their actions accordingly. This strategy aims to guarantee the best possible result without relying on the rationality of the other players, and even making the most pessimistic assessment of their potential behavior.

Let u_i denote the utility function for player i. Then, let S_i be the set of possible strategies of player i, and S_{-i} be the set of possible strategies for rest of the players. Similarly, let s_i denote the selected strategy for player i and s_{-i} denote the selected strategy vector of all the players except player i.

If player i chooses strategy s_i, the utility he/she will recieve is going to depend on the strategies of the other players. Given that s_i is selected, the worst possible utility player i can obtain is $\min_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i})$. And, under the maximin strategy, the payoff value is calculated as $\max_{s_i \in S_i} \min_{t_{-i} \in S_{-i}} u_i(s_i, t_{-i})$.

The calculation of the best actions under the maximin strategy of both players is given in Table 6.2.

Table 6.2 Maximin calculation in payoff table

		Player II		
		B1	B2	$\min_{s_{II} \in S_{II}} u_I(s_I, s_{II})$
Player I	A1	2, 1	2, -20	2
	A2	3, 0	-10, 1	-10
	A3	-100, 2	3, 3	-100
	$\min_{s_I \in S_I} u_{II}(s_I, s_{II})$	0	-20	(2,0)

There could be applications of this strategy in situations with a single decision-maker. Next, we propose the interpretation of the maximin strategy in terms of Robust MCLP-P.

6.2 Maximin Approach in Robust MCLP-P

In this section, we demonstrate with a practical example that if we define the possible worst-case scenarios instead of players, we can compute the maximin value through payoff tables. This allows utilizing matrix operations while computing the solution rather than solving the actual model.

Assume that there are two demand points to be covered. We are going to open two facilities, $y_I = 1$, $y_2 = 1$, and let $\Gamma = 1$. So, the problem is to decide the coverage assignments that will yield the highest maximin values and find the facility that will be in its worst-case scenario.

For this example, we randomly generated the coverage and variation values given in Table 6.3.

Table 6.3 Generated coverage data

	c_{ij}	$\hat{c}_{ij}$
$i = 1, j = 1$	0.10568	0.06775
$i = 2, j = 1$	0.16325	0.07056
$i = 1, j = 2$	0.18202	0.07175
$i = 2, j = 2$	0.19219	0.07036

Let a_{ij} denote the coverage assignment of demand point i to facility j. If two facilities are to be opened and if each demand point is covered at most by one facility, there could be only four possible instances of coverage assignments, which are {(a_{11}, a_{21}), (a_{11}, a_{22}), (a_{21}, a_{12}), (a_{12}, a_{22})} (E.g., (a_{11}, a_{21}) represents the case where $x_{11} = 1, x_{21} = 1, x_{12} = 0, x_{22} = 0$.).

Now, if we calculate the objective values under these scenarios, we obtain:

Table 6.4 Objective values under different scenarios

	w_1	w_2
(a_{11}, a_{21})	0.13062	0.26893
(a_{11}, a_{22})	0.23012	0.22751
(a_{21}, a_{12})	0.27471	0.27352
(a_{12}, a_{22})	0.37421	0.23210

Each row (a_{ij}, a_{kl}) of the Table 6.4 represents a possible coverage assignment instance, and the column w_j represents the facility with the worst-case scenario. E.g. the cell in (a_{11}, a_{22}) row and w_1 column is the instance that $x_{11} = 1, x_{21} = 1, x_{12} = 0, x_{22} = 0$, and $w_1 = 1, w_2 = 0$. (a_{11}, a_{22}) row values are calculated as respectively:

$$(c_{11} * x_{11} - \hat{c}_{11} * x_{11} * w_1) + (c_{22} * x_{22} - \hat{c}_{22} * x_{22} * w_2)$$

$$= (0.10568 * 1 - 0.06775 * 1 * 1) + (0.19219 * 1 - 0.07036 * 1 * 0)$$

$$= 0.23012$$

and

$$(c_{11} * x_{11} - \hat{c}_{11} * x_{11} * w_1) + (c_{22} * x_{22} - \hat{c}_{22} * x_{22} * w_2)$$

$$= (0.10568 * 1 - 0.06775 * 1 * 0) + (0.19219 * 1 - 0.07036 * 1 * 1)$$

$$= 0.22751$$

Now, if we apply the maximin strategy, we obtain the results given in Table 6.5.

Table 6.5 Maximin value calculation

	w_1	w_2	$\min_{s \in U} u(x,s)$	$\max_{x \in X} \min_{s \in U} u(x,s)$
(a_{11}, a_{21})	0.13062	0.26893	0.13062	0.27352
(a_{11}, a_{22})	0.23012	0.22751	0.22751	
(a_{21}, a_{12})	0.27471	0.27352	0.27352	
(a_{12}, a_{22})	0.37421	0.23210	0.23210	

Even though there actually is a single decision maker in making a location decision, this robust problem has a nature of a game involving two players (Sniedovich, 2016). The robustness variable w, embodies uncertainty and the realized value of w is not controlled by the decision maker. So, in this game, the decision maker plays first by selecting a decision $x \in X$. As a consequence, a worst-case scenario takes place associated with this decision. Hence, the goal of the decision maker is to select an $x \in X$ that provides the greatest payoff by anticipating the worst-case scenario response.

We found that the resulting value of the payoff table is equivalent to the model solution. We demonstrated the relation between payoff tables and discrete robust optimization only for the maximin approach which is valid for the Robust MCLP-P. Yet, by using the maximax payoff tables instead of maximin tables, this is directly applicable to Semi-Robust MCLP-P as well. This approach is beneficial as it enables us to utilize matrix operations rather than solving the actual model. We observed that it is especially useful for the Semi-Robust MCLP-P.

CHAPTER 7

CONCLUSION

In this thesis, we address the MCLP-P under coverage uncertainty. In MCLP-P, coverage depends on the distance between each demand-facility point pair and the critical distances defined for each facility. In our model, we assume that the critical distances are subject to change and upper and lower bounds of critical distances are the only available information. We adopt the discrete robust optimization framework to deal with this problem and we propose two different approaches to hedge against uncertainty, namely Robust MCLP-P and Semi Robust MCLP-P.

In the Robust MCLP-P, we maximize the overall coverage while assuming that in the worst-case scenario the variation is going to be maximized as well. For the Robust MCLP-P we propose two exact solution approaches, a primal dual approach and a Benders decomposition algorithm.

In the Semi Robust MCLP-P, we aim to select facilities that are going to maximize the coverage while being affected by the worst-case scenario variation at the least. For the Semi-Robust MCLP-P we propose two heuristic and two exact solution approaches. We propose a mixed integer program, a Benders decomposition algorithm, greedy neighborhood search algorithm and random neighborhood search algorithm.

We also investigate the process of finding robust solutions from the perspective of game theory using payoff tables. We present the relation between discrete robust optimization and game theory. We demonstrate that we can utilize payoff tables for computation instead of solving the actual model.

In computational experiments section, we show the factors that have influence in our problem and their effect on both solutions and computational cost. We compare each solution method and present their performance under many different instances. In the Semi-Robust MCLP-P experiments, we demonstrate that our proposed way of obtaining Benders cuts allows us to significantly outperform the linear model for the realistic large instances. Proposed heuristic algorithms, the GNS and the RNS provide 1.49% and 1.08% optimality gaps in average respectively and we were able to obtain optimal or near-optimal solutions for most of the cases in very short time frames. Then, we present the impact of the proposed models and make practical comparisons among themselves and the classical MCLP-P. If there is coverage uncertainty, we show that our models outperform the classical MCLP-P up to 18.73%.

This study can be extended along several directions. We define the coverage uncertainties over the critical distances in the MCLP-P. Future studies may change the source of uncertainty. Another future research direction could be to utilize the stochastic optimization framework to solve the MCLP-P under coverage uncertainty, if there is available data. For instance, the parameters of the partial coverage function of the classical MCLP-P could be replaced with random variables that are assumed to be distributed according to the available data. Then, the expected coverage could be maximized. This could provide a different perspective for the decision-makers. However, this could increase the computational difficulty and enforce developing other solution approaches. Another interesting future study could be to implement the robust approach developed in this thesis to different location problems. As an example, double coverage models could be taken into consideration, which is commonly used in ambulance location problems, where handling uncertainty carries critical importance.

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APPENDIX A

Robust MCLP-P Experiment

Computational experiments made for the Robust MCLP-P can be found in Table 8.1.

Table 8.1 Robust MCLP-P Experiment

						$MCLP-P$		$R-LP$				Benders				
														t_{sol}		
Case	$\|I\|$	$\|J\|$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	# of cuts	t_{sub}	t_{master}	t_{total}
1	500	8	2	1	0.70	246.63	0.09	219.04	246.24	0.16	0.38	0.00	3	0	0.46	1.05
2	500	8	2	1	0.35	237.29	0.07	220.74	234.60	0.13	0.34	0.00	3	0	0.31	0.86
3	500	8	4	1	0.70	254.22	0.07	216.82	244.67	0.19	0.41	0.00	5	0	0.86	1.57
4	500	8	4	1	0.35	344.09	0.1	334.92	343.88	0.11	0.34	0.00	2	0	0.26	0.75
5	500	8	4	2	0.70	378.73	0.05	340.56	374.65	0.18	0.4	0.00	5	0	0.64	1.55
6	500	8	4	2	0.35	311.52	0.05	297.07	306.96	0.1	0.37	0.00	1	0	0.09	0.59
7	500	8	4	3	0.70	334.78	0.09	279.77	325.37	0.25	0.51	0.00	8	0	1.62	2.63
8	500	8	4	3	0.35	375.47	0.09	345.43	375.28	0.27	0.5	0.00	4	0	0.77	1.57
9	500	8	6	2	0.70	396.97	0.08	374.83	394.97	0.13	0.43	0.00	2	0	0.23	0.76
10	500	8	6	2	0.35	374.55	0.04	366.31	373.63	0.04	0.34	0.00	2	0	0.08	0.55
11	500	8	6	3	0.70	430.82	0.05	398.56	428.11	0.1	0.36	0.00	4	0	0.67	1.48
12	500	8	6	3	0.35	371.21	0.05	352.97	367.61	0.05	0.33	0.00	1	0	0.05	0.55
13	500	8	6	5	0.70	387.43	0.07	348.32	380.21	0.49	0.82	0.00	6	0	0.82	1.82
14	500	8	6	5	0.35	411.72	0.04	393.34	410.01	0.06	0.33	0.00	1	0	0.07	0.49
15	500	16	4	1	0.70	208.84	0.07	185.41	208.48	0.25	0.52	0.00	4	0	0.94	1.84
16	500	16	4	1	0.35	385.29	0.17	374.08	385.16	0.69	1.23	0.00	5	0	2.16	3.96
17	500	16	4	2	0.70	389.41	0.25	350.69	386.46	1.65	2.31	0.00	12	0.01	7.63	11.28
18	500	16	4	2	0.35	387.88	0.31	367.80	386.41	0.45	0.88	0.00	7	0.01	2.85	4.83
19	500	16	4	3	0.70	386.58	0.19	333.53	377.09	0.71	1.13	0.00	14	0.01	7.14	10.34
20	500	16	4	3	0.35	388.19	0.19	367.63	386.92	0.36	0.83	0.00	4	0.01	1.04	2.53
21	500	16	8	2	0.70	447.44	0.1	427.66	445.46	0.42	0.85	0.00	4	0	1.06	2.37
22	500	16	8	2	0.35	443.79	0.16	434.34	443.06	0.39	0.89	0.00	4	0	1	2.35
23	500	16	8	4	0.70	451.28	0.11	433.34	448.19	0.31	0.77	0.01	6	0.01	1.72	3.62
24	500	16	8	4	0.35	468	0.08	459.22	466.12	0.1	0.59	0.00	1	0	0.09	0.81
25	500	16	8	6	0.70	442.34	0.12	406.82	437.32	0.39	0.89	0.00	13	0.01	5.64	8.64
26	500	16	8	6	0.35	455.42	0.12	444.06	455.19	0.13	0.73	0.00	1	0	0.11	0.95
27	500	16	12	3	0.70	484.79	0.09	474.70	483.86	0.13	0.77	0.00	4	0.01	0.71	2.17
28	500	16	12	3	0.35	430.67	0.09	421.58	430.08	0.08	0.66	0.00	2	0	0.16	1.12
29	500	16	12	6	0.70	440.03	0.09	412.65	437.91	0.12	0.72	0.00	3	0	0.36	1.5
30	500	16	12	6	0.35	462.9	0.08	452.95	462.82	0.08	0.59	0.00	1	0	0.07	0.76

Table 8.1 (Cont'd)

Case	$\|I\|$	$\|J\|$	P	Γ	δ	$MCLP-P$ z	$MCLP-P$ t_{sol}	$R-LP$ z	$R-LP$ z_{nom}	$R-LP$ t_{sol}	$R-LP$ t_{total}	Benders Gap (%)	Benders # of cuts	Benders t_{sol} t_{sub}	Benders t_{sol} t_{master}	Benders t_{total}
31	500	16	12	9	0.70	461.29	0.07	437.80	456.86	0.11	0.6	0.00	3	0	0.58	1.62
32	500	16	12	9	0.35	439.89	0.07	426.57	439.71	0.07	0.49	0.00	1	0	0.07	0.83
33	500	32	8	2	0.70	384.42	0.14	363.93	383.15	0.47	0.9	0.00	6	0	2.31	4.01
34	500	32	8	2	0.35	490.13	0.21	488.43	490.04	0.23	1.1	0.00	1	0	0.2	1.54
35	500	32	8	4	0.70	480.77	0.3	464.78	478.59	1.42	2.49	0.01	8	0	7.91	12
36	500	32	8	4	0.35	487.37	0.28	483.24	487.33	0.6	1.59	0.01	2	0	0.81	2.98
37	500	32	8	6	0.70	478.94	0.26	455.06	472.01	3.45	4.34	0.00	22	0.02	60.5	69.56
38	500	32	8	6	0.35	491.26	0.25	487.71	491.21	0.6	1.56	0.00	2	0	0.86	2.81
39	500	32	16	4	0.70	475.45	0.19	463.03	470.06	0.29	1.16	0.00	6	0	1.55	4.55
40	500	32	16	4	0.35	490.09	0.19	487.10	489.71	0.27	1.11	0.00	1	0	0.26	1.8
41	500	32	16	8	0.70	494.6	0.12	486.66	493.74	0.15	0.99	0.00	3	0	0.45	2.74
42	500	32	16	8	0.35	486.17	0.15	482.00	485.96	0.13	1.21	0.00	1	0	0.15	1.61
43	500	32	16	12	0.70	488.03	0.16	478.41	486.99	0.27	1.41	0.00	3	0	0.75	3.06
44	500	32	16	12	0.35	499.27	0.32	498.96	499.27	0.27	1.29	0.00	1	0	0.23	2
45	500	32	24	6	0.70	492.94	0.16	485.53	490.59	0.73	1.7	0.00	10	0.01	4.29	9.37
46	500	32	24	6	0.35	493.55	0.13	490.79	493.55	0.16	1.13	0.00	2	0	0.38	2.33
47	500	32	24	12	0.70	478.83	0.28	456.79	472.63	0.58	1.7	0.01	7	0.01	2.23	6.24
48	500	32	24	12	0.35	493.31	0.13	490.44	493.30	0.13	0.95	0.00	2	0	0.36	2.27
49	500	32	24	18	0.70	496.54	0.15	492.34	496.28	0.15	1.07	0.00	2	0	0.32	2.13
50	500	32	24	18	0.35	487.44	0.13	483.64	487.32	0.12	1	0.00	1	0	0.12	1.39
51	2000	8	2	1	0.70	1033.28	0.33	875.64	1030.54	1.3	2.23	0.00	3	0	3.06	5.46
52	2000	8	2	1	0.35	881.81	0.26	810.19	881.68	0.93	1.93	0.00	2	0	1.09	3.04
53	2000	8	4	1	0.70	1547.76	0.26	1463.76	1539.42	0.34	1.37	0.00	2	0	0.61	2.73
54	2000	8	4	1	0.35	1508.54	0.3	1474.69	1507.53	0.73	1.77	0.00	4	0	2.61	5.2
55	2000	8	4	2	0.70	1505.22	0.28	1329.90	1466.39	1.15	2	0.00	6	0	5.92	9.39
56	2000	8	4	2	0.35	1475.06	0.3	1408.57	1469.60	0.75	1.65	0.00	3	0	1.67	3.95
57	2000	8	4	3	0.70	1364.87	0.36	1182.66	1346.96	0.58	1.58	0.00	3	0	1.23	3.84
58	2000	8	4	3	0.35	1548.04	0.26	1468.47	1539.00	0.51	1.36	0.00	4	0	2.1	4.68
59	2000	8	6	2	0.70	1478.69	0.27	1391.92	1464.30	0.51	1.38	0.00	4	0	2.55	5.85
60	2000	8	6	2	0.35	1652.07	0.41	1609.16	1649.22	0.34	1.7	0.01	3	0	1.18	3.76

Table 8.1 (Cont'd)

						$MCLP-P$		$R-LP$				Benders				
Case	$\|I\|$	$\|J\|$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	# of cuts	t_{sol} t_{sub}	t_{master}	t_{total}
61	2000	8	6	3	0.70	1247.17	0.28	1126.00	1231.23	0.76	1.73	0.00	7	0.01	4.08	8.01
62	2000	8	6	3	0.35	1499.37	0.2	1436.18	1496.69	0.23	1.16	0.00	2	0	0.42	2.17
63	2000	8	6	5	0.70	1477.87	0.43	1296.25	1401.06	0.47	1.47	0.00	2	0	0.74	2.64
64	2000	8	6	5	0.35	1460.31	0.23	1379.49	1454.61	0.32	1.27	0.00	2	0	0.5	2.34
65	2000	16	4	1	0.70	1465.38	0.64	1390.42	1463.36	2.78	4.75	0.00	5	0	9.06	15.9
66	2000	16	4	1	0.35	1513.3	0.85	1466.47	1511.77	3.98	6.17	0.00	2	0	3.69	10.45
67	2000	16	4	2	0.70	1502.48	1.19	1331.54	1464.19	11.1	13.04	0.00	20	0.02	153.86	171.53
68	2000	16	4	2	0.35	1542.6	0.69	1470.89	1526.81	2.9	4.75	0.00	7	0	18.32	25.91
69	2000	16	4	3	0.70	1497.16	0.76	1266.23	1448.04	10.84	12.68	0.00	16	0.01	143.72	157.77
70	2000	16	4	3	0.35	1501.58	0.7	1404.20	1498.55	4.79	6.49	0.00	10	0.01	51.02	60.76
71	2000	16	8	2	0.70	1927.73	0.78	1892.36	1924.06	0.94	2.68	0.00	4	0	3.43	9
72	2000	16	8	2	0.35	1898.62	0.69	1879.11	1897.06	0.69	2.57	0.00	1	0	0.77	4.07
73	2000	16	8	4	0.70	1821.66	1.06	1717.77	1800.60	3.15	5.02	0.00	13	0	56.15	68.21
74	2000	16	8	4	0.35	1959.68	0.7	1946.13	1959.25	1.71	3.73	0.00	3	0	3.76	8.32
75	2000	16	8	6	0.70	1698.54	0.75	1564.76	1658.13	21.44	23.32	0.00	75	0.04	2193.98	2252.32
76	2000	16	8	6	0.35	1769.44	0.79	1721.19	1760.76	1.56	3.5	0.00	5	0	5.99	12.82
77	2000	16	12	3	0.70	1902.58	0.72	1849.01	1892.13	0.9	2.83	0.00	5	0.01	3.82	9.97
78	2000	16	12	3	0.35	1773.44	0.51	1732.52	1770.23	0.54	2.45	0.00	1	0	0.55	3.7
79	2000	16	12	6	0.70	1692.49	0.59	1591.58	1680.17	1.75	3.63	0.00	7	0	12.78	20.29
80	2000	16	12	6	0.35	1863.5	0.74	1840.07	1856.38	1.44	3.17	0.00	4	0	3.32	8.89
81	2000	16	12	9	0.70	1874.91	0.65	1757.95	1858.67	1.51	3.4	0.00	5	0	5.26	11.51
82	2000	16	12	9	0.35	1907.16	0.58	1877.10	1906.52	0.66	2.54	0.00	1	0	0.56	3.9
83	2000	32	8	2	0.70	1884.15	1.95	1823.22	1873.78	4.99	8.82	0.00	4	0	12.95	25.72
84	2000	32	8	2	0.35	1951.39	2.14	1939.86	1950.99	2.27	5.99	0.00	2	0	4.44	13.1
85	2000	32	8	4	0.70	1864.85	2.13	1763.45	1849.97	41.15	44.87	0.00	28	0.01	1523.07	1569.46
86	2000	32	8	4	0.35	1922.94	2.52	1898.79	1921.28	13.57	17.66	0.00	10	0	93	115
87	2000	32	8	6	0.70	1963.77	1.92	1906.75	1960.04	15.99	19.77	0.00	10	0	130.15	150.3
88	2000	32	8	6	0.35	1940.45	2.89	1919.78	1939.48	2.28	6.7	0.00	2	0	3.83	11.51
89	2000	32	16	4	0.70	1989.21	1.46	1972.21	1981.97	1.82	5.35	0.00	3	0	4.77	13.63
90	2000	32	16	4	0.35	1974.1	2.17	1963.48	1973.25	1.8	5.54	0.00	1	0	1.79	8.47

Table 8.1 (Cont'd)

						$MCLP-P$		$R-LP$				Benders				
Case	$\|I\|$	$\|J\|$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	# of cuts	t_{sol} t_{sub}	t_{sol} t_{master}	t_{total}
91	2000	32	16	8	0.70	1957.87	1.57	1928.20	1951.38	1.76	5.16	0.00	1	0	1.4	7.44
92	2000	32	16	8	0.35	1970.86	1.52	1958.64	1969.55	1.54	4.91	0.00	2	0	3.66	11.4
93	2000	32	16	12	0.70	1893.06	1.44	1851.14	1880.72	5.22	8.63	0.00	17	0	257.95	286.51
94	2000	32	16	12	0.35	1991.47	1.74	1987.81	1991.46	1.47	5.18	0.00	1	0	1.32	7.41
95	2000	32	24	6	0.70	1872.9	1.3	1794.08	1831.18	1.54	4.87	0.00	4	0	5.68	15.25
96	2000	32	24	6	0.35	1953.5	1.16	1938.40	1951.26	1.35	4.69	0.00	4	0	5.47	14.94
97	2000	32	24	12	0.70	1900.26	1.27	1839.62	1883.35	1.54	4.88	0.00	8	0	12.9	27.35
98	2000	32	24	12	0.35	1942.65	1.24	1920.10	1940.16	1.32	4.73	0.00	1	0	1.16	6.78
99	2000	32	24	18	0.70	1949.62	1.24	1882.15	1924.95	1.29	4.69	0.00	2	0	2.42	9.32
100	2000	32	24	18	0.35	1962.89	1.17	1949.40	1961.57	1.24	4.63	0.00	2	0	2.43	9.54
101	8000	8	2	1	0.70	3700.57	1.79	3188.80	3665.31	12.45	16.03	0.00	4	0	17.49	28.31
102	8000	8	2	1	0.35	3736.3	1.69	3516.13	3732.27	11.32	14.77	0.00	3	0	13.67	23.15
103	8000	8	4	1	0.70	6041.93	1.55	5735.44	6016.73	16.83	20.36	0.00	4	0	38.92	49.81
104	8000	8	4	1	0.35	5944.82	2.44	5762.67	5939.51	3.09	6.69	0.00	2	0	3.39	11.89
105	8000	8	4	2	0.70	5375.36	1.62	4749.78	5291.33	18.8	22.05	0.00	10	0.02	187.44	205.95
106	8000	8	4	2	0.35	6089.04	1.55	5848.58	6077.31	1.94	5.51	0.00	3	0	4.99	13.79
107	8000	8	4	3	0.70	6051.4	1.66	5082.88	5932.61	23.8	27.27	0.00	8	0	167.73	183.81
108	8000	8	4	3	0.35	6162.22	1.49	5753.27	6127.54	4.06	7.51	0.00	2	0	5.74	13.22
109	8000	8	6	2	0.70	5880.52	1.4	5558.59	5852.02	4.56	8.17	0.00	5	0	13.9	25.63
110	8000	8	6	2	0.35	6559.99	1.44	6371.33	6528.33	1.91	5.62	0.00	2	0	3.39	11.41
111	8000	8	6	3	0.70	4561.49	1.37	4216.96	4466.66	3.1	6.57	0.00	3	0	5.66	14.77
112	8000	8	6	3	0.35	6672.08	1.47	6478.97	6640.70	2.01	5.65	0.00	2	0	3.22	11.03
113	8000	8	6	5	0.70	6537.99	1.22	5785.99	6283.52	1.65	4.96	0.00	2	0	2.68	10.23
114	8000	8	6	5	0.35	5869.48	1.41	5642.67	5818.13	1.76	5.09	0.00	2	0	3.6	11.31
115	8000	16	4	1	0.70	6146.5	5.38	5738.37	6122.71	51.14	57.75	0.00	7	0	164.28	197.36
116	8000	16	4	1	0.35	6094.82	5.49	5952.89	6087.4	22.45	29.65	0.00	3	0.02	39.21	60.46
117	8000	16	4	2	0.70	6139.47	5.38	5620.11	6127.67	66.54	73.68	0.00	8	0	345.49	380.26
118	8000	16	4	2	0.35	6202.6	5.77	5912.05	6189.67	38.22	44.39	0.00	8	0	232.14	265.72
119	8000	16	4	3	0.70	6150.76	6.41	5292.13	6061.61	208.39	215.16	6.74	20	2.68	3794.26	>3600
120	8000	16	4	3	0.35	6165.88	5.87	5784.55	6158.36	245.42	252.61	0.00	13	1.74	1459.14	1534.01

Table 8.1 (Cont'd)

						$MCLP-P$		$R-LP$				$Benders$				
														t_{sol}		
Case	$\|I\|$	$\|J\|$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	$\#$ of $cuts$	t_{sub}	t_{master}	t_{total}
121	8000	16	8	2	0.70	7267.23	5.62	7035.01	7214.45	39.85	46.46	0.01	6	0.81	112.21	152.47
122	8000	16	8	2	0.35	7273.87	4.32	7155.68	7255.56	5.19	12.35	0.00	2	0.26	9.61	30.99
123	8000	16	8	4	0.70	7121.4	4.72	6738.05	7026.97	19.08	26.61	0.00	9	1.2	128.1	180.55
124	8000	16	8	4	0.35	7231.91	4.35	7055.45	7202.66	5.73	12.68	0.00	2	0.26	11.29	32.79
125	8000	16	8	6	0.70	7644.68	4.55	7291.69	7537.62	17.97	24.29	0.00	11	1.31	203.75	262.51
126	8000	16	8	6	0.35	7153	4.41	6969.14	7126.22	10.86	17.33	0.00	3	0.36	27.61	52.28
127	8000	16	12	3	0.70	7555.01	4.03	7346.16	7519.52	5.09	11.28	0.00	5	0.59	23.42	56.25
128	8000	16	12	3	0.35	7885.49	4.05	7855.22	7882.06	5.49	12.02	0.00	2	0.22	9.8	30.1
129	8000	16	12	6	0.70	6690.35	4.02	6153.78	6383.33	6.38	12.92	0.00	6	0.69	33.24	69.6
130	8000	16	12	6	0.35	7206.19	4.07	7008.25	7158.04	4.94	11.49	0.00	2	0.23	8.99	28.73
131	8000	16	12	9	0.70	7365.3	3.86	6944.16	7279.31	6.09	12.3	0.00	5	0.63	25	57.65
132	8000	16	12	9	0.35	7496.93	4.14	7312.22	7470.25	7.75	14.29	0.00	2	0.23	9.83	29.98
133	8000	32	8	2	0.70	7632.36	31.52	7457.89	7617.45	251.32	265.27	0.01	13	3.52	2407.34	2581.92
134	8000	32	8	2	0.35	7816.92	19.14	7787.19	7816.02	25.84	39.62	0.00	2	0.51	49.39	105.03
135	8000	32	8	4	0.70	7703.62	18.43	7443.79	7633.59	255.97	269.64	0.00	13	3.35	3222.75	3383.37
136	8000	32	8	4	0.35	7652.47	20.97	7559.50	7646.54	25.88	39.66	0.00	1	0.26	25.07	74.45
137	8000	32	8	6	0.70	7603.89	20.55	7202.90	7518.33	2450.82	2464.83	0.98	8	2.12	>3600	>3600
138	8000	32	8	6	0.35	7687.25	21.25	7584.96	7676.11	41.83	55.85	0.00	4	1.03	189.16	265.89
139	8000	32	16	4	0.70	7688.84	19.32	7422.32	7540.07	22.23	36.08	0.00	5	1.26	106.56	188.76
140	8000	32	16	4	0.35	7670.46	20.3	7577.34	7657.84	21.49	35.64	0.00	1	0.27	21.71	66.95
141	8000	32	16	8	0.70	7940.8	22.75	7869.91	7923.09	26.24	40.25	0.00	6	1.52	145.37	240.51
142	8000	32	16	8	0.35	7919.69	20.38	7885.99	7918.02	22.41	36.3	0.00	1	0.25	20.06	65.83
143	8000	32	16	12	0.70	7694.57	18.61	7360.69	7535.23	598.44	612.39	0.02	19	5.18	3726.85	>3600
144	8000	32	16	12	0.35	7703.94	21.58	7610.53	7695.13	25.53	39.27	0.00	2	0.52	44.08	101.47
145	8000	32	24	6	0.70	7999.77	12.83	7999.35	7999.77	15.51	29.54	0.00	1	0.27	15.26	54.17
146	8000	32	24	6	0.35	7912.09	13.26	7877.24	7908.73	15.59	29.41	0.00	2	0.52	32.71	82.23
147	8000	32	24	12	0.70	7870.34	13.64	7680.82	7801.73	17.96	31.93	0.00	8	2.06	144.41	249.71
148	8000	32	24	12	0.35	7423.45	13.53	7317.88	7403.94	20.7	34.44	0.01	4	1.04	69.11	136.43
149	8000	32	24	18	0.70	7484.29	13.3	7119.40	7367.03	17.01	30.89	0.00	5	1.29	83.5	159.82
150	8000	32	24	18	0.35	7918.43	13.09	7886.44	7915.10	15.85	29.67	0.00	2	0.53	31.42	80.6

APPENDIX B

Semi-Robust MCLP-P Experiment

Computational experiments made for the Semi-Robust MCLP-P can be found in Table 8.2.

Table 8.2 Semi-Robust MCLP-P Experiment

						$MCLP-P$		$SR-LP$				Benders					GNS		RNS	
Case	$\|I\|$	$\|J\|$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	# of cuts	t_{sol} t_{sub}	t_{sol} t_{master}	t_{total}	Gap (%)	t_{total}	Gap (%)	t_{total}
1	500	8	2	1	0.7	244.7	0.1	214.4	244.7	0.2	0.5	0.0	14.0	0.0	0.0	2.7	0.00	0.01	0.00	0.12
2	500	8	2	1	0.35	263.0	0.1	246.5	259.6	0.2	0.5	0.0	8.0	0.0	0.0	1.7	0.00	0.01	0.00	0.07
3	500	8	4	1	0.7	216.4	0.0	199.1	216.4	0.1	0.4	0.0	9.0	0.0	0.0	1.7	0.43	0.01	0.43	0.09
4	500	8	4	1	0.35	371.9	0.1	369.1	371.2	0.1	0.4	0.0	16.0	0.0	0.0	3.0	0.16	0.02	0.16	0.16
5	500	8	4	2	0.7	347.9	0.1	331.5	341.5	0.2	0.6	0.0	33.0	0.0	0.0	5.7	0.00	0.01	0.00	0.14
6	500	8	4	2	0.35	388.9	0.1	380.8	387.4	0.1	0.4	0.0	15.0	0.0	0.0	2.7	3.68	0.01	2.44	0.09
7	500	8	4	3	0.7	362.9	0.1	327.6	346.2	0.2	0.6	0.0	44.0	0.0	0.0	7.5	0.00	0.01	0.00	0.14
8	500	8	4	3	0.35	382.3	0.1	366.5	379.2	0.2	0.5	0.0	25.0	0.0	0.1	4.7	0.20	0.01	0.20	0.11
9	500	8	6	2	0.7	367.4	0.1	360.3	361.2	0.2	0.5	0.0	27.0	0.0	0.0	4.9	1.41	0.02	0.00	0.10
10	500	8	6	2	0.35	421.5	0.0	417.9	420.8	0.1	0.4	0.0	12.0	0.0	0.0	2.2	1.25	0.04	1.25	0.14
11	500	8	6	3	0.7	368.6	0.0	366.4	367.9	0.1	0.4	0.0	9.0	0.0	0.0	1.7	1.96	0.02	3.74	0.15
12	500	8	6	3	0.35	386.2	0.0	382.3	385.3	0.1	0.4	0.0	15.0	0.0	0.0	2.6	1.80	0.01	1.40	0.17
13	500	8	6	5	0.7	367.5	0.0	340.9	360.0	0.1	0.4	0.0	20.0	0.0	0.0	3.3	1.01	0.02	1.01	0.14
14	500	8	6	5	0.35	382.6	0.0	366.8	380.9	0.1	0.4	0.0	11.0	0.0	0.0	1.9	0.73	0.05	0.73	0.14
15	500	16	4	1	0.7	223.4	0.1	199.8	208.2	0.2	0.5	0.0	13.0	0.0	0.0	2.4	0.00	0.01	0.00	0.11
16	500	16	4	1	0.35	386.9	0.2	383.7	386.0	0.3	1.0	0.0	72.0	0.0	0.0	23.7	0.44	0.03	0.44	0.18
17	500	16	4	2	0.7	372.8	0.1	350.4	355.2	0.5	1.1	0.0	150.0	0.1	0.1	51.1	4.42	0.01	4.42	0.17
18	500	16	4	2	0.35	375.6	0.2	365.3	371.9	0.4	1.1	0.0	62.0	0.0	0.0	20.1	2.29	0.01	2.29	0.23
19	500	16	4	3	0.7	392.3	0.2	356.5	376.0	0.5	1.1	0.0	215.0	0.0	0.1	75.1	1.03	0.02	1.03	0.17
20	500	16	4	3	0.35	389.2	0.1	375.9	386.4	0.4	1.1	0.0	92.0	0.0	0.1	31.2	1.00	0.02	1.65	0.23
21	500	16	8	2	0.7	460.3	0.1	459.7	459.8	0.2	0.8	0.0	68.0	0.0	0.0	22.0	2.36	0.12	0.30	0.36
22	500	16	8	2	0.35	455.9	0.1	455.7	455.9	0.1	0.8	0.0	41.0	0.0	0.0	13.4	2.96	0.07	0.96	0.21
23	500	16	8	4	0.7	459.3	0.1	458.6	459.2	0.2	0.8	0.0	137.0	0.0	0.1	44.5	2.68	0.05	2.55	0.31
24	500	16	8	4	0.35	443.4	0.1	441.7	443.0	0.2	0.8	0.0	57.0	0.0	0.0	18.0	2.29	0.06	1.95	0.28
25	500	16	8	6	0.7	463.7	0.1	448.3	457.5	0.2	0.8	0.0	487.0	0.0	0.4	226.0	4.02	0.06	1.30	0.35
26	500	16	8	6	0.35	470.6	0.1	467.1	470.4	0.2	0.9	0.0	118.0	0.0	0.1	37.9	1.27	0.08	0.71	0.27
27	500	16	12	3	0.7	459.7	0.1	459.7	459.7	0.1	0.7	0.0	16.0	0.0	0.0	5.4	0.01	0.14	0.01	0.27
28	500	16	12	3	0.35	414.7	0.1	414.7	414.7	0.1	0.8	0.0	8.0	0.0	0.0	3.0	0.16	0.13	0.19	0.25
29	500	16	12	6	0.7	473.4	0.1	473.3	473.3	0.1	0.7	0.0	35.0	0.0	0.0	11.0	1.10	0.09	0.83	0.22
30	500	16	12	6	0.35	429.2	0.1	429.1	429.2	0.1	0.8	0.0	14.0	0.0	0.0	4.7	0.70	0.11	1.04	0.24

Table 8.2 (Cont'd)

						$MCLP-P$		$SR-LP$				Benders					GNS		RNS	
Case	$\|I\|$	$\|J\|$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	# of cuts	t_{sol} t_{sub}	t_{sol} t_{master}	t_{total}	Gap (%)	t_{total}	Gap (%)	t_{total}
31	500	16	12	9	0.70	482.44	0.06	480.66	481.45	0.17	0.75	0.00	226	0.03	0.05	69.91	1.05	0.13	0.36	0.25
32	500	16	12	9	0.35	453.37	0.06	451.96	452.73	0.14	0.82	0.00	39	0.05	0.03	11.82	0.82	0.11	0.82	0.24
33	500	32	8	2	0.70	389.61	0.13	384.01	385.93	0.34	0.91	0.00	57	0.05	0.03	18.80	2.22	0.02	2.22	0.19
34	500	32	8	2	0.35	491.66	0.19	491.58	491.60	0.38	1.70	0.00	128	0.06	0.20	94.91	2.01	0.20	1.89	0.45
35	500	32	8	4	0.70	492.50	0.17	491.77	492.30	0.42	1.68	0.00	545	0.08	5.15	1748.14	3.53	0.11	2.24	0.49
36	500	32	8	4	0.35	478.50	0.16	476.41	478.33	0.41	1.80	0.00	101	0.06	0.08	66.79	3.15	0.16	0.82	0.42
37	500	32	8	6	0.70	486.27	0.35	480.50	483.58	1.08	2.34	6.42	366	0.09	20.69	>3600	2.89	0.08	2.89	0.49
38	500	32	8	6	0.35	485.07	0.23	484.94	485.02	0.40	1.66	0.05	105	0.07	0.20	76.35	4.10	0.11	1.33	0.39
39	500	32	16	4	0.70	493.80	0.12	493.80	493.80	0.25	1.56	0.03	57	0.07	0.01	36.48	0.04	0.53	0.02	0.58
40	500	32	16	4	0.35	471.29	0.14	471.29	471.29	0.19	1.54	0.00	23	0.05	0.01	15.14	0.01	0.52	0.04	0.61
41	500	32	16	8	0.70	493.81	0.19	493.81	493.81	0.29	1.62	0.04	76	0.07	0.03	46.94	1.10	0.44	0.04	0.47
42	500	32	16	8	0.35	485.75	0.16	485.75	485.75	0.25	1.51	0.06	95	0.07	0.03	59.39	0.40	0.37	0.11	0.50
43	500	32	16	12	0.70	492.57	0.14	492.38	492.50	0.21	1.53	0.10	647	0.07	0.30	506.17	1.50	0.27	0.40	0.36
44	500	32	16	12	0.35	499.13	0.13	499.13	499.13	0.20	1.56	0.08	161	0.07	0.17	116.86	0.71	0.25	0.85	0.37
45	500	32	24	6	0.70	485.61	0.14	485.61	485.61	0.23	1.62	0.00	5	0.06	0.01	4.28	0.00	0.69	0.00	0.53
46	500	32	24	6	0.35	489.53	0.17	489.53	489.53	0.17	1.47	0.10	5	0.07	0.01	4.26	0.00	0.75	0.00	0.52
47	500	32	24	12	0.70	498.04	0.12	498.04	498.04	0.18	1.46	0.01	8	0.07	0.01	5.93	0.00	0.55	0.00	0.63
48	500	32	24	12	0.35	468.15	0.14	468.15	468.15	0.18	1.52	0.00	7	0.06	0.01	5.33	0.00	0.80	0.00	0.56
49	500	32	24	18	0.70	494.80	0.12	494.76	494.80	0.19	1.51	0.08	49	0.07	0.03	29.43	1.30	0.58	0.57	0.44
50	500	32	24	18	0.35	481.02	0.12	481.02	481.02	0.17	1.49	0.09	5	0.07	0.01	4.14	0.38	0.58	0.30	0.53
51	2000	8	2	1	0.70	912.55	0.28	816.96	857.61	0.89	2.40	0.00	14	0.07	0.01	10.43	2.18	0.02	2.18	0.30
52	2000	8	2	1	0.35	861.63	0.21	820.75	850.48	0.46	1.97	0.00	9	0.08	0.01	7.14	0.00	0.02	0.00	0.28
53	2000	8	4	1	0.70	1507.55	0.26	1478.62	1491.54	0.55	2.03	0.00	12	0.08	0.01	9.29	0.00	0.06	0.00	0.39
54	2000	8	4	1	0.35	1447.06	0.29	1428.23	1441.95	1.12	2.57	0.00	31	0.07	0.02	22.05	1.75	0.06	0.35	0.47
55	2000	8	4	2	0.70	1520.56	0.31	1436.18	1471.81	1.00	2.36	0.00	29	0.08	0.01	20.27	1.93	0.02	1.93	0.44
56	2000	8	4	2	0.35	1478.79	0.29	1440.35	1469.36	1.31	2.66	0.00	20	0.07	0.01	14.38	4.42	0.05	2.59	0.52
57	2000	8	4	3	0.70	1369.09	0.30	1254.86	1320.09	1.17	2.51	0.00	32	0.08	0.02	21.70	1.74	0.05	1.74	0.66
58	2000	8	4	3	0.35	1507.26	0.28	1432.20	1493.83	1.00	2.35	0.00	27	0.08	0.01	18.58	4.86	0.03	4.86	0.36
59	2000	8	6	2	0.70	1762.08	0.25	1747.80	1754.71	0.31	1.69	0.00	12	0.07	0.00	8.95	2.71	0.11	2.73	0.45
60	2000	8	6	2	0.35	1510.67	0.25	1508.99	1510.58	0.30	1.66	0.00	15	0.06	0.01	10.92	0.25	0.11	0.06	0.56

Table 8.2 (Cont'd)

						$MCLP-P$		$SR-LP$				Benders					GNS		RNS	
Case	$\lvert I\rvert$	$\lvert J\rvert$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	# of cuts	t_{sol} t_{sub}	t_{sol} t_{master}	t_{total}	Gap (%)	t_{total}	Gap (%)	t_{total}
61	2000	8	6	3	0.70	1504.08	0.20	1452.88	1489.10	0.48	1.88	0.00	16	0.07	0.01	11.37	0.00	0.09	0.00	0.45
62	2000	8	6	3	0.35	1714.08	0.23	1685.51	1712.76	0.34	1.85	0.00	11	0.08	0.01	8.25	0.46	0.03	0.44	0.75
63	2000	8	6	5	0.70	1451.95	0.27	1329.13	1412.73	0.71	2.15	0.00	22	0.07	0.01	14.72	4.54	0.08	4.54	0.39
64	2000	8	6	5	0.35	1736.73	0.24	1677.77	1733.11	0.43	1.87	0.00	15	0.07	0.01	10.52	0.95	0.08	0.95	0.37
65	2000	16	4	1	0.70	1512.19	0.66	1498.73	1505.52	2.60	5.37	0.00	56	0.17	0.02	75.55	2.75	0.13	1.82	0.70
66	2000	16	4	1	0.35	1459.29	0.68	1449.19	1457.39	2.56	5.21	0.00	33	0.17	0.02	44.99	1.63	0.16	0.00	0.63
67	2000	16	4	2	0.70	1540.30	0.58	1450.24	1491.88	2.98	5.74	0.00	114	0.18	0.06	152.16	4.01	0.06	1.50	0.58
68	2000	16	4	2	0.35	1597.74	0.61	1564.76	1590.14	1.88	4.68	0.00	39	0.14	0.02	52.80	4.05	0.06	4.05	0.52
69	2000	16	4	3	0.70	1537.95	0.60	1386.73	1451.46	6.04	8.69	0.00	319	0.17	0.14	436.18	3.00	0.03	3.00	0.52
70	2000	16	4	3	0.35	1540.68	0.64	1475.62	1529.67	4.02	6.64	0.00	134	0.21	0.04	176.99	3.43	0.03	3.43	0.62
71	2000	16	8	2	0.70	1865.28	0.62	1862.53	1863.51	0.74	3.68	0.00	80	0.17	0.02	109.20	2.11	0.20	0.58	0.94
72	2000	16	8	2	0.35	1837.83	0.62	1837.34	1837.81	1.31	4.44	0.00	31	0.17	0.02	44.36	0.80	0.28	0.72	0.92
73	2000	16	8	4	0.70	1810.11	0.65	1802.10	1807.69	1.30	4.10	0.00	198	0.18	0.09	257.74	4.74	0.16	3.02	0.92
74	2000	16	8	4	0.35	1783.46	0.64	1779.35	1782.59	0.90	3.71	0.00	43	0.17	0.02	58.18	1.97	0.13	1.78	1.06
75	2000	16	8	6	0.70	1858.89	0.63	1800.15	1831.77	1.66	4.64	0.00	261	0.19	0.15	345.76	4.57	0.14	2.69	0.91
76	2000	16	8	6	0.35	1555.09	0.61	1527.51	1550.13	1.19	3.97	0.00	50	0.15	0.03	63.43	4.52	0.13	4.52	0.95
77	2000	16	12	3	0.70	1733.54	0.56	1733.37	1733.54	0.61	3.39	0.02	14	0.14	0.01	20.23	0.08	0.42	0.08	1.19
78	2000	16	12	3	0.35	1916.41	0.55	1916.39	1916.41	0.60	3.23	0.00	20	0.15	0.01	27.58	0.25	0.45	0.03	0.75
79	2000	16	12	6	0.70	1922.66	0.49	1917.39	1919.39	0.92	3.53	0.01	67	0.15	0.02	83.86	2.51	0.30	1.42	1.03
80	2000	16	12	6	0.35	1766.38	0.56	1763.54	1765.09	0.73	3.55	0.01	20	0.15	0.01	26.95	0.98	0.33	0.50	0.83
81	2000	16	12	9	0.70	1798.91	0.55	1780.25	1789.35	1.27	3.93	0.00	152	0.15	0.04	181.29	1.60	0.31	1.63	0.92
82	2000	16	12	9	0.35	1883.02	0.56	1873.01	1881.61	0.87	3.71	0.02	29	0.15	0.02	36.39	0.33	0.25	0.12	0.53
83	2000	32	8	2	0.70	1913.01	1.72	1912.55	1912.85	2.62	7.91	0.00	194	0.35	0.20	513.93	1.05	0.59	3.05	1.53
84	2000	32	8	2	0.35	1910.63	2.20	1909.28	1910.37	4.07	9.29	0.00	211	0.31	0.25	567.73	2.53	0.59	1.79	1.31
85	2000	32	8	4	0.70	1936.35	1.70	1927.96	1932.58	3.14	8.45	13.83	448	0.43	10.73	>3600	2.43	0.42	2.18	1.42
86	2000	32	8	4	0.35	1943.76	1.65	1940.78	1943.51	2.70	8.02	5.88	349	0.42	22.27	>3600	4.28	0.30	1.71	1.64
87	2000	32	8	6	0.70	1936.06	1.70	1904.54	1927.58	18.94	23.98	21.72	314	0.42	16.43	>3600	0.74	0.31	1.46	1.94
88	2000	32	8	6	0.35	1933.02	1.72	1923.72	1932.72	3.33	8.55	10.57	260	0.47	23.93	>3600	3.87	0.31	3.87	2.08
89	2000	32	16	4	0.70	1933.45	1.53	1933.45	1933.45	1.89	7.14	0.01	37	0.30	0.02	97.75	0.00	1.64	0.01	2.22
90	2000	32	16	4	0.35	1995.43	1.56	1995.43	1995.43	1.80	7.16	0.02	78	0.30	0.03	195.77	0.01	1.89	0.01	2.47

Table 8.2 (Cont'd)

						$MCLP-P$		$SR-LP$				$Benders$					GNS		RNS	
Case	$\lvert I\rvert$	$\lvert J\rvert$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	# of cuts	t_{sol} t_{sub}	t_{sol} t_{master}	t_{total}	Gap (%)	t_{total}	Gap (%)	t_{total}
91	2000	32	16	8	0.70	1923.35	1.63	1923.35	1923.35	2.44	7.41	0.00	71	0.33	0.02	173.86	0.07	1.11	0.08	1.70
92	2000	32	16	8	0.35	1927.27	1.47	1926.88	1927.04	2.41	7.56	0.02	20	0.30	0.02	53.18	0.27	1.03	0.17	1.44
93	2000	32	16	12	0.70	1885.46	1.58	1867.96	1873.53	2.80	8.00	0.39	1040	0.31	5.03	>3600	2.28	1.30	0.88	1.59
94	2000	32	16	12	0.35	1928.17	1.50	1927.07	1927.84	2.55	7.67	0.00	44	0.23	0.00	113.67	0.39	0.80	0.91	1.91
95	2000	32	24	6	0.70	1978.57	1.19	1978.57	1978.57	1.22	6.27	0.00	6	0.23	0.00	19.61	0.00	2.83	0.00	1.95
96	2000	32	24	6	0.35	1958.50	1.13	1958.50	1958.50	1.28	6.63	0.00	6	0.30	0.00	19.50	0.00	2.13	0.00	2.02
97	2000	32	24	12	0.70	1981.28	1.13	1981.28	1981.28	1.31	6.75	0.00	7	0.30	0.02	20.86	0.00	1.72	0.00	2.27
98	2000	32	24	12	0.35	1982.22	1.19	1982.22	1982.22	1.41	6.34	0.00	8	0.31	0.02	23.72	0.01	2.06	0.00	1.88
99	2000	32	24	18	0.70	1872.07	1.08	1871.87	1872.07	1.25	6.47	0.02	12	0.33	0.02	31.89	0.60	1.70	0.19	1.64
100	2000	32	24	18	0.35	1963.45	1.20	1963.45	1963.45	1.63	6.83	0.00	12	0.33	0.00	31.91	0.29	1.42	0.19	1.44
101	8000	8	2	1	0.70	4082.59	1.67	3475.81	3695.05	20.17	25.58	0.00	21	0.45	0.00	60.52	3.40	0.06	3.40	1.64
102	8000	8	2	1	0.35	3811.29	1.83	3634.65	3766.82	13.19	18.28	0.00	18	0.41	0.02	52.57	0.00	0.06	0.00	0.81
103	8000	8	4	1	0.70	6014.36	1.67	5840.77	5925.08	22.50	28.29	0.00	22	0.23	0.03	63.34	0.76	0.14	0.61	1.66
104	8000	8	4	1	0.35	5982.86	1.44	5912.42	5966.76	2.08	7.19	0.00	8	0.25	0.02	25.52	0.00	0.19	0.00	2.05
105	8000	8	4	2	0.70	6099.52	1.44	5824.74	6004.53	5.28	10.96	0.00	27	0.39	0.02	75.49	0.00	0.13	0.00	1.53
106	8000	8	4	2	0.35	5185.63	1.52	5076.02	5142.58	10.03	15.13	0.00	13	0.36	0.02	38.54	0.76	0.13	0.76	1.80
107	8000	8	4	3	0.70	5313.18	1.38	4676.74	5083.46	8.77	13.86	0.00	44	0.42	0.02	115.80	0.00	0.19	0.00	2.83
108	8000	8	4	3	0.35	5759.43	1.45	5521.39	5727.52	4.84	10.72	0.00	21	0.36	0.02	58.10	0.00	0.23	0.00	1.80
109	8000	8	6	2	0.70	6333.91	1.30	6274.78	6300.76	3.58	8.63	0.00	8	0.31	0.02	25.21	1.84	0.36	1.51	2.13
110	8000	8	6	2	0.35	6524.94	1.39	6509.44	6519.73	1.97	7.84	0.00	6	0.25	0.00	19.96	0.00	0.31	0.00	2.04
111	8000	8	6	3	0.70	6787.93	1.25	6680.87	6713.17	4.39	9.56	0.00	11	0.39	0.00	32.96	0.47	0.45	0.00	1.55
112	8000	8	6	3	0.35	6941.02	1.39	6847.96	6928.09	2.73	8.13	0.00	7	0.23	0.02	22.16	0.42	0.23	0.42	2.73
113	8000	8	6	5	0.70	5898.12	1.31	5170.11	5552.75	11.44	16.67	0.00	21	0.38	0.02	55.27	1.57	0.45	0.93	1.84
114	8000	8	6	5	0.35	6577.42	1.39	6364.06	6561.93	5.38	10.75	0.00	14	0.34	0.02	38.66	2.15	0.25	2.15	1.53
115	8000	16	4	1	0.70	5913.14	5.43	5843.00	5879.18	28.75	38.85	0.00	111	0.84	0.03	582.38	3.06	0.44	0.00	2.77
116	8000	16	4	1	0.35	5965.18	4.71	5902.86	5946.13	21.71	32.39	0.00	71	0.77	0.03	371.45	4.08	0.31	0.00	2.48
117	8000	16	4	2	0.70	6272.26	5.52	5958.04	6107.29	59.51	69.53	0.00	409	0.83	0.22	2126.86	1.49	0.19	1.49	2.69
118	8000	16	4	2	0.35	6166.53	10.13	6014.67	6128.72	74.51	84.45	0.00	288	0.80	0.12	1479.30	3.99	0.16	3.99	2.34
119	8000	16	4	3	0.70	5851.18	5.06	5235.96	5553.32	120.84	130.89	0.00	396	0.83	0.22	2007.63	0.00	0.35	0.00	3.24
120	8000	16	4	3	0.35	6017.53	5.14	5787.22	5988.39	60.73	70.62	0.00	249	0.84	0.08	1255.84	2.12	0.20	2.00	2.60

Table 8.2 (Cont'd)

						MCLP − P		SR − LP				Benders					GNS		RNS	
Case	$\|I\|$	$\|J\|$	P	Γ	δ	z	t_{sol}	z	z_{nom}	t_{sol}	t_{total}	Gap (%)	# of cuts	t_{sol} t_{sub}	t_{sol} t_{master}	t_{total}	Gap (%)	t_{total}	Gap (%)	t_{total}
121	8000	16	8	2	0.70	7709.83	4.80	7701.06	7704.52	6.91	16.88	0.00	28	0.84	0.02	152.58	2.82	1.21	0.64	3.71
122	8000	16	8	2	0.35	7413.22	4.02	7397.85	7411.42	6.61	16.50	0.00	14	0.48	0.02	78.90	1.44	1.34	0.17	4.68
123	8000	16	8	4	0.70	7462.90	4.05	7408.32	7433.60	9.17	19.53	0.00	70	0.48	0.03	357.00	4.17	0.50	1.44	4.57
124	8000	16	8	4	0.35	7645.37	4.58	7639.71	7644.73	7.58	17.35	0.00	23	0.47	0.02	122.23	2.12	0.71	1.05	4.29
125	8000	16	8	6	0.70	7023.13	5.06	6832.76	6914.94	20.64	30.47	6.83	608	0.91	2.16	>3600	1.81	0.43	1.81	4.42
126	8000	16	8	6	0.35	7478.34	4.64	7450.14	7471.44	7.66	17.77	0.00	21	0.62	0.02	114.17	3.49	0.65	3.49	4.69
127	8000	16	12	3	0.70	6558.73	4.08	6558.56	6558.56	6.61	16.41	0.00	10	0.66	0.00	59.54	0.10	2.26	0.04	4.54
128	8000	16	12	3	0.35	7381.23	4.00	7378.14	7380.33	6.41	17.61	0.00	9	0.66	0.00	54.12	0.18	2.29	0.24	4.43
129	8000	16	12	6	0.70	7545.00	3.53	7518.64	7530.29	6.95	16.97	0.00	24	0.80	0.02	124.52	2.17	1.19	1.37	3.58
130	8000	16	12	6	0.35	6816.34	3.66	6814.51	6815.96	6.14	17.08	0.00	9	0.78	0.02	53.52	0.47	1.53	0.15	3.05
131	8000	16	12	9	0.70	7227.59	3.78	7218.89	7227.06	6.29	16.10	0.00	33	0.80	0.02	160.46	3.25	0.73	2.88	3.01
132	8000	16	12	9	0.35	7371.53	3.86	7312.80	7360.26	7.22	17.19	0.00	16	0.77	0.02	83.71	0.52	1.10	0.47	3.80
133	8000	32	8	2	0.70	7770.36	19.32	7752.74	7762.95	33.74	53.53	6.30	311	1.78	4.13	>3600	2.93	5.30	1.91	7.32
134	8000	32	8	2	0.35	7761.17	20.08	7759.15	7760.86	18.18	37.67	5.82	233	1.78	10.27	>3600	2.75	5.80	2.04	5.87
135	8000	32	8	4	0.70	7803.62	24.74	7775.38	7790.09	77.11	96.98	15.16	243	3.52	8.67	>3600	2.46	1.89	2.29	5.41
136	8000	32	8	4	0.35	7795.09	46.83	7785.45	7794.62	24.88	47.54	8.54	167	2.27	10.78	>3600	1.82	2.49	1.37	6.13
137	8000	32	8	6	0.70	7743.12	22.38	7592.06	7704.84	100.42	121.76	23.26	224	1.75	9.35	>3600	4.08	1.27	2.92	7.30
138	8000	32	8	6	0.35	7606.26	19.07	7556.67	7601.53	39.39	60.41	11.38	210	1.88	9.41	>3600	1.79	1.82	1.79	5.76
139	8000	32	16	4	0.70	7626.01	18.34	7626.01	7626.01	18.99	49.87	0.00	54	1.84	0.01	587.33	0.01	12.71	0.01	9.51
140	8000	32	16	4	0.35	7921.83	21.80	7921.83	7921.83	20.57	43.69	0.00	59	0.99	0.01	621.09	0.00	12.64	0.00	12.43
141	8000	32	16	8	0.70	7912.95	26.15	7912.95	7912.95	20.66	44.73	0.00	116	2.54	0.02	1267.92	0.59	7.93	0.41	7.78
142	8000	32	16	8	0.35	7922.22	20.17	7922.17	7922.21	21.22	44.12	0.00	73	1.73	0.01	781.46	0.22	7.08	0.28	9.03
143	8000	32	16	12	0.70	7845.32	24.60	7831.86	7837.54	28.89	50.47	36.99	174	2.21	19.77	>3600	2.90	5.16	1.16	8.48
144	8000	32	16	12	0.35	7897.06	21.64	7893.36	7896.44	26.91	49.50	0.00	263	1.63	0.06	>3600	1.60	4.25	1.28	5.78
145	8000	32	24	6	0.70	7220.98	15.62	7220.98	7220.98	19.25	41.68	0.00	5	1.08	0.01	76.02	0.00	20.23	0.00	13.42
146	8000	32	24	6	0.35	7874.28	16.80	7874.28	7874.28	10.79	35.71	0.00	5	1.15	0.01	71.93	0.00	22.69	0.00	15.31
147	8000	32	24	12	0.70	7914.81	17.89	7914.81	7914.81	11.73	34.35	0.00	6	1.08	0.01	86.74	0.01	20.66	0.01	12.82
148	8000	32	24	12	0.35	7856.02	14.04	7856.02	7856.02	13.84	39.88	0.00	6	1.81	0.01	87.33	0.00	15.96	0.00	11.18
149	8000	32	24	18	0.70	7919.87	12.57	7919.66	7919.85	15.08	37.20	0.00	11	1.74	0.00	123.51	0.51	7.89	0.36	6.16
150	8000	32	24	18	0.35	7942.24	12.80	7942.24	7942.24	10.93	31.25	0.00	6	1.70	0.02	75.90	0.39	10.63	0.03	8.40

APPENDIX C

Impact of Robustness on Solutions

Investigation of the impact of robustness on solutions can be found in Table 8.3.

Table 8.3 Impact of Robustness on Solutions

	δ																				
	25%							65%							90%						
Γ	MCLP	R_{nom}	SR_{nom}	$MCLP_R$	R	$MCLP_{SR}$	SR	MCLP	R_{nom}	SR_{nom}	$MCLP_R$	R	$MCLP_{SR}$	SR	MCLP	R_{nom}	SR_{nom}	$MCLP_R$	R	$MCLP_{SR}$	SR
2	281.6	281.6	281.5	275.3	275.4	280.1	280.2	269.3	268.3	267.3	251.7	253.0	264.0	265.8	277.9	276.5	270.0	254.2	254.5	268.5	270.0
4	281.6	281.5	280.4	270.2	270.5	276.9	277.3	269.3	264.8	261.0	236.6	240.0	255.4	258.1	277.9	270.8	261.2	233.2	235.8	256.1	259.4
6	281.6	281.5	280.2	265.9	266.4	273.4	274.1	269.3	263.1	256.1	222.5	228.1	245.0	250.1	277.9	266.9	250.0	215.6	219.7	241.1	247.8
8	281.6	280.8	278.8	261.9	262.4	269.4	270.4	269.3	260.6	250.0	210.0	217.3	233.2	239.8	277.9	266.4	237.5	199.2	204.6	225.2	233.9
10	281.6	280.1	277.0	258.1	259.0	265.3	266.5	269.3	259.5	244.2	198.8	208.2	220.0	228.2	277.9	257.1	226.2	183.8	193.5	208.6	219.4
12	281.6	280.3	276.8	254.8	256.2	260.8	262.0	269.3	257.9	238.3	189.4	199.7	205.5	215.1	277.9	257.9	213.8	170.1	183.2	189.1	203.4
14	281.6	280.4	275.0	252.4	253.8	255.1	256.4	269.3	260.0	226.7	181.6	193.7	189.7	199.6	277.9	253.7	211.6	158.9	173.4	167.3	198.7