EXPLORING THE UNDERLYING MECHANISMS OF A SIXTH GRADE STUDENT'S FIRST STEPS TOWARDS THE DEDUCTIVE PROOF SCHEME

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ABSTRACT

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This study used an individual teaching experiment methodology to investigate a sixth-grade student's first steps towards the deductive proof scheme. The study consisted of two major parts. The first part aimed to explore the processes by which an individual student may (1) come to understand a deductive argument as an instance of mathematical proof for the first time (Learning Goal 1). The second part aimed to (2) develop the main idea of this first instance proof as a transferrable main idea to the tasks of proving conjectures analogous to the first one proved and then explore (3) how the student's developed understandings enabled or constrained her attempts to prove other similar and novel theorems. The main focus of the second part was on definitional reasoning as a crucial way of thinking associated with the development of the deductive proof scheme (Learning Goal 2). The study used constructs from DNR-Based Instruction in Mathematics and Learning Through Activity (LTA) theoretical frameworks complementarily to achieve its purposes. The study's instructional approach to the first learning goal successfully explicated the mechanisms behind the student's understanding of the first instance proof of the study. Its approach to the second learning goal, on the other hand, did not result in the intended definitional reasoning yet informed empirically-based hypotheses regarding its development. The study concluded with a second elaboration of how the DNR and LTA constructs can be used together to provide a theoretical foundation for the learning and teaching of proof.

Keywords: Deductive Proof Scheme, Definitional Reasoning, DNR-Based Instruction in Mathematics, Learning Through Activity, Individual Teaching Experiments

ÖZ

BİR ALTINCI SINIF ÖĞRENCİSİNİN TÜMDENGELİMLİ İSPAT ŞEMASINA DOĞRU İLK ADIMLARININ ALTINDA YATAN MEKANİZMALARIN BİR İNCELEMESİ

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Bu çalışma bir altıncı sınıf öğrencisinin tümdengelimli ispat şemasını geliştirmeye yönelik ilk adımlarını araştırmaktır. Bireysel öğretim deneyi yönteminin uygulandığı bu çalışma, iki ana bölümden oluşmaktadır. İlk bölümde, katılımcı öğrencinin (1) tümdengelimli bir argümanı matematiksel ispatın ilk örneği olarak anlama süreci incelenmektedir (Öğrenme Hedefi 1). İkinci bölümde, öğrencinin (2) ispatın ilk örneği olarak anladığı bu argümanın ana fikrini analojik varsayımları kanıtlama görevlerinde uygulanabilir bir ana fikir olarak geliştirmesi süreci ve ardından (3) öğretim deneyi sırasında edindiği bilgilerin diğer benzer veya yeni kanıtlama girişimlerini nasıl desteklediği veya kısıtladığı araştırılmaktadır. İkinci bölümün ana odak noktası, tümdengelimli ispat şemasının (Öğrenme Hedefi 2) gelişmesiyle doğrudan ilişkili önemli bir düşünme şekli olan tanımsal muhakemedir. Matematikte DNR-Tabanlı Öğretim ve Etkinlik Yoluyla Öğrenme (LTA) çerçevelerinin farklı teorik yapıları bir arada kullanmıştır. İlk öğrenme hedefine yönelik olarak tasarlanan öğretim yaklaşımı, öğrencinin ilk örnek ispatını anlama sürecini açıklayan mekanizmaları başarılı bir şekilde ortaya çıkarmıştır. İkinci öğrenme hedefine yönelik olarak tasarlanan yaklaşım ise, amaçlanan tanımsal muhakeme ile sonuçlanmasa da tanımsal muhakemenin gelişimine ilişkin ampirik temelli hipotezlerin oluşturulmasına olanak sağlamıştır. Çalışma, ispatın öğretimini teşvik etmek için gerekli teorik altyapının oluşturulmasında DNR ve LTA teorik yapılarının kullanımına dair ikinci bir yorum geliştirmiştir.

Anahtar Kelimeler: Tümdengelimli İspat Şeması, Tanımsal Akıl Yürütme, DNR'ye Dayalı Matematik Öğretimi, Etkinlik Yoluyla Öğrenme, Bireysel Öğretim Deneyleri

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LIST OF ABBREVIATIONS

AoC	Abstraction of Commonality
CoA	Coordination of Actions
DNR	DNR-Bases Instruction in Mathematics
HLT	Hypothetical Learning Trajectory
LTA	Learning Through Activity
METU	Middle East Technical University
MoNE	Ministry of National Education
NAEP	National Assessment of Educational Progress
NCTM	National Council of Teachers of Mathematics

CHAPTER 1

INTRODUCTION

Instructional efforts to make proving a routine practice of school mathematics are grounded in its importance to the discipline (Harel, 2008a; 2008b; Stylianides & Stylianides, 2017). Proving is the mental act through which the truth of mathematical assertions is established (Harel & Sowder, 2007; Harel, 2008c). By employing this mental act, individuals (or communities) propose arguments to ascertain themselves or to persuade others. The proposed arguments may or may not qualify as mathematical proof depending on their adherence to the nature of the mathematics discipline (Harel, 2008a). An argument, as a connected sequence of assertions for or against a mathematical claim, is accepted as valid proof only if it uses previously accepted statements, valid forms of reasoning, and forms of representation that are acceptable to the mathematics community (Stylianides, 2007). However, individuals' conceptions of what constitutes proof in mathematics, called their proof schemes, are not always consistent with this rigorous notion of the discipline. For instance, individuals may rely on a limited number of confirming examples to establish the validity of a general statement or may judge the validity of an argument strictly by its appearance rather than evaluating its underlying structure. In a research-based taxonomy of proof schemes, which was proposed by Harel and Sowder (1998), the two erroneous ways of thinking were characterized, among other categories, as the empirical proof scheme and the ritual proof scheme, respectively, while the rigorous notion of proof was associated with the deductive proof scheme.

Based on the robust research finding that the empirical proof scheme is pervasive among students and largely retained even after individuals learn about secure methods of proving (Education Committee of the European Mathematical Society, 2011), recent efforts in mathematics education aim to foster students' transition from empirical to deductive modes of reasoning at the early grade levels of education

(Stylianides & Stylianides, 2009). This is a two-fold learning goal. Since individuals' proof schemes are not mutually exclusive, people can hold multiple proof schemes concurrently (Harel & Sowder, 1998). The empirical proof scheme does not disappear or fade away effortlessly. Rather, it takes substantial effort for students to understand the deficiencies of empirical evidence in mathematics (Harel, 2008b). This makes eliminating the empirical proof scheme one aspect of the learning goal and promoting the deductive proof scheme another.

An empirically-based approach to the first aspect was offered by Stylianides and Stylianides (2009). The authors developed an instructional sequence that successfully helped undergraduate mathematics students realize the limitations of empirical arguments and develop an intellectual need for a more secure method for validating mathematical generalizations. This study aims to explore the second aspect of the learning goal following the elimination of the empirical proof scheme—how to promote students' transition toward the deductive proof scheme. To clarify the challenge undertaken, it is crucial to understand Harel's (2007) triple proving-proof-proof scheme from a theoretical point of view, which will be discussed in the following section.

Proving as a mental act in DNR-based instruction in mathematics

DNR-based instruction in mathematics (DNR) is a theoretical framework that aims to answer two major questions of mathematics education: "(1) What is the mathematics that we should teach in school?" (Harel, 2008a, p. 487), and "(2) How should we teach it?" (Harel, 2008b, p. 893). The name DNR is an acronym for the three instructional principles foundational to the framework: *duality*, *necessity*, and *repeated reasoning*. The three principles will be introduced after stating the DNR's definition of mathematics. The definition emphasizes an important but mostly ignored component of mathematics and hence provides a shared conceptual basis for the study of topics such as justification, proof and definitional reasoning (Simon, 2013). Harel (2008a, 2008b) defines mathematics as a collection of all the *products* and all the *cognitive characteristics* of *mental acts* that the mathematics community has accepted throughout history. Mental acts, such as interpreting, modeling, defining, or proving, are the actions by which mathematics is practiced. Proving, which is the focus of the current study, is considered one of the mental acts in DNR-based instruction. According to Harel, a cognitive product of a mental act is an individual's *way of understanding* associated with that particular act. A cognitive characteristic shared by the individual's multitude of products of the same mental act is the individual's *way of thinking* associated with that mental act. In other words, mathematics is the discipline consisting of all the *ways of understanding* and the *ways of thinking* that the mathematics community has accepted throughout history (Harel, 2008a). Practitioners of the discipline do this "by carrying out mental acts with particular characteristics—ways of thinking—to produce particular constructs—ways of understanding" (Harel, 2008a, p.490). This triad of *mental act-way of understanding-way of thinking* is a generalization of the more specific one, the *proving act-proof-proof scheme* (Harel, 2007; Harel 2008a). From this perspective, one's *proof scheme* is her *way of thinking* associated with the act of *proving,* and the particular *proofs* she produces are her *ways of understanding* associated with the act of *proving*.

The duality principle asserts a reciprocal relationship between the development of ways of understanding and ways of thinking. The necessity principle states that students must see an intellectual need to learn what mathematics instruction aims to teach them, and this need is not the same as a psychological or social need. And repeated-reasoning principle sets continuous practice as a key component to the internalization, organization, and retention of mathematics knowledge (Harel, 2008a). Duality is the main principle which explains why developing a deductive proof scheme is a challenging task. It is a two-part principle referred to sometimes by Harel as "Duality I" and "Duality II."

> *Duality principle: (a) Learners come with a set of ways of thinking, some desirable and some undesirable, that inevitably affect the ways of understanding we intend to teach them. (b) Learners develop desirable ways of thinking only through repeated application of proper ways of understanding. (Harel, 2021, p. 711).*

In the above-mentioned study, Stylianides and Stylianides (2009) challenged and successfully eliminated undergraduate mathematics students' empirical proof schemes by using the constructivist notion of cognitive conflict "as a mechanism for supporting developmental progressions in students' knowledge" (Stylianides & Stylianides, 2009,

p. 317). Engaging in the task sequence, the participants of the study expressed concerns about the empirical arguments they had constructed previously and wondered how a mathematical generalization could ever be securely validated. That is, the participants developed an intellectual need for a more secure method of validating mathematical generalizations. Despite being triggered to learn about secure methods of validation, when the forthcoming instruction offered them the opportunity to construct a proof for the first time, they were not able to produce one without the researchers' scaffolding, as with the authors' expectation. In the first place, the participants showed no evidence of anticipating what a valid proof would look like (Stylianides & Stylianides, 2009). A DNR-based explanation of the case can be given as follows: The first part of the duality principle, Duality I, suggests that the empirical proof scheme no longer influences the participants' subsequent proving attempts because it has now been eliminated. This was also true for the deductive proof scheme because the participants did not yet possess it. The second part of the duality principle, Duality II, suggests that the students develop the deductive proof scheme by repeatedly experiencing proper ways of understanding proofs. This suggests that the students must undergo a cognitive shift at some point from not recognizing to recognizing a mathematical argument, a product of their own or someone else's activity, as an example of a valid proof for the first time. Without such a shift, the students cannot be expected to anticipate what kind of response they need to construct "to prove" a given theorem. In this study, I will refer to this point as the first instance of mathematical proof.

Understanding the first example of mathematical proof as a proper member of the category without yet possessing the deductive proof scheme is a paradoxical learning goal. This study primarily aims to explicate the mechanism behind individuals' attainment of this learning goal, which has not been the focus of educational research thus far. As a step towards this, the following section introduces the idea of a "learning paradox" (Pascual-Leone, 1976) and explains why seeing an argument as a proof for the first time is subject to this paradox, as any other case of learning is.

The learning paradox (Pascual-Leone, 1976)

The learning paradox is a consequence of what Piaget called assimilation, one central idea of constructivist theory. According to Piaget's theory of equilibration, assimilation and accommodation are two key processes by which individuals maintain equilibrium while cognizing their experiential world. Experiences new to the individual are assimilated into one's existing concepts and operations (one's existing cognitive schemes) only if they produce expected results for the individual's actions based on the individual's prior experiences. The unfitting experience, which is contradictory to the individual's existing assimilatory structures, creates a perturbation (a disequilibrating experience) in the individual's cognitive structures and leads to an accommodation. Referring to a rearrangement of one's existing schemes, accommodation is the process that reestablishes the desired state of equilibrium (von Glasersfeld, 1995; Simon, Tzur, Heinz, & Kinzel, 2004).

New experiences an individual can recognize and act on (to accomplish a goal) are dependent on one's assimilatory conceptions (Simon, Tzur, Heinz, & Kinzel, 2004). The following claim is adapted from Simon et al.'s (2004) example: the notion of assimilation inherently implies that if the conception of deductive proof is not available to the learner, she cannot recognize an instance of the category in the way someone who has that conception does, no matter how transparently displayed the proof is. This makes the challenge undertaken in the current study a specific case of the general learning paradox: how learners can "get from a conceptually impoverished to a conceptually richer system by anything like a process of learning" (Fodor, 1980, p. 149, cited in Bereiter, 1985).

A disequilibrating experience may stimulate a process of accommodation, as in the case of Stylianides and Stylianides's (2009) study. But it does not necessarily result in a constructive reorganization in the desired direction as illustrated in Simon (1995). One possibility for the solution to the learning paradox was postulated by Piaget (2001) as the process of "reflective abstraction" and followed by other researchers (Bereiter, 1985; Bickhard, 1991; Campbell & Bickhard, 1986; Smith, diSessa, & Roschelle, 1993). Simon, Tzur, Heinz, and Kinzel (2004) elaborated on the construct as the learner's reflection on the activity-effect relationship, which later, along with other original research (Simon, 1995; Simon & Tzur, 2004; Simon, Tzur, Heinz, & Kinzel, 2004; Tzur, 1996; Tzur & Simon, 2004), constituted the basis for the Learning Through Activity (LTA) research program. The LTA's view on learning as reflective abstraction was the main source of inspiration for this study, which looks into how to reach a difficult learning goal: moving toward a deductive proof scheme.

Learning Through Activity (LTA)

LTA is a research program that aims to explicate the mechanisms of mathematics conceptual learning by studying individual students' transition from one conceptual state to another. The basis for studying conceptual change here is the close observation of individual students' goal-directed activities, engendered by purposefully designed task sequences that are also evolving as the research study progresses. The program uses its own (emerging) instructional design framework (Simon, Kara, Placa, & Avitzur, 2018) and LTA-modified teaching experiment methodology (Simon, 2018) to fulfill its two broad and interrelated research goals: to generate a unified theory of conceptual learning and instructional design; and to generate empirically-based hypothetical learning trajectories (HLTs; Simon, 1995) for specific mathematics concepts. This study utilizes both of the LTA research outputs to fulfill its specific purposes.

A key contribution of the LTA theoretical framework is Simon's (2017) definition of "mathematical concept" as a researcher's construct that is revisable in the service of organizing instructional pedagogy. By articulating a mathematical concept as a theoretical construct for mathematics education research purposes, one can accurately define the "goal understandings" that a particular instructional design intends to foster. Simon (2017) designated a mathematical concept as "a researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (p. 123). Hence, it suggests that concepts include not only the objects but also the logical necessity of the relationships among them. Based on this interpretation of a mathematical concept as a researcher's construct, the study of mechanisms by which an individual can construct a conceptual understanding of a specific proof (a first proper way of understanding associated with the act of proving) can be interpreted as how any mathematics concept is learned (in LTA terms, abstracted by the student from her activity). The intended concept of this study can be viewed as understanding the logical necessity of the relationships involved in a particular deductive proof.

Recently, Simon (2020) made a distinction between two types of concepts resulting from two different elaborations of reflective abstraction. The above definition (particularly its emphasis on the logical necessity involved in a relationship) was then restricted to refer to only one category of concepts. Simon (2020) named the two types of concepts based on the processes through which they are reflectively abstracted by the individual learner from her activity. Namely, he referred to the concepts that are constructed through a *coordination of actions* type of reflective abstraction as "CoA concepts" and to those that are constructed through an *abstraction of commonality* type of reflective abstraction as "AoC concepts." The two processes that account for the construction of these concepts may also be the mechanisms to explain one's transition toward the deductive proof scheme.

1.1. The purposes of the study

This study was initiated in the hope that the ideas of CoA and AoC concepts and the related types of reflective abstraction might help explain one's first steps toward the deductive proof scheme. An individual sixth-grade student's earliest encounters with proper instances of deductive proofs were investigated as a two-part learning goal (hereafter, Learning Goal 1 and Learning Goal 2) after her empirical proof scheme was successfully eliminated by adapting and using the task sequence developed in Stylianides and Stylianides (2009). The two constructs, the CoA and the AoC concepts, were not perfectly compatible with the two learning goals of the study. Inevitably, the researcher's initial articulation of the study's concepts differed in nature from the ones that appeared so far in LTA research reports. Learning Goal 1 of the study was articulated as a CoA concept, and Learning Goal 2 was investigated based on the study of an AoC concept and its two different extensions. Below are brief articulations of the intended concepts specific to this study. Operational definitions of the concepts from the LTA theoretical perspective are provided later in the Definition of Important Terms section.

Intended CoA concept

The initial hypothesis of the study was that, in line with the LTA instructional design approach (Simon, Kara, Placa, & Avitzur, 2018), an individual student, in response to a given task, would engage in a sequential activity that would result in a flowchart proof when completed. Reflecting on the result of her activity, the student would have a chance to realize that she indeed created a proof for a specific mathematical generalization. An essential condition for the student to categorize this unfamiliar end-product of her activity as a new type of object, as a first instance of mathematical proof (a first member of a not-yet-constructed class of objects), was her first seeing this product as an object. Therefore, understanding this first instance of proof as an object, in fact as a structural object (Miyazaki, Fujita, & Jones, 2017), constituted Learning Goal 1 of the study. It was articulated as the CoA concept to be constructed by the student through her activity.

Miyazaki, Fujita, and Jones (2017) define the structure of deductive proofs as the relational network of singular and universal propositions connected via two types of deductive reasoning: universal instantiation and hypothetical syllogism. A proof deduces singular propositions (the premises, conclusions, and intermediate propositions between them) from universal propositions (theorems, definitions, and axioms) by means of universal instantiation and connects these singular propositions by hypothetical syllogism. A structural understanding of a particular proof involves understanding both universal instantiations and hypothetical syllogisms contained in the proof. With this certain structure, proofs are connected chains of sequential assertions (Stylianides, 2007), and the students may not have a picture of wholeness in them immediately. They may not even relate a produced proof to a product of one's carrying out the mental act of proving, including their own (Harel, 2008a). Indeed, empirical research shows that students may see a written proof text simply as a "task solution," rather than an argument for or against a mathematical claim (Iversen, 2022).

As Miyazaki, Fujita, and Jones (2017) put it, "Seeing a *proof as an object [emphasis added]* enables appreciation of the components of a proof and their interconnections, how a proof is composed of these components, and why a proof needs the structure that it has" (p. 226). This suggests that, if achieved, a structural understanding of a particular proof may help the student put this new object (the product of her activity) into a class of “arguments that try to validate a mathematical claim” and hopefully “valid proofs” instead of “invalid arguments.”

Intended AoC concept

After a first instance of proof is understood as a structural object, the next part of the problem is about how to support students’ repeated practice of constructing other proper instances to develop the character of the deductive proof scheme. In fact, the *deductive proof scheme* consists of two sub-categories. From the two sub-categories, the *transformational proof scheme* is accessible to individuals long before the other, *modern axiomatic proof scheme* (Harel, 2007; Harel & Sowder, 1998, 2007). The *transformational proof scheme* is marked by three essential characteristics of *generality*, *operational thought*, and *logical inference*. An individual who holds this scheme understands that a proof is a “for all” argument without exception (generality), formulates goals and sub-goals and anticipates their consequences (operational thought), and acknowledges the necessity of employing logical inference rules for justifying in mathematics. In the current study, the generality and logical inference characteristics of the deductive proof scheme would already be instantiated in the first instance of proof (Learning Goal 1), when the intended CoA concept was constructed by the student. The next part of the study, along with the application of these two characteristics, focused on fostering the operational thought characteristic, that is, how to set goals and sub-goals when starting a proof to arrive at the desired conclusion. The *modern axiomatic proof scheme*, in addition to the characteristics of *generality*, *operational thought*, and *logical inference*, includes an understanding that proving must ultimately be based on undefined terms and axioms, which constitute one’s mathematical reality (Harel, 2008a). Since it is beyond the conceptual reach of students at the early grade levels of the school, this study did not intend to reach up to this scheme (Harel, 2008b).

Given that a proof scheme is the common cognitive characteristic shared by one’s multitude of proofs (2008a), the second purpose of the study was to support the student’s practice of understanding multiple instances of valid proofs through her own creating, after a first instance of it was understood as a structural object. At this point, the study was built on an aspect of what Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff (2012) called a "holistic" understanding of a particular proof. In an assessment model for undergraduate mathematics students' proof comprehension, authors defined *holistic comprehension* as understanding a particular proof as a whole "in terms of its main ideas, methods, and application to other contexts" (p. 10). The latter aspect of the application to other contexts indicated an ability to transfer or adapt general ideas or methods from the particular proof to other proving tasks (Mejia-Ramos et al., 2012). This suggests, if understood holistically, the first instance of mathematical proof in the student's experience could be a base for her subsequent act of proving similar theorems, hence for her practicing the act of proving as necessitated by the Duality II principle.

In this study, the student's practice was designed around proving theorems that are similar (at first analogous and then slightly different) to the one whose proof was understood as a structural object for the first time. The "first instance proof" of the study was purposefully designed to be a short direct proof. The proof began with the statement of a single definition and logically followed from this definition by the employment of a single type of logical inference rule, modus ponens (one that is evidenced by psychological research to be accessible earliest to the individuals: $[(p \longrightarrow q) \land p] \vdash q$. By this decision, the study first reduced the complexity of the structure of the first proof to be understood as an object (intended CoA concept) and then helped with confronting potential difficulties that the student could experience when she subsequently practiced proving herself.

Determining how to start proofs, how to define terms (Moore, 1994), and which premises to use (Miyazaki, Fujita, & Jones, 2017; Weber, 2001) are major difficulties that students experience with proving. The specific "first instance proof" of this study condensed these difficulties (that in another proof could come up several times and concurrently) into a single one: defining the key object of interest (only once) to start proving the theorem. However, this was still not an easy task for a sixth-grade student. Harel (2008a) defines definitional reasoning as "the way of thinking by which one defines objects and proves assertions in terms of mathematical definitions" (p. 495). This crucial way of thinking, integral to the development of the deductive proof scheme, cannot be assumed to exist among middle school students. It is not until adulthood that (if at all) most students make sense of the notion of mathematical definition and appreciate its use and value in proving theorems (Harel, 2008a). Therefore, in our case, understanding one specific concept definition in relation to the theorem being proved and appreciating its use in proving other similar theorems, and hence the ability to transfer it, constituted a first step towards the second learning goal of the study. This was the "main idea" of the proof intended to be transferrable to other proving tasks and was articulated as an AoC concept of the study. Such a conception could result from the student's abstracting of the commonality in her activity through proving a certain class of theorems that share the same structure with the first instance proof; hence, it could be similar to an AoC concept created through an AoC type of reflective abstraction (Simon, 2020).

It is important to note that a structural understanding of the "first instance proof" (intended CoA concept), might imply aspects of a holistic understanding such as understanding the main idea and the method of proof. The second part of the study, by focusing on the student's ability to transfer what was understood from the first instance activity, can be viewed as an attempt to build on this structural understanding to achieve a holistic understanding. For this reason, and ease of reference, this study refers to its intended AoC concept as a holistic understanding of the first instance proof.

1.2. Research questions

This study aimed to explore the process by which an individual sixth-grade student's first steps towards the deductive proof scheme can be promoted (to be explained) by building on a first structural and then holistic understanding of a particular proof. More specifically, the study explored the processes by which an individual student may (1) come to understand a deductive argument as an instance of mathematical proof for the first time (intended CoA concept) and then (2) develop a holistic understanding of this "first instance proof" that is transferable to the practice of proving conjectures analogous to the first one (intended AoC concept). The intended CoA and AoC concepts together aimed to promote the characteristics of the deductive proof scheme in the student's way of thinking associated with the act of proving. Therefore, an additional aspect of the study explored (3) how the developed understandings enabled or constrained the student's act of proving other conjectures, both similar (not analogous) to the first one and novel. The following research questions guided the study using an individual teaching experiment methodology.

1. How does (or may) an individual student come to understand a deductive argument as an instance of mathematical proof for the first time? Can a coordination of actions (CoA) type of reflective abstraction explain the process of understanding the first instance proof as a structural object?

2. How does (or may) the student develop a holistic understanding of this first instance proof so that she can transfer its main idea to the analogous proving tasks? Can an abstraction of commonality (AoC) type of reflective abstraction explain the process of constructing such a holistic understanding?

3. How does the student (attempt to) start proving (a) conjectures that are similar (but not analogous) to the first instance conjecture, (b) the target conjecture of the study and (b) a novel conjecture?

1.3. Selection of the content area and the grade level

The sixth-grade level was selected for the study based on methodological considerations. First, conjectures to be proved were formulated in the number theory content area. Then, the distribution of concepts and operations critical to the hypothesized learning process of the study was examined based on the Turkish Middle School Mathematics Curriculum (MoNE, 2018) by the grade levels. Since the goal of the study was to capture an individual student's transition from not knowing to knowing particular mathematical understandings (Tzur, 2018), it was critical for the student participant to have only certain conceptual structures and operations available, and not certain others (Simon, 2018). The sixth-grade level was selected for meeting this criterion in the context of the Turkish Middle School Mathematics Curriculum (MoNE, 2018).

The number theory content area was selected for its particular affordances and questions relevant to the study's interest. Due to the widespread recognition that number theory provides a fruitful basis for mathematical reasoning, the study of its concepts in relation to mathematical proof has spread down into the early years of school (Campbell & Zazkis, 2002). Especially with its introductory concepts restricted to the domain of whole numbers, the content area is regarded to offer rich opportunities to young students (National Council of Teachers of Mathematics [NCTM], 1989). Examples of the introductory concepts, factors, multiples, divisibility, and whole number patterns (Campbell & Zazkis, 2002), frequently appear in recent research investigations in which students' proof-related conceptions and abilities are at the focus. Given that this study aimed to investigate an individual student's process of coming to understand the very first example of mathematical proof as a structural object (Learning Goal 1), number theory content had the potential to offer her the earliest encounter with the idea of proof.

Stylianides (2009) examined the opportunities that a mathematics textbook series offered to middle school students to engage in *reasoning-and-proving* (Stylianides, 2008). By the hyphenated term, Stylianides (2008) defined the overarching activity that embraces four major activities of making sense of and constructing mathematical knowledge: "identifying patterns," "making conjectures," "providing non-proof arguments," and "providing proofs." The first version of the Connected Mathematics Project (CMP) (Lappan et al., 1998/2004) textbook series was selected for the study for its renowned attempt to follow recommendations outlined in the NCTM (1989, 2000) Standards, including that reasoning-and-proving be emphasized across the grade levels and content areas. Findings of the analysis showed that, compared to geometry and algebra units, number theory units contained the highest proportion of designed opportunities for students to engage in reasoning-and-proving in general and "providing proofs" in particular.

Stylianides (2009) offered two possible explanations for the unexpected finding (the largest proportion was expected to be in geometry units due to the long historical tradition of associating proof in school mathematics with Euclidean geometry), as in the following: First, as basic number theory concepts are accessible to young students and receive a great deal of attention in their textbooks, definitions of these concepts, which are essential components of mathematical proofs, are expected to be well developed in the middle grades. The author stated that, in algebra and geometry, on the other hand, systematic development does not begin before middle school. Second, many results of the domain are sensible to young students sparking their interest in investigations that invite the need for proof. “In middle school geometry, for example, it is not easy to find properties that are not evident as being true or false through suitable, simple to perform drawings” (Stylianides, 2009, p. 282).

The last sentence of the above explanation implies that the students are more likely to find properties evident to them in number theory compared to other content areas. However, an important aspect of the issue (with proving) is that some of the analytic and semantic implications of properties may not be as evident to all students as they are to others (Campbell, 2002). A case to illustrate this is when Zazkis (1998) asked preservice elementary school teachers to determine the parity of a given list of numbers such as $7^{50} \cdot 3^{40}$ and $1234567 \cdot 2^{40}$. While for many of the participants, the presence of a factor of 2 indicated evenness of the number in question, for remarkably less of the participants a lack of it indicated oddness. In other words, the concept of parity did not have the same set of implications for all the participants of the study, which were in that case problem-solving strategies (Campbell, 2002; Zazkis, 1998). Brown, Thomas, and Tolias (2002) made a similar point regarding the strategic use of prime factorizations concerning another empirical study’s findings: The simplicity of factorizing small numbers may lead to the misbelief that using these factorizations as a structural tool for solving problems is equally simple (Brown, Thomas, & Tolias, 2002). Examples like these point to the difference, in our case, between recognizing the evident properties of mathematical objects and goal-directedly using them with the purpose of proving. The latter is implicated in Harel’s (2008a) definitional reasoning, as a way of thinking, and is central to Learning Goal 2 of this study.

Stylianides (2009) makes the following comment on the pictorial argument presented below in Figure 1.1. In a context where the definitions of odd numbers and multiples of 4 are available to a classroom community, the argument would not count as proof if it made no explicit reference to these two definitions. Because, for an argument to meet the standard of proof, it needs to address the accepted truths of the classroom community appropriately. However, the term “accepted truths” also includes the “modes of reasoning” that are available to the classroom community at a given time, including the “use of definitions to derive general statements” (Stylianides, 2007, p. 292). This suggests the same argument may not properly address the reasoning modes of students who lack definitional reasoning, and hence may not count as proof for them. The second and third research questions of this study were asked in an attempt to explore how this mode of reasoning may (or may not) develop in students.

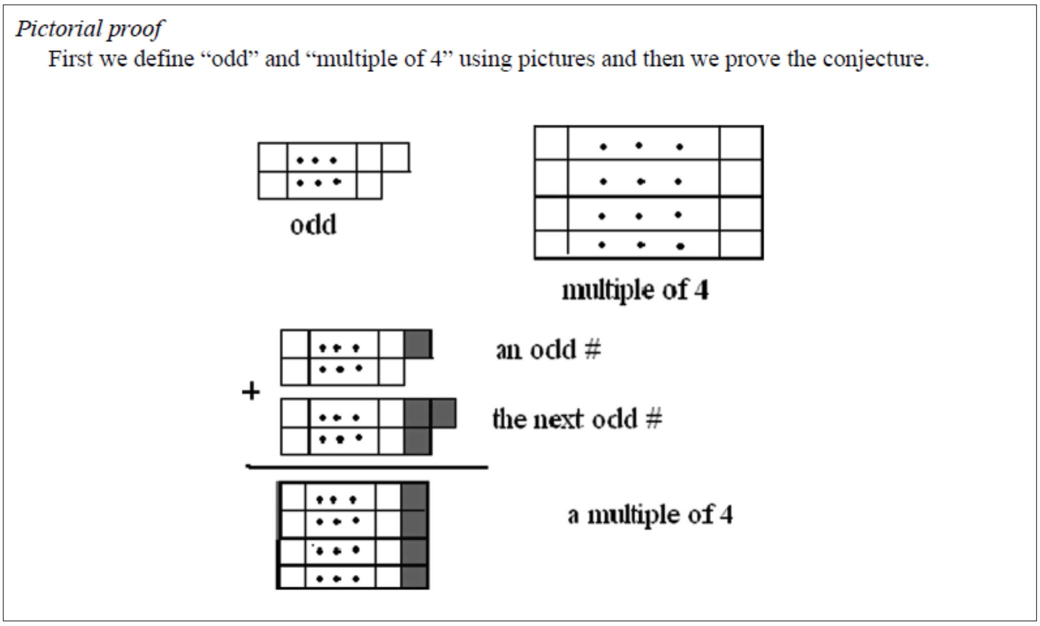


Figure 1.1 A pictorial proof of the claim "The sum of any two consecutive odd natural numbers is a multiple of 4." (Stylianides, 2009, p. 266).

Unlike its approach to the investigation of the first research question, this study did not design a perturbational trigger for definitional reasoning before the implementation of the LTA-guided task sequence. Remember that before engaging the student in a task sequence to promote her structural understanding of the first instance of mathematical proof (Learning Goal 1), the student's attention was attracted to her way of thinking associated with the act of proving (empirical reasoning) as was triggered by the mechanism of cognitive conflict. This time, the study did not attract the student's attention to her way of thinking associated with the act of defining (in relation to proving). Rather, it aimed to observe whether or not repeatedly using particular definitions in proving theorems (experiencing proper ways of understanding definitions) could foster definitional reasoning (a way of thinking) at a level that would allow the student to engage in other simple proving tasks. This reflected the current unstated expectation of the mathematics education field from the students to propose their own proofs while lacking definitional reasoning. In this respect, the current study questions the feasibility of "proving" activities when the instruction does not attend to the role of definition use explicitly; that is when the instruction treats definitions involved in particular proofs as ways of understanding only and ignores definitional reasoning as a way of thinking.

Here an important distinction is to be made on the nature of proof-related tasks students are recommended to engage in, in their classrooms. Among the *reasoning-and-proving* activities envisioned for school mathematics, providing proof for a given conjecture may be the last one, as with the disciplinary practice, after identifying patterns, making conjectures, and providing non-proof arguments. However, this does not mean that creation of proofs will always pass through this same path of activities. Tasks may directly ask students to prove a given statement (Stylianides, 2008). For instance, in the textbook analysis study mentioned above, about one-sixth of the reasoning-and-proving activities expected the students to produce valid proofs to verify the truth of readily formulated conjectures. Some others yet supported the creation of valid proofs that explain why a conjecture is true after students make the conjecture themselves by identifying patterns (Stylianides, 2009). In other words, the tasks that result in the creation of valid proofs may differ in the routes they take toward those endpoints.

Now consider the conjecture that is accepted accessible even to the youngest students: The sum of two odd numbers is always an even number. Instructional approaches such as those that base the students' conception of even and odd numbers on meaningful grounds (such as the concept of a "pair" and the action of "pairing" concrete objects, as in Blanton, Gardiner, Ristroph, Stephens, Knuth and Stroud, 2022) may gradually lead to representational justifications that might be accepted as proofs at their level. Such approaches are valuable to the students' construction of new mathematical knowledge. However, they serve different research investigation purposes than the current study's interest. This is because the major reasoning-and-proving activity the students engage in is not "providing proof" in those cases. When the students initiate their activities for purposes other than proving a given conjecture but end up with a produced proof, the process of universally instantiating the key concept definition to begin to prove is bypassed (or at least obscured). It is not the students who decide to define or search for the defining characteristics of even and odd numbers to arrive at the desired proofs. Even if students develop fine arguments for parity conjectures, what exactly students learn about the act of proving from this experience and to what extent they will be able to prove other conjectures is questionable. Therefore, studies employing these approaches do not provide (if not vicariously) information on what is involved in understanding a mathematical proof as an object and developing definitional reasoning as a way of thinking associated with the act of proving.

This study set the above conjecture of parity as its target conjecture to be proved by the individual student at the end of the teaching experiment study. One of the tasks of post-assessment provided the conjecture (as a novel task for the student) and asked for its proof. Up to that point, a variety of conjectures was needed for studying throughout the experiment to both foster and collect data on the student's development. This need was ensured by taking parity from a broader perspective – as a specific case of divisibility by any number. This approach was also in line with Zazkis's (1998) suggestion that the concept of evenness (equivalently of divisibility by two) is better understood if it is seen as a specific case of divisibility by any natural number. This was the main reason why the sixth-grade level was selected for the study. As the Turkish Middle School Mathematics Curriculum (MoNE, 2018) covered the concepts of divisibility in the first semester of the sixth-grade level, the study was planned for the second semester of this grade level. Therefore, the study had the chance to observe whether the student would come to define odd numbers to prove a novel task on her initiative based on her broader conception of divisibility. In addition, the seventh-grade level covered basic pattern-generalization activities that, due to their connection to the development of the deductive proof scheme, the study preferred the student not been familiarized with. Therefore, the second semester of the sixth-grade level seems to have offered an ideal timing for the current study.

1.4. Significance of the study

An important objective of the field is to work out how the growing number of intervention-based studies in the area of proof relate to each other and how they can fit into an entire instructional program to design students' learning of the concept of proof itself, alongside the construction of new mathematical knowledge (Stylianides & Stylianides, 2017). This study aims to contribute to the research strand by examining what is involved in attaining the eventual objective of such an overarching instructional effort – developing the deductive proof scheme. In other words, the study aims to articulate the “goal understandings” that an instructional program of proof might want the students to eventually achieve. Articulation of the “goal understandings” is a difficult task for such a crosscutting notion of the discipline as proof, as it is with the other concepts of mathematics education (Simon, 2017). Simon (2017) contends that it is generally impossible for educational researchers to remember what it was like to not have particular understandings and what it had taken them to develop those understandings. He suggests observing how individuals’ mathematical functioning differs from each other or how a single individual’s mathematical functioning differs at two different points in time as a generative way of researching what it means to understand a mathematical concept and what components are involved in understanding it (Simon, 2017). By adopting the individual teaching experiment methodology of the LTA research program, this study may help formulate the goal understandings that the proof instruction might want to achieve within school mathematics.

Findings of the study may inform future decisions regarding where to place the obtained (and open to improvement) goal understandings in a broad instructional program of proof and how to connect them with the other *reasoning-and-proving* activities. It is important to note that this study was conducted at the sixth-grade level for methodological purposes only. Thus, the sixth-grade level does not indicate the exact time when students should learn the intended concepts of the study. Rather, the findings of the study have the potential to inform realistic connections to prior knowledge of students at other (both earlier and later) grade levels when planning for the attainment of similar learning goals. Information gathered on the issues of feasibility addressed in the previous section, some of which are specific to the number theory content area and others are general to the development of the deductive proof scheme, may serve this purpose well.

The primary contributions of the study are theoretical rather than practical. This study does not claim to have the purpose of generating a practical HLT for deductive proof at the sixth-grade level. Instead, the main purpose is to explicate the processes by which an individual student, who does not have the concept of deductive proof yet, may conceptually understand a very first example of proof and then build on an understanding of this first example to learn proving within the deductive proof scheme.

The task sequence used to achieve the purposes of the study might offer new approaches to studying a variety of conjectures for students who have access to the concept of divisibility. However, the task sequence includes only the last one of the four types of *reasoning-and-proving* activities students need to engage in within school mathematics: providing proofs. Identifying patterns, making conjectures, and providing non-proof arguments (2008a) were not the major goals of the students' activities, as they should have also been. Therefore, the study provides only a limited variety of tasks as practical instructional resources.

Extant literature lacks theoretical knowledge of the mechanisms by which students' ways of thinking associated with the act of proving change (hence can be promoted), except for the notion of cognitive conflict as a mechanism to eliminate the empirical proof scheme (Harel, 2008a). Available frameworks of proof comprehension, two of which guided the current study (Ahmadpour, Reid, & Fadaee, 2019; Miyazaki, Fujita, & Jones, 2017), capture differing levels of students' understanding of particular proofs. In the model of students' understanding of written proof texts, Ahmadpour, Reid, and Fadaee (2019) also capture the transitions between the states of understanding a particular proof. However, in neither of the frameworks are the mechanisms by which the desired transitions take place explicated. The current study might offer potential ways of promoting the intended transitions in students' thinking by explicating the mechanisms that underlie intended learning.

For achieving its purposes, this study treats one's development of the deductive proof scheme as a learning paradox to be resolved. As Bereiter (1985) states, taking the learning paradox seriously has the potential to inspire the creation of teaching methods that are adapted to the difficulties students face. Although this study does not claim such immediate practical significance, it aims to serve as the foundation for long-term efforts to design methods of teaching proof, a concept widely acknowledged to be difficult to teach. One possibility, for example, could be the following. Stylianides and Stylianides (2017) argue that in a comprehensive instructional program of proof like the one mentioned above, the unit of intervention should be "as small as it is realistically possible, with each individual intervention targeting a well-defined and possibly modest set of learning goals" (p. 125). This is envisaged to produce interventions of short duration, allowing for the flexible placement of each component within the larger program or the simple incorporation into existing curricula. In the case that the current study successfully fulfills its purposes, the HLT that was used primarily as a methodological tool for investigation purposes can be adapted, through reasonable effort, to propose a more realistic HLT of short duration, as is desirable.

Regarding the transition from the empirical to the deductive proof scheme, Harel (2013) compared the consequences of relying on students' two types of intellectual needs: the need for certainty and the need for causality. Harel (2013) proposed the idea of attracting students' attention to the cause (causes) of why an assertion is true rather than challenging the legitimacy of empirical reasoning to obtain certainty. This proposition was derived from two empirical observations: the first being the difficulty of generating an intellectual perturbation to produce the need for certainty, and the second being the possibility of developing a habit of looking for cause(s) in students by repeatedly attending to causal explanations (and comparing them to empirical alternatives) (Harel, 2013). On the other hand, Stylianides and Stylianides (2009) reported a case where an instructional task sequence successfully challenged undergraduate mathematics students' need for certainty by using a mechanism of cognitive conflict. The current study, by adapting the task sequence from Stylianides and Stylianides (2009), adds to the comparison of the roles of the two types of intellectual needs in the development of the deductive proof scheme by students.

In fact, whether the intellectual need (of one type or another) is indispensable for the development of the deductive proof scheme is of more interest to the present study. This study uses constructs from the DNR and LTA theoretical frameworks, which adopt contrasting but complementary perspectives on learning mathematics. DNR defines learning as "a continuum of disequilibrium–equilibrium phases manifested by (a) intellectual and psychological needs that instigate or result from these phases and (b) ways of understanding or ways of thinking that are utilized and newly constructed during these phases" (Harel, 2008b, p. 897). The notion of intellectual need, as a type of learned human need, is central to this view of learning because it constitutes perturbational experience for the construction of new disciplinary knowledge (Harel & Koichu, 2010), along with other psychological needs referring to the motivational drives to confront a problem and to pursue an effort to solve it (Harel & Koichu, 2010). DNR asserts that a perturbational state (hence the intellectual need) does not necessarily bring about the intended learning. The individual may not be able to construct the intended knowledge piece and remain in a state of disequilibrium (Harel, 2008b). However, DNR accepts that "problem solving is the only means of learning" (Harel, 2008b, p. 896) and views perturbation as a necessary condition for constructing a desired piece of knowledge (Harel, 2010).

On the other hand, the LTA theoretical framework builds on Piaget's notion of reflective abstraction to explain mathematical conceptual learning. Simon (2013) argues that "disequilibrium is neither sufficient nor necessary to explain mathematics conceptual learning" (p. 291). The fact that instances of disequilibrium can be observed in many cases where new learning takes place does not mean that perturbation is the mechanism that explains the learning (Simon, 2013). Simon, Saldanha, McClintock, Akar, Watanabe, and Zembat (2010) illustrate a case from a teaching experiment study where conceptual learning is explained without any reference to a disequilibrating experience. The LTA instructional approach, as well, is not grounded in problem solving where the learner does not have an available solution strategy. Rather, it offers an instructional design approach to foster students' reinvention of intractable mathematics concepts through goal-directed activity, reflection, and anticipation. Simon (2013) considers the work of the LTA and DNR programs to use different methodologies to serve different goals. "[W]hereas DNR is an attempt to look broadly at mathematics, mathematics learning, and mathematics teaching, ... [the LTA] research program can be seen as going more deeply into a narrow slice of the broader DNR program" (Simon, 2013, p. 284). On the other hand, he considers their two corresponding approaches to mathematics instruction—one grounded in problem solving and the other not—as complementary. However, the degree to which they can be combined or whether they are just different instructional tools to achieve distinct ends needs to be clarified. This study contributes to answering this question by combining constructs from the DNR and LTA.

For the current study, given the paradoxical learning goals it aimed to achieve (due to Duality I and II principles), it was not possible to fully implement the LTA instructional design approaches. However, the study built on the participating student's (partially) goal-directed activity, reflection, and anticipation as inspired by the LTA approach. As a result, the study's findings may contribute to the LTA research program's ultimate goal of performing analyses across studies that report accounts of learning through activity (Simon et al. 2010). In addition, Simon (2020) indicated that further research may generate new types of AoC concepts that might be quite different from the ones investigated by the LTA research up to now. This study attempts to define new CoA and AoC-type concepts to study the construction of DNR's ways of thinking (Harel, 2008a).

1.5. Definition of important terms

Mathematical proof: The following definition of proof proposed by Stylianides (2007) was adopted for the study, with an unavoidable adaptation to its social aspect.

> Proof is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community (Stylianides, 2007, p. 291, italics in original).

Table 1.1 illustrates the three components of any argument that are central to the definition.

The definition is an attempt to do justice to both mathematics as a discipline and to students as learners of mathematics. The term "classroom community" refers primarily to the students, while the definition assigns to the teacher a distinct role of representing the discipline to the students and facilitating their connection with a broader knowledge of mathematics (Stylianides, 2007). In the current study, in addition to the representation of the discipline, the agency of the classroom community was also reflected in the role of the researcher. Both roles were restricted by the principles of the LTA research methodology (Simon, 2018).

Table 1.1 Examples of the Three Components of a Mathematical Argument Mentioned in the Definition of Proof (Stylianides, 2007, p.292)

Component of an argument	Examples
Set of accepted statements	Definitions, axioms, theorems, etc.
Modes of argumentation	Application of logical rules of inference (such as modus ponens and modus tollens), use of definitions to derive general statements, systematic enumeration of all cases to which a statement is reduced (given that their number is finite), construction of counterexamples, development of a reasoning that shows that acceptance of a statement leads to a contradiction, etc.
Modes of argument representation	Linguistic (e.g., oral language), physical, diagrammatic/pictorial, tabular, symbolic/algebraic, etc.

The target proof of the study: The study used a flowchart proof format adapted from Miyazaki, Fujita, and Jones's (2015) to visually present the underlying structure of deductive proofs. The flow-chart argument presented in Figure 1.2 was accepted as a mathematical proof for the sixth-grade level. It was designated as the "target proof" of the study meaning that the whole instructional sequence of the teaching experiment aimed to equip the student with essential knowledge and skills so that she would be able to create it individually at the end of her participation. The proof constituted a correct response to a post-assessment task where the conjecture "The sum of two odd numbers is an even number" was given and its proof was requested. In order to preserve the novelty of the task for the student, the instructional sequence up to that point carefully planned when and how to touch on the definitions of even and odd numbers and their specific way of representation developed for this study.

The representation system called "general representation" treated parity as a special case of divisibility by any natural number (Zazkis, 1998) and made a connection to the numbers' modular structure (Campbell, 2002). Basically, to represent the modular structure of a given natural number according to some modulus n (which is 2 for the case of parity), that number of identical "counting items" were thought to be equally distributed into n number of "cups" with the appropriate number of leftovers (remainders according to the division theorem) (Campbell, 2002). The representation is communicated as “n equal groups” and some number of “singles” (of counting items).

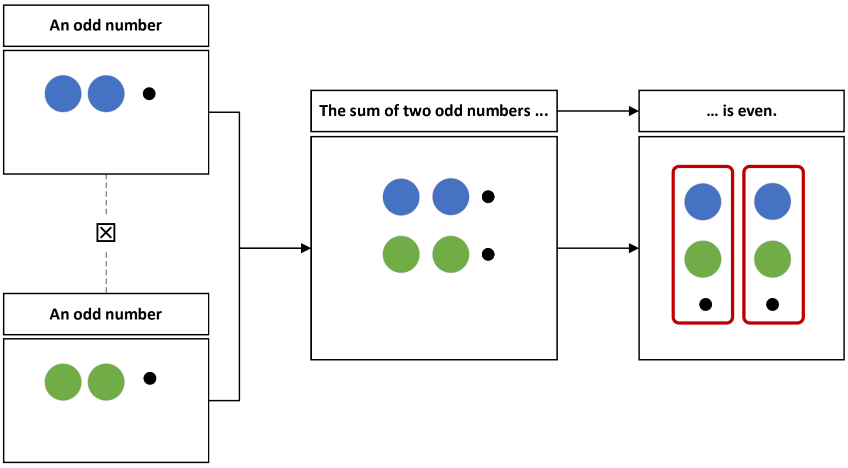


Figure 1.2 A flowchart proof of the theorem “The sum of two odd numbers is an even number.”

Understanding a proof as a structural object: Miyazaki, Fujita, and Jones (2017) define the “structure of deductive proofs” as the relational network of singular and universal propositions connected via two types of deductive reasoning: universal instantiation (two different applications of $[(p \longrightarrow q) \wedge p] \vdash q)$ and hypothetical syllogism ($[(p \rightarrow q) \wedge (q \rightarrow r)] \vdash (p \rightarrow r)$). A proof deduces singular propositions (the premises, conclusions, and intermediate propositions between them) from the universal propositions (theorems, definitions, and axioms) by means of universal instantiation and connects these singular propositions by use of hypothetical syllogism. Hypothetical syllogism here refers to a “syllogism whose premises belong to not less than one compound proposition including conditionals ‘if, then’” (Miyazaki, Fujita, & Jones, 2017, p.226). Table 1.2 explicates the relational network of the current study’s target proof, which makes the reasoning involved valid.

Note that the flowchart proof in Figure 1.2 does not reveal the universal instantiations involved in Table 1.2 explicitly. The flowchart proof format used in the current study, although adapted from Miyazaki, Fujita, and Jones (2015), differed from theirs in this respect. A more precise adaptation would label the following universal instantiations; however, this was not preferred because it would cause additional complexity in the proof format.

Universal instantiation from the property of odd numbers:

Universal proposition: If a number is odd then it is not divisible by two. (The property of odd numbers)

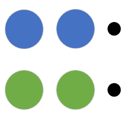
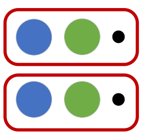
Singular proposition: Let A be an odd number, then it is of the form 2n+1 for integer n.

Universal instantiation from the condition of being even:

Singular proposition: A+B is a number of the form 2k, then it is even.

Universal proposition: If a number is divisible by two then it is even. (Condition of being even) (More explanation on distinction between the two types of universal instantiations are found in the method section).

Table 1.2 Deductive relations underlying the proof of the conjecture "the sum of two odd numbers is an even number."

Proof	Deductive relations
Let A and B be two odd numbers. A is of the form:  (Definition of odd number) B is of the form:  (Definition of odd number) The sum of these two numbers is of the form:  Because this sum is divisible by 2  It is an even number.	1) Two singular propositions, (i) 'If A is an odd number, then it is of the form 2n+1', and (ii) 'If B is another odd number, then it is of the form 2m+1', are deduced by universal instantiations of the definition of odd numbers (a universal proposition). 2) A singular proposition, (iii) 'If the sum A+B is of form 2(n+m+1), then it is an even number', is deduced by universal instantiation of the definition of even numbers (a universal proposition). 3) These three propositions (i), (ii), and (iii) are connected by using hypothetical syllogism, and we obtain 'If two odd numbers A and B are summed, the sum is an even number, which is equivalent to the singular proposition to be proved.

However, it is believed that, in a sense, the label "an odd number" in the target proof (in Figure 1.2) might point to the idea of a universal proposition, because it does not make reference to a fixed value (do not state a singular proposition such as "A is an odd number" as alternatively would be), but points implicitly to any and/or all such numbers (the property of odd numbers). This approach is also more compatible with the pictorial proofs similar to the one in Figure 1.1.

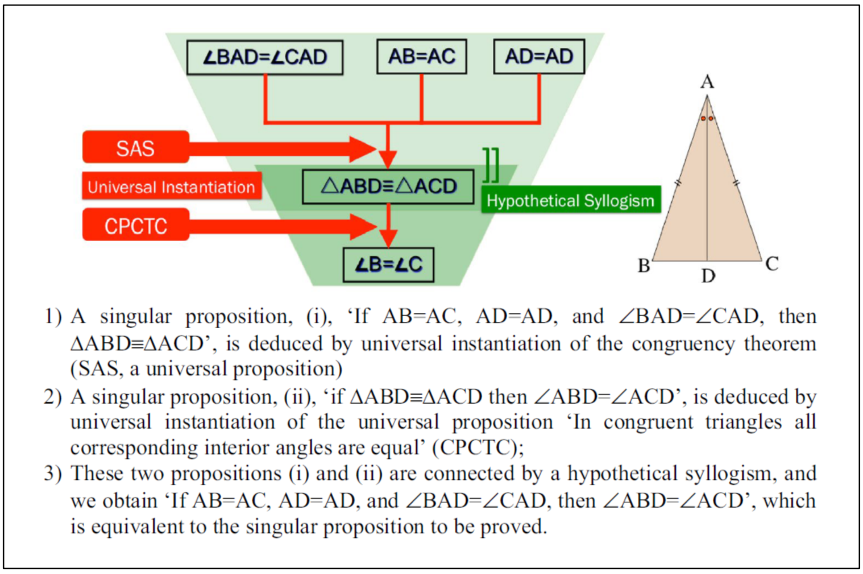


Figure 1.3 Proof of 'the base angles of isosceles triangles are equal' (taken from Miyazaki, Fujita, & Jones, 2015, p.1214).

Miyazaki, Fujita, and Jones (2017) interpret presence or absence of an understanding of universal instantiations (as part of one's structural understanding of proofs) based on the students' ability to supply appropriate labels for the universal propositions involved in flowchart proofs (as in Figure 1.3). As explained, this was not preferred in our case. Instead, this study relied on a model of proof comprehension to operationally define the student participant's understanding of universal instantiations.

In the model of students' understanding of written proof texts, Ahmadpour et al. (2019) define the "Path of Structure" to describe possible states of understanding the logical structure of a formally acceptable proof text (such as the target proof of the current study). Along this path, through the transitions of *generalization*, *abstraction*, and *formalization*, one may develop from the state of Naïve Experience (in which existence of confirming examples are read by the reader of the proof to validate generalizations) to those of General Procedure, Abstract Structure and Formulated Proof sequentially. This study posits that understanding a case of universal instantiation in a proof is the same as understanding the (not explicitly stated) universal proposition related to a particular singular proposition as an abstract structure. Thus, identifying abstract structures in one’s understanding of a proof is critical to the procedures of the study, especially to the procedure of data analysis.

To identify abstract structures in one’s understanding of a proof it is helpful to distinguish between *generalization* and *abstraction.* Generalization occurs when the subject realizes that all the members in a class of objects share a particular property (Ahmadpour et al. 2019). Abstraction “occurs when the subject focuses attention on specific properties of a given object and then considers these properties in isolation from the original” (Harel & Tall, 1991, p. 39). The latter is a process of extracting the defining properties of a class of objects whose cases have been observed. Therefore, abstract structures are independent of the specific forms of representation in which they are embodied (Ahmadpour et al. 2019). Below excerpt from the empirical data of the current study illustrates a case where the sixth-grade student Beren makes an abstraction about the set of numbers “15, 24, 33, 42, 51, …”

> *Beren* : *So, think of it this way. You know, the number that is right here (Points to the number 24 shown in the specific representation system of the study as 9 groups of 2 singles plus 6 more singles), how can I explain it? It is* <u>*not*</u> *the number 24 here when we actually look at it. It, I mean, consists of 9 equal groups and 6 singles.*

Note that the abstract structure indicated in Beren’s thinking is an indication of the following universal instantiation.

> <u>Universal instantiation (the property of the numbers in the pattern 15, 24, 33, 42, 51, …)</u>
> Universal proposition: If a number is a member of the pattern 15, 24, 33, 42, 51, …, then it is of the form 9k+6 for whole number k. (The definition of the number pattern)
> Singular proposition: 24 is a member of the pattern 15, 24, 33, 42, 51, …, then it is of the form 9k+6 for some whole number k.

In short, this study takes any indication of the student’s understanding of an abstract structure as an indication of her understanding of a related universal proposition that is not visible in the flowchart proof; her understanding of a universal instantiation. A structural understanding of a proof also includes understanding of hypothetical syllogism, information on which is left to the next term of the study to be defined.

The first instance proof: The "first instance proof" of the study (given in Figure 1.4) was selected to be a direct proof that logically followed from a single definition (Stylianides & Stylianides, 2008). It used *modus ponens*, which was evidenced by psychological research to be mastered before the other forms of deductive reasoning (like modus tollens), as its only logical inference rule. "Modus ponens (MP) states that if the antecedent p of a conditional claim (rule) $p \rightarrow q$ is true, then the consequent q is also true. MP can be summarized as follows: $[(p \longrightarrow q) \wedge p] \vdash q$" (Stylianides & Stylianides, 2008, p.108). This means, the first instance proof of the study had a relatively simple structure for one's first encounter with a deductive proof.

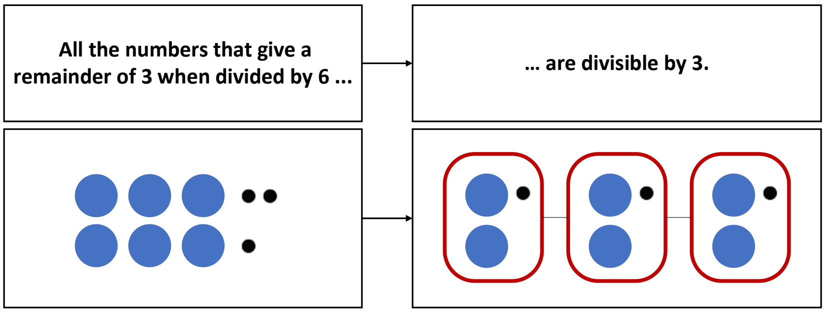


Figure 1.4 The first instance proof

In fact, the term was used to indicate a class of proofs analogous to each other that the student was to experience repeatedly until developing the intended AoC concept of the study. The related class of conjectures each concerned whether all of the numbers in infinite arithmetic number pattern was divisible by a given natural number or gave a certain remainder. For example, the proof in Figure 1.4 was expected for the theorem "All the numbers in the pattern 9, 15, 21, 27, 33, 39, 45, … are divisible by 3".

Intended CoA concept of the study is to understand the above proof (or any analogous proof) as a structural object, that is, to understand the universal instantiations and hypothetical syllogism involved. The previous definition of the target proof of the study illustrated universal instantiations that are applicable to the first instance proof. Here, Table 1.3 illustrates hypothetical syllogism (double modus ponens) in relation to the first instance proof of the study.

Table 1.3 Hypothetical Syllogism $((P \to Q) \land (Q \to R)) \to (P \to R)$ underlying the proof in Figure 1.4

$\boldsymbol{P \to Q}$	If "A" is a number that (belongs to the set 9, 15, 21, 27, 33, 39, 45, …) gives a remainder of 3 when divided by 6, then it is of the form 6k+3, for integer k.
P	A is a number that gives a remainder of 3 when divided by 6.
Q	Then, it is of the form 6k+3.
$\boldsymbol{Q \to R}$	If a number is of the form 3n for integer n, then it is a number "divisible by 3".
Q	6k+3 is a number of the form 3(2k+1) =3m.
R	Then, it is divisible by 3.
$\boldsymbol{P \to R}$	The number A, that gives a remainder of 3 when divided by 6, is divisible by 3.

Intended CoA and AoC concepts of the study are each defined in relation to the first instance proof. Before defining the two terms of the study, the LTA's notion of mathematical concept (Simon, 2017) is explained first.

Mathematical concept: Simon (2017) defines a mathematical concept as a researcher's construct that characterize the learner's present knowledge or her potential knowledge aimed to be fostered through an instructional design effort. That is, an articulation of a concept is either "inferred" or "intended" student knowledge (p. 123). Although it is not meant to be so, the researcher may articulate the goal understandings of an instructional design in the language of the student. This is not to capture what the student would say about her understanding, but to specify most accurately what might the student get to know, think about and use (Simon, 2017; Simon et al, 2018). The following example from Simon (2017) illustrates a mathematical concept as a researcher's construct.

> *Understanding cardinality: When I count, I say my counting words in order and touch a different object for each word. When I have no more objects to touch I stop counting. If there are more objects I can say more counting words. So depending on how far I get in my counting, that reflects how many objects there are. The last number tells me how many objects there are. [The reader will note that young children who develop this concept will not be able to articulate any of this. This is a researcher's attempt to articulate their understanding.] (Simon, 2017, p. 134).*

It is important to note that any articulated concept can be further developed to better formulate different aspects or components of understanding, or to provide greater detail about it. Understanding can be characterized in many ways that has no end.

Intended CoA concept of the study: The subset of mathematical concepts resulting from the coordination of actions (CoA) type of reflective abstraction (which is indeed regarded a coordination of concepts) are categorized as CoA concepts. In this study, understanding a first instance of mathematical proof as a structural object was considered to be the result of the coordination of following two actions: (1) given an arithmetic number pattern, identify its defining property as a modular structure shared by all its elements and (2) given a modular structure, make an appropriate divisibility inference. The following CoA concept was articulated in terms of the first instance proof of the study (Figure 1.4):

> All the numbers in the pattern 9, 15, 21, 27, 33, 39, 45, … can be shown as 6 equal groups and 3 singles of identical counting items. They are each a number that gives a remainder of 3 when divided by 6. 6 equal groups and 3 singles of identical counting items can be equally distributed into 3 groups without a remainder. Independent of the number of counting items contained in each of the initial groups of six, the total number of counting items is always divisible by 3. Therefore, all the numbers in the pattern are divisible by 3 without exception. This is a proof to the conjecture “All the numbers that gives a remainder of 3 when divided by 6 are divisible by 3.”

Note that the articulation entails the two types of deductive reasoning: universal instantiation (a particular modular structure is abstracted from the elements of a class of objects) and hypothetical syllogism (outlined in Table 1.3); hence, captures the intended structural understanding. It also includes an emphasis on the generality characteristic of the deductive proof scheme in contrast to the empirical arguments students accepted as valid before. The methods section provides more details on the LTA theoretical perspective on the formulation of a CoA concept as a goal-action composite, its abstraction from the learner’s already available concepts through her activity (Simon, 2020), and the two stages of the participatory and the anticipatory stages of a concept being constructed (Simon, Placa, & Avitzur, 2016; Tzur & Simon, 2004).

The main idea of the first instance proof (Intended AoC concept): "An AoC concept is a mathematical structure that can be used to sort mathematical experiences into examples and non-examples" (Simon, 2020, p. 4). It is the result of an AoC type of reflective abstraction—the learner's abstracting of the commonality in her activity. This study's AoC concept was defined as sorting the conjectures into cases when looking for a shared modular structure in number patterns would and would not be useful to validate a conjecture. In the student's language, it was articulated as:

> If I can identify that all of the elements in a number pattern can be represented as the same number of equal groups of counting items with the same number of single leftovers, I can say, in some cases, that all of the numbers in the pattern are divisible (or indivisible) by a given whole number.

This is a superficial statement of the main idea of the proof, focusing only on the role of a particular definition as a way of understanding and leaving the relevant *way of thinking* implicit. The main idea of the particular proof can be stated more completely as using modular structure as the *defining property* of arithmetic number patterns for making divisibility inferences about all the numbers in the pattern. To reflect a more realistic expectation from the student, this study aimed to develop the main idea of the first instance proof as it was superficially stated in the former expression, in the first place. Developing this main idea as an understanding transferable to the task of proving analogous theorems was expected to be a result of the student's abstracting of the common structure shared by the set of analogous conjectures and the produced proofs. Then, the study extended its AoC concept to the study of other similar conjectures (not analogous, but similar to the first instance proof), as in Table 1.4**,** to provide the student with other examples of definitional reasoning.

The extensions was done both to promote definitional reasoning (a way of thinking) through repeated application of proper ways of understanding (in line with the Duality II principle of the DNR) and to observe the student's reliance on her CoA and AoC concepts as the learning process continues with new ways of understanding. Two related but new AoC concepts were extended from the original AoC concept of the study by stating new conjectures in terms of the sums of elements from distinct arithmetic number patterns. The distinction between the structures to be abstracted by the student was reflected in the flow-chart looks corresponding to each type. The new concepts were each considered extensions of the original AoC concept because they both built on the relationship between an arithmetic number pattern and a modular structure as the defining property of the pattern. For ease of reference, related concepts are labeled AoC-Extension 1 (AoC-E1) and AoC-Extension 2 (AoC-E2) concepts. Table 1.4 presents a representative conjecture for each of the AoC, AoC-E1 and AoC-E2 concepts.

Table 1.4 Conjectures studied in relation to AoC, AoC-E1 and AoC-E2 concepts of the study

(AoC)	Conjectures about a single arithmetic number pattern *Absolute divisibility* All the numbers that give a remainder of 4 when divided by 8 are divisible by 4. *Absolute indivisibility* All the numbers that give a remainder of 3 when divided by 8 are divisible by 2. All the numbers in the pattern 14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94, 102, … gives a remainder of 2 when divided by 4. *Indecisiveness of neither divisibility nor indivisibility* What can be said if anything about all the numbers that give a remainder of 5 when divided by 8 in relation to being divisible by 5? Conjectures about sums of (two or more) numbers from arithmetic patterns
(AoC-E1)	*Properties of addends instantiated dependent on one another* If a number that gives a remainder of 1 when divided by 6 is added up with 2 more of itself, the sum is always divisible by 3.
(AoC-E2)	*Properties of addends instantiated independent of one another* If a number that gives a remainder of 1 when divided by 6 is summed with another number that gives a remainder of 2 when divided by 3, the sum is always divisible by 3.

CHAPTER 2

LITERATURE REVIEW

This study uses constructs from the DNR and the LTA theoretical frameworks complementarily. Some of the constructs that are foundational to the study were already introduced in the previous chapter. This chapter provides additional information on the aspects that are needed for a thorough interpretation of the methodological decisions and the empirical findings of the study. Particularly regarding DNR, the broader structure of the framework is presented with its underlying premises, concepts, and claims. The taxonomy of proof schemes (Harel, 2008) and the notions of intellectual need and definitional reasoning are further explained. The distinction between two special cases of the empirical and deductive proof schemes, namely process pattern generalization and result pattern generalization, is addressed. With the LTA theoretical framework, the process of reflective abstraction is distinguished from an empirical learning process. The formal characterization and construction processes of the CoA and AoC concepts are detailed, and the notion of mental run is described. Lastly, the model of students' ways of understanding a proof (Ahmadpour et al., 2019), which is the framework used in the data collection and analysis procedures of the study, is explained.

2.1. DNR-based Instruction in mathematics

DNR-based instruction in mathematics (DNR for short) is a conceptual framework for what mathematics to teach in schools and how to teach it (Harel, 2008a; 2008b). It is a comprehensive system of three types of constructs: premises, concepts, and claims. Figure 2.1 presents the DNR structure. The framework is named through the initials of *Duality*, *Necessity* and *Repeated Reasoning,* which are the instructional principles of the framework. Duality, necessity and repeated reasoning principles are the foundational claims of the framework regarding how student learning is influenced by teaching behavior. They also constitute the basis for derivation or organization of other, not explicitly labeled, principles of the system.

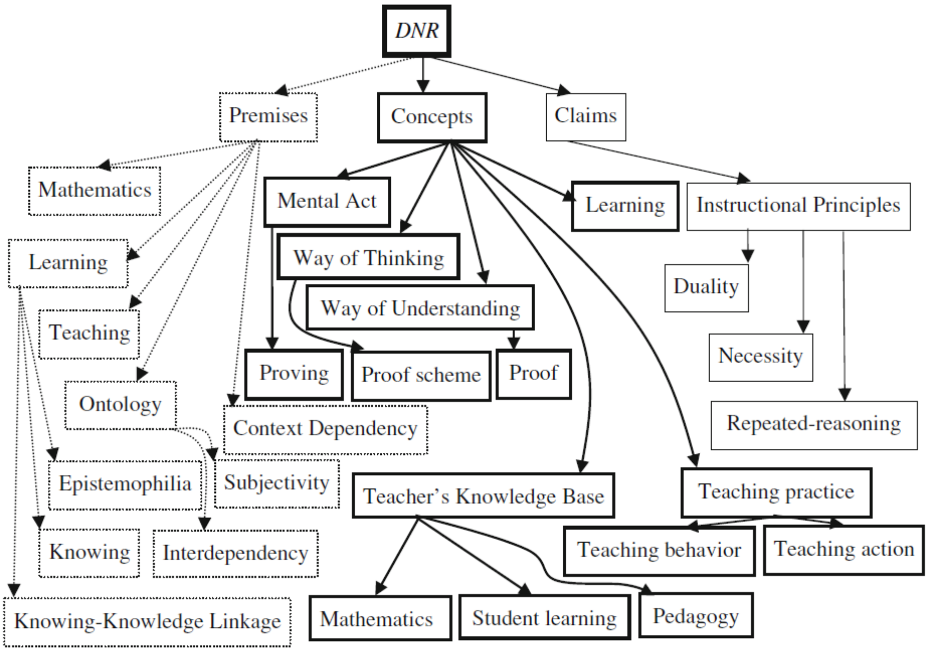


Figure 2.1 DNR structure (Harel, 2008b, p. 903).

DNR accepts eight premises all but the first one taken from or derived from known theories. These premises state the assumptions based on which the concepts of the framework are defined and the claims of it are formulated. The eight premises are roughly divided into four categories of mathematics, learning, teaching and ontology:

1. Mathematics
    - Mathematics: Mathematics is a collection of all the ways of understanding and ways of thinking that have been institutionalized (accepted by the mathematic community) throughout history.
2. Learning
    - *Epistemophilia: Humans—all humans—possess the capacity to develop a desire to be puzzled and to learn to carry out mental acts to solve the puzzles they create. Individual differences in this capacity, though present, do not reflect innate capacities that cannot be modified through adequate experience (Harel, 2008b, p. 894).*

- Knowing: Knowing is a developmental process characterized by an ongoing tension between assimilation and accommodation, aimed at achieving (temporary) equilibrium.
- *Knowing-Knowledge Linkage: Any piece of knowledge humans know is an outcome of their resolution of a problematic situation (Harel, 2008b, p. 894).*
- Context Dependency: Learning is influenced by contextual factors.

3. Teaching
    - Teaching: Mathematics is not learned spontaneously. An individual's capabilities will change based on the presence or absence of assistantship by an expert or a more knowledgeable peer.
4. Ontology
    - Subjectivity: Anything that people claim to have observed are a result of what their mental structure assigns to their surroundings.
    - Interdependency: Humans' perceptions of the world influence and direct their behavior. In turn, their actions shape their views of the world.

By the Mathematics Premise, DNR defines mathematics as a discipline consisting of all the ways of understanding and ways of thinking that have been accepted by mathematics community at large throughout the history. Ways of understanding refer to the "subject matter" which is the collection of all axioms, definitions, theorems, proofs, problems and their solutions, whereas ways of thinking (which are in fact conceptual tools) denote the characteristics of the mental acts whose products comprise the ways of understanding (Harel, 2008a; 2008b). Ways of thinking are currently classified into three categories, which are problem solving approaches, proof schemes, and beliefs about mathematics (Harel, 2008a). To illustrate, consider the act of problem solving. Any solution (either correct or incorrect) to a particular problem is a way of understanding, whereas problem solving approaches such as "looking for a simpler problem for solving the original one" or "looking for just the key words in the problem statement" are ways of thinking associated with the act of problem solving. Any way of understanding or thinking that is institutionalized, i.e., accepted by the contemporary mathematics community, is "desirable." The intermediate conceptions learners develop towards the institutionalized ones are considered desirable as well, from a pedagogical standpoint.

DNR accepts a reciprocal relationship between the development of ways of understanding and ways of thinking. The Duality Principle states that: “Students develop ways of thinking through the production of ways of understanding, and conversely, the ways of understanding they produce are impacted by the ways of thinking they possess” (Harel, 2008b, p. 889). As the Duality principle is the major source of the problem situation being investigated in the current study, it was explained in detail in the previous chapter.

The Necessity Principle, on the other hand, needs further attention as the study touches on two distinct types of intellectual needs. DNR defines learning to be driven by a perturbational (disequilibrating) experience:

> *Learning is a continuum of disequilibrium–equilibrium phases manifested by (a) intellectual and psychological needs that instigate or result from these phases and (b) ways of understanding or ways of thinking that are utilized and newly constructed during these phases (Harel & Koichu, 2010, p. 118).*

As stated in the above definition, DNR acknowledges psychological (motivational) factors in one’s drive to resolve her experienced perturbations. However, the main focus of the necessity principle lies in the concept of intellectual need, which is essentially connected with the concept of epistemological justification. Epistemological justification is an awareness of how a piece of knowledge resolves an intellectual perturbation in the learner’s view (Harel & Soto, 2017). It is “the learner’s perceived cause for the birth of knowledge” (Harel, 2008a, p. 488). Therefore, the concept of intellectual need is defined in relation to the knowledge piece being created to satisfy that need, just as disequilibrium (or perturbation) can only be thought of in relation to the corresponding state of equilibrium that resolves it (Harel, 2008b; 2010). Accordingly, DNR accepts any piece of knowledge to emerge out of an individual’s (community’s) confronting a problematic situation that is unsolvable with one’s current knowledge (Harel, 2008b; 2010). While psychological needs relate to the conditions that drive or hinder one’s learning in general, intellectual need is specific to the *epistemology* of the particular discipline (Harel, 2008b). Epistemology here is used for the structure of the knowledge domain from both the individual learner's and experts' point of view, the latter embracing also the historical development of the discipline (Harel, 2013). For students to develop an epistemological justification for a knowledge piece, instruction should intellectually necessitate the construction of this knowledge. Methodologically, the best way to identify one's intellectual need is to observe that one's engagement in a particular problematic situation has led her construct the intended knowledge piece and that the individual has an understanding of how that knowledge piece resolves that problematic situation (Harel, 2013).

So far, Harel classified intellectual needs into five types, illustrated each of them from historical development of the discipline and discussed their pedagogical implications. The five types are (1) *need for certainty*, (2) *need for causality*, (3) *need for computation*, (4) *need for communication*, and (5) *need for structure*. The classification is not mutually exclusive as the needs are inseparably connected to each other rather than isolated. It is neither complete as other needs can be introduced with the development of mathematical practices in the discipline. Harel (2013) highlights that these classes are not fixed entities. Rather they are the result of centuries of mathematical practice that will continue to evolve in the future. According to him, this suggests the following hypothesis: "Intellectual needs are learned, not innate" (p. 145). If the hypothesis is true, that means the intellectual needs must "have cognitive primitives, whose role is to orient us to the intellectual needs we experience when we learn mathematics" and they "cannot be taken for granted in mathematics teaching. A continual and sustained instructional effort is necessary for students' mathematical behaviors to become oriented within and driven by these needs" (p.145).

Two types of intellectual needs are foundational to the current study: the need for certainty and the need for causality. The need for certainty is an individual's desire to determine the truth of a conjecture "by whatever means he or she deems appropriate" (Harel, 2013, p. 124). Those subjective "means" that the individuals may deem appropriate to obtain certainty, independent of the ones accepted by the mathematics community, are the individuals' proof schemes. The community's proof schemes are subject to changes as well. Indeed, it evolved throughout history before the deductive proof scheme dominated the discipline.

The need for certainty is satisfied when the individual identifies a conjecture as valid. However, the need for causality goes beyond obtaining a conviction. It is the individual's "desire to know why the assertion is true—the cause that makes it true. Thus, the need for causality is one's desire to explain, to determine a cause of a phenomenon." (Harel, 2013, p. 126). While the act of *proving* is an individuals' drive to achieve certainty, *explaining* refers to her attempt to understand the cause of that certainty. Accordingly, the distinction between searching for certainty and causality is a crucial aspect of *reasoning-and-proving* activities in general and of the current research study in particular.

Lastly, the repeated reasoning principle aims to make sure that students internalize, retain, and organize the mathematics knowledge they learn (Harel, 2008b). The repeated reasoning principle does not merely advise drill or practice of routine problems. Rather, it aims to foster the process of internalization through which the students come to apply desirable ways of understanding and thinking autonomously and spontaneously. To this end, students' evolving intellectual needs should be reflected in the variety and complexity of the problems they are assigned, and students should be invited to engage in thinking as they work through problem situations provided sequentially.

2.1.1. Ways of thinking associated with the act of proving

Proving is the mental act at the center of the current study. Therefore, individuals' potential ways of thinking associated with the act of proving are reviewed here. At first, Harel and Sowder's (1998) taxonomy of proof schemes is introduced. After that, two ways of thinking related to the development of the deductive proof scheme are presented.

Harel and Sowder (1998) offered the term "proof schemes" as referring to "what the person offers to convince others" (p. 275) and distinguished between major categories of *external conviction*, *empirical* and *deductive* proof schemes (Harel, 2008). Each category has subordinate categorizations as indicated in Table 2.1.

Table 2.1 A partial taxonomy of proof schemes (Harel, 2008, p.491).

External conviction	Empirical	Deductive
Authoritative	Inductive	Transformational
Ritual	Perceptual	Modern axiomatic
Non-referential		

Authoritative proof scheme refers to relaying on the consent of an outside authority while deciding on whether an argument is a valid proof or not. Teachers' comments and books generally serve as such authority for students. *Ritual proof scheme* is judging the validity argument strictly by its appearance. Harel (2008) illustrates this proof scheme by the case of a student's accepting a proof in geometry as necessarily having a two-column format (a traditional way of proving in American school geometry). The *non-referential (symbolic) proof scheme* is characterized with "symbol manipulations" where "the symbols or the manipulations hav[e] no coherent system of referents in the eyes of the student" (Harel, 2008, p.491).

The *empirical proof scheme* includes two subcategories: *inductive proof scheme* in which evidence is derived from the use of a limited number of confirming examples, and the *perceptual proof scheme* in which "observations are made by means of rudimentary mental images" (Harel & Sowder, 1998, p.255).

> *Empirical arguments provide inconclusive evidence by verifying its truth only for a proper subset of all the cases covered by the generalization, whereas proofs provide conclusive evidence for its truth by treating appropriately all cases covered by the generalization. Thus, we may say that, in contrast to empirical arguments, proofs offer secure methods for validation of mathematical generalizations (Stylianides & Stylianides, 2009, p.315).*

The *deductive proof* scheme is associated with the mathematical notion of proof. It consists of two main subcategories: the *transformational proof scheme* and the *modern axiomatic proof scheme. Transformational proof scheme* is characterized by generality, operational thought, and logical inference; where, generality implies considering all of the possible cases applicable without exception, operational thought means organizing of the whole process by setting goals and sub-goals and anticipating their outcomes. Logical inference characteristic is ensured when it is just accepted that one needs to apply logical inference rules for reaching mathematical conclusions.

Modern axiomatic proof scheme possesses all the three characteristics of the transformational proof scheme and an additional one: "the attribute that … in principle any proving process must start from undefined terms and axioms" (Harel, 2008, p.491).

In the current study, the focus is on promoting an individual student's transition from the empirical to the deductive proof scheme. Two other ways of thinking associated with the development of the deductive proof scheme are relevant to the study. Namely, they are process pattern generalization and definitional reasoning.

A process pattern generalization (PPG) can be defined in relation to its less desirable counterpart, a result pattern generalization (RPG). Within the two ways of thinking, PPG refers to one's proving by attending to *regularity in the process*, while RPG refers to one's proving by attending *merely* to *regularity in the result,* such as relying on the substitution of numbers. Harel (2008a) illustrates the two ways of thinking in the following example:

> *Proving that the general term of the sequence 1, 2, 4, 8, 16, …, is* 2^n *because this is consistent with the given numerical values is RPG; proving the same fact by demonstrating that the process which generated this sequence is equivalent to repeated multiplication by two is PPG (p. 491).*

While the former act belongs to the deductive proof scheme, the latter belongs to the empirical proof scheme. RPG is a less desirable way of thinking, but it is still desirable because inadequate or even erroneous ways of thinking inevitably exist in one's learning of mathematics in the long term. And more importantly, it might initiate one's finding regularity in the process.

Definitional reasoning is the other crucial way of thinking which was actually built into the second part of the current study. Namely, the study investigated whether a proper understanding of the particular main idea of a first instance proof (and its two distinct extensions) initiates its development as a way of thinking in the individual student or not. "Definitional reasoning is the way of thinking by which one defines objects and proves assertions in terms of mathematical definitions" (Harel, 2008a, p. 489). A mathematical definition applies universally to all objects being defined and exclusively to them (Harel, 2013). An important characteristic of definitional reasoning is that a mathematical concept must necessarily be uniquely defined within a given theory. If there are multiple definitions for a concept, they must be logically equivalent, meaning that each one must logically imply the other. Typically, such understanding of definitions is not developed easily among students. For the students to develop a conception of mathematical definition, including an awareness for their usefulness and value to the proving process, repeated experience of the intellectual need for this is required (Harel, 2013). It is not expected to manifest itself in young students, as most students may not reach this stage of growth even at the college level or after (Harel, 1999). For this reason, the current study aims to take an initial step towards the study of definitional reasoning at the middle school level.

2.2. The Learning Through Activity Theoretical Framework

The Learning Through Activity theoretical framework defines a mathematical concept as a researcher's construct and elaborates on Piaget's notion of reflective abstraction as the key mechanism accounting for the students' construction of these concepts through their activity. Building on these major constructs, the framework develops instructional design principles for creating task sequences that promote guided reinvention of the intended concepts (Simon et al., 2018). As a starting point for clarifying what a conceptual understanding is, The LTA accepts the following view on mathematical concepts:

> *Mathematical definitions and theorems do not specify mathematical concepts (Vergnaud, 1997). For example, rational numbers are defined as a/b, where a and b are integers, and b is not equal to 0. However, this definition offers no specification of what it means to understand (have concepts of) fractions and ratio. Mathematical concepts consist of mathematical (mental) objects and the relationships among them (Simon et al., 2018, p. 96).*

According to LTA, a mathematical concept is a researcher's construct that characterize an individual learner's present knowledge or her potential knowledge aimed to be fostered through an instructional design effort. It is an "inferred" or "intended" student knowledge (Simon, 2017, p. 123). It is important to note that there is no unique way of articulating a concept and any articulated concept can be further developed to better formulate different aspects or components of understanding, or to provide greater detail about it. The next section describes two types of concepts that the LTA defines and studies how to promote them in students. However, before that the following paragraph explains how learning a mathematics concept is conceptualized differently than learning in general, such as learning a fact or vocabulary (Simon et al., 2018).

Simon (2006) built on Piaget's notion of reflective abstraction as the type of abstraction that would uniquely explain the construction of mathematical concepts. Piaget (1980) recognized it as a distinctive construct compared to a classical view of abstraction referred to as empirical or simple abstraction. Empirical abstraction is the process of taking out a property from objects where "the quality abstracted from the object is already recognized in the object, in the same form, before the abstraction" (Piaget, 1950, p. 75, quoted in Montangero & Maurice-Naville, 1997, p. 56) and "does not account for logico-mathematical construction" (Simon, 2020, p.2). Reflective abstraction, on the contrary, "is an abstraction from the learner's activity and explains the construction of new, higher-level knowledge" (Simon, 2020, p. 2).

Simon (2020) made a distinction between two types of concepts resulting from two different elaborations of reflective abstraction. The coordination of actions and the abstraction of commonality are two types of reflective abstraction that produce what are called "CoA concepts" and "AoC concepts" respectively. The next two sections describe theoretical background of these concepts each.

2.1.1. A CoA concept and its construction

LTA theoretical framework assumes a recursive structure in conceptual learning by explaining construction of new concepts from previously learned concepts. A newly constructed concept at one level becomes a building block of another concept to be constructed at the next level (Simon et al., 2018). A concept is a goal-action composite, represented as G_n-A_n. The goal refers to the learner's goal, and the action refers to the one that the learner uses to achieve that goal. The following notation is used to represent the student's existing concepts and concepts just being constructed: Two different concepts already available to the student are shown as G_{0a}-A_{0a} and G_{0b}-A_{0b}, where the primary subscript "0" signals that each of the concepts is already available to the student, and the secondary subscripts, "a" and "b," signals that each pair of goal-action indicates a distinct concept from one another. A primary subscript of "1", as in G_1-A_1, is used to signify the new, higher-level concept whose construction is being investigated. An account of the conceptual learning, then, aims to explain how the concept G_1-A_1 is developed from the student's two prior concepts G_{0a}-A_{0a} and G_{0b}-A_{0b}.

The learning process consists of two consecutive stages of a concept: the participatory stage and the anticipatory stage (Tzur & Simon et al., 2004). The participatory stage of a concept is the result of a single reflective abstraction. Simon et al. (2018) considers Piaget's (2001) construct of reflective abstraction as a coordination of concepts instead of a coordination of actions. In the above notion of a mathematical concept, actions are always a part of a concept and are linked to a goal. Accordingly, in the above formulation, building the more advanced concept G_1-A_1 is a coordination of the concepts G_{0a}-A_{0a} and G_{0b}-A_{0b}, not only the actions A_{0a} and A_{0b}. Still, the phrase coordination of actions is used to indicate the coordination of concepts of which these actions are a part (Simon et al., 2018). Abstraction of the higher-level concept results from a single reflective abstraction and includes learning the logical necessity involved.

Reflective abstraction occurs during the learner's engagement in a multi-step activity. In order to solve a novel task, the learner sets a new goal that is attainable by her existing knowledge of concepts. This goal is a 0-level goal, called the task goal and signified by GT. To meet the task goal, the learner calls on a sequence of actions (A_{0a} and A_{0b}), each corresponding to a sub-goal for the individual steps in the activity. The activity, then, is a goal and a sequence of actions combined for solving a novel task, GT ($A_{0a} \rightarrow A_{0b}$), and more completely, GT (G_{0a}–$A_{0a} \rightarrow G_{0b}$–$A_{0b}$). The arrow means that the actions in the activity occur in a particular sequence allowing the intended coordination to take place. When the two 0-level actions are coordinated into a new (single) higher-level action, A_1, this means the new concept, GT–A_1 (a ready-at-hand pair of goal and action), is abstracted. Having abstracted the new concept, the student can now anticipate the result of the activity without carrying it out step by step; but only if she is thinking about the original activity.

The anticipatory stage of the concept is constructed when the student does not need to think about the original activity to call on the abstraction learned any more. The participatory and anticipatory stages of the concept differ from each other in terms of the goal to which the new abstraction is linked. While in the former stage, the higher-level action A_1 is linked to the task goal, in the latter stage, the same action is linked to another goal, G_1, decontextualized from the task goal. Therefore, the transition from the former stage to the latter is a transition from GT – A_1 to G_1-A_1. The theoretical framework is currently unable to fully explain how this transition occurs, but it is mainly through the process of connecting the participatory-stage abstraction to the higher-level goal, represented as G_1 –GT –A_1. The anticipatory stage of the concept, decontextualized from the activity engaged, is the intended learning goal.

An HLT for a new CoA concept is specified through the following steps:

1. *Assess the relevant understanding of the learner.*
2. *Specify the learning goal (intended abstraction).*
3. *Identify an activity or activity sequence that the learner already has available that could be the basis for the new abstraction.*
4. *Design a sequence of tasks that is likely to bring forth the learners' use of this activity and lead to the intended abstraction (Simon, Placa & Avitzur, 2016, p. 67).*

A critical strategy for designing a task sequence with the intended characteristics bears on the notion of mental run where the student is asked to narrate a hands-on solution without actually executing its steps. The following is an illustration from Simon (2010; 2018), regarding a prospective teacher participant's reinvention of a common-denominator algorithm for division of fractions. Based on her already available solution strategy for division of fractions with the same denominators (based on drawing rectangle diagrams), such as 7/8 ÷ 3/8, the subsequent tasks asked her to engage in mental runs to determine answers to problems with larger denominators, such as 7/103 ÷ 2/103. In an attempt to anticipate the answers to the particular problems, the student eventually comes to realize that she could solve such division tasks with common denominators by dividing the numerators. This marks the point where the coordination of the actions involved in her activity takes place and the intended CoA concept is constructed. As articulated by the authors the concept involves understanding *the logical necessity of the idea that division of two numbers that are measured in the same unit or partial unit is invariant as the unit of measure varies. In terms of division of fractions (with common denominators), it meant knowing that the quotient of the two numerators is the quotient of the fractions regardless of the size of the common denominator (Simon et al., 2018, p. 99).*

2.1.2. An AoC concept and its construction

"An AoC concept is a mathematical structure that can be used to sort mathematical experiences into examples and non-examples" (Simon, 2020, p. 4). It is an abstraction of *common structure* through the learner's abstracting of the commonality in her activity (the AoC type of reflective abstraction). To illustrate, what is the "same" for word problems of partitive division is a common mathematical structure. Recognizing the presence or absence of this structure (the sameness) in problem situations is an ability determined by the knowledge of the concept, the partitive division (although the student would not say so about her conception). Simon (2020) outlined the basic design principles for promoting an AoC concept as establishing an informal activity for the student and making the student use this activity to solve word problems by representing the activity using arrow diagrams. Word problems are important because they present the situations of the common mathematical structure that the student is about to learn to identify. The arrow diagrams, by representing the activity to the student, help the student consider cases and non-cases of the situations for which the activity is applicable. When the student comes to use the diagram for appropriate word problems only, this means the student comes to recognize the intended mathematical structure. This is when the abstraction of commonality is completed (Simon, 2020).

2.3. A model of students' ways of understanding a proof (Ahmadpour et al., 2019)

Ahmadpour et al. (2019) proposed a model of students' ways of understanding a written proof text. The model accepts three possible end-states for students' understanding of a formally acceptable proof and explains the states and transitions in students' understanding towards these end-states. The three end-states in one's reading of a proof are named "Formulated Proof," "Procedural Proof," and "Formulaic Proof." A proof is said to be "read" as a Formulated Proof, if the reader comprehends its underlying deductive structure. The theoretical pathway towards this end-state is called the Path of Structure. On the other hand, a proof is said to be read as a Procedural Proof if the reader perceives the proof "as a general procedure that can be applied to any number, as a sort of recipe for producing examples, rather than a deductive structure applicable to all numbers" (Ahmadpour et al., 2019, p. 87). The route towards this end-state is called the Path of Procedure. Lastly, a proof is said to be read as a Formulaic Proof, if the reader evaluates only its surface-level form syntactically. The pathway to this end-state is the Path of Form. Figure 2.2 presents the three pathways described in the model where ovals show the states of understanding and arrows indicate the transition processes between the states.

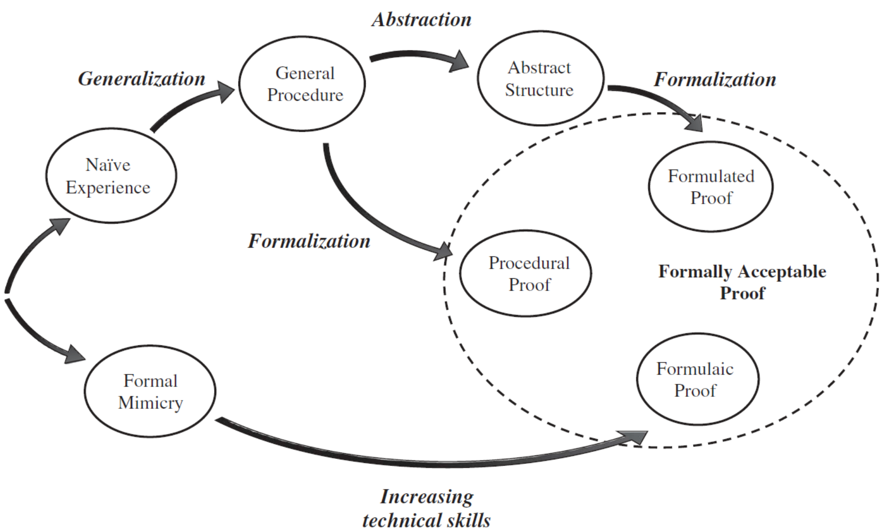


Figure 2.2 The model of students' ways of understanding a proof (Ahmadpour et al., 2019, p. 86).

Ahmadpour et al. (2019) consider understanding a dynamic process affected by the shifts in the learner's attention. Therefore, the transitions of how one state of understanding develops into another and possibility of switching between the path-ways (especially linking the form and structure of a proof) are both central to its focus. Therefore, it allows description of students' progression over time beyond marking the states of understanding at fixed points in time.

Form of argument representation constitutes an inseparable aspect of the instructional design approaches utilized in the current study. However, since the main focus of the study is on promoting students' understanding of the deductive structure of mathematical proof (Learning Goal 1), the Path of Structure is of additional importance. Figure 2.3 presents the Path of Structure that theorizes students' development from the state of Naïve Experience to the state of Formulated Proof, as intended to be explored by the current study. This path theorizes that the students' understanding of a proof may develop from the state of Naïve Experience to those of General Procedure, Abstract Structure and Formulated Proof sequentially, through the transition processes of *generalization*, *abstraction*, and *formalization*.

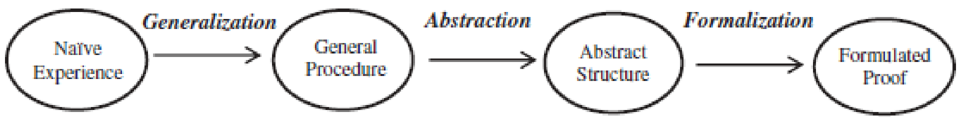


Figure 2.3 The path of the structure (Ahmadpour et al., 2019, p. 90).

As explained before (in Section 1.5), the current study's intended understanding of the deductive structure of the particular proofs is the state of Abstract Structure. Ahmadpour et al. (2019) consider *abstraction* as "the construction of an idea that stands for a class" and define "abstraction in proving" as "the construction of a structure that 'reifies' (Sfard, 1991) the underlying idea of a general procedure" (Ahmadpour et al., 2019, p. 93). The data analysis procedures explained in the findings section include more references to Sfard (1991) and Sfard and Linchevski (1994), in line with the abstraction-related goals of the study.

CHAPTER 3

METHOD

This study used an individual teaching experiment methodology to investigate a sixth-grade student's first steps towards the deductive proof scheme, after eliminating her empirical proof scheme first. The study consisted of two major parts. The first part of the study aimed to explore the processes by which an individual student may (1) come to understand a deductive argument as an instance of mathematical proof for the first time (intended CoA concept). The second part of the study first aimed to (2) develop the main idea of this first instance proof as a transferrable main idea to proving theorems analogous to the first one (intended AoC concept) and then explored (3) how the students' understandings developed (CoA and AoC concepts), enabled or constrained her attempts to prove other similar (but not analogous to first one) and novel conjectures. The study did not claim a purpose of generating a practical HLT for deductive proof at the sixth grade level. Instead, the main purpose of the study was to explicate the mechanisms by which an individual student, who does not have the concept of deductive proof yet, might come to understand a very first example of the category as a mathematical proof, and then practice the act of proving within the deductive proof scheme by building on an (first structural and then holistic) understanding of this first example. The study aimed to answer the following research questions in the context of proving basic number theory conjectures.

1. How does (may) an individual student come to understand a deductive argument as an instance of mathematical proof for the first time? Can a coordination of actions (CoA) type of reflective abstraction explain the process of understanding the first instance proof as a structural object?

2. How does (may) the student develop a holistic understanding of this first instance proof so that she can transfer its main idea to the analogous proving tasks? Can an AoC type of reflective abstraction explain the process of constructing such a holistic understanding?

3. How does the student (attempt to) start proving (a) conjectures that are similar (but not analogous) to the first instance conjecture, (b) the target conjecture of the study and (b) a novel conjecture?

This chapter describes the research design, the specific context and participants of the study, and the data collection and analysis procedures. The data collection procedures section introduces instructional design approaches used in the teaching experiment sessions and data collection procedures. The chapter closes with information on trustworthiness, ethics, and the researcher's role in conducting the study.

3.1. Design of the study

This study used the LTA modified teaching experiment methodology (Simon, 2018). A teaching experiment is conducted when the researcher is interested in constructing models of students' mathematical realities that may differ from the researchers' consensual knowledge of mathematics (Steffe & Thompson, 2000). Through a sequence of teaching episodes, the researcher examines a record of what the students reveal about their reasoning with an understanding of concepts and operations; and looks for ways of making them the conceptual bases of students' progress in learning. Analyses of the data can be conducted after each teaching session to progressively refine the intervention for subsequent sessions, and after the completion of the teaching experiment to document an account of the whole intervention. Through the recursive cycles of hypothesis formulation, experimental testing, and hypothesis reconstruction, the methodology allows for the generation and testing of set-before and on-the-fly hypotheses of student learning (Steffe & Thompson, 2000).

The LTA teaching experiment is a modified version of the teaching experiment methodology (Steffe & Thompson, 2000) that emerged as a product of the LTA research program (Simon et al., 2010). The methodology provides a way of integrating the study of conceptual learning and instructional design with the process of creating HLTs (Simon, 1995) for mathematics concepts, which are the two overarching goals of the research program. Based on the results of the former, researchers gain a greater knowledge of how student learning occurs and use this knowledge for producing HLTs. The study of resulting HLTs, in turn, feeds back as empirical evidence for further elaboration of the former.

The LTA research methodology is built on the cyclic nature of the teaching experiment (Steffe & Thompson, 2000) because of two main reasons. First, progressive refinement of the intervention between cycles increases the possibility of capturing successful learning that may occur within the period of data collection. This is of vital importance to LTA research since the methodology aims to study how learning occurs from one conceptual state to another, rather than merely outlining students' thinking as a set of discrete steps. Second, it enables formulating and testing hypotheses, which serve the two overarching goals of the research program. On the other hand, the LTA research methodology made a number of modifications to the original teaching experiment methodology in order to meet its more specific goals, especially for overcoming the methodological difficulty of studying learning–a change in one's thinking.

1. LTA teaching experiments have to be single-subject studies. Studying with a single student at a time increases the possibility of gathering evidence of the individual student's mental activity through tracing her/his overt activity continuously, which otherwise would be interrupted and obscured at times by a second student's talking or acting. The single-subject nature of the experiments allows attributing the learning to the student's activity elicited by the task sequence rather than to interaction with a peer.

2. The researcher-teacher acts as a strict problem poser who is only allowed to decide when to pose a next type of task in the planned sequence, adjust the ordering of tasks in response to the student's activity, and ask probing questions. The researcher does not tell or show solutions, offer hints, or ask leading questions to avoid her/his extraneous influence on the student's actions.

3. Whether the student has the concept to be learned or not is assessed right before posing the intended task sequence. For a segment of data to be analyzed with the purpose of accounting for the student's learning, two endpoints of data must be ensured: first, when the student did not have the concept, and second, when she/he learned the concept.

4. LTA teaching experiments employ the specific LTA instructional design approach. The researcher creates an initial HLT based on the LTA theoretical framework and modifies it through the ongoing teaching experiment

analyses. This approach enables collection of data on the learning process, which may not be ensured (or may not be aimed at all) in an original teaching experiment study. The instructional design approach has the following affordances: (1) Depending on the success of the HLT, it produces the intended learning to be studied. (2) It allows the researcher to focus on the student's activity elicited by the task sequence to collect evidence about the learning process. (3) It integrates the study of the learning and the instructional design, and (4) provides a testable model of the outcome in the form of a task sequence (Simon, 2018).

The distinctive characteristic of the LTA teaching experiment methodology is its focus on explicating mechanisms of conceptual learning (the transition process) that takes place from one conceptual state to another, unlike the other studies aiming to describe the sequence of the conceptual states (the products) only (Simon, 2013). This purpose is fulfilled by studying students' making new abstractions through their activity, guided by the LTA theoretical framework. (Simon et al., 2018). Given this particular characteristic of the LTA teaching experiment methodology, it seems to be a promising research approach to investigate how a sixth-grade student comes to understand a very first instance of proof (a connected chain of reasoning) as a mathematical proof (an argument that establishes the validity of a conjecture), while not yet possessing the disciplinary concept of proof.

Before designing the LTA teaching experiment study, a preliminary phase of "exploratory teaching" (Steffe & Thompson, 2000) was conducted to gain a first-hand experience of how a typical sixth-grade student might operate with the concepts and operations relevant to the current study. Steffe and Thompson (2000) recommended that "Any researcher who hasn't conducted a teaching experiment independently, but who wishes to do so, should engage in exploratory teaching first" (p. 274). The purpose of engaging in such experiential work is to develop familiarity with students' mathematics. Accordingly in a teaching experiment, the researcher does not insist that students should learn the concepts and operations in the ways the researcher knows them (Steffe & Thompson, 2000, p. 274). Therefore, a year before the LTA teaching experiment study reported here, an eight-week-long exploratory teaching was conducted with an individual sixth-grade student. The experiential knowledge gained through the exploratory teaching, along with the further review of the relevant literature, informed the key decisions of the LTA teaching experiment study. The data collection procedures section includes a unit on instructional decisions influenced by the results of exploratory teaching.

3.2. Context and participants of the study

The teaching experiment and the exploratory teaching were each conducted one-on-one with the individual sixth-grade students. Melis (pseudonym) participated in the eight-week exploratory teaching through the end of the Spring Semester of 2019. Beren (pseudonym) participated in the ten-week teaching experiment initially scheduled for the same period of the following year, 2020, but postponed to the summer vacation due to the Covid-19 pandemic. The two students were enrolled in public schools in Ankara, attended mostly by students from medium socioeconomic backgrounds. They had no previous interaction with the concept of proof. The Turkish Middle School Mathematics Curriculum (MoNE, 2018) emphasizes students' explaining their reasoning and evaluating others' in the classroom but does not explicitly address the concept of proof at the middle school level. Therefore, it can be assumed that the deductive nature of mathematically valid arguments is not explicated to the students at these grade levels.

Sixth-grade students were selected for the study because they had learned the key concepts of basic number theory at this grade level. The curriculum (MoNE, 2018) addresses the concept of parity at the third-grade level and the concepts of factors, multiples, and divisibility of natural numbers at the sixth-grade level. Distributive property, being an essential concept for proving the extended conjectures of the study, is addressed at the sixth-grade level. Competence in these concepts constituted one criterion for selecting individual students to participate in the study. The others included the student's ability to express mathematical ideas clearly and willingness to learn new (out of curriculum) mathematics. Based on these criteria, Melis was approached through her school mathematics teacher's opinion and Beren was approached through her private tutor's suggestion. Both students volunteered for the study, along with their parents' consent.

3.2.1. The participant Beren

Apart from meeting the conditions for participation in the study, some of Beren's personal characteristics played an important role in gathering of a useful data set for the LTA teaching experiment study. Throughout the experiment, she demonstrated a high level of perseverance for the problems she was asked to solve and the ones she formulated herself.

Beren : *Wait a minute, I'll find out.*
Researcher : *I think you will find out.*
Beren : *I've found all of them so far, and I'll find this one too.*

She was highly motivated to engage in the designed task sequence and involved in guided discovery processes as intended. She also appreciated the process of creating mathematical knowledge through her own effort without the researcher telling or showing to her. In the below case, unable to solve a task, Beren was handed in a new one to address her difficulty. After solving that task, she turned back to the original one and commented:

Beren : *You know what, this is (now) much easier. After solving this one (the latter task), this (the former task) came right away… This makes me very happy all of a sudden. You know, I say 'aha!' to myself and solve it.*
…
Researcher : *Any questions?*
Beren : *No, not. Well, there was at first, but then I figured out these ones (refers to the both of the tasks). I solved the other one. Then I took a look and figured out this one too. But believe me, this is the same thing in every lesson. I don't understand how I come to do this any time, how I succeed in the end. (Sounds enthusiastic)*

In addition to these, she frequently reflected on her own thinking and acting, without being asked, and even at the outside of the teaching sessions. For example, in the middle of a totally unrelated task, she commented:

Beren : *For a moment, while labeling this (box), I felt something. How hard I drew the flowchart the first day we started this, and after you left, (I asked myself) why was that did we draw the chart like this. Where did we get from here? I remember asking this.*

The findings section presents more instances of how spontaneous actions and comments of Beren, especially the connections she made to the previous tasks encountered in prior sessions, enriched the data set of the study.

3.3. Data collection procedures

3.3.1. Instructional design procedures

The teaching experiment consisted of five major phases outlined in Table 3.1. The first three phases made up the first part of the study to construct the intended CoA concept. The subsequent two phases corresponded to the second part of the study which aimed to develop the AoC and extended AoC concepts.

Table 3.1 The instructional intervention

Phases of the Intervention		Learning Goals	Sessions	Dates
1	Basic number theory concepts	Developing the concepts to be coordinated in Phase 3	1^{st} - 4^{th}	June 22, 29 July 02, 08
2	Monstrous counterexample illustration	Feeling an intellectual need for a secure method of validation (Need for certainty)	5^{th}	July 13
3	The CoA concept	Understanding a first instance of mathematical proof	5^{th} - 6^{th}	July 13, 14
4	The AoC concept	Modular structure in relation to proving a certain class of theorems	7^{th}	July 21
5	The extended AoC concepts	Building on CoA and AoC concepts to practice proving	8^{th} - 10^{th}	August 24, 27, 31

In the first part of the study, the first two phases aimed to promote the prerequisite conceptions needed for constructing the intended CoA concept. Namely, the task sequence in first phase aimed to develop the concepts to be coordinated in the third phase. The second phase applied Monstrous Counterexample Illustration (Stylianides & Stylianides, 2009) to challenge the student's existing proof scheme, the empirical proof scheme, by cognitive conflict. The third phase, then initiated the student's learning towards the deductive proof scheme by constructing the first intended concept of the study. In the second part of the study, Phase 4 fostered the construction of the main idea of the first instance proof as an AoC concept. Phase 5, extended this second concept of the study to the task of proving other conjectures. Table 3.1 presents the five phases of the teaching experiment together with the specific times of the sessions devoted to each phase.

3.3.1.1. Designing for the intended CoA and AoC concepts of the study

Investigation of the study's first purpose as a learning process comparable to the construction of a CoA concept was initiated based on the following hypothesis: Understanding the first instance of mathematical proof without yet having an accurate conception of what a proof is may require a special type of reflective abstraction, and such reflective abstraction can be of the type coordination of actions. Considering "one particular proof" as a CoA concept may not seem appropriate. However, Simon's definition of "mathematical concept" as a researcher's construct motivates this initiative. Simon (2017) highlights that concepts include not only the objects but also the logical necessity of the relationships among them. Understanding a proof, similarly, can be viewed as a coordination of concepts that are chained via logical necessities of the relationships involved. Hence, the process of conceptually understanding a proof can be similar to that of learning a concept (in the LTA sense) "as a result of a single reflective abstraction" (Simon et al., 2018, p. 98). The reflective abstraction here can be of the type coordination of actions, because Simon (2020), later, associated the notion of logical necessity with the CoA type of concepts only. Moreover,

> *LTA researchers work from the assumption that the process of intramental reorganization that leads to new mathematical concepts in the individual is an inherent ability and tendency of human learners and is invariant as the situations for mathematics learning vary. (Simon, 2018, p. 114)*

Based on the LTA hypothesis "that students learn through their activity (mental and physical) in all situations—with and without pedagogical interventions" (Simon, 2013, p. 284), this study makes use of the LTA elaboration of how new concepts are constructed from students' mathematical activity (Simon, 2013; Simon & Tzur, 2004). The hypothesis particular to this study is that the process of carrying out a sequential activity that produces a mathematical proof (that the student does not yet perceive as proof) and reflecting on the activity-effect relationship may produce the intended learning by the student's coordinating the actions carried out sequentially (Simon, 2013; Simon & Tzur, 2004). This frames the first purpose of the study: Understanding the first instance of mathematical proof as a CoA concept.

The second purpose of the study was to help the student develop an appreciation for the key concept "modular structure" in relation to the act of proving theorems analogous to the first instance proof. The intended concept was an instance of Harel's (2008a) *definitional reasoning,* "the *way of thinking* by which one defines objects and prove assertions in terms of mathematical definitions" (p. 495) and was articulated as the AoC concept of the study.

The CoA and AoC concepts together aimed to help the student prove the target conjecture (i.e., the sum of two odd numbers is an even number) of the study individually at the end of her participation in the teaching experiment. Student's achievement of this particular end-goal, as well as her constructing the CoA and AoC concepts intended, necessitated her understandings and using at least one way of representing the number theory concepts involved. Therefore, this study developed and used the new representation system explained in the next section.

3.3.1.2. The representation system of the study

This study treats parity, which is the key concept needed for the target proof of the study, from the broader perspective of divisibility by any natural number. The new representation system called the "general representation" symbolizes the modular structure of a particular natural number according to a specified modulus n, to indicate the result of the number's division by that number n (Campbell, 2002). Therefore, it provides a way of representing any and all of the numbers in an arithmetic pattern through their shared modular structure. With this representation system, the following meaning is created for the modular structure of an unknown quantity: an unknown number of identical "counting items" evenly distributed into identical "cups" with (or without) some leftovers. In the flowchart proof presented in Figure 1.2 (the target proof of the study), odd numbers are represented using two identical cups (large circles) with one remainder counting item (little circle). The same meaning extends to the case of grouping things (cups and counting items) into equal amounts. In the last step of the proof, when the two groups of the same number of counting items are obtained, the new groups constitute a new set of identical cups (referred to as "packages" in the contextual situation that introduced the representation to the student). The resulting configuration conveys the meaning that grouping the total amount of counting items (both inside and outside of the cups) into two equal parts, in this case, is possible without any remainder. Hence, this new configuration also represents an even number.

The representation system differs from the other examples available in the research literature in order to support the conception of modular congruence as an abstract structure (Ahmadpour et al., 2019) detached from the specific examples of numbers sharing a particular structure. Moreover, it allows a way of differentiating between distinct fixations of unknown quantities by using different looks (colors or markings) for identical cups for each object represented. The flowchart proof format of the study aims to ensure that different cup looks are used appropriately by placing checkpoints between relevant boxes. In Figure 1.2, the cross-mark between the two leftmost boxes indicates that the two objects shown in the boxes are not dependent on each other; that is, each of the cups does not have to contain the same number of counting items even if they may. In other cases where different objects are dependent on each other (e.g., in proving that the sum of three consecutive numbers is divisible by three), identical cups are used to represent these objects, and the checkpoints are filled with a check-mark. Obtaining the structure of a sum of two (or more) numbers in terms the addends' modular structures and the resulting object's being subject to the distributive law for division was specifically studied with this representation, in the Phase 5.

3.3.1.3. Instructional decisions made based on the exploratory teaching results

Observations and intermediate conclusions from the exploratory teaching have been instrumental in designing the instructional sequence that would allow the application of the theoretical framing of the study's learning goals. This section presents important results from the exploratory teaching together with the explanations of how they shaped the Phase 1 and Phase 2 hypothesized learning processes and related task design procedures. Next, the task sequences of Phase 1 and Phase 2 are presented. After that, design procedures followed in Phase 3 for promoting the intended CoA concept is explained.

The flowchart proof format

The exploratory teaching did not use a structured proof format. It was due to the concern that the student might develop a ritual proof scheme (Harel & Sowder, 1998);

that is, attend to the surface-level form of the proof while overlooking its underlying structure (Ahmadpour et al., 2019). Intention of the exploratory teaching was restricted to gaining a first-hand experience on how a sixth-grade student would reason with the specific way of representation provided to her. The expectation -the loose hypothesis- was that a student of grade six would know about the concept of parity but would not have a way of representing all the even/odd numbers in a single instantiation, hence, she would lack the ability to start a proof based on this concept or reason with this concept.

Figure 3.1 Melis's reasoning about the conjecture "Three times an odd number plus one is an even number": "three times an odd number plus one" created as 6 cups and 4 counting items (left) and parity determined by distributing the total items into two equal groups (right).

The exploratory teaching provided Melis with concrete manipulatives which were identical cups (available in different colors) and identical counting items, just as the ones used in the representation system introduced above. Before each session, it was made sure that the cups of same color contained the same number of counting items that the researcher knew (because she prepared them ahead of the session) but the student did not. The student was encouraged to think about the variation of the amounts contained in the cups while learning to make logical inferences. Student responses to the questions and tasks posed by the researcher included use of the manipulatives and/or their representation on paper. Figure 3.1 depicts two sequential video scenes illustrating Melis's reasoning process about the conjecture "Three times an odd number plus one is an even number".

Through the end of the experiment, Melis was guided to construct tables such as the one given in Figure 3.2, in order to help her keep track of her thinking. However, it was observed that, while going through the steps towards the conclusion she repeatedly lost track of her activity, especially the overarching purpose of doing the activity. This created a distraction for her to view the entire chain of reasoning as a single object, making it difficult for her to realize how a proof is composed of the components she created sequentially and how these components are connected to each other (Miyazaki, Fujita, & Jones, 2017).

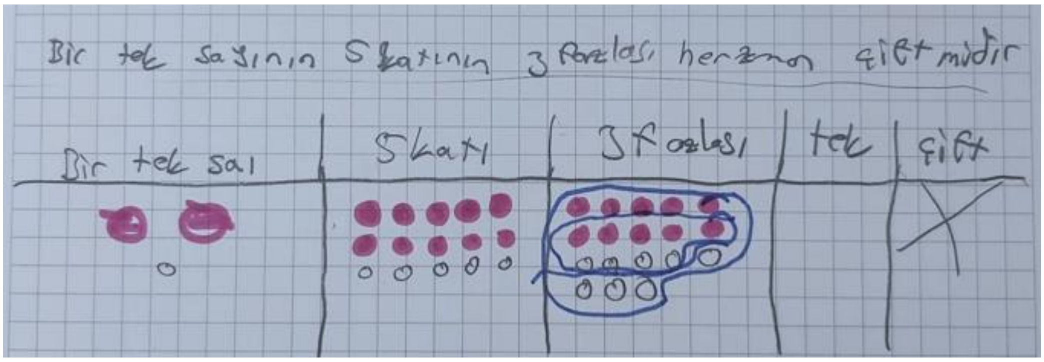


Figure 3.2 Melis’s table proof for the conjecture “Five times an odd number plus three is always an even number”.

This observation suggested the idea of shifting the focus of intervention from the view of proof as a process, an intellectual activity (Harel & Sowder, 2007), to the view of proof as a structural object (Miyazaki & Yumoto, 2009). To promote an understanding of proof as a structural object, the flowchart proof format of this study was adapted from Miyazaki, Fujita, and Jones (2017). The specific proof format was expected to support the student’s understanding of the deductive relations underlying the proof (universal instantiation and hypothetical syllogism), through explicit visualization. Because of the concern explained previously (the possibility of developing a ritual proof scheme), the instructional design procedure endeavored to maintain a careful balance between the visual and structural aspects of the flowchart proofs. Having studied the new representation system with a sixth-grade student in the exploratory teaching provided confidence for the integration of the new aspect -the flowchart proof format- to the investigation of the complex learning goal in the actual teaching experiment.

Defining parity: Evenness and divisibility by two

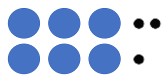
An important observation regarding Melis's conception of parity was that she knew the odd and even numbers as the respective number patterns of 1, 3, 5, 7, 9, 11… and 2, 4, 6, 8, 10, 12, …, but, she did not relate the numbers' parity with their divisibility by two. For the two- or three-digit numbers, she determined parity as a function of the last digit only. Similar disconnection has been reported in previous research findings. Zazkis (1998) suggested that the equivalence of being 'even' and being 'divisible by two' should be recognized and emphasized explicitly because "[t]he knowledge of this equivalence is not applied naturally and spontaneously even among students with a relatively solid mathematical background" (p. 88). Turkish middle school mathematics curriculum (MoNE, 2018) does not make such emphasis on the equivalence of the two concepts.

Phase 1 of the LTA teaching experiment set the construction of this relationship as a learning goal. An understanding of odd numbers, for example, was considered to include relating the numbers in the pattern 1, 3, 5, 7, 9, 11… with not being perfectly divisible by two, and more specifically, with giving a remainder of one when divided by two. Understanding this relationship would be critical for the student's making sense of the specific representation used for odd numbers, which is algebraically equivalent to $2n+1$, $n \in Z$. However, the study of odd numbers, especially the label "odd" was purposefully avoided until it appeared in the tasks used with the purposes of data collection. This was not to destroy novelty of the target conjecture to be proved at the end of the teaching experiment study. The phrase "numbers that give a remainder of one when divided by two" was intentionally taught to the student in order to help the student explain her thinking in later tasks of the trajectory. The next section includes further considerations about use of parity definitions within the specific mathematical proof targeted in this study.

Differentiating premises from conclusions:

The target proof uses two closely-related premises (definitions of being odd or even) thrice (see Table 1.2). In algebraic terms, the following expression is constructed for the sum of two distinct odd numbers: $(2n+1) + (2m+1)$, where n and m are positive integers (twice use of a premise-the definition of an odd number), then the sum is expressed as $2(n+m+1) = 2k$, a multiple of 2 for the integer k (third use of a premise-the definition of an even number) (Stylianides & Stylianides, 2008). A difference to observe in the two uses, apart from the even-odd variation, is that in the first use, one way of the two-directional relationship indicated in the definition is used, and in the second use, the other way. That is, in designating any odd number as 2n+1 (as 2m+1 also), the relation “If an integer is odd, then it is not divisible by two” is used, and then in identifying 2(n+m+1) as an even number, the oppositely-directed relation “If an integer is divisible by two, then it is even” is used.

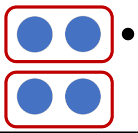
Table 3.2 The (arithmetic) number pattern, verbal description and representation illustration

The number pattern	Verbal description	Representation(s)
1, 3, 5, 7, 9, 11…	The numbers that give a remainder of one when divided by two (The odd numbers)	
3, 9, 15, 21, 27, …	The numbers that give a remainder of three when divided by six	

Similar analysis applies to the first instance proof of the study (Figure 1.4). Understanding or constructing this proof requires an ability to use the relations of both directions. Given the expression “the numbers that give a remainder of three when divided by six” (describing the number pattern 3, 9, 15, 21, 27, …), the representation 6k+3, k ∈ Z is created. Then, for the resulting object 6k+3=3(k+1), the expression “a number divisible by three” which is equivalent to “a number that gives a remainder of zero when divided by three” is constructed. For this reason, Phase 1 of the LTA teaching experiment extended the above defined learning goal (equalizing evenness and divisibility by two). The extended learning goal included making conversions between a given arithmetic number pattern, the verbal description of the common characteristic (a modular structure) shared by all the numbers in the pattern, and its general representation. Given any one of the three, the student would learn to construct the other two. Table 3.2 exemplifies the case for two different number patterns.

Understanding parity definitions as two-directional relationships is important for differentiating between premises and conclusions in the proof. In the sixth week of the exploratory teaching, Melis was able to determine the parity of various configurations created with cups and counting items; meaning that she was able to use the relation “If a number is divisible by two, then it is even" (condition of being even) successfully. However, while starting to prove the conjecture given in Figure 3.2, she thought the representation 4k+1 would represent the odd numbers. This was (probably) because she determined this configuration to be an odd number in a previous task. That is, instead of starting with the property of odd numbers, she relied on her previous knowledge that another set of numbers (congruent to 4k+1) once had satisfied the condition of being odd. She did not employ the oppositely-directed (to the above) relation "If a number is odd, then it is not divisible by two" appropriately.

Table 3.3 Universal instantiations of the property and the condition of being odd

A premise / A conclusion	Two singular propositions		
A universal instantiation of the *premise*: "If a number is odd, then it is not divisible by two". (property)	If a given number A is an odd number	→	Then it is of the form 
A *conclusion* based on the universal instantiation of the *premise*: "If a number is not divisible by two, then it is odd." (condition)	If a given number B is of the form: 	→	Then, it is odd.

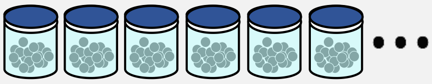
This was not because Melis did not know the pattern of odd numbers. Rather, this was an issue of (not) attending to the "defining property" of the set. The question for her to engage in should have been "which universal proposition should be instantiated to represent any and all odd number?" The answer to this question is not just the definition of odd number. It is the one-way of the "if and only if" relationship stated in the definition, the one we called above "property of being odd" (not the condition): "If a number is odd, then it is not divisible by two." The Table 3.3 illustrates the difference between two cases of universal instantiation, in relation to a premise and a conclusion. The first case given in the table indicates a more sophisticated understanding compared to the second. One can arrive at the conclusion in the second line, as Melis was able to do, without being aware that a universal proposition is in use. This is not the case for the first line because this time a conscious attention to the universal proposition, defining property of odd numbers, is needed.

In order to develop "property" aspect of the relationship, which enables the more difficult universal instantiation, the following set of tasks were designed (See Table 3.4 for the Phase 1 task sequence). For the modular structure 6k+3 represented in cups and counting items system, consecutive tasks asked the student to determine the numerical values it can and cannot take (Task 3), to explore the covariation of the number of counting items in each cup (k) and the total number of counting items (6k+3) (Task 4), and to list the set of numbers that share this modular structure orderly (3, 9, 15, 21, 27, …) (Task 5). The three tasks together aimed the first way of the two-directional relationship: Given the modular structure, list the numbers in the pattern (given the defining property find the elements of the set). Note that the description "the numbers that give a remainder of three when divided by six" was not vital to this procedure. But, it played an important role in the following three tasks designed to develop the other way of the relationship: Given the number pattern, identify the modular structure shared by all its elements (given a set of elements, finding defining property of the set).

After the completion of Task 5, the term "general representation" was introduced to the student using the contextual situation of cups and tiny cookies on a bakery shelf (see Figure 3.3). The description "the numbers that give a remainder of … when divided by …" was provided to the student at this point of the teaching experiment. Then, the following three sets of tasks were applied to establish the reverse-order relationship. First, given a representation of the modular structure, the student was asked to identify the number pattern and the verbal description (Task 6a). Second, given a description, the student was asked to construct the number pattern and the representation of the related modular structure (Task 6b). Third, given a number pattern, the student was asked to state the relevant description and represent the related modular structure (Task 6c). It was believed that the verbal description would help the student transition to the third type of task; hence the first- and second-type tasks were placed before the third. The expectation was that tasks of type-one would help the student consider a modular structure as a result of a division operation by using the specific language provided to her. Then, tasks of type-two would help the student observe the relationship of the constant difference between the consecutive elements of the pattern to the number of cups used in its general representation and that of the first number in the pattern to the remaining counting items. The two types of tasks would equip the student with the necessary skills for describing (and then representing) a number pattern through the common characteristic shared by all its elements.

Tiny Cookies on a Bakery Shelf

It is not known how many cookies there are on the bakery shelf. However, it is certain that when Ms. Esin previously distributed cookies equally into six cups, three were left.



Let us denote a cup with an unknown number of cookies inside with "" and single-packed cookies with "●". In this case, we show the total number of cookies as:

This notation represents all numbers that give a remainder of _3_ when divided by _6_ . That is, the number of cookies in total can be any number that fits this representation.

Figure 3.3 The General representation

From "parity generalized to divisibility by any number" to "parity as a specific case of divisibility by any number"

Note that there is no difference between divisibility by two and divisibility by any other number, other than labeling the former differently as evenness/oddness (Zazkis, 1998). **Error! Reference source not found.** above already illustrates the analogy. While the exploratory teaching started with the study of parity first and then extended the ideas to the more general case of divisibility by other numbers, the LTA teaching experiment preferred the reverse order. That is, divisibility by numbers other than two was studied first, where the concept of modular structure as a result of division operation was constructed. Parity, then, was addressed as one of the many possible cases of divisibility. The reverse order from the general to the specific was preferred to enable the study of conjectures such as "Numbers that give a remainder of 3 when divided by 6 are divisible by 3" before those that include sums or multiples of even/odd numbers. This generated the following affordances. First, it created a chance to study a reduced flowchart proof (like in the first instance proof) before the complete one in the hope of making the student's reflection on the activity-effect relationship more focused. Second, it eliminated universally instantiating nearly the same definition (based on divisibility by two) in two different ways (from the universal to the singular proposition, as a property, and from the singular to the universal proposition, as a condition) in the very first instance of proof. Although not the only factor, it was believed to help avoid confusion between premises and conclusions, as opposed to what was observed in exploratory teaching. Third, it would provide the student with a problematic situation where she would have some degree of uncertainty compared to the case of even/odd conjectures. Turkish middle school mathematics curriculum covers the concept of parity at the third-grade level. One of the objectives states that the students "examine the sums of odd and even natural numbers on models and expresses whether the sums are odd or even" (MoNE, 2018, p. 38). Even if the two participants of this study were not able to prove the theorems in the pre-assessment, they revealed certainty of the theorems during teaching experiment sessions.

Intellectual need for a secure method of validation (the need for certainty)

The following dialog took place in the exploratory teaching when Melis was asked to answer the true/false question: All the numbers that give a remainder of 6 when divided by 9 are divisible by 3. True or False?

Melis:	*Let us start with the smallest (number). (picked up 24 as the least such number mistakenly, determined that 24 was a multiple of 3, decided that the statement was true)*
Researcher:	*How do you know that the statement is true for all such numbers?*
Melis:	*We need to use the cups! (drew 9 cups and 6 counting items). We will distribute them to three groups equally. Then we will add two singles to each group.*

The dialog shows that Melis learned the procedure to be followed for proving the given conjecture. However, she carried it out only after the researcher's (probably leading) probing question. This suggested she did not feel an intellectual need for mathematical proof as a secure method of validating the generalization (Harel, 2013; Stylianides & Stylianides, 2009). Even if she might have developed some sense of deductive proof scheme during her participation in the study (if she had not applied the procedure just to please the researcher), she maintained the empirical proof scheme concurrently. This was an example of a robust research finding in the proof literature. Many students, even after learning about secure methods of proving, are reported to retain the empirical proof scheme (Education Committee of the European Mathematical Society, 2011). Phase 2 of the teaching experiment aimed to help the student move toward a non-empirical proof scheme before Phase 3 initiated her learning of the structure of the deductive proof. For this purpose, Phase 2 utilized an HLT developed by Stylianides and Stylianides (2009) to help undergraduate students realize the limitations of empirical arguments and see an intellectual need to learn about secure methods of validation, i.e., mathematical proofs (p. 348).

3.3.1.4. Phase 1 Basic Number Theory Concepts

Instructional decisions underlying the creation of the Phase 1 task sequence were already explained before in the section related to the exploratory teaching results. This section presents a summary of the task sequence in Table 3.4 below.

Table 3.4 Phase 1 Task Sequence

Task Sequence	Task Purpose(s)
1. Do you agree with Ms. Esin? (Esin Hanım'a katılıyor musunuz?)	Providing an example of a modular structure (6k+3) in a contextual situation, actuating the idea of making a divisibility inference about this structure-an unknown quantity, (dividing an algebraic expression by a number)
2. Can three kids share all the cookies evenly? (Kurabiyelerin tamamını üç çocuk eşit paylaşabilir mi?)	Partitioning configurations of cups and counting items representing various modular structures into a given number of equal groups (dividing an algebraic expression by a number), determining cases of perfect divisibility, indivisibility, and uncertainty
3. Can the number of cookies be 125? (Kurabiyelerin sayısı 125 olabilir mi?)	Investigating the number values a modular structure (6k+3) can and cannot take
4. Let's check out the situation in the cookie rack! (Kurabiye rafının durumunu inceleyelim!)	Investigating the effect of change in each cup (k) on the total amount of counting items (6k+3), and vice versa
5. What numbers can be the total number of cookies? (Kurabiyelerin toplam sayısı hangi sayılar olabilir?)	Listing in order the elements of the number pattern (3, 9, 15, 21, 27, …) that share the same modular structure (6k+3)
6. Let's learn the general representation (Genel Gösterimi Tanıyalım)	Given one of the three, determining two others of the associated (i) cups and counting items configuration, (ii) verbal description of the modular structure shared by the number pattern (iii) (ordered) elements of the number pattern Givens in the tasks followed the above sequence (i, ii, iii).

3.3.1.5. Phase 2 The Monstrous Counterexample Illustration

Stylianides and Stylianides (2009) developed an HLT to help undergraduate students realize the limitations of empirical arguments and see an intellectual need to learn about secure methods of validation, i.e., mathematical proofs (p. 348). Among the other theoretical considerations, the notion of cognitive conflict played a key role in participants' learning of the intended conceptions. Undergraduate students first worked on two problems. The Squares Problem fostered identification of a numerical pattern and posed a problem whose solution would be simple if the pattern was a true generalization. The next posed problem, known as the Circle and Spots problem (Brown, 2003; Harel & Brown, 2008), allowed generalizations for the first five cases but not for the forthcoming ones. Then, the Monstrous Counterexample Illustration that the Stylianides and Stylianides (2009) adapted from Davis (1981) was presented to the students to provoke a cognitive conflict in their empirical proof schemes. Lastly, the squares problem was revisited, where the students searched for a more secure way of validating the generalization involved and developed a proof with the help of the instructor.

> Consider the following statement:
>
> The expression $1 + 1141n^2$ (where n is a natural number) *never* gives a square number.
>
> People used computers to check this expression and found out that it does ***not*** give a square number for any natural number from 1 to 30,693,385,322,765,657,197,397,207.
>
> BUT
>
> It *gives* a square number for the next natural number!

Figure 3.4 The Monstrous Counterexample Illustration (Stylianides & Stylianides, 2009, p. 330; adapted from Davis, 1981).

The same approach was used in Phase 2 of this study, except that the construction of a first proof was left to the next phase. The two tasks preceding the Monstrous Counterexample Illustration were adapted to the sixth-grade level and the basic number theory content domain. However, a similar adaptation was not possible for the Monstrous Counterexample Illustration. The illustration in Figure 3.4 presented a conjecture that was validated for natural numbers up to an extremely large count (on the order of septillions) but failed for the next number. The concept of square root was not available to the participating student because it is a topic of the eighth-grade level (MoNE, 2018). Therefore, the process of squaring and square rooting was examined with the student until she prepared to make sense of the conjecture in the illustration. As Stylianides and Stylianides (2009) did, the name Monstrous Counterexample was not shared with the student.

Table 3.5 Phase 2 Task Sequence

Task Sequence	Task Purpose(s)
1. Problems/Let's think and decide! (Problemler/Düşünelim, Karar verelim!)	Familiarizing with conjectures as general statements to be proved or refuted, revealing the empirical proof scheme
2. The Monstrous Counterexample Illustration True or False? (Doğru mu? Yanlış mı?)	Realizing the limitations of empirical arguments
3. Revisit of arguments created for Task 1	Reflecting on the limitations of empirical arguments of one's own, seeing an intellectual need to learn about secure methods of validation

After an indication of cognitive conflict was observed, the student was asked to go back to and reflect on the empirical arguments she proposed before she was exposed to the illustration. Table 3.5 presents the task sequence with purposes underlying the inclusion of each task.

3.3.1.6. Phase 3 The CoA concept: Understanding the first instance of mathematical proof

In line with the LTA guidelines, an HLT for the intended CoA concept of the study was specified through the following steps (Simon, Placa, & Avitzur, 2016, p. 67):

1. Assess the relevant understanding of the learner.

At the beginning of Phase 3, the student demonstrated the understanding that a limited number of confirming examples do not offer a secure method for proving mathematical generalizations. The cognitive conflict approach used in Phase 2 seemed to trigger an intellectual need (the need for certainty) in the student for a more secure method of validation, which aligned with the necessity principle of the DNR instructional framework (Harel, 2008b). The necessity principle states, "For students to learn the mathematics we intend to teach them, they must see a need for it, where 'need' means intellectual need, not social or cultural need" (Harel, 2008b, p. 900).

Beren, although demonstrated this intellectual need for a more secure method for validating generalizations, was not able to prove any theorems, as expected. The results section includes evidence of Beren's thinking and understanding at the beginning of Phase 3.

2. *Specify the learning goal (intended abstraction).*

The learning goal for the student was to understand a connected chain of reasoning (any of the proofs analogous to the first instance proof) as an instance of mathematical proof. Understanding the proof involves seeing the whole chain of reasoning as a single united entity (a structural object) that proves the particular theorem. Based on the *duality principle* of the DNR instructional framework (Harel, 2021), the student's *way of* understanding the proof at hand, at this point, could not be supported by the deductive proof scheme (*the desirable way of thinking* that the student did not possess yet). *Duality Principle I* states that: "Learners come with a set of ways of thinking, some desirable and some undesirable, that inevitably affect the ways of understanding we intend to teach them" (Harel, 2021, p. 711). Although she had learned about the limitations of empirical arguments (in Phase 2) and eliminated her faulty way of thinking (the empirical proof scheme), she had no idea what kind of arguments would offer a secure method of validating mathematical generalizations. Indeed, the study aimed to help her develop the deductive proof scheme (the desirable way of thinking) by providing her a first instance of a proper way of understanding. Because, "[l]earners develop desirable ways of thinking only through repeated application of proper ways of understanding" (*Duality Principle II*, Harel, 2021, p. 711).

3. *Identify an activity or activity sequence that the learner already has available that could be the basis for the new abstraction.*

The activity guided the student to carry out the following two actions sequentially:

A_{0a}: Given an arithmetic number pattern, identify a modular structure shared by all its elements.

A_{0b}: Given a modular structure, make appropriate divisibility inferences.

The activity was expected to be the basis for the student's understanding of the logical necessity that the divisibility inference made about the modular structure is valid for all the numbers that share the same structure. Hence, the result of her activity ($A_{0a} \rightarrow A_{0b}$) is a proof to a particular theorem.

4. *Design a sequence of tasks that is likely to bring forth the learners' use of this activity and lead to the intended abstraction.*

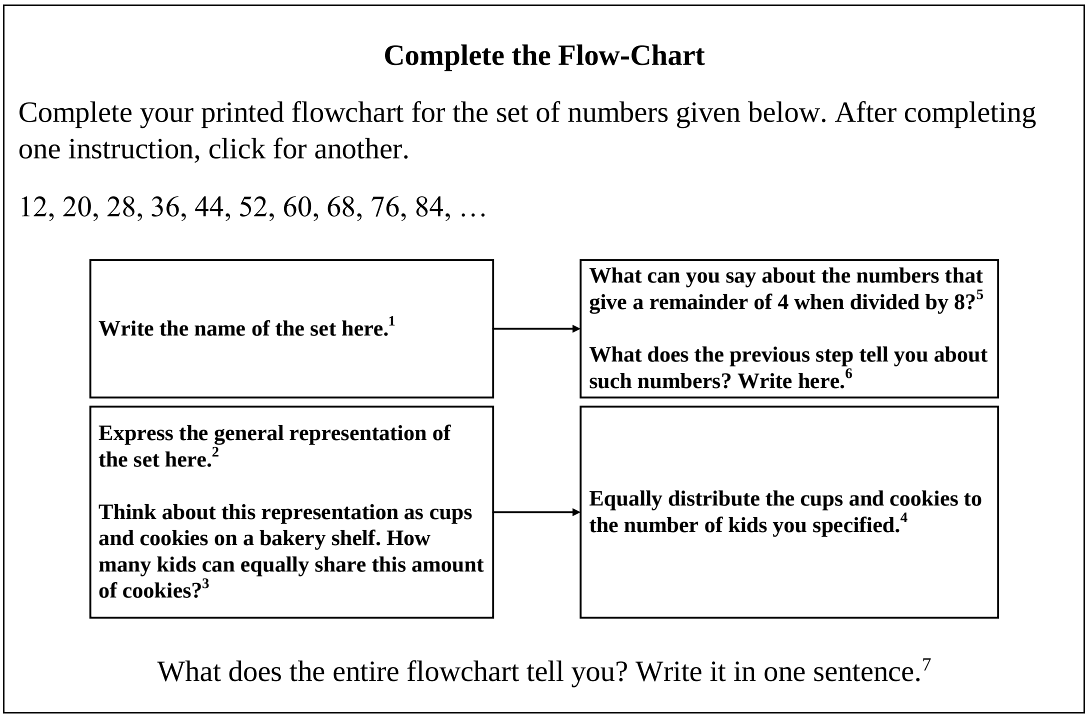


Figure 3.5 The PowerPoint slide presenting the task to the student

*: The student was given a flowchart printed on paper. The instructions to be followed were provided to the student in a PowerPoint slide. The student animated the flow of the steps at her own pace. The instructions appeared in the order indicated by the superscripts.

In the current study, the design of the task sequence was slightly different from the procedure outlined in the LTA instructional design framework. LTA research poses to students tasks that they are already able to solve by setting the task goal themselves. In the current study, however, no task goal available to the student seemed to help her call on the two actions identified above sequentially. Therefore, the task did not allow the student to set a single task goal herself. Instead, as given in Figure 3.5, the task asked her to complete a flowchart template by meeting the individual goals (sub-goals making up the entire activity) provided to her by the help of a PowerPoint slide, step-by-step. In other words, the task goal for the student was to complete the flowchart by following the instructions provided. Nevertheless, similar to the LTA approach, engaging in the activity GT ($A_{0a} \rightarrow A_{0b}$) alone did not require new learning on the part of the student. The hypothesized process was based on the student's available concepts.

The flowchart proof resulting from the student's activity was a proof of the conjecture "The numbers that give a remainder of 4 when divided by 8 are divisible by 4" (Figure 3.6). The hypothesis was that the student reflecting upon the complete flowchart would realize that the flowchart indeed conveyed a new type of connected meaning beyond the distinct actions carried out (participatory stage of the proof instance), and this meaning was related to the goal of proving a particular conjecture (anticipatory stage of the concept proof). Note that the latter part of the hypothesis concerned connecting the student's abstraction from her activity to a new goal G_1, namely, the goal of validating a mathematical generalization. Table 3.6 presents the hypothesized learning process from the available concepts to the construction of the CoA concept through the student's activity.

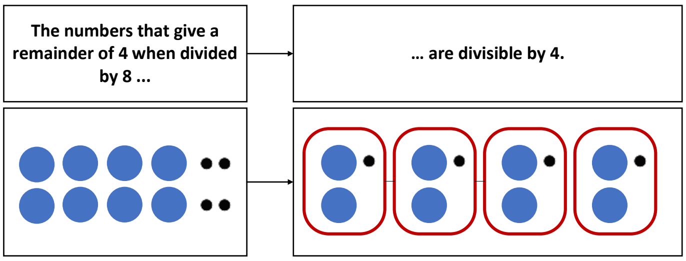


Figure 3.6 Expected product of the student's activity

Originally in LTA definition of abstracting a CoA concept, the result of the coordination of actions replaces the sequence of actions coordinated with a single higher-level action. This is viewed as anticipating the result of the activity without carrying it out sequentially (Simon, 2020). However, in the current study, replacement of the activity was neither likely nor desired. Instead, the study aimed to help the student anticipate that carrying out the two actions sequentially creates a chain of reasoning (not a single action but a single connected object) that meets the goal of proving a specific conjecture. Therefore, the intended learning might be compatible with the notion of anticipating the result of an activity without carrying it out and can be considered a case of coordination of actions (Simon, 2020). But this coordination has a serious limitation in that the first concept, a general representation (the modular structure) of a given number pattern, is not a concept that the student knows when to call on. The student uses this concept in her activity because she was instructed to do so. This is the main issue Phase 4 (and Phase 5) aimed to deal with.

Table 3.6 The hypothesized learning process for Phase 3

The sequence		The concept
An available concept from Phase 1 (G_{0a}-A_{0a})	G_{0a}:	To create a general representation of a number pattern given (12, 20, 28, 36, 44, 52, 60, …)
	A_{0a}:	Use cups in the number of the constant difference between each term plus counting items in the number of the first term (8k+4, k ∈ Z.)
Another available concept from Phase 1 (G_{0b}-A_{0b})	G_{0b}:	To check the divisibility of a given cups and counting items configuration by a number (determine divisibility of 8k+4 by 4)
	A_{0b}:	Equally distribute the given configuration to the given number of groups (divide 8k+4 by 4, check the remainder)
The activity (G_T (G_{0a}-A_{0a} →G_{0b}-A_{0b})/ G_T (A_{0a}→A_{0b}))	G_T:	To complete a given flowchart (The participatory stage)
	A_1:	Apply the result of A_{0b} to all the numbers that allow A_{0a} (The result of the divisibility check applies to all the elements that share the same cups-counting items structure)
The CoA concept being constructed (G_1-A_1)	G_1:	To validate a generalization (The anticipatory stage)
	A_1:	Apply the result of A_{0b} to all the numbers that allow A_{0a} (Conduct the actions A_{0a}→A_{0b}) *

*: This formulation is new and is used in the context of this study only.

The instructional design did not include separate tasks for promoting the transition from the participatory stage of the concept to the anticipatory stage. The transition was thought of as more likely to be an instant shift of attention from the process of carrying out the activity to the perception of the resulting object. To shift the student's attention as intended, the seventh step in the PowerPoint slide asked the student to reflect on the result of her activity: "What does the entire flowchart tell you? Write it in one sentence." Especially the last direction, "Write it in one sentence" was expected to foster the student's connecting the flowchart proof to the conjecture it is a proof to. Another task in the sequence, from the perspective of gradual sophistication (instead of an instant shift), followed Harel's (2013) observation regarding students' intellectual need for causality. Harel (2013) reported that through repeated experiences of comparing two types of solutions to problems, students gradually come to realize that causal solutions are of more intellectual value than empirical solutions. In the current study, the student was asked to compare the emerging flowchart proof resulting from her activity to an empirical argument proposed by a fictional student for validating the same conjecture.

During the teaching experiment, Beren first encountered the task in Figure 3.5, but did not learn all the aspects of the learning goal. Then, after including a modified version of the task in the task sequence that uses case-by-case flowcharts and integrates Simon et al. (2010)'s notion of mental run, she developed a better understanding of the intended CoA concept. Decisions regarding the inclusion of the new task are explained in the findings section, close by the analysis of the learning process.

3.3.1.7. Phase 4 The AoC Concept: Modular structure as a way of representing all the numbers in an arithmetic number pattern to prove a class of conjectures

Assume that the student coordinated the two sequential actions and connected the result of this coordination ($A_{0a} \rightarrow A_{0b}$) to a new goal ($G_1$: proving a conjecture); the student now understands a first example of mathematical proof that proves one particular conjecture. However, she may not know how to go about proving another given conjecture, because she may not yet be aware of the role of the concept definition used to start the proof. In the first instance activity, she does not call on the concept modular structure (as a way of defining the particular set of numbers) herself, but uses it properly only because she is instructed to do so by directions such as "Write the name of the set here" and "Express the general representation of the set here". Thus, it may not be immediate for her to understand the purpose of beginning the activity, the proof, with this particular concept (G_{0a}-A_{0a}).

> *In the LTA elaboration of reflective abstraction, the goal component of a concept determines the situations for which the related action can be called on. That is, only when the learner sets a goal (GT) compatible with the goal component of the concept (G_0 of G_0-A_0) can that concept be called on. (Simon, Kara, Placa, & Avitzur, 2018, p. 101)*

The concept of modular structure, as the student is expected to learn in Phase 1, is not a concept that she can call on to solve any given task. Why should one look for modular congruence in a set of numbers (set the goal G_{0a}: to create general representation of a number pattern) if she is not to define the pattern? Why should one define a set of numbers by using the concept of modular structure if she is not to make an inference about all the elements in the pattern? The purpose of defining (in our case) gains its meaning only if one has a purpose of generality in her argument. This points to the following dilemma. One of the following must be accomplished before the other: Either the concept modular structure, which takes on its meaning in a particular proof, must be constructed first, or a particular proof that makes use of the concept should be understood as a proof. This study was initiated based on the latter possibility. The particular flowchart proof in Phase 3 used a rudimentary concept of modular structure developed in Phase 1 to construct what a proof is first.

Phase 4 aimed to help the student develop the concept "modular structure" in relation to the act of proving theorems analogous to the first instance proof. It was considered to be the main idea of the first instance proof: using the modular structure shared by all the elements of an arithmetic number pattern (if exists), for making divisibility inferences about all the numbers in the pattern. The intended understanding was defined as the AoC concept of the study. "An AoC concept is a mathematical structure that can be used to sort mathematical experiences into examples and non-examples" (Simon, 2020, p. 4). The goal for the student in the present study was to learn to sort the world into cases when using general representation (standing for the modular structure of numbers) would and would not be useful to make inferences. If it is possible to identify a modular structure shared by all the individual numbers in a given conjecture, then it might be possible to make a divisibility inference about the whole set of numbers. The hypothesis was that by abstracting the commonality in her repeated activity of proving analogous conjectures, the student may come to understand the role of starting the particular type of proofs with the concept modular structure. This would be a first set of experiences of definitional reasoning for the student.

Specific to this study, the AoC concept was defined within the context of the proof within which it became a useful concept, and more precisely became the proof's main idea. Therefore, the basis for the student's reflective abstraction was the entire activity sequence of creating the same-structure proofs. Simon (2020) outlined the basic design principles for promoting an AoC concept as establishing an informal activity for the student and making the student use this activity to solve word problems by representing the activity using arrow diagrams. Word problems are important because they present the situations of the common mathematical structure that the student is about to learn to identify. The arrow diagrams, by representing the activity to the student, help the student consider cases and non-cases of the situations for which the activity is applicable. When the student comes to use the diagram for appropriate word problems only, this means the student comes to recognize the intended mathematical structure. This is when the abstraction of commonality is completed (Simon, 2020). In the current study, the activity of creating a connected chain of reasoning was already established in Phase 3. Phase 4 posed tasks where the student was asked to prove a sequence of conjectures repeatedly. However, in contrast to the related design principle, really few of these tasks used word problems to ask for proofs. This was, first, due to the belief that the conjectures given in the form of full sentences could do the same function of presenting common mathematical structure, and second, because this way of requesting the activity would be more compatible with the purpose of the reasoning-and-proving activity engaged (given a conjecture provide a proof). The specific flowchart proof format used in this study played the role of an arrow diagram, helping the student keep track of her activity. The task sequence engaged the student in arguing for or against conjectures of three types, where the inferences of absolute divisibility or absolute indivisibility could be made, or no inferences could be made at all (see Table 1.4 for a list of the conjectures representative of each type).

3.3.1.8. Phase 5 The extended AoC Concepts (AoC-E1 and AoC-E2)

The last phase of the study had two-fold purposes. On the one hand, this phase aimed to provide the student with more cases of definitional reasoning. On the other hand, it aimed to provide a useful data set to answer the third research question of the study:

> How does the student (attempt to) start proving (a) conjectures that are similar (but not analogous) to the first instance conjecture, (b) the target conjecture of the study and (b) a novel conjecture?

At the end of Phase 4, the student was expected to be able to prove theorems analogous to the first one. But, for proving any novel theorem she would need to develop an appreciation of the notion of mathematical definition. However, this study did not design a way of triggering an intellectual need for this specific way of thinking. Rather, the study applied the approach of engendering repeated ways of understanding to promote its development. Therefore, Phase 5 of the teaching experiment promoted new learning to enable student's proving of new classes of conjectures, by introducing the extended AoC concepts gradually. Since the research question aimed to be answered here is not about how the new learning takes place on the part of the extended AoC concepts, instructional approaches used in this Phase are explained in the findings section, in close proximity to interpretations of the relevant data. The task sequence engaged the student in arguing for or against many conjectures. These conjectures mostly used the extended AoC concepts of the study (see Table 1.4 for a list of the conjectures representative of each type) and some others used non-examples of the intended structure.

3.3.2. Data collection tools

The major data sources for the study were the audio and video recordings of the one-on-one teaching experiment sessions and the records of participating student's written work. The majority of the tasks requested the student to prove conjectures and think aloud while working on any of the given tasks. Probing (and occasionally prompting) questions were used to capture details of the student's understanding and thinking. The most important aspect of the data collection process was regarding the sequencing of the tasks so that each task carried to potential to capture pre-assessment evidence before any subsequent learning instance. The researcher's field notes on her evolving conjectures of the student's learning, and a second elaboration of these conjectures resulting from her meetings with her supervisor and co-advisor provided a secondary data set to support the retrospective data analysis procedures.

3.4. Data analysis

This study reports retrospective analyses of data from the third, fourth and fifth phases of the LTA teaching experiment. Evidences from the first and the second phases are consulted only when needed for supporting interpretations of the data from the primarily focused phases. Phase 3 data on the construction of the CoA concept was analyzed based on the LTA theoretical account of conceptual learning (Simon et al., 2018). The process of how the student constructed a new higher-level concept from two initial concepts available to her was described based on the goal-action definition of a CoA concept. Any indications of understanding or lack of understanding of the deductive relations involved in the proof were interpreted in terms of the resulting goal-action complex. The model of students' ways of understanding a proof

(Ahmadpour et al., 2019) that was introduced in the literature review section guided capturing of appropriate indications. Data from Phase 4 and Phase 5 were analyzed with a focus on the student's goal-setting behavior in her attempts to prove the given conjectures. The resulting accounts of data were mostly descriptive of the structures the student seemed to have abstracted from her repeated experience of proving. Analyses of the two phases primarily aimed to capture the existence or non-existence of any indications regarding the target definitional reasoning in the student's thinking.

3.5. Trustworthiness

In order to ensure the trustworthiness of the findings, this study employed the following procedures offered for qualitative research methods: researcher role, peer review, and rich and thick description (Merriam, 2009).

Researcher role: The LTA research methodology restricts the role of the researcher to that of a problem poser who cannot present task solutions to the participating student or direct her towards the expected solutions by influencing her thinking beyond what the posed task elicits. She is only allowed to conduct the sequencing and timing of the tasks included in the designed task sequence in response to the student's response. The retrospective analysis aims to capture and eliminate the points at which the researcher's actions (unintentionally) might have caused impacts on the student's thinking before creating models of the student's learning (Simon, 2018, p. 116).

The first four phases of the current study endeavored to meet this LTA restriction on the role of the researcher strictly. However, the fifth phase involved more flexibility in this respect in an effort to capture any instance(s) of definitional reasoning for which an accurate assessment tool was missing. In terms of the second learning goal of the study, several times the researcher exceeded her prescribed role and made comments that might promote definitional reasoning. The Findings section addresses these points clearly, none of which produced the desired way of thinking. Therefore, it did not harm the credibility of the findings. Instead, it might be considered to provide empirical evidence to Harel's (2008c) claim that "[p]reaching ways of thinking to students would have no effect on the quality of the ways of understanding they would produce" (p. 20).

An important aspect of the researcher's role included building mutual trust between the researcher and the student so that the student would explain her thinking without hesitation and not engage in potential behaviors to please the researcher. For this purpose, the researcher mostly refrained from indicating her own thinking or dictating her own mathematical truth and left final decisions to the student. For example, in response to a critical yes-or-no question from the student (regarding the truth of a conjecture), she said she did not know the answer. It is believed that her consistent behavior throughout the phases improved the mutual trust between them.

Additionally, the nature of the LTA teaching experiment methodology required the researcher to document her hypotheses regarding the student’s knowledge before the data collection process and her learning during the teaching sessions. Any potential bias in the data collection and analysis process should have already been noted in this research report.

Peer review: The peer review process includes a knowledgeable peer inspecting the plausibility of conclusions made by reviewing sections of the raw data (Merriam, 2009). In the current study, this process was conducted by the two supervisors of the researcher. In addition, any potential reader of this report necessarily engages in a peer review process because the findings section presents almost complete documentation of the raw data to support claims of learning between the LTA’s two end-points: "a point when the student did not have the concept and a later point when the student had learned the concept... No claim about the learning process is trustworthy without evidence of these endpoints” (Simon, 2018, p. 116).

Rich and thick description: Similar to the findings section, the other sections of this report include detailed information on the critical aspects of the study. These aspects include the design, theoretical framing, initial hypotheses, context, the individual participants of the study, and the procedures followed to reach conclusions. This level of detailing is instrumental to the comparison of the learning examined here (both successful and unsuccessful) to other similar research, thereby aiding in the construction of a theoretical foundation for the learning and teaching of proof.

CHAPTER 4

FINDINGS

This chapter presents the findings of the study in three main sections. Section 4.1 explains the sixth-grade student Beren's process of coming to understand a first instance of mathematical proof—the intended CoA concept of the study. After that, Section 4.2 presents her process of developing the main idea of this first instance proof in such a way that it is transferable to the act of proving analogous conjectures—the intended AoC concept of the study. Lastly, Section 4.3 describes her process of learning to prove a set of new conjectures, with the major focus being on the extent to which her CoA and AoC concepts serve her knowledge base (enable or constrain her goal-setting behavior). This last section provides data on how she approaches the tasks of proving (a) conjectures that are similar but not analogous to the first instance conjecture, (b) the target conjecture of the study, and (c) a novel conjecture.

Data for Section 4.1 comes from the first part of the study, consisting of the first three phases of the teaching experiment. Data for Sections 4.2 and 4.3 come from the second part of the study, namely the fourth and fifth phases focusing on definitional reasoning.

4.1. The construction of the CoA concept: Understanding the first instance of mathematical proof

This section presents Beren's process of coming to understand the first instance of mathematical proof as a result of her participation in the first three phases of the teaching experiment. Phase 3 contained the major task designed for constructing the intended CoA concept and was preceded by Phase 1 and Phase 2, both of which aimed to prepare the student for participation in this major task. Phase 1 introduced the basic number theory concepts needed for her engagement with the major Phase 3 task successfully, and Phase 2 triggered her intellectual need for the concept to be learned in Phase 3—a (non-empirical) secure method of validating mathematical generalizations. Beren's learning from the originally planned Phase 3 task, compared to the learning goal, led to a redesign of the major task into a new set of tasks. The modified task sequence, using a different (case-based) flowchart structure and integrating Simon et al. (2010)'s notion of mental run, fostered her abstraction of the learning goal as intended and introduced modifications to the hypothesized learning process.

The section begins with a description of Beren's prior knowledge and skills at the beginning of Phase 3. Particularly, the sections 4.1.1 and 4.1.2 provide evidence that right before her participation in the Phase 3 task, Beren demonstrated the prerequisite understandings needed (the ways of understandings developed during Phase 1) but not the intended CoA concept of the study (a way of proving conjectures by using those understandings appropriately). The evidence for the latter is of vital importance to the analysis of Phase 3 data because the LTA teaching experiment methodology sets such evidence as a requirement before attributing students' learning to the activity elicited by the task sequence.

After Beren's prior knowledge and skills are introduced in the next two sections, the third one describes her interaction with the originally planned task of Phase 3. An account of her learning is elaborated in relation to her overt activity, followed by an explanation of how this elaboration led to the design of a more sequential task for the same purpose. The modified task sequence is introduced, and Beren's engagement with this new task sequence is described. Afterwards, a second account of Beren's learning is generated based on her engagement with the modified task sequence. The data for this first part of the study comes mostly from Phase 3 of the teaching experiment, including both the originally planned task and the modified task sequence. Conceptions generated in Phase 1 and Phase 2 are referred to only to enable an explanation of the learning that occurred in Phase 3. It is important to note that during Phase 1 and Phase 2, the LTA methodological principle of avoiding an influence on the student's interaction with the tasks was followed with the same level of rigor as in the other phases of the study. At the points when concepts were not fully available to the student, i.e., the goal component of the concept modular structure, the gap was filled by reminding her of the representation system externally. In any other task, Beren maintained the role of an individual problem solver, similar to what the following parts of this section demonstrate about her participation. The findings section might seem to include more detail than needed about the student's activity and speech. This level of detail was seen as necessary to provide accurate evidence of the inferences made. This was particularly due to the fact that the data analysis procedure included frequent references to previous sections of the data set.

4.1.1. Beren's prior knowledge at the beginning of Phase 3

4.1.1.1. Her knowledge of the basic number theory concepts (Phase 1 outcomes)

The intended CoA concept of the study was to structurally understand a flowchart proof of the conjecture "The numbers that give a remainder of 4 when divided by 8 are divisible by 4" as a first instance of mathematical proof. Proving the conjecture included the steps of creating a general representation of the numbers listed in 12, 20, 28, 36, 44, 52, … and making the related divisibility inference out of this representation. At the end of Phase 1, Beren was able to make conversions between three images of an arithmetic number pattern: the ordered list of its elements, its general representation, and its description as "the numbers that give a remainder of … when divided by …" As exemplified in Figure 4.1, given any one of these three images, Beren was able to provide the other two correctly.

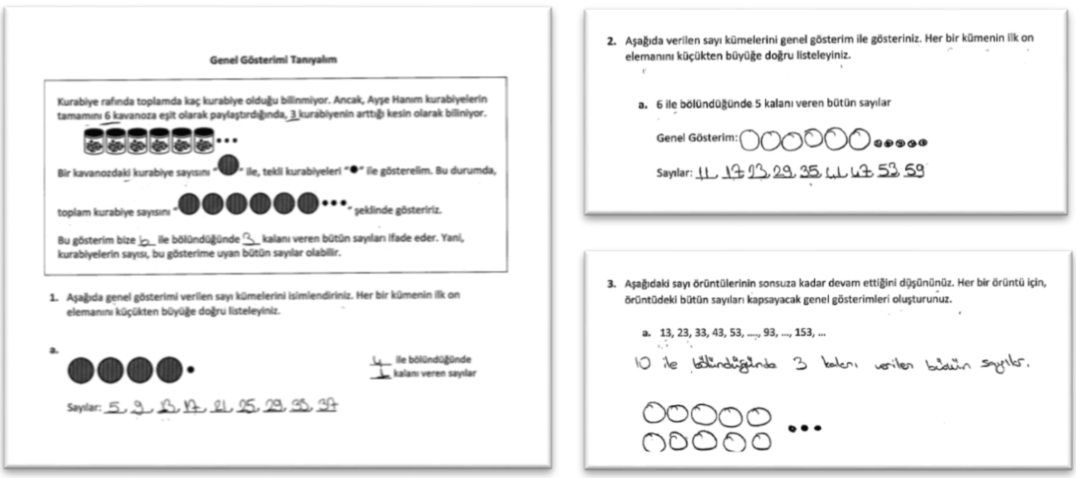


Figure 4.1 Beren's responses to the three exemplary tasks from the Phase 1 task sequence: Let's learn the general representation (Genel Gösterimi Tanıyalım)

Beren was not expected to know when to "call on" the above concepts and use them purposefully in solving a task. Her concepts, at that moment, were at the participatory level only, since the goal of defining a set of objects was not yet available to her.

Nevertheless, her competence with the conversion tasks was thought to support her future engagement with the major Phase 3 task. Namely, she would have the ability to create the “name” and the “general representation” for a given arithmetic number pattern when the major Phase 3 task later instructed her to do so.

Despite her skills in executing the three actions (listing, naming, and representing) correctly, she was not expected to use them goal-directedly to prove a given conjecture as well. As expected, for validating the two conjectures given below, Beren did not spontaneously think to employ the related actions but provided empirical arguments. Conjecture A was administered as the last task of the Phase 1 task sequence. It was placed in this specific order with the purpose of assessing the presence or absence of the intended CoA concept of the study. Immediately before this task, Beren had just practiced the general representation (given in Figure 4.1) which could offer her a way of dealing with all the numbers in the conjecture domain. Right after that, this last task aimed to assess whether or not she would use this way of representation purposefully to prove the related conjecture. The second conjecture below, Conjecture B, was mathematically equivalent to Conjecture A but could be pedagogically easier due to its way of presentation. It was one of the conjectures examined before and after the Monstrous Counterexample Illustration in Phase 2 (necessarily before Phase 3). In both cases, Beren demonstrated the empirical proof scheme. That is, she selected a few convenient cases and examined the truth of the statements for her selections only.

Conjecture A
“All the numbers in the sequence 12, 20, 28, 36, 44, 52, 60, 68, … are divisible by ___.” What are the numbers that can fit in the blank space in the sentence? (Beren’s written answer: 2 and 4.)

Beren : *12 is divisible by 2. 20 is divisible by 2. All the numbers here are divisible by 2. I am writing the 2. (Evaluated other alternatives.) I think only 2 and 4.*
Researcher : *Well, let us consider that you fill in the blank with 2. We say ‘are divisible by 2’. And I'll ask you how can you explain this to me? Why are they all divisible?*
Researcher : *Are you going to count them all one by one?*
Beren : *No, we don’t need to count them all. We look at the pattern here, how much it gets increases (up to the last number listed before ellipsis) and after that, we add a few more (steps forward) as I did (she applied this method for alternatives other than 2), and*

if they follow the same rule, that proves it. […] Well, one can find it by trying, in short.

Conjecture B

"Numbers that give a remainder of 4 when divided by 8 are divisible by 4 without a remainder." Is the statement true or false? Why? (Beren's written answer: It's true. Because every number that gives a remainder of 4 when we divided by 8 are divisible by 4 without a remainder.)

Beren : *Hmm… Gives a remainder of 4 when divided by 8. For instance, 28.*
Researcher : *How did you find?*
Beren : *Well, I counted as 8, 16, 24, and from 24, for example, if it is 8, 16, it could have been 20 as well, but first, I thought of 28 for a moment, so let me continue from that. 8, 16, 24. I added 4 over 24 and, 28. When I divided 28 by 8, the remainder was 4. Then, when I divided 28 by 4 it is divided without a remainder.*
Researcher : *OK.*
Beren : *The same thing happens with 20, 4, 8, 12, 16, 20, and with 4, 8, 9, 10, 11, 12, I mean 4, 8, 12.*
Researcher : *What do you think?*
Beren : *I think it is true. I am writing here again. Well, I will cross out the "false".*
Researcher : *OK. Could you briefly explain why?*
Beren : *True, because in fact I tried […] Every number that gives a remainder of 4 when we divide it by 8 already divisible by 4 without a remainder. I think, this is a rule. I am writing.*

One may argue that Beren might not have used her concepts of modular structure because her empirical proof scheme was prominent. However, even after her empirical proof scheme was eliminated through the Phase 2 task sequence, Beren was not able to prove Conjecture B above.

4.1.1.2. Her proof scheme (Phase 2 outcomes)

At the beginning of Phase 3, Beren demonstrated a *non-empirical proof scheme*—the conception that empirical arguments do not offer a secure method for validating mathematical generalizations (Stylianides & Stylianides, 2009). She developed this conception during Phase 2, after being exposed to the Monstrous Counterexample Illustration (Stylianides & Stylianides, 2009).

The researcher mediated the flow of the illustration on a single PowerPoint slide (as in Figure 3.4). Her first hit on the enter key introduced a conjecture to be explored: "The expression $1+1141a^2$ (where n is a natural number) never gives a square number”. Although the concept of a square number was reviewed before the illustration, the conjecture was not fully accessible to Beren due to her prior knowledge. She was not expected to instantly develop the reverse process of taking the square root of $1+1141a^2$. The researcher turned her attention from her way of understanding the particular conjecture to her faulty way of thinking associated with the act of proving the conjecture—her *empirical proof scheme*.

Researcher : This is a bit of difficult for you right now, but I want to show you something else. Now, what do we do to try this? We take this “a” to be 1...
Beren : We take it as 2. We take it as 3. We try to put in some number, if it holds, it’s true. If not, it’s wrong. But we will try more than five. (In the previous Phase 2 session, she had made the decision that the truth of conjectures must be checked for at least five cases.)
Researcher : You say more than five.
Beren : Exactly.
Researcher : Now (hits the enter key) if you wish, you can read yourself.
Beren : People used computers to decide whether this sentence was true or not, and from 1 to thirty, six hundred ninety-three, three hundred eighty-five ... (thought the twenty-six-digit number was a comma-separated list of three-digit numbers.)
Researcher : This is a single number.
[...]
Researcher : No, not that. This is one single number. It has digits for millions, and trillions and ...
Beren : Huh! I get it. So, this is just one number (looks amazed). Then, ... up to that number ... they found that for all numbers up to that number, this expression does not equal to a perfect square number. I mean, they tried from 1 to this number.
Researcher : Yes, and because it is very difficult to try it manually, they try it with a computer program.
Beren : It is not possible to try manually at all.

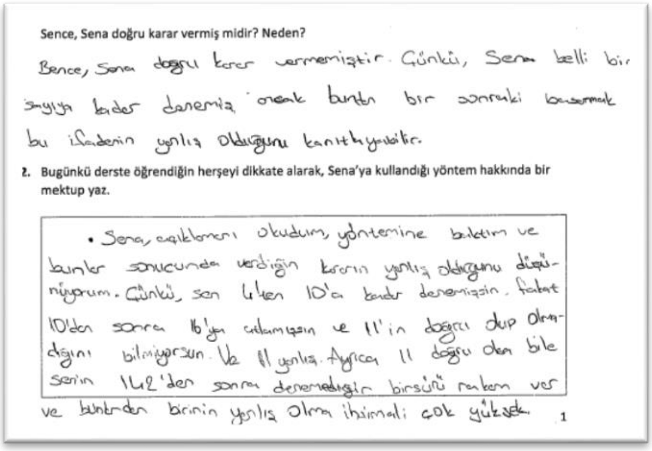
With the last two hits on the enter key, Beren continued reading the expressions that sequentially appeared on the computer screen: (i) “But” (accompanied by a sound effect to attract the student’s caution to the next coming item) ... (ii) “This expression gives a square number for the next natural number (the number after the twenty-six-digit number, which is a counterexample to the conjecture in question).” After a short silence of thinking and consternation (7 seconds), she continued:

Beren : Well, it says... try up to a certain point. But you know ... I can’t try till forever. (Sounds helpless)
Researcher : What does that mean?
Beren : Something should be devised for this urgently.

Researcher	:	*What kind of a thing?*
Beren	:	*Well, something that will try for all these numbers, quickly, but all the numbers that mind…I mean all the numbers that are found up to now and decide whether [it is] true or not. Because, the number that I could not read, I mean, think about it, I thought these were each a number apart, and read them so… if after trying up to [a] this [big] number, the forthcoming [number] gives a perfect square, that means I will wipe out on the exam.*
Researcher	:	*What do we understand from here?*
Beren	:	*So, trying for five of them is not enough. That is, I guess.*
Researcher	:	*[What about] ten of th-…*
Beren	:	*Even that is not [enough].*
Researcher	:	*What do we take from this?*
Beren	:	*From this, that … we need to find a way other than trying.*

Besides marking a switch to the *non-empirical poof scheme*, the above dialog also demonstrates that Beren developed an intellectual need for a secure method for validating mathematical generalizations. The two particular lines, "Something should be devised for this urgently" and "we need to find a way other than trying," indicated her feeling of the intellectual need. Moreover, her emphasis on the great extent of the domain of the conjecture indicated that she made sense of the "generality" characteristic of the *deductive proof scheme*. Her phrase "something that will try for all these numbers, quickly, but all the numbers that mind... I mean all the numbers that are found up to now," aligns well with the mathematical notion of proof as a "for all" argument without exception (Harel, 2008a). However, Beren had no clue what a valid argument would look like.

Following the Monstrous Counterexample Illustration, Beren was asked to evaluate a fictional student Sena's empirical argument for the conjecture "any integer greater than 3 can be expressed as the sum of two prime numbers" and write a letter to Sena about her method of validation, considering what she had learned in the particular teaching session. Her letter, given in Figure 4.2, pointed to the existence of a true counterexample that Sena has not treated due to the skip-selection of cases and the potential existence of others that might not be captured through case-by-case examination of a finite number of cases from an infinite domain. When the researcher asked, "What would you recommend to Sena?" Beren responded, "I can't recommend anything, because you know I myself do the same as hers, unfortunately." She was not able to prove the Conjecture B either, as stated before. These observations together provide evidence that Beren had no available way of proving conjectures, including the specific class of conjectures referred to as the "first instance proof" of the study.


Sence, Sena doğru karar vermiş midir? Neden?

Bence, Sena doğru karar vermemiştir. Çünkü, Sena belli bir sayıya kadar denemiş ancak bundan bir sonraki basamak bu ifadenin yanlış olduğunu kanıtlayabilir.

2. Bugünkü derste öğrendiğin herşeyi dikkate alarak, Sena'ya kullandığı yöntem hakkında bir mektup yaz.

• Sena, açıklamanı okudum, yöntemine baktım ve bunlar sonucunda verdiğin kararın yanlış olduğunu düşünüyorum. Çünkü, sen 4'ten 10'a kadar denemişsin. Fakat 10'dan sonra 16'ya atlamışsın ve 11'in doğru olup olmadığını bilmiyorsun. Ve 11 yanlış. Ayrıca 11 doğru olsa bile senin 142'den sonra denemediğin birsürü rakam var ve bunlardan birinin yanlış olma ihtimali çok yüksek.

1

1. I think Sena didn't make the right decision. Because Sena tried up to a certain number, but the next digit may prove that this statement is wrong.

2. Sena, I read your explanation, looked at your method, and as a result of these, I think your decision was wrong. Because you tried from 4 to 10. But you jumped to 16 after 10 and you don't know if 11 is correct or not. And 11 is wrong. Also, even if 11 is correct, there are a lot of numbers that you haven't tried after 142, and it's very likely that one of them is wrong.

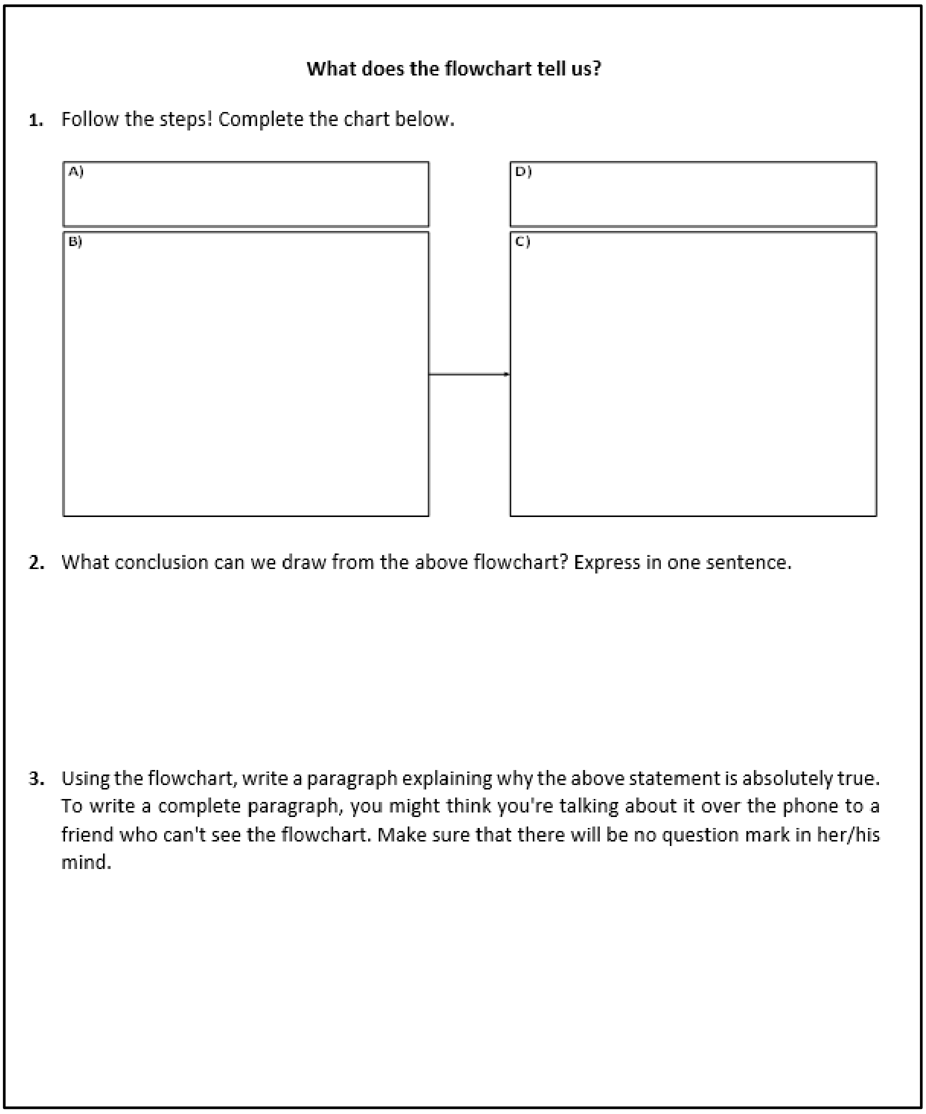
Figure 4.2 Beren's response to a fictional student Sena about her empirical argument

4.1.2. Beren's engagement with the original task designed for the CoA concept

The original Phase 3 task asked Beren to complete a blank flowchart printed on an activity sheet (see Figure 4.3) for a given set of numbers (12, 20, 28, 36, 44, 52, 60, 68, 76, 84...). The instructions to be carried out to complete the flowchart were provided via a PowerPoint slide (Figure 3.5). The following items sequentially appeared on the screen as Beren completed a given step and pressed the enter key for the next one.

Instruction#

1. Write the name of the set here. [Box A]
2. Express the general representation of the set here. [Box B]
3. Think about this representation as cups and cookies on a bakery shelf. How many kids can equally share this amount of cookies? [Box B]
4. Equally distribute the cups and cookies to the number of kids you specified. [Box C]
5. What can you say about the numbers that give a remainder of 4 when divided by 8? [Box D]
6. What does the flowchart tell you about such numbers? Write it here. [Box D]



What does the flowchart tell us?

1. Follow the steps! Complete the chart below.

A) D)

B) C)

2. What conclusion can we draw from the above flowchart? Express in one sentence.

3. Using the flowchart, write a paragraph explaining why the above statement is absolutely true. To write a complete paragraph, you might think you're talking about it over the phone to a friend who can't see the flowchart. Make sure that there will be no question mark in her/his mind.

Figure 4.3 The originally planned task

Instruction 1 was not immediate for Beren. She was allowed to press the enter key a second time and think about Instructions 1 and 2 together. She did not spontaneously recall the actions related to arithmetic number patterns that she successfully used in Phase 1, especially the term “general representation.”

Beren : (trying to name the set) Should I say ‘the set of numbers increasing by eight, and even’? How do we do the sets?

Researcher : You can think about the number sequence. You don’t have to think of it as a set.

Beren : Okay.

Researcher : These numbers have one characteristic in common. (Several times, including this one, the researcher exceeded her prescribed role, and Beren received the help of the questions to adjust her relevant way of understanding but did not develop the related way of thinking.)

Beren : *Okay. The difference between them is eight. They're all even numbers. But there's one more thing. It can't be that simple.*
Researcher : *We've done similar things with you over the past few weeks.*
…
Researcher : *Well, we were writing like this. Let me give you a little reminder.*
Beren : *Would be great.*
Researcher : *We would put these numbers in a group called "the numbers that give 'some' remainder when divided by 'some number'."*
Beren : *Oh! OK thanks. Then, it can be numbers that are perfectly divisible by 4, for example. Because, 12 is perfectly divisible by 4. 20 is divisible. 28 is already divisible. It is 9 in 36, and 44 also. Because 36, 40, 44.*
Researcher : *Well, is 8 there in the list? (The researcher here attempted to provide her with a counterexample. But, unclear how, this made Beren remember the strategy she had been using to name the given arithmetic number patterns.)*
Beren : *8? 8, 16. Numbers that give a remainder of 4 when divided by 8. (Sounds sure she has found the correct description) Because, 8, 16… 8; there is a difference of 4 between 8 and 12. 8, 16…16; there is a difference of 4 between 16 and 20. 8, 16, 24… 24; there is a difference of 4 between 24 and 28. 8,16, 24, 32; there is a difference of 4 with 36. In this way. It is the same for 44 as well, for 84 as well.*

The last line of Beren is representative of the strategy she consistently used for finding out the verbal description corresponding to the given arithmetic number patterns, but only when she remembered about this procedure. Specifically, for arithmetic number patterns that share the same modular structure, such as $qk+r$, in order to find the right description, "the numbers that give a remainder of r when divided by q," she counted by (candidate) *q's* (as with by 8's in the above case) and added a constant remainder r (4 in the above case) each time to her count to see if the numbers she got matched the numbers in the pattern.

As described above, she has incidentally found the verbal description of the set of numbers 12, 20, 28, 36, 44, 52, 60, 68, 76, 84... as "the numbers that give a remainder of 4 when divided by 8." However, the description did not lead to the creation of the related general representation 8k+4 with ease. As she struggled to recall the representation system, the researcher helped her within the boundaries of Phase 1 learning goals.

Beren : *(re-reads Instruction 2: Express the general representation of the set here. [Box B]) How was the general representation?*

		Well, I remember. Now, we will divide 12 by 8 and show that there remains 4. Is not that?
Researcher	:	*Do you mean the (long) division operation?*
Beren	:	*Yes. I don't quite remember the general representation. I am so sorry.*
Researcher	:	*No, please. I'll remind you right away.*
Beren	:	*Would be great.*
Researcher	:	*We called the general representation the following, like we were thinking about cookies are in the cups and the single-packed ones outside of the cups. We drew the cups in circles.*

Beren continued thinking and trying for a while, but failed to remember the idea of cups and cookies. Then, the researcher showed her "General Representation" introductory statement (Figure 3.3). Since her concepts related to the general representation (i.e., goal: to name a given arithmetic number pattern; action: provide the description "numbers that give a remainder of... when divided by..."; goal: to represent a given arithmetic number pattern; action: provide its cups and counting items configuration) were participatory stage concepts each, they were expected to be available to her only when Phase 1 activities, especially the contextual situation of the bakery story, were present. Beren talked through the representations of the specific cases: 12, 20, 28, 36, and 44.

Beren	:	*Here (in the "General Representation" introductory statement) it is divided equally into 6 cups, but here (in the current task) our number (12, the number of cookies) is known. We'll be distributing 12 cookies and I have to have 4 leftovers.*
Researcher	:	*Yes. And how many cups?*
Beren	:	*I will draw my 8 cups. There will be one for each. I will have 4 leftovers.*
Researcher	:	*You do this for the number 12.*
Beren	:	*Yes.*
Researcher	:	*What was the next number after 12?*
Beren	:	*12… well, 20.*
Researcher	:	*It was 20. How many will there be in each cup when it's 20?*
Beren	:	*2. In others it will continue like 3, 4, 5.*
Researcher	:	*Well, I don't want you to think about the [cookies] inside, just whatever the most general representation is...*
Beren	:	*OK. I got it.*

The researcher then showed Beren the three types of conversion tasks that she had engaged in during Phase 1, and then went back to the current task of Phase 3. Beren drew the representations of four specific numbers of the pattern separately, as given in Figure 4.4. This was an indication that the general representation did not mean to her "any number" sharing the same abstract structure, as it was thought at the end of Phase

1, where she had successfully used the representation without indicating any specific number. Even though she was aware of the covariation between the number of items inside the cups and the numbers listed in the pattern (the below excerpt reveals that she relied on this relationship wisely), and despite the researcher's insistence on the "general" character of the representation system, she could not or did not separate the common structure of the numbers she represented from the specific cases she considered. She most probably did not have any goal that required her to take such an action.

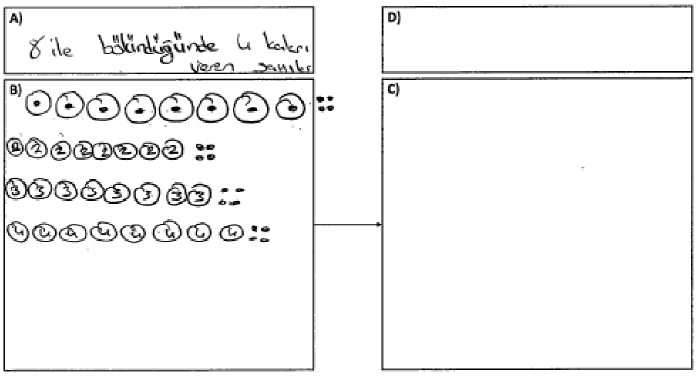


Figure 4.4 Beren's first attempt on the original task of the Phase 3

Researcher	:	*Now, we remember the general representation.*
Beren	:	*Then, I am beginning with 12.*
Beren	:	*I found these (the specifics of each number represented) without trying. I could have tried it, too.*
Researcher	:	*What do you mean?*
Beren	:	*Well, I could also try and find out how many go into each cup (for each number in the pattern), but why didn't I try, because it was the week before last week, or the last week, I don't remember exactly, but we did something. There was a list (Refers to the issue of covariation explored in Task 5 of the Phase 1 task sequence) How many of ... the leftovers were given, the number in the cup and the leftover number. The leftover was 3. I drew the number in the cups (increasing) one by one. And! Also, I said that it looks very nice one under another. I did from there.*
Researcher	:	*OK. Now I want to ask you about this, Beren. Look at each of the numbers you drew. Are they different from each other?*
Beren	:	*No, [they are] actually the same. Only the number inside the cups change. Actually, the number of the cups is the same. The number of leftovers is the same.*
Researcher	:	*Yes. That is why we call this the general representation.*
Beren	:	*Yes.*
Researcher	:	*If we don't think about what is inside, it can be used in the place of 12, and of 20, and of 28, okay?*

Beren : *Yeah okay.*
Researcher : *Now, keep this page. I will give you the same of that page.*
Beren : *The same? (Seems confused)*
Researcher : *Yeah, because I want you to write one single general representation for all the numbers, not separately like here.*

On a new blank flowchart, Beren filled in boxes A and B as in Figure 4.5, just because she was asked to do so. After drawing the representation, she explained her thinking, and the researcher once again took the opportunity to emphasize the general characteristics of the representation system.

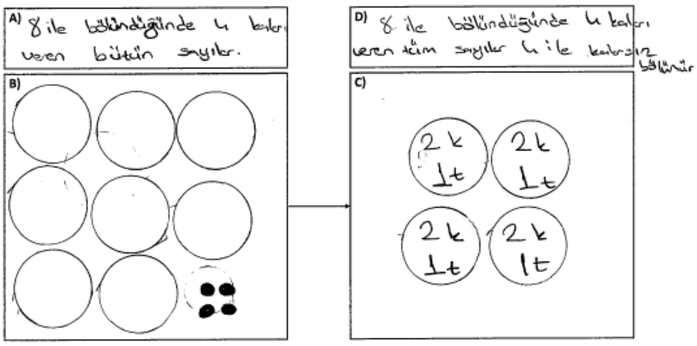


Figure 4.5 Beren's second attempt on the original task of the Phase 3 (*k* for cups and *t* for single cookies)

Beren : *Now, I explain. Here are the cookies in the 1, 2, 3, 4, 5, 6, 7, 8 cups, I mean, eight circles showing the number of cups, the cups. In any way, I will have eight cups already. One, for 12, one goes into cups (each cup). For 20, 2, for 28, 3, for 36, 4, for 44, 5, for 52, 6, for 60, 7, for 68, 8, for 76, 9, for 84, 10 goes into. (Points to the elements of the number pattern sequentially.) My leftover 4 will never change in any way.*
Researcher : *What if this number goes on forever?*
Beren : *Will be the same again. The cup count is the same. The leftover count, the number of cookies left out will be the same.*
Researcher : *Yes, that is why we call this the general representation.*
Beren : *Yeah okay.*

Beren continued with reading Instruction 3 given in the PowerPoint slide. In response to the first part of it, "Think about this representation as cups and cookies on a bakery shelf," she responded, "It already is." Then she answered, "How many kids at most can equally share this amount of cookies?"

Beren : *If a maximum of 4 children come, each child gets 2 cups and 1 single cookie.*

In response to Instruction 4: "Equally share the cups and cookies to the number of kids you specified [Box C]," she responded, "So simple." "Again, I will have eight cups already," and filled in Box C as in Figure 4.5. When she came to Instruction 5, she was asked the question, "What can you say about the numbers that give a remainder of 4 when divided by 8?" She responded:

Beren	*: About the numbers that give a remainder of 4 when divided by 8, I can say with certainty that they are even. (Not looking at the flow-chart, looking at the list of numbers 12, 20, 28, 36, 44, 52, 60, 68, 76, 84, ...)*
Researcher	*: What does the box C tells you about this?*
Beren	*: The box C, huh…how, what it tells?*
Researcher	*: What do you see in the (box) C?*
Beren	*: In the (box) C, I see that I can equally share to four kids. (Looks at the box C only)*
Researcher	*: What does that mean?*
Beren	*: I mean, div… by eight (suddenly directs her gaze to the box A, and seems to experience an a-ha moment). Hhh… I remember this! We have done this! (She appeared to remember an example of a "conjecture" she was asked to validate during Phase 2.) For example, all the numbers that give a remainder of two when divided by three could be equally shared among two persons, it was.*
Researcher	*: Let's drop it, and see what's here.*
Beren	*: Exactly, right! I think so. All the numbers that give a remainder of 4 when divided by 8 are divisible by 4. I remember this. We have done something like this, but the example of three…is not quite.*
Researcher	*: Well, where do you see that 'divisible by four' situation?*
Beren	*: In here. (Points to the box C). Should I write it? Huh. It (the Instruction 6) says 'write it'. All the…that give a remainder of four when divided by eight… (Filling in the box D, stops for a second) Is this proven?*
Researcher	*: I don't know.*

Thinking about the questions directed in Instructions 5 and 6, Beren remembered about a mathematical generalization she was asked to validate in one of the previous teaching sessions. Then, she pieced together the conjecture appeared on the flowchart as a complete statement: "All the numbers that give a remainder of 4 when divided by 8 are divisible by 4". This was only a vague manifestation of the hypothetical syllogism, represented in Table 4.1, targeted by the major Phase 3 task.

The conjecture Beren stated was equivalent to the conclusion ($P \rightarrow R$) of the above hypothetical syllogism. But her expressions about this conclusion implied an understanding of the second modus ponens ($Q \rightarrow R$) only (shown in Box C of the flowchart), while overlooking the first one ($P \rightarrow Q$). Note that the two instances of modus ponens contained in Table 4.1 are closely related to the two universal instantiations involved in the desired proof. The methods section already clarified that the first one of these universal instantiations is not as straightforward as the second one can be for the students. Particularly in the first one, modus ponens is employed to assume that a fixed but not specified number A gives a remainder of 4 when divided by 8 (P is true) and deduce the singular proposition that “such a number A is of the form 8k+4” (Q) from the universal proposition $P \rightarrow Q$. It requires the student to anticipate the need to make this particular assumption to start with. In the second universal instantiation, on the other hand, no such process of making an assumption is needed since the image of a number of the form 8k+4 is already present at that point. This time, it is sufficient for the student to just remember and use the universal proposition $Q \rightarrow R$.

Table 4.1. Target Hypothetical Syllogism $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightarrow (P \rightarrow R)$ for the specific case

$\boldsymbol{P \rightarrow Q}$	If “A” is a number that (belongs to the set 12, 20, 28, 36, 44, 52, 60, 68, 76, 84, …) gives a remainder of 4 when divided by 8, then it is of the form 8k+4, for integer k.
P	A is a number that gives a remainder of 4 when divided by 8.
Q	Then, A is of the form 8k+4.
$\boldsymbol{Q \rightarrow R}$	If a number is of the form 4n for integer n, then it is a number “divisible by 4”.
Q	A number of the form 8k+4 is of the form 4(2k+1) = 4m.
R	Then, a number of the form 8k+4 is divisible by 4.
$\boldsymbol{P \rightarrow R}$	The number A, that gives a remainder of 4 when divided by 8, is divisible by 4.

Beren’s stating the above conjecture correctly (especially the generality aspect involved), was interpreted to be a consequence of the resulting flowchart’s evoking the image of an analogous conjecture that she had encountered before in Phase 2. Her explanation of the resulting flowchart did not demonstrate an understanding of the logical necessity relationship the conjecture indicated. When she was asked to tell to a hypothetical friend why the conjecture was true, she procedurally explained the steps she had followed to complete the activity. Despite her borrowing the phrases “but this is the general representation” and “so that it expresses all the numbers that give a remainder of 4 when divided by 8,” no indicative evidence was observed that she was aware of the abstract structure underlying the particular representation she used or the abstract structure underlying the whole argument.

Beren	*: This flowchart shows me that all the numbers that give a remainder of 4 when divided by 8 are perfectly divisible by 4. Well, why? In our first step in this chart… (it) expresses all the numbers that give a remainder of 4 when divided by 8. In step B it says draw me a representation of this, express it visually. I draw 8 cups here. It's like it has cookies in it. I draw 8 circles and show them as cups. Since I have to have 4 leftovers, I draw 4 of another thing and show them as cookies. For example, I give an example of 12. When 12 is divided by 8, it gives a remainder of 4. When I divide 12 by 8 and gives the remainder of 4, the result is 1. I draw 1 cookie inside each circle, and that exactly expresses all the numbers that give a remainder of 4 when divided by 8.*
Researcher	*: Doesn't it express only for 12?*
Beren	*: It expresses only for 12. For 84, it does not do that. For 84, there will be 10 cookies in the cups, but this is the general representation, I say. I draw 8 circles; I draw 4 more so that it expresses all the numbers that give a remainder of 4 when divided by 8.*

The modular structure represented by 8 cups and 4 counting items, for Beren, was not a stand-alone property of any and all of the numbers in the conjecture domain. Rather, it was tied to the specific examples she referred to, such as the cases of 12, 84, and others; and the number inside cups were still at her focus. Her understanding of the whole argument was not at the level of abstract structure either according to the students' ways of understanding a proof framework (Ahmadpour et al., 2019). It was considered "close to" the state of general procedure because Beren explained the whole flowchart as a procedure that applies to different cases of interest. But it was "close" because she perceived no use in expressing the particular cases in general representation to start the procedure. She checked the result of division by eight just because Instruction 2 suggested doing that, and (understandably) she did not consider this step a component of the process of validating the conjecture. When evaluating the truth of the conjecture for particular cases, such as 12 and 84, her focus was on dividing that case directly by 4 and was not on the numbers' modular congruence. The last sentence of the below excerpt exemplifies her that she sees no need to use the general representation of numbers to make the intended decision.

Researcher	*: So why does this show us for all the numbers?*
Beren	*: What do you mean?*

Researcher : Does this show for 12… its divisibility by 4? … How does it show?
Beren : Already…
Researcher : 12 is a number that gives a remainder of 4 when divided by 8.
Beren : Yes.
Researcher : And we say that it has to be divisible by 4.
Beren : But, we can find already if we try.

The absence of the abstract structure (as an indicator of the universal instantiation) and the related vague manifestation of the hypothetical syllogism provided evidence that Beren did not abstract the intended CoA concept of the study. That is, she did not develop a structural understanding of the particular proof obtained as a result of her activity. Her question towards the end of the activity (while filling in box D of the flowchart), "Is this proven?" suggested that even if she felt a need for the certainty of the conjecture stated, she did not perceive the result of her activity as a proof of that conjecture. Therefore, the originally planned task failed to support Beren's understanding of the first instance of a mathematical proof.

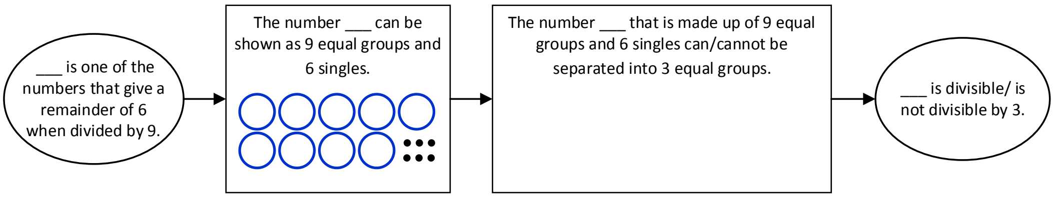
4.1.3. The modified task sequence

The originally planned task failed to foster Beren's understanding of the structure 8k+4 as an abstract structure detached from the particular examples. Beren demonstrated a solid understanding of the one-to-one correspondence between the elements of the number pattern and the number of counting items contained in each of the cups for each of these elements. However, the intended shift of attention from seeing the representation 8k+4 as an expression of generality to seeing it as a defining property of the set 12, 20, 28, 36, 44, 52, 60, 68, 76, 84, … had not been realized. This was not an unexpected result at all, given the difficulty of transitioning from computational processes to abstract objects. According to Sfard (1991), the first two steps of this transition, *interiorization* and *condensation,* are gradual, quantitative changes; but the last step, *reification,* is "an instantaneous quantum leap" (p. 20) and "an ontological shift—a sudden ability to see something familiar in a totally new light" (p. 19). Along with the LTA hypothesis, the following conjecture also supported Phase 3's initial task design: If Phase 1 of the teaching experiment study would provide Beren with enough experience of interiorization and condensation for the concept of modular congruence, encountering an instance of mathematical proof could offer to her the required shift of attention and result in the intended reification. During the time of data collection, it was not possible to identify the exact reason why the desired shift of attention did not occur. It could either be the failure of Phase 3 task design. Or, procedural fluency Beren demonstrated in Phase 1 (the conversion tasks—among three images of arithmetic number patterns—and contextual fair sharing situations involving general representation) might not be an indication of sufficient quantitative experience needed before reification. Considering both possibilities, the modified task sequence, given in Figure 4.6, has built on the operational rather than structural origins of the concept of modular congruence (Sfard, 1991). While the activity elicited by the original task started with constructing the general representation of a given arithmetic number pattern (the structural conception), the modified task sequence treated the elements of the pattern individually. The modified task started with, after identifying particular elements of a particular arithmetic number pattern (described as "the numbers that give a remainder of 6 when divided by 9"), determining the specifics of individual elements (i.e., the quotient) in terms of the readily presented general representation of the pattern (the operational conception). In terms of the LTA theoretical framework, the modified task sequence was expected to elicit an activity more accessible to Beren because it employed the concept of division (by 9), which was an anticipatory level concept for her. The original task, however, attempted to use a participatory-level concept (the general representation that Beren had struggled to retain), for which it was challenging to develop the related goal component without yet developing the need for "defining."

The major focus of the modified task sequence was universal instantiation rather than hypothetical syllogism. Therefore, the modified task sequence began with the statement of the open conjecture to be validated (all the numbers that give a remainder of 6 when divided by 9 are divisible by 3). The student was asked to select appropriate cases to test the conjecture and fill in the given case-by-case flowchart for each of the cases selected consecutively. By integrating the LTA notion of mental run (reflecting on the repeated flowchart activity), the task sequence aimed to bring the intended universal proposition to the student's attention. Overall, the modified task sequence aimed to spotlight a particular modular structure (9k+6) shared by the elements of an arithmetic number pattern (15, 24, 33, 42, 51, 60, 69, …) as a (defining) property of *any and all* of the numbers in the pattern, which was the missing component needed for Beren's attainment of a structural understanding of a first instance proof.

What do the flowcharts tell?

Is the statement “all the numbers that give a remainder of 6 when divided by 9 are divisible by 3” true or false? Choose the cases you wish and fill in the charts given below for each of the cases. [To be repeated for multiple cases.]



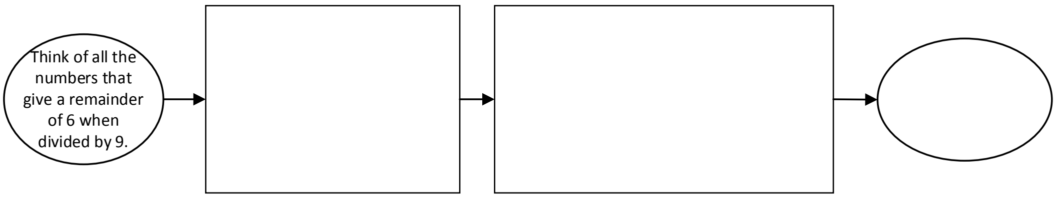
Let us think about the flowcharts!

Think aloud and explain:

1. One of the numbers that gives a remainder of 6 when divided by 9 is 4794. What result would you get if you were to fill in the chart for 4794?
2. What result would you get if you were to fill in the chart for a number larger than ten million that gives a remainder of 6 when divided by 9?
3. What does this result show us?

Now let's write the thoughts and draw a new flowchart!

4. Think of all the numbers that give a remainder of 6 when divided by 9 and fill in the flowchart below <u>without using any examples</u>.



5. Using the flowchart write a paragraph to explain why the statement “All numbers that give a remainder of 6 when divided by 9 are divisible by 3 without a remainder” is certainly true.
6. Below is an explanation by Öykü. Compare the paragraph you wrote with Öykü’s explanation.

   Statement: Numbers that give a remainder of 6 when divided by 9 are divisible by 3.
   Öykü’s Explanation: The statement is absolutely true. Because, the numbers 15, 24, 33, 42, 51, 60, 69, ve 78 are numbers that give a remainder of 6 when divided by 9. All these numbers, as can be seen below, are divisible by 3.

   15 ÷ 3 = 5
   24 ÷ 3 = 8
   33 ÷ 3 = 11
   42 ÷ 3 = 14
   51 ÷ 3 =17
   60 ÷ 3 = 20
   69 ÷ 3 = 23
   78 ÷ 3 = 26

   a. What are the differences between the two methods (yours and Öykü’s)?
   b. Which one of these two methods do you think is more reliable? Why?

Figure 4.6 The modified task sequence

The hypothesis of the modified task sequence was similar to the original one stated in Method Section 3.3.1.6. Namely, reflecting upon the case-based flowchart activity, Beren was expected to realize that any one of the resulting flowcharts indeed conveyed a new type of connected meaning beyond the distinct actions carried out (participatory stage of the proof instance), and this meaning was related to the goal of proving the conjecture being investigated (anticipatory stage of the concept proof). From the two actions to be coordinated, A_{0a} was changed and A_{0b} remained the same as follows:

$A_{0a\ (original)}$:	Given an arithmetic number pattern, identify a modular structure shared by all its elements.
A_{0b}:	Given a modular structure, make appropriate divisibility inferences.

$A_{0a\ (modified)}$:	Given a particular element of an arithmetic number pattern, identify its modular structure according to the given modulus n.
A_{0b}:	Given a modular structure, make appropriate divisibility inferences.

4.1.4. Beren's engagement with the modified task sequence

For the first case-by-case flowchart (see Figure 4.7), she selected the number 15 as a case that gives a remainder of 6 when divided by 9. An abridged version of her explanations regarding her process of filling in the first case-by-case flowchart is as follows:

Beren : *I will start with the simple one. 9. How did I find it? You know, we count as 9, 18, 27, there I added 6 to 9, and 15 came out. So, when I divide 15 by 9, the remainder is 6. (Filled in the blank space in Step 1 with "15.") The number 15 can also be shown in the same way... 9 equal groups and 6 singles. Yes, that's true.*
Researcher : *How is that represented?*
Beren : *There is one in each of these (cups). We are thinking of the cookie thing (Wrote '1' inside each of the cups given in Step 2). The number ... consisting of 9 equal groups and 6 singles can be, cannot be separated into 3 equal groups. The resulting is the number 15, again. Is this a room for me to draw here too? (Step 3)*
Researcher : *Yes.*

Initially, her focus was on dividing the number 15 by 3, independently of its configuration in the general representation.

Beren : 15… Let me say this… Separable.
Researcher : How?
Beren : We separate five by five.

But then her focus shifted from the value of the number 15 to its specific form in general representation.

Beren : But if it has to necessarily be something like a multiple of 9... No, I gave up. Not like that (probably considers the remainder of 6). Three many, but the same again it can only be separated five by five. Christmas baskets again to be distributed among three persons. (Drew 3 baskets with 3 cups and 2 singles in each, keeping an eye on the total number of cookies in every basket). Inside these ones each (points to one of the cups) there is one. [The basket] consists of five. Others likewise. The number 15, one second, then, we should mark 'can be separated.' The number 15 is perfectly divisible by 3.

For the second flowchart, Beren picked up the number 24.

Beren : Again, it will give a remainder of 6 when divided by 9. Should I find a larger number this time?
Researcher : You can choose anyone you want. Because you're going to do this exactly for three more (flowcharts).
Beren : Okey then let me go with 18 this time. (Counted 6 over 18). 24 is divisible [sic]. (Filled in the related blanks in the flowchart with '24'.) And this time two of them goes into each one of them (Filled in the cups in the second step with "2" each, and went on reading the third step.) The number 24 that is made up of 9 equal groups and 6 singles can, cannot be separated into three equal groups. I am counting right now, 3, 6, 9, 12, 15, 18, 21, 24, it's ok (Her focus is again on dividing the number 24 by 3). (Drew three baskets.) Here (points to the third step of the previous flowchart) five (cookies) go in each (basket, in the case of 15). How many go in here (in the case of 24)?

She first structurally mapped the third step look from the case of 15 to the case of 24. While doing this structural mapping, she demonstrated an instance of the intended abstract structure for the first time. That is, for Beren, 9k+6 instantly became a structure "standing for itself," separated from the specific case of 24 being examined.

Beren : Three equal groups. Wait a minute, 3 okay. Same two of singles thing we will have. Two more will be in each of these (cups) and three of these (cups)… But it is not 24 then? (Probably thought each cup to contain a single cookie as in the previous case.)
Researcher : Why not?

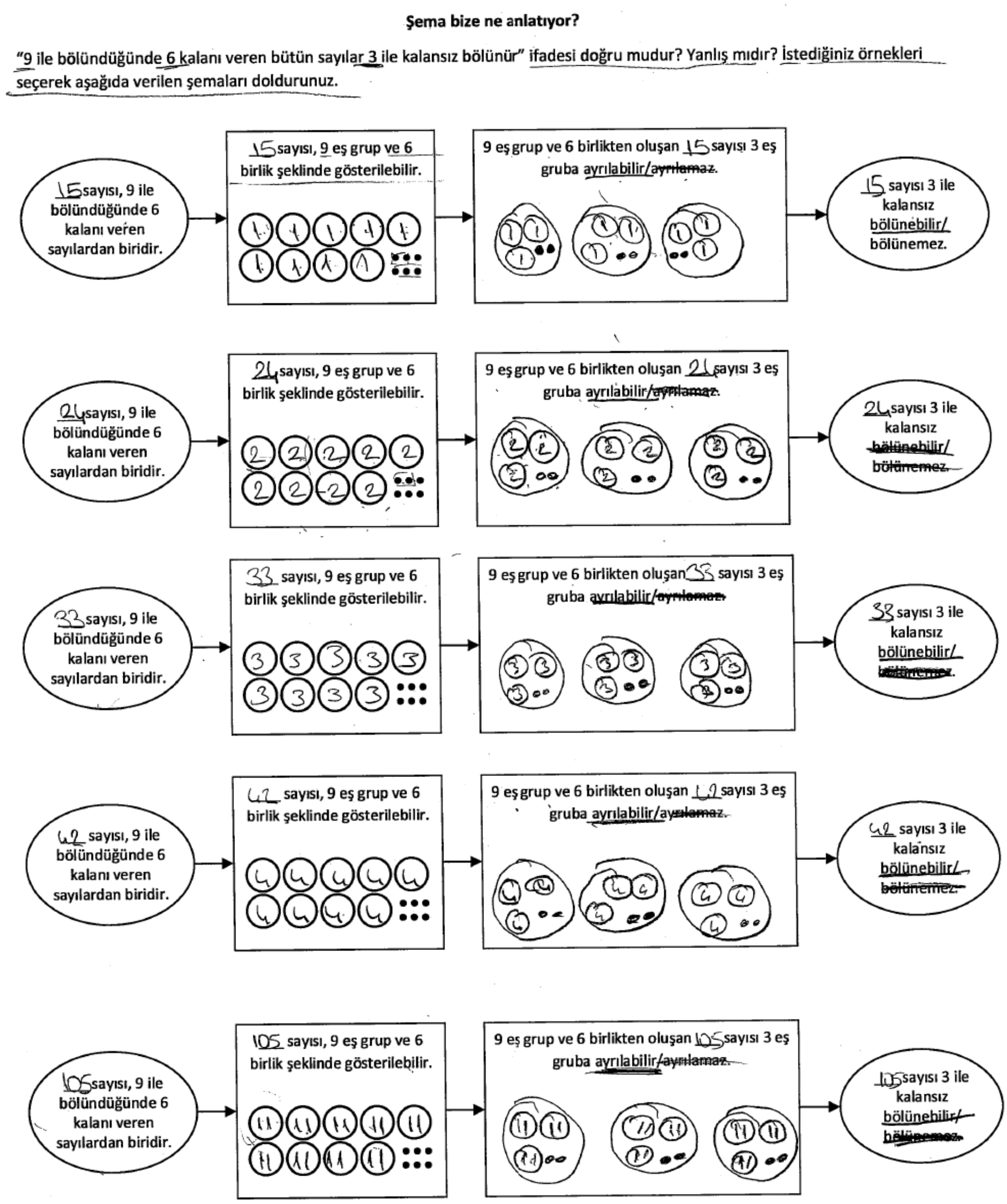


Figure 4.7 Beren's work on the case-based flowcharts of the modified task sequence

Beren	*: No. Wait a minute. <u>Whatever the number is.</u> Okay I now understand (Seems to experience an a-ha moment). No matter what the number it is, my things here (points to the items in the second step) are the same. That means, again 3 cups and 2 single packs go in (each basket). Only the number inside the cup(s) changes. Okay, I got it. So, in total, this time, there will be [...] 8 cookies.*
Researcher	*: What is your decision?*
Beren	*: Can be separated. <u>They all can be separated</u>, in fact.*
Researcher	*: What do you mean they all can be separated?*
Beren	*: Well, think of it this way. You know, the number that is right here (Points to the number 24 shown in general representation*

as 9 groups of 2 singles plus 6 more singles), how can I explain it… It is not that the number 24 here when we actually look at it. I mean, you know it is (a thing) consisting of 9 equal groups and 6 singles.
Researcher : *Yes.*
Beren : *It already. Well, let me say this way. This one is separable, in any way, it is separable into three equal parts and it is again three cups and two single packs. I can't put it into words but… is divisible by three.*

Even if the intended abstract structure was observed in the second flowchart, the researcher encouraged Beren to work on further appropriate cases. She picked up the numbers 33 and 42 for the next two flowcharts. Increasing the number inside the cups by one at a time, she consciously and purposefully recited the steps of the flowchart to match the same structure as in the first two ones.

Beren : *The number 33 is divisible by 9. 33. There are three in each of them (cups). You know, I just recite it directly. I mean, because from here (points to the first two flowcharts completed) I understood what it is that, what is going on.*

Beren : *42 is one of the numbers that gives a remainder of 6 when divided by 9. This time four goes into [each cup].*

For each pick, she mentally did the division operation first (33:3 and 42:3) to check the result of the divisibility decision. She didn't seem as confident as she had just been when transitioning from the second to the third flowchart about the steps she was following. Only after making sure that the number selected was perfectly divisible by three, she drew three baskets in the third steps by rote, equally distributed cups and singles to each of the baskets and then marked the number of items in each of the cups to obtain the correct quotient (11 and 14).

Beren : *But wait a minute… 33… divisible, yes, can be separated. Why? How? Like this. This time there be 11 cookies in each one (basket). Like this. 3, 6, 9, 10, 11; 3, 6, 9, 11; 3, 6, 9, 11 (counts by threes for three cups, adds two singles, repeats for three baskets) The number 33 is perfectly divisible by 3.*

Beren : *The number 42 that is made up of 9 equal groups and 6 singles can be separated into three equal groups. Now, it is like this. 30, 33, 36, 42 to that 39, 42.*
Researcher : *I don't understand, what did you do?*
Beren : *I tried if it is divisible by 3.*
Researcher : *Oh ok, now I get it.*

Beren	:	*And this one by the way… Let me tick off this first (crossed out the option 'not divisible' in the last step). Because I already do that by looking at here first. The number 42 is divisible.*

Then she considered choosing an "upper number" (üst rakam), as she put it.

Beren	:	*Should I go orderly again? Should I go a little to the upper numbers?*
Researcher	:	*Choose an upper number if you wish.*
Beren	:	*Okay. Now, 9, 18, 27, 36, 45, 54, 63, 72, 81, 90, 99. 99, 100, 101, 102, 103, 104, 105 it is. 105 is divisible. I mean that not divisible but one of the numbers that give a remainder of 6 when divided… Now, 9,1; 18,2; 9,18,27, 3. (Points to three previous flowcharts). Let me count myself. Shall I tell it later?*
Researcher	:	*Alright.*
Beren	:	*9,18, 27, 36, 45, 54, 63, 72, ... 11 already! (Sounds like: Why am I even searching?). Well, 9, 90, 99 11. (Reasons based on 9k) Eleven go into each one.*

This time she was about to make her decision without making the division operation 105:3. She attempted to reason based on the modular structure of the number 105 (refers to this structure as a "rule" in the below except. As she was not fully certain, Beren questioned whether the conjecture was a proven one or not. Unlike in the originally planned task of Phase 3, this time she ascertained herself by making a connection to the Monstrous Counterexample Illustration.

Beren	:	*The number 105 can (or) cannot be separated into 3 equal groups (reads the step 3). Can be separated. Why it can be separated? Because, in the end, it goes again with the same rule and all of them that go with this rule necessarily have to be. Well, it might not be proven, but I think it holds… It holds, isn't it? But wait a minute, it did hold, because that number… that day there was a number when I tried to read it, I couldn't read, starting with 30.*
Researcher	:	*Uh-huh…*
Beren	:	*That in fact was a single number but I read as multiple numbers. Even if it goes up to those numbers, <u>based on this thing</u> (points to the representation in step two), it already is divisible. (Completed the division action in the third step) Alright "divisible", that is it.*

When she was asked to mentally run the flowchart activity for the number 4794 and a number that is greater than ten million, her reasoning was based on the same abstract structure again—the property of all the numbers in the conjecture domain.

Beren : *You know this (step one) and this one (step two) already holds. This one (step three), as well. This one (step four) certainly holds as well. If, in here, one of them holds (points to the statement 'One of the numbers that give a remainder of 6 when divided by 9 is 4794.') … Only in here…again divisible is this, because based on the digits (numbers) here (points to the configuration in box 2). Just I would have some trouble figuring out the number of cookies in each cup.*

Beren : *For a number greater than ten million […] I would again obtain the same result. If for example it was something around eleven million, again that number would be one of those numbers that gives a remainder of 6 when divided by 9. Again, that number can be shown as 9 equal groups and 6 singles. I just wouldn't know the number in the equal groups, well, which I would be able to find if I think and try.*

She was certain of the truth of the conjecture. In the rest of the dialog, she used the expression "we prove" for the first time. Compared to her previous uses of the verb, where she attributed absolute certainty to an outside authority, here she expressed confidence in her decision.

Researcher : *Do we need to know that (the number inside the cups)?*
Beren : *No, not that much.*
[…]
Researcher : *What conclusion are we trying to draw? What is the final result we want to reach?*
Beren : *Whether it is (divisible) or not.*
Researcher : *No, look at this whole flowchart.*
Beren : *Whether that number is divisible by 3 or not. After all, if we add 6 to one of the numbers that is a multiple of 9, it is already divisible by 3 without a remainder.*
Researcher : *Why?*
Beren : *We have already divided them into three here. And here we prove that it is divisible. And the reason is, well, for me there is not a reason for this. That just means all numbers that give a remainder of 6 when divisible by 9 can also be divided by 3 without a remainder.*

As Beren seemed to have the intended CoA concept of the study—the first instance of mathematical proof, the next task in the sequence (Task 4) promoted her transition to a general flowchart (see Figure 4.8).

Beren : *(Reads part 5 of the modified task) Think of all the numbers that give a remainder of 6 when divided by 9 and fill in the flowchart below without using any examples (Seems unsure). So how do I do it?*

Researcher : *Now here we thought about 15 first, then we thought about 24. Then we thought about 33, 42, 105, and we always came to the similar conclusion.*

Beren : *Uh-huh.*

Researcher : *Now here we think about all the numbers that give a remainder of 6 when divided by 9. What would we say when it was like this: any number that gives a remainder of 6 when divided by 9?*

Beren : *Ok I found it!*

Step by step, she adapted the sentences from the particular flowchart created for the case of 105 and directly recited the representations used. She systematically replaced the phrase “the number 105” each time it appeared with that of “all the numbers that give a remainder of 6 when divided by 9.”

Figure 4.8 Beren’s transition to the general flowchart

She was guided by the researcher to piece together (Task 5) the steps of the resulting flowchart to obtain a proof text. The following text was obtained, where she added the last sentence to convince her hypothetical friend that she had not seen the flowchart.

In this chart, we have considered all the numbers that give a remainder of 6 when divided by 9. And in the second step of our flowchart, it explains that these numbers can be represented as 9 equal groups and 6 singles. In our third step, it shows that these numbers, consisting of 9 equal groups and 6 singles, can be divided into 3 equal parts. In our last step, says that these numbers are divisible by 3 without a remainder. If you are asking me how I came to the conclusion in the last step, you can try drawing the step 3.

When she was asked to compare her own argument to the empirical argument given in Task 6, she referred to her own as “proving for all.”

Beren : *I have thought about all the numbers that give a remainder of 6 when divided by 9. Did not think about some numbers as she did. She could not fully prove that all of them are (divisible by 3). But I have proven that they are all, in the flowchart I made. (Reads part b) Which one of these two methods do you think is more reliable? Why? I am not saying this because this is my way, but my method is more reliable.*

Researcher : *Why?*

Beren : *Because here, as I said, she made it based on eight numbers only, but I meant all the numbers. Now, a person reading this first would say that it would be (true) for these numbers, but not for the rest. But when I read it (mine), I believed it is (true for) all.*

Lastly, the researcher introduced Beren to the concept of mathematical proof and its scope in the teaching experiment she would continue to participate in.

Researcher : *You said something like this in the previous lesson, remember? You said that mathematicians urgently need to find a program and we need to be able to try (conjectures) for all numbers.*
Beren : *Yeah.*
Researcher : *Such a thing is not possible. Because, computers work just like us. […] So, there is no way to try for all the examples. Instead, we use methods like this, where we can think about all the examples, and that's how mathematicians actually do proofs. This (flowchart) is a proof for someone your age.*
Beren : *I see.*
Researcher : *I mean, our whole point with you was to learn to do proofs. Now that we understand what proof is, from now on, we're going to do other proofs.*

4.1.5. An account of Beren's learning

The below account of learning explains Beren's abstraction of the logical necessity relationship stated in the conjecture as a new way of understanding—a new, connected meaning beyond the distinct actions carried out in the flowchart activity—and then her relating this newly abstracted meaning to the goal of proving the particular conjecture being examined. The former aspect was defined as the participatory stage of the intended CoA concept as the coordinated actions were connected to the task goal of completing the flowchart activity (the result of a coordination of actions process), and the latter was defined as the anticipatory stage of the intended CoA concept as they were connected to the actual goal of the intended concept: proving a conjecture.

During her participation in the modified task sequence, when Beren first detached the modular structure 9k+6 from the particular case being examined, the number 24, she also seemed to instantly have abstracted the logical necessity of the relationship whose correctness was being investigated. We interpret the two abstractions as having occurred simultaneously since the continuous data did not reveal any time lag in between. In her expression, the two instances of invariance appeared immediately after one another. First, she revealed an understanding of the structure 9k+6 as an invariant property of any and all of the numbers in the pattern "15, 24, 33, 42, 51, 60, 69, …"

Beren : *No. Wait a minute. Whatever the number is. Okay I now understand (Seems to experience an a-ha moment). No matter what the number it is, my things here (cups and counting items) are the same.*

Second, she referred to this abstraction as the reason why the relationship between being a member of the pattern and being divisible by 3 was invariant.

Beren : *That means, again 3 cups and 2 single packs go in (each basket). Only the number inside the cup(s) changes. Okay, I got it.*

This marked a momentary "construction of a structure that 'reifies' (Sfard, 1991) the underlying idea of a general procedure" (Ahmadpour et al., 2019, p. 93); hence, the abstraction of the intended logical necessity relationship. Beren produced the same reasoning when the researcher asked her about her final decision about the number 24. She utilized both abstractions together to explain why, in her words, "they (the elements of the pattern) *all* can be separated." This time, her understanding of the resulting flowchart included the aspect of universal instantiation (the abstract structure defining the number pattern) that was missing in her response to the original Phase 3 task. However, this was only an instantaneous emergence of the intended structural understanding of the first instance of proof. While working on the subsequent case-based flowcharts for the numbers 33, 42, and 105, Beren did not maintain the logical necessity relationship that seemed to have been abstracted.

Only after making a connection to Monstrous Counterexample Illustration, while working on the case 105, the intended coordination of actions was completed and the participatory level of the intended CoA concept was constructed. After then, Beren consistently demonstrated immediate conviction of perfect divisibility for *any and all* of the elements that belong to the given number pattern. She no longer needed to go through individual division operations to determine divisibility of the class of numbers. She was able to anticipate the result of the whole activity without carrying it out sequentially (Simon, 2020). In her explanation, Beren focused only on the invariant divisibility relationship and played down the number of counting items contained in each of the cups in distinct cases. Her learning, we claim, is not the result of an empirical abstraction but of a reflective abstraction because her justification was present without hesitation and went beyond demonstrating a numerical pattern alone Also, it did not take place until she engaged in the modified task sequence (as in Simon, Kara, & Placa, 2018). We conclude that her learning was the result of a *coordination of actions* process.

Her feeling of the need for certainty while working on case 105 and then thinking about the Monstrous Counterexample Illustration caused Beren to engage in a mental run earlier than the time it was planned for by the task sequence.

Beren : *The number 105 can, cannot be separated into 3 equal groups (reads the step 3). Can be separated. Why it can be separated? Because, in the end, it goes again with the same rule and <u>all of them that go with this rule</u> necessarily have to be. Well, it <u>might not be proven</u>, but I think it holds… <u>It holds, isn't it? But wait a minute, it did hold</u>, because that number… that day there was a number when I tried to read it, I couldn't read, starting with 30…. that in fact was a single number but I read as multiple numbers. <u>Even if it goes up to those numbers, based on this thing</u> (points to the representation in step two), it already is divisible. Alright "divisible", that is it.*

Beyond playing the role of a mental run task, this connection seems to have unexpectedly fostered a second reflection on Beren's way of thinking about the act of proving and, as a result, fostered her relating the whole activity she engaged in to the goal of proving. That is, it seems to have triggered Beren's transition from the participatory to the anticipatory stage of the concept. Since a second moment of reflection was not identifiable in the above data set, where such a transition might have come about, we speculate that she might have attained both levels of the CoA concept in response to the same notion of mental run she spontaneously engaged in. The speculation is limited by the observation that no indicator of a distinct moment of reflection has been captured in the current study's data set. However, the following aspects provide supporting evidence.

Just after the emergence of the participatory level concept, when Beren was thinking about the number 4794 and a number that is greater than ten million, she demonstrated the intended anticipatory level concept—stated the conjecture in her own words and claimed that the flowchart activity offered a proof to that conjecture. During her speech, she did not sound as if she was thinking this for the first time.

Beren	*: After all, if we add 6 to a number that is a multiple of 9, it is already (we already know that) divisible by 3 without a remainder.*
Researcher	*: Why?*
Beren	*: We have already divided them into three here. And here <u>we prove</u> that it is divisible.*

In addition, when Beren demonstrated an understanding of the argument at the level of abstract structure for the first time (working on case 24, she viewed perfect divisibility as a certain result "[n]o matter what the number is"), she did not compare her argument to a mathematical proof. If maintained, her understanding of the logical necessity relationship alone would probably mark the participatory level of the concept. We do not know whether she would engage in such a comparison between a faulty and a valid way of thinking and (as a result) develop the anticipatory level concept if she had engaged in the mental run designed in the modified task sequence instead of the one that made a connection to the Monstrous Counterexample Illustration. We speculate the twenty-six-digit number might have caused Beren to both better envisage the extent of the conjecture domain to reflect on her abstraction as a for all argument and reflect on her proof scheme to resolve the cognitive conflict she experienced previously in Phase 2.

4.2. The construction of the AoC concept: The main idea of the first instance proof

This section presents evidence that soon after understanding the first instance proof, Beren demonstrated the intended AoC concept of the study: using the modular structure shared by all the elements of an arithmetic number pattern for making divisibility inferences about all the numbers in the pattern. This was the main idea of the first instance proof and Beren was able to use it to prove conjectures that she recognized to share the same structure as her first instance proof. However, none of her proving experiences was a conscious instance of definitional reasoning for Beren. She was not aware that by using the concept modular structure (general representation in her language) she was actually defining the given number patterns with the purpose of proving. Below is reported how Beren approached proving tasks of Phase 4, including cases where the researcher's triggering about the act defining did not make sense to her. Just after Beren constructed an understanding of the first instance proof, she was asked to evaluate two analogous conjectures. The first conjecture was about an unfortunate arithmetic number pattern with a relatively large constant difference of twelve among its consecutive members.

Conjecture C

One of the numbers in the pattern 18, 30, 42, 54, 66, 78, 90, 102, 114, 126, 138, 150, … is 354. The number 354 is perfectly divisible by 6. Does the same result apply to all the numbers in the pattern? Why?

The task requested that she state the conjecture to be proven explicitly before completing the flowchart (see Figure 4.9). The expected statement was as follows: The numbers that give a remainder of 6 when divided by 12 are divisible by 6. When the researcher attempted to guide Beren towards the relevant modular structure, her question about the defining characteristics of the number pattern did not make sense to her.

(Abridged excerpt)

Beren : *The statement that we want to prove… that, up to 354, all the numbers in this number sequence are divisible by 6.*
Researcher : *Not up to 354 actually. It continues after 354.*
Beren : *There are the next ones, uh-huh.*
Researcher : *What numbers are they? (Silence) If you wish we can look at the flowchart first, and then go back to the above part.*
Beren : *The numbers that are multiples of 6.*
[…]
Beren : *The statement that we want to prove (is)… […] that the numbers in this chart are divisible by 6 without a remainder, we are trying to prove here.*
Researcher : *Now we have numbers, we are trying to prove that they are divisible by 6 without a remainder. What is the name of the set of numbers we have? Let's think about it. What do these numbers have in common?*
Beren : *That being divisible by 6. Whether or not the numbers in this number sequence are divisible by 6, we are trying to prove, I mean, are divisible.*

She remembered what to do as a first step when the researcher said to her, "I wonder if these are the numbers that give a remainder of something when divided by something?" After a few attempts, she came up with the correct modular structure.

Researcher : *How much do they increase each time?*
Beren : *Oh yes, I could find it directly from there. They increase twelve by twelve. Then 12, 24; these are all the numbers that give a remainder of 6 when divided by 12. There was an easier method from there. It hadn't occurred to me before.*

As soon as she figured out the correct modular structure, she asked:

Beren : Now, can I do it by looking at one of the charts over here?

She was allowed to structurally map her first instance proof to the current task. In fact, even if she had not asked herself, if she were not able to do the activity, she would be encouraged to do the same. This was to encourage her abstraction of the commonality between the conjecture-proof relationship defined as the AoC concept of the study. Beren completed the flowchart as in Figure 4.9 and went back to the task of stating the conjecture to be proved.

Beren : All the numbers that give a remainder of six when divided by 12 can be divided into six equal parts […] divisible by six without a remainder. The statement that we want to prove is whether or not all the numbers that give a remainder of six when divided by 12 are divisible by six, that they are divisible.

Her understanding of the resulting proof was at the level of abstract structure. It was especially important that the property of the arithmetic number pattern be observable in her expression as an abstract structure.

Researcher : So, are we sure it will be (true) for all numbers?
Beren : Yes.
Researcher : Even for the numbers beyond billions?
Beren : (Probably misled by the question) No, we are not sure about that.
Researcher : Why are not we?
Beren : But here we take all the numbers as a basis and it is affirmed. That would be true, I think. Because, anyway, the amount inside these doesn't matter. The amount that goes into these [cups] doesn't matter. What matters is the number here. (Circles around the general representation with her finger)

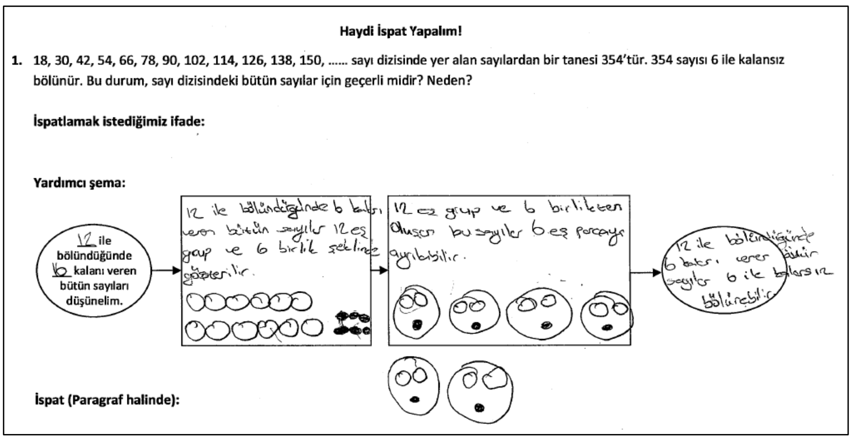


Figure 4.9 Beren's flowchart proof of Conjecture C

In the next task, she was watching to see if the same structure would apply to the given conjecture.

Conjecture D

Is the statement that numbers that give a remainder of 5 when divided by 8 are odd numbers, true or false?

This was the first time that being odd was integrated into a learning task. (Others were encountered previously in Phase 2, where Beren's empirical examinations were not interfered with.) Since Beren did not yet relate being odd to giving a remainder of one when divided by two, she did not perceive this conjecture as analogous to the one proved in the first instance.

Beren : *(Reads the conjecture) But this is now a little bit different (glanced for a moment at the previous flowchart she had set aside, seems disappointed)*
Researcher : *May be. We will think about this step by step.*
Beren : *But it won't be like the one here… you know (Shows her previous flowchart).*
Researcher : *Why not?*
Beren : *You know, here we did it differently. In here, we found out whether [they] were perfectly divisible or not. [Now] here, being odd or not… I did not like it.*

The researcher encouraged her to try engaging in the activity.

Researcher : *What does it mean to be an odd number?*
Beren : *It means not being divisible by two.*
Researcher : *So?*
Beren : *So, it means being odd, as you know.*
Researcher : *So, we will examine it with respect to being divisible by two.*

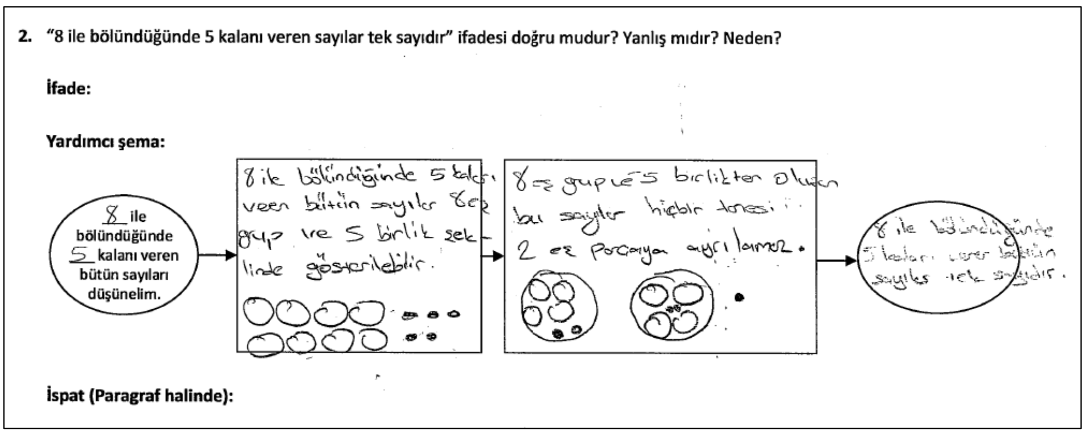


Figure 4.10 Beren's flowchart proof of Conjecture D

Beren thought silently for some time without attempting to take any steps. What happened next revealed that even though she was able to successfully take the first action to initiate the proof, she resisted taking this first step because she could not figure out the second action needed to obtain the proof structure she expected. This might be supporting evidence for how the AoC concept of the study articulated at the outset—defining a class of objects (arithmetic number patterns) to prove a particular class of conjectures—may frame the development of operational thought characteristic of the deductive proof scheme.

Beren	: *When divided by eight… let me think. But, now how can I write it?*
Researcher	: *You can think.*
Beren	: *I will be thinking a lot.* *…*
Researcher	: *Did you do the first step?*
Beren	: *No, I could not.*
Researcher	: *I will ask why. What did you have trouble with? Can you tell me?*
Beren	: *About how to draw the flowchart. Not a problem about proving… Well, it's a very different chart.*

When the researcher told her to take only the first step of the activity, she was quick in filling in the related part correctly (see Figure 4.10).

Researcher	: *You can do the first step in the same way. Let me give you such a tip.*
Beren	: *Okay, then. We will do like this. By eight, now I know by heart, and all the numbers that give five. Eight equal parts. Can be shown as eight equal groups and five singles, we would say.*

However, the second action did not follow with ease. She paid much attention to her observation that the numbers five and eight had no common divisors and concluded that no number could perfectly divide the structure 8k+5. Trying to understand why she was considering division by numbers other than two, the researcher revoiced Beren's thinking to her.

Researcher	: *Here is how I understand you: For a number in this structure, there is no number for which we can say it is "divisible by."*
Beren	: *Exactly.*

Later, it turned out that what constrained her thinking was her difficulty with defining odd numbers in terms of their constant *remainder* with respect to division by two. As can be seen in her expression reported above, Beren knew that being odd meant not being divisible by two. But she did not consider that a remainder of one as a result of division by two could be purposefully used to reveal that something is not divisible by two. This was not about her lack of understanding of division operation. Rather, it was about her definition of being odd. Although she mentioned that dividing into two equal groups would result in a remainder of one, she did not consider drawing the result of an equally sharing situation as the second action to be used in the flowchart proof. When she carried out this action by following the researcher's direction and observed the single remainder, she called the structure 2(4k+2) +1 "odd" without hesitation.

Beren	*: Well, in here (case-based flowchart used in the modified task sequence) it said "can/cannot be separated." Can't I say this way? All the numbers consisting of 8 equal groups and 5 singles cannot be divided into 3 equal groups, can't I say? Because, there it says "can/cannot be separated." So, shall I do this same again?*
Researcher	*: Okay, you can say "cannot be separated", there is nothing wrong with it. So, why are you dividing by 3?"*
Beren	*: No, I can try another number as well, not 3.*
Researcher	*: Which number?*
Beren	*: But it is not divisible by any number. Why? Because it does not fall equally into any of them, even if it's two, because, in any way <u>one more</u> will go into one of them.*

The above excerpt reveals that she was also constrained by the language used in the case-based flowchart of the modified task sequence, where the third box requested selection between two options only: can or cannot be separated into three equal groups. We interpret that she was confused about both how to represent a "not divisible" decision and how to label an appropriate second action before attempting to draw it. After completing the flowchart under the researcher's guidance, she summarized the difficulty she experienced.

Beren	*: I was surprised to see a different question than the ones here, let me say that.*

In the next teaching experiment section (ten days later), she was asked to evaluate the three conjectures given below. Due to the ten-day break, the conjectures were selected from the previously addressed ones to help her remember the flowchart activity.

Conjecture E

All the numbers that give a remainder of 4 when divided by 8 are divisible by 4. (True/False)

Conjecture F

All the numbers that give a remainder of 3 when divided by 8 are divisible by 2. (True/False)

Conjecture G

All the numbers that give a remainder of 5 when divisible by 8 are divisible by 5. (True/False)

First, she reviewed why it would not be appropriate to give examples. Then she tried to remember about the flowchart proofs she did before and, thinking step by step, successfully described the whole activity.

Beren	:	*Aah! We would do that thing; we would do a chart. Wonder if a chart works? Can I do it with a chart?*
Researcher	:	*You can try.*
Beren	:	*But I do not remember the chart.*
Researcher	:	*What would we do in a chart? Do you remember the things we did?*
Beren	:	*Not exactly. In the first, the rule was stated.*
Researcher	:	*Uh-huh.*
Beren	:	*In the second would draw a shape and say that rule… if or not… Huh! would show like a general representation. In the third would do that you know […] would divide into equal groups or something. I do not remember exactly what would do in the third step. Well, would write the text and leave some space and we would into three groups, four groups… Alright! I remember. Now I will do that. In the first one, I will write the rule. In the second, I will draw the general representation. In the third, I will say that all the numbers that give a remainder of 4 when divided by 8 can be divided into 4 equal groups. If… Okay well, that's easy. I figured it out by thinking.*
Researcher	:	*Let's do it.*
Beren	:	*Do we have an exemplary chart? Can I do it by looking at…*

The researcher provided Beren with a blank flowchart template. She filled it in fluently to obtain the proof given in Figure 4.11. Then she went back to the original activity sheet and selected the option "true" for the conjecture. In the provided space, she was guided to write a proof text by putting together the ideas involved in the flowchart.

Beren	:	*So, here I'm marking that it is 'true'. Do I need to write anything here?*
Researcher	:	*Now that the flowchart helped you (think) and you have explained in there. What would you write here?*
Beren	:	*Well, I would again tell the chart's thing here, I mean, how I did that. 'I drew a flowchart like this and this'. At the first place I showed what I would do… that I would do that by thinking of*

what, what to take as a basis. In the second step, well, shall I tell you directly what I am going to write? I can write that too.

Researcher : *Rather than telling the flowchart you can do like this. You know the part in here, you can write that, then continue with this one. So, you can combine these (parts) all. A paragraph appears.*

Beren : *Aha! Yes, I write that.*

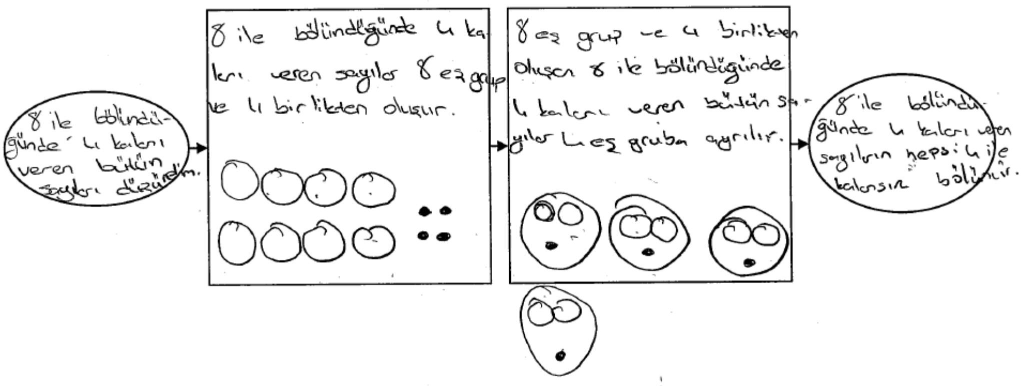
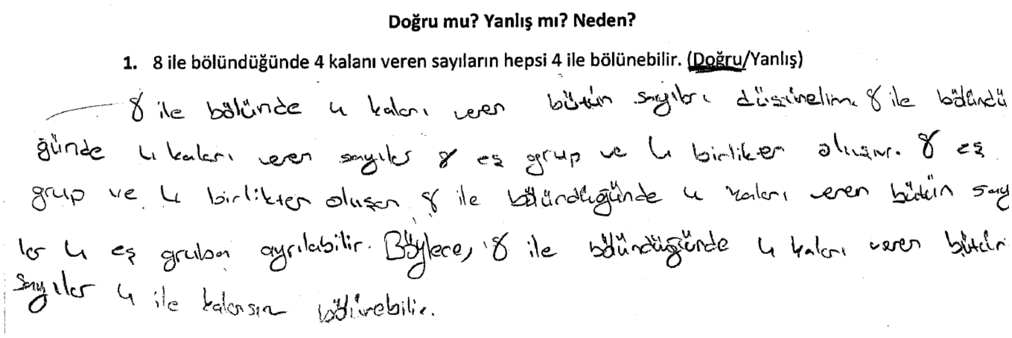


Figure 4.11 Beren's flowchart proof of the Conjecture E

The resulting text is given in Figure 4.12.



Doğru mu? Yanlış mı? Neden?

1. 8 ile bölündüğünde 4 kalanı veren sayıların hepsi 4 ile bölünebilir. (Doğru/Yanlış)

8 ile bölünde 4 kalanı veren bütün sayıları düşünelim. 8 ile bölündüğünde 4 kalanı veren sayılar 8 eş grup ve 4 birlikten oluşur. 8 eş grup ve 4 birlikten oluşan 8 ile bölündüğünde 4 kalanı veren bütün sayılar 4 eş gruba ayrılabilir. Böylece, 8 ile bölündüğünde 4 kalanı veren bütün sayılar 4 ile kalansız bölünebilir.

Figure 4.12 Beren' written proof of Conjecture E

The whole connected chain of reasoning was an abstract structure separated from the particular way of representation.

Researcher : *Now, can a friend who has never seen this flowchart understand this paragraph?*

Beren : *She can. Even… well if she thinks a little, if does … having never seen the chart, not exactly the same one of the chart of course but… For example, if we say her 'show it by drawing it as a chart, and if we say show these by drawing', I think, you know she understands what kind of thing she should do. I think, she can do a similar thing.*

She quickly determined that the next conjecture was false.

Conjecture F

All the numbers that give a remainder of 3 when divided by 8 are divisible by 2. (True/False)

Her reasoning, at first, was based on distributive law for division, which can be seen as further evidence for her possession of the intended AoC concept. Distributive law for division, as an abstract structure, was part of the commonality to be abstracted among the class of conjectures targeted by the AoC concept. Beren revealed this abstract structure several times in the data set.

Beren : Now, I will say that I think there is a logical error in this sentence first of all.
Researcher : Why do you think so?
Beren : After all, not all numbers that give a remainder of 3 when divisible by 8 are divisible by 2. Well, it's okay here (compares to the previous conjecture), a person who didn't know about this beforehand could do something. It gives the remainder of 4 and knows that all of them are divisible by 4. 8 is a multiple of 4. But 8, 3 and 2 are numbers that have nothing to do with each other.
Researcher : How are they need to be related?

She did not explain more but concluded with a counterexample.

Beren : Now, 8, 9, 10, 11 for example is the simplest example. 11 cannot be divided by 2. Falsified even at the first example.

For the conjecture G similarly, she provided a counterexample.

Conjecture G

All the numbers that give a remainder of 5 when divided by 8 are divisible by 5. (True/False)

Beren : Again, let me try this for a first example to see whether it holds or not. 8, 9, 10, 11, 12, 13. 13 cannot be divided by five. You know, there is no need to draw a long chart and try for these ones. They are already gone from the first example.

In summary, Beren's thinking and acting during Phase 4 revealed a variety of indications that she abstracted the intended (rudimentary) main idea of the first instance proof. She demonstrated a quick ability to distinguish between the conjectures for which the related flowchart structure was appropriate and those for which it was not (spontaneously attending to the distributive law of division). For the conjectures sharing the same structure as the first instance proof, she created same structure proofs without much difficulty. For the Conjecture D that she perceived "different" due to involving the label “odd” instead of “divisible by two,” she did not think to use the flowchart activity. For the false conjectures, as well, she did not engage in the flowchart activity but provided counterexamples. In addition to these instances, she demonstrated awareness that the flowchart activity she learned was not compatible with an extended AoC conjecture (reported at the beginning of the next section) that was directed at the end of Phase 4 as a pre-cursor to Phase 5. Her ability to distinguish between the cases when the flowchart activity was appropriate and when it was not was an indication that Beren possessed the intended AoC concept of the study. In addition, this section documented the difficulty Beren experienced in defining odd numbers in particular and in recognizing the defining act in general.

Before moving on with Phase 5, the form of the flowchart was changed back to the originally planned one as in Figure 4.13 and Figure 4.14. This was needed for extending the flowchart format to address the extended AoC concepts of the study. Beren seemed to have no problem adapting to the new form. For the flowchart proof in Figure 4.13, she said the following:

Beren	*: I got this one. I figured it out the first moment I saw the flowchart.*
Researcher	*: What did you see there?*
Beren	*: All the numbers that give a remainder of 4 when divided by 8 are divisible by two. The chart conveys this.*

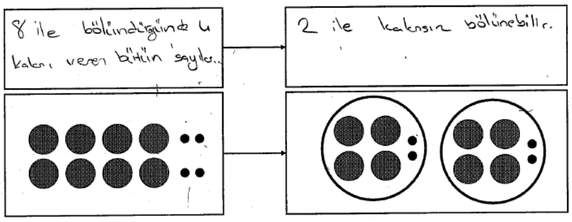


Figure 4.13 A first task to transition to the new flowchart format

In a next task (given in Figure 4.14), she used the new type of flowchart to prove a conjecture she completed herself.

4.3. Construction of the extended AoC concepts and further practice of proving

The main purpose of this section is to present an analysis of Beren’s goal-setting behavior reflected in her way of approaching to a variety of proving tasks. During Phase 5, Beren learned about proving conjectures that had structures other than the AoC concept of the study – the only conjecture-proof structure that was available to her up to that time. Therefore, the Phase 5 carried the potential of providing Beren with further instances of definitional reasoning; and concurrently providing us useful data on whether diversifying her ways of understanding (new structures of conjecture-proof pairs) would influence her way of thinking at some point. This section analyzed how Beren approached to the tasks of proving conjectures during and after learning about the extended AoC concepts of the study (see Table 4.3); how she responded to unfamiliar conjectures, each time a new one was posed. The analysis aims to answer the third research question of the study:

> How does the student (attempt to) start proving other conjectures that are (a) similar (but not analogous) to the first instance conjecture (including the target conjecture of the study) and (b) novel?

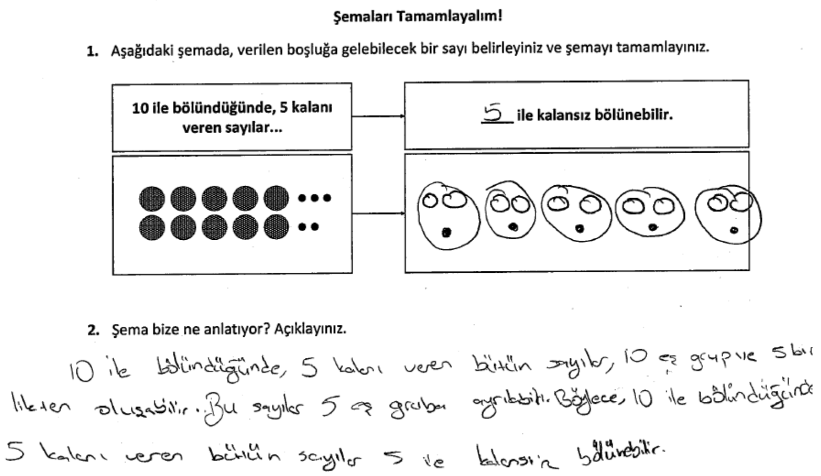


Figure 4.14 A second task for transitioning to the new flowchart format

The study of extended AoC concepts necessitated two specific advancements in the flowchart structure used so far. First, the sums being modeled were connected to separate boxes for visualizing the addends that make up the sum. Second, the relationship between the addends (whether they are instantiated depending on one another or not) was reflected in a checkbox in between.

The first aspect of the advanced flowchart was introduced to Beren, in the last session of Phase 4, when she was asked to prove a conjecture including a sum as the first time. The placing of the task was preferred to capture her first reaction to a related but new type of conjecture immediately after mastering the class of conjectures analogous to the first instance proof. Also, it was before taking a one-month break before Phase 5 began. The AoC-E1-type conjecture stated that: “A number that gives a remainder of 1 when divided by 6 is added up with 2 more of itself. What is the remainder after dividing this sum by 6?” At first, the conjecture sounded complex to Beren. To understand the relation correctly, she constructed the following example: 7 + 9 = 16, 16 = 6(2) + 4. But she did not know how to prove the relationship for all such cases. She was given the extended flowchart template and invited to guess about its use. However, the new structure did not make sense to her until told by the researcher.

Beren : *Does the chart start from here? (The middle box) Or from here? (The first addend)*
Researcher : *I don’t know. Just think about it, how it can be (used)? ... What do you see in there?*
[…]
Researcher : *When you look at here (the whole flowchart), is there anything you say “I know” about?*
Beren : *Yes.*
Researcher : *What is that?*
Beren : *A number, that here (points to the middle box for sum) I can understand that the sum of a number that gives a remainder of one when divided by 6 and two more of this number is 16 when I do it for a single example. The only thing that I know for now is this. I have no idea what to do.*

Table 4.3 AoC-Extension 1 and AoC-Extension 2 concepts

(AoC-E1)	Conjectures about sums of (two or more) numbers from arithmetic patterns *Properties of addends instantiated dependent on one another* If a number that gives a remainder of 1 when divided by 6 is added up with 2 more of itself, the sum is always divisible by 3.
(AoC-E2)	*Properties of addends instantiated independent of one another* If a number that gives a remainder of 1 when divided by 6 is summed with another number that gives a remainder of 2 when divided by 3, the sum is always divisible by 3.

After the researcher explained to her that the first two box shows two addends of the sum contained in the middle box, she completed the flowchart as in Figure 4.15, for the specific case of 19+21 only. Even if she used the cups and counting items system in her work, her explanation of the resulting flowchart did not make a connection to the desired abstract structures. She merely addressed the specific sum being examined and did not compare her work to a mathematical proof. What Beren took out of doing this task was a rudimentary understanding of the flowchart structure in terms of representing the sum of two addends. This understanding was maintained when she later responded to a Phase 5 task.

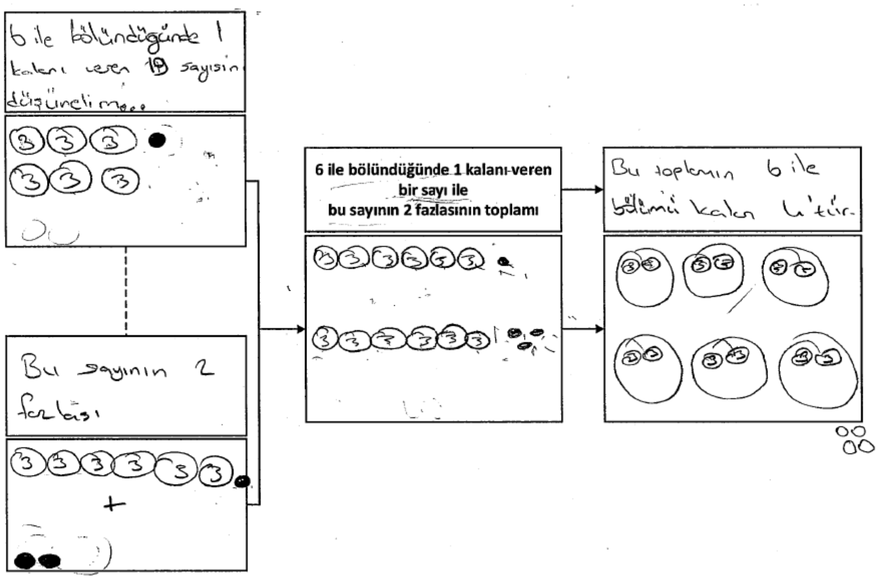


Figure 4.15 Beren’s flowchart for the case of 19+21

4.3.1. Starting Phase 5

Since Phase 5 started after a one-month break from Phase 4 (due to the summer vacation), its first task aimed to briefly review key ideas involved in the previous phases. Beren was asked to evaluate an empirical argument (by the hypothetical student Eren) for the AoC conjecture “All the numbers in the sequence 14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94, 102, … gives a remainder of 2 when divided by 4,” and then prove it herself.

> **Question:** “All the numbers in the sequence 14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94, 102, … gives a remainder of 2 when divided by 4,” Do you agree with this statement? Explain why.
>
> **Eren’s response:** I agree with this statement. Because:

14	4	22	4	46	4	102	4
-12	3	-20	5	-40	11	- 8	25
2		2		6		22	
				-4		-20	
				2		2	

> When all these numbers are divided by 4, the remainder is 2. I think the result is the same for the rest of the numbers.

Beren did not accept Eren's argument as a valid proof and constructed the below flowchart given in Figure 4.16. She remembered and applied the first action herself after thinking for some time, but completed the second action only after receiving the researcher's guidance. Her understanding of the flowchart argument was at the level of abstract structure.

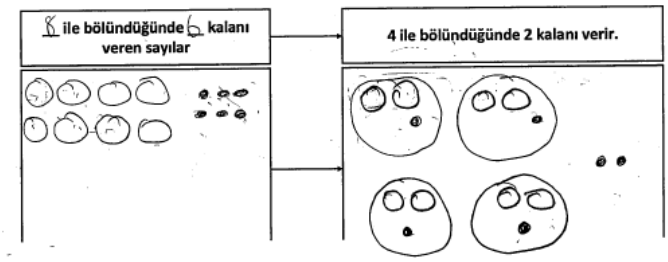


Figure 4.16 Beren's proof for the conjecture "All the numbers in the sequence 14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94, … gives a remainder of 2 when divided by 4."

4.3.2. An obstacle to the development of the extended AoC concepts: Difficulty in accepting the lack of closure

At the beginning of Phase 5, a modular structure ($qk+r$) was an abstract structure for Beren, representing any and all elements in an arithmetic number pattern; however, it was not a "fully-fledged mathematical object" (Sfard, 1991) yet. As expected, Beren was not able to do addition operation with two such structures. Given the following conjecture, she spontaneously determined two modular structures corresponding to the given number patterns. But she could not figure out what to do with these two structures and how to think about the intended sum.

Conjecture 1

Two different number patterns are given below. Emir picks one number from each of the two number patterns and adds them up. He observes that these sums are a multiple of 4.

The first number pattern: 6, 10, 14, 18, 22, 26, 30, 34, 38, …

The second number pattern: 14, 22, 30, 38, 46, 54, 62, 70, …

The numbers Emir selected: 6 + 30 = 36 36 is perfectly divisible by 4. ✓

10 + 22 = 32 32 is perfectly divisible by 4. ✓

18 + 54 = 72 72 is perfectly divisible by 4. ✓

Do you think this result is always true? Prove your answer.

Beren	:	*There is a difference of four among them. In the other one as well there is a difference of eight … I think Emir's conclusion is not always correct. This is actually the same as Eren's (empirical argument). It starts from two or three examples. I think I can solve this again with a small (minor, simple) graph (flowchart).*
Researcher	:	*OK, let's try it.*
Beren	:	*I can even do this directly by speaking.*

In her speech, she attempted to check if each of the two structures were individually divisible by four. Her goal was to apply distributive property to determine the final decision. Inspecting the defining property of the sum seemed not to be part of her goal.

Beren	:	*Let me say so. The numbers that give a remainder of 2 when divided by 4 the first number sequence. Now we have to prove its divisibility by 4 without a remainder, but that didn't work either. Ugh!*
Researcher	:	*What did not work?*
Beren	:	*Numbers that give a remainder of 2 when divided by 4 cannot be divided by 4 without a remainder. I'm down.*

Since the two structures were not individually divisible by four, distributive property was not applicable. She asked for a flowchart template to try to prove the conjecture using the flowchart structure. Her expression about the middle box allotted to the sum revealed that Beren expected the addition operation to result in another modular structure; that is, she considered the sum of the two structures an exercise to be "finished." This marked a phenomenon addressed in algebra learning literature as a cognitive difficulty in "accepting lack of closure" (Collis, 1974).

Beren	:	*Do we have the graph thing?*
Researcher	:	*Which one do you want?*
Beren	:	*It doesn't matter, I mean a graph.*
Researcher	:	*Now I am giving you a slightly different graph. Do you know why do I give this one?*
Beren	:	*Didn't we make a graph like this? There was a similar one, I remember it…. OK, this is how I do it. Numbers that give a remainder of 2 when divided by 4, and numbers that give a remainder of 2 when divided by 8… Sorry 6. I write numbers that give a remainder of 6 and draw their graphs here. After that, I come here (the middle box for sum) and <u>there is something else here in the form of that gives whatever the remainder when divided by whatever</u>. If so, can I do the [labels of the addends]?*
Researcher	:	Of course.
Beren	:	*Well, I think so. (Labels the addends and draws related representations) There is to be something here (the middle box)*

and the result of this is to be something divisible by 4 without a remainder… I can't figure it out…. Ugh! But this is very difficult. This never used to happen.

She did not know how to proceed with the middle box. The flowchart activity was quit to return later, after solving the lack of closure problem. This result was expected based on the last task of the Phase 4 (about an AoC-E1 type of conjecture) and was re-assessed here for an AoC-E2 type of conjecture. Presentation of the conjectures differed in two cases, and the latter case better revealed the source of the student's difficulty.

4.3.3. A contextual task for promoting the acceptance of lack of closure and variable distinction

One way of overcoming difficulty in accepting the lack of closure is outlined in the algebra learning literature as "[p]roviding students with a broader context in which not completing the expression makes sense" (Tirosh, Even, & Robinson, 1998, p.62). The contextual task, in Figure 4.17, aimed to overcome this difficulty and simultaneously address the issue of variable distinction. The second aspect was necessary to support the same structure as the flowchart activity that was just left to be returned. The task included examination of the contextual situation for potential cases by using case-by-case flowcharts in succession. This was to promote the student's ability to switch between the structural and operational conceptions of the same construct – the sum of two modular structures. Similar to the modified task sequence of Phase 3, after realizing the invariant structure repeated in case-by-case flowcharts, construction of a general flowchart was promoted.

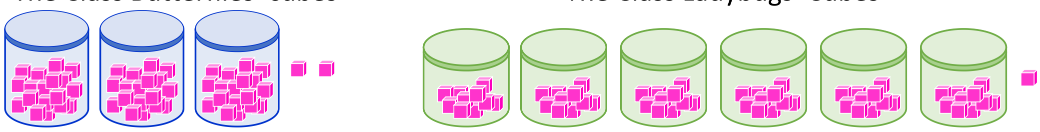
Beren correctly determined the answer to the contextual situation.

Researcher : How many groups?
Beren : Three, so easy.
Researcher : How?
Beren : If there are three groups, they share equally. Now they get one of these (blue boxes), two of these (green boxes) to each group. One from here, two from here, One from here, two from here. And one to each… these extra cubes are given to each group. That's all.

The Linking Cubes Game

Butterflies and Ladybugs are two kindergarten classes at Atatürk Primary School. The two classes keep their linking cubes in their toy boxes, with an equal number of cubes distributed into each box. The remaining cubes are placed on their toy shelves. Below are the cubes of the Butterfly Class and the Ladybugs Class.

The Class Butterflies' cubes   The Class Ladybugs' Cubes



Butterflies and Ladybugs are designing an event to play the Linking Cubes Game together. With how many groups can this game be played; the groups can share all the cubes equally without any leftovers?

What numbers can be the numbers of cubes these two classes have? Let's think.

(To be repeated for a number of flowcharts)

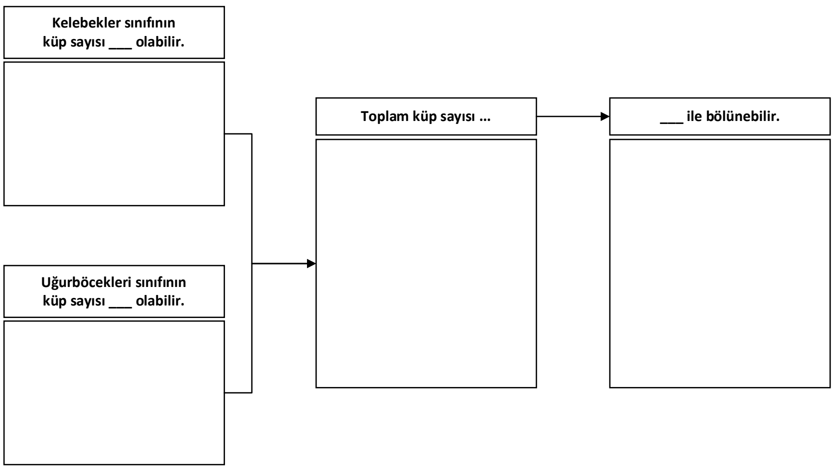


Figure 4.17 The contextual task used to overcome the difficulty in accepting the lack of closure and address variable distinction

Then, she filled in two case-by-case flowcharts as in Figure 4.18.

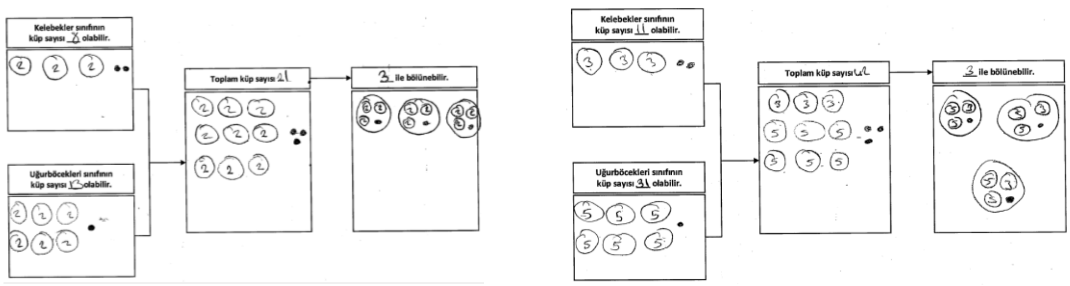


Figure 4.18 Beren's case-by-case flowcharts

When she struggled with the middle box of the second case-based flowchart (on the right) (she attempted to and failed to make cups contain an equal number of counting items by redistributing threes and fives into "some number of" cups), the contextual situation helped her represent the intended addition operation correctly.

Beren	:	*You know, it was easy in this one (when all the cups contained two items – on the left), but it got challenging as the numbers got different (on the right).*
Researcher	:	*Let's think this way. Cubes of the Butterfly class stand on a table. Exactly like this… the way we see it. The cubes of the Ladybugs class are on another table… just like this. Imagine bringing them all together and putting them on the same table. This is, say, the teacher's desk.*
Beren	:	*Ok. There will be three inside these (cups) if I think so. (Continued writing threes and fives in the related cups in the middle box.)*

The last line above was an indication that she accepted the lack of closure in adding two modular structures. After the completion of the middle box, she focused on the sameness between the last steps of the two case-based flowcharts.

Researcher	:	What are we trying to do now?
Beren	:	That… what the number this (sum) is divisible by… It's divisible by three, the same.
Researcher	:	Why is it the same?
Beren	:	Because we put (into the three groups) one… two of these one of these; two of these one of these; two of these one of these. Yes, then divisible by three. Aha! This is easy actually.

When she transitioned to doing a general flowchart, she seemed to be mentally keeping track of the distinct cups contributing to the sum coming from distinct addends. When she was talking about distributing the total of nine same-looking cups (not color-distinguished yet) in the middle box to three equal groups in the last step of the activity, she said "We put one of these and put two of these" to each group while she respectively pointed to the first line of the three cups and the remaining two lines of the six cups. This was further indication that she accepted the lack of closure as an abstract structure. The researcher offered to her differentiate between the two types of cups to make remembering it easier. Thus, she colored the cups she used for only one of the addends, as in Figure 4.19. She also was aware that the two types of cups were not meant to contain necessarily different numbers of counting items.

Beren : (The number) inside these (colored cups) are all the same and these (non-colored cups) as well. But it is not certain whether these two (a colored and a non-colored cup) will be the same.
Researcher : That's exactly why we made such a small distinction between the two so that we wouldn't confuse them.
Beren : Exactly.

Thus, both objectives of the task were successfully fulfilled. This made Beren ready for practicing AoC-E2 concepts.

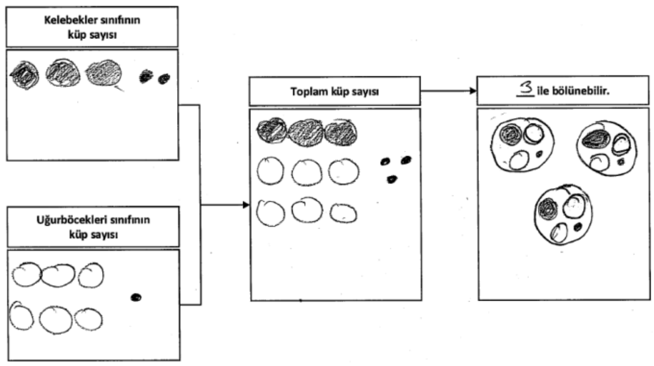


Figure 4.19 Beren's general flowchart

4.3.4. The AoC-E2 concept

Turning back to Conjecture 1, she reviewed the conjecture statement and completed her awaiting flowchart activity (given in Figure 4.20 below). Only help by the researcher was about the language of the middle box.

Beren : I think (Emir's argument) is not always true. I've already drawn this (addends) here. I am also writing here… is it okay if I write the sum of the 2 charts directly?
Researcher : It works. Not a chart but what are they? From the two number sequences…
Beren : The sum of two number sequences.
Researcher : It is like the sum of one element each from the two number sequences, isn't it?
Beren : Hah okay. (Writes the phrase and repeats) The sum of one element each from the two number sequences. What can it be? I will combine them directly. (Completed the middle box)

She spontaneously applied the variable distinction and anticipated that the resulting flowchart would qualify as a for-all argument.

Beren : Now in this [last step] I will draw that four of these and equally distribute into them and one who will look at and see this understands I think. One doesn't understand it all from here

		(Emir's confirming examples), but does understand from here. I am coloring the inside of these (cups), right?
Researcher	:	*Why do you want to color?*
Beren	:	*Because may be different.*
Researcher	:	*Don't bother painting them all if you want. Just put a mark.*
Beren	:	*Let me draw a line like this. (Marked all the cups related to the first number pattern across the flowchart steps)*

Beren reflected that her last two flowcharts shared the same structure. This was an indicator for the presence of the intended AoC-E2 concept.

Beren	:	*Actually, this is very easy (compared to what I expected). After solving this one (points to the previous three flowcharts – two case-by-case and one general), this came along right away. […] This makes me very happy all of a sudden. You know, when I myself say aha and I find it.*

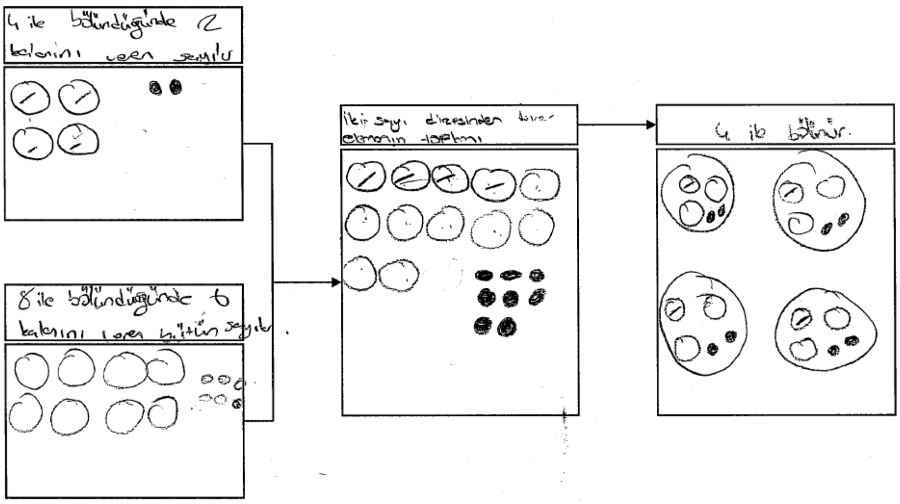


Figure 4.20 Beren's flowchart proof for Conjecture 1

The next session started with (making and) proving three AoC-E2 conjectures in a row.

Conjecture 2

Two different number patterns are given below. Eylül will pick one number from each of the two number patterns and adds them up. This sum is always divisible by ___ without a remainder.

The first number pattern: 9, 16, 23, 30, 37, 44, 51, …

The second number pattern: 12, 19, 26, 33, 40, 47, 54, …

Conjecture 3

<u>Two numbers</u> are picked from the number pattern that continues as 10, 17, 24, 31, 38, 45, 52, 59, 66, …, and added up. When this sum is divided by 7, the remainder is always ___.

Conjecture 4

<u>Three numbers</u> are picked from the number pattern that continues as 10, 17, 24, 31, 38, 45, 52, 59, 66, …, and added up. When this sum is divided by 7, the remainder is always ___.

Given the Conjecture 2, Beren first examined three cases 9+12=21, 37+33 =70, 23+26=49 and determined that they are all divisible by seven. Then, she seemed convinced that they were all divisible.

Beren : *It is perfectly divisible by seven. Can I prove it? I can't prove it but, in a way, though, (they are) divisible.*

Contrary to the researcher's expectations (because Beren generally seemed to attribute her success in proving to "doing [it] with a graph"), she did not request a flowchart template. The underlying reason was not identifiable at the time of data collection, or during the retrospective analysis process either. The researcher encouraged her to remember the previous session's AoC concept proof (Conjecture 0) to see whether she would ask for an extended AoC flowchart template or not. She did not request one, but the researcher provided an extended flowchart template. Beren's thinking about the Conjecture 0 and the extended flowchart for Conjecture 2 revealed that she was struggling with remembering the expression "the numbers that give a remainder of ___ when divided by ___" to start the flowchart activity.

Researcher : *Yesterday, if you remember, you confronted Eren about something like this.*
Beren : *Yes.*
Researcher : *What did you do? What did Eren do?*
Beren : *Eren had done something like this (points to her three confirming examples). I had directly made a graph of it.*
Researcher : *Yes, what did you do on that graph?*
Beren : *The first graph, oh! But I forget those things. The thing (in) the first step (I) don't have it, I mean. […] If there was (a graphic) right now I would draw it, but I can't think of it.*
Researcher : *Well, let me give you a chart. (The researcher handed in the extended flowchart template.)*
Beren : *How do I start with this… nine… a second… 10,11,12,13,14,15,16 what… there is a difference of seven*

numbers. Between 12 and 19 as well there is a difference of seven numbers. Now I will start up here (The label of the first addend), this one (the representation of the first addend) and then this (the label of the second addend). I will try something […] No it wasn't. We used to make a grouping.

Researcher : What would we group? ... what do you remember?

Beren : I mean, you know… that we were going to draw here shapes or something, these came to my mind (the representations). But I don't remember how I filled in these (the labels). I have these in mind (the representations).

Researcher : If you want, you can start with the shapes and then write their names.

Beren : But wouldn't it be wrong ("ters")? […] I'm so confused. How is it going to get done? (Thought in silence for about a minute.)

Researcher : Then… Shall I remind you of that question the last time we made a comparison with Eren?

Beren : Okay.

Researcher : Alright. Now this the question was. Let's remember the question first.

Beren : (Browsed the conjecture "All the numbers in the sequence 14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94, 102, ... gives a remainder of 2 when divided by 4") gives a remainder of 2 when divided by 4. Huh! Always, we would write it here in the first place. It gives a remainder of 2 when divided by 4. We would draw it. You see, I get confused when it's not there.

Remembering how to name and represent the two given number patterns may seem to be useful first steps in creating the requested proof. However, Beren's purpose in remembering these components was not connected to the goal of obtaining the invariant (defining) characteristic of the intended sums. Rather, she was aiming to create a flowchart similar to those previously encountered.

Beren : We will find out by which number these (numbers) are divisible without a remainder. (On the one hand, she seemed to consider applying the distributive law of division by having the two addends individually divisible by seven.) Then, we will write here, you know. How should I say? This one… when divided by this number, or, the numbers divisible without a remainder. Here we will draw its graph (representation). In here as well we will do that for the second number sequence. Then, we will come to this part and show that thing of its. And this one is its result… that it is divisible by seven.

Researcher : What we will show in the middle?

Beren : I haven't found it yet either. You know… I'll find it after doing. (Completed the components related to the two addends and thought about the middle step for a while.) I will find it. Here, if I found these (the addends), I would find it quite (easily).

Researcher : Yes, what does the question ask for us (to do)?

Beren : *The question from us… This is a sum, hah! Sum! I will write here the sum of the numbers that give a remainder of 2 when divided by 7 and the numbers that give a remainder of 5 when divided by 7. But can't I just shorten it and write the sum of these numbers? Let me write the sum of these two numbers.*

She completed the rest of the flowchart as in Figure 4.21.

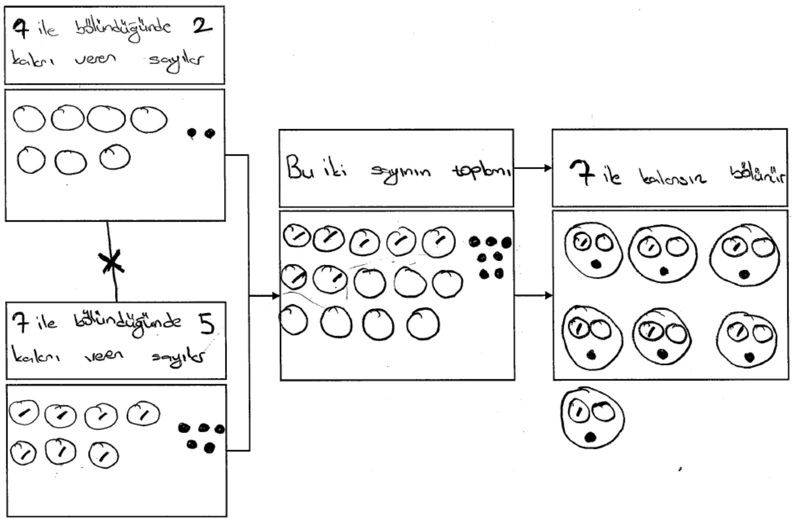


Figure 4.21 Beren's proof of Conjecture 2

Beren's understanding of the resulting flowchart proof was at the level of abstract structure except for an unidentifiable point. The abstract structure of the *sum* was not explicitly addressed in her expressions as *the invariant characteristic of every possible sum*. More accurately, it was not possible to claim its presence or absence though Beren's reported speech.

To summarize, Beren's work on Conjecture 2 revealed that she did not spontaneously set the goal of describing the invariant characteristic(s) of all possible sums. Her attempts to create the proof were mostly based on recalling the previous flowcharts. Even if she expressed an understanding of the resulting argument at the level of abstract structure (to some extent), she was not able to create it through her own goal-directed activity.

Conjecture 3

<u>Two numbers</u> are picked from the number pattern that continues as 10, 17, 24, 31, 38, 45, 52, 59, 66, …, and added up. When this sum is divided by 7, the remainder is always ___.

Conjecture 3 was slightly different than the Conjecture 2 as it required the two addends selected from exactly the same number pattern. We speculate, at first, Beren did not consider that a second box was needed for a second addend in the flowchart structure. Then, she found out that the two addends would look the same (except coloring).

Beren	*: This is easy. This can be done like this. (Points to the flowchart proof of Conjecture 2 that she put aside) […] But not on this chart, I guess.*
Researcher	*: Why not?*
Beren	*: Like this… No, it is on this chart, but how?*
Researcher	*: What did you think of?*
Beren	*: For a moment you know I thought like that only 1, 2, 3-step thing (respectively points to three points on the flowchart: a point in between the two addends – as if they were replaced by a single box, the middle box, and the final decision box) would be enough.*
Researcher	*: Why did you think that? I have a guess but…*
Beren	*: You know… I'll say the remainder is always [3] when (the numbers in the given sequence are) divided by 7 (points to the imaginary first step in between the two addends). Here I will add up those numbers. But there is something else here, isn't it?*
Researcher	*: No nothing else. I guessed the following. Well, here (in Conjecture 2) we see two number sequences, we write one here, the other one here, and add them up. But there (in Conjecture 3) when we see one (number sequence) one may get confused.*
Beren	*: Uh-huh. Yes, it is.*
Researcher	*: One may think this second one (addend) is not needed, just like in the old (AoC) charts.*
Beren	*: Is needed, right? (Sounds not knowing how to use a second one)*
Researcher	*: But here, too, (in Conjecture 3) we add up two numbers. (This line suggested Beren the following reasoning.)*
Beren	*: Aha! Yes. Again, in both of them I will write 'the numbers that give a remainder of six when divided by seven'. (Fixes her mistake with the remainder) The numbers that give a remainder of three when divided by seven (reads aloud the first addend). In this one as well I will write the same and draw the same [representation]. Next, I will say 'the sum of these two numbers' and in here I will find the result. Alright, now I am doing, drawing it. I will finish this and then will be explaining.*

She decided to make a difference between the two sets of cups as in Figure 4.22.

Beren	*: Shall I differentiate these (cups) now? Do I need to mark by lines? Because both are the same, gives a remainder of three when dividing by seven.*
Researcher	*: Let us see the question and decide if it is needed or not.*
Beren	*: But the numbers are different.*
Researcher	*: Can be different right?*
Beren	*: Uh-huh. That's why I am marking.*

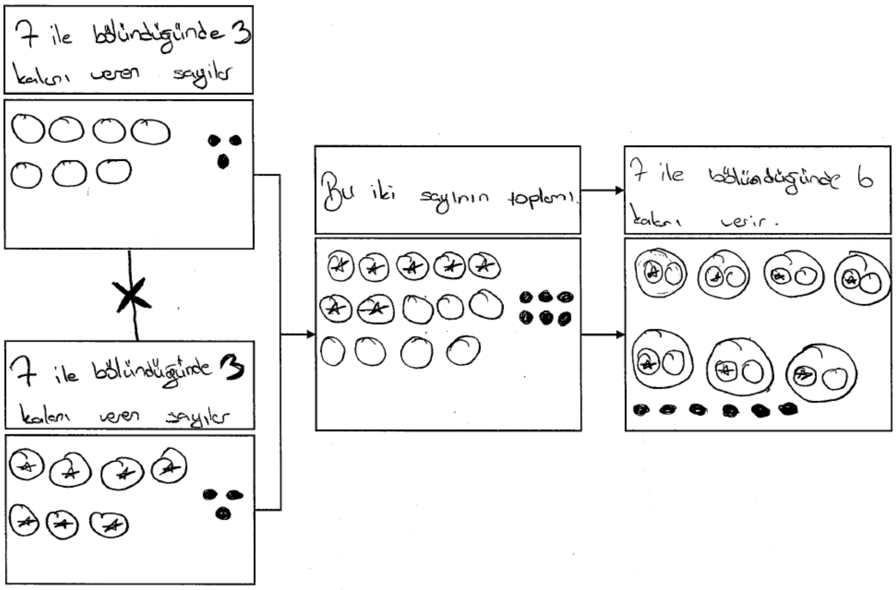


Figure 4.22 Proof of Conjecture 3

Her comment on the resulting flowchart indicated the presence of the intended AoC-E2 concept.

Beren : *I think I've proven enough. The same reasoning as here. Shall I tell you again?*

Conjecture 4

<u>Three numbers</u> are picked from the number pattern that continues as 10, 17, 24, 31, 38, 45, 52, 59, 66, …, and added up. When this sum is divided by 7, the remainder is always ___.

She was asked to tell the proof without using a flowchart.

Beren : *In here three numbers are selected. Well, the chart would be difficult with 3 numbers.*

Researcher : *What do we need to do? What is to change in the chart?*

Beren : *What is to change in the chart? Nice question. One more of these things at the beginning (addends) to be added, would not be? Here (in the middle) it is said the sum of these three numbers. The numbers (of cups) changes in this sum, not 7, 14, but 21 it becomes. And here (in the last step) the more… again seven boxes (equal groups). The remainder changes only.*

Next, she wrote the following proof text in Figure 4.23.

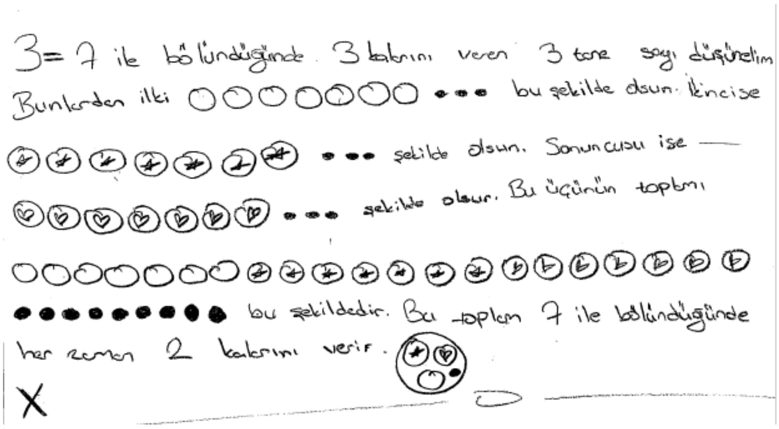


Figure 4.23 Proof of the Conjecture 4

The only guidance by the researcher was about the use of language (such as "let us think of three numbers") and using the general representation in a text, somewhere other than a flowchart. After she was encouraged to use the representation for the first selected number, just as in the flowchart, she completed the rest herself.

Beren : *I am directly like telling the triple of this chart. A remainder of three when divided by seven… well this one will be so difficult.*
Researcher : *Well, I'll help you.*
Beren : *Alright. …that give a remainder of three when divided by seven.*
Researcher : *... let us think about three numbers.*
Beren : *Okay. (Writes the full sentence 'let us think about three numbers that give a remainder of three when divided by seven.)*
Researcher : *Now the first of these…*
Beren : *Huh alright. Oh! Very nice. The first of these is … ten.*
Researcher : *Hmm. I thought more like… Not with the numbers but…*
Beren : *Huh! Ughh!!!*
Researcher : *Just like a chart you can directly draw there. It's not a problem at all, it might be even more understandable.*
Beren : *Should I say 'the first of these… what'?*
Researcher : *How can we show the first one? What would we write in the first box in the chart? (Beren shows the representation for the first addend) Alright we can draw the same there.*
Beren : *Should I draw this one here? (Affirmed) For the second, I am writing this by putting inside a different mark […] now let me put stars here, and I will be putting something else something like a heart (for the third one).*

Beren successfully adapted the main idea of the Conjecture 3 (a way of understanding) to Conjecture 4. Her above work indicated the abstract structure of the argument to some extent (with no evidence on the abstract structure of the sum, as before) was separated from the flowchart look.

4.3.5. The AoC-E1 concept

Reading the Conjecture 5 she examined cases and thought to complete the conjecture with a nine, as the highest possible divisor. Then she tried to decide whether to use a flowchart or not for proving the conjecture. She decided writing a proof text as she did with the previous conjecture.

> **Conjecture 5**
>
> A number that is a multiple of 3 is added up with twice that number. This sum is always divisible by ___ without a remainder.

She started the proof by writing "Let us think of the numbers that are a multiple of three." She listed the numbers 3, 6, 9, 12, 15, 18, 21, 24, 27, 30. After her trials close in meaning, the researcher helped her continue with "and let us add up (each of) these numbers with twice the number <u>itself</u>." Beren noted a confirming example even if she knew it would not qualify as proof.

Beren	:	*Shall I give an example? (Affirmed) (Noted her exemplary case as 9x2=18; 9+18=27; 27/9=3). So, it's like this and so will the other operations.*
Researcher	:	*What do you mean?*
Beren	:	*I mean with nine it is like this. <u>The operation thing with 27 will follow the same order as this</u>. I am talking about that.*
Researcher	:	*Hmm.*
Beren	:	*As can be seen in the example 9… Oh ugh! There has to be proof of this. How will I do that? (Sounds helpless) Oh, it's so difficult.*

She knew the steps to be followed (as implied in the above excerpt) but did not know what representation to use to start the process. The researcher exceeded her prescribed role to create an opportunity for her to engage in the act of defining multiples of three. At first, she attempted to influence Beren's way of thinking. (This is not problematic for the case of the current study because no matter what the researcher told her; she did not demonstrate the act of defining.)

Researcher	:	*Let us check the [Conjecture 4]. I have given you a sequence of numbers. And you to account for all the numbers here...*
Beren	:	*Uh-huh.*
Researcher	:	*…used something like this. Now in here what can we use to account for all the numbers … that through which we can think about all the numbers. Something has to help us. (An attempt to influence her way of thinking while her focus is on the general representation used in Conjecture 4)*

Beren : *We draw a graph like this but we do not know how many of these (cups) to have. […] Well, but for a multiple of three what…*

Researcher : *Which numbers are the numbers that are multiples of three? You wrote it here, didn't you? (Points to her list 3, 6, 9, 12, 15, 18, 21, 24, 27, 30)*

Beren : *Yes, I did.*

Researcher : *Do you think is there anything that we can use to account for these (numbers)? Something that will express/mean all of these to us (A second attempt to influence her way of thinking while her focus is on the list of numbers to be described – to support a more operational conception than structural and her knowledge of arithmetic number patterns)*

Beren : *We will be drawing such a figure again.*

Researcher : *How does it [look] this time?*

Beren : *By drawing the number that is a multiple of three… But no, I really can't do that. […] I don't know how to draw that.*

The researcher next attempted to influence Beren's way of understanding, which resulted in her construction of the visual proof in Figure 4.24 successfully.

Researcher : *What remainder does a multiple of 3 give when divided by 3? (An attempt to influence her way of understanding the particular situation)*

Beren : *[…] there is no remainder. No, there is no remainder. (Seems puzzled.)*

Researcher : *How can we show then?*

Beren : *How can we show this? (Thinks for about three seconds) Okay! I will draw three of circles, then by saying 'equals to twice of this,' I will draw twice of it. No yes…I will write that this way. Three of them, writing a plus, writing twice of that, add them up, equals, I will write the sum. Isn't that proof already? Aha! This is very nice.*

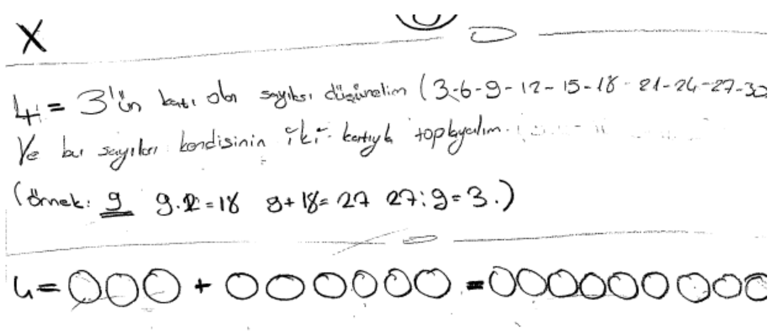


Figure 4.24 Proof of Conjecture 5

She did not make a differentiation between the cups appropriately. This was her first proof of the type AoC-E1.

Beren : *I am drawing three of them here, I only do not know how many are inside, plus. Yay! I am so happy. Plus one two three ... should I draw six here?*

Researcher : *Think about it.*

Beren : *I will draw six because I do not know how many are in these ones that I drew here and (it is) twice this, will be twice the shape here. Therefore, I will draw six of them. Actually, this is very easy (compared to what I expected). Result (is) When a number that is a multiple of 3 is added up with 2 times this number, the result is divisible by 9 without a remainder… always this sum is divisible by 9 without a remainder.*

Researcher : *Yes. Where do you see it is divisible by nine?*

Beren : *That there are 9 of them here. This is actually proof.*

Note that Beren was able to determine divisibility of a variety of modular structures by three in the last step of the proofs she created up to this point. She knew that distributing cups and counting items into three equal groups indicated divisibility by three, which is equivalent to being a multiple of three. However, she was not able to spontaneously use the same structure to begin with the above visual proof. (She might be constrained by her knowledge of the concept modular structure -with a non-zero remainder- as the only way of representing generality.)

Conjecture 6

An even number is added up with twice itself. This sum is always divisible by ___ without a remainder.

General representation of an even number was not immediately accessible to Beren. For Conjecture 6, she constructed the following argument (given in Figure 4.25) by chance, not as a product of correctly defining even numbers.

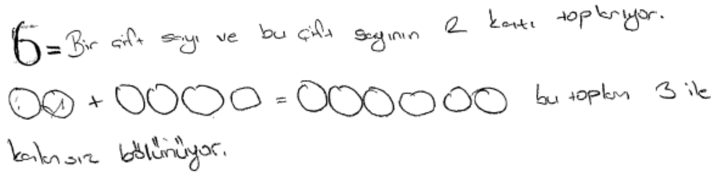


Figure 4.25 Argument for the Conjecture 6

Beren : *If I write here 'When we add up any even number with two times this number, we see that the sum is divisible by three'… Not that alright! An even number means… I will make something like this (Representation used in Conjecture 5, 3k+6k=9k) again, but something like this one (Representation used in Conjecture 4, 7k+3+7m+3+7n+3=7(k+m+n+1) +2). Is that okay if I start from two?I mean, does it show a definite result?*

Researcher : *What do you mean?*
Beren : *Well, let me tell you like this. I will write 'an even number and twice this number is added up' Then I will write here two plus four equals six. Can I do a proof writing like that?*
Researcher : *Let's write that first. Then we talk about whether that is proof or not.*

She was encouraged to write what was in her mind to be discussed after. The way she made sense of the notations she used was not always consistent. More specifically, she started with thinking about specific cases of even numbers but then ended up with the general representation. The change was triggered with the researcher's question.

Beren : *(Completed her writing) I think isn't that a good enough proof?*
Researcher : *Why do you think so?*
Beren : *Now we say even numbers and do not even numbers start at two?*
Researcher : *Uh-huh.*
Beren : *Inside this two… Not a two for example, this can be a four and this can be an eight.*
Researcher : *Yes.*
Beren : *Well, I spoke as if it was valid for every number, but of course not.*
Researcher : *Why not?*
Beren : *Because I started from two, I set out by meaning two.*
Researcher : *Is this a box? (Because her circles in Figure 4.25 look like the cups she used so for, but she interprets as singles occasionally in this data section.)*
Beren : *Yes, a box.*
Researcher : *Inside… in each of them have a cookie for example.*
Beren : *Yes.*
Researcher : *Then it shows us a two.*
Beren : *Yes Aha! (Aha moment) but then can be two shows a fou… Okay okay yes this is correct I think is a good enough proof.*
Researcher : *Can you explain more?*
Beren : *I explain. Here let us think for example cookies. Let me say now these have each two inside. Two-two makes four. These ones also have two each but makes what this makes eight. I added up [eight and four] makes twelve and by chance at that moment how should I say… to the patisserie comes three kids and I can equally distribute these (cookies) to those three kids… no matter how many are inside… in a way two packages go to each one.*

4.3.6. Introduction of the checkbox to the flowchart structure

Conjecture 7

From the two number patterns given below, a number is selected from each and added together. This sum is always divisible by ___ without a remainder.

2, 4, 6, 8, 10, 12, 14, 16, 18, …

4, 8, 12, 16, 20, 24, 28, 32, 36, …

At first, Beren worked on a number of examples to determine if their sum was divisible by 2 or 6 as the highest divisor possible. Encountered a counterexample for the latter alternative (32, the sum of 12 and 20, is not divisible by 6), she asked for a flowchart.

Researcher	:	*Aha! 32 is not divisible for example. I added up these 12 and 20, they are not divisible. Can I do a chart?*
Beren	:	*This (the first pattern) is the numbers that are divisible by two and this one (the second pattern) is the numbers that are divisible by four without a remainder. There was one example here, can I do it by looking at it? (Affirmed; from among the papers Beren had set aside before she found the flowchart made for Conjecture 3, see Figure 4.22). (Browsed the Conjecture 7 statement again and then checked her current flowchart where she already labeled the two addends) (For the first addend) I have said 'numbers that are divisible by 2 without a remainder'. (Checked the representation used in Conjecture 3) It's enough actually if I draw two (cups) here, right? I will draw those divisible by 4 without a remainder with 4 many (cups).*

She did not spontaneously think to differentiate between the cup types used for the two number patterns; however, answered correctly when the researcher asked about.

Beren	:	*The sum of these two numbers makes six, makes six of that. What did I do next (in proving Conjecture 3)? Can't I write 'always divisible by six without a remainder'? Because look at this.*
Researcher	:	*Look. Before you tried for some...32; it was not divisible by six.*
Beren	:	*Yes.*
Researcher	:	*Why did that happen?*
Beren	:	*(Checked again her operations for the case of 12+20) It is not six.*
Researcher	:	*Then, why there are six boxes here?*
Beren	:	*Because it is the sum of these two but we do not know how many are in them?*
Researcher	:	*So? (Irrelevant response) Do they all contain the same (number of items)?*
Beren	:	*No. Huh! Mark these two with a line, a star.*
Researcher	:	*That's why we use those stars as you know.*
Beren	:	*Yes.*
Researcher	:	*So that we don't get confused.*

Lastly, by comparing Conjecture 7 to Conjecture 6, the researcher introduced Beren to using either a checkmark or a cross-mark to indicate the presence or absence of a relation between the addends of any sum. (see Figure 4.26)

Researcher : Now I will ask you a last thing to do. Conjecture 6 and Conjecture 7 are somewhat similar, actually.
Beren : Uh-huh.
Researcher : Can you tell me the similarity?
Beren : Here (in Conjecture 6) an even number and twice that number is added up. This sum is always divisible by three. Here, let us write two in here. I mean here in fact two times the two is four. [Two times] the four is eight. [Two times] the six is 12. [Two times] the eight is 16.
Researcher : Contrary to that, is there a relationship between [the addends] here (in Conjecture 7)?
Beren : Isn't between these two.
Researcher : Then, let us put a cross-mark on this. This means inside the cups of these two boxes (boxes of the flowchart) need not to the same number of items. (The cross-mark introduced)
Beren : Yea.
Researcher : It might be the same… if as a coincidence.
Beren : Yea.

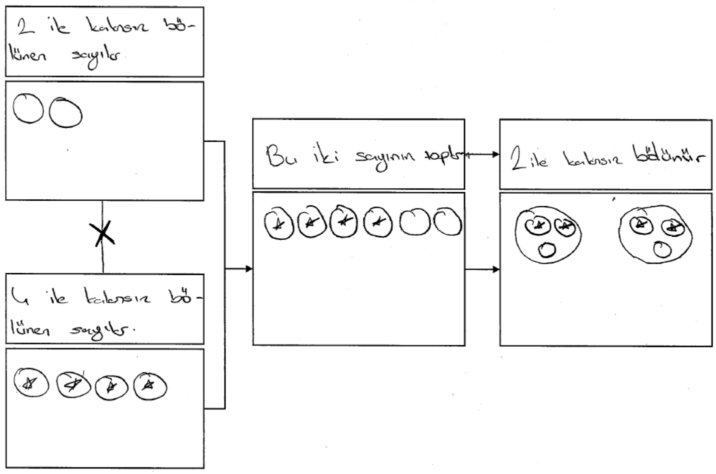


Figure 4.26 Proof of Conjecture 7

After marking Conjecture 7 with a cross-mark, Beren constructed a flowchart proof of Conjecture 6 and this time the checkmark between the addends was introduced as in Figure 4.27.

Researcher : What if we made a flowchart for the [Conjecture 6] Shall we draw that too? (Beren constructed the two addends) Is there a relationship between these [addends]?
Beren : There is.
Researcher : Why is there?
Beren : Because this is even number and this one is twice that (number) (that emphasized). In [Conjecture 7] for example, there are different (amounts inside the cups). But (in Conjecture 6), if there is one (item) in this [cup], it is one again in this one as well. If there is five, in this one is five as well.

Researcher : *Yes. Let's put some other mark here, like a tick, if you wish. Let us tick it when [the addends are] related and cross it when [they are] not related, okay?*

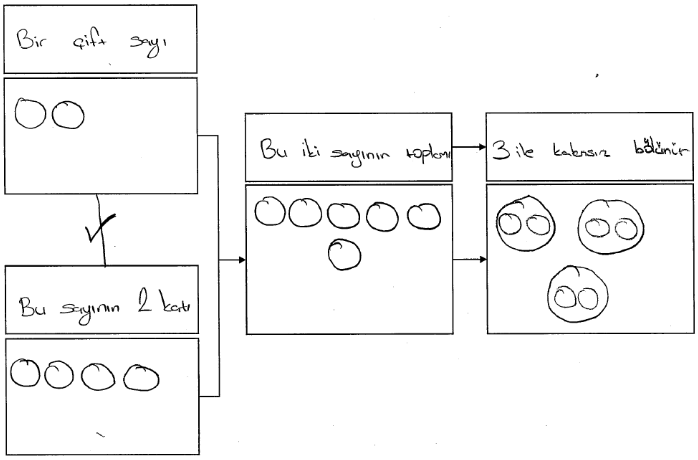


Figure 4.27 Flowchart proof of Conjecture 6

Just after the introduction, Beren was asked to review the previous conjectures of the day, numbered from 2 to 5. For all the conjectures she correctly indicated when to use a checkmark and a cross-mark. Her responses demonstrated an understanding of the differentiation between the type of conjectures appropriate for using same-value cups and not-necessarily-same-value cups.

Beren : *(Used a cross-mark for Conjecture 3) For these… will be crossed out, there is already a shape (difference) in between.*
Researcher : *Why comes a cross-mark?*
Beren : *Let me explain right away. Just like [Conjecture 7] actually. Because, if there are five in this [cup] we do not know whether or not there are also five in this one, we are not sure. But can it be (five), yes, can be. But it is not, ninety percent it is not.*

In fact, this last activity was a reinforcement for her recognition of AoC-E1 and AoC-E2 concepts as two different structures. Beren's above work indicated that she abstracted the commonality of both extended AoC concepts.

4.3.7. Proving the target conjecture of the study and a novel conjecture

The above findings reported so far provides evidence that Beren had learned all the planned concepts of the study as intended: a first instance of proof (the CoA concept) and three different conjecture-proof structures (the AoC, the AoC-E1 and the AoC-E2 concepts). As a post-assessment she was asked to evaluate the truth of the target conjecture of the study and a novel conjecture. When she was asked to prove the target conjecture "The sum of two odd numbers is an even number", she successfully determined the appropriate flowchart type to be used – one with a cross mark between the addends. But she did not know how to represent the two addends in general terms and did not think to use the concept of modular structure for this purpose. Moreover, her attempts to create a general representation of any odd number revealed that she did not associate being odd with being not divisible by two. This was despite using the same relation previously in an AoC conjecture (Conjecture D). She demonstrated the same difficulty with thinking in terms of the remainder of a division operation. Trying to find out the correct number of cups to use, she eliminated option two as a non-plausible candidate.

Beren	*: If we make two (cups) even in the first place when there is one inside, we collapse. I mean it is canceled right away. Not two.*
Researcher	*: What numbers do two (cups) show us?*
Beren	*: Even.*
Researcher	*: Two, four, six, … I am increasing (the item per cup) one by one.*
Beren	*: Yes.*
Researcher	*: How do the odd numbers differ from that?*

Despite the researcher's prompting questions, she did not relate the odd numbers to the modular structure of $2k+1$; hence could not start an argument.

Similarly, for the novel conjecture "The sum of two consecutive numbers is always an ____ number. (Odd/Even)", Beren selected the appropriate flowchart type – one with a checkmark between the addends.

Beren	*: I take this one. One with the checkmark.*
Researcher	*: Why?*
Beren	*: Here is one consecutive number, here is the other consecutive number… So for example, a five (and) six, here the sum and this one is the proof it's being odd (or) even? Because both of them are consecutive, I mean. (If) one is five the other is six.*
Researcher	*: That means? Means they are… with each other…*
Beren	*: Exactly, the same… are related.*

However, after then she did not engage in any observable action to find out how to represent the sum of two consecutive numbers.

Beren	*: What do I do now? […] I am thinking. I will find out… To here the first consecutive number and to here… by writing the second consecutive number, here making addition, here I will prove its*

being odd or even. […] This will turn out to be an odd number. For example, the sum of five (and) six makes 11, odd number. Again seven, eight… that makes what… makes fifteen. They are odd numbers. I am sure that (the result is an) odd number but I have to prove this in this way.

For both conjectures, the focal observation was that she did not list all the cases to examine their invariant property. All in all, during Phase 5, Beren learned to prove two new AoC structures and, hence, understood two new cases of defining objects (as a way of understanding). However, each time she attempted to prove a conjecture, she tried to construct a flowchart similar to those she encountered before. Neither during the learning process nor afterwards, in the above two questions, did she tend to state definitions to prove a conjecture that she considered unfamiliar. Inspecting the defining property of objects never seemed to be part of her goal-setting behavior.

CHAPTER 5

DISCUSSION

This study investigated an individual sixth-grade student's first steps towards the deductive proof scheme by building on a structural and then holistic understanding of a particular proof. The five-phase individual teaching experiment study explored the processes by which the participating student Beren came to (1) understand a deductive argument as an instance of mathematical proof for the first time (intended CoA concept) and then (2) developed an (imperfect) holistic understanding of this first instance proof that is transferable to the practice of proving conjectures analogous to the first instance proof (intended AoC concept). The learning of the CoA and AoC concepts of the study has been completed at the end of the fourth phase of the teaching experiment. At that point, Beren's holistic understanding of the class of analogous proofs was only rudimentary due to instantiating one of the intended ways of thinking (the deductive proof scheme) while lacking the other (definitional reasoning). More specifically, Beren demonstrated the way of thinking that the sequential actions she learned to follow offered a secure method for validating the set of conjectures she has been studying so far. However, she did not recognize that the first of the two actions she used, what she knew as showing in "general representation," was a product of the defining act. Phase 5, on the one hand, extended the AoC concept of the study to the concepts referred to as AoC-E1 and AoC-E2 to provide Beren with more instances of proper ways of understanding associated with definitional reasoning. On the other hand, data from Phase 5 served the exploration of (3) how the developed understandings enabled or constrained Beren's act of proving other conjectures, both similar (not analogous) to the first one and novel. Neither during Phase 5 nor in the post-assessment did Beren demonstrate definitional reasoning. What she had learned from her AoC concepts did not go beyond rudimentary main ideas that she successfully used to prove conjectures with the same exact structures previously encountered.

5.1. The Learning Goal 1: The first instance of mathematical proof as a CoA concept

The initial hypothesis of the study was that by carrying out the sequential activity designed in accordance with the LTA instructional design approach (to the extent possible), the student would end up with a flowchart argument that she could recognize as a mathematical proof for the first time. Even though the intended learning took place only after introducing a modification to the initially designed task sequence, Beren's learning from her activity provided supporting evidence for the initial hypothesis of the study. Her abstraction of the logical necessity relationship in the resulting proof, the technically defined participatory-level CoA concept, was a result of the coordination of actions mechanism. And her earlier-than-expected transition to the technically defined anticipatory-level CoA concept was a result of a second reflection on her newly constructed abstraction as an entity that satisfied her need for certainty. However, it is believed that the true anticipatory level of the concept, according to the LTA theoretical framework, has never been attained during the teaching experiment study. Since Beren did not view starting the proof with the concept of modular structure as an instance of definitional reasoning, it was questionable whether she would be able to prove a given analogous conjecture by setting the goal of defining herself when she would not remember about the specific concept at a later point in time.

5.1.1. The participatory stage of the concept

What Beren has learned from the modified task sequence may seem to deviate from the intended outcome of the original Phase 3 task, since the general flowchart did not result from the modified activity but got introduced externally after the CoA concept was constructed. With the modified task sequence, the first of the two actions being coordinated changed from creating the general representation of the given arithmetic number pattern (in a single flowchart), $A_{0a\ (original)}$, to dividing the individual elements of the number pattern by a certain divisor (in consecutive case-based flowcharts), $A_{0a\ (modified)}$. Therefore, one may think (as we were concerned about before the retrospective data analysis), the modified action may not offer an accurate example of defining act as the initially preferred action would do; because it is not acted on all the numbers contained in the pattern at once. ("A mathematical definition is a description that applies to all objects to be defined and only to them" (Harel, 2013, p. 139)). Contrary to our concern, through the end of the activity, Beren demonstrated an instance of understanding the modular structure 9k+6 as the defining characteristic of the given arithmetic number pattern when she said "After all, <u>if we add 6 to a number that is a multiple of 9</u>, it already is divisible by 3 without a remainder." This was a new way of describing the pattern beyond showing merely the abstract structure in her mind. With this advancement in Beren's understanding, an *unnecessary first step* of the originally planned activity turned into being a *defining property* (although not maintained). Particularly, in the original activity Beren thought there was no need to divide the numbers in the sequence 12, 20, 28, 36, 44, 52, … by 8 first, to determine whether they were all divisible by 4 or not.

Researcher : 12 is a number that gives a remainder of 4 when divided by 8.
Beren : Yes.
Researcher : And we say that it has to be divisible by 4.
Beren : But, we can find already if we try.

The truth of the conjecture was established (i.e., the logical necessity relationship was abstracted) only when Beren realized that the structure 9k+6 was an invariant characteristic of the elements in number pattern. The originally planned task, where she lacked such awareness of the invariant property, did not foster the truth of the conjecture beforehand. This is only natural because understanding the related property as an abstract structure constitutes a part of the learning goal (i.e., universal instantiation)—understanding the resulting proof as a structural object. This was the key advancement needed for promoting the intended CoA concept of the study. It was achieved due to the modified task sequence building on an operational conception of the concept modular structure in the sense of Sfard (1991) (based on division operation), rather than its structural conception. Note that at the end of Phase 1, Beren possessed the operational conception of modular structure. She was able to use the one-to-one correspondence relationship between the elements of the number pattern and their quotient with respect to a given modulus n (as a process). However, its structural correspondent as an abstract invariant property (as an object) was not immediate for her when she was engaged in the original task of Phase 3, unlike in the modified task sequence.

We speculate that her construction of the structural conception of modular structure as a property was fostered by the situation of examining the elements of a number pattern with respect to division by a whole number divisor. It came about as an abstraction from her activity (as a result of the reflective abstraction mechanism) when Beren anticipated the result of filling in the case-based flowchart for the number 24 before actually doing so. This suggests the coordination of the two division actions, A_{0a} $_{(modified)}$ and A_{0b}, (i.e., dividing the elements of the number pattern 15, 24, 33, 42, 51, 60, 69, … by 9, and dividing the resulting number in the form of 9k+6 by 3), resulted in two distinct abstractions occurring almost simultaneously. First, the particular modular structure as an abstract property of the set of numbers was abstracted as an invariant property. Second, the divisibility relationship stated in the related conjecture was abstracted as the intended CoA concept of the study. This supports the loose hypothesis of the study which was stated in the methods section earlier: The purpose of defining (in our case) gains its meaning only if one has a purpose of generality in her argument. Otherwise, why should one look for modular congruence in any and all of the elements in a set of numbers, if she is not to define that set of elements? Why should one define a set of numbers if she is not to make an inference about all those elements at a time?

Occurrences of the two abstractions so close to each other takes us to the point that the first instances of defining and proving acts as proper ways of understanding, without yet possessing the related ways of thinking, can be studied and promoted in relation to each other. Our data explicated a case where the two abstractions were closely connected with the student's transitioning from an operational to a structural conception of the key concept of the proof. In particular, Beren's transitioning from an operational to a structural conception of the concept modular structure *9k+6* was promoted by the nature of the modified task sequence (situated in an LTA-inspired task of examining a conjecture, unlike in the originally planned task) and the resultant structural conception in turn promoted the abstraction of the relationship stated in the conjecture, the intended CoA concept of the study. That is, the modified task sequence both promoted reification of the particular modular structure (Beren perceived the particular modular structure "in a totally new light;" Sfard, 1991, p. 19) and benefited from this reified property of numbers in promoting the abstraction of the logical necessity relationship involved in the whole proof. This suggests *reasoning-and-proving* activities (or only those of the type *providing proofs* as in our case) may offer a potential framing to the reification of simple arithmetic ideas; which in turn means, students can engage in *reasoning-and-proving* activities as early as they are ready to reify arithmetical processes. One of the purposes of the current study was to gain insight about when to place *reasoning-and-proving* activities (*providing proofs* in our case) in school mathematics earliest. Sfard and Linchevski (1994) wrote that “the transition from purely operational outlook to the dual, process-product interpretation of algebraic formulae occurs in close vicinity to the point at which arithmetic meets with algebra” (p. 206). In line with the findings of the current study, we add to this claim of the authors that the same point at which arithmetic meets with algebra might be a good place to bring instantiations of defining and proving acts jointly to the students’ attention.

Any instructional task that encourages validating a generalization based on *regularity in the process*, what Harel (2008a; 2008b) refers to as a *process pattern generalization* way of thinking, may be thought to support the same learning as the modified task sequence. However, without the invariant property of the object about which an inference is to be made abstracted, understanding a general argument (an end product) as a proper instance of proof (as a structural object) seems unattainable. Hence, the resulting product of such activities does not necessarily produce a quick influence on the student’s way of thinking. The student may probably read the argument as a general procedure rather than as an abstract structure (Ahmadpour et al., 2019). On the other hand, tasks that aim to foster the coordination of actions mechanism might potentially enable time-effective ways of stepping into the development of the deductive proof scheme. This is only if the two abstractions learned can be effectively brought to the students’ attention.

In the current study, Beren’s need for certainty triggered by the Monstrous Counterexample Illustration proved effective in bringing the resulting argument to her attention as a first instance of proof, which marked the birth of a new way of thinking for her-the deductive proof scheme. However, the similar birth of definitional reasoning as a second way of thinking did not come along without an intellectual need triggered. The construction of the AoC concepts of the study did not result in such a focus spontaneously.

5.1.2. The anticipatory stage concept

The motivation of the present study when using the Monstrous Counterexample Illustration before the LTA task sequence was not to trigger an intellectual need for certainty. Rather, it was a methodological requirement for ensuring the trustworthiness of the study's findings in line with the LTA teaching experiment method. After developing the essential number theoretical concepts, at the end of Phase 1, Beren provided an empirical argument to an assessment task requesting a proof for a first instance conjecture. That she provided an empirical argument as a response to the task did not mean she was not able to use her concepts goal-directedly to produce a deductive argument. If she were to hold the two proof schemes concurrently, as would be possible according to Harel and Sowder (1998), her empirical response could only be an indication of her possibly dominant proof scheme. Hence, in order to gather reliable evidence that Beren did not have the intended CoA concept before engaging in the LTA task sequence, it was crucial to make a second assessment after eliminating her empirical proof scheme.

On the one hand, the LTA teaching experiment methodology necessitated this lack of knowledge evidence before reliably attributing the student's learning to her participation in the designed task sequence. On the other hand, it was acknowledged that the student's triggered need for certainty might introduce a potential validity threat to the results of the study in case its contribution to learning had not been identifiable in the data analysis procedure. Luckily, the role of the cognitive conflict Beren experienced in Phase 2 came out spontaneously and overtly in her process of constructing the intended CoA concept of the study in Phase 3. Thus, Beren's remarkable input created a unique data set for the study's interest.

Beren's spontaneous connection to the Monstruous Counterexample Illustration during her engagement with the modified task sequence stimulated her transitioning from the technically defined participatory stage to the anticipatory stage concept. (Note that such a connection was not made during her participation in the original task of Phase 3.) This connection was surprisingly early, even before the purposefully designed task requested that she engaged in a mental run. It was a consequence of her feeling the need for certainty, as indicated in her overt behavior when she questioned the truth of the conjecture being examined before thinking about the extremely large number seen in the illustration. Certainty followed at the moment Beren explained causality in the conjecture. Therefore, the construction of the participatory-level concept cannot be claimed to be independent of her need for causality. However, it is not possible to anticipate whether the same transition to the anticipatory level would take place or not without Beren recalling the illustration and engaging in the mental run offered by the task sequence (for the number 4794 and for a number larger than ten million, both of which were known to give a remainder of 6 when divided by 9). That is, the anticipatory-level concept would naturally follow the participatory-level concept following Beren's abstraction of the for all relationship (may be to fulfill an untriggered need or as a result of reflecting on one's proper way of understanding). Or, if not, would it result from a last task included in the modified task sequence that aimed to address the student's need for causality by requesting that the student compare the argument resulting from her activity to a given empirical argument?

Nevertheless, the findings of the study provide a contribution to the discussion on the role of intellectual need in learning about ways of thinking. Based on his experiential knowledge of the field, Harel (2013) recognized the difficulty of generating an intellectual perturbation in students to trigger their need for certainty, and put forth the idea of developing, instead, a habit of looking for cause(s) in students' explanations (by repeatedly attending to causal explanations and comparing them to empirical arguments). Therefore, regarding the transition from the empirical to the deductive proof scheme, Harel (2013) promoted the idea of attracting students' attention to the cause (or causes) of why an assertion is true rather than challenging the legitimacy of empirical reasoning. However, the current study provides an incident of learning where triggering an intellectual need for certainty followed by the LTA-inspired task sequence produced an instantaneous shift in the student's proof scheme. This suggests the underlying mechanisms of cognitive conflict and coordination of actions might be used complementarily to ensure the learning of new ways of thinking by designing interventions of short duration.

A point not to be underestimated is the success of using Stylianides and Stylianides (2009)'s cognitive conflict-grounded approach with a sixth-grade student who did not even have knowledge of the concepts involved. Beren did not have the square root concept needed to understand the Monstrous Counterexample Illustration (as a way of understanding), but still benefited from the task in terms of reflecting on her way of thinking. In addition, if it had not been preceded by the Monstrous Counterexample Illustration, the modified task sequence would not be a meaningful task to be repeated several times; after all, it started with an act of defining that initially seemed pointless to the student, and she did not know to call on to meet a goal that she set herself. The indispensability of such a defining action in attaining the learning goal of the current study (a structural understanding of a mathematical proof) leaves students with no chance to create their own proofs to experience repeated instances of causality without an outer influence (without a first example at least). In other words, the LTA-based approach used in this study (unless redesigned for meaningful engagement) may be appropriate only to use with limited repetitions to create short, direct influences on students' ways of thinking, whereas building on more causal explanations accessible to the students necessarily bypasses a pure act of defining that can be reflected on as part of a structural object.

Currently, the LTA theoretical framework is not able to adequately explain the mechanism of transition from the participatory to the anticipatory stage of a concept. In fact, in the LTA instructional approach, the student is not expected to state the logical necessity relationship learned as a mathematical theorem. At the participatory stage, the student is able to anticipate the result of the activity through which she abstracted the new concept (GT-A_1) without actually carrying it out. At the anticipatory stage, she can call on this new abstraction herself to meet a decontextualized goal (G_1) without thinking about the original activity. In the current study, however, the technically defined participatory stage concept aimed at no meaningful task goal (other than following a list of sequential instructions to complete a flowchart activity), and the transition to the anticipatory stage was defined as connecting the learned abstraction to the goal of proving, a goal that is totally new to the student. Given the difference in theoretic formulation of the learning goals, the current study does not offer a direct contribution to the LTA theoretical framework regarding the transition process. However, attending to how the intended transition took place in the specific case of this study can provide valuable insight into mathematics learning in general by demonstrating complementary aspects of the LTA and DNR theories.

The DNR and LTA theoretical frameworks accept contrasting views on mathematics learning. DNR views problem solving as "the only means of learning" (Harel, 2008b, p. 896) and claims that perturbation is, although not sufficient, a necessary condition for the construction of new disciplinary knowledge (Harel & Koichu, 2010). The notion of intellectual need is central to this view of learning as a source of one's perturbational state (Harel, 2013). On the contrary, the LTA argues that "disequilibrium is neither sufficient nor necessary to explain mathematics conceptual learning" (Simon, 2013, p. 291) and instead builds on Piaget's notion of reflective abstraction. As supporting evidence for his claim, Simon (2013) points to a previous study (Simon, Saldanha, McClintock, Akar, Watanabe, & Zembat, 2010) where conceptual learning was explained without any reference to a disequilibrating experience.

However, Simon (2013) points out that the two approaches to mathematics instruction—one grounded in problem solving and the other in guided reinvention of hard-to-learn concepts through goal-directed activity, reflection, and anticipation—can be used complementarily to foreground different aspects of mathematics learning. Simon (2013) concludes, "To what extent the two approaches can be combined or to what extent they are just different instructional tools employed to accomplish different goals still needs to be worked out" (p. 293). By building on both perturbation (cognitive conflict) and reflective abstraction, the current study presents an early example of using the two theories complementarily. The findings of the study suggest an advancement on the two contrasting claims regarding the necessity of perturbation in learning. The following is a piecewise hypothesis combining the DNR and LTA perspectives on learning, informed by the empirical data of the current study.

> In constructing a way of understanding, perturbation is neither necessary nor sufficient, as the LTA theoretical framework suggests. However, in learning about a way of thinking, perturbation is not sufficient but a necessary condition (a trigger for one's reflection on her way of thinking), as the DNR suggests.

The hypothesis is informed by the following empirical observations: Beren's construction of the logical necessity relationship "all the numbers that give a remainder of 6 when divided by 9 are divisible by 3" was explained by the mechanism of reflective abstraction alone without reference to the perturbation she experienced when exposed to the Monstrous Counterexample Illustration. However, as speculated above, without the mechanism of cognitive conflict triggering a reflection on her way of thinking (bringing the learned abstraction to the student's attention), it might not be certain that Beren would relate the activity giving rise to the abstraction learned, GT ($A_{0a} \rightarrow A_{0b}$), to the goal of proving. This leads to the view that perturbation might be a necessary condition for learning about disciplinary ways of thinking. The same position can be supported by the empirical research finding that students may not see a written proof text as an argument for or against a mathematical claim but simply as a "task solution" (Iversen, 2022). Findings from the study of AoC concepts as well provide (weak) support for the above hypothesis in that without a reflection on the learned abstraction—the invariant property of the numbers in an arithmetic number pattern—Beren did not relate her abstraction to the act of defining, hence did not recognize an instance of the related way of thinking. The evidence here is only weak because the AoC concepts of the study introduced their own limitations to her learning in terms of building on rudimentary main ideas already.

If the above hypothesis is true, the following retrospective interpretation may be plausible: The LTA's initial claim might have been motivated by its focus on ways of understanding, while DNR's claim might have been influenced by its emphasis on ways of thinking mostly (although DNR accepts ways of understanding and thinking as equally important to mathematics learning). It is vital to note that the hypothesis is applicable only when mathematical knowledge is defined as including both ways of understanding and thinking, as introduced by the DNR theoretical framework.

5.2. The Learning Goal 2: Instances of definitional reasoning as AoC concepts

At the end of Phase 4, Beren was capable of producing flowchart proofs for conjectures analogous to the first one proved in her view. Her activity ($A_{0a\ (original)} \rightarrow A_{0b}$) involved describing the arithmetic number pattern given in the conjecture through the numbers' shared modular structure (what she accepted as "general representation") and making a divisibility inference out of that representation about all the numbers in that pattern. This was an indication that Beren demonstrated the intended AoC concept of the study, i.e., transferred the rudimentary main idea of the first instance proof to analogous proving tasks. However, as she repeatedly produced analogous proofs (to promote definitional reasoning, in line with Duality II), her knowledge of the AoC concept did not go beyond the rudimentary main idea. That is, she did not step into perceiving the concept of modular structure she used to begin proofs as a way of defining arithmetic number patterns, and she did not learn to set the goal of defining herself to start her proofs. During the extension of the ideas to AoC-E1 and AoC-E2 concepts, to experience more instances of proper ways of defining, Beren did not move beyond such rudimentary understandings of the main ideas. Neither in proving the target conjecture nor a novel one did she engage in any attempt indicative of the intended definitional reasoning. The amount of repeated experience might be insufficiently low, and it is a limitation that the tasks offered cases of defining a limited variety of sets of objects since the two structures were extensions of the same concept.

Scrutinizing back to the original nature of an AoC concept and the type of learning it promises helps explain the study's failure to generate the intended definitional reasoning. In a sense, the Duality II principle wishes for the students' abstraction of the commonality in their proper ways of understanding: "Learners develop desirable ways of thinking only through repeated application of proper ways of understanding" (Harel, 2021, p. 711). The initial hypothesis of the current study was that by abstracting the commonality in her repeated activity of proving analogous conjectures, the student may come to understand the role of starting the particular type of proofs with the concept of modular structure. This was thought to offer the student a first set of experiences with definitional reasoning, similar to the idea of a first instance proof. If a first set alone would not produce the intended abstraction, more instances of it might. This was the reason why the AoC concept of the study was extended to the study of two other concepts. However, these loose articulations were beyond what an AoC concept originally might promise.

An AoC concept is the "ability to recognize problem situations that can be solved by the same activity," which "creates for the learner a new category with which to view and make sense of her world" (Simon, 2022, p. 9). Similar to its idiosyncratically defined CoA concept, the AoC concept of the study was articulated differently than the usual LTA practice. Since it was considered to be the main idea of the first instance proof (with modular structure as "our" focus), the basis for the intended reflective abstraction was the entire activity sequence of creating the same-structure proofs for same-structure conjectures. Accordingly, what Beren has abstracted from her activity is restricted to the common structure (the relationship between a given arithmetic number pattern and its whole number divisors) of the conjectures that her learned activity was able to prove. As an example, Beren immediately disproved Conjecture F: “All the numbers that give a remainder of 3 when divided by 8 are divisible by 2,” relying on the absence of the relation she abstracted (i.e., being object to the distributive law of division): “Eight, three, and two are numbers that have nothing to do with each other.” The problem situation she came to recognize was proving a given conjecture with a certain structure, while the actual learning goal (Learning Goal 2) for her was to recognize “a part” of this problem as a sub-problem. This can be thought of as if there should be two different AoC concepts targeted, just as the first instance proof involved two different abstractions. The next section presents a reinterpretation of how the constructs of CoA and AoC concepts can (or cannot) frame the study of students’ learning about ways of thinking.

5.3. The CoA and AoC concepts in development of the deductive proof scheme

This study was initiated in the hope that the ideas of CoA and AoC types of reflective abstraction mechanisms might help explain one’s first steps toward the deductive proof scheme. Regarding the two types of reflective abstraction, Simon (2020) wrote:

> CoA seems to be useful for explaining the construction of mathematical objects and relationships between them. AoC so far has proven useful in explaining the construction of arithmetic operations. Further work might identify additional types of concepts produced by AoC, perhaps quite different concepts that are constructed through abstraction of commonality in one’s activity (p. 9).

Considering the particular first instance proof of the study, the CoA mechanism seems to explain the construction of two distinct logical necessity relationships involved: one between a given arithmetic number pattern and its modular structure, and the other between a given arithmetic number pattern and its whole number divisor. The nature of the activities through which the two abstractions were constructed in this study is not exactly the same as the task design approach outlined in the LTA. This is because the goal components of these so-called CoA concepts (goal-action composites), the former of which is "to define" and the latter is "to prove," do not match any task goal available to the students to engage in any designed activity. Nevertheless, as the study documents, the CoA mechanism accounts for the learning of the two abstractions. The next part of the question is: what mechanism can promote the goals of "to define" and "to prove" in students?

Knowledge of an AoC concept (e.g., a quantitative operation) is the "ability to recognize problem situations that can be solved by the same activity" (Simon, 2022, p. 9). Inspired by the above quote from Simon (2020), this ability to recognize problem situations can be extended to include the ability to recognize disciplinary problem situations that can be solved by carrying out the same mental act. This might seem to frame students' recognition of the mental act of proving (a particular class of conjectures) as an AoC concept—an anticipation that a learned activity sequence will execute that act if enacted. However, the similarity between the cases of constructing an arithmetical operation (such as division) and a mental act (such as proving) is in the knowledge piece constructed as an end-product—what is similar to having an AoC concept. Contrary to what is expected, the mechanism that can account for the construction of these two knowledge pieces is not the abstraction of commonality.

The findings section reports that Beren demonstrated the intended AoC concept of the study during her participation in Phase 4. Although she demonstrated an understanding of the conjecture structure that her activity was able to prove, she did not learn this abstraction as a result of the AoC mechanism. (What she has abstracted was a way of understanding that did not have a connection to the related way of thinking and was not associated with the proving act.) It was earlier in Phase 3 when her participatory-level CoA concept was connected to the goal of proving, with the mechanism of cognitive conflict doing the work. If else, learning about the proving act would not be possible through the intended AoC mechanism in Phase 4, just as she did not learn about the defining act. A revised hypothesis is that only after students are introduced to the disciplinary acts of proving and defining (promisingly by building on their intellectual need), the LTA notion of an AoC concept might frame the study of proving a particular class of conjectures and defining a particular class of objects (as ways of understanding only). An AoC type of reflective abstraction seems not sufficient in itself to account for the construction of disciplinary acts of proving and defining without another influence on their ways of thinking. Once the disciplinary problem situation is recognized (a need for a proof or a definition is realized), knowledge of which concept can do that work for one and how that work can be done can be framed by the actual CoA and AoC concepts of the LTA theoretical framework (as ways of understanding). A more elaborated revision can be offered in future work.

5.4. Implications for task design and the number theory content area

This study provides an initial characterization of the "goal understandings" that the proof instruction might aim to achieve within school mathematics. It also explains the mechanisms by which some of these understandings are constructed by the student, while falling short in explaining others and pointing to directions for future research. Related theoretical implications are already discussed in the previous sections of the chapter. The task design approach used in the study is not a practical contribution to the instructional approaches to teaching proof. The tasks were merely used as the researcher's tools to serve the theoretical purposes of the study: to explicate the processes by which an individual student, who does not have the concept of deductive proof yet, may conceptually understand a very first example of proof and then build on an understanding of this first example to learn proving within the deductive proof scheme. Still, a few practical implications can be stated regarding the plausible use of the number theory content area in early proof instruction.

Definitions of number theory concepts are expected to be well developed in the middle grades. The introductory concepts, such as factors, multiples, divisibility, and whole number patterns, are accessible to young students and receive a great deal of attention in their textbooks (Stylianides, 2009). However, empirical data from the current study reveals that, probably unlike the concepts themselves, their definitions may not be accessible to the students when they do not recognize what a definition is. Despite the purposeful sequencing of the conjectures during Phase 5 of the current study, Beren did not come to define odd numbers herself. She did not recognize that the number sequence 1, 3, 5, 7, 9... was one of those arithmetic number patterns that she got used to working with. Thinking based on division by two and thinking based on a constant remainder were neither immediately accessible to her as invariant properties. This highlights important questions regarding the feasibility of even the simplest proofs given the students’ available ways of understanding and thinking. Accessibility is characterized by the goal understandings initiated in the current study and to be further explored in future work.

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APPENDICES

A. APPROVAL OF THE METU HUMAN SUBJECTS ETHICS COMMITTEE

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

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T: +90 312 210 22 91
F: +90 312 210 79 59
ueam@metu.edu.tr

Sayı: 28620816 / 106

06 MART 2019

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Prof.Dr. Erdinç ÇAKIROĞLU

Danışmanlığını yaptığınız Merve DİLBEROĞLU'nun **"Ortaokul Öğrencilerinin Matematiksel İspat Kavramını Anlamalarına ve İspat Yapma Becerilerini Edinmelerine Yönelik Bir Öğrenme Rotasının Geliştirilmesi: Bireysel Öğretim Deneyleri"** başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **097-ODTÜ-2019** protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

Prof. Dr. Tülin GENÇÖZ
Başkan

Prof. Dr. Ayhan SOL
Üye

Prof. Dr. Ayhan Gürbüz DEMİR
Üye

Prof. Dr. Yaşar KONDAKÇI
Üye

Doç. Dr. Emre SELÇUK
Üye

Doç. Dr. Pınar KAYGAN
Üye

Dr. Öğr. Üyesi Ali Emre TURGUT
Üye

B. OFFICIAL PERMISSIONS TAKEN FROM THE MINISTRY OF NATIONAL EDUCATION/MEB ARAŞTIRMA İZNİ ONAYI

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

Sayı : 14588481-605.99-E.7418319 11.04.2019
Konu : Araştırma izni

ORTA DOĞU TEKNİK ÜNİVERSİTESİ REKTÖRLÜĞÜNE
(Öğrenci İşleri Daire Başkanlığı)

İlgi: a) MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2017/25 nolu Genelgesi.
b) 19/03/2019 Tarihli ve E. 104 sayılı yazınız.

Üniversiteniz, İlköğretim Anabilim Dalı Doktora Programı öğrencisi Merve DİLBEROĞLU'nun **"Ortaokul Öğrencilerinin Matematiksel İspat Kavramını Anlamalarına ve İspat Yapma Becerilerini Edinmelerine Yönelik Bir Öğrenme Rotasının Geliştirilmesi: Bireysel Öğretim Deneyleri"** konulu çalışması kapsamında uygulama yapma talebi Müdürlüğümüzce uygun görülmüş ve uygulamanın yapılacağı İlçe Milli Eğitim Müdürlüklerine bilgi verilmiştir.

Uygulama formunun (9 sayfa) araştırmacı tarafından uygulama yapılacak sayıda çoğaltılması ve çalışmanın bitiminde bir örneğinin (cd ortamında) Müdürlüğümüz Strateji Geliştirme Şubesine gönderilmesini rica ederim.

Turan AKPINAR
Vali a.
Milli Eğitim Müdürü

Güvenli Elektron...
Aslı İle Aynı...
12/04/2019

Mahmut ÖZDEMİR

Adres: Alparslan Türkeş cad. Emniyet Mah. 4/A Yenimahalle/ANKARA
Elektronik Ağ: ankara.meb.gov.tr
e-posta: istatistik06@meb.gov.tr

Bilgi için: Ayşe ARDA
Tel: 0 (312) 212 36 00
Faks: 0 (312) 221 02 16

Bu evrak güvenli elektronik imza ile imzalanmıştır. https://evraksorgu.meb.gov.tr adresinden 75e3-f934-312f-9be3-c3bf kodu ile teyit edilebilir.

C. THE INSTRUCTIONAL SEQUENCE/ ETKİNLİK DİZİSİ

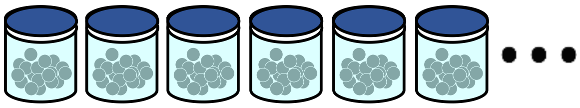
PHASE 1 BASIC NUMBER THEORY CONCEPTS

TASK 1

Esin Hanım'a katılıyor musunuz?

Armağan Pastanesi'nde çalışan Esin Hanım, fırsat buldukça çocuklar için kurabiye pişiriyor. Bu kurabiyeleri onlar için ayırdığı 6 kavanoza, her bir kavanozda eşit sayıda kurabiye olacak şekilde yerleştiriyor. Kalan kurabiyeleri tek tek paketliyor. Okul çıkışında çocuklar pastaneye uğradığında her birine eşit sayıda kurabiye veriyor.

Pazartesi günü Esin Hanım pişirdiği kurabiyeleri 6 kavanoza eşit paylaştırdığında artan 3 kurabiyeyi tekli paketlere koydu ve çocukları beklemeye başladı.



O sırada, Esin Hanım ve bir müşteri konuşmaya başladılar:

Esin Hanım: Eğer bugün pastaneye 3 çocuk gelirse kurabiyelerin tamamını eşit paylaşabilirler.
Müşteri: Nereden biliyorsunuz? Tahminimce burada 100'den fazla kurabiye var. Yoksa bütün bu kurabiyeleri saydınız mı?
Esin Hanım: Hayır, saymadım. Ama raftaki duruma bakınca anlayabiliyorum.
… (devamı gelecek).

Esin Hanım'ın düşüncesine katılıyor musunuz? Neden?

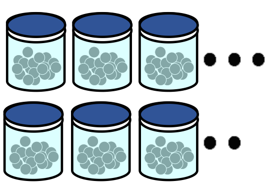
TASK 2

Kurabiyelerin <u>tamamını</u> üç çocuk eşit paylaşabilir mi?

Esin Hanım'ın farklı günlerde çocuklar için pişirdiği kurabiyeler aşağıda gösterilmiştir.

1. Hangi günlerde pastaneye üç çocuk gelirse kurabiyeleri hiç artan olmadan eşit olarak paylaşabilirler?

Pazartesi



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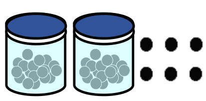
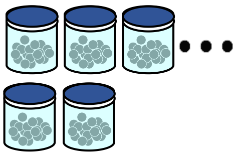
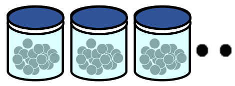


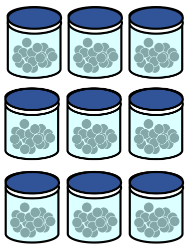
Salı



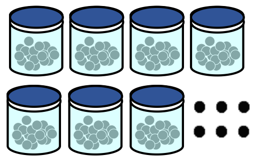
Cuma



Çarşamba



Cumartesi



2. Hangi günlerde pastaneye üç çocuk gelirse kurabiyeler eşit paylaşıldığında mutlaka artan olur?

3. Hangi günlerde pastaneye üç çocuk gelirse kurabiyeler eşit paylaşıldığında artan kurabiye olup olmayacağını bilemeyiz?

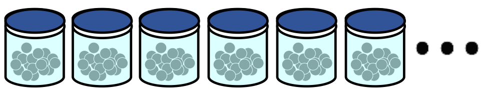
4. İstediğiniz bir gün ve sayı seçerek, aşağıdaki cümleyi kesinlikle doğru olacak şekilde tamamlayınız.

"_____________ günü, pastaneye ____ çocuk gelirse, kurabiyeleri hiç artan olmadan, eşit olarak paylaşabilirler".

TASK 3

Kurabiyelerin sayısı 125 olabilir mi?

Esin Hanım ile müşterinin diyalogunu hatırlayalım.



Esin Hanım:	Eğer bugün pastaneye 3 çocuk gelirse kurabiyelerin tamamını
eşit olarak	paylaşabilirler.
Müşteri:	Nereden biliyorsunuz? Yoksa bütün bu kurabiyeleri saydınız mı?
Esin Hanım:	Hayır saymadım ama raftaki duruma bakınca anlayabiliyorum.
Müşteri:	Ya kurabiyelerin sayısı 125 ise? O zaman 3 kişi eşit paylaşamaz ki!
Esin Hanım:	125 olamaz.
Müşteri:	Neden olmasın? Kurabiyelerin sayısı birçok faklı sayı olabilir.
Esin Hanım:	Doğru. Birçok sayı olabilir. Örneğin, 123 olabilir, 129 olabilir…
ama 125	olamaz.
Müşteri:	…

1. Esin Hanım’a katılıyor musunuz? Neden? Aşağıdaki tabloyu doldurunuz.

Toplam kurabiye sayısı	123	124	125	126	127	128	129
Bir kavanozdaki kurabiye sayısı							
Dışarıda kalan kurabiye sayısı							

2. Aşağıda verilen cümleleri yukarıdaki tabloya göre uygun şekilde tamamlayınız.

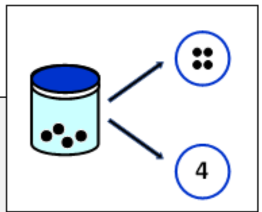
   a. Kurabiyelerin sayısı 123 olabilir/olamaz. Çünkü, …

   b. Kurabiyelerin sayısı 125 olabilir/olamaz. Çünkü, …

   c. Kurabiyelerin sayısı 129 olabilir/olamaz. Çünkü, …

TASK 4

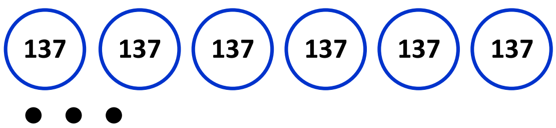
Kurabiye rafının durumunu inceleyelim!



Kavanozları “○” ile, tekli kurabiyeleri “●” ile gösterelim.

İçerisinde 4 kurabiye olan bir kavanozu “(::)” veya “(4)” şeklinde gösterebiliriz.

Aşağıda gösterilen toplam kurabiye sayısı 825’tir.

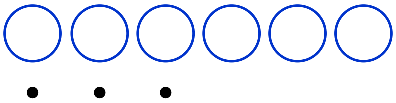
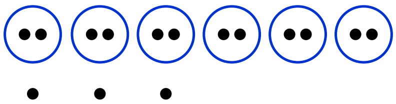
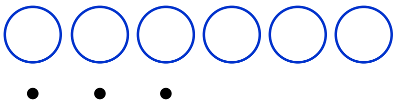
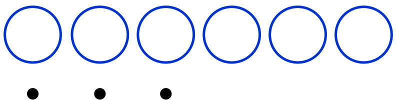
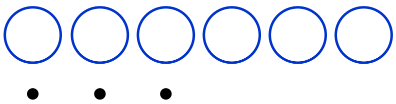
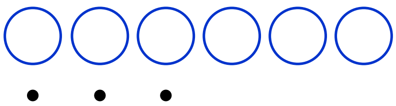
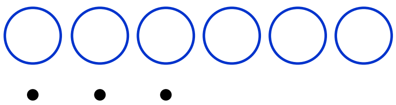
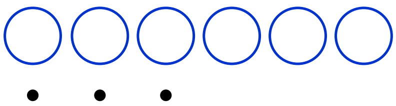
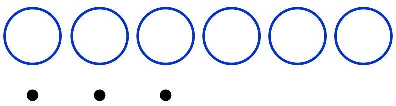
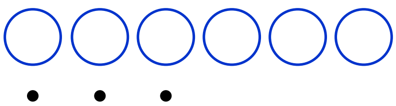
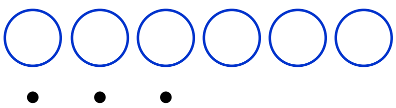
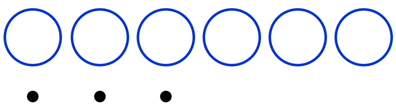


1. Eğer her bir kavanozda 139 kurabiye olsaydı, toplam kurabiye sayısı kaç olurdu?
2. Eğer toplam kurabiye sayısı 833 olsaydı kurabiye rafının son durumu nasıl olurdu? Çizerek gösteriniz.

TASK 5

Kurabiyelerin toplam sayısı hangi sayılar olabilir?

Aşağıdaki tablodan yararlanarak kurabiyelerin sayısı olabilecek sayıları listeleyiniz.

Kurabiye rafının durumu	Toplam kurabiye sayısı	Bir kavanozdaki kurabiye sayısı	Dışarıda kalan kurabiye sayısı
			
	15	2	3
			
			
			
			
			
			
			
			
			
			

3. Aşağıdaki tablodan yararlanarak 135'ten büyük ilk 5 sayıyı söyleyiniz.

Toplam kurabiye sayısı	135					
Bir kavanozdaki kurabiye sayısı	22					
Dışarıda kalan kurabiye sayısı	3					

4. 255 kurabiye için rafın durumunu çiziniz. Kurabiyelerin sayısı olabilecek, 255'ten küçük en büyük sayıyı bulunuz.

5. 854 kurabiye için rafın durumunu çiziniz. Kurabiyelerin sayısı olabilecek, 854'ten büyük, en küçük sayıyı bulunuz.

6. Diyalogu tekrar hatırlayınız, boşlukları doldurunuz!

Esin Hanım:	Birçok sayı olabilir. Örneğin, 123 olabilir, 129 olabilir… ama 125 olamaz.
Müşteri:	Nasıl bu kadar emin olabiliyorsunuz?
Esin Hanım:	Çünkü, kurabiyelerin sayısı sadece ___ ile bölündüğünde ___ kalanı veren sayılar olabilir.
Müşteri:	Sizi şimdi anladım! Haklısınız. Eğer 3 çocuk gelirse, kurabiyeleri eşit paylaşabilirler, çünkü ___ ile bölündüğünde ___ kalanı veren bütün sayılar ___ ile kalansız bölünebilir.

7. Müşterinin, "__ ile bölündüğünde ___ kalanı veren bütün sayılar ___ ile kalansız bölünebilir." sözüne katılıyor musunuz? Nedenini açıklayınız.

TASK 6

Genel Gösterimi Tanıyalım

Raftaki Mini Kurabiyeler

Kurabiye rafında toplamda kaç kurabiye olduğu bilinmiyor. Ancak, Esin Hanım kurabiyelerin tamamını 6 kavanoza eşit olarak paylaştırdığında, 3 kurabiyenin arttığı kesin olarak biliniyor.

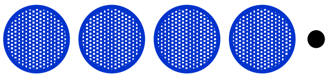
Bir kavanozdaki kurabiye sayısını "" ile, tekli kurabiyeleri "●" ile gösterelim. Bu durumda,

toplam kurabiye sayısını "" şeklinde gösteririz.

Bu gösterim bize ___ ile bölündüğünde ___ kalanı veren bütün sayıları ifade eder. Yani, kurabiyelerin sayısı, bu gösterime uyan bütün sayılar olabilir.

1. Aşağıda genel gösterimi verilen sayı kümelerini isimlendiriniz. Her bir kümenin ilk on elemanını küçükten büyüğe doğru listeleyiniz.

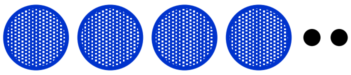
a.



____ ile bölündüğünde ____ kalanı veren sayılar

Sayılar: ____, ____, ____, ____, ____, ____, ____, ____, ____

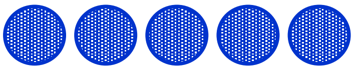
b.



____ ile bölündüğünde ____ kalanı veren sayılar

Sayılar: ____, ____, ____, ____, ____, ____, ____, ____, ____

c.



____ ile bölündüğünde ____ kalanı veren sayılar

Sayılar: ____, ____, ____, ____, ____, ____, ____, ____, ____

2. Aşağıda verilen sayı kümelerini genel gösterim ile gösteriniz. Her bir kümenin ilk on elemanını küçükten büyüğe doğru listeleyiniz.

a. 6 ile bölündüğünde 5 kalanı veren bütün sayılar

Genel Gösterim:

Sayılar: ____, ____, ____, ____, ____, ____, ____, ____, ____

b. 4 ile bölündüğünde 3 kalanı veren bütün sayılar

Genel Gösterim:

Sayılar: ____, ____, ____, ____, ____, ____, ____, ____, ____

c. 3 ile bölündüğünde 0 kalanı veren bütün sayılar

Genel Gösterim:

Sayılar: ____, ____, ____, ____, ____, ____, ____, ____, ____

d. 2 ile bölündüğünde 1 kalanı veren bütün sayılar

Genel Gösterim:

Sayılar: ___, ___, ___, ___, ___, ___, ___, ___, ___

3. Aşağıdaki sayı örüntülerinin sonsuza kadar devam ettiğini düşününüz. Her bir örüntü için, örüntüdeki bütün sayıları kapsayacak genel gösterimleri oluşturunuz.

a. 13, 23, 33, 43, 53, …, 93, …, 153, …

b. …,33, 38, 43, 48, 53, 58, …, 83, 88, 93, 98, …

c. 9, 16, 23, 30, 37, 44, 51, 58, …

4. "12, 20, 28, 36, 44, 52, 60, 68, …. sayı dizisindeki bütün sayılar ___ ile bölünür." cümlesinde, verilen boşluğa gelebilecek sayılar nelerdir?

PHASE 2 THE MONSTROUS COUNTEREXAMPLE ILLUSTRATION

Conjectures asked to be proved/refuted before the illustration, and the arguments revisited after the illustration:

Problemler

1. "8 ile bölündüğünde 4 kalanı veren sayılar 4 ile kalansız bölünür." ifadesi doğru mudur? Yanlış mıdır? Neden?
2. "8 ile bölündüğünde 5 kalanı veren sayılar tek sayıdır" ifadesi doğru mudur? Yanlış mıdır? Neden?
3. 18, 30, 42, 54, 66, 78, 90, 102, 114, 126, 138, 150, …… sayı dizisinde yer alan sayılardan bir tanesi 354'tür. 354 sayısı 6 ile kalansız bölünür. Bu durum, sayı dizisindeki bütün sayılar için geçerli midir? Neden?
4. 12, 15, 18, 21, 24, 27, ………, 51, 54, 57, 60, 63, 66, 69 …… sayı dizisinde yer alan sayılardan bir tanesi 354'tür. 354 sayısı 6 ile kalansız bölünür. Bu durum, sayı dizisindeki bütün sayılar için geçerli midir? Neden?

Düşünelim, Karar Verelim!

1. 27 ve 28 ardışık sayılardır. 27+28 işleminin sonucu 55'tir ve 55 bir TEK sayıdır. Bu durum, herhangi iki ardışık sayının toplamı için geçerli midir? Neden?

2. 21, 22 ve 23 üç ardışık sayıdır. Bu üç sayının toplamı 22'nin (ortadaki sayının) 3 katıdır. Bu durum, herhangi üç ardışık sayı için geçerli midir? Neden?

3. 43 ve 47 ardışık asal sayılardır. 43+47-1 işleminin sonucu 89'dur ve 89 da bir asal sayıdır. Bu durum, herhangi iki ardışık tek asal sayı için geçerli midir? Neden?

Sena'nın Kararı

1. Aşağıda, arkadaşın Sena'nın yazdığı bir not verilmiştir.

3'ten büyük her tam sayı, iki asal sayının toplamı olarak ifade edilebilir. Aşağıdaki örnekler bunun doğru olduğunu gösteriyor. Çünkü, birçok farklı sayı için bunu denedim ve 100'den büyük sayıları bile dikkate aldım.

4=2+2 ✓	16=5+11✓	50=7+43 ✓	100=11+89 ✓
5=2+3 ✓	19=2+17 ✓	54=13+41 ✓	115=2+113 ✓
6=3+3 ✓	25=2+23 ✓	66=5+61 ✓	126=23+103 ✓
7=2+5 ✓	26=13+13 ✓	68=7+61 ✓	138=11+ 127 ✓
8=3+5 ✓	30=7+23 ✓	70=11+59 ✓	142=137+5 ✓
9=2+7 ✓	42=5+37 ✓	88=29+59 ✓	…
10=3+7 ✓	46=23+23 ✓	91=2+89 ✓	…

Sence, Sena doğru karar vermiş midir? Neden?

The monstrous counterexample illustration:

Doğru mu? Yanlış mı?

"a" bir doğal sayı olsun. $1+1141a^2$ ifadesi asla bir tam kare sayıya eşit değildir.

Bu cümlenin doğru olup olmadığına karar vermek için insanlar bilgisayarları kullandılar ve 1'den 30,693,385,322,765,657,197,397,207'ye kadar olan bütün sayılar için, bu ifadenin bir tam kare sayıya eşit olmadığını buldular.

AMA

Bu ifade bir sonraki doğal sayı için bir tam kare sayı verir!!!

2. Bu derste öğrendiğin her şeyi dikkate alarak, Sena'ya kullandığı yöntem hakkında bir mektup yaz.

PHASE 3 THE CoA CONCEPT: UNDERSTANDING THE FIRST INSTANCE OF MATHEMATICAL PROOF

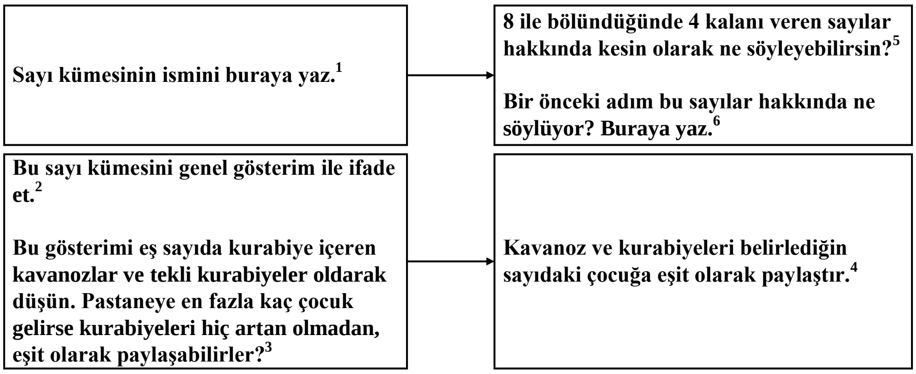
(The original task)

Şema ne anlatıyor?

Şemayı tamamla!

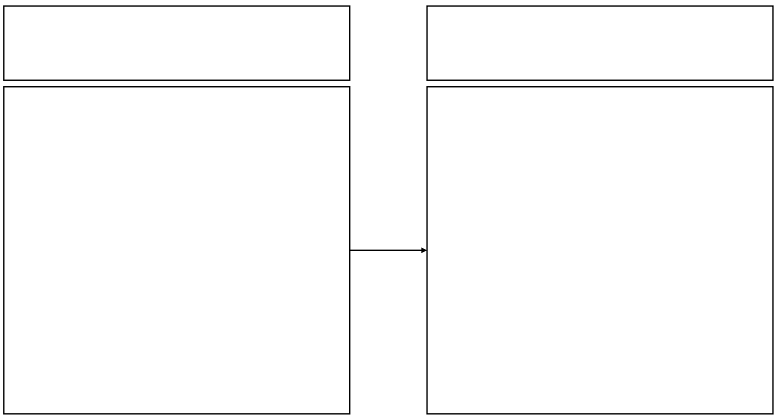
Elindeki şemayı aşağıda verilen sayı kümesi için tamamla. Adımların birini tamamladıktan sonra diğerine geçmek için tıkla.

12, 20, 28, 36, 44, 52, 60, 68, 76, 84, …



Şemanın bütünü ne anlatıyor? Bir cümle ile ifade et.[7]

1. Adımları takip et! (Öğrenci yukarıdaki şekilde verilen PowerPoint sayfasını kullanır.)



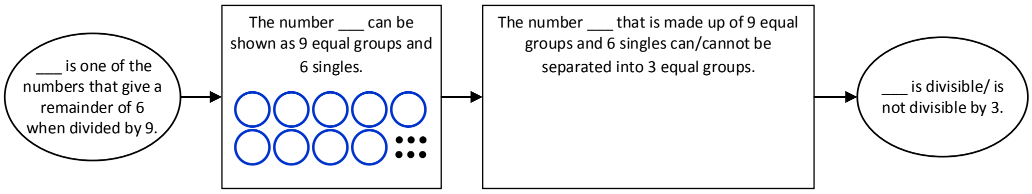
2. Bu şemadan hangi sonucu çıkarabiliriz? Bir cümle ile ifade et.

3. Şemadan yararlanarak, yukarıdaki cümlenin neden kesinlikle doğru olduğunu anlatan bir paragraf yaz. Eksiksiz bir paragraf yazmak için, bu paragrafı, şemayı göremeyen bir arkadaşına telefonda anlatacağını düşünebilirsin. Kafasında soru işareti kalmayacağından emin olmalısın.

(The modified task)

What do the flowcharts tell?

Is the statement "all the numbers that give a remainder of 6 when divided by 9 are divisible by 3" true or false? Choose the cases you wish and fill in the charts given below for each of the cases. [To be repeated for multiple cases.]



Let us think about the flowcharts!

Think aloud and explain:

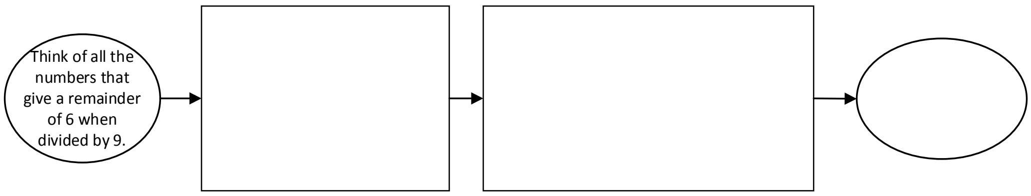
7. One of the numbers that gives a remainder of 6 when divided by 9 is 4794. What result would you get if you were to fill in the chart for 4794?

8. What result would you get if you were to fill in the chart for a number larger than ten million that gives a remainder of 6 when divided by 9?

9. What does this result show us?

Now let's write the thoughts and draw a new flowchart!

10. Think of all the numbers that give a remainder of 6 when divided by 9 and fill in the flowchart below <u>without using any examples</u>.



11. Using the flowchart write a paragraph to explain why the statement "All numbers that give a remainder of 6 when divided by 9 are divisible by 3 without a remainder" is certainly true.

12. Below is an explanation by Öykü. Compare the paragraph you wrote with Öykü's explanation.

Statement: Numbers that give a remainder of 6 when divided by 9 are divisible by 3.

Öykü's Explanation: The statement is absolutely true. Because, the numbers 15, 24, 33, 42, 51, 60, 69, ve 78 are numbers that give a remainder of 6 when divided by 9. All these numbers, as can be seen below, are divisible by 3.

15 ÷ 3 = 5	51 ÷ 3 =17
24 ÷ 3 = 8	60 ÷ 3 = 20
33 ÷ 3 = 11	69 ÷ 3 = 23
42 ÷ 3 = 14	78 ÷ 3 = 26

c. What are the differences between the two methods (yours and Öykü's)?

d. Which one of these two methods do you think is more reliable? Why?

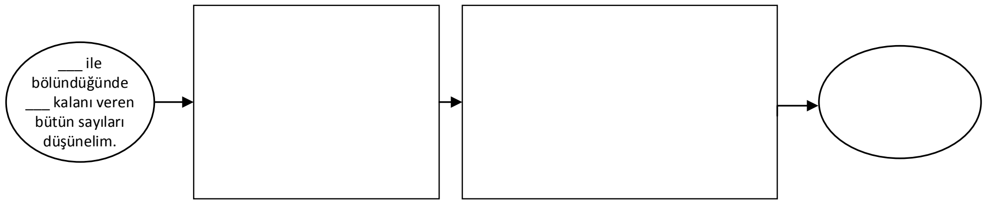
PHASE 4 THE AoC CONCEPT: MODULAR STRUCTURE AS A WAY OF REPRESENTING ALL THE NUMBERS IN AN ARITHMETIC NUMBER PATTERN TO PROVE A CLASS OF CONJECTURES

Haydi İspat Yapalım!

5. 18, 30, 42, 54, 66, 78, 90, 102, 114, 126, 138, 150, …… sayı dizisinde yer alan sayılardan bir tanesi 354'tür. 354 sayısı 6 ile kalansız bölünür. Bu durum, sayı dizisindeki bütün sayılar için geçerli midir? Neden?

İspatlamak istediğimiz ifade:

Yardımcı şema:

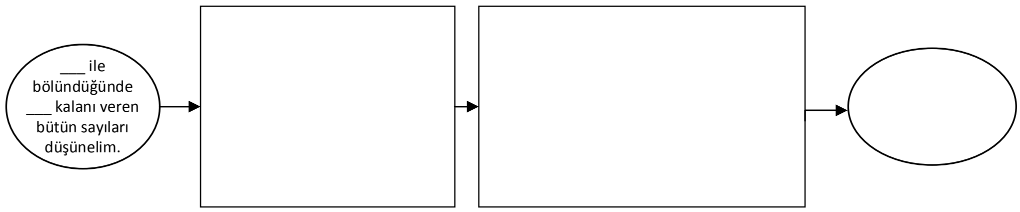


İspat (Paragraf halinde):

6. "8 ile böündüğünde 5 kalanı veren sayılar tek sayıdır" ifadesi doğru mudur? Yanlış mıdır? Neden?

İfade:

Yardımcı şema:



İspat (Paragraf halinde):

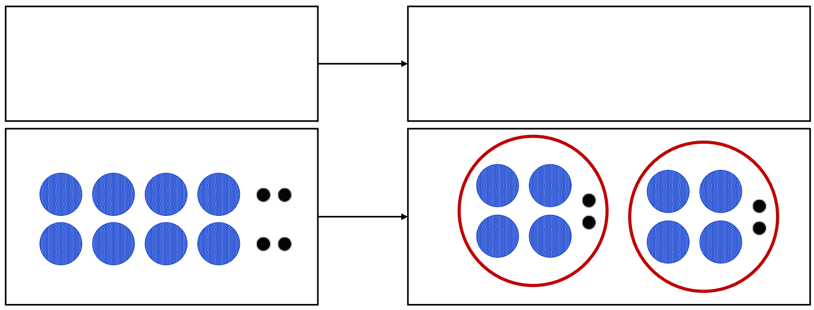
Doğru mu? Yanlış mı? Neden?

1. 8 ile bölündüğünde 4 kalanı veren sayıların hepsi 4 ile bölünebilir. (Doğru/Yanlış)

2. 8 ile bölündüğünde 3 kalanı veren sayıların hepsi 2 ile bölünebilir. (Doğru/Yanlış)

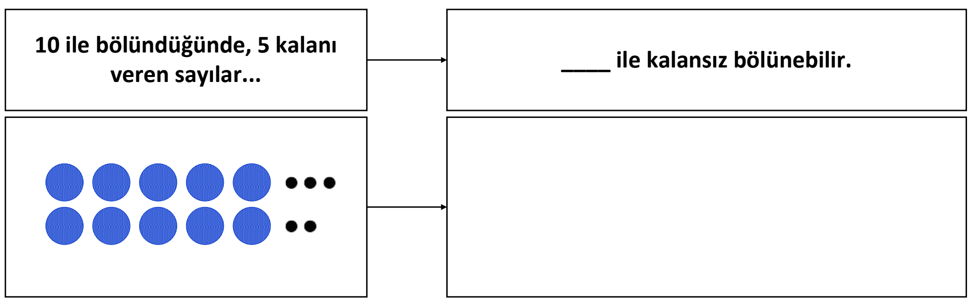
3. 8 ile bölündüğünde 5 kalanı veren sayıların hepsi 5 ile bölünebilir. (Doğru/Yanlış)

Şema bize ne anlatıyor?



Şemayı Tamamlayalım!

1. Aşağıdaki şemada, verilen boşluğa gelebilecek bir sayı belirleyiniz ve şemayı tamamlayınız.
2.



PHASE 5 THE EXTENDED AoC CONCEPTS (AoC-E1 AND AoC-E2)

Hatırlayalım ☺

Aşağıdaki soruyu ve Eren'in bu soruya verdiği yanıtı inceleyiniz.
Soru: "14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94, 102, … sayı dizisindeki bütün sayılar 4 ile bölündüğünde 2 kalınını verir." Bu ifadeye katılıyor musunuz? Nedenini Açıklayınız.

Eren'in Yanıtı: Bu ifadeye katılıyorum. Çünkü,

14	4
-12	3
2	

22	4
-20	5
2	

46	4
-40	11
6	
-4	
2	

102	4
- 8	25
22	
-20	
2	

Bu sayıların hepsi 4 ile bölündüğünde kalan 2 oluyor. Bence geriye kalan sayılar için de sonuç aynıdır.

1. Eren'in yanıtı verilen cümlenin doğruluğunu kesin olarak gösterir mi? Neden?

2. Bu ifadenin doğruluğunu kesin olarak gösteren bir açıklama yazınız.

"14, 22, 30, 38, 46, 54, 62, 70, 78, 86, 94, 102, … sayı dizisindeki bütün sayılar 4 ile bölündüğünde 2 kalınını verir."

3. Aşağıda iki farklı sayı dizisi verilmiştir. Emir her iki sayı dizisinden birer tane sayı seçip bu sayıları topluyor ve bu toplamların 4'ün bir katı olduğunu gözlemliyor.

Birinci sayı dizisi: 6, 10, 14, 18, 22, 26, 30, 34, 38, …

İkinci sayı dizisi: 14, 22, 30, 38, 46, 54, 62, 70, …

Emir'in seçtiği sayılar: 6 + 30 = 36   36 sayısı 4 ile kalansız bölünür. ✓

10 + 22 = 32   32 sayısı 4 ile kalansız bölünür. ✓

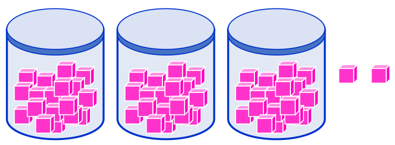
18 + 54 = 72   72 sayısı 4 ile kalansız bölünür. ✓

Sizce bu sonuç her zaman doğru mudur? Yanıtınızı ispatlayınız.

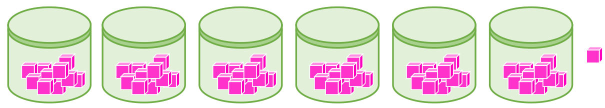
Geçmeli Küp Oyunu

Kelebekler ve Uğurböcekleri, Atatürk İlkokulu'ndaki iki anasınıfıdır. Bu iki sınıf geçmeli küplerini sınıflarındaki kutulara her kutuda eşit sayıda küp olacak şekilde yerleştirmektedir. Kalan küpler ise oyuncak rafına konulmaktadır. Aşağıda Kelebekler Sınıfı ve Uğurböcekleri Sınıfının küpleri verilmiştir.

Kelebekler Sınıfına ait küpler



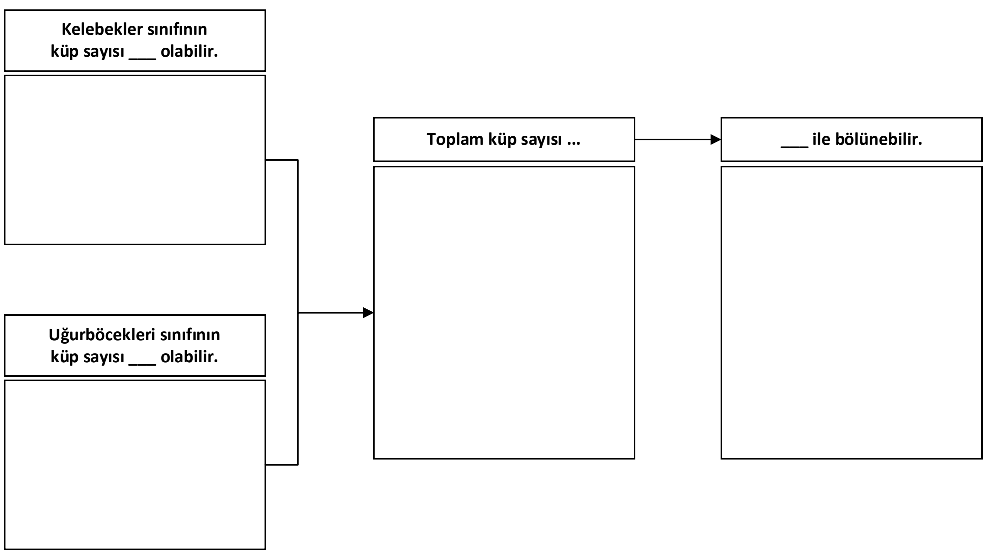
Uğurböcekleri Sınıfına ait küpler



Kelebekler ve Uğurböcekleri birlikte oyun oynayacakları bir etkinlik tasarlıyorlar. Bu oyun kaç grup ile oynanırsa, gruplar küpleri hiç artan küp olmadan eşit paylaşabilirler?

Bu iki sınıfın küp sayıları hangi sayılar olabilir? Düşünelim.

(Repeated for a number of flowcharts)



Problemler

Aşağıda verilen ifadelerde boşluk yerine gelebilecek en büyük sayıyı belirleyiniz. İfadenin doğruluğunu ispatlayınız.

---

4. Aşağıda iki farklı sayı dizisi verilmiştir. Eylül her iki sayı dizisinden birer tane sayı seçip bu sayıları toplayacak. Bu toplam her zaman ___ ile kalansız bölünür.

Birinci sayı dizisi: 9, 16, 23, 30, 37, 44, 51, …
İkinci sayı dizisi: 12, 19, 26, 33, 40, 47, 54, …

5. 10, 17, 24, 31, 38, 45, 52, 59, 66, … şeklinde devam eden sayı dizisinden iki sayı seçilerek toplanıyor. Bu toplam 7 ile bölündüğünde kalan her zaman ___ olur.

6. 10, 17, 24, 31, 38, 45, 52, 59, 66, … şeklinde devam eden sayı dizisinden üç sayı seçilerek toplanıyor. Bu toplam 7 ile bölündüğünde kalan her zaman ___ olur.

7. Üçün katı olan bir sayı ile bu sayının 2 katı toplanıyor. Bu toplam her zaman ___ ile kalansız bölünür.

8. 3, 6, 9, 12, 15, 18, 21, 24, 27, … sayı dizisinden bir sayı seçiliyor ve kendisinin 2 katı toplanıyor. Bu toplam her zaman ___ ile kalansız bölünür.

9. Bir çift sayı ile bu sayının 2 katı toplanıyor. Bu toplam her zaman ___ ile kalansız bölünür.

10. Aşağıda verilen iki sayı dizisinden birer sayı seçilerek toplanıyor. Bu toplam her zaman ___ ile kalansız bölünür.

2, 4, 6, 8, 10, 12, 14, 16, 18, …
4, 8, 12, 16, 20, 24, 28, 32, 36, …

D. CURRICULUM VITAE

Merve Dilberoğlu

EDUCATION

Ph.D.	Elementary Education
(2015-2023)	Middle East Technical University, Faculty of Education, Ankara, Turkey.
M.Sc.	Elementary Science and Mathematics Education
(2012-2015)	Middle East Technical University, Faculty of Education, Ankara, Turkey. Thesis: *An investigation of pre-service middle school mathematics teachers' ability to connect the mathematics in content courses with the middle school mathematics*
B.Sc.	Elementary Mathematics Education
(2007-2012)	Middle East Technical University, Faculty of Education, Ankara, Turkey.
Minor Degree	
(2007-2012)	Mathematics Middle East Technical University, Faculty of Arts and Science, Ankara, Turkey.

ACADEMIC JOB EXPERIENCE

(2012-2022)	Teaching and Research Assistance Middle East Technical University, Faculty of Education, Elementary Education, Ankara, Turkey.

PUBLICATIONS

Papers in International Conference Proceedings

Dilberoğlu, M., Çakıroğlu, E., & Haser, Ç. (2022). A sixth-grade student's growth in understanding written proof texts: pre- and post-interview analyses of a teaching experiment study. In J. Hodgen, E. Geraniou, G. Bolondi, & F. Ferretti (Eds.), *Proceedings of the Twelfth Congress of the European Society for Research in Mathematics Education (CERME12)* (pp. 236-243). Free University of Bozen-Bolzano, Italy and ERME.

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E. TURKISH SUMMARY / TÜRKÇE ÖZET

BİR ALTINCI SINIF ÖĞRENCİSİNİN TÜMDENGELİMLİ İSPAT ŞEMASINA DOĞRU İLK ADIMLARININ ALTINDA YATAN MEKANİZMALARIN BİR İNCELEMESİ

1. GİRİŞ

İspatı okul matematiğinin rutin bir parçası haline getirmeye yönelik yapılan çalışmaların temelinde ispatın matematik disiplinindeki önemi vardır (Harel, 2008a; 2008b; Stylianides & Stylianides, 2017). İspatlama, matematiksel iddiaların doğruluğunu belirleyen "zihinsel eylem"dir (Harel & Sowder, 2007; Harel, 2008c). Bu zihinsel eylem ile bireyler varsayımları doğrulamak için argümanlar önerirler. Önerilen argümanların matematiksel ispat olarak nitelendirilip nitelendirilmeyeceği argümanın matematik disiplininin doğasına uygunluğuna bağlı olarak belirlenir (Harel, 2008a). Bir argüman, yalnızca matematik topluluğu tarafından önceden kabul edilmiş ifadeler, geçerli muhakeme biçimleri ve kabul edilen temsil biçimlerini kullanılıyorsa geçerli ispat olarak kabul edilir (Stylianides, 2007). Bununla birlikte, *ispat şemaları* olarak adlandırılan, bireylerin matematikte ispatı neyin oluşturduğuna dair kavramaları, disiplinin bu titiz anlayışı ile her zaman uyumlu değildir. Örneğin, bireyler genel bir ifadenin geçerliliğini tesis etmek için sınırlı sayıda doğrulayıcı örneğe güvenebilir veya bir argümanın geçerliliğini, temel yapısını değerlendirmek yerine sadece görünümüne göre yargılayabilir. Harel ve Sowder (1998) tarafından önerilen, araştırmaya dayalı bir ispat şeması sınıflandırmasına göre bu iki hatalı düşünme biçimi sırasıyla *deneysel ispat şeması* ve *ritüel (alışkanlık edinilmiş) ispat şeması* olarak nitelendirilirken, matematiksel ispat kavramı, *tümdengelimli ispat şeması* ile ilişkilendirilmiştir.

Deneysel ispat şemasının öğrenciler arasında yaygın olduğunu ve bu şemanın, güvenli ispat yöntemleri öğrenildikten sonra dahi kalıcı olduğunu gösteren araştırma bulguları bulunmaktadır (Avrupa Matematik Derneği Eğitim Komitesi, 2011). Bu nedenle matematik eğitiminin paydaşları okulun ilk yıllarından başlayarak öğrencilerin deneysel akıl yürütme biçimlerinden tümdengelimli akıl yürütme biçimlerine geçişini teşvik etmeyi amaçlamaktadır (Styianides & Stylianides, 2009). Bu geçiş, iki adımlı bir öğrenme hedefidir. Bireylerin ispat şemaları birbirini dışlamadığından, bir kişi aynı anda birden fazla ispat şemasına sahip olabilir (Harel & Sowder, 1998). Yani, tümdengelimli ispat şeması edinildiğinde, deneysel ispat şeması kendiliğinden ortadan kaybolmaz. Aksine, öğrencilerin deneysel ispatların sınırlılıklarını anlamaları için önemli derecede çaba gereklidir (Harel, 2008b). Bu durum, deneysel ispat şemasının ortadan kaldırılmasını öğrenme hedefimizin bir adımı haline getirir. Ardından izleyen ikinci adım ise tümdengelimli ispat şemasının gelişimini teşvik etmektir.

İlk adıma yönelik araştırma temelli bir yaklaşım, Stylianides ve Stylianides (2009) tarafından ortaya konulmuştur. Yazarlar, matematik lisans öğrencilerinin deneysel argümanların sınırlılıklarını fark etmelerine ve matematiksel genellemeleri doğrulamak için daha güvenli bir yöntem aramak için zihinsel bir ihtiyaç (intellectual need; Harel, 2013) geliştirmelerine yönelik bir öğretim dizisi geliştirmiştir. Çalışmanın katılımcıları, burada hedeflenen zihinsel ihtiyacı hissetmelerine rağmen, araştırmacıların desteği olmadan bir ispatı kendi kendilerine üretememişlerdir. Daha da önemlisi, katılımcılar geçerli bir ispatın nasıl görüneceğini tahmin etme konusunda herhangi bir anlayışa sahip olduklarına dair belirti göstermemişlerdir (Styianides & Stylianides, 2009). Bu çalışma, deneysel ispat şemasının ortadan kaldırılmasının ardından, öğrenme hedefinin ikinci adımını incelemeyi amaçlamakta, yani öğrencilerin tümdengelimli ispat şemasına geçişini sağlamanın yollarını araştırmaktadır. Burada karşılaşılan zorluğu anlamak, Harel'in (2007) *ispatlama-ispat-ispat şeması* üçlüsünü teorik bir bakış açısından ele almak gerekir.

DNR tabanlı matematik öğretiminde zihinsel bir eylem olarak ispatlama

DNR tabanlı öğretim (DNR), matematik eğitiminin iki ana sorusunu yanıtlamayı amaçlayan teorik bir çerçevedir: "(1) Okulda öğretmemiz gereken matematik nedir?" (Harel, 2008a, s. 487) ve “(2) Nasıl öğretmeliyiz?” (Harel, 2008b, s. 893). DNR adı, çerçevenin temelini oluşturan üç öğretim ilkesinden türetilmiştir: İkilik (Duality), Gereklilik (Necessity) ve Tekrarlı düşünme (Repeated Reasoning). Harel (2008a, 2008b) matematiği, tarih boyunca matematik camiasının kabul ettiği zihinsel eylemlerin ürünlerinin ve bu ürünlerin bilişsel özelliklerinin tümü olarak tanımlamaktadır. Matematiksel bilgi yorumlama, modelleme, tanımlama gibi zihinsel eylemlerin uygulanmasıyla üretilir. Bu çalışmanın odak noktası olan ispatlama da bu zihinsel eylemlerden biridir. Harel'e göre zihinsel bir eylemin bilişsel bir ürünü, bireyin o eylemle ilişkili *anlama biçimidir* (a way of understanding). Bireyin aynı zihinsel eyleminin çok sayıda ürünün gösterdiği herhangi bir bilişsel ortak özelliği ise bireyin o zihinsel eylemle ilişkili *düşünme biçimidir* (a way of thinking). Başka bir deyişle matematik, tarih boyunca matematik camiasının kabul ettiği tüm anlama ve düşünme biçimlerini içeren disiplindir (Harel, 2008a). Bu *zihinsel eylem-anlama biçimi-düşünme biçimi* üçlüsü, daha önce adı geçen *ispatlama-ispat-ispat şeması* üçlüsünün genellemesidir (Harel, 2007; Harel 2008a). Bu açıdan bakıldığında, kişinin ispat şeması, ispatlama eylemiyle ilgili düşünme biçimidir ve ürettiği ispatlar, ispatlama eylemiyle ilgili anlama biçimleridir.

İkilik ilkesi, anlama biçimlerinin gelişimi ile düşünme biçimlerinin gelişimi arasında çift yönlü bir ilişki olduğunu ileri sürer. Gereklilik ilkesi, öğrencilerin bir şeyi öğrenmek için zihinsel bir ihtiyaç duymaları gerektiğini belirtir. Ve tekrarlı düşünme ilkesi, tekrarlı uygulamaların matematik bilgisinin içselleştirilmesi, düzenlenmesi ve akılda kalması için önemli bir bileşen olduğunu ileri sürer (Harel, 2008a). İkilik ilkesi, tümdengelimli ispat şemasını geliştirmenin neden zorlu bir görev olduğunu açıklayan ana ilkedir.

> İkilik ilkesi: (I) Öğrenciler, onlara öğretmeyi amaçladığımız anlama biçimlerini kaçınılmaz olarak etkileyen, bazıları istenen ve bazıları istenmeyen bir dizi düşünme biçimine halihazırda sahiptirler. (II) Öğrenciler, yalnızca uygun anlama biçimlerinin tekrarlı olarak uygulanması yoluyla arzu edilen düşünme biçimlerini geliştirirler (Harel, 2021, s. 711).

İkilik ilkesinin ikinci kısmı, öğrencilerin matematiksel ispatları anlamanın uygun biçimlerini deneyimleyerek tümdengelimli ispat şemasını geliştirebileceğini söyler. Bu iddia, öğrencilerin matematiksel bir argümanı geçerli bir ispat örneği olarak tanımama halinden tanımaya bilişsel bir değişim geçirmeleri gerektiği anlamına gelir. Böyle bir değişimin deneyimlenmediği durumlarda, (yukarıda bahsedilen çalışmada olduğu gibi) öğrencilerin belirli bir teoremi ispatlamak için ne tür bir argüman oluşturmaları gerektiğini öngörebilmeleri beklenemez. Bu çalışmada, bu değişimin deneyimlendiği ilk andan *matematiksel ispatın ilk örneği* olarak bahsedeceğim.

Henüz tümdengelimli ispat şemasına sahip olmadan, matematiksel ispatın ilk örneğini bu kategorinin uygun bir üyesi olarak anlamak paradoksal bir öğrenme hedefidir. Bu çalışma, bugüne kadar eğitim araştırmalarının odak noktası olmayan bu öğrenme hedefine bireylerin ulaşma sürecinin ardındaki mekanizmayı irdelemeyi amaçlamaktadır. Buna yönelik bir adım olarak, aşağıdaki bölüm "öğrenme paradoksu" fikrini (Pascual-Leone, 1976) tanıtır ve bir argümanı ilk kez ispat olarak görmenin, herhangi bir öğrenme durumunda olduğu gibi, bu paradoksun bir örneği olduğunu açıklar.

Öğrenme Paradoksu (Pascual-Leone, 1976)

Öğrenme paradoksu, yapılandırmacı kuramın ana fikirlerinden biri olan, Piaget'nin *özümleme* (assimilation) olarak isimlendirdiği sürecin bir sonucudur. Piaget'in *denge* (equilibration) teorisine göre, özümleme ve *uyum* (accommodation), bireylerin deneyimsel dünyalarını kavrarken dengede olma halini sürdürmek için kullandığı iki temel süreçtir. Yeni deneyimler, bireyin önceki deneyimleriyle uyumlu sonuçlar doğurmaları halinde, kişinin mevcut kavram ve işlemlerine (mevcut bilişsel şemalarına) asimile edilir. Var olan özümleme yapılarıyla çelişen yeni bir yaşantı, kişinin bilişsel yapılarında bir pertürbasyon (dengesizleştirici bir deneyim) yaratır. Kişinin mevcut şemalarının yeniden düzenlenmesi anlamına gelen uyum, arzu edilen denge durumunun yeniden kurulmasını sağlar (von Glasersfeld, 1995; Simon, Tzur, Heinz ve Kinzel, 2004).

Bir bireyin tanıyabileceği ve (bir hedefe ulaşmak için) harekete geçebileceği yeni durumlar, kişinin özümlenen kavramlarına bağlıdır (Simon, Tzur, Heinz ve Kinzel, 2004). Simon, Tzur, Heinz ve Kinzel'in (2004) iddiasından yola çıkarak şöyle denilebilir: Özümleme kavramının doğası gereği, bir öğrenci tümdengelimli ispat şemasına sahip değilse, ne kadar şeffaf bir şekilde gösterilirse gösterilsin bir ispatı bu kategorinin bir örneğini olarak tanıyamaz. Bu, mevcut çalışmanın hedefini öğrenme paradoksunun özel bir durumu haline getirir. Öğrenme paradoksu, öğrenme süreci denilen herhangi bir şeyin nasıl mümkün olduğunu, yani bireylerin kavramsal olarak zayıf bir sistemden daha güçlü bir sisteme geçişini nasıl sağladığını sorgular (Fodor, 1980, s. 149, aktaran Bereiter, 1985).

Dengesizleştirici bir deneyim, Stylianides ve Stylianides'in (2009) çalışmasında olduğu gibi, bir düzenleme sürecini tetikleyebilir. Ancak, Simon'ın (1995) örneklendirdiği gibi, sürecin istenen yönde bir düzenleme ile sonuçlanacağını garanti etmez. Öğrenme paradoksuna çözüm sunabilecek bir olasılık, Piaget'nin (2001) öne sürdüğü "yansıtıcı soyutlama"dır. Simon vd. (2004), bu yapıyı, öğrenen kişinin *etkinlik-etki* ilişkisi üzerindeki *yansıtıcı düşüncesi* olarak yorumlamıştır. Bu düşünce daha sonra, diğer orijinal araştırmalar dizisi ile birlikte (Simon, 1995; Simon & Tzur, 2004; Simon, Tzur, Heinz, & Kinzel, 2004; Tzur, 1996; Tzur & Simon, 2004), Etkinlik Yoluyla Öğrenme (Learning Through Activity) araştırma programının temelini oluşturmuştur. Etkinlik Yoluyla Öğrenme teorik çerçevesinin yansıtıcı soyutlama yorumu, bir bireyin tümdengelimli ispat şemasına nasıl ulaşılabileceğini inceleyen bu çalışma için ilham kaynağı olmuştur.

Etkinlik Yoluyla Öğrenme (EYÖ)

EYÖ, öğrencilerin bir kavramsal anlayıştan diğerine geçişini birey düzeyinde inceleyerek kavramsal matematik öğreniminin mekanizmalarını açıklamayı amaçlayan bir araştırma programıdır. Burada kavramsal değişimi incelemenin temel dayanağı, hedeflenen amaca yönelik görev dizilerine yanıt olarak öğrencinin sergilediği zihinsel etkinliklerin yakından gözlemlenmesidir. Araştırma programın kapsamlı ve birbiriyle ilişkili iki amacı vardır: (1) bütünleşik bir kavramsal matematik öğrenimi ve öğretim tasarımı teorisi oluşturmak ve (2) çeşitli matematik kavramları için *varsayıma dayalı öğrenme rotaları* (hypothetical learning trajectories; Simon, 1995) geliştirmek. Bu amaçları yerine getirmek için, EYÖ kendi araştırma programının çıktıları olan öğretim tasarımı çerçevesini (Simon, Kara, Placa ve Avitzur, 2018) ve EYÖ-uyarlanmış öğretim deneyi metodolojisini (Simon, 2018) kullanır. Burada anlatılan çalışma, ispat öğrenimine yönelik kendine özgü amaçlarını gerçekleştirmek için her iki çıktıdan yararlanmaktadır.

EYÖ teorik çerçevesi kapsamında, Simon (2017) "matematik kavramı" ifadesini öğretim pedagojisini organize etme amacıyla kullanılan, geliştirilmeye açık bir teorik yapı olarak tanımlar. Farklı şekillerde tanımlanabilen matematik kavramları, çeşitli öğretim tasarımlarının geliştirmeyi amaçladığı hedef anlayışları doğru bir şekilde ifade etmek için bir araç olarak kullanılır. Simon (2020) şu ana kadar iki ayrı kavram türü öne sürmüştür. Her iki kavram da öğrencinin gerçekleştirdiği etkinlik üzerinde yansıtıcı soyutlama yapması yoluyla öğrenilir ve o sırada gerçekleşen yansıtıcı soyutlamanın türüne göre isimlendirilir. "Eylemlerin koordinasyonu" (coordination of actions) yoluyla inşa edilen kavramlara "EK kavramları" (CoA concepts), "aynılığın soyutlanması" yoluyla inşa edilen kavramlara "AS kavramları" (AoC concepts) denir. Bu çalışmanın temel motivasyonu, eylemlerin koordinasyonu ve aynılığın soyutlanması süreçlerinin, kişinin tümdengelimli ispat şemasına geçişini açıklayan mekanizmalar olabileceği düşüncesidir.

1.1. Çalışmanın Amacı

Bu çalışma, EK ve AS kavram yapılarının ve ilişkili yansıtıcı soyutlama türlerinin, herhangi bir bireyin tümdengelimli ispat şemasına doğru ilk adımlarını atma sürecini açıklayabileceği varsayımından yola çıkılarak tasarlanmıştır. Bireysel bir altıncı sınıf öğrencisinin deneysel ispat şeması, Stylianides ve Stylianides (2009)'in geliştirdiği etkinlik dizisinin uyarlanmasıyla, başarılı bir şekilde ortadan kaldırılmıştır. Ardından, öğrencinin tümdengelimli ispatın ilk örneği ile karşılaşması iki adımlı bir öğrenme hedefi olarak tasarlanmış ve araştırılmıştır. Birinci öğrenme hedefi bir EK kavramı olarak tanımlanmıştır. İkinci öğrenme hedefi ise bir AS kavramı ve bu kavramın iki farklı uzantısı üzerinden tanımlanarak araştırılmıştır. Aslında, EYÖ'nün EK ve AS kavram yapıları burada söz edilen iki öğrenme hedefiyle tam olarak uyumlu değildir. Bu nedenle, çalışmanın hedeflenen kavramlarını ifade etme şekli, şimdiye kadar EYÖ araştırma raporlarında görülen kavramlardan biraz farklıdır. Aşağıda, bu çalışmaya özgü amaçlanan kavramların kısa açıklamaları bulunmaktadır.

Çalışmanın başlangıç hipotezi, EYÖ öğretim tasarımı yaklaşımına (Simon, Kara, Placa ve Avitzur, 2018) uygun olarak tasarlanan bir görev dizisine yanıt olarak, katılımcı öğrencinin sıralı bir etkinliği gerçekleştireceği ve etkinlik tamamlandığında bir akış şeması ispatı elde edeceği beklentisi üzerine kurulmuştur. Öğrenci, yaptığı etkinliğin sonucunda ortaya çıkan bu şema üzerinde düşünerek, şemanın belirli bir matematiksel genelleme için ispat oluşturduğunu fark etme şansına sahip olacaktır. Öğrencinin, etkinliğinin bu ürününü matematiksel ispatın ilk örneği olarak anlaması, yani henüz inşa edilmemiş olan tümdengelimli ispat şemasının ilk üyesi olarak kategorize etmesi için temel koşul, onu ilk kez bir bütün, bir nesne olarak görmesi olarak düşünülmüştür. İspatın bu ilk örneğini bir nesne, daha doğrusu yapısal bir nesne olarak anlamak (Miyazaki, Fujita ve Jones, 2017), çalışmanın birinci öğrenme hedefini oluşturmaktadır ve bir EK kavramı olarak tanımlanmıştır. Burada, ispatı yapısal bir nesne olarak anlamak, ispatta yer alan *evrensel örneklemeler* (universal instantiation) ve *varsayımsal kıyasların* (hypothetical syllogism) tamamını anlamayı içerir (Miyazaki, Fujita ve Jones, 2017).

İspatın ilk örneği yapısal bir nesne olarak anlaşıldıktan sonra, çalışmanın ikinci öğrenme hedefi, tümdengelimli ispat şemasının karakterini geliştirmek, yani öğrenciyi diğer uygun ispat örneklerini oluşturması için desteklemekle ilgilidir. Bu noktada çalışma, Mejia-Ramos, Fuller, Weber, Rhoads ve Samkoff'un (2012) matematik bölümü lisans öğrencilerinin ispatı anlamalarına ilişkin ortaya koydukları değerlendirme modelinin bileşenlerinden biri olan, bir ispatın "bütüncül" olarak anlaşılması üzerine inşa edilmiştir. Yazarlar bu anlayışı, belirli bir ispatı "ana fikirleri, yöntemleri ve diğer bağlamlara uygulanabilirliği açısından" bir bütün olarak anlamak şeklinde tanımlarlar (Mejia-Ramos, Fuller, Weber, Rhoads ve Samkoff, 2012, s. 10). Bir ispatın diğer bağlamlara uygulanması, içinde bulundurduğu genel fikirlerin veya yöntemlerin diğer ispat yapma görevlerine aktarılmasını veya uyarlanmasını içerir (Mejia-Ramos vd., 2012). Bu da bize, bütüncül olarak anlaşılırsa, öğrencinin deneyimlediği ilk matematiksel ispat örneğinin, sonraki benzer teoremleri ispatlama eylemi için temel oluşturabileceğini ima eder. Dolayısıyla öğrenciye, İkilik II ilkesinin gerektirdiği şekilde ispatlama eylemini tekrarlı olarak uygulayabilmesi için bir temel oluşturabilir.

Bu hedef aynı zamanda, tümdengelimli ispat şemasının gelişimi ile doğrudan ilişkili olan diğer bir düşünme biçiminin, *tanımsal akıl yürütmenin*, gelişimini teşvik etmek için ilk adım olarak görülmektedir. Harel (2008a) tanımsal akıl yürütmeyi "kişinin nesneleri tanımlayarak, iddiaları matematiksel tanımlar üzerinden ispatladığı düşünme biçimi" olarak ifade eder (s. 495). Çalışmanın ilk örnek ispatı olarak, oldukça kısa bir doğrudan ispat seçilmiştir. Bu ispat, bir tanım ifadesiyle başlar ve bu tanımdan modus ponens mantıksal çıkarım kuralının uygulanması ile devam eder. Bu nedenle, ispatlanan teorem ile ilgili olarak belirli bir kavram tanımını anlamak ve bu kavramın benzer diğer teoremleri ispatlamadaki kullanımını takdir etmek ve dolayısıyla onu aktarabilmek, çalışmanın ikinci öğrenme hedefi olan tanımsal akıl yürütmeye yönelik ilk adımı oluşturmaktadır. Buradaki hedef öğrenme, ilk örnek ispatının “ana fikri”dir ve çalışmanın AS kavramı olarak tanımlanmıştır. Bu kavram, öğrencinin aynı yapıyı paylaşan (analojik) teoremleri ispatlamak için ilk örneğin ana fikrini uygulamasından yola çıkarak elde ettiği deneyimlerdeki aynılığı soyutlamasıyla öğrenilebilir. Yani, hedeflenen anlayış bir AS kavramına benzer olabilir (Simon, 2020). AS kavramının iki ayrı uzantısı olan AS-Ek1 ve As-Ek2 kavramları tanımsal akıl yürütmenin örneklerini çeşitlendirerek, öğrencinin bu anlama biçimleri üzerinden ilgili düşünme biçimini geliştirip geliştirmeyeceğini gözlemlemek amacıyla çalışılmıştır.

1.2. Araştırma soruları

Bu çalışma, bireysel bir altıncı sınıf öğrencisinin tümdengelimli ispat şemasına yönelik ilk adımlarını araştırmaktadır. İlk bölüm, katılımcı öğrencinin (1) tümdengelimli bir argümanı matematiksel ispatın ilk örneği olarak anlama (hedeflenen EK kavramı) sürecini incelemektedir (Öğrenme Hedefi 1). İkinci bölüm, öğrencinin (2) ispatın ilk örneği olarak anladığı bu argümanın ana fikrini analojik varsayımları ispatlama sürecinde uygulanabilir bir ana fikir olarak geliştirmesi (hedeflenen AS kavramı) ile başlar. Amaçlanan EK ve AS kavramları birlikte, öğrencinin ispat eylemiyle ilişkili düşünme biçiminde tümdengelimli ispat şemasının özelliklerini geliştirmeyi amaçlar. Bu nedenle, çalışmanın ikinci bölümü (3) öğrenilen kavramların, öğrencinin hem ilk örneğe benzer (analojik olmayan) hem de yeni olan diğer varsayımları ispatlama eylemini nasıl desteklediği veya kısıtladığını inceler. İkinci bölümün ana odak noktası, tümdengelimli ispat şemasının gelişmesiyle doğrudan ilişkili önemli bir düşünme biçimi olan tanımsal akıl yürütmedir (Öğrenme Hedefi 2). Bireysel öğretim deneyi yöntemini kullanılarak yapılan araştırmaya aşağıdaki araştırma soruları rehberlik etmiştir.

1. Bireysel bir öğrenci, tümdengelimli bir argümanı ilk kez matematiksel ispatın bir örneği olarak nasıl anlayabilir? Eylemlerin koordinasyonu (EK) tipi bir yansıtıcı soyutlama, ilk örnek ispatını yapısal bir nesne olarak anlama sürecini açıklayabilir mi?
2. Öğrenci, ilk örnek ispatının ana fikrini benzer ispatlama görevlerine aktarabileceği bir şekilde nasıl geliştirebilir? Aynılığın soyutlanması (AS) tipi bir yansıtıcı soyutlama, burada hedeflenen bütüncül anlayışı inşa etme sürecini açıklayabilir mi?

3. Öğrenci (a) ilk örnek varsayıma benzer (ancak analojik olmayan) varsayımları, (b) çalışmanın hedef varsayımını ve (b) tamamen yeni bir varsayımı ispatlama işine nasıl başlar (hangi girişimlerde bulunur)?

1.3. Öğrenme alanı ve sınıf düzeyi seçimi

Altıncı sınıf düzeyi, metodolojik değerlendirmelere dayalı olarak belirlenmiştir. İlk olarak, ispatlanacak varsayımlar sayılar teorisi içerik alanında oluşturulmuştur. Daha sonra Ortaokul Matematik Öğretim Programı (MEB, 2018) temel alınarak çalışmanın varsayılan öğrenme süreci için kritik olan kavram ve işlemlerin sınıf düzeylerine göre dağılımı incelenmiştir. Çalışmanın amacı, bireysel bir öğrencinin belirli matematiksel anlayışları bilmemekten bilmeye geçiş sürecini incelemek olduğundan (Tzur, 2018), katılımcı öğrencinin yalnızca belirli kavramları ve işlemleri biliyorken, diğerlerini bilmiyor olması kritik önem taşımıştır (Simon, 2018). Ortaokul Matematik Dersi Öğretim Programı (MEB, 2018) kapsamında altıncı sınıf düzeyi bu kriteri sağladığı için seçilmiştir.

1.4. Çalışmanın Önemi

Okul matematiği için dört tip *akıl yürütme-ve-ispatlama* etkinliği öngörülmüştür (Stylianides, 2008). *Akıl yürütme-ve-ispatlama* terimini kullanarak Stylianides (2008), matematiksel bilgiyi anlamlandırma ve yapılandırmaya yönelik dört ana etkinliği kapsayan etkinlikler bütününe atıfta bulunur. Bu dört ana etkinlik, "örüntü belirleme", "varsayımda bulunma", "ispatlamayan argüman üretme" ve "ispat üretme" olarak ifade edilir. Alanyazında, bu dört tip etkinlik ile ilgili birçok çalışma bulunmaktadır. Matematik eğitimi alanının önemli bir amacı, ispat alanında yapılan tasarım odaklı çalışmaların birbirleriyle nasıl ilişkilendirileceğini ve ispat öğretimini tasarlamak üzere bütüncül bir öğretim programı içerisine nasıl yerleştirilebileceklerini belirlemektir (Styianides & Stylianides, 2017). Bu çalışma, bu kadar kapsamlı bir çabanın tümdengelimli ispat şemasının geliştirilmesi nihai amacına ulaşmasının bireysel düzeyde nasıl mümkün olabileceğini inceleyerek araştırmalar bütününe katkıda bulunmayı amaçlamaktadır. Başka bir deyişle çalışma ispata yönelik bir öğretim programının öğrencilerin ulaşmasını isteyebileceği "hedef anlayışları" belirlemeyi amaçlamaktadır.

Çalışmanın katkıları uygulamadan çok teoriye yöneliktir. Bu çalışma, altıncı sınıf düzeyinde tümdengelimli ispatın öğretimi için bir varsayıma dayalı öğrenme rotası üretme amacı taşımamaktadır. Bununla birlikte, araştırma hedeflerine ulaşmak amacıyla kullanılan etkinlik dizisi, bölünebilirlik kavramına erişimi olan öğrencilerin çeşitli varsayımları ispatlamayı deneyimleyebilmesi konusunda yeni yaklaşımlar sunabilir. Ancak yine de sunulan öğretim kaynakları sınırlılıklara sahiptir. Benzer şekilde, çalışma altıncı sınıf düzeyinde gerçekleştirilmiş olsa da bu düzey öğrencilerin ilgili hedef anlayışları öğrenmeleri gereken zamanı tam olarak göstermez. Daha önce de belirtildiği gibi, bu sınıf düzeyi yalnızca metodolojik değerlendirmelere dayalı olarak belirlenmiştir. Diğer yandan, araştırmanın bulguları ispatın öğrenilmesi konusunda farklı sınıf düzeylerindeki öğrencilerin var olan bilgileriyle gerçekçi bağlantılar kurulmasına olanak sağlama potansiyeline sahiptir.

Mevcut alanyazın, deneysel ispat şemasını ortadan kaldıran bir mekanizma olarak bilişsel çatışma kavramının dışında, öğrencilerin ispatlama eylemiyle ilişkili düşünme biçimlerinin gelişimini açıklayan mekanizmalara ilişkin teorik bilgiden yoksundur (Harel, 2008a). Matematiksel ispatları anlama biçimlerine yönelik olarak ortaya konan teorik çerçeveler, bireylerin var olandan daha iyi anlayışlara geçişlerinin hangi öğrenme mekanizmaları yoluyla gerçekleştiğini açıklamamıştır. Bu çalışma, ilgili mekanizmaları açıklayarak hedeflenen geçişleri teşvik etmenin yollarını sunabilir. Burada metodolojik bir araç olarak kullanılan varsayıma dayalı öğrenme rotası, mevcut müfredatın olabildiğince erken dönemlerine dahil edilebilecek, kısa süreli bir öğrenme rotasının geliştirilmesine temel oluşturabilir.

1.5. Tanımlar

EK kavramı: Bu kavram türü bir hedef-eylem bileşimidir. Bir EK kavramının yansıtıcı soyutlaması, öğrencinin çok adımlı bir etkinliği izlemesi sırasında gerçekleşir. Verilen yeni bir görevi çözmek için öğrenci, mevcut kavram bilgisi ile ulaşabileceği bir hedef belirler. Görev hedefine ulaşmak için, her biri etkinlikteki bireysel adımlar için bir alt hedefe karşılık gelen bir dizi eylemi (A_{0a} ve A_{0b}) çağırır ve işe koşar. Etkinlikteki eylemler, amaçlanan koordinasyonun gerçekleşmesini sağlayan belirli bir sırayla gerçekleşir. İki sıralı eylem yeni bir üst düzey eylem oluşturacak şekilde koordine edildiğinde, bu, yeni kavramın soyutlandığı, yani ilgili ilişkinin mantıksal gerekliliğinin öğrenildiği anlamına gelir. Yeni kavramı soyutlayan öğrenci, adım adım gerçekleştirmeden etkinliğin sonucunu tahmin edebilir (Simon, 2020).

AS Kavramı: Bu kavram türü matematiksel deneyimleri örnek olan ve örnek olmayanlar olarak gruplandırmak için kullanılabilen matematiksel yapılardır. AS kavramları, öğrencinin birden fazla etkinliğindeki aynılığı soyutlaması yoluyla inşa edilir. Örneğin, bölmenin paylaştırma anlamını içeren problem durumlarında "aynı" olan şey ortak bir matematiksel yapıdır. Karşılaşılan problem durumlarında bu yapının (aynılık) varlığının veya yokluğunun fark edilmesi, bireyin paylaştırma (bölme) kavram bilgisine sahip olmasına bağlıdır (Simon, 2020).

2. YÖNTEM

Bu çalışmada, öğretim deneyi (Steffe & Thompson, 2000) yönteminden yola çıkılarak geliştirilmiş olan EYÖ-uyarlanmış öğretim deneyi uygulanmıştır (Simon, 2018). Bu yöntem, kavramsal öğrenme ve öğretim tasarımı çalışmalarını, matematik kavramları için varsayıma dayalı öğrenme rotaları oluşturma süreciyle bütünleştirmenin bir yolunu sunar. EYÖ öğretim deneyi metodolojisinin ayırt edici özelliği, öğrenci düşüncesindeki kavramsal aşamaların sırasını tanımlamayı amaçlayan diğer çalışmalardan farklı olarak, bir kavramsal aşamadan diğerine geçiş sürecinde gerçekleşen öğrenme mekanizmalarını açıklamaya çalışmasıdır (Simon, 2013). Bu amaç, öğrencilerin EYÖ teorik çerçevesinin rehberliğinde hazırlanmış etkinlikler üzerinden kendi soyutlamalarını yapma süreçlerinin incelenmesiyle yerine getirilir (Simon vd. 2018). EYÖ öğretim deneylerinin, araştırmacı ile öğrencinin bire bir çalıştığı deneyler şeklinde gerçekleştirilmesi zorunludur. Araştırmacı, öğrencinin eylemleri üzerinde, tasarlanan etkinlik dışında bir etki oluşturmaktan kaçınmak adına çözüm yollarını söylemez, göstermez, ipuçları vermez veya yönlendirici sorular sormaz. Tasarlanan etkinlik dizisini uygulamadan hemen önce, öğrencinin hedef kavrama sahip olup olmadığı değerlendirilir. Bu değerlendirmede, öğrencinin hedef kavramı bilmiyor olması, çalışmanın amacı için uygun veri setinin elde edilmesi açısından kritik öneme sahiptir. Öğrenci kavramı öğrenmeden önce yapılan ilk değerlendirme ve kavramı öğrendikten sonra yapılan ikinci bir değerlendirme, istenen veri setinin iki uç noktasını oluşturur. Veri seti, bu iki nokta arasında gerçekleşen öğrenme sürecini açıklamak amacıyla analiz edilir. EYÖ öğretim deneyi yönteminin bu özelliği göz önüne alındığında, tümdengelimli ispat şemasına sahip olmayan bir öğrencinin ispatın ilk örneğini anlama sürecini araştırmak için uygun bir araştırma yaklaşımı olduğu düşünülebilir.

Öğretim deneyinin gerçekleştirilmesinden bir yıl önce, tipik bir altıncı sınıf öğrencisinin ilgili kavramlar ve işlemlerle nasıl çalışabileceğine dair ilk elden bir deneyim kazanmak amacıyla "keşfedici öğretim" (Steffe & Thompson, 2000) yürütülmüştür. Sekiz haftalık keşfedici öğretim sonucunda kazanılan deneyimsel bilgi, ilgili literatürün tekrar ve daha ayrıntılı bir şekilde gözden geçirilmesiyle birlikte, EYÖ öğretim deneyi çalışmasının temel yaklaşımlarını şekillendirmede belirleyici olmuştur.

2.1. Çalışmanın bağlamı ve katılımcılar

Öğretim deneyi ve keşfedici öğretim çalışmaları iki ayrı altıncı sınıf öğrencisiyle yürütülmüştür. Melis (takma isim) 2019 yılı bahar dönemi sonuna doğru sekiz haftalık keşfedici öğretime katılım sağlamıştır. Beren (takma isim), 2020 yılının aynı dönemi için planlanan, ancak Covid-19 salgını nedeniyle aynı yılın yaz tatiline ertelenen on haftalık öğretim deneyine katılmıştır. Her iki katılımcı öğrenci, Ankara'da, çoğunlukla orta seviyede sosyoekonomik düzeyden gelen öğrencilerin kayıtlı olduğu devlet okullarından seçilmiştir. Ortaokul Matematik Dersi Öğretim Programı (MEB, 2018), öğrencilerin sınıflarında muhakemelerini açıklamalarının ve sunulan muhakemeleri değerlendirmelerinin önemine vurgu yapmakta, ancak ortaokul düzeyinde ispat kavramına açıkça değinmemektedir. Bu nedenle, çalışmanın öncesinde katılımcı öğrencilerin ispat kavramıyla bir etkileşimlerinin olmadığı kabul edilmiştir.
Altıncı sınıf düzeyi, sayılar teorisinin temel kavramlarını kapsadığı için seçilmiştir. Ortaokul müfredatı (MEB, 2018) tek ve çift kavramlarını üçüncü sınıf düzeyinde, çarpanlar, katlar ve doğal sayıların bölünebilirliği kavramlarını ise altıncı sınıf düzeyinde ele almaktadır. Çalışmanın AS-Ek1 ve As-Ek2 kavramlarının öğrenilmesi için gerekli olan dağılma özelliği, altıncı sınıf düzeyinde ele alınmaktadır. Katılımcı öğrenciler, öncelikle çalışma için önemli olan bu kavramlardaki yeterliliklerine ve daha sonra matematiksel fikirlerini net bir şekilde ifade etme becerilerine ve müfredat dışında matematik öğrenmeye istekli olmalarına bağlı olarak seçilmiştir. Her iki öğrenci de ebeveynlerinin onayı doğrultusunda çalışma için gönüllü olmuştur.

2.2. Öğretim Deneyi

Öğretim deneyi beş ana aşamadan oluşmaktadır. İlk üç aşama, amaçlanan EK kavramını oluşturmak üzere çalışmanın ilk bölümünü oluşturmaktadır. Sonraki iki aşama, AS ve genişletilmiş AS kavramlarını geliştirmeyi amaçlayan ikinci bölüme karşılık gelmektedir. İlk iki aşama, hedeflenen EK kavramını oluşturmak için gerekli olan önkoşul kavramları hedeflemektedir. Birinci aşamadaki görev dizisi, üçüncü aşamada koordine edilecek kavramları geliştirmeyi amaçlamıştır. İkinci aşamada, öğrencinin mevcut ispat şeması olan deneysel ispat şemasını bilişsel çatışma yoluyla geçersiz kılmak için Canavar Karşı Örnek İllüstrasyonu (Stylianides & Stylianides, 2009) uygulanmıştır. Üçüncü aşama, tümdengelimli ispat şemasına yönelik öğrenmeyi başlatmayı, yani ispatın ilk örneğinin anlaşılmasını (EK kavramı) hedeflemiştir.

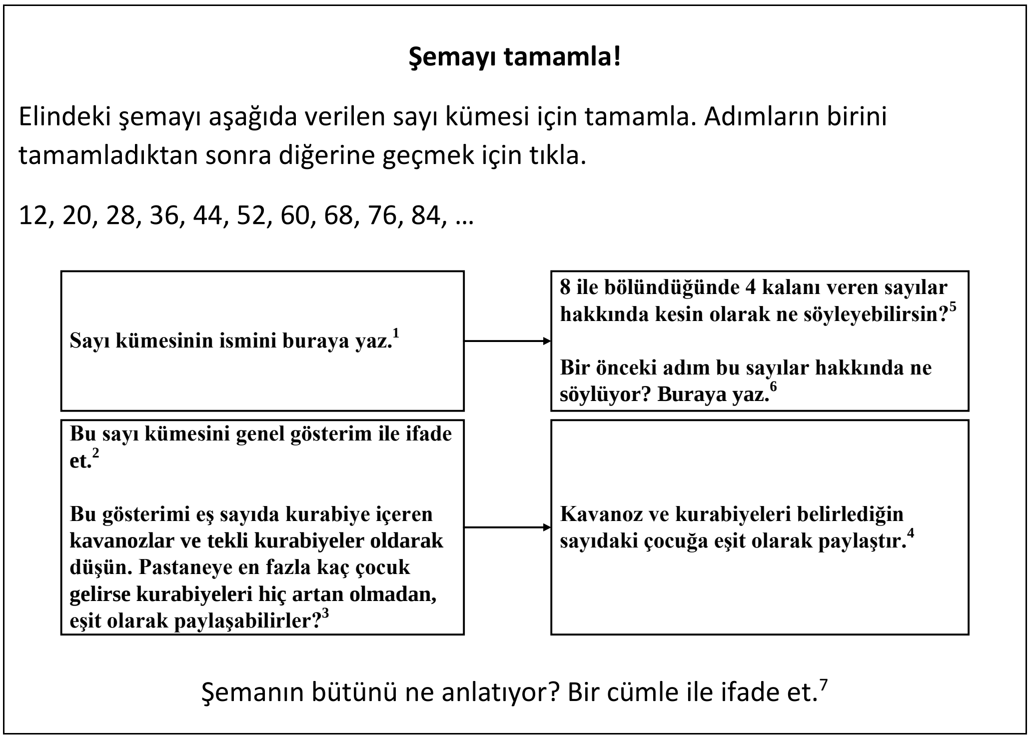
Tablo 2.0.1 Öğretim Deneyinin Aşamaları

	Aşamalar	Öğrenme Hedefleri	Oturumlar
Aşama 1	Sayılar teorisinin temel kavramları	Aşama 3'te koordine edilecek kavramların geliştirilmesi	1 - 4
Aşama 2	Monstrous karşı örnek illüstrasyonu	Güvenli bir doğrulama yöntemi için entelektüel bir ihtiyaç hissetmek (Kesinlik ihtiyacı)	5
Aşama 3	EK kavramı	Matematiksel ispatın ilk örneğini anlama	5 - 6
Aşama 4	AS kavramı	Belirli bir teorem sınıfını ispatlama işine ilişkin modüler yapı kavramı	7
Aşama 5	Genişletilmiş AS kavramları	EK ve AS kavramlarını kullanarak ispat yapmayı deneyimleme	8 - 10

Çalışmanın ikinci bölümünde, dördüncü aşama, ilk örnek ispatının ana fikrini bir AS kavramı olarak geliştirmeyi, beşinci aşama ise bu AS kavramını diğer varsayımların ispatlanması sürecine genişletmeyi hedeflemiştir. Tablo 2.1, öğretim deneyinin beş aşamasını ve her aşamaya ayrılan çalışma süresini göstermektedir. Veri analizi sürecinin odağında olan üçüncü, dördüncü ve beşinci aşamaların içerdiği etkinlikler aşağıda sunulmuştur.

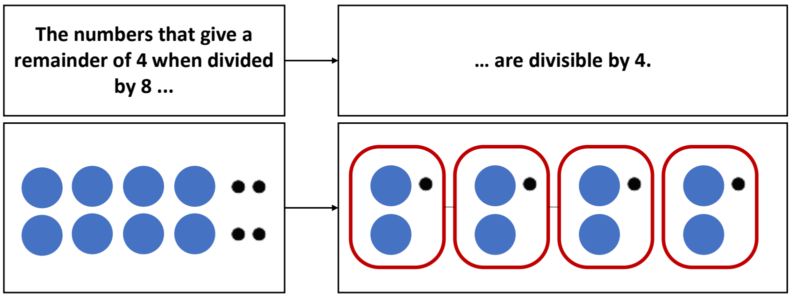
Aşama 3: Hedeflenen EK kavramı

Çalışmanın ilk hipotezine göre, Şekil 2.1’de verilen, (mümkün olan ölçüde) EYÖ öğretim tasarımı yaklaşımına uygun olarak tasarlanan etkinliği tamamladığında, öğrenci Şekil 2.2’deki gibi bir akış şemasına ulaşacaktır. Öğrencinin ortaya çıkan bu argümanı matematiksel ispatın ilk örneği olarak fark etmesi mümkün olabilir.



Şekil 2.1 Ana etkinlik

*: Öğrenciye basılı bir akış şeması verilir. İzlenmesi gereken yönergeler bir PowerPoint slaydı içinde verilir. Öğrenci, adımların akışını kendi hızında takip eder. Yönergeler, üst simgelerde belirtilen sırada görünür.



Şekil 2. 2 Öğrenci etkinliğinin sonucunda beklenen akış şeması ispatı

Etkinlik, aşağıdaki iki eylemin sırayla gerçekleştirmesini teşvik etmek amacıyla tasarlanmıştır:

A_{0a}: Verilen bir aritmetik sayı örüntüsünün tüm elemanları tarafından paylaşılan bir modüler yapı belirleme

A_{0b}: Verilen bir modüler yapı için uygun bölünebilme çıkarımları yapma

Etkinliğin, modüler yapı ile ilgili yapılan bölünebilirlik çıkarımının aynı yapıya sahip tüm sayılar için geçerli olmasının mantıksal gerekliliği konusunda öğrencinin anlayışına temel oluşturması; dolayısıyla, öğrencinin ortaya çıkan sonucu ($A_{0a} \rightarrow A_{0b}$) belirli bir teoremin ispatı olarak görmesi beklenmiştir.

Aşama 4 ve 5: Hedeflenen AS kavramı ve genişletilmiş AS kavramları

Çalışmanın hedeflenen AS kavramı, ilk ispat örneğinin ana fikri olarak tanımlanmıştır. Bu nedenle, öğrencinin yansıtıcı soyutlamasının temeli, aynı yapıdaki ispatları üretme sürecinin tamamı olarak düşünülmüştür. Simon (2020), bir AS kavramının inşasını teşvik etmek için temel tasarım ilkelerini, öğrenci için enformel bir etkinlik oluşturmak, öğrencinin bu etkinliği ok diyagramları kullanarak temsil etmesini sağlamak ve bu diyagramları sözel matematik problemlerini çözmek için kullanmasını teşvik etmek olarak belirtmiştir. Mevcut çalışmada, sıralı bir akıl yürütme zinciri oluşturma etkinliği öğretim deneyinin üçüncü aşamada kurulmuştur. Dördüncü aşama, öğrenciden ilk örnek ispatına analojik olan varsayımları ispatlamasını istemiştir. Çalışma boyunca kullanılan akış şeması formatı, öğrencinin etkinliğini takip etmesine yardımcı olan bir ok diyagramının rolünü oynamıştır.

Beşinci aşama, genişletilmiş EK kavramlarını kademeli olarak tanıtarak öğrencinin yeni varsayım çeşitlerini ispatlamasını sağlamak üzere yeni öğrenmelere sebep olmuştur. Bu aşama, bir yandan öğrenciye daha fazla tanımsal akıl yürütme örneği sunarken, diğer yandan çalışmanın üçüncü araştırma sorusunu yanıtlamak için yararlı bir veri seti oluşturmuştur.

2.3. Veri Toplama Süreci

Uygulama sırasında araştırmacı, tasarlanan etkinlikleri öğrenciye sırayla sunmuş ve öğrencinin yanıt oluşturma sürecini çoğunlukla müdahale etmeden gözlemlemiştir. Öğrencinin anlama ve düşünme biçimlerinin ayrıntılarını yakalayabilmek için zaman zaman irdeleyici (ve nadiren yönlendirici) sorular yöneltmiştir. Çalışmanın başlıca veri kaynakları, bire bir öğretim deneyi oturumlarının ses ve görüntü kayıtlarını ve katılımcı öğrencinin yazılı çalışmalarını içermektedir. Etkinliklerin büyük çoğunluğu öğrenciden kendisine verilen varsayımları ispatlamasını istemiştir. Yanıt oluşturma süreci boyunca öğrencinin yüksek sesle düşünmesi teşvik edilmiştir. Araştırmacının, öğrencinin öğrenmesiyle ilgili gelişen varsayımları hakkında aldığı notları ve bu varsayımların, araştırmacının danışmanı ve eş danışmanıyla yaptığı toplantılarda geliştirilen ikinci bir yorumu, geriye dönük veri analizi sürecini desteklemek amacıyla veri seti olarak kullanılmıştır.

2.4. Veri Analizi

Mevcut çalışma, EYÖ öğretim deneyinin üçüncü, dördüncü ve beşinci aşamalarından elde edilen verilerin geriye dönük analizlerini içermektedir. Birinci ve ikinci aşamalardan elde edilen veriler, yalnızca asıl odak noktası olan veri analizi süreçlerini desteklemek amacıyla kullanılmıştır. Üçüncü aşamada gerçekleşen EK kavramının inşa edilme süreci, EYÖ teorik çerçevesinin kavramsal öğrenme açıklamasına dayalı olarak analiz edilmiştir (Simon vd. 2018). Öğrencinin sahip olduğu iki kavramdan yeni bir üst düzey kavram oluşturma süreci, EK kavram türünün hedef-eylem (goal-action) tanımına dayalı olarak açıklanmıştır. Öğrencinin, kendi etkinliği sonucunda ortaya çıkan ispatta yer alan tümdengelim ilişkilerine dair anlayışı *öğrencilerin bir ispatı anlama biçimleri modeline* (Students' ways of understanding a proof; Ahmadpour ve diğerleri, 2019) göre irdelenmiştir.

Dördüncü ve beşinci aşamalardan elde edilen veriler, öğrencinin verilen varsayımları ispatlama girişimi sırasında ortaya koyduğu hedef belirleme davranışına göre analiz edilmiştir. Bu iki aşama için gerçekleştirilen analizlerin temel odak noktası öğrencinin düşünmesinde tanımsal muhakeme ile ilgili herhangi bir göstergenin varlığını veya yokluğunu tespit etme amacı olmuştur.

3. BULGULAR

3.1. Öğrenme Hedefi 1: İspatın ilk örneğini anlama

Beren verilen adımları takip ederek akış şemasını tamamladığında, ortaya çıkan ispatın tümdengelimli yapısını tam olarak anlayamamıştır. Beren'in anlayışının ispatta yer alan *varsayımsal kıyası* bir derece içerdiği, ancak *evrensel örneklemeyi* içermediği görülmüştür. Bu nedenle, öğretim deneyinin bir sonraki oturumunda ana etkinliğin ikinci bir versiyonu tasarlanmış ve uygulanmıştır. Sfard'ın (1991) süreç-nesne (process-object) ayrımını ve EYÖ'nün zihinsel işletim (mental run; Simon vd. 2010) tekniğini uygulayan etkinlik hedeflenen EK kavramını oluşturmada başarıya ulaşmıştır. Her ne kadar ilk başta tasarlanan etkinlik üzerinde değişiklik yapılmış olsa da gerçekleşen öğrenmenin altında yatan mekanizmanın yine eylemlerin koordinasyonu olduğu saptanmıştır. Ayrıca, Beren'in ortaya çıkan ispatı matematiksel ispat olarak anlamlandırma sürecinde DNR'nin *kesinlik ihtiyacı* kavramının rol oynadığı görülmüştür.

3.2. Öğrenme Hedefi 2: Tanımsal akıl yürütmeyi geliştirme

Dördüncü aşama süresince Beren'in ilk örnek ispatının ana fikrini tanıdığı ve bu fikri analojik varsayımları ispatlama girişiminde başarı ile uyguladığı görülmüştür. Ancak, bu deneyimlerden hiçbiri Beren için tanımsal muhakemenin bir örneğine dönüşmemiştir. Yani, Beren kullandığı modüler yapı gösterimlerinin, varsayımlarda verilen aritmetik sayı dizilerini tanımlama işini karşıladığını fark etmemiştir. Bu gözlem öğretim deneyinin beşinci aşamasında da kendini tekrar göstermiştir. Beren, hedeflenen AS-Ek1 ve AS-Ek2 kavramları için birçok akış şeması ispatı oluşturmuştur. Ancak her yeni ispat yapma girişiminde, kendisine oluşturduğu hedef daha önce karşılaştığı akış şemasına benzer bir akış şeması oluşturmaya çalışmaktan öteye gitmemiştir. Ne öğretim deneyi sürecinde ne de sonrasında yapılan değerlendirme sırasında, o ana kadar öğrendiği yapılarından farklı görünen varsayımları ispatlamak için tanım yapma eğilimi göstermemiştir.

4. TARTIŞMA VE ÖNERİLER

Çalışmanın bulguları, öğretim deneyinin dördüncü aşamasının sonunda Beren'in hedeflenen EK ve AS kavramlarının her ikisini de sergilediğini göstermiştir. Ancak Beren'in kendi etkinlikleri sonucunda ortaya çıkan ispatlara dair anlayışının istenen yetkinlikte olmadığı saptanmıştır, çünkü bu anlayış çalışmada amaçlanan iki düşünme biçiminden birini örneklendirirken (tümdengelimli ispat şeması) diğerini es geçmiştir (tanımsal muhakeme). Beren, uygulamayı öğrendiği sıralı eylem dizisini üzerinde çalıştığı varsayımları doğrulamak için güvenli bir yöntem olarak gördüğünü ortaya koymuştur. Ancak, kullandığı eylemlerden ilkinin tanımlama eylemi olduğunu fark etmemiştir. Beren'in tanımsal muhakeme ile ilişkili uygun anlama yollarına dair deneyimini artırmak amacıyla tasarlanan beşinci aşamadan ve aynı zamanda öğretim deneyi sonrasında yapılan değerlendirmeden elde edilen veriler de Beren'in tanımsal akıl yürütme davranışını göstermediğini desteklemiştir.

Çalışmanın amacı EK ve AS türü yansıtıcı soyutlama mekanizmalarının, bireylerin tümdengelimli ispat şemasına doğru ilk adımlarını açıklama olasılığını incelemektir. Bu iki tür yansıtıcı soyutlama ile ilgili olarak Simon (2020) şunları yazmıştır: EK, matematiksel nesnelerin yapısını ve aralarındaki ilişkileri açıklamak konusunda yararlı görünüyor. AS şimdiye kadar aritmetik işlemlerin yapısını açıklamada yararlı olduğunu kanıtladı. İlerideki çalışmalar, AS tarafından soyutlanan yeni kavram türlerini, hatta bireyin etkinliğindeki ortaklığın soyutlaması yoluyla inşa edilen oldukça farklı kavramları tanımlayabilir (Simon, 2020). Çalışmanın kendine özgü ilk örnek ispatı düşünüldüğünde, EK mekanizmasının ispatın içerdiği iki ayrı mantıksal gereklilik ilişkisinin soyutlanmasını açıkladığı görülmüştür. Ancak beklenenin aksine, Beren'in sergilediği (hedeflenen) AS kavramı, gerçekten bir AS mekanizmasının sonucu olarak ortaya çıkmamıştır. Çalışmanın bulgularından yola çıkarak aşağıdaki ampirik temelli yorum oluşturulmuştur.

Öğrenciler matematik disiplininin ispatlama ve tanımlama eylemleriyle tanıştırıldıktan sonra ancak, EYÖ teorik çerçevesinin AS kavram yapısı belirli bir varsayım sınıfını ispatlama ve belirli bir nesne sınıfını tanımlama çalışmalarının çerçevesini oluşturabilir. Bunlar, DNR'ye göre düşünme biçimleri değil, anlama biçimleridir. AS tipi bir yansıtıcı soyutlama, öğrencilerin düşünme biçimleri üzerinde başka bir etki olmaksızın (hatta entelektüel ihtiyaçları üzerine inşa edilmeksizin) ispatlama ve tanımlama eylemlerinin inşasını açıklamak için tek başına yeterli değildir. EYÖ'nün EK ve AS kavramları, bir ispat veya tanım yapma ihtiyacı fark edildiğinde, bu işin nasıl yapılabileceğini, hangi kavramın işe koşulabileceğini ayırt etme becerilerini çerçeveleyebilir. Gelecekteki çalışmalar daha ayrıntılı bir revizyon önerebilir.

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