# ON THE JACOBIAN MATRICES OF GENERALIZED CHEBYSHEV POLYNOMIALS

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# ABSTRACT

## ON THE JACOBIAN MATRICES OF GENERALIZED CHEBYSHEV POLYNOMIALS

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Any semisimple complex Lie algebra admits a root system, and a multivariable generalization of Chebyshev polynomials is attached to each root system. In this thesis, a practical way to compute the determinant and each entry of the Jacobian matrix of these generalized Chebyshev polynomials in terms of characters of irreducible Lie algebra representations, using the theory of exponential invariants and Weyl character formula, is described and explicit results for rank two cases is given.

Keywords: Lie algebra, exponential invariants, Chebyshev polynomials, Weyl character formula

# ÖZ

## GENELLEŞTİRİLMİŞ CHEBYSHEV POLİNOMLARININ JACOBİAN MATRİSLERİ ÜZERİNE

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Her yarı basit kompleks Lie cebiri bir kök sistemi meydana getirir ve her kök sistemine Chebyshev polinomlarının bir çok değişkenli genelleştirilmesi bağlanmıştır. Bu tezde, bu genelleştirilmiş Chebyshev polinomlarının Jacobian matrisinin determinantının ve her bir elemanının indirgenemez Lie cebiri temsillerinin karakterleri cinsinden hesaplanması için, üstel değişmezlerin teorisinden ve Weyl karakter formülünden yola çıkılan pratik bir yöntem gösterilmiştir ve ikinci mertebeden örnekler için açık formüller sunulmuştur.

Anahtar Kelimeler: Lie cebirleri, üstel değişmezler, Chebyshev polinomları, Weyl karakter formülü

To my mother, who made it all possible.

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# CHAPTER 1

# LIE ALGEBRAS

The aim of this chapter is to introduce Lie algebras and describe the classification of complex matrix Lie algebras. Throughout this chapter, we follow [8]. We shall begin with the definition of a Lie algebra.

**Definition 1.1.** Let $F$ be a field. An $F$-vector space $L$ endowed with an operation $L \times L \to L$ (called bracket or commutator), denoted by $(x, y) \mapsto [x, y]$, is a Lie algebra if it satisfies the following conditions:

(L1) Bracket operation is bilinear.

(L2) $[x, x] = 0$ for all $x \in L$.

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

The condition (L3) is called the Jacobi identity. An important remark is that if (L1) and (L2) are applied to the bracket $[x + y, x + y]$, we obtain $[x, y] = -[y, x]$, which we will call (L2'). The conditions (L2) and (L2') are equivalent unless $F$ has characteristic 2. We continue with the usual definitions that arise in common algebraic structures.

**Definition 1.2.** For a Lie algebra $L$, a subspace $L' \subset L$ is called a subalgebra of $L$ if $[x, y] \in L'$, $\forall x, y \in L'$. The subspace $L'$ is called an ideal of $L$ if for all $x \in L, a \in L'$ we have $[x, a] \in L'$. For an ideal $I \subset L$, the quotient algebra $L/I$ is defined to be the Lie algebra of the cosets $x + I = \{x + a : a \in I\}$, where the Lie bracket operation of $x + I$ and $y + I$ is defined to be $[x + I, y + I] = [x, y] + I$.

**Definition 1.3.** Let $L$ be a Lie algebra.

1. The center of $L$, denoted by

$$Z(L) := \{z \in L \mid [x, z] = 0,\ \forall x \in L\},$$

is the set of elements that commutes with every element in $L$.

2. The derived algebra of $L$, denoted by

$$[L, L] = \{[x, y] \mid \forall\, x, y \in L\},$$

is the set of commutators of all possible pairs in $L$.

It is straightforward to check that both these subsets of $L$ are in fact ideals of $L$.

**Definition 1.4.** A non-abelian Lie algebra is called simple if it has no ideals other than 0 and itself.

**Definition 1.5.** For any Lie algebras $L, L'$, the map $\varphi : L \to L'$ is called a *Lie algebra homomorphism* if it preserves the bracket operation, in other words, $\varphi([x, y]) = [\varphi(x), \varphi(y)]$. If $\varphi$ is a bijection, it is called a *Lie algebra isomorphism.*

We will briefly call these maps homomorphism and isomorphism when it is clear from the context that they are between Lie algebras. It requires little effort to show that for any Lie algebra homomorphism $\varphi$, Ker $\varphi$ is an ideal of $L$ and Im $\varphi$ is a subalgebra of $L'$.

We will continue with some explicit examples of Lie algebras. Let $V$ be a finite dimensional vector space over a field $F$, and let End$(V)$ be the set of all linear transformations $V \to V$. When $V$ has dimension $n$, its elements can be denoted by $n \times n$ matrices. If we define the bracket operation as $[x, y] = xy - yx$, it can be checked that End$(V)$ is a Lie algebra under this operation. This Lie algebra is denoted $\mathfrak{gl}(V)$. It consists of all $n \times n$ matrices with entries from $F$ and its dimension is $n^2$. The Lie algebra structure should not be confused with the group of the invertible $n \times n$ matrices with entries from $F$ under the standard matrix product, which is denoted as $GL(V)$. Sometimes $\mathfrak{gl}(n, F)$ is also used to denote the set $\mathfrak{gl}(V)$ to put the emphasis on the dimension of the underlying field.

We set $e_{ij},\ 1 \leq i, j \leq n$ as the usual standard basis, where $e_{ij}$ has 1 as the entry in the intersection of row $i$ with column $j$ and 0 everywhere else. By using the identity

$e_{ij}e_{kl} = \delta_{jk}e_{il}$ where the $\delta$ is the usual Kronecker-Delta function ($\delta_{ij} = 1$ whenever $i = j$ and 0 otherwise), the bracket operation turns out for the basis elements as

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{il}e_{kj}$$

We define new families of matrix Lie algebras $A_\ell, B_\ell, C_\ell, D_\ell$ for $\ell \geq 1$, which occur as subalgebras of $\mathfrak{gl}(n, F)$. They are called the classical algebras and they provide examples of Lie algebras and that are central to the classification problem.

$A_\ell$: Let $\dim V = \ell + 1$. We denote by $\mathfrak{sl}(V)$ (or $\mathfrak{sl}(\ell+1, F)$) the set of endomorphisms of $V$ with zero trace (or $(\ell+1) \times (\ell+1)$ matrices with zero trace). Since $tr(xy - yx) = tr(xy) - tr(yx) = 0$, this set is a Lie subalgebra of $\mathfrak{gl}(V)$. It is called the *special linear algebra* and it has dimension $(\ell+1)^2 - 1$.

$B_\ell$: Let $\dim V = 2\ell + 1$. Define the $(2\ell+1) \times (2\ell+1)$ matrices in the form of

$$s = \begin{pmatrix} 0 & v_1 & v_2 \\ -v_2^t & p & m \\ -v_1^t & n & -p^t \end{pmatrix}$$

where $p, m, n$ are $\ell \times \ell$ matrices with $m^t = -m$ and $n^t = -n$. Set of these matrices forms a matrix Lie algebra, called the *odd-dimensional orthogonal Lie algebra* and denoted $\mathfrak{so}(V)$ or $\mathfrak{so}(2\ell+1, F)$. Its dimension is $2\ell^2 + \ell$.

$C_\ell$: Let $\dim V = 2\ell$. Define the $2\ell \times 2\ell$ matrices in the form of

$$s = \begin{pmatrix} p & m \\ n & -p^t \end{pmatrix}$$

where $p, m, n$ are $\ell \times \ell$ matrices with $m^t = m$ and $n^t = n$. Set of these matrices forms a matrix Lie algebra, called the *symplectic Lie algebra* and denoted $\mathfrak{sp}(V)$ or $\mathfrak{sp}(2\ell, F)$. Its dimension is $2\ell^2 + \ell$.

$D_\ell$: Let $\dim V = 2\ell$. Define the $2\ell \times 2\ell$ matrices in the form of

$$s = \begin{pmatrix} m & p \\ -p^t & n \end{pmatrix}$$

where $p, m, n$ are $\ell \times \ell$ matrices with $m^t = -m$ and $n^t = -n$. Set of these matrices forms another form of an orthogonal matrix Lie algebra, called the *even-dimensional orthogonal Lie algebra* and denoted $\mathfrak{so}(V)$ or $\mathfrak{so}(2\ell, F)$. Its dimension is $2\ell^2 - \ell$.

A common property of the classical algebras is that they are all simple Lie algebras. There are several other Lie algebras that are worth noting, namely, $\mathfrak{t}(n, F)$, the Lie algebra of all upper triangular $n \times n$ matrices; $\mathfrak{n}(n, F)$, the Lie algebra of all strictly upper triangular $n \times n$ matrices; and $\mathfrak{d}(n, F)$, the Lie algebra of all diagonal $n \times n$ matrices. It is a straightforward check to verify that each described set defines a Lie algebra.

**Definition 1.6.** Let $L$ be a Lie algebra. The derived series of $L$ is defined as $L^{(0)} = L,\ L^{(1)} = [L, L]$ and $\forall n \geq 1,\ L^{(n+1)} = [L^{(n)}, L^{(n)}]$. The Lie algebra $L$ is called solvable if $L^{(n)} = 0$ for some positive integer $n$.

**Proposition 1.7.** *Let $L$ be a Lie algebra.*

1. *If $L$ is solvable, then all of its subalgebras and homomorphic images are solvable.*
2. *If $I$ is a solvable ideal of $L$ and $L/I$ is solvable, then $L$ is solvable as well.*

**Definition 1.8.** Every Lie algebra admits a unique maximal solvable ideal, called the radical of $L$, and denoted by Rad$L$. The Lie algebra $L$ is called semisimple if its radical is 0.

**Theorem 1.9.** *For any Lie algebra $L$, $L/RadL$ is semisimple.*

As a result of Theorem 1.9, any Lie algebra consists of two parts, namely, the solvable part Rad$L$ and the semisimple part $L/$Rad$L$. The following theorem characterizes the elements of the solvable matrix Lie algebras.

**Theorem 1.10.** *(Lie's Theorem) Let $V$ be a finite dimensional vector space and $L$ be a solvable subalgebra of $\mathfrak{gl}(V)$. Then, $L$ consists of upper triangular matrices relative to some basis of $V$.*

Hence, any solvable Lie algebra is a subalgebra of $\mathfrak{n}(V)$. We also want to understand the structure of semisimple Lie algebras.

**Definition 1.11.** For any $x \in L$, the adjoint map is a map $L \to L$ given by ad $x(y) = [x, y]$.

**Definition 1.12.** Let $L$ be a Lie algebra, an element $x \in L$ is called semisimple if ad $x$ is diagonalizable.

The following theorem states that any semisimple Lie algebra can be represented as a direct sum of finitely many simple Lie algebras, hence simple Lie algebras are building blocks of semisimple Lie algebras.

**Theorem 1.13.** *Let L be a semisimple Lie algebra. Then there exists ideals $L_1, \ldots, L_k$ ideals of L such that each $L_i$ for $1 \leq i \leq k$ is a simple Lie algebra itself and $L = L_1 \oplus \ldots \oplus L_k$.*

In order to understand the structure of semisimple complex Lie algebras, it is sufficient to understand the simple Lie algebras. We begin with the following definition.

**Definition 1.14.** A Lie subalgebra $H$ of a Lie algebra $L$ is called a Cartan subalgebra if $H$ is abelian, every element of $H$ is a semisimple element and $H$ is maximal with these conditions.

The choice of a Cartan subalgebra $H$ is not unique, but when we fix a Cartan subalgebra $H$, $L$ turns out to be a direct sum of subspaces

$$L_\alpha = \{x \in L : [h, x] = \alpha(h)x, \ \forall h \in H\}$$

which are called the root spaces and the functionals $\alpha : H \to \mathbb{C}$ are called roots. This representation is called the root space decomposition. For any root space decomposition

$$L = H \oplus \big( \oplus_{\alpha \in \Phi} L_\alpha \big)$$

of a Lie algebra $L$, the corresponding root set $\Phi$ is called a root system under a special inner product. The root space decomposition is independent of the choice of the Cartan subalgebra hence any two such decompositions is isomorphic.

The classification of all simple complex Lie algebras turns out to be equivalent to classifying all possible root systems. The following famous theorem is known due to the results of several mathematicians:

**Theorem 1.15.** *Any simple complex Lie algebra is isomorphic to either one of the classical Lie algebras* $A_\ell$ *for* $\ell \geq 1$*,* $B_\ell$ *for* $\ell \geq 2$*,* $C_\ell$ *for* $\ell \geq 3$*,* $D_\ell$ *for* $\ell \geq 4$*, or one of the five exceptional Lie algebras:* $G_2, F_4, E_6, E_7, E_8$.

The notation corresponds to a structure called root systems which each simple complex Lie algebra admits. These roots can be identified as vectors in a Euclidean space with a certain inner product and each root system can be identified with the associated Cartan matrix or Dynkin diagram. The axiomatic definitions of the root systems and their properties are studied in the next chapter.

# CHAPTER 2

# ROOT SYSTEMS

The aim of this chapter is to establish root systems describing certain aspects of the Lie algebras starting with axioms in Euclidean space only. Throughout the entire chapter, we follow [8]. Let $E$ be a finite dimensional real vector space endowed with a positive definite symmetric bilinear form $(\alpha, \beta)$. A reflection with respect to a vector $\alpha$ is a linear transformation fixing $H_\alpha$ (the hyperplane orthogonal to $\alpha$) pointwise and sending any vector orthogonal to $H_\alpha$ to its negative. An explicit formula for the reflection associated with $\alpha$ is

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha$$

For the ease of notation, we will denote $\frac{2(\beta, \alpha)}{(\alpha, \alpha)}$ as $\langle \beta, \alpha \rangle$.

**Definition 2.1.** A collection of vectors $\Phi$ of $E$ is called a *root system* if it satisfies the following axioms:

(R1) $\Phi$ is finite, spans $E$, and does not contain 0.

(R2) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$.

(R3) $\Phi$ is invariant under $s_\alpha$ for any $\alpha \in \Phi$.

(R4) $\langle \beta, \alpha \rangle \in \mathbb{Z}, \ \forall \alpha, \beta \in \Phi$.

**Definition 2.2.** Two root systems $\Phi$ in the Euclidean space $E$ and $\Phi'$ in the Euclidean space $E'$ are said to be isomorphic if there exists a vector space isomorphism $\phi : E \to E'$ sending $\Phi$ to $\Phi'$ such that $\langle \beta, \alpha \rangle = \langle \phi(\beta), \phi(\alpha) \rangle$ for all pairs of roots $\alpha, \beta \in \Phi$.

We give examples of root systems of rank one and two. There is a unique 1 dimensional root system up to isomorphism as a result of the axiom R2, since there are at most 2 vectors on a given line. This root system is denoted $A_1$, which is shown in Figure 2.1.



Figure 2.1: The root system $A_1$.

The two dimensional examples are $A_1 \times A_1$, $A_2$, $B_2$ and $G_2$. It is straightforward to check that all these sets of vectors satisfy the root system axioms.

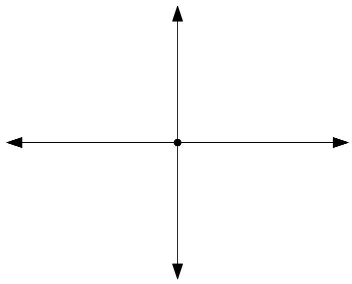

Figure 2.2: The root system $A_1 \times A_1$.

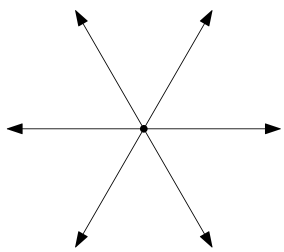

Figure 2.3: The root system $A_2$.

We aim to understand the classification of root systems up to isomorphism. We have

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = \frac{4(\alpha, \beta)^2}{\|\alpha\|^2 \|\beta\|^2} = 4\cos^2\theta$$

where $\theta$ is the angle between the non-parallel vectors $\alpha$ and $\beta$. By the axiom (R4), both values are integers and their product is, therefore, a positive integer less than 4 by the equation. This information, combined with the fact that $\langle \beta, \alpha \rangle$ and $\langle \alpha, \beta \rangle$ have the same sign, puts a limitation on the number of possibilities. All possible pairs for these values are given in Table 2.1, assuming $\beta$ is the longer vector.

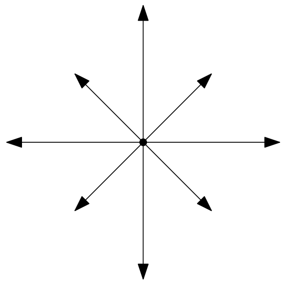

Figure 2.4: The root system $B_2$.

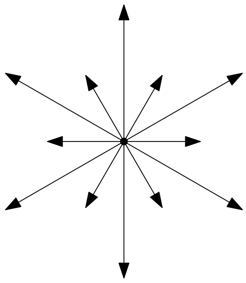

Figure 2.5: The root system $G_2$.

Table 2.1: All possibilities for the relations between any root pairs.

| $(\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle)$ | $\theta$ | $\|\beta\|^2/\|\alpha\|^2$ |
|---|---|---|
| $(0,0)$ | $\pi/2$ | undetermined |
| $(1,1)$ | $\pi/3$ | 1 |
| $(-1,-1)$ | $2\pi/3$ | 1 |
| $(1,2)$ | $\pi/4$ | 2 |
| $(-1,-2)$ | $3\pi/4$ | 2 |
| $(1,3)$ | $\pi/6$ | 3 |
| $(-1,-3)$ | $5\pi/6$ | 3 |

We will continue with several definitions that will have central importance to understand the structure of the root systems and to this thesis.

**Definition 2.3.** The Weyl group associated with a root system $\Phi$ is the group generated by the reflections $s_\alpha$, for all $\alpha \in \Phi$.

**Remark 2.4.** Since any such reflection permutes the finite set $\Phi$, the Weyl group is naturally a subset of a finite permutation group, thus it is a finite group for any root system.

**Example 2.5.** In the root system $B_2$, the reflecting hyperplanes are the lines perpendicular to the roots. All these reflections are elements of the dihedral group $D_8$, and the composition of two reflections belonging to two of the closest roots generates rotations. Therefore, the Weyl group of $B_2$ is the dihedral group with $8$ elements.

**Definition 2.6.** A subset $\Delta$ of $\Phi$ is called a base if it is a vector space basis for $E$ and each root $\beta$ can be written as $\beta = \sum k_\alpha \alpha$ with integer coefficients $k_\alpha$ are all nonnegative or nonpositive. The roots in $\Delta$ are called simple.

**Remark 2.7.** A base is not unique. In the Figure 2.6 we can either choose $\alpha_1, \alpha_2$ or choose $\beta_1, \beta_2$ to form a base for the root system $B_2$.

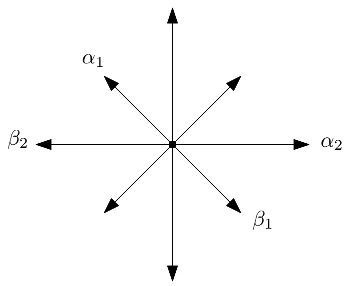


Figure 2.6: Two possible choices of base for $B_2$.

**Example 2.8.** In the figures 2.7 and 2.8, the roots of $A_2$ and $B_2$ are written as a linear combination of the chosen basis $\alpha_1$, $\alpha_2$. It can be seen that all coefficients are either nonpositive or nonnegative for each root.

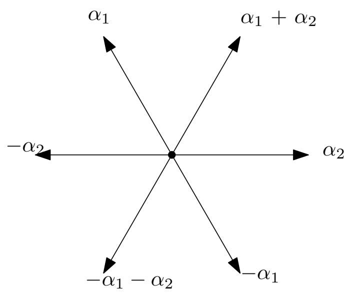


Figure 2.7: $A_2$ with a base $\alpha_1$, $\alpha_2$ chosen.

The cardinality of $\Delta$ is the same as the span of the dimension of $\Phi$. Since $\Delta$ is a basis for $E$, the representation $\beta = \sum k_\alpha \alpha$ is unique. We can define the height of a

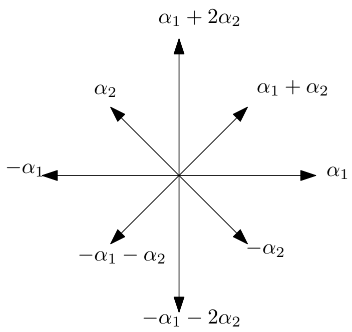


Figure 2.8: $B_2$ with a base $\alpha_1$, $\alpha_2$ chosen.

root as ht$(\beta) = \sum k_\alpha$. The root $\beta$ is called positive if all coefficients are nonnegative (denoted $\beta \succ 0$) relative to the base $\Delta$, and negative otherwise (denoted $\beta \prec 0$). The collection of all positive (respectively, negative) roots is denoted $\Phi^+$ (respectively, $\Phi^-$). Moreover, since the sum of two positive elements is again positive this structure defines a partial order on the roots, namely, $\beta \succ \alpha$ whenever $\beta - \alpha \succ 0$.

**Lemma 2.9.** *If $\Delta$ is a base for $\Phi$, then all the angles between its elements are obtuse, in other words, $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta \in \Delta$.*

A base always exists and it is unique up to transformations under the Weyl group.

**Theorem 2.10.** *Any root system $\Phi$ has a base $\Delta$, moreover, for any two bases $\Delta$, $\Delta'$, there exists an elements of Weyl group $w$ such that $\Delta = w(\Delta')$.*

**Example 2.11.** In Figure 2.6, the two bases $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$ are images of each other under a reflection.

**Theorem 2.12.** *The reflections $s_\alpha$ for all $\alpha \in \Delta$, called simple reflections, generate the Weyl group.*

This theorem provides a smaller set of generators for the Weyl group, and one advantage of these simple reflections is that we can keep track of the images of positive roots in an easier fashion:

**Lemma 2.13.** *Let $\alpha$ be a simple root. Then $s_\alpha$ permutes the positive roots other than $\alpha$.*

The hyperplanes $H_\alpha$ partition $E$ into finitely many regions; each of the connected components of $E - \cap_\alpha H_\alpha$ is called the (open) Weyl chambers of $E$. It is a well known fact that for the 2 regions separated by any $H_\alpha$, one of them contains all elements $\gamma$ such that $(\alpha, \gamma) > 0$ and the other region contains all elements $\gamma$ such that $(\alpha, \gamma) < 0$. This motivates the following definition.

**Definition 2.14.** The fundamental Weyl chamber relative to base $\Delta$ is the set of all elements $\gamma \in E$ such that $(\alpha, \gamma) > 0$ for all $\alpha \in \Delta$. It is denoted by $\mathfrak{C}(\Delta)$, or just by $\mathfrak{C}$ when $\Delta$ is clear from the context.

The following lemma shows that the closure of the fundamental Weyl chamber $\overline{\mathfrak{C}(\Delta)}$ is a fundamental domain for the action of the Weyl group. In other words, each vector in $E$ has a unique conjugate under the action of the Weyl group in $\overline{\mathfrak{C}(\Delta)}$.

**Lemma 2.15.** *Let $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$. If $w(\lambda) = \mu$ for some $w$ in the Weyl chamber, then $\lambda = \mu$*

Weyl chambers are also permuted under the action of the Weyl group, and bases are in one to one correspondence with the relative fundamental Weyl chambers.

**Example 2.16.** A base and the relative fundamental Weyl chamber is demonstrated in the Figure 2.9. The highlighted open region is the fundamental Weyl chamber and the other 7 similar regions are all the fundamental regions.

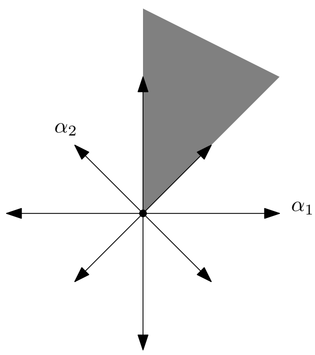


Figure 2.9: A basis $\Delta = \{\alpha_1, \alpha_2\}$ and the relative $\mathfrak{C}(\Delta)$ for $B_2$.

It is possible to generate a new root system $\Phi$ from the cross product of root systems $\Phi_1$ of dimension $k$ and $\Phi_2$ of dimension $l$ in a $k+l$ dimensional Euclidean space when each root of $\Phi_1$ is perpendicular to each root of $\Phi_2$. Hence the root systems which cannot be obtained by such operations (which will be called irreducible) are building blocks for all the root systems. It turns out that such root systems can be classified as well and each irreducible root system corresponds to a simple complex Lie algebra.

**Definition 2.17.** A root system $\Phi$ is called irreducible if it cannot be written as a partition of two of its proper subsets such that both subsets are root systems and any two roots in different subsets are orthogonal to each other.

For example, the root systems $A_1, A_2, B_2, G_2$ are all irreducible while $A_1 \times A_1$ is not.

**Definition 2.18.** Fix an ordering of the simple roots $\alpha_1, \ldots, \alpha_\ell$. The matrix $(\langle \alpha_i, \alpha_j \rangle)$ is called the Cartan matrix of $\Phi$, and its entries are called the Cartan integers.

**Example 2.19.** For the rank 2 cases, one can obtain the Cartan matrix using the values obtained in Table 2.1. For instance, the Cartan matrix for the $G_2$ root system is $\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$. It depends on the ordering of the simple roots, but it does not depend on the choice of the base.

The Cartan matrix of a root system defines its roots uniquely up to isomorphism. There is an alternative way to store and represent the identical information.

**Definition 2.20.** The Coxeter graph of a root system $\Phi$ is a graph having $\ell$ vertices, each standing for the corresponding simple root, where $i$th and $j$th vertices are joined by $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ many edges. When a double or triple edge occurs in the graph, this means that one of the roots is longer than the other. In this case, an arrow can be added on the edges pointing to the shorter root and the resulting figure is called the Dynkin diagram of $\Phi$.

Therefore, in a Coxeter graph, between any two vertices, there can be 0,1,2, or 3 edges can occur. In this case, we only draw the multiple edges. When there are no edges, roots are perpendicular and no conclusion about their lengths can be made from this information. When there is a single edge, the roots have equal length. When the number of edges is at least 2, this means that one of the edges is longer than the other, and it is highlighted with an arrow on the edges pointing towards the shorter root, as shown in Figure 2.10. The number of edges between nodes is 0,1,2,3 when the angle between them is $\pi/2, 2\pi/3, 3\pi/4, 5\pi/6$ respectively. The arrow between the edges points towards the shorter root. In addition, these edges are the Dynkin diagrams of all rank 2 root systems $A_1 \times A_1$, $A_2$, $B_2$, $G_2$ respectively.

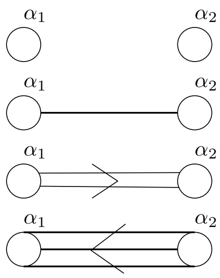


Figure 2.10: All possible edges in a Dynkin diagram.

It turns out that irreducible root systems are building blocks for any root system in a sense:

**Proposition 2.21.** *Any root system $\Phi$ decomposes uniquely as a union of irreducible root systems $\Phi_1, \dots \Phi_k$. In this case, it is denoted $\Phi = \Phi_1 \oplus \dots \oplus \Phi_k$ as an orthogonal direct sum.*

Since Dynkin diagrams uniquely determine a root system, we will present all possible Dynkin diagrams that can occur for an irreducible root system.

**Theorem 2.22.** *Any irreducible root system of rank $\ell$ has a Dynkin diagram isomorphic to one of the diagrams shown in Figure 2.11, hence all possible irreducible root systems are $A_\ell, B_\ell, C_\ell, D_\ell, E_6, E_7, E_8, F_4, G_2$.*

**Remark 2.23.** This theorem is compatible with Theorem 1.15.

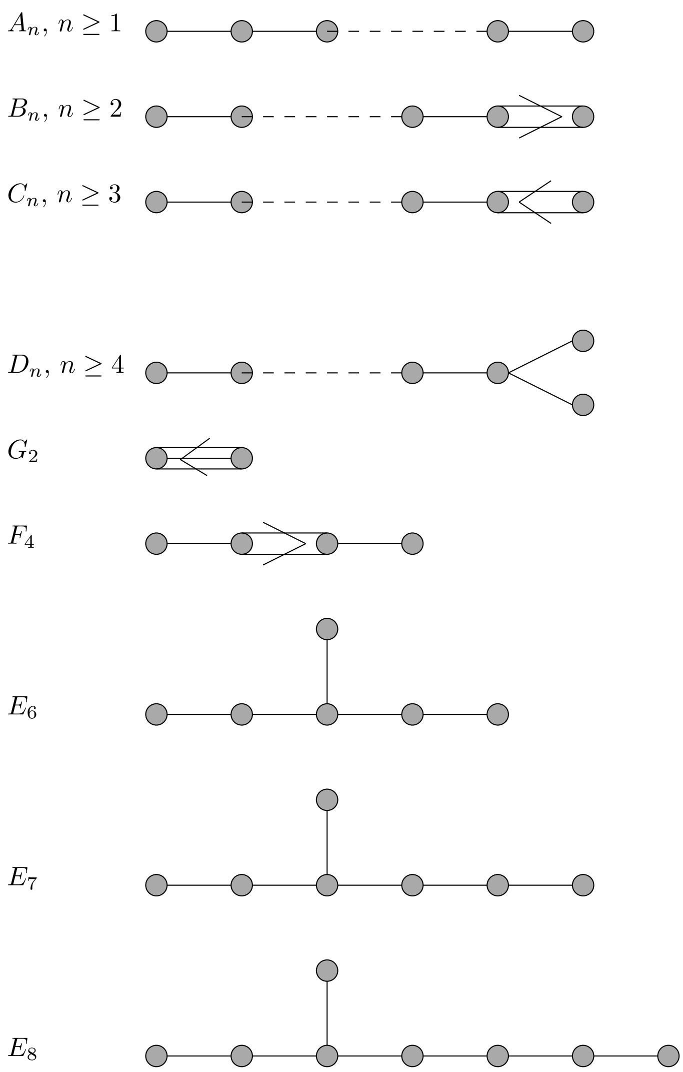


Figure 2.11: All possible irreducible root systems

# CHAPTER 3

# WEIGHTS AND REPRESENTATIONS

The aim of this chapter is to describe the structure of weights of a root system that arises from representations. First, we define weights and related concepts arising from a root system. Then, we will describe irreducible Lie algebra representations. We follow [8] throughout this chapter.

## 3.1 Weights

Let $\mathfrak{g}$ be a Lie algebra of rank $n$ and let $\Phi$ be the associated root system lying in the Euclidean space $E$. Let $\Delta$ be a base and let $\alpha_1, \ldots, \alpha_n$ be the simple roots.

**Definition 3.1.** The elements $\lambda \in E$ satisfying $\langle \lambda, \alpha_i \rangle \in \mathbf{Z}$ for each $1 \leq i \leq n$ are called weights.

Let $\Lambda$ denote the set of all weights. By the root system axioms, all roots satisfy this property therefore all roots lie in the set $\Lambda$. We denote the subgroup of $\Lambda$ generated by $\Phi$ as $\Lambda_r$, which is called the root lattice. It is indeed a lattice because it is the $\mathbf{Z}$-span of simple roots.

An element $\lambda \in \Lambda$ is called dominant if $\langle \lambda, \alpha_j \rangle$ is a nonnegative integer for any simple root $\alpha_j$, and it is called strictly dominant if these integers are positive. We denote the set of dominant weights as $\Lambda^+ := \Lambda \cap \overline{\mathfrak{C}}$, and strictly dominant weights are equivalent to the set $\Lambda \cap \mathfrak{C}$.

Weights also form a lattice, and the basis is the dual lattice of the simple roots, defined as the following:

**Definition 3.2.** The fundamental (dominant) weights of $\Phi$ relative to the basis $\Delta$ are the vectors $\omega_1, \ldots, \omega_n$ satisfying the relation

$$\langle \omega_i, \alpha_j \rangle = \delta_{ij}, \; \forall \, 1 \leq i, j \leq n$$

where $\delta_{ij}$ is the Kronecker delta function.

We can write any $\lambda \in \Lambda$ as $\lambda = \sum \langle \lambda, \alpha_i \rangle \alpha_i$. Therefore, $\Lambda$ is a **Z**-basis of fundamental weights, and the dominant and strictly dominant elements are characterized by those with the coefficients $m_i := \langle \lambda, \alpha_i \rangle$ being all nonnegative and positive, respectively. Under this definition, we can define a partial order on the set $\Lambda$.

**Definition 3.3.** Let $\lambda_1, \lambda_2 \in \Lambda$. We call $\lambda_1$ higher than $\lambda_2$ whenever $\lambda_1 - \lambda_2$ is a dominant weight.

**Example 3.4.** Let us consider $\mathfrak{g} = A_2$. Its root system and corresponding weights are shown in Figure 3.1. Here, $\omega_1$, $\omega_2$ are the fundamental dominant weights and the

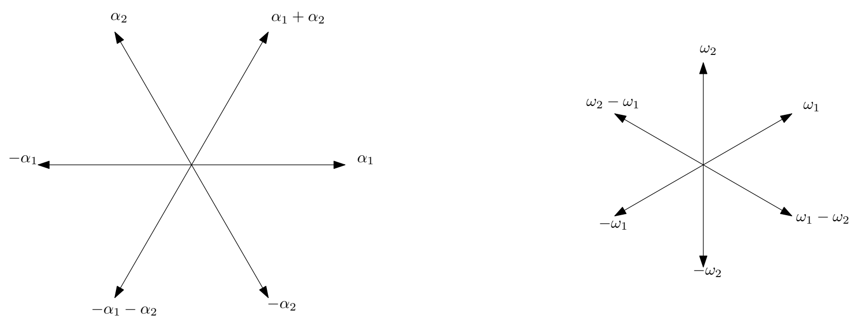


Figure 3.1: Weights corresponding to the root system $A_2$.

other weights shown in the figure are the images of these fundamental weights under the action of the Weyl group. Also, these vectors are the reflecting lines corresponding to the reflections of the Weyl group, hence the image of $\omega_1$ under an element of the Weyl group can only be $\omega_1$, $-\omega_2$ and $\omega_2 - \omega_1$. Similarly, orbit of $\omega_2$ under this action is $\omega_2$, $-\omega_1$ and $\omega_1 - \omega_2$. The orbit of each fundamental weight is disjoint from others in the general case as well.

Using the definition of a reflection, we obtain $s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_i$. This means $\Lambda$ is invariant under the action of Weyl group. The next two lemmas bring key information about the orbits of weights under the Weyl group, which will play a key role in the study of representations.

**Lemma 3.5.** *Each weight is conjugate to exactly one dominant weight under the action of the Weyl group. If $\lambda$ is dominant, then $w(\lambda) \prec \lambda$ for all $w \in W$, and if $\lambda$ is strictly dominant, $w(\lambda) = \lambda$ only when $w = 1$.*

**Lemma 3.6.** *Let $\lambda \in \Lambda^+$. Then, the number of dominant weights $\mu \prec \lambda$ is finite.*

### 3.2 Representations

In this section, our aim is to describe the structure of representations of semisimple complex Lie algebras determined by their weights. We will start with the general properties of Lie algebra representations.

**Definition 3.7.** Let $L$ be a Lie algebra over a field $F$. A representation of $L$ is a Lie algebra homomorphism $\varphi : L \to \mathfrak{gl}(V)$ where $V$ is a finite dimensional vector space over $F$.

**Example 3.8.** The adjoint map, defined as ad : $L \to \mathfrak{gl}(L)$, $\text{ad}_x(y) = [x, y]$ is a Lie algebra homomorphism, therefore it is a representation of $L$.

We can use the module notion in order to study the same structure:

**Definition 3.9.** Let $L$ be a Lie algebra over a field $F$. A Lie module, or equivalently an $L$-module, is a finite dimensional $F$-vector space $V$ together with a bilinear map $L \times V \to V$, denoted $(x, v) \mapsto x.v$ satisfying the condition $[x, y].v = x.(y.v) - y.(x.v)$ for all $x, y \in L$ and $v \in V$.

These two definitions are equivalent. A representation $\varphi$ can be made into an $L$-module by defining $x.v := \varphi(x)(v)$.

**Definition 3.10.** A submodule of $V$ is a subspace $W$ which is invariant under the action of $L$. In other words, for each $x \in L$ and $w \in W$, we have $x.w \in W$. $V$ is called irreducible (or simple) if it has no nonzero submodules other than itself.

In the representation language, a submodule is called subrepresentation. The same definition for irreducibility applies too.

**Definition 3.11.** An $L$-module $V$ is called completely reducible if it can be written as a direct sum of irreducible $L$-modules.

The next theorem tells us that irreducible representations are the building blocks for all finite dimensional representations, as every finite dimensional representation turns out to be a direct sum of irreducible representations.

**Theorem 3.12.** *(Weyl) Let* $L$ *be a semisimple Lie algebra. Then, every finite dimensional representation of* $L$ *is completely reducible.*

For the rest of this chapter, we restrict our attention to complex semisimple Lie algebras. Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}$.

**Definition 3.13.** An element $\lambda \in \mathfrak{h}^*$ is called a weight if there exists a nonzero $v \in V$ such that $\pi(h)v = \lambda(h)v$ for all $h \in \mathfrak{h}$. The corresponding weight space is denoted

$$V_\lambda = \{v \in V : \pi(h)v = \lambda(h)v \text{ for } h \in \mathfrak{h}\}$$

**Remark 3.14.** Recall that $\mathfrak{h}$ is abelian and finite dimensional, hence, similar to the root space decomposition, all the elements of $\mathfrak{h}$ are simultaneously diagonalizable therefore simultaneous eigenvectors exist and these vectors correspond to weights.

Now we begin exploring the structure of weights of representations.

**Proposition 3.15.** *Let* $(\pi, V)$ *be a finite dimensional representation of* $\mathfrak{g}$*. Then, every weight* $\lambda$ *of* $\pi$ *is an integral element, i.e., all entries of* $\lambda$ *are integers.*

Consider the Weyl group action on the root system of $\mathfrak{g}$. Weyl group also acts on the weights and the following theorem states that the set of weights remains invariant under the action of $W$.

**Theorem 3.16.** *The weights of* $\pi$ *are invariant under the action of* $W$*, including multiplicities.*

Recall that the root system axioms carry the properties of the roots of a root space decomposition of $\mathfrak{g}$. Therefore, we can define a set of positive and negative roots for the root system $\Phi$ of $\mathfrak{g}$. When we fix a set of positive roots, the partial order of weights is defined in the same manner as 3.3. A weight $\lambda$ of a representation $V$ is called the highest weight if $\lambda \prec \lambda_0$ for any other weight $\lambda_0$ of $V$.

The following theorem describes the structure of the weights of any irreducible finite dimensional representation of $\mathfrak{g}$.

**Theorem 3.17.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra.*

1. *Every irreducible, finite-dimensional representation of $\mathfrak{g}$ has a highest weight.*
2. *There exists a unique irreducible finite dimensional representation of $\mathfrak{g}$ with highest weight $\mu$ up to isomorphism.*
3. *If $\mu$ is an irreducible, finite dimensional representation of $\mathfrak{g}$ with highest weight $\mu$, then $\mu$ is a dominant integral element.*

# CHAPTER 4

# EXPONENTIAL INVARIANTS

Before studying the generalized Chebyshev polynomials, firstly we shall define the necessary tools that will allow us to construct such polynomials, which are exponential invariants and anti-invariants. We follow [2] throughout this chapter.

Let $\mathfrak{g}$ be a Lie algebra of rank $n$, $\Phi$ its associated root system, and $\Lambda$ the weight lattice, which is a free **Z**-module of rank $n$. For a ring $A$, the group algebra $A[\Lambda]$ denotes the additive group $\Lambda$ over $A$, which means, its elements are linear combinations of the terms $ae^p$ with $a \in A$ and $p \in \Lambda$. The exponential notation is used to distinguish between two additive groups, therefore we have $e^{p+p'} = e^p + e^{p'}$, $(e^p)^{-1} = e^{-p}$ and $e^0 = 1$ for all $p, p' \in \Lambda$. Throughout this thesis, we will set $A = \mathbf{Z}$, and it is well known that whenever $A$ is a UFD, $A[\Lambda]$ is also a UFD.

We recall the partial order structure on $\Lambda$ given in the definition 3.3. Under this partial order, whenever $x = \sum_{p \in P} x_p e^p \in \mathbf{Z}[\Lambda]$, we can define the set of maximal elements of $x$, which are the elements in the index set $p \in P$ with nonzero coefficients $x_p$ and maximal in $P$. A term $x_p e^p$ is called a maximal term in this case. The next lemma will be helpful in our future calculations:

**Lemma 4.1.** *Let $x, y \in \mathbf{Z}[\Lambda]$ such that $x$ has a family of maximal terms $(x_p e^p)_{p \in X}$ and $y$ has a unique maximal term $e^q$. Then, the family of maximal terms of $xy$ is $(x_p e^{p+q})_{p \in X}$*

Since the Weyl group acts on $\Lambda$, it also acts on $\mathbf{Z}[\Lambda]$ as $w(e^p) = e^{w(p)}$ for all $w \in W$ and $p \in \Lambda$.

**Definition 4.2.** The set of elements that are invariant under the action of the Weyl group is denoted by $\mathbf{Z}[\Lambda]^W$. In other words $w(x) = x$ for all $w \in W$. For any $p \in \Lambda$, we denote the orbit of $p$ in $\Lambda$ under $W$ as $Wp$, and $\sum_{q\in Wp} e^q$ is denoted by $S(e^p)$.

The elements of the form $S(e^\lambda)$ for $\lambda \in \Lambda$ are called $S$-type elements due to the notation. They are by definition invariant elements. Moreover, considering any invariant element in $\mathbf{Z}[\Lambda]^W$, each term will have a unique image under the action of Weyl group lying in $\Lambda \cap \overline{\mathfrak{C}}$. The elements $S(e^p)$ with $p \in \Lambda \cap \overline{\mathfrak{C}}$ form a basis for the $\mathbf{Z}$-module $\mathbf{Z}[\Lambda]^W$. Since the set of fundamental weights is a basis for $\Lambda$, the next theorem will play a significant role in our study of invariant elements.

**Theorem 4.3.** *(Bourbaki) Let $\omega_1, \dots, \omega_n$ be the fundamental weights and let $x_i = S(e^{\omega_i})$. Define the homomorphism*

$$\varphi : \mathbf{Z}[X_1, \dots, X_n] \to \mathbf{Z}[\Lambda]^W$$

*be the homomorphism taking $X_i$ to $x_i$ for all $1 \leq i \leq n$. Then, the map $\varphi$ is an isomorphism.*

In other words, any invariant element can be written as a polynomial of the elements $x_i$.

Another class of elements we will study is the anti-invariant elements. Recall that any element $w \in W$, can be obtained by simple reflections since simple reflections generate the Weyl group. Let $l(w)$ be the minimum number of simple reflections that shall be applied consecutively to obtain $w$. We define $\varepsilon(w) = (-1)^{l(w)}$, it is an equivalent definition to the determinant of the element $w \in W$ since each simple reflection has determinant $-1$ as a linear transformation.

**Definition 4.4.** An element $x \in \mathbf{Z}[\Lambda]$ is called anti-invariant if $w(x) = \varepsilon(w)x$ for all $w \in W$.

The *J-type* elements are defined to be

$$J(x) = \sum_{w\in W} \varepsilon(w)w(x)$$

and they are anti-invariant almost by definition. At this stage, it is important to emphasize the difference between the indexes of the summation in the $S$-type and $J$-type elements. While the former one prevents possible multiplicities from appearing in the sum, this is not used in the $J$-type elements. The following set of propositions shows that this is never the case, as all anti-invariant elements with unique maximal term already have their maximal term in the fundamental Weyl chamber.

**Theorem 4.5.** *The sum of all fundamental weights is equal to half the sum of all positive roots. This sum is denoted by $\rho = \omega_1 + \ldots + \omega_n \in \Lambda$.*

The element $\rho$ is minimal in the set $\mathfrak{C} \cap \Lambda$ in the sense that for any $\lambda \in \mathfrak{C} \cap \Lambda$, we have $\lambda \prec \rho$. Moreover, the element $J(e^\rho)$ has $e^\rho$ as its unique maximal term. Following theorem reveals that the element $\rho$ plays a significant role in the structure of the set of $J$-type elements.

**Theorem 4.6.** *For any $p \in \Lambda$, the element $J(e^p)$ is divisible by $J(e^\rho)$, in other words, $J(e^p) = J(e^\rho)A$ for an element in the group algebra $A \in \mathbb{Z}[\Lambda]$. Moreover, quotient $J(e^p)/J(e^\rho)$ is an invariant element.*

Therefore, we conclude that any anti-invariant element is a product of an invariant element and $J(\rho)$. Moreover, the product of an invariant element with the anti-invariant element $J(e^\rho)$ is anti-invariant, hence this product turns out to be sum of $J$-type elements as $S$-type elements form a basis for the $\mathbb{Z}$-module of invariant elements. The quotients $J(e^p)/J(e^\rho)$ specifically turns out to be special elements that are characters of irreducible representations by the Weyl Character Formula, which we will study in the following chapters.

# CHAPTER 5

# GENERALIZED CHEBYSHEV POLYNOMIALS

In this chapter, we define a generalization for the well-studied Chebyshev polynomials. This generalization was first given in [14], and somewhat later in [7]. The Chebyshev polynomials are defined as the polynomials satisfying the relation

$$T_k(2\cos\theta) = 2\cos(k\theta).$$

In the literature, it is usually defined as the polynomials satisfying $\tilde{T}_k(\cos\theta) = \cos(k\theta)$ and $2^{k-1}$ appears as the leading coefficient in these terms, however, the previous definition gives a normalized version with leading coefficient 1, therefore it will be appropriate to use the normalized family for the rest of this chapter. It is a well-studied family of polynomials and they're known to satisfy the recurrence relation

$$T_{n+1}(x) = xT_n(x) - T_{n-1}(x).$$

By definition, they commute with each other as we have

$$T_n(T_m(x)) = 2\cos(mn\theta) = T_m(T_n(x))$$

for $x = 2\cos\theta$. The first few Chebyshev Polynomials are

$$T_0(x) = 2$$
$$T_1(x) = x$$
$$T_2(x) = x^2 - 2$$
$$T_3(x) = x^3 - 3x$$
$$T_4(x) = x^4 - 4x^2 + 2$$

with the general recursion formula $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$.

Moreover, an easy algebraic manipulation shows that these polynomials satisfy the identity

$$T_k\left(x + \frac{1}{x}\right) = x^k + \frac{1}{x^k}.$$

Lidl and Wells managed to generalize this identity and construct polynomial mappings from $\mathbf{R}^n \to \mathbf{R}^n$ in [12]. If $P(x) = x^2 - ax + b$ is a quadratic polynomial with integer coefficients having $z, b/z$ as roots, one can define the polynomial $P_k(x) = x^2 - a_k x + b^k$ having $z^k, (b/z)^k$ and there exists a polynomial with integer coefficients that satisfy $g_{k,b} = a_k$. The $b = 1$ case coincides with the usual Chebyshev polynomials. An important property of these polynomials is given in [11] that they permute the finite field $\mathbf{F}_q$ if and only if $(k, q^2 - 1) = 1$. Moreover, it is proven that any polynomial that produces a permutation of $\mathbf{F}_q$ for infinitely many $q$ is a composition of Chebyshev polynomials and linear polynomials in [5]. Lidl and Wells extend $P$ to a polynomial of degree $n$ with

$$P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + b$$

where all coefficients are integers. Let $z_1, \ldots, z_n$ be its roots. Then, similarly, we can take the polynomial having $z_1^k, \ldots, z_n^k$ as roots, and their coefficients can be written as integral polynomials of coefficients of $P$. These multivariable polynomials of [12] were constructed as a generalization of Chebyshev polynomials, and they permute $\mathbf{F}_q^n$ if and only if $(k, q^s - 1) = 1$ for all $1 \leq s \leq n + 1$.

On the other hand, as a motivation for the Lie algebra-based generalized definition, we can identify $x, 1/x$ with the root system $A_1$ as $e^{\omega} + e^{-\omega}$ since it is $S(e^{\omega})$ in this root system. Therefore, using Chevalley's Theorem as well, we can make the following definition.

**Definition 5.1.** The generalized Chebyshev polynomial attached to a semisimple Lie algebra $\mathfrak{g}$ of rank $n$ is the polynomial map $P_{\mathfrak{g}}^k : \mathbf{C}^n \to \mathbf{C}^n$ satisfying the relation

$$P_{\mathfrak{g}}^k(S(e^{\omega_1}), \ldots, S(e^{\omega_n})) = (S(e^{k\omega_1}), \ldots, S(e^{k\omega_n}))$$

It turns out that the family of polynomials defined in Lidl and Wells's article coincides with the polynomials defined by the Lie algebras of type $A_n$. Moreover, it is proven in [10] new polynomials obtained in the generalized Chebyshev polynomials also satisfy the property of being permutation polynomials $\mathbf{F}_q^n \to \mathbf{F}_q^n$ with the necessary and sufficient conditions being $(k, q^s - 1) = 1$ for all $1 \leq s \leq n+1$. Explicit calculations of these polynomials can be found in [1].

The Chebyshev polynomials of the second kind can be defined by the relation

$$\frac{\partial T_k(x)}{\partial x} = kU_{k-1}(x)$$

as well as the polynomials satisfying $U_k(2\cos\theta)\sin\theta = 2\sin((k+1)\theta))$. Notice that $\cos x$ is an even function and $\sin x$ is an odd function, therefore they are invariant and anti-invariant respectively with respect to the $y$-axis. This situation can be interpreted as that Chebyshev polynomials are defined for the invariant elements, and we can study the derivation in terms of the division of anti-invariant elements, which corresponds to irreducible representation characters by the Weyl Character Formula, which motivates the main problem of this paper.

# CHAPTER 6

# MAIN RESULTS

In this chapter, we're ready to prove our main results. Let $\mathfrak{g}$ be a semisimple Lie algebra of rank $n$. The main problem we deal with in this thesis is to express the Jacobian matrix of the generalized Chebyshev polynomials $P_{\mathfrak{g}}^{k}$ in terms of characters of irreducible representations of $\mathfrak{g}$. In this manner, Weyl Character Formula is a very important tool that reveals a connection between the characters and anti-invariant elements.

**Theorem 6.1.** *(Weyl Character Formula) Let $\chi_\lambda$ be the character of the irreducible representation of $\mathfrak{g}$ with the highest weight $\lambda \in \Lambda$. Then, $\chi_\lambda = \dfrac{J(e^{\lambda+\rho})}{J(e^{\rho})}$, where $\rho = \omega_1 + \ldots + \omega_n$.*

Let $P_{\mathfrak{g}}^{k} = (g_1, \ldots, g_n)$ be the generalized Chebyshev polynomial, our goal is to compute the Jacobian of $P_{\mathfrak{g}}^{k}$. Let $y_1, \ldots, y_n$ be the variables for the polynomials $g_i = g_i(y_1, \ldots, y_n)$, then we have

$$J(P_{\mathfrak{g}}^{k}) = \frac{\partial(g_1, \ldots, g_n)}{\partial(y_1, \ldots, y_n)} = \begin{pmatrix} \frac{\partial(g_1)}{\partial(y_1)} & \frac{\partial(g_1)}{\partial(y_2)} & \cdots & \frac{\partial(g_1)}{\partial(y_n)} \\ \frac{\partial(g_2)}{\partial(y_1)} & \frac{\partial(g_2)}{\partial(y_2)} & \cdots & \frac{\partial(g_2)}{\partial(y_n)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(g_n)}{\partial(y_1)} & \frac{\partial(g_n)}{\partial(y_2)} & \cdots & \frac{\partial(g_n)}{\partial(y_n)} \end{pmatrix}$$

as the Jacobian. In order to understand this matrix, we recall the definition of $P_{\mathfrak{g}}^{k}$, which is

$$P_{\mathfrak{g}}^{k}(S(e^{\omega_1}), \ldots, S(e^{\omega_n})) = (S(e^{k\omega_1}), \ldots, S(e^{k\omega_n})).$$

Therefore, both $y_i$'s and $g_i$'s can be realized as functions of $\omega_i$'s. We can define a derivative operation on these sums with respect to the fundamental weights to apply the chain rule as the following: We define a formal derivation operator $D_j$ on $A[\Lambda]$ for each $1 \leq j \leq n$ as the linear operator satisfying $D_j(ae^{k\omega_i}) = \delta_{ij}ake^{k\omega_i}$ for any $1 \leq i \leq n$, $k \in \mathbf{Z}$ and $a \in A$. By linearity, this formula is the same as

$$D_j\left(\sum_\lambda a_\lambda e^\lambda\right) = \sum_\lambda \left(\lambda, \alpha_j^\vee\right) a_\lambda e^\lambda$$

In order to realize these operators as partial derivatives, we can consider formal sums of the group algebra $A[\Lambda]$ as a complex-valued function with the evaluation $e^\lambda(\gamma) \mapsto e^{-2\pi i(\lambda,\gamma)}$. Hence, we're in a position to apply the chain rule to obtain

$$\frac{\partial(g_1,\ldots,g_n)}{\partial(\omega_1,\ldots,\omega_n)} = \frac{\partial(g_1,\ldots,g_n)}{\partial(y_1,\ldots,y_n)}\frac{\partial(y_1,\ldots,y_n)}{\partial(\omega_1,\ldots,\omega_n)}$$

where in terms of the fundamental weights, we have $y_j = S(e^{\omega_j})$ and $g_j = S(e^{k\omega_j})$. The following exercise in [2] reveals that we can calculate the determinant of the last matrix, hence the first one in a similar fashion and we will have the first result on the determinant of the Jacobian:

**Theorem 6.2.** *Keeping the notation above, we have* $\det(D_i(S(e^{\omega_j})) = J(e^\rho)$.

Before completing the proof, we need two lemmas that we will use.

**Lemma 6.3.** *The inner product* $(\lambda, \gamma)$ *can be computed as*

$$(\lambda, \gamma) = \sum_{m=1}^{n}(\lambda, \alpha_m^\vee)(\omega_m, \gamma)$$

*Proof.* By definition, we have $(\alpha_i^\vee, \omega_j) = \delta_{ij}$, and both coroots and fundamental weights form a basis hence if we write $\lambda = \sum_{m=1}^n a_m\omega_m$ and $\gamma = \sum_{m=1}^n b_m\alpha_m^\vee$ we get $(\lambda, \gamma) = \sum_{m=1}^n a_m b_m = \sum_{m=1}^n(\lambda, \alpha_m^\vee)(\omega_m, \gamma)$. □

The second lemma is about realizing the Weyl group as a matrix group since its elements are linear transformations of a Euclidean space. It turns out that all its entries are integers and its entries can be calculated in a useful fashion.

**Lemma 6.4.** *Let* $T_w = [(\omega_i, w(\alpha_j^\vee))]$ *for each* $w \in W$*. Then, the map* $w \mapsto T_w$ *is an injective group homomorphism from* $W$ *into* $GL(n, \mathbf{Z})$ *and* $\det(w) = \det(T_w)$.

*Proof.* First, we will show that the map is a group homomorphism. We have

$$T_w T_{w'} = [(\omega_i, w(\alpha_j^\vee))][(\omega_i, w'(\alpha_j^\vee))]$$

and reflections are isometries, therefore each $w \in W$ is an isometry and we can apply $w^{-1}$ to each entry of $T_w$ to say

$$T_w T_{w'} = [(\omega_i, w(\alpha_j^\vee))][(\omega_i, w'(\alpha_j^\vee))] = [(w^{-1}(\omega_i), \alpha_j^\vee)][(\omega_i, w'(\alpha_j^\vee))]$$

since it preserves inner product. By Lemma 6.3 this is nothing but

$$T_w T_{w'} = [(w^{-1}(\omega_i), w'(\alpha_j^\vee)]$$

and applying $w$ to each entry of this matrix we obtain

$$T_w T_{w'} = [(\omega_i, ww'(\alpha_j^\vee)] = T_{ww'}$$

Injectivity follows from the fact that

$$T_w = [(\omega_i, w(\alpha_j^\vee))] = I_{n\times n} \Rightarrow (\omega_i, w(\alpha_j^\vee)) = \delta_{ij} = (\omega_i, \alpha_j^\vee) \Rightarrow w(\alpha_j^\vee) = \alpha_j^\vee$$

for each $1 \leq i, j \leq n$ and since coroots span $E$, $w$ must be the identity transformation. For the final part, we recall that the Cartan matrix transforms the fundamental weights into the simple roots, and $s_{\alpha_i}(\omega_j) = \omega_j - \delta_{ij}\alpha_i$ for each $\alpha_i \in \Delta$. Thus, the matrix $T_w$ for $w = s_{\alpha_i}$ is obtained by subtracting the $i$th row of the Cartan matrix from the identity matrix. Such a matrix has integer entries and the main diagonal has a unique $-1$ entry with all the remaining entries equal to 1, thus its determinant is equal to $-1$. The Weyl group is generated by $s_{\alpha_i}$, and therefore $T_w$ has integer entries for each $w \in W$ and $\det(w) = \det(T_w)$ follows. □

Now we're ready to prove Theorem 6.2.

*Proof.* First, we will prove that $\det(D_i(S(e^{\omega_j})))$ is an anti-invariant element of $\mathbf{Z}[\Lambda]$. Let $S(e^{\omega_i}) = x_i = \sum_{\lambda\in\Lambda} a_\lambda^i e^\lambda$. We can express the action of $D_j$ on $x_i$ as $D_j(x_i) = \sum_{\lambda\in\Lambda} a_\lambda^i (\lambda, \alpha_j^\vee) e^\lambda$. Therefore, the action of Weyl group on the group algebra is

$$w(D_j(x_i)) = \sum_{\lambda\in\Lambda} a_\lambda^i (\lambda, \alpha_j^\vee) e^{w(\lambda)}.$$

Since each $w \in W$ is an isometry, we can replace $(\lambda, \alpha_j^\vee)$ with $(w(\lambda), w(\alpha_j^\vee))$ and apply Lemma 6.3 to find

$$w([D_j(x_i)]) = \left[ \sum_{m=1}^{n} \left( \sum_{\lambda \in \Lambda} a_\lambda^i (w(\lambda), \alpha_m^\vee) e^{w(\lambda)} \right) (\omega_m, w(\alpha_j^\vee)) \right].$$

Right-hand side of this equation is nothing but the product of the two matrices

$$\left[ \left( \sum_{\lambda \in \Lambda} a_\lambda^i (w(\lambda), \alpha_j^\vee) e^{w(\lambda)} \right) \right] \quad \text{and} \quad \left[ (\omega_i, w(\alpha_j^\vee)) \right]$$

where $B = T_w$ as in Lemma 6.4. Thus, we have

$$w([D_j(x_i)]) = [D_j(w(x_i))] T_w.$$

By definition, each $x_i$ is an invariant element hence $w(x_i) = x_i$. The determinant function is multiplicative, and $\det(w) = \det(T_w)$ by Lemma 6.4 so we conclude that $\det(D_j(x_i))$ is anti-invariant.

All that is left to show is that $\det(D_j(x_i))$ has a unique maximal term $e^\rho$. We first note that $D_j(x_i)$ has maximal terms less than or equal to $e^{\omega_i}$, because each $D_j$ operator lefts the exponent invariant and (possibly) changes the coefficients and each $x_i$ has a unique maximal term $e^{\omega_i}$. Moreover, the derivation $D_j$ satisfies the following property

$$D_j(e^{\omega_i}) = \delta_{ij} e^{w_i}.$$

So, $D_j(x_i)$ has unique maximal term $e^{\omega_i}$ if and only if $i = j$. Thus the diagonal summand $\prod D_j(x_j)$ of $\det(D_j(x_i))$ has unique maximal term $e^\rho = e^{\omega_1 + \ldots + \omega_n}$ by Lemma 4.1.

Each $n!$ summand of the determinant is a product of $n$ terms and each derivation $D_j$ occurs exactly once. Moreover, $e^{\omega_i}$ is the unique maximal term of $S(e^{\omega_i})$ for each $1 \leq i \leq n$ and this term appears on the entry $D_i(S(e^{\omega_j}))$ if and only if as stated above. As a corollary of the previous lemma, the maximal weight of the product of some terms in **Z** is a subset of the sum of maximal term sets, therefore each summand has terms with weights less than $\rho$, and $\rho$ only appears on the summand obtained from the main diagonal. This completes the proof of the fact that $\det(D_j(x_i))$ has a unique maximal term $e^\rho$.

The elements $J(e^\lambda)$, with $\lambda \in \Lambda \cap \mathfrak{C}$, form a basis of the submodule of anti-invariant elements. Thus we must have $\det\left([D_j(x_i)]\right) = J(e^\rho)$. $\square$

**Theorem 6.5.** $\det(J(P_{\mathfrak{g}}^k)) = k^n \chi_{(k-1)\rho}$

*Proof.* Looking at the equation

$$\frac{\partial(g_1,\dots,g_n)}{\partial(\omega_1,\dots,\omega_n)} = \frac{\partial(g_1,\dots,g_n)}{\partial(y_1,\dots,y_n)} \frac{\partial(y_1,\dots,y_n)}{\partial(\omega_1,\dots,\omega_n)}$$

obtained by the chain rule, from the theorem 6.2 we find $\dfrac{\partial(y_1,\dots,y_n)}{\partial(\omega_1,\dots,\omega_n)}$ has determinant $J(e^\rho)$. Moreover, $\dfrac{\partial(g_1,\dots,g_n)}{\partial(\omega_1,\dots,\omega_n)} = \dfrac{\partial(S(e^{\omega_1}),\dots,S(e^{k\omega_j}))}{\partial(\omega_1,\dots,\omega_n)}$ and since each term in the sum $S(e^{k\omega_j})$ corresponds to a term in the sum $S(e^{\omega_j})$ raised to the power $k$, we will have a coefficient $k$ in each entry and the exponents will be same as in the expression $\det(D_i(S(e^{\omega_j}))$, with dilation by $k$. Therefore this Jacobian will have determinant $k^n J(e^{k\rho})$. Finally, dividing these two terms and using the Weyl character formula we find $\det(J(P_{\mathfrak{g}}^k)) = k^n \dfrac{J(e^{k\rho})}{J(e^\rho)} = k^n \chi_{(n-1)\rho}$. □

In order to calculate the entries, we will again use the chain rule formula in a similar fashion. Trivially, each entry is an invariant element of $A[\Lambda]$ since they are polynomials of $S(e^{\omega_j}) = y_j$'s, and let's define Jac$(k) = \det(D_i(S(e^{k\omega_j}))$. Our strategy is to work with the expression $J(P_{\mathfrak{g}}^k) =$ Jac$(k)$Jac$(1)^{-1}$ to determine each entry. For the inverse of Jac$(1)$ we will work with the adjugate matrix Adj(Jac$(1)$) to be able to make calculations. Following example for $\mathfrak{g} = A_2$ will summarize our method:

**Example 6.6.**

$$J(P_{A_2}^k) = k \begin{pmatrix} \chi_{k-1,0} & -\chi_{k-2,0} \\ -\chi_{0,k-2} & \chi_{0,k-1} \end{pmatrix}$$

for $k \geq 2$ where $\chi_{a,b}$ stands for $\chi_{a\omega_1 + b\omega_2}$

*Proof.* We have $S(e^{\omega_1}) = e^{\omega_1} + e^{-\omega_2} + e^{\omega_2 - \omega_1}$ and $S(e^{\omega_2}) = e^{\omega_2} + e^{-\omega_1} + e^{\omega_1 - \omega_2}$. So, we can calculate

$$\text{Jac}(1) = \begin{pmatrix} e^{\omega_1} - e^{\omega_2 - \omega_1} & -e^{-\omega_2} + e^{\omega_2 - \omega_1} \\ -e^{-\omega_1} + e^{\omega_1 - \omega_2} & e^{\omega_2} - e^{\omega_1 - \omega_2} \end{pmatrix}$$

and similarly

$$\text{Jac}(k) = k \begin{pmatrix} e^{k\omega_1} - e^{k\omega_2 - k\omega_1} & -e^{-k\omega_2} + e^{k\omega_2 - k\omega_1} \\ -e^{-k\omega_1} + e^{k\omega_1 - k\omega_2} & e^{k\omega_2} - e^{k\omega_1 - k\omega_2} \end{pmatrix}$$

Since the adjugate of a $2 \times 2$ matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and we have $\text{Jac}(k)\text{Adj}(\text{Jac}(1)) = J(e^\rho)J(P_{A_2}^k)$. Calculating the left-hand side, we obtain

$$\text{Jac}(k)\text{Adj}(\text{Jac}(1)) = \begin{pmatrix} e^{k\omega_1+\omega_2} + \dots & -e^{(k-1)\omega_1+\omega_2} + \dots \\ -e^{(k-1)\omega_2+\omega_1} + \dots & e^{k\omega_2+\omega_1} + \dots \end{pmatrix}$$

where only the elements lying on the fundamental Weyl chamber are shown. We know that entries of $J(P_{A_2}^k)$ are composed of polynomials of invariant elements and $J(e^\rho)$ is an anti-invariant element, therefore each entry above is anti-invariant and since these are the only elements in the fundamental Weyl chamber we get

$$\text{Jac}(k)\text{Adj}(\text{Jac}(1)) = \begin{pmatrix} J(e^{k\omega_1+\omega_2}) & -J(e^{(k-1)\omega_1+\omega_2}) \\ -J(e^{(k-1)\omega_2+\omega_1}) & J(e^{k\omega_2+\omega_1}) \end{pmatrix} = J(e^\rho)J(P_{A_2}^k)$$

Dividing both sides by $J(e^\rho)$ and using the Weyl character formula, the result follows.

□

Now, we can state our main theorem:

**Theorem 6.7.** *The entries of the Jacobian matrix $J(P_{\mathfrak{g}}^k)$ are given by*

$$\frac{\partial g_i}{\partial y_j} = \sum_{w_1 \in W} \sum_{w_2 \in W} \frac{d_{ij}^k(w_1, w_2)}{2s_i} \chi_{w_1(k\omega_i)+w_2(\rho-\omega_j)-\rho}$$

*where $d_{ij}^k(w_1, w_2)$ is defined as*

$$d_{ij}^k(w_1, w_2) = \begin{cases} \det(w_2)(w_1(k\omega_i), w_2(\alpha_j^\vee)) & \text{if } w_1(k\omega_i) + w_2(\rho - \omega_j) \in \Lambda \cap \mathfrak{C}, \\ 0 & \text{otherwise.} \end{cases}$$

*for each pair $(w_1, w_2) \in W^2$.*

The proof requires explicit calculations on the formula

$$\text{Jac}(k)\text{Adj}(\text{Jac}(1)) = J(e^\rho)J(P_{\mathfrak{g}}^k)$$

hence we shall write down the expressions for each matrix on the left-hand side. First, we need to express the entries of the Jacobian matrix explicitly.

**Theorem 6.8.** *The Jacobian matrix* $\text{Jac}(1) = [D_j(S(e^{\omega_i}))]$ *is given by*

$$\text{Jac}(1) = \left[ \frac{1}{s_i} \sum_{w \in W} (w(\omega_i), \alpha_j^\vee) e^{w(\omega_i)} \right]$$

*where* $s_i$ *is the size of the stabilizer group* $\text{Stab}(\omega_i)$.

*Proof.* The result follows immediately, once the derivation $D_j$ is applied to

$$S(e^{\omega_i}) = \sum_{\mu \in W(\omega_i)} e^\mu = \frac{1}{s_i} \sum_{w \in W} e^{w(\omega_i)}.$$

since it is a linear operator given by $D_j(S(e^{\omega_i})) = (w(\omega_i), \alpha_j^\vee) e^{w(\omega_i)}$ □

Next, we will compute the entries of the adjugate matrix $\text{Adj}(\text{Jac}(1))$. We need the following lemma for the proof.

**Lemma 6.9.** *Let* $\lambda = w_1(\omega_i) + w_2(\rho - \omega_j)$ *for some* $w_1, w_2 \in W$. *The element* $\lambda$ *is in the fundamental Weyl chamber* $\mathfrak{C}$ *if and only if* $\lambda = \rho$. *This is possible if only if* $\omega_i = \omega_j$, $w_1 \in \text{Stab}(\omega_i)$, *and* $w_2 \in \text{Stab}(\rho - \omega_i)$.

*Proof.* The element $\lambda = w_1(\omega_i) + w_2(\rho - \omega_j)$ is trivially in the weight lattice. If $\lambda \in \mathfrak{C}$, then we have $\rho \leq \lambda$ since $\rho$ is the minimal element in $\mathfrak{C}$. On the other hand, we know $w(\mu) \leq \mu$ for $\mu \in \overline{\mathfrak{C}}$, $w \in W$. Thus

$$\rho \leq \lambda = w_1(\omega_i) + w_2(\rho - \omega_j) \leq \omega_i + \rho - \omega_j.$$

However, $\omega_i \leq \omega_j$ holds if and only if $i = j$ so we must have $i = j$ and equalities are satisfied in both $w_1(\omega_i) \leq \omega_i$ and $w_2(\rho - \omega_j) \leq \rho - \omega_j$. Therefore, $\lambda = \rho$ and $w_1(\omega_i) = \omega_i$ which implies $w_1 \in \text{Stab}(\omega_i)$ and $w_2(\rho - \omega_i) = \rho - \omega_i$ which again implies $w_2 \in \text{Stab}(\rho - \omega_i)$. □

**Theorem 6.10.** *The adjugate matrix of* $\text{Jac}(1)$ *is given by*

$$\text{Adj}(\text{Jac}(1)) = \left[ \frac{1}{2} \sum_{w \in W} \det(w)(\omega_i, w(\alpha_j^\vee)) e^{w(\rho - \omega_j)} \right]$$

*Proof.* Let $A = \text{Jac}(1)$ and let $B$ be the matrix in the hypothesis. We will show that $AB = J(e^\rho)I_{n\times n}$. Let

$$A_{im} = \frac{1}{s_i}\sum_{w_1\in W}(w_1(\omega_i), \alpha_m^\vee)e^{w_1(\omega_i)}, \text{ and}$$
$$B_{mj} = \frac{1}{2}\sum_{w_2\in W}\det(w_2)(\omega_m, w_2(\alpha_j^\vee))e^{w_2(\rho-\omega_j)}.$$

The entries of the product $AB = C$ are given by $C_{ij} = \sum_{m=1}^n A_{im}B_{mj}$. By Lemma 6.3, we have

$$\sum_{m=1}^n (w_1(\omega_i), \alpha_m^\vee)(\omega_m, w_2(\alpha_j^\vee)) = (w_1(\omega_i), w_2(\alpha_j^\vee)).$$

Therefore, the entries of the matrix $C$ are given by

$$C_{ij} = \frac{1}{2s_i}\sum_{w_1\in W}\sum_{w_2\in W}\det(w_2)(w_1(\omega_i), w_2(\alpha_j^\vee))e^{w_1(\omega_i)+w_2(\rho-\omega_j)}.$$

We claim that $C_{ij} = \delta_{ij}J(e^\rho)$, and we start by showing that each entry is anti-invariant. Let $w \in W$, then

$$w(C_{ij}) = \frac{1}{2s_i}\sum_{w_1\in W}\sum_{w_2\in W}\det(w_2)(w_1(\omega_i), w_2(\alpha_j^\vee))e^{ww_1(\omega_i)+ww_2(\rho-\omega_j)}$$

Since each $w \in W$ is an isometry of the space we apply $w$ to the inner product to obtain

$$w(C_{ij}) = \frac{1}{2s_i}\sum_{w_1\in W}\sum_{w_2\in W}\det(w_2)(ww_1(\omega_i), ww_2(\alpha_j^\vee))e^{ww_1(\omega_i)+ww_2(\rho-\omega_j)}$$

multiplying both sides with $\det(w)$ and using that the Weyl group is a finite group hence $wW = W$, we get $\det(w)w(C_{ij}) = C_{ij}$. Since $\det(w) = \pm 1$ for each $w \in W$ we get the desired result $w(C_{ij}) = \det(w)C_{ij}$.

Now we follow the maximal elements to determine each entry. From Lemma 6.9, we know that $w_1(\omega_i) + w_2(\rho - \omega_j) \in \mathfrak{C}$ is not true when $i \neq j$, hence $C_{ij} = 0$ for $i \neq j$. Moreover, for $i = j$, it is in the fundamental Weyl chamber only when $w_1 \in \text{Stab}(\omega_i)$ and $w_2 \in \text{Stab}(\rho - \omega_i)$. Since $s_i = |\text{Stab}(\omega_i)|$ and $2 = |\text{Stab}(\rho - \omega_i)|$ there are $2s_i$ many $e^\rho$ terms appearing in the sum and after dividing by $2s_i$, we get that $C_{ii} = J(e^\rho)$.

□

Now we are ready to prove our main result:

*Proof of Theorem 6.7.* Recall that $J(P^k_{\mathfrak{g}}) = \text{Jac}(k)\text{Jac}(1)^{-1}$. Moreover $\text{Jac}(1)\text{Adj}(\text{Jac}(1)) = J(e^\rho)$. It follows that

$$J(P^k_{\mathfrak{g}}) = \frac{1}{J(e^\rho)}\text{Jac}(k)\text{Adj}(\text{Jac}(1)).$$

All entries of the matrix $J(P^k_{\mathfrak{g}})$ are invariant under $W$ since they are polynomials of the invariant elements $y_1, \ldots, y_n$. On the other hand, each entry of the matrix $\text{Jac}(k)\text{Adj}(\text{Jac}(1))$ is anti-invariant since it is obtained by multiplying the matrix $J(P^k_{\mathfrak{g}})$ with the anti-invariant element $J(e^\rho)$.

Similar to our previous results, we want to understand the entries of the matrix $\text{Jac}(k)\text{Adj}(\text{Jac}(1))$ in terms of the anti-invariant basis elements $J(e^\lambda)$, with $\lambda \in \Lambda \cap \mathfrak{C}$. We have an explicit description for $\text{Jac}(1)$ by Theorem 6.8. Moreover, the chain rule implies that

$$\text{Jac}(k) = \left[\frac{1}{s_i}\sum_{w\in W}(w(k\omega_i), \alpha_j^\vee)e^{w(k\omega_i)}\right].$$

On the other hand, by Theorem 6.10, we have

$$\text{Adj}(\text{Jac}(1)) = \left[\frac{1}{2}\sum_{w\in W}\det(w)(\omega_i, w(\alpha_j^\vee))e^{w(\rho-\omega_j)}\right]$$

The rest of the proof follows closely the pattern of the proof of Theorem 6.10. Applying Lemma 6.3 to the product of these two matrices, we see that the $ij$-th entry of the product is given by

$$\frac{1}{2s_i}\sum_{w_1\in W}\sum_{w_2\in W}\det(w_2)(w_1(k\omega_i), w_2(\alpha_j^\vee))e^{w_1(k\omega_i)+w_2(\rho-\omega_j)}.$$

Each entry is anti-invariant and can be expressed as a linear combination of $J$-type basis elements $J(e^\lambda)$, with $\lambda \in \Lambda \cap \mathfrak{C}$. The coefficients are captured by using the function $d^k_{ij}$. We finally divide everything by $J(e^\rho)$, a common divisor of anti-invariant elements. Using the Weyl character formula, we get the desired expression. □

**Remark 6.11.** If $k = 1$, the integer $d^k_{ij}(w_1, w_2)$ is rather well understood by Lemma 6.9. We have $d^1_{ij}(w_1, w_2) = 1$ if and only if $\omega_i = \omega_j$, $w_1 \in \text{Stab}(\omega_i)$, $w_2 \in \text{Stab}(\rho - \omega_i)$ are all satisfied and $d^1_{ij}(w_1, w_2) = 0$ otherwise. The situation changes dramatically if $k > 1$, but the computation of $d^k_{ij}$ can be done practically with the help of a computer by using the matrices $T_w$ of Lemma 6.4.

**Remark 6.12.** Calculation of matrices in the Weyl group only requires the knowledge of the Cartan matrix. When the $n \times n$ Cartan matrix is given, the $n$ matrices obtained by subtracting a column of Cartan matrix from the identity matrix generates the Weyl group as a matrix group. Thus, calculation of the $d_{ij}^k$ values only involves checking whether $w_1(k\omega_i) + w_2(\rho - \omega_j)$ lies in the fundamental Weyl chamber (which is equivalent to $n$ inner products being positive) and calculating the values $\det(w_2)(w_1(k\omega_i), w_2(\alpha_j^\vee))$, again a straightforward matrix operation.

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