AN ANALYSIS ON THE MEMBRANE PARADIGM OF BLACK HOLES

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ABSTRACT

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Smooth geometries of spacetime generically give a boundary where the spacetime is at infinity. According to Einstein-Hilbert's action, the extra term coming from this boundary topology cannot change the field equations. Still, the action must be aided by a new surface-term action which cancels the contribution coming from the original boundary at infinity. However, black hole geometries induce a different boundary surface which is unique in its properties. It is called the horizon, a null surface of no return, where time becomes spatial and vice-versa. Since the inside of the black hole is not available for an observer outside, one can construct a fake or stretched horizon just outside the true horizon with similar properties. A fiducial observer sees it as a time-like hypersurface, a.k.a a membrane, which has no pathologies in the kinds of the true horizon. The stretched horizon is indeed a $D-1$ dimensional time-like surface. This surface can be linked through the true horizon by ingoing null congruence as an injection. Since it is not null by nature, it has a non-singular metric on it which brings a healthy approximation to the behaviour of the true horizon. This approach to black hole dynamics was pinned with the name "Membrane Paradigm of Black Holes". The paradigm is known to be valid for static and rotating black holes. We have tested the theory with different conserved charges allowed by the no-hair theorem. Moreover, we test it by using the Johannsen-Psaltis black hole. It is a deviation from the Kerr metric. This metric has no pathology up to higher spin solutions with a set of parameters that clearly indicates linear parametric deviations from the Kerr metric. They have the same metric structure as Kerr in Boyer-Lindquist coordinates such that geodesic tests are also comparable with the Kerr Spacetime.

Keywords: Kerr Black Holes, Kerr-like Black Holes, Membrane Paradigm, Johannsen-Psaltis Metric

ÖZ

KARADELİKLERİN ZAR PARADİGMASI ÜZERİNE BİR ANALİZ

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Uzayzamanın pürüzsüz geometrileri genel olarak onun sonsuzda olduğu bir sınır verir. Einstein-Hilbert'in eylemine göre, bu sınır topolojisinden gelen fazladan terim, alan denklemlerini değiştiremez. Yine de eylemin varyasyonunun sıfır olabilmesi için, sonsuzdan gelen katkıyı iptal eden yeni bir yüzey terimli eylem yardımcı olmalıdır. Bununla birlikte, kara delik geometrileri, benzersiz olan farklı bir sınır yüzeyi oluşturur. Buna olay ufku denir. Zamanın uzamsal hale geldiği ve uzayın zamansal olduğu, geri dönüşü olmayan boş bir yüzey. Kara deliğin içi dışarıdaki bir gözlemci için uygun olmadığından, gerçek ufkun hemen dışında benzer özelliklere sahip sahte veya gerilmiş bir ufuk inşa edilebilir. Hareket etmediği varsayılan bir gözlemci, onu zamansal bir hiper yüzey, yani bir zar olarak görür.Bu zar, gerçek olay ufkuna nazaran hiçbir patolojisi olmayan bir sahte yüzeydir. Uzatılmış, yalancı ufuk aslında $D-1$ boyutlu zamansal bir yüzeydir. Bu yüzey, bir enjeksiyon olarak boş kongrüansa girerek gerçek ufuk boyunca bağlanabilir. Doğası gereği sıfır olmadığı için, üzerinde gerçek ufuk davranışına sağlıklı bir yaklaşım getiren tekil olmayan bir metriğe sahiptir. Kara delik dinamiğine yönelik bu yaklaşım, "Kara Deliklerin Zar Paradigması" adıyla sabitlenmiştir. Paradigmanın statik ve dönen kara delikler için geçerli olduğu biliniyor,

ancak biz onu farklı, global olarak korunan yükler ile de test ediyoruz. Ayrıca dönen karadeliklere eklenen lineer bir deformasyonun sonucu oluşan Johannsen-Psaltis kara deliğini kullanarak da test ediyoruz. Bu metriğin yüksek spin çözümleri de dahil olmak üzere hiçbir patolojisi yoktur. BL koordinatlarında Kerr ile aynı metrik yapıya sahiptirler, öyle ki jeodezik testler de Kerr uzayzamanı ile karşılaştırılabilir.

Anahtar Kelimeler: Kerr Karadelikleri, Kerr-tipi Karadelikler, Zar Paradigması, Johannsen-Psaltis metriği

To my beloved family and my friends.

"You shouldn’t trust the storyteller; only trust the story."

-The Sandman

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LIST OF ABBREVIATIONS

$\mathbb{R}$	Real numbers
$C^\infty(M)$	Infinitely differentiable functions
M	Manifold
Σ	Hypersurface
H	Horizon
H_s	Fake horizon
O	Topology
A	Atlas
T_pM	Tangent space of the manifold at point p
T_p^*M	Cotangent space of the manifold at point p
$\mathcal{B}$	Basis set
∇	Covariant derivative
∂	Ordinary derivative
$\vert$	3-Covariant derivative
$\vert\vert$	2-Covariant derivative
EE	Einstein's Equations
GR	General Relativity
$(\mu, \nu, \dots)$	Spacetime index
$(a, b, \dots)$	Vierbein indices
$(A, B, \dots)$	2-surface indices
$\Omega^p(M)$	Space of p-forms
Sch	Schwarzschild black hole
ModSch	Modified Schwarzschild black hole
JP	Johannsen-Psaltis black hole

RN	Reissner-Nordstrom black hole
KN	Kerr-Newman black hole
(A)dS	(Anti)-de Sitter spacetime
KNA	Kerr-Newman-(A)dS black hole
NP	Newman-Penrose formalism
NJA	Newman-Janis Algorithm
EM	Einstein-Maxwell equations
CTC	Closed Time-like Curves
EF	Eddington-Finkelstein coordinates
BL	Boyer-Lindquist coordinates
FFO	Free-Falling Observer
FIDO	Fiducial Observer
OHO	Outer Horizon Observer
GHY	Gibbons-Hawking-York Boundary Term
LIF	Locally Lorentz Frame

CHAPTER 1

INTRODUCTION

Black holes are fascinating solutions of general relativity, which excited generations of physicists. After Schwarzschild [1] found the first spherically symmetric solution of a point mass, it was understood that the solutions also allow singularities with a surface. The surface is called the " event horizon, a perfect unidirectional surface: causal influences can cross it in only one direction" [2]. It is conjectured that the universe cannot allow naked singularities, hence they must both mathematically and experimentally be dressed with horizons [3]. After a while, different solutions for black hole geometries were found. The static solutions of Einstein-Maxwell's theory gave the Reisner-Nordström black holes, which are static black holes with electromagnetic charge. In 1961, Roy Kerr discovered a solution for rotating black holes [4]. In 1965, with Schild, he rewrote the charged rotating (so-called Kerr-Newman) black hole in covariant coordinates [5]. Together with the Schwarzschild metric, it is understood that solutions are unique up to their conserved charges, mass M, charge Q, and angular momentum a. The works on black hole thermodynamics open up new ideas on the global properties of black holes. In the end, it was conjectured that these were the only properties a black hole can have, i.e. the black hole has no-hair [6]. The no-hair theorem highly constrained possible black hole solutions. This opened up a new line of questioning in quantum gravity research. For an outside observer, a black hole tends to have only 1 microstate, its mass, charge and angular momentum. However, calculating with respect to its horizon area it seems it should have $e^{10^{90}}$ microstates [7]. Different approaches were tried to solve these issues. Relativists tried to construct extended actions instead of the Einstein-Hilbert action that should represent the dynamics of the universe [8]; or they tried to induce parametric deviations that allow new hair on the black hole geometry. It is known that several types of parametric deviations are also considered to be pathological. They have naked singularities in other regions of spacetime or they endure closed time-like curves (CTCs) outside the horizon.[9]. One might expect such pathologies since a direct violation of the no-hair theorem requires such an unphysical phenomenon. These problematic structures within the parametrically deviated solutions can be advantageous. One might align the CTCs to unwanted regions so that observational tests can be directed to non-pathological regions where they intended to analyze. Johannsen-Psaltis [10] constructed a deviation from the Kerr metric which is now abbreviated as the JP metric. This metric has no pathology up to higher spin solutions with a set of parameters that clearly indicates linear parametric deviations from the Kerr metric. The JP metric has the same metric structure as the Kerr metric in Boyer-Lindquist (BL) coordinates [11] such that geodesic tests are also comparable with the Kerr Spacetime [12]. The JP metric is built so that it is regular outside the horizon, hence astrophysical tests can be aligned arbitrarily close to its horizon.

In the supergravity approach, black holes are modelled as solitonic lumps, with smooth horizonless structures which are supported by topology [13]. In 1996, Maldacena worked on black holes in string theory [14] and in the following years he introduced the AdS/CFT correspondence [15] and ever since holographic principles [16] dominated the black hole survey in theoretical physics. However, there was a different type of approach that gave astonishing approximations for the black hole probes. In 1979, Damour published a paper on black holes titled "Eddy's Currents" [17]. It was the first idea to use the black hole solutions while mimicking the horizon surface as a regular surface that encloses the black hole singularity. Later, the name for the approach was coined by Thorne and Price, in their book as "The Membrane Paradigm" [18]. The structure of the black hole membrane paradigm was introduced. Since the inside of the black hole is not available for an observer outside, it is understood that for a time-like observer outside of the horizon, there should be a surface that mimics the black hole as a "fluid bubble" [19] with electrical conductivity, shear $\&$ bulk viscosities through transport coefficients with the calculations on the surface gravity [17]. A careful calculation on the horizon led to Ohm's law, Joule's law and the non-relativistic Navier-Stokes equation [20]. Early works of Damour, followed by Thorne and Price [18], showed a way to mimic the horizon without its pathologies. They constructed a fake horizon or *streched horizon* just outside of the *true horizon* with similar properties. A fiducial observer sees it as a time-like hypersurface, a.k.a membrane, which has no pathologies in the kinds of the true horizon.[1] The *streched horizon*, as an arbitrarily close surface to the true horizon is indeed a 2+1 time-like surface. A rigorous approach to the aforementioned paradigm was built by Parikh and Wilczek [22] who considered the Gibbons-Hawking-York (GHY) boundary term, not as a term to be added to the boundary at infinity but as a boundary term on a time-like hypersurface that envelopes the horizon. This surface can be linked through the true horizon by ingoing null congruence as an injection while acting as a fluid bubble. Since it is not null by nature, it has a non-singular metric on it which brings an outstanding approximation to the behaviour of the true horizon. Recall that the null event horizon of the black hole has a degenerate metric.

This thesis is structured as follows: In the first Chapter (1), we summarized the preliminaries for a physical-geometric theory. In the second Chapter (2), we mostly focused on spacetime splittings that should help us realize the hydrodynamic correspondence induced by the membrane paradigm. Moreover, the second Chapter encapsulates the works on different spacetimes and we generalized them under one metric to rule them all. We showed a way of transforming from static spacetimes to rotating spacetimes which is called the Newman-Janis Algorithm [23] and we gave the specifics on the JP spacetime and its possible generalization. In the third Chapter (3), we use the full machinery of the membrane paradigm and found its results for different types of black holes. The last Chapter contains: (un)-charged static black holes with (non)-zero cosmological constant,(un)-charged rotating black holes with (non)-zero cosmological constant and their parametrically deviated versions. We showed their transport coefficients and checked their forms while testing their limiting values of vanishing global charges to see if the web of metrics is transparent to transition from one to another. We also added an Appendix (A) specific to the Johannsen-Psaltsi spacetime. In this Appendix, we look at its double-copy structure. The double copy phenomenon is basically "square rooting" a known gravity solution in the Kerr-

[1] This new bubble is in analogy with one of the early works of electrodynamics the "method of images"[21].

Schild form to produce a gauge field of a corresponding self-dual Yang-Mills theory. The Appendix aims to find single-copy and corresponding parametric corrections of a parametrically deviated metric proposed by Johannsen-Psaltis [24]. In the second Appendix (B), we investigated the membrane paradigm's results in the presence of a background spacetime. The background spacetime was added by hand to produce the Einstein-Hawking-Horowitz theory on the stretched horizon stress tensor. To have the dynamics of the background spacetime and main spacetime compatible we applied our paradigm in the vicinity of Schwarzschild-(anti)-de Sitter as main spacetime and (anti)-de Sitter spacetime as background spacetime.

1.1 Vectors on a Manifold

Let us assume we have a manifold $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m)$ with a smooth assignment of curves γ, $C^\infty(\mathcal{M})$ functions f and coordinate charts ϕ. One can then construct tangent vectors on a point p of the manifold $\mathcal{M}$. Here, $\mathcal{O}_m$ denotes the topology and the $\mathcal{A}_m$ is the maximal atlas.

We need a curve $\gamma : \mathbb{R} \longrightarrow \mathcal{M}$ to define tangent vectors. One can parametrize the curve as a domain of an open set (a, b) with parameter t.

Then, the tangent vector at p is a directional derivative of a function f with argument $\gamma(t)$ at $t = 0 \in (a, b)$:

$$\frac{df(\gamma(t))}{dt}\Big|_{t=0} = \frac{\partial f}{\partial x^\mu}\frac{dx^\mu(\gamma(t))}{dt}\Big|_{t=0}. \tag{1.1}$$

The right-hand side of the equality is the coordinate-induced notation of the directional derivative. We can define the differential operator χ:

$$\chi = \chi^\mu \frac{\partial}{\partial x^\mu} \equiv \chi^\mu \partial_\mu. \tag{1.2}$$

Hence, one can write $\chi[f] \equiv \chi^\mu \frac{\partial f}{\partial x^\mu}$.

Furthermore, an equivalence class of curves that pass point p under parametrization $\gamma(0) = p$ can be defined such that all tangent vectors at p are said to live in the tangent space T_pM [25].

A chart-induced basis of $T_p(U \subseteq M)$ is the directional derivatives at the point $p \in M$. Let $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m)$ be a m-dimensional differentiable manifold. A chart-induced basis is a complete set:

$$\mathcal{B}(\chi) \equiv \left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^m} \right\}$$

which spans the whole tangent space such that dim(T_pM)=dim(M)=m. A vector is an abstract notion. It is defined as the element of the vector space. How we should represent the vector in different patches of $\mathbb{R}^d$ should not change the vector itself.

Let us take two coordinate patches (U, x) and (V, y) on $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m)$ such that $p \in (U \cap V)$. An element $\chi \in T_pM$ can be represented in two different charts:

$$\chi = \chi^{\mu}_{(x)} \frac{\partial}{\partial x^{\mu}}\Big|_p, \qquad \chi = \chi^{\mu}_{(y)} \frac{\partial}{\partial y^{\mu}}\Big|_p.$$

Then a smooth transformation between charts can be found as:

$$\chi^{\mu}_{(y)} = \chi^{\nu}_{(x)} \frac{\partial y^{\mu}}{\partial x^{\nu}}. \tag{1.3}$$

which is the rule of how the components of a vector change under coordinate changes.

1.2 Covectors on a Manifold

One can construct a map that takes an element from the vector space T_pM, $p \in M$ and gives an element of $\mathbb{R}$. The space of all such maps constitutes T_p^*M and is said to be the cotangent vector space or dual space of T_pM [25].

$$T_p^*M \ni \omega : T_pM \longrightarrow \mathbb{R}.$$

The action of ω on χ can be understood as a contraction

$$\langle , \rangle = T_p^*M \otimes T_pM \longrightarrow \mathbb{R},$$

$$\langle \omega, \chi \rangle = \chi^{\mu} \langle \omega, \frac{\partial}{\partial x^{\mu}} \rangle. \tag{1.4}$$

If ω is expressed in a chart-induced basis one-forms

$$\chi^{\mu} \langle \omega_{\nu} dx^{\nu}, \frac{\partial}{\partial x^{\mu}} \rangle = \chi^{\mu} \omega_{\nu} \langle dx^{\nu}, \frac{\partial}{\partial x^{\mu}} \rangle,$$

$$\chi^{\mu} \omega_{\nu} \delta^{\nu}_{\mu} = \chi^{\mu} \omega_{\mu} \in \mathbb{R}.$$

This concludes that a cotangent space can be spanned by basis one-forms $\Big\{dx^{\mu}\Big\}$. Like vectors, a change in the coordinate basis should not affect the covector itself. For (U, x),$(V, y)$$\in M$ when $p \in (U \cap V)$

$$\omega = \omega^{(x)}_{\mu} dx^{\mu}\Big|_{p} = \omega^{(y)}_{\nu} dx^{\nu}\Big|_{p},$$
$$\omega^{(x)}_{\nu} = \omega^{(y)}_{\mu} \frac{dx^{\mu}}{dy^{\nu}}. \tag{1.5}$$

where (1.5) shows how one-forms transform under a change of coordinate charts.

1.3 Tensors

A tensor is a multilinear map which is a generalization of vectors and covectors [26]. It takes m # of covectors and n # of vectors and maps them to $\mathbb{R}$ multilinearly. These objects are said to be tensors of type (m, n) and abbreviated as $\mathbf{T}^{m}_{(n),p}(M)$ where $p \in M$. On a chart-induced basis, the geometric object T reads as

$$T = T^{\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} \frac{\partial}{\partial x^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{\mu_m}} dx^{\nu_1} \otimes \cdots \otimes dx^{\nu_n}\Big|_{p}.$$

From one chart to another, a tensor transforms like the products of vectors and covectors simultaneously.

$$T^{(y)\mu_1 \dots \mu_m}{}_{\nu_1 \dots \nu_n} = \frac{\partial y^{\mu_1}}{\partial x^{\alpha_1}} \otimes \cdots \otimes \frac{\partial y^{\mu_n}}{\partial x^{\alpha_m}} \frac{\partial x^{\beta_1}}{\partial y^{\nu_1}} \otimes \cdots \otimes \frac{\partial x^{\beta_n}}{\partial y^{\nu_n}} T^{(x)\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}. \tag{1.6}$$

1.4 Connections

A connection on $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m)$ is a map that takes two vector fields and gives a new vector field. Its action on functions is identical to the directional derivative's action. It is a $C^{\infty}(\mathcal{M})$ operation which obeys the Leibniz Rule. In Einstein's Gravity, the connection itself is induced by the properties of the metric manifold i.e. the Levi-Civita Connection, but in general, it is not the case [27].

An affine manifold is a manifold with ∇, such that it is a quartet $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m, \nabla)$.

Let us investigate the action of a vector field χ on Y in coordinate charts:

$$\nabla_\chi Y = \nabla_{\chi^\mu \frac{\partial}{\partial x^\mu}}(Y^\nu \frac{\partial}{\partial x^\nu}) = \chi^\mu \nabla_{\frac{\partial}{\partial x^\mu}}(Y^\nu \frac{\partial}{\partial x^\nu})$$
$$= \chi^\mu \Big((\nabla_{\frac{\partial}{\partial x^\mu}} Y^\nu)\frac{\partial}{\partial x^\nu} + Y^\nu (\nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu}) \Big).$$

In total, $\nabla_\chi Y$ is a vector field hence $(\nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu})$ must be a vector field, too. We define the connection coefficient Γ as:

$$(\nabla_{\frac{\partial}{\partial x^\mu}} \frac{\partial}{\partial x^\nu}) = \Gamma^\alpha_{\mu\nu} \frac{\partial}{\partial x^\alpha}. \tag{1.7}$$

We will find the connection coefficient function $\Gamma^\alpha_{\mu\nu}$ of the covariant derivative along vector field χ.

Similarly, if one contracts a vector with a covector and acts on it by ∇, upon a bit of manipulation, it is easy to see the following [28]:

$$\nabla_{\frac{\partial}{\partial x^\mu}} dx^\nu = -\Gamma^\nu_{\mu\alpha} dx^\alpha. \tag{1.8}$$

Until now, we defined Γ in the neighbourhood of some coordinate chart (U, x), one should consider other local charts to check the compatibility. such that we have a $C^\infty(\mathcal{M})$-covering of that manifold [25].

$(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m, \nabla)$ with two compatible charts $(U, x), (V, y)$ such that $U \cap V \neq \{\}$ then the transformation between the connection coefficient functions are [25]:

$$\Gamma^\alpha{}_{(y)\mu\nu} = \frac{\partial y^\alpha}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial y^\mu \partial y^\nu} + \frac{\partial y^\alpha}{\partial x^\beta}\frac{\partial x^\gamma}{\partial y^\mu}\frac{\partial x^\xi}{\partial y^\nu}\Gamma^\beta{}_{(x)\gamma\xi}. \tag{1.9}$$

As one can realize, (1.9) has an additional term $\frac{\partial y^\alpha}{\partial x^\beta}\frac{\partial^2 x^\beta}{\partial y^\mu \partial y^\nu}$. It prevents Γ's to be a tensorial object. Let's say, $\Gamma^\alpha_{(x)\mu\nu} = 0$ in one specific chart, it shows that there might be some chart such that $\Gamma^\alpha_{(y)\mu\nu} \neq 0$ since the residue term survives.

Even though Γ's are not tensorial objects, we will see their derivatives and anti-symmetrization yield tensorial objects. One can anti-symmetrize the connection co-efficient function to find the torsion tensor on M.

$$T^\alpha{}_{\mu\nu} \equiv \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu},$$

so if the torsion tensor is zero or $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$, it means that the connection coefficients are symmetric in lower indices. Torsion-freeness of a connection is one of the requirements for a Levi-Civita connection which is the case in GR [27].

1.4.1 Curvature and Parallel Transport

Let us use the affine connection definition to bring up the parallel transportation of two vector fields with respect to each other. Given a manifold $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m, \nabla)$, a curve $\gamma : \mathbb{R} \longrightarrow \mathcal{M}$ one can define a vector field χ along $\gamma(t)$ then the auto parallel transportation along χ itself gives a relation:

$$\nabla_{\chi}\chi = \frac{d}{dt}x^{\mu} + \Gamma^{\mu}_{\alpha\beta}\frac{dx^{\alpha}}{dt}\frac{dx^{\beta}}{dt} = 0. \tag{1.10}$$

(1.10) has a deep meaning for physics as well as mathematics. From a mathematical perspective, this equation shows ∇ is important to realize the intrinsic curvature of the manifold. From a physical perspective, If we choose χ as the velocity field then this equation should be equal to "acceleration", i.e. $a_{\chi} = \nabla_{\chi}\chi$. This equation shows the straightest possible path a vector with respect to itself can follow in $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m, \nabla)$. However, we still do not have the notion of length in our manifold. If this was a metric manifold, as we will introduce soon, the shortest path $\cong$ straightest path iff $a_{\chi} = 0$. That's why in Einstein's Gravity free falling object which follows the shortest and straightest path does not feel any acceleration. This also corresponds to a geometrical realization of Newton's First Law of Motion [27].

Since we are now closer to defining the curvature and the Einstein tensors, let us change our notation into a physicist's notation without losing any generality.

$$\nabla_{\frac{\partial}{\partial x^{\mu}}} \equiv \nabla_{\mu}, \qquad\qquad \frac{\partial}{\partial x^{\mu}} \equiv \partial_{\mu}.$$

Even though we gave hints about the need for a metric connection in GR, we have not yet introduced a metric connection with reasonable assumptions:

- On the differentiable manifold $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m, \nabla)$, one can add a metric structure such that it is a non-singular $(0, 2)$ tensor field g [29]:

$$g : T_p\mathcal{M} \otimes T_p\mathcal{M} \longrightarrow \mathbb{R}.$$

On the coordinate-induced basis, one can write this as follows:

$$g = g_{\mu\nu} dx^\mu dx^\nu,$$

the metric can be used to find the lengths of $\chi \in T_p\mathcal{M}$ as

$$g(\chi, \chi) = g_{\mu\nu} \chi^\mu \chi^\nu.$$

Also, it gives the line element as the magnitudes of infinitesimal displacement vectors $\vec{dl}|_p$,

$$ds^2 = g(\vec{dl}, \vec{dl}) = g_{\mu\nu} dx^\mu dx^\nu,$$

The metric tensor by definition also has an inverse.

$$g^{-1} = g^{\mu\nu} \partial_\mu \partial_\nu.$$

The metric needs two vectors to give an element of real numbers. However, one can go around that definition to realize an isomorphism between tangent and cotangent spaces. It is called the *musical isomorphism*.

$$g : T_p\mathcal{M} \longrightarrow T_p^*\mathcal{M} \iff \chi \mapsto g(\chi, \cdot) \equiv \chi,$$

or inversely,

$$g^{-1} : T_p^*\mathcal{M} \longrightarrow T_p\mathcal{M} \iff \omega \mapsto g^{-1}(\omega, \cdot) \equiv \chi.$$

In physical terms, musical isomorphism is just the raising and lowering of the index of the generic tensors. General Relativity must be reduced to Special Relativity in the vicinity of a point $p \in \mathcal{M}$ as a requirement of physical consistency. The symmetric $(0, 2)$ tensor should be diagonalized at each point $p \in \mathcal{M}$ while allowing the below transformation.

$$g_{\mu'\nu'}|_p = \Lambda^\alpha{}_{\mu'} \Lambda^\beta{}_{\nu'} g_{\alpha\beta}|_p, \tag{1.11}$$

where $\Lambda^\alpha{}_{\mu'}$ is an $n \times n$ invertible matrix. In Lorentizian manifolds [2], one can diagonalize the metric tensor as follows:

$$g_{\mu'\nu'}|_p = \eta_{\mu'\nu'} = \text{diag}(-1, 1, \ldots, 1)$$

[2] These are the metric manifolds with at least one negative eigenvalue.

This is possible since an $n\times n$ symmetric matrix has $\frac{n(n+1)}{2}$ independent components while a generic invertible matrix has n^2 components. In total, the number of independent components can be found as:

$$n^2-\frac{n(n+1)}{2}=(n-1)+\frac{(n-1)(n-1)}{2}.$$

These independent components refer to symmetry transformations of spacetime. From physics perspective we should have $(n-1)$ number of boosts and $\frac{(n-1)(n-1)}{2}$ number of rotations. These are the Lorentz transformations of the restricted Lorentz Group $SO(1,n)^+$ in the vicinity of a point p.

It should be understood that the transformation $\Lambda^{\alpha}{}_{\mu'}$ need not to be a coordinate transformation $\Lambda^{\alpha}{}_{\mu'}=\frac{\partial x^{\alpha}}{\partial x^{\mu'}}$. The integrability condition of a coordinate transformation requires:

$$\partial_{[\sigma'}\Lambda^{\alpha}{}_{\mu']}=0,$$

but this might not hold for a generic transformation matrix $\Lambda^{\alpha}{}_{\mu}\in GL(1,n)$. The local flatness construction for a Riemannian manifold with a metric compatible connection (i.e. $\nabla_{\chi}g=0\implies\nabla_{\alpha}g_{\mu\nu}=0$) together with these specific coordinates called Locally Lorentz Frame (LIF) gives:

$$g_{\mu'\nu'}|_p=\eta_{\mu'\nu'}\implies\nabla_{\sigma'}g_{\mu'\nu'}|_p=\nabla_{\sigma'}\eta_{\mu'\nu'}=\partial_{\sigma'}\eta_{\mu'\nu'}=0,$$

This yields,

$$\partial_{\sigma'}g_{\mu'\nu'}|_p=0,$$

but

$$\partial_{\alpha'}\partial_{\sigma'}g_{\mu'\nu'}|_p\neq 0.$$

This requirement can be justified by counting the total degrees of freedom that are fixed by transformations. For $\Lambda^{\mu'}{}_{\alpha}=\frac{\partial x^{\mu'}}{\partial x^{\alpha}}$ has more than enough parameters to diagonalize $g_{\mu\nu}$. To make $\partial_{\sigma'}g_{\mu'\nu'}|_p=0$ we need $\frac{n^2(n+1)}{2}$ parameters which is fixed by the object $\frac{\partial x^{\alpha}}{\partial x^{\mu'}\partial x^{\nu'}}$ that has $\frac{n^2(n+1)}{2}$ parameters.

In the next order, we see the second derivative of the metric tensor $\partial_{\alpha'}\partial_{\sigma'}g_{\mu'\nu'}$ has $\frac{n^2(n+1)^2}{2}$ parameters which should be fixed by $\frac{\partial x^{\alpha}}{\partial x^{\mu'}\partial x^{\nu'}\partial x^{\lambda'}}$ which has $\frac{1}{3!}n^2(n+1)(n+2)$ independent components. Under transformations, the components cannot be totally fixed. Moreover,

$$\frac{n^2(n+1)^2}{4}-\frac{1}{6}n^2(n+1)(n+2)=\frac{1}{12}n^2(n^2-1),$$

which has exactly the same number of independent components as the Riemann Tensor. In LLF, metric compatibility must arise. Since the metric must be locally flat in GR,

$$\partial_{\alpha'} g_{\mu'\nu'}|_p = 0 \iff \Gamma^{\lambda'}_{\mu'\nu'} = 0,$$

which can be written in a covariant way since in physics all coordinate systems should be equally valid.

$$\nabla_{\alpha'} g_{\mu'\nu'} = 0.$$

Consider two vector fields $\chi = \partial_\mu$, $Y = \partial_\nu$ with a metric $g : T_pM \otimes T_pM \longrightarrow \mathbb{R}$, then:

$$\nabla_\alpha g(\partial_\mu, \partial_\nu) = 0,$$
$$= (\nabla_\alpha g)(\partial_\mu, \partial_\nu) + g(\nabla_\alpha \partial_\mu, \partial_\nu) + g(\partial_\mu, \nabla_\alpha \partial_\nu).$$

$$0 = (\nabla_\alpha g)_{\mu\nu} = \partial_\alpha g_{\mu\nu} - \Gamma^\sigma{}_{\nu\alpha} g_{\sigma\mu} - \Gamma^\sigma{}_{\mu\alpha} g_{\sigma\nu}, \tag{1.12}$$

if we write down the cyclic permutations of (1.12) and take its linear combination:

$$0 = -\partial_\alpha g_{\mu\nu} + \partial_\mu g_{\alpha\nu} + \partial_\nu g_{\mu\alpha} + T^\sigma{}_{\alpha\mu} g_{\sigma\nu} + T^\sigma{}_{\alpha\nu} g_{\sigma\mu} - 2\Gamma^\sigma{}_{(\mu\nu)} g_{\sigma\alpha},$$

where $T^\sigma{}_{\alpha\nu}$ is the torsion tensor. By contracting the metric with itself, one can decompose metric connection coefficients in symmetric and anti-symmetric parts:

$$\Gamma^\sigma{}_{(\mu\nu)} + \Gamma^\sigma{}_{[\mu\nu]} = \frac{1}{2} g^{\sigma\alpha} \Big(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\nu\mu} \Big) + \frac{1}{2} \Big(T_{\nu}{}^{\sigma}{}_{\mu} + T_{\mu}{}^{\sigma}{}_{\nu} + T^\sigma{}_{\mu\nu} \Big).$$

There is a special kind of connection that can reveal up to a constraint. It is demanding $T_{\mu\nu}{}^\sigma = 0$, a torsion-free metric connection which is called the Levi-Civita connection. It is an a priori assumption of Einstein's Gravity [30]. That means, in Einstein's Gravity the metric of the manifold itself designates how one can transport a vector intrinsically [27]. Using the covariant derivative along the manifold gives us its intrinsic properties [28].

Let us define the Riemann Curvature Tensor $Riem$, as a multi-linear map such that $Riem : T_pM \otimes T_pM \otimes T_pM \longrightarrow T_pM$.

On a non-coordinate basis, we define a map

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} = R(X,Y)Z. \tag{1.13}$$

On coordinate induced basis its coefficients read

$$\begin{aligned} R^{\alpha}{}_{\beta\mu\nu} &= \langle dx^{\alpha}, R(\partial_{\mu}, \partial_{\nu})\partial_{\beta} \rangle, \\ &= \Big\langle dx^{\alpha}, \nabla_{\mu}(\Gamma^{\sigma}{}_{\nu\beta}\partial_{\sigma}) - \nabla_{\nu}(\Gamma^{\sigma}{}_{\mu\beta}\partial_{\sigma}) \Big\rangle, \\ &= \partial_{\mu}\Gamma^{\alpha}{}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\sigma}{}_{\nu\beta}\Gamma^{\alpha}{}_{\mu\sigma} - \Gamma^{\sigma}{}_{\mu\beta}\Gamma^{\alpha}{}_{\nu\sigma}. \end{aligned} \tag{1.14}$$

1.5 Gravity

Since in the previous section, we dedicated ourselves to explaining the mathematics of general relativity, we are ready to define the spacetimes as a geometric object:

"A four-dimensional *differentiable manifold* with a *smooth atlas* equipped with a *Levi-Civita Connection* with a time-oriented *Lorentzian Metric* is called spacetime if and only if it satisfies *Einstein's Equations*" [27].

For this section, our purpose is to define both sides of Einstein's equations (EE). It can be defined heuristically or by using Hilbert's way. In the next sections, we will talk about constant time foliations of a given spacetime. In this section, it is wise to follow Hilbert's way, meaning " The Lagrangian Formulation of GR" [28].

Firstly, one has to construct a Lagrangian, it must be a Lorentz invariant scalar and upon variation, it must give the correct Einstein's equations. Let's say we are given the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$ its trace with first and third indices will give Ricci tensor.

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}. \tag{1.15}$$

Ricci tensor can be contracted with the metric, which results in the Ricci Scalar:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.16}$$

The easiest, linear action one can construct is to take this Ricci scalar as a Lagrangian density. By following the footsteps of Hilbert one can construct an action. One might guess that the theory inherits the Levi-Civita connection so that every term in the field equations of the action can be derived from the metric itself, as in the standard General Relativity. This connection also gives a torsionless [25], symmetric Γ's; (i.e, $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$).

Up to now, we just dealt with the gravity field terms and rather ignored the matter terms in the field equations. The full action of the theory is given together with the gravity and matter fields

$$\mathcal{S} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}R + \mathcal{S}_M(g_{\mu\nu}, \psi), \tag{1.17}$$

where ψs are the matter fields.

Let's start with gravity field terms, and do a variation of the metric

$$\delta\mathcal{S}_{met} = \frac{1}{2\kappa}\int d^4x\big(\delta(\sqrt{-g})R + \sqrt{-g}\delta R\big),$$

$$= \frac{1}{2\kappa}\int d^4x\sqrt{-g}(-\frac{1}{2}g_{\mu\nu}R\delta g^{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}). \tag{1.18}$$

The metric variation on the determinant of the metric is:

$$\delta\sqrt{-g} = \frac{-1}{2\sqrt{-g}}g g_{\mu\nu}\delta g^{\mu\nu}. \tag{1.19}$$

Then we have,

$$\delta_g S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu})\delta g^{\mu\nu} + \frac{1}{2\kappa}\int d^4x\sqrt{-g}(g^{\mu\nu}\delta R_{\mu\nu}). \tag{1.20}$$

The first part of the variation gathered nicely, however, the second part is more tedious. By definition, we know that tensors are not changing their forms with respect to different coordinate systems. A result expressed in one coordinate system should generalize in all coordinate systems [28].

For the variation of $R_{\mu\nu}$ if we pick normal coordinates such that whole Γ's will vanish then we have $\partial_\mu \equiv \nabla_\mu$. Let us restate Ricci tensor $R^{\alpha}{}_{\mu\alpha\nu}$:

$$R^{\alpha}{}_{\mu\alpha\nu} = \partial_\mu\Gamma^{\alpha}{}_{\nu\beta} - \partial_\nu\Gamma^{\alpha}{}_{\mu\beta} + \Gamma^{\sigma}{}_{\nu\beta}\Gamma^{\alpha}{}_{\mu\sigma} - \Gamma^{\sigma}{}_{\mu\beta}\Gamma^{\alpha}{}_{\nu\sigma}. \tag{1.21}$$

Since we are using a torsion-free connection

$$T^{\tau}{}_{\rho\nu} = 2\Gamma^{\tau}{}_{[\rho\nu]} = 0.$$

In normal coordinates, Γ's vanish but there is no reason for their derivatives also to vanish [26]. In fact, a normal coordinate system is a definition of a locally flat chart. If derivatives of connection coefficients vanish, it means that the spacetime is globally flat in all other coordinate systems.

$$R_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\rho\mu}, \tag{1.22}$$

$$\delta R_{\mu\nu} = \partial_\rho \delta\Gamma^\rho_{\mu\nu} - \partial_\nu \delta\Gamma^\rho_{\rho\mu}, \tag{1.23}$$

where $[\partial_\alpha, \delta] = 0$. Even though Γ's are functions, their variations are tensors. Hence we need to upgrade the partial derivatives into covariant derivatives

$$\delta R_{\mu\nu} = \nabla_\rho \delta\Gamma^\rho_{\mu\nu} - \nabla_\nu \delta\Gamma^\rho_{\rho\mu}. \tag{1.24}$$

This expression is sometimes called the Palatini identity. The second part of (1.20) reads as:

$$\sqrt{-g}g^{\mu\nu}\delta R_{\mu\nu} = \sqrt{-g}\Big\{\nabla_\alpha(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^\alpha_{\mu\alpha})\Big\}, \tag{1.25}$$

$$= \sqrt{-g}\Big\{\nabla_\sigma[g^{\mu\nu}\delta^\sigma{}_\alpha\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\nu}\delta^\sigma{}_\nu\delta\Gamma^\alpha_{\mu\alpha}]\Big\}, \tag{1.26}$$

where the metricity condition is applied in (1.25). It can be seen as a total derivative. Let us call the term (1.26) as J^μ,

$$J^\sigma = \nabla_\sigma[g^{\mu\nu}\delta^\sigma{}_\alpha\delta\Gamma^\alpha_{\mu\nu} - g^{\mu\nu}\delta^\sigma{}_\nu\delta\Gamma^\alpha_{\mu\alpha}].$$

We can clearly see the total derivative structure in the action:

$$\frac{1}{2\kappa}\int d^4x\sqrt{-g}(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu})\delta g^{\mu\nu} + \frac{1}{2\kappa}\int d^4x\partial_\mu(\sqrt{-g}J^\mu), \tag{1.27}$$

where $\partial_\mu(\sqrt{-g}J^\mu) = \sqrt{-g}\nabla_\mu J^\mu$. The total derivative allows us to apply Stokes' theorem

$$\frac{1}{2\kappa}\int_{\mathcal{M}} d^4x\sqrt{-g}(-\frac{1}{2}g_{\mu\nu}R + R_{\mu\nu})\delta g^{\mu\nu} + \frac{1}{2\kappa}\int_{\partial\mathcal{M}} d^3y\varepsilon\sqrt{|h|}n_\mu J^\mu, \tag{1.28}$$

where n_μ is the normal vector to the hypersurface. The Lorentzian signature of the hypersurface is determined by $\varepsilon = \pm 1$.[3] Even though it looks like this action is not

[3] If a spacetime has a black hole, the cosmic-censorship conjecture [3] creates a null-hypersurface as a horizon. In that sense, doing calculations on null surfaces as boundary terms create problems. We will try to get around this problem with the Membrane Paradigm [20] in the next sections.

stabilized because of the surface term, one can add an external Lagrangian which can cancel the boundary contribution upon variation. [4] When the boundary action is cancelled out, the remaining bulk term will correspond to the field equations. This means, if General Relativity is a genuine model of gravity then the field equations should be:

$$\boxed{G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu},} \tag{1.29}$$

where $G_{\mu\nu}$ is the so-called Einstein Tensor. The variations of the matter sector with respect to the metric yield a $(0,2)$ symmetric tensor field given on

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}} \frac{\delta \mathcal{S}_M}{\delta g^{\mu\nu}}. \tag{1.30}$$

Hence the total field equation will stay covariantly constant:

$$\nabla^\mu G_{\mu\nu} = \kappa \nabla^\mu T_{\mu\nu} = 0.$$

To understand the outcomes of (1.29) better, one might take the trace of this equation to find the structural equation as:

$$\boxed{-R = \kappa T} \tag{1.31}$$

where $g^{\mu\nu} g_{\mu\nu} = 4$ in 4 dimensions so that one can work out the trace of (1.29).
In General Relativity, the trace equation gives us a better look at the nature of the theory itself. Firstly, the trace equation depends on R and T algebraically. It leads that General Relativity admits a unique flat solution for maximally symmetric vacuum spacetimes. So, for maximally symmetric spacetimes as (R: constant while $T_{\mu\nu} = 0$) (2.1) reduces to $R = 0$ then one has Minkowski spacetime Moreover, since the matter is minimally coupled to metric one can show the diffeomorphic invariance i.e, the energy conservation linked by Levi-Civita connection as $\nabla_\mu T^{\mu\nu} = 0$ [32].

Note that we could also add a cosmological constant to the action by just shifting the Hilbert action as $R \longrightarrow R - 2\Lambda$, which yields the field equations as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{1.32}$$

This theory is called the cosmological Einstein's gravity.

[4] This type of Lagrangians is generically called Gibbons-Hawking-York boundary terms [31].

CHAPTER 2

PRELIMINARY DISCUSSIONS: A BRIEF REVIEW

2.1 Review of 3+1 Splitting of Spacetime

Up to now, we have talked about the construction of GR with the coordinate basis formalism. However, in the language of differential forms and Cartan's structure equations, gravity exhibits itself more elegantly [25]. The structure of this section will include the geometry of vierbeins and the 3+1 splitting of spacetime via cleverly chosen foliated vierbeins.

2.1.1 Non-Coordinate Basis

In contrast to the coordinate basis, we choose our basis in such a way that the metric becomes flat, the Lorentzian metric. In that sense, we are actually ortho-normalizing our basis vectors. To fix our notation we will use $\{a, b, c, d \dots\}$ for vierbein indices and $\{\mu, \nu, \gamma, \lambda \dots\}$ for spacetime indices.

One can remember that, in the coordinate basis, $\mathcal{B}(\{\partial_\mu\})$ spans T_pM and $\mathcal{B}(\{dx^\mu\})$ spans T_p^*M of the manifold $(\mathcal{M}, \mathcal{O}_m, \mathcal{A}_m, \nabla, g)$. A clever choice of basis can be:

$$e_a = e_a{}^\mu \partial_\mu,$$

where $\det(e_a{}^\mu > 0)$ In this sense,

$$g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab},$$

where $e^\mu{}_a e_\mu{}^b = \eta^b{}_a$. As one can remember, the vector is an abstract object that exists independently from the chosen basis on the chart of the manifold.[27]

In Coordinate Basis: $$V = V^\mu \partial_\mu,$$

In non-Coordinate Basis: $$V = V^a {e_a}^\mu \partial_\mu,$$

such that the components of the vectors transform with the vielbeins as:

$$V^\mu = V^a {e_a}^\mu, \qquad V^a = {e^a}_\mu V^\mu.$$

The $\mathcal{B}(\{e_a\}) \in (T_pM)$ is a valid basis on tangent vector space at $p \in M$. As happened in previous sections, one can always find a dual basis for T_pM. Let some $\omega \in T_p^*M$ such that $\omega : T_pM \longrightarrow \mathbb{R}$

$$\begin{aligned} \langle \omega, V \rangle &= \langle \omega, e_a e^a_\mu V^\mu \rangle, \\ &= e^a_\mu V^\mu \langle \omega, e_a \rangle. \end{aligned}$$

Let $\omega = \omega_\mu {e^\mu}_a \theta^a$ such that $\langle \theta^a \, , \, e_b \rangle = {\delta^a}_b$. Hence, the dual basis, T_p^*M is spanned by $\theta^a = {e^a}_\mu dx^\mu$. This means that one can redefine metric tensor as:

$$g = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} \theta^a \otimes \theta^b.$$

Furthermore, by using this basis one can write down Cartan's structure equations and find Riemann and Torsion tensors in a more elegant way. Firstly, one can define the connection coefficients as [28]:

$$\nabla_a e_b \equiv {\Gamma^c}_{ba} e_c.$$

If we expand this relation:

$${e_a}^\mu (\partial_\mu {e_b}^\lambda + {e_b}^\lambda {\Gamma^\nu}_{\mu\lambda}) e_\mu = {\Gamma^c}_{ba} {e_c}^\nu e_\nu,$$

such that

$${\Gamma^c}_{ba} = {e^c}_\nu e_{a\mu} (\partial_\mu {e_b}^\nu + {e_b}^\lambda {\Gamma^\nu}_{\mu\lambda}) = {e^c}_\nu {e_a}^\mu \nabla_\mu {e_b}^\nu$$

This equation corresponds to connection coefficients on a non-coordinate basis. Now, one can define matrix-valued one form by contracting the connection coefficient in a non-coordinate basis with basis one-form:[25]

$${\omega^a}_b = {\Gamma^a}_{bc} \theta^c.$$

This prepares us to declare Cartan's first and second structure equations:

$$d\theta^a + \omega^a{}_b \wedge \theta^b = \frac{1}{2} T^a{}_{bc} \theta^b \wedge \theta^c, \tag{2.1}$$

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} \theta^c \wedge \theta^d. \tag{2.2}$$

where $d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ such that $\Omega^p(M)$ is the space of p-forms. The operator d is called the exterior derivative. The exterior product $\wedge$ is an anti-symmetric product of p-forms. T^a is a torsion two-form and $R^a{}_b$ is the curvature 2-form.

Promoting Einstein's gravity should lead us to Levi-Civita Connection, in this formalism metric compatibility and torsion-free properties of the connection reveal themselves as [25]:

- $T^a = 0$ (torsion-free).
- $\Gamma_{abc} = \eta_{dc} e^d{}_\nu e_a{}^\mu \nabla_\mu e_b{}^\nu = -\eta_{dc} e_{b\nu} e_a{}^\mu \nabla_\mu e^d{}_c = -\Gamma_{abc}$ (metric compatibility) these two assumptions constraint matrix-valued one forms:
- $\omega_{ab} = -\omega_{ba}$,
- $d\theta^a + \omega^a{}_b \wedge \theta^b = 0$.

2.1.2 3+1 Splitting of General Relativity

While we are getting closer to the Membrane Paradigm of black holes, we should understand spacetime foliations better and drift away from 4D pictures and choose some useful foliations to split space and time, as well as the tensors on the manifold.

As we gave the intuition for the importance of differential geometry for gravity, we started to use words *spacetime* and *manifold* interchangeably. One should not forget that splitting space with constant time foliations is not canonical but there is an obvious choice for static spacetime and a judicious choice for rotating spacetime [33].

Let our spacetime $\mathcal{M}$ with dim($\mathcal{M}$) $= 4$ and sign($\mathcal{M}$) $= 2$ has a constant hypersurface Σ with dim(Σ) $= 3$, sign(Σ) $= 3$, this hypersurface can be thought as an immersion Φ such that $\Phi : \tilde{\Sigma} \longrightarrow M$. [1]

$$\forall \gamma : \mathbb{R} \longrightarrow \tilde{\Sigma} \ \exists \ \Phi : \gamma \longrightarrow \Phi(\gamma) \in \Sigma_t.$$

Also, one can define a push-forward mapping Φ_* such that

$$\Phi_* : T_p(\tilde{\Sigma}_t) \longrightarrow T_p M$$
$$\chi = (\chi^x, \chi^y, \chi^z) \mapsto \Phi_* \chi = (0, \chi^x, \chi^y, \chi^z)$$

and its pull-back Φ^*, allows us to find the first fundamental form of Σ_t or the induced metric on Σ_t.

This embedding can be stated as $g_{\mathcal{M}} = \Phi^* g_{\Sigma_t}$ such that $g_{\Sigma_t} = h$ or in component form:

$$g^{(\Sigma)}_{\mu\nu} = g^{(\Sigma)}_{\alpha\beta} \frac{\partial \Phi^\alpha}{\partial x^\mu} \frac{\partial \Phi^\beta}{\partial x^\nu},$$

where $\Phi : x^i \mapsto \Phi^\mu(x^i)$. This form of the metric on the hypersurface is called the first fundamental form on Σ_t.

One can argue that with respect to the spacetime $\mathcal{M}$, the variations on Σ_t can be extrinsically understood. In this sense, defining a normal vector to Σ_t is customary.

If Σ_t=constant, then its gradient with respect to the field t should give the normal vector. $\nabla_\mu t$ such that its dual dt manifests as gradient one-form. One should understand that constant time slices mean there is a scalar field t acting as an equipotential surface such that $\forall t$ is manifestly constant [33].

Since $\nabla_\mu t$ is a normal vector on Σ_t, if Σ_t is space-like, $\nabla_\mu t$ must be time-like to be an orthogonal vector or vice versa. [2] One can also find the unit normal to Σ_t as:

$$\hat{n}_\mu = \frac{\nabla_\mu t}{\sqrt{-\nabla^\nu t \nabla_\nu t}}. \tag{2.3}$$

Furthermore, the norm of (2.3) should be timelike:

$$\langle n_\mu dx^\mu, n^\nu \partial_\nu \rangle = n_\mu n^\nu \delta^\mu{}_\nu$$
$$n^\mu n_\mu = -1$$
$$n_\mu dx^\mu = -\alpha dt$$

[1] Since this is an immersion, we are safe from intersections of foliations. This is also a diffeomorphism.

[2] Something winks at the "Membrane Paradigm" from this realization is if Σ_t is null then $\partial_\mu t$ is also null which makes the metric h on Σ_t degenerate. Since the horizon is a null hypersurface, it has a degenerate metric. Hence, it is clever to mimic the effects of the horizon from outside with a "fake horizon" to capture the physics

such that α is a $C^{\infty}(\mathcal{M})$ scalar field. It is called the *lapse function*.

Since the unit normal vector, $\hat{n}$ has a non-trivial component α, we define a new vector $m = \alpha\hat{n}$ as *normal evolution vector*. This is a special vector since the *Lie dragging* of Σ_t along m does not affect the elements of $T_p(\Sigma_t)$.

Let $\pounds_m : \Sigma_t \longrightarrow \Sigma_{t+\delta t} \,|\, \forall V \in T_p(\Sigma_t)$, $\pounds_m V \in T_p(\Sigma_{t+\delta t})$ where in component form:

$$\pounds_m h_{\mu\nu} = m^{\gamma}\nabla_{\gamma}h_{\mu\nu} + h_{\mu\gamma}\nabla_{\nu}m^{\gamma} + h_{\gamma\nu}\nabla_{\mu}m^{\gamma}, \tag{2.4}$$

where $m^{\gamma} = \alpha\hat{n}^{\gamma}$.

$$\pounds_m h_{\mu\nu} = -2\alpha K_{\mu\nu} \equiv \hat{n}\pounds_{\hat{n}}h_{\mu\nu}, \tag{2.5}$$

where in the last equality we used the C^{∞}-linearity for tensors. $K_{\mu\nu}$ is defined as the extrinsic curvature on the surface Σ_t.

$$\hat{n}\pounds_{\hat{n}}h_{\mu\nu} = \frac{1}{\alpha}\pounds_n h_{\mu\nu} \implies K_{\mu\nu} = -\frac{1}{2}\pounds_{\hat{n}}h_{\mu\nu},$$

while the Lie dragging of the orthogonal projector to the hypersurface Σ_t should be zero.

$$\pounds_m h^{\mu}{}_{\nu} = m^{\gamma}\nabla_{\gamma}h^{\mu}{}_{\nu} + h^{\mu}{}_{\gamma}\nabla_{\nu}m^{\gamma} - h^{\gamma}{}_{\nu}\nabla_{\gamma}m^{\mu}$$
$$\pounds_m h^{\mu}{}_{\nu} = 0. \tag{2.6}$$

The projector can be decomposed into orthogonal and parallel parts as $h^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \hat{n}^{\mu}\hat{n}_{\nu}$. It provides the condition of orthogonally projected tensor T of type (r, q) on $\underset{r}{\otimes} T_p M \underset{q}{\otimes} T_p^* M$ with the push-forward map $h^* : \underset{r}{\otimes} T_p M \underset{q}{\otimes} T_p^* M \longrightarrow \underset{r}{\otimes} T_p(\Sigma_t) \underset{q}{\otimes} T_p^*(\Sigma_t)$ will not be dragged outside of Σ_t by $\pounds_m$.

We can find a coordinate system adopted foliation by constructing a chart such that $\varphi(p) \in \Sigma_t$,$\varphi(p) = x^i$ where $i : 1, 2, 3 \ldots$. This specific chart on Σ_t should be smooth along $\Sigma_t \longrightarrow \Sigma_{t+\delta t}$ such that $x^{\mu} = \{t, x^i\}$ will be a valid coordinate system on $\mathcal{M}$. These charts naturally induce a coordinate basis on $\mathcal{M}$ such that $\{\partial_t, \partial_i\} \in T_p\mathcal{M}$. One can see that ∂_t also drags spacelike hypersurfaces Σ_t however in general they are not co-linear with m.

So we define the *shift* vector $\vec{\beta}$ such that:

$$\vec{\partial_t} = m + \vec{\beta}, \tag{2.7}$$

where $m = \alpha\hat{n}$ identifies *lapses* of normal evaluation between hypersurfaces. They lapse $t \mapsto t + \delta t$ from one Euclidean observer on Σ_t to another on $\Sigma_{t+\delta t}$ whereas $\vec{\beta}$ measures shifts from starting point p of the evaluation such that $\vec{\beta} \in_p (\Sigma_t)$. [3]

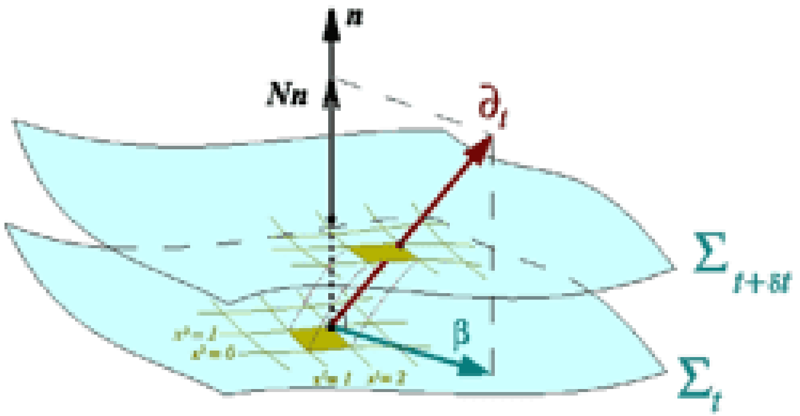

Figure 2.1: This figure shows the evolution from one Euclidean foliation of spacetime to the other one with respect to the normal evaluation vector, and shift vector. The lapses throughout different slices are controlled by shifts.

Upon this construction of ∂_t, one can see that the metric component can be written as [34]:

$$g(\partial_t, \partial_t) = -\alpha^2 + \beta_i\beta^i,$$

which clearly shows ∂_t is not inherently time-like. ∂_t is:

- Time-like $\iff$ $\alpha^2 > \beta_i\beta^i$,
- Space-like $\iff$ $\alpha^2 < \beta_i\beta^i$,
- Null-like $\iff$ $\alpha^2 = \beta_i\beta^i$.

Now, let us find other parts of the $3 + 1$ split:

$$\begin{aligned} g_{00} &= g(\partial_t, \partial_t) = -\alpha^2 + \beta_i\beta^i, \\ g_{0i} &= g(\partial_t, \partial_i) = \langle \partial_t, \alpha n^t \partial_t + \beta^i \partial_i \rangle = \beta_i, \\ g_{ij} &= h_{ij}, \end{aligned} \tag{2.8}$$

[3] In this sense $\vec{\beta} = 0 \iff$ chosen coordinate system establishes an orthogonal basis on Σ_t

or in terms of block-matrix form:

$$g_{\mu\nu} = \begin{bmatrix} -\alpha^2 + \beta_i\beta^i & \beta_i \\ \beta_j & h_{ij} \end{bmatrix}, \tag{2.9}$$

such that the metric on manifold $\mathcal{M} = \Sigma_t \otimes \mathbb{R}$ is:

$$g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + h_{ij}\Big\{[dx^i + \beta^i dt][dx^j + \beta^j dt]\Big\}. \tag{2.10}$$

And the dual of this metric can be written as:

$$g^{\mu\nu}\partial_\mu\partial_\nu = -\frac{1}{\alpha^2}\partial_t^2 + 2\frac{\beta^i}{N^2}\partial_t\partial_i + \big[h^{ij} - \frac{\beta^i\beta^j}{n^2}\big]\partial_i\partial_j, \tag{2.11}$$

where, $\sqrt{-g} = \alpha\sqrt{h}$. Also, the determinants of g and h are coordinate dependent since the choice of α and β are slicing dependent to the spacetime.

2.1.3 3+1 Splitting of Electrodynamics

The rules we have given above should apply to Electrodynamics. First, we can assume that $\mathcal{M} = \Sigma \otimes \mathbb{R}$ and the time coordinate $t \in \mathbb{R}$ produces a Killing field ∂_t, hence the decomposition of normal and parallel components with respect to (Σ_t, h) should behave as:

$$\partial_t = \alpha\hat{n} + \vec{\beta} \iff \partial_t = m + \vec{\beta}$$

One can choose a non-coordinate basis such that the dual basis to the splitting is orthonormal.

$$\theta^0 = \alpha dt, \tag{2.12}$$

$$\theta^i = \vartheta^i + \beta^i dt, \tag{2.13}$$

[4] where new dual pair on Σ_t is $\{e_i, \vartheta^i\}$ instead of $\{\partial_i, dx^i\}$.

Then, the normal evaluation vector becomes:

$$m = \alpha\hat{n} = (\partial_t - \beta^i).$$

FIDO observers give us a recipe to decompose the field tensor F into 3+1:

$$F = E_i\theta^i \wedge \theta^0 + \frac{1}{2}B_{ij}\theta^i \wedge \theta^j,$$

[4] The new choice of vierbeins is called FIDO observers. They are said to be at rest with respect to some other observer.

where $F \in \Omega^2(M)$ while $E \equiv E_i\theta^i \in \Omega^1(M)$ and $B \in \Omega^2(M)$ are called electromagnetic curvature 2-form, electric 1-form and magnetic 2-form respectively [33].

The decomposition also hints at the topology of space (2.13).[5]

Two parts of curvature 2-form F are:

1. $E = E_i\theta^i = E_i(\vartheta^i + \beta^i dt) \equiv \epsilon + \iota_\beta \epsilon dt$,
2. $B = \frac{1}{2}B_{ij}\big[(\vartheta^i + \beta^i dt) \wedge (\vartheta^j + \beta^j dt)\big] = \varrho + \big(dt \wedge \iota_\beta \varrho\big)$.

such that $\epsilon = E_i\vartheta^i$ and $\varrho = \frac{1}{2}B_{ij}\vartheta^i \wedge \vartheta^j$ and ι is a map called the interior product, $\iota : \Omega^{p+1}(\mathcal{M}) \longrightarrow \Omega^p(\mathcal{M})$. If χ is a vector field the interior product acts as:

$$\iota_\chi \omega(\chi_1, \ldots, \chi_{p+1}) = \omega(\chi, \chi_1, \ldots, \chi_p),$$

where $\omega \in \Omega^{p+1}(\mathcal{M})$ or in coordinate basis:

$$\begin{aligned} \iota_\chi \omega &= \iota_\chi\Big(\frac{1}{(p+1)!}\omega_{\mu_1\ldots\mu_{p+1}} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{p+1}}\Big) \\ &= \frac{1}{p!}\chi^\nu \omega_{\nu\mu_1\ldots\mu_p} dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{p+1}} \end{aligned}$$

where $\iota_\chi \omega \in \Omega^p(M)$. Through these definitions, we arrive at the 3+1 decomposition of F.

$$F = \varrho + (\alpha\epsilon - \iota_\beta \varrho) \wedge dt, \tag{2.14}$$

such that electromagnetic curvature 2-form is a closed 2-form, i.e. $dF = 0$. One should also consider that this decomposition should be also valid for the differential d.

$$\mathbf{d}\omega = d\omega + dt \wedge \partial_t \omega,$$

$$\begin{aligned} dF &= d(\varrho + (\alpha\epsilon - \iota_\beta \varrho) \wedge dt), \\ &= d\varrho + dt \wedge \partial_t \varrho + d(\alpha\epsilon) \wedge dt - d(\iota_\beta \varrho) \wedge dt = 0, \end{aligned} \tag{2.15}$$

where d is a differential on Σ. If it acts on Σ, also, it is a differential on $\mathcal{M}$ if it acts on an element of $\mathcal{M}$. This shows that we have $3+1$ decomposition of *Bianchi Identities* [33].

$$d\varrho = 0, \qquad\qquad d(\alpha\epsilon) + \partial\varrho = d(\iota_\beta \varrho).$$

[5] Magnetic 2-forms are generically integrated on a purely Euclidean-geometric side that encrypts the topology of the background fields.

One can define these equations with *Lie derivatives*. Let ι_χ be the interior product, d be the differential operator and $\omega \in \Omega^1(M)$. Then, the combination [25],

$$
\begin{aligned}
(d\iota_\chi + \iota_\chi d)\omega &= d(\iota_\chi \omega) + \iota_\chi(d\omega) \\
&= d(\chi^\mu \omega_\mu) + \iota_\chi\big[\frac{1}{2}(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\mu \wedge dx^\nu\big] \\
&= (\omega_\mu \partial_\nu \chi^\mu + \chi^\mu \partial_\nu \omega_\mu) dx^\nu + \chi^\mu(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\nu \\
&= (\omega_\mu \partial_\nu \chi^\mu + \chi^\mu \partial_\mu \omega_\nu) dx^\nu \\
&\equiv \pounds_\chi \omega \\
\therefore \pounds_\chi &= (d\iota_\chi + \iota_\chi d) \iff d\iota_\chi = \iota_\chi d - \pounds_\chi .
\end{aligned}
$$

The equations become:

$$
d\varrho = 0, \qquad d(\alpha\epsilon) + \partial_t \varrho = \iota_\beta d\varrho - \pounds_\beta \varrho. \tag{2.16}
$$

Let us also find the 3+1 decomposition of gauge connection 1-form A:

$$
\begin{aligned}
A = A_\mu \theta^\mu &= \alpha A_0 dt + A_i(\vartheta^i + \beta^i dt) \\
&= (\alpha A_0 + \beta^i A_i) dt + A_i \vartheta^i .
\end{aligned} \tag{2.17}
$$

Let $\mathcal{A} = A_i \vartheta^i$ such that $A_0 = \phi, A_i = \iota_\beta \mathcal{A}$ corresponds to the natural setting $\phi = -(\alpha A_0 + \iota_\beta \mathcal{A})$. Since $F \in \Omega^2(\mathcal{M})$ is exact and closed, one can always find an $A = \Omega^1(\mathcal{M})$ such that:

$$
\begin{aligned}
F = dA &= d(-\phi dt + \mathcal{A}) \\
&= -d\phi \wedge dt + d\mathcal{A} + dt \wedge \partial_t \mathcal{A} \\
&= -(d\phi + \partial_t \mathcal{A}) \wedge dt + d\mathcal{A} = dA.
\end{aligned} \tag{2.18}
$$

One can identify the equations as follows:

$$
\varrho = d\mathcal{A}, \qquad \alpha\epsilon = -d\phi + \partial_t \mathcal{A} + \iota d\mathcal{A}.
$$

We found the *Bianchi Identities* by decomposing $F = dA$ to find the Maxwell's equations with sources. We have a duality transformation for spacetime $\mathcal{M}$ with the metric g.

Let, $*$ be the Hodge dual operator on dual basis $\Omega^r(\mathcal{M})$ or on a coordinate basis:

$$
*(\theta^{a_1} \wedge \theta^{a_2} \wedge \cdots \wedge \theta^{a_r}) = \frac{1}{(m-r)!} \epsilon^{a_1 \ldots a_r}{}_{b_{r+1} \ldots b_m} \theta^{b_{r+1}} \ldots \theta^{b_n},
$$

where $\epsilon_{\cdot\cdot}^{\cdot}$ is Levi-Civita tensor. Hence, if $F = E_i\theta^i \wedge \theta^0 + \frac{1}{2}B_{ij}\theta^i \wedge \theta^j$, its dual is $*F = -B_i\theta^i \wedge \theta^0 + \frac{1}{2}\epsilon^i{}_{jk}E_i \wedge \theta^j\theta^k$.

With the 3+1 decomposition procedure, it will become:

$$*F = \mathcal{D} - (\alpha\mathcal{H} + \iota_\beta\mathcal{D}) \wedge dt, \qquad \mathcal{H} = *\mathcal{B},$$
$$\mathcal{D} = *\epsilon.$$

Also the current 3-form will be $\mathcal{J} = *\mathcal{S}$ such that

$$\rho = \mu + (\iota_\beta\mu - \alpha J) \wedge dt, \qquad *j = *(j_k\vartheta^k),$$
$$\rho = \mu\vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3,$$

such that

$$d * F = 4\pi\rho, \qquad d\mathcal{D} = 4\pi\rho,$$
$$d(\alpha\mathcal{H}) = (\partial_t - \pounds_\beta)\mathcal{D} + 4\pi\alpha J.$$

2.1.4 2+1+1 Split of Spacetime

2.1.4.1 Spacelike foliation of $h_{\mu\nu}$

As we discussed in the previous section, a spacetime $(\mathcal{M}, \mathcal{O}, \nabla, g)$ can be decomposed into $\mathcal{M} = \Sigma \otimes \mathbb{R}$ with an immersion Φ such that the induced metric can be defined as (Σ, h) on 3-dimensional space-like hypersurface Σ. Activating the entire machinery of *the Membrane Paradigm* requires a further decomposition on Σ. We have $3 + 1 \implies 2 + 1 + 1$ split with a spacelike hyper-2-surface [22].

Let $\Sigma^{(2)}$ be a space-like cross-section of Σ with the metric γ on $\Sigma^{(2)}$. We have a space-like normal k just like n is the time-like normal of Σ.

This composite immersions act as a successive inductions from $\mathcal{M} \xrightarrow{\Phi^{-1}} \Sigma \xrightarrow{\phi^{-1}} \Sigma^{(2)}$ hence the doubly induced metric can be written as:

$$g_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu, \qquad \gamma_{\mu\nu} = h_{\mu\nu} + k_\mu k_\nu,$$

such that,

$$g_{\mu\nu} = h_{\mu\nu} + n_\mu n_\nu - k_\mu k_\nu. \tag{2.19}$$

If we define $\{A, B, \dots\}$ as coordinate indices on $\Sigma^{(2)}$ the metric on $\Sigma^{(2)}$ can be written as [34]:

$$\gamma_{AB} = e^{\mu}{}_{A} e^{\nu}{}_{B} h_{\mu\nu}.$$

where $k_{\mu} n^{\mu} = 0$.

2.1.4.2 Time-like foliation of the Hypersurface Σ

This decomposition of hypersurface constitutes the backbone of the *Membrane Paradigm*. Let us review its geometrical construction.

Let y^i be the coordinates on Σ where Σ is a time-like hypersurface of $\mathcal{M}$ such that $y^i = \{t, \xi^A\}$ such that $\{A, B, \dots\}$ runs from $(1, 2)$. Then, the induced metric on (Σ, h) can be written as [35]:

$$h_{ij} = g_{\mu\nu} e^{\mu}{}_{i} e^{\nu}{}_{j},$$

such that:

$$g_{\mu\nu} = n_{\mu} n_{\nu} + h_{ij} e^{i}{}_{\mu} e^{j}{}_{\nu},$$

where n^{μ} is space-like. Also, one can further decompose time-like metric $h_{\mu\nu}$ as:

$$\gamma_{AB} = h_{\mu\nu} e^{\mu}{}_{i} e^{\nu}{}_{j} e^{i}{}_{A} e^{j}{}_{B},$$

which shows a further decomposition in terms of space-like cross-section of $\Sigma = \mathbb{R} \otimes \Sigma^{(2)}$. This new hypersurface $(\Sigma^{(2)}, \gamma)$ is space-like with decomposition:

$$\gamma_{\mu\nu} = h_{\mu\nu} + U_{\mu} U_{\nu}.$$

For every surface, we need to define different connections ∇ [34]. However, those connections should be equivalent on charts $(U, x^i) \in M$,$(V, y^i) \in \Sigma$,$(W, \xi^A) \in \Sigma^{(2)}$ iff

$$U \cap V \neq \varnothing, \qquad V \cap W \neq \varnothing,$$
$$U \cap W \neq \varnothing.$$

The notation can be fixed as:

If $(\mathcal{M})$ is a phenomenological 4D spacetime with the 4-covariant derivative ∇_{μ}, then (Σ, h) is 3D time-like hypersurface with 3-covariant derivative $|_{a}$ and a space-like normal n^μ where $h^\mu{}_\nu$ acts as a projector from $T_p(M) \longrightarrow T_p(\Sigma)$. Furthermore, a 2D space-like cross-section of Σ is denoted as $(\Sigma^{(2)}, \gamma)$ with 2-covariant derivative $||_A$ and a time-like normal U^μ where $\gamma^\mu{}_A$ acts as a projector from $T_p(\Sigma) \longrightarrow T_p(\Sigma^{(2)})$.

$$\forall v \in T_p(\Sigma) \cap T_p(M) : \nabla_\mu v^\mu = v^\nu{}_{|\mu},$$
$$\forall w \in T_p(\Sigma) \cap T_p(\Sigma^{(2)}) : w^n u_{|\mu} = w^n u_{||\mu}.$$

This construction assumes $n^\mu U_\mu = 0$, also,

$$h^\mu{}_\nu \nabla_\mu \omega^\alpha = w^\alpha{}_{|\nu} - K^\gamma{}_\nu w_\gamma n^\alpha,$$

where we have already defined the second fundamental form as:

$$K^\mu{}_\nu = h^\gamma{}_\nu \nabla_\gamma n^\mu, \qquad K^\mu{}_\nu = K^\nu{}_\mu,$$
$$K_{\mu\nu} n^\nu = 0.$$

Since the spacetime foliation Σ is time-like, its geodesics through normal evolution should be *affine*, in contrast to *null-hypersurface*[6] [36]

$$a^\nu = n^\mu \nabla_\mu n^\nu = 0.$$

2.2 A Brief Introduction to Black Holes

2.2.1 A review of Black Holes

In this section, we will review some generic properties of black holes. The discussion consists of a specific path to follow which will serve our motivations through *Membrane Paradigm*. One can find solutions to Einstein's equations such that the metric of spacetime is singular at some point $p \in M$. By *Cosmic Censorship Conjecture* [3], which was proposed by Penrose, there should be a specific topology on the spacetime in the vicinity of the singularity such that the singularity can not be detected by an outside observer. This topological boundary, a null-hypersurface of no return, is

[6] This constitutes robustness of membrane paradigm since the event horizon metric as a projected surface is *null* and *degenerate* such that geodesics are not affinely transported.

called the event horizon. Horizon breaks the time symmetry through its nullness since an observer outside can not be informed by an observer crossing that boundary.

4D black holes motivated by Einstein's Gravity also endure a beautiful *no-go theorem* called the *no-hair theorem* [6]. This theorem states that the only degrees of freedom a black hole are its global charges, i.e. "Mass, Angular Momentum, Electric Charge". The structure of this section can be stated as follows, we will start with the most generic spherically symmetric black hole solution. It might consist of a mass M, charge Q and cosmological constant Λ. After that, we will turn to the *Newman-Janis formalism* [23] to produce rotating black holes. These rotating holes without charge are called *Kerr Black Holes* [4]. Adding a charge to metric will give the *Kerr-Newman* [37] type black hole solutions.

One can notice that the NJ algorithm is valid for parametrically deviated black holes, too [9]. Starting from a modified Schwarzschild metric, one can find *Johannsen-Psaltis* [38] metric. The aforementioned metric is designed for strong-field tests of the no-hair theorem. It constitutes a parametrical deviation from Kerr's black hole.

2.2.2 The Algorithm on Bussines

Let us write the steps of the Newman-Janis Algorithm.

1. Spherically symmetric metrics have a *seed function*. One can make the seed function visible by transforming to the Eddington-Finkelstein coordinates $(t, r, \theta, \phi) \longrightarrow (u, r, \theta, \phi)$
2. Complexify both the seed function and the corresponding coordinates $f(r) \longrightarrow f(r, \bar{r})$, find the Newman-Penrose null tetrads.
3. Transform the metric to the Boyer-Lindquist coordinates.

Let us start with a generic, static, spherically symmetric spacetime with the Schwarzschild-type seed metric [39].

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_2^2 \tag{2.20}$$

The Gaussian Curvature κ of $d\Omega_2^2 = d\theta^2 + K(\theta)d\phi^2$ can be fixed by choice:

$$\kappa = \begin{Bmatrix} +1 \text{ if } S^2 \\ -1 \text{ if } H^2 \end{Bmatrix} \implies K(\theta) = \begin{Bmatrix} \sin\theta \text{ if } \kappa = 1 \\ \sinh\theta \text{ if } \kappa = -1 \end{Bmatrix}.$$

Let $\kappa = 1$ such that we have a spherical topology on $\mathcal{M} = \Sigma \otimes S^2$. Then the metric is

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{2.21}$$

with the coordinates $x^\mu = \{t, r, \theta, \phi\}$. Transforming this into Eddington-Finkelstein coordinates gives:

$$du = dt + f^{-1}(r)dr \implies dt = du - f^{-1}(r)dr,$$

such that $dt^2 = du^2 - 2f^{-1}(r)dudr + (f^{-1}(r))^2dr^2$ transforms the metric to

$$\begin{aligned} ds^2 &= -f(r)\big(du^2 - 2f^{-1}(r)dudr + (f^{-1}(r))^2dr^2\big) + f^{-1}(r)dr^2 + r^2d\Omega_2^2 \\ &= -f(r)du^2 - 2dudr - f^{-1}(r)dr^2 + f^{-1}(r)dr^2 + r^2d\Omega_2^2 \\ ds^2 &= -f(r)du^2 - 2dudr + r^2d\Omega_2^2. \end{aligned} \tag{2.22}$$

Now, let us pick a complex null tetrad. This null tetrad should obey the Newman-Penrose formalism [40]. If one can fix $\phi = \phi_0$ and $\theta = \theta_0$ such that $ds^2 = 0$:[7]

$$\begin{aligned} -f(r)du^2 - 2dudr &= 0 \\ du &= \frac{-f(r)}{2}dr. \end{aligned} \tag{2.23}$$

Hence if one picks, the set of null basis as $(l^a, n^a, m^a, \bar{m}^a)$ where $\bar{m}^a = (m^a)^*$. This choice should obey the following rules:

$$\begin{aligned} m^a\bar{m}^a &= +1, \qquad\qquad l^an_a = -1, \\ l^al_a = n^an_a = m^am_a = \bar{m}^a\bar{m}_a &= 0, \\ l^am_a = l^a\bar{m}_a = n^am_a = n^a\bar{m}_a &= 0. \end{aligned}$$

In the NP formalism, the tetrad basis consists of two real basis vectors $\{l^a, n^a\}$ and two complex basis vectors $\{m^a, \bar{m}^a\}$. They are self-orthogonal and cross-orthogonal

[7] We are considering the vanishing (t, r) cross-section of the metric at constant spherical angles to find the null tetrads in (t, r) directions

while real and complex subsets are normalized to $\{-1, 1\}$ respectively in the $(-, +, +, +)$ signature. A real subset of the basis can be seen from (2.23) easily.

$$l^a = \delta_r{}^a, \qquad n^a = \delta_u{}^a - \frac{f(r)}{2}\delta_r{}^a, \tag{2.24}$$

the imaginary part can be detected by choosing $t = t_0$ and $r = r_0$ where $ds^2 = 0$.

$$\begin{aligned} d\Omega_2^2 &= d\theta^2 + \sin^2\theta d\phi^2 = 0, \\ rd\theta &= rd\phi^2, \end{aligned}$$

such that:

$$m^a = \frac{1}{\sqrt{2}r}(\delta_\theta{}^a + \frac{i}{\sin\theta}\delta_\phi{}^a), \qquad \bar{m}^a = \frac{1}{\sqrt{2}r}(\delta_\theta{}^a - \frac{i}{\sin\theta}\delta_\phi{}^a). \tag{2.25}$$

In general, one can also find the null-tetrad by finding the orthonormal tetrad in (t, r, θ, ϕ) coordinates. For spherically symmetric metrics:

$$\begin{aligned} ds^2 &= g^{\mu\nu}\partial_\mu\partial_\nu \\ &= \frac{-1}{f(r)}\partial_t^2 + f(r)\partial_r^2 + \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 \\ ds^2 &= \eta^{ab}e_a{}^\mu e_b{}^\nu\partial_\mu\partial_\nu. \end{aligned}$$

One can choose the vielbeins in such a way that the metric can be orthogonalized

$$e_a{}^t = -\sqrt{\frac{1}{f(r)}}, \qquad e_a{}^r = \sqrt{f(r)}, \tag{2.26}$$

$$e_a{}^\theta = \frac{1}{r}, \qquad e_a{}^\phi = \frac{1}{r\sin\theta}. \tag{2.27}$$

such that the basis can be a linear combination of (2.26) and (2.27)

$$l_a = \frac{e_a{}^t + e_a{}^r}{\sqrt{2}} \qquad n_a = \frac{e_a{}^t - e_a{}^r}{\sqrt{2}} \tag{2.28}$$

$$m_a = \frac{e_a{}^\theta + ie_a{}^\phi}{\sqrt{2}} \qquad \bar{m}_a = \frac{e_a{}^\theta - ie_a{}^\phi}{\sqrt{2}} \tag{2.29}$$

The main part of the formalism is to complexify (r, u) subspace [41]. Let $r \longrightarrow \tilde{r} - ia\cos\theta$, $u \longrightarrow \tilde{u} + ia\cos\theta$,$\theta = \tilde{\theta}$, and $\phi = \tilde{\phi}$. The seed function $f(r) \in \mathbb{R}$ is naturally generalized into $f(r, \bar{r}) \in \mathbb{C}$, also the null-tetrad adapts into new coordinates. The transformation can be written as $\tilde{Z}^a = \frac{\partial\tilde{x}^\mu}{\partial x^\nu}Z^a$ where $Z^a = \{l^a, n^a, m^a, \bar{m}^a\}$. The real part of the tetrad analytically continued to complex space.

$$\tilde{l}^a = \delta^a{}_r, \qquad \tilde{n}^a = \delta^a{}_{\tilde{u}} - \frac{f(r, \bar{r})}{2}\delta^a{}_r.$$

The complex part is more tricky but it boils down to a straightforward algebraic calculation.

$$m^a = \frac{1}{\sqrt{2}(\tilde{r} + ia\cos\theta)}\Big(\delta^a{}_\theta + \frac{i}{\sin\theta}\delta^a{}_\phi - ia\big(\delta^a{}_{\tilde{u}} - \delta^a{}_{\tilde{r}}\big)\sin\theta\Big),$$
$$\bar{m}^a = \frac{1}{\sqrt{2}(\tilde{r} - ia\cos\theta)}\Big(\delta^a{}_\theta - \frac{i}{\sin\theta}\delta^a{}_\phi + ia\big(\delta^a{}_{\tilde{u}} - \delta^a{}_{\tilde{r}}\big)\sin\theta\Big).$$

For convenience, we will drop the "tilde" from now on. Moreover, the seed function should also be complexified [41].

$$r \longrightarrow r - ia\cos\theta, \qquad \bar{r} \longrightarrow r + iacos(\theta),$$
$$|r|^2 = r\bar{r} = r^2 + a^2\cos(\theta)^2 = \Sigma(r,\theta).$$

The seed function reads as:

$$f(r,\bar{r}) = 1 - \frac{2M(r)}{r}\Big(\frac{r^2}{\Sigma(r,\theta)}\Big), \tag{2.30}$$

such that, $M(r)$ might include mass M, charge Q or cosmological constant Λ.

$$M(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^3}{6}. \tag{2.31}$$

The line element can be written as

$$\begin{aligned} ds^2 = {} & f(r,\theta)du^2 + 2dudr + 2a\sin^2\theta[1 - f(r,\theta)]dud\phi - 2a\sin^2\theta drd\phi \\ & - \Sigma(r,\theta)d\theta^2 - \sin^2\theta\big[(r^2 + a^2) + a^2\sin^2\theta[1 - f(r,\theta)]\big]d\phi^2. \end{aligned} \tag{2.32}$$

Actually, this line element is not finalized, we can gauge away $g_{r\phi}$ and $g_{u\phi}$ components by appropriate coordinate transformation. Let us transform our metric into Boyer-Lindquist type such that it becomes *Hamilton-Jacobi Separable* [12], whereas, in the spherically symmetric form, it is also *Klein-Gordon separable* [42]. One has to find a transformation $(u, r, \theta, \phi) \longrightarrow (t, r, \theta, \phi)$:

$$du = dt + g(r)dr,$$
$$d\phi = d\phi + h(r)dr.$$

Hence the line element becomes:

$$\begin{aligned} ds^2 = {} & f(r,\theta)[dt + g(r)dr]^2 + 2(dt + g(r)dr)dr \\ & + 2a\sin^2\theta[1 - f(r,\theta)](dt + g(r)dr)(d\phi + h(r)dr) \\ & - 2a\sin^2\theta dr(d\phi + h(r)dr) - \Sigma(r,\theta)d\theta^2 \\ & - \sin^2\theta\big[(r^2 + a^2) + a^2\sin^2\theta(1 - f(r,\theta))\big](d\phi + h(r)dr)^2. \end{aligned} \tag{2.33}$$

If $g_{rt} = g_{r\phi} = 0$, one can uniquely solve $g(r)$ and $h(r)$ as:

$$g(r) = \frac{r^2 + a^2}{f(r,\theta)\Sigma(r,\theta) + a^2 \sin^2\theta}, \tag{2.34}$$

$$h(r) = \frac{-a}{f(r,\theta)\Sigma(r,\theta) + a^2 \sin^2\theta}. \tag{2.35}$$

For simplicity, let the denominator be $\Delta(r,\theta) \equiv f(r,\theta)\Sigma(r,\theta) + a^2 \sin^2\theta$ which is called the *discriminant* for Kerr-like metrics. By substituting (2.34) and (2.35) into (2.33) the metric can be simplified.

$$ds^2 = -f(r,\theta)dt^2 + \frac{\Sigma(r,\theta)}{\Delta(r,\theta)}dr^2 + 2a(f(r,\theta) - 1)\sin^2\theta dt d\phi + \Sigma(r,\theta)d\theta^2 + \left\{r^2 + a^2 + (1 - f(r,\theta))a^2\sin^2\theta\right\}\sin^2\theta d\phi^2 \tag{2.36}$$

There are few other coordinate systems, however, the orthogonal coordinate system might be a nice choice because in these coordinates off-diagonal terms in the metric are invisible. Let $(t,r,\theta,\phi) \longrightarrow (T,r,\theta,\phi)$ such that the first coordinate patch is the Boyer-Lindquist patch and the second one is the orthogonal ellipsoidal coordinates. These two coordinate charts are diffeomorphic to each other with the transformation:

$$dT = dt - a\sin\theta d\phi,$$
$$d\phi = d\varphi - \frac{a}{a^2 + r^2}dt.$$

The metric becomes:[8]

$$ds^2 = -\frac{\Delta(r,\theta)}{\Sigma(r,\theta)}dT^2 + \frac{\Sigma(r,\theta)}{\Delta(r,\theta)}dr^2 + \Sigma(r,\theta)d\theta^2 + \frac{(r^2 + a^2)^2\sin^2\theta}{\Sigma(r,\theta)}d\varphi^2. \tag{2.37}$$

This solution depends on a generic choice of $f(r,\theta)$ seed. Hence, one can find a vast amount of applications that correspond to different spacetimes. We will now specify the functions $f(r,\theta)$ by judiciously choosing their value [41].

2.3 Different Spacetimes for Different Global Charges

2.3.1 Minkowski Spacetime

It is the easiest solution of Einstein's Equations. The unique vacuum, maximally symmetric solution for Einstein's gravity. To find this solution and its rotating version

[8] It is interesting to see that in ellipsoidal coordinates (T,r) subspace is isomorphic to $AdS_{1,1}$ and the rest is spherodial shape.

one can assume $f(r,\theta) = 1$. In this situation, static, spherically symmetric metric transforms into [39]:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2,$$

which is a flat space with non-vanishing Γ's but a vanishing Riemann tensor. After NJA, the algorithm changes the metric as:

$$ds^2 = -dt^2 + \frac{\Sigma(r,\theta)}{r^2+a^2}dr^2 + \Sigma(r,\theta)d\theta^2 - (r^2+a^2)\sin^2\theta d\phi^2,$$

which is again a flat space with the rotating observer.[9]

2.3.2 Kerr Spacetime

It is a model of spacetime [43] such that there exists a rotating black hole with mass M. It has a singularity caused by the event horizon (2.2)

In the static case, the Schwarzschild spacetime line element is:

$$ds^2 = -(1-\frac{2M}{r})dt^2 + (1-\frac{2M}{r})^{-1}dr^2 + r^2 d\Omega_2^2.$$

After NJA, the Schwarzschild metric becomes

$$f(r,\theta) = 1 - \frac{2M}{r}\Big(\frac{r^2}{\Sigma(r,\theta)}\Big) = 1 - \frac{2Mr}{\Sigma(r,\theta)}, \tag{2.38}$$

$$\Delta(r) = f(r,\theta)\Sigma(r,\theta) + a^2\sin^2\theta = r^2 + a^2 - 2Mr. \tag{2.39}$$

In the Boyer-Lindquist Coordinates, the Kerr metric reads

$$\begin{aligned} ds^2 = (1-\frac{2Mr}{\Sigma(r,\theta)})dt^2 - \frac{4Mra\sin^2\theta}{\Sigma(r,\theta)}dtd\phi + \frac{\Sigma(r,\theta)}{\Delta(r)}dr^2 \\ + \Sigma(r,\theta)d\theta^2 + \big[r^2 + a^2 + \frac{2Mra^2\sin^2\theta}{\Sigma(r,\theta)}\big]\sin^2\theta d\phi^2. \end{aligned} \tag{2.40}$$

One can detect the event horizon of this spacetime by looking at $g^{rr} = 0$.

$$\begin{aligned} \Delta(r) = 0 \implies r^2 - 2Mr + a^2 = 0 \\ = M \pm \sqrt{M^2 - a^2}. \end{aligned}$$

[9] As in the case of Classical Mechanics there are no wrong coordinates or badly placed observers, all observers should find the same equation of motion however, the way to find it might be harder in the specific application or different coordinates system reveals different properties of the given system.

Kerr spacetime has various features. Classically, a point cannot have a rotation hence, as a classical solution, Kerr BH has a *ringularity* inside its horizons [43]. It has inner, and outer horizons and an ergosphere that drags the spacetime as the rotation happens [43]. Its ergospheres can be shown as:

$$g_{tt} = 1 - \frac{2Mr}{\Sigma(r,\theta)} = 0 \implies r_{1,2} = M \pm \sqrt{M^2 - a^2\cos^2\theta}.$$

One should understand that Kerr black hole has a naked singularity if it overspins, i.e. $a > M$, hence this condition is prevented by the Cosmic Censorship Conjecture.[35]

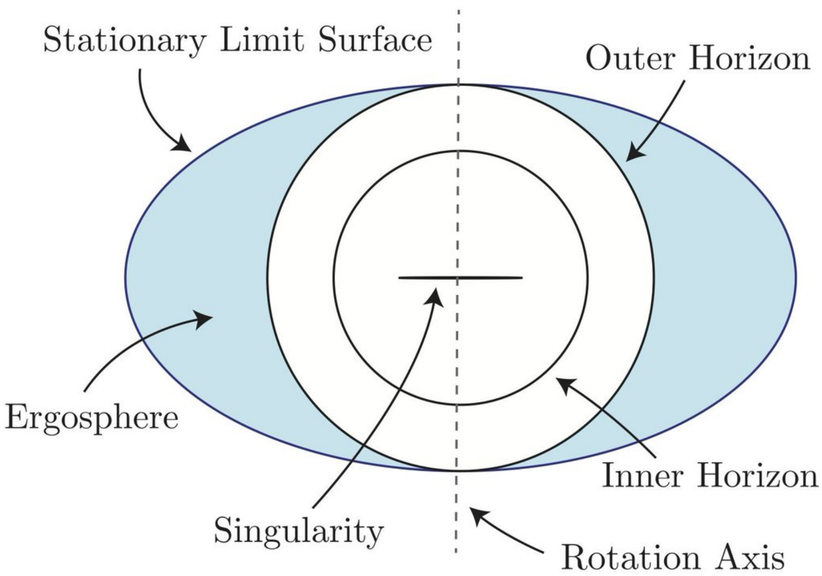

Figure 2.2: Schematics of the Kerr black hole's horizons and its important regions.

2.3.3 When Reissner-Nordstrom Press Charges Against Schwarzschild

A virtue of the no-hair theorem black holes also have electrical charges to spare [44]. Luckily, NJA does not overlook electrical charges. One can find a charged version of the Kerr metric which is called the Kerr-Newman spacetime through the charged version of the Schwarzschild metric also called Reissner-Nordstrom (RN) spacetime.

Recall our seed function for static spacetimes:

$$f(r) = 1 - \frac{2M(r)}{r}$$

If we let $M(r) = M - \frac{Q^2}{2r}$ such that the line element becomes:

$$ds^2 = -(1 - \frac{1}{r}(M - \frac{Q^2}{2r}))dt^2 + (1 - \frac{1}{r}(M - \frac{Q^2}{2r}))^{-1}dr^2 + r^2 d\Omega_2^2 \tag{2.41}$$

which is called the Reisner-Nordstrom metric and it is the electrovacuum solution of Einstein-Maxwell equations.

After NJA, one can arrive at the Kerr-Newman metric with the seeds:

$$f(r,\theta) = 1 - \frac{2Mr - Q^2}{\Sigma(r,\theta)}, \tag{2.42}$$

$$\Delta(r) = r^2 + a^2 - 2Mr + Q^2. \tag{2.43}$$

Then the rotating solution of the Einstein-Maxwell equations is:

$$\begin{aligned} ds^2 = -(1 - \frac{2Mr - Q^2}{\Sigma(r,\theta)})dt^2 + \frac{\Sigma(r,\theta)}{\Delta(r)}dr^2 + 2a\frac{2Mr - Q^2}{\Sigma}\sin^2\theta dt d\phi \\ + \left[r^2 + a^2 + (\frac{2Mr - Q^2}{\Sigma})a^2 \sin^2\theta\right]\sin^2\theta d\phi^2, \end{aligned} \tag{2.44}$$

which is the unique Kerr-Newman metric.

2.3.4 Rotating (anti)-de Sitter Spacetime

One can choose the seed function as the (anti)-de Sitter spacetime by adding a non-zero cosmological constant Λ,

$$\begin{Bmatrix} \text{dS if } \Lambda > 0 \\ \text{AdS if } \Lambda < 0 \end{Bmatrix} \tag{2.45}$$

Its static seed can be stated as:

$$f(r) = 1 - \frac{2M(r)}{r} \text{ where } M(r) = \frac{\Lambda r^2}{6} \tag{2.46}$$

such that

$$f(r) = 1 - \frac{\Lambda r^2}{3} \tag{2.47}$$

One should keep in mind that the NJ is an algorithm and the choice of complexification is rather arbitrary upon higher degrees of $\mathcal{O}(r^2)$. This arbitrariness might lead to wrong answers. If one writes $r^2 \equiv |r|^2 = \Sigma$ then the complexification seed is wrongly stated as:

$$f(r,\theta) = 1 - \frac{\Lambda\Sigma(r,\theta)}{3} \text{ (which is clearly wrong).} \tag{2.48}$$

This does not correspond to the true rotating AdS solution. True complexification can be achieved by the function:

$$f(r,\theta) = 1 - \frac{2M(r)}{r}\Big(\frac{r^2}{\Sigma(r,\theta)}\Big) \iff M(r) = \frac{\Lambda r^3}{6} \tag{2.49}$$

such that the seed "cracks" as:

$$f(r,\theta) = 1 - \frac{\Lambda}{3}(\frac{r^4}{\Sigma(r,\theta)}), \tag{2.50}$$

$$\Delta(r) = f(r,\theta)\Sigma(r,\theta) + a^2\sin^2\theta = r^2 + a^2 - \frac{\Lambda r^4}{3}, \tag{2.51}$$

where the line element of (A)dS can be visible with ellipsoidal coordinates:

$$ds^2 = -\frac{\Delta(r)}{\Sigma(r,\theta)}dT^2 + \frac{\Sigma(r,\theta)}{\Delta(r)}dr^2 + \Sigma(r,\theta)d\theta^2 - \frac{(r^2+a^2)^2\sin^2\theta}{\Sigma(r,\theta)}d\phi^2 \tag{2.52}$$

If $a = 0$, we have static de-Sitter cosmology:

$$ds^2 = -(1 - \frac{\Lambda r^2}{3})dt^2 + (1 - \frac{\Lambda r^2}{3})^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{2.53}$$

which topologically corresponds $\mathcal{M} = AdS_2 \otimes S^2$.

2.3.5 The Kerr-Newman-(Anti) de Sitter Spacetime

If one commits to the definitions we had through this chapter, one can use all the global properties to find a rotating solution for cosmological Einstein-Maxwell Equations [37]. Let Reisner-Nordström-de Sitter seed be:

$$f(r,\theta) = 1 - \frac{2M(r)}{r}\Big(\frac{r^2}{\Sigma(r,\theta)}\Big) \iff M(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^3}{6},$$

such that we have a static, charged spacetime with a non-zero cosmological constant. RN(A)dS spacetime reads under NJA as:

$$1 - \frac{2M(r)}{r}\Big(\frac{r^2}{\Sigma(r,\theta)}\Big) = 1 - \frac{2}{r}\Big(M - \frac{Q^2}{2r} + \frac{\Lambda r^2}{6}\Big(\frac{r^2}{\Sigma(r,\theta)}\Big)\Big) \tag{2.54}$$

$$f(r,\theta) = \Big[1 - \frac{2Mr}{\Sigma(r,\theta)} + \frac{Q^2}{\Sigma(r,\theta)} - \frac{\Lambda r^4}{3\Sigma(r,\theta)}\Big] \tag{2.55}$$

$$\Delta(r) = r^2 + a^2 - 2Mr + Q^2 - \frac{\Lambda r^4}{3}, \tag{2.56}$$

and the metric can be written according to the above equations.

2.3.6 Parametrically Deviated Kerr-Like Black Holes

The theory for astrophysical black holes is important to understand the cosmological applications and analyse the data. No-hair theorem [6] strictly constrains the behaviour of the black holes and identifies them only by their masses, spin, and electrical charge. However, rotating black holes get freed from their charges easily and the no-hair theorem focuses on Kerr black holes as the only stationary, axisymmetric, asymptotically flat vacuum solutions of Einstein's equations. Together with cosmic censorship conjecture, Kerr black holes are also considered to be a unique solution with no Closed Time-like Curves (CTCs) [45], and a ringularity enclosed with a horizon. Current observational data also probe violations of the no-hair theorem in strong-field regions. This violation requires a correction to total global degrees of freedom [38]. The correction of this type can be captured by parametric corrections to Kerr Black Holes which are classified as Kerr-Like black holes. There are vast amounts of solutions in the literature.

It is known that several types of parametric deviations are also considered pathological. They have naked singularities in other regions of spacetime or they endure CTCs outside the horizon [9].

One might expect such pathologies since a direct violation of the no-hair theorem requires such an unphysical phenomenon. These problematic structures within the parametrically deviated solutions can be advantageous. One might align the CTCs to unwanted regions so that observational tests can be directed to non-pathological regions.

Johannsen-Psaltis [10] constructed a deviation from the Kerr metric which is now abbreviated as the JP metric. This metric has no pathology up to higher spin solutions with a set of parameters that clearly indicates linear parametric deviations from the Kerr metric. They have the same metric structure as the Kerr in BL coordinates such that geodesic tests are also comparable with the Kerr Spacetime [12].

Johannsen-Psaltis metric is built in such a way that it is regular outside the horizon, hence astrophysical tests can be aligned arbitrarily close. Construction of this metric follows the same steps as the rotating black hole metrics in previous chapters. To derive its components, one should first modify the Schwarzschild black hole metric, such that it corresponds to parametrically deviated, static, spherically symmetric metric. The line element of the Schwarzschild-like metric can be stated as [10]:

$$ds^2 = -f(r)[1+h(r)]dt^2 + f(r)^{-1}[1+h(r)]dr^2 + r^2 d\Omega_2^2, \tag{2.57}$$

where $h(r) = \sum_{k=0}^{\infty} \epsilon_k \left(\frac{M}{r}\right)^k$. Let us restate the form of the metric as a linear perturbation:

$$\begin{aligned} ds^2 &= g_{\mu\nu}^{Schwarzschild} dx^\mu dx^\nu + h_{AB}^{perturb.} dx^A dx^B \\ &= -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_2^2 - h(r)dt^2 + h(r)dr^2. \end{aligned}$$

If we also remember the generic seed form of spherically symmetric, static null-basis $Z^\mu = \{l^\mu, n^\mu, m^\mu, \bar{m}^\mu\}$ chosen basis can be captured by the deviation with a correction to n^μ in the Eddington-Finkelstein Coordinates:

$$l^\mu = \delta^\mu{}_r, \qquad n^\mu = \frac{1}{1+h(r)}\big(\delta^\mu{}_u + \frac{f(r)}{2}\delta^\mu{}_r\big), \tag{2.58}$$

$$m^\mu = \frac{1}{\sqrt{2}r}\big(\delta^\mu{}_\theta + \frac{i}{\sin\theta}\delta^\mu{}_r\big), \qquad \bar{m}^\mu = \frac{1}{\sqrt{2}r}\big(\delta^\mu{}_\theta - \frac{i}{\sin\theta}\delta^\mu{}_r\big). \tag{2.59}$$

Again using the same recipe, one can find the NJA transformed new basis as:

$$l^\mu = \delta^\mu{}_r, \tag{2.60}$$

$$n^\mu = \frac{1}{1+h(r,\bar{r})}\left(\delta^\mu{}_u - \frac{1}{2}\Big(1 - \frac{2M(r)}{r}\big(\frac{r^2}{\Sigma(r,\theta)}\big)\Big)\delta^\mu{}_r\right), \tag{2.61}$$

$$m^\mu = \frac{1}{\sqrt{2}(r - ia\cos\theta)}\left(\delta^\mu{}_\theta + \frac{i}{\sin\theta}\delta^\mu{}_\phi - ia\Big(\delta^\mu{}_u - \delta^\mu{}_r\Big)\right), \tag{2.62}$$

$$\bar{m}^\mu = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\left(\delta^\mu{}_\theta - \frac{i}{\sin\theta}\delta^\mu{}_\phi + ia\Big(\delta^\mu{}_u - \delta^\mu{}_r\Big)\right), \tag{2.63}$$

where $\Sigma(r,\theta) = r^2 + a^2\cos^2\theta$, if we complexify r and u, we find $h(r,\bar{r})$ as:

$$h(r,\bar{r}) = \sum_{k=0}^{\infty}\Big(\epsilon_{2k} + \epsilon_{2k+1}\frac{M}{2}(\frac{r+\bar{r}}{r\bar{r}})\Big)\big(\frac{M^2}{r\bar{r}}\big)^k.$$

or in polar coordinates:

$$h(r,\theta) = \sum_{k=0}^{\infty}\Big(\epsilon_{2k} + \epsilon_{2k+1}\frac{Mr}{\Sigma(r,\theta)}\Big)\big(\frac{M^2}{\Sigma(r,\theta)}\big)^k. \tag{2.64}$$

Similar transformations to BL coordinates in the case of the Kerr metric will yield the JP metric:

$$\begin{aligned} ds^2 = -\Big[1+h(r,\theta)\Big]f(r,\theta)dt^2 + 2a\big(-\frac{2M(r)}{r}\frac{r^2}{\Sigma(r,\theta)}\sin^2\theta[1+h(r,\theta)]\big)dtd\phi \\ + \frac{\Sigma(r,\theta)[1+h(r,\theta)]}{\Delta(r)+a^2\sin^2\theta h(r,\theta)}dr^2 + \Sigma(r,\theta)d\theta^2 \\ + \Bigg[\sin^2\theta\Big(r^2+a^2+a^2\sin^2\theta\big(\frac{2M(r)}{r}(\frac{r^2}{\Sigma(r,\theta)})\big)\Big) \\ + h(r,\theta)a^2\sin^4\theta\big(1+\frac{2M(r)}{r}\big)\big(\frac{r^2}{\Sigma(r,\theta)}\big)\Bigg]d\phi^2. \end{aligned} \tag{2.65}$$

As stated before by using this approach one can easily upgrade the JP metric into different types. Let $M(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^3}{6}$ then we have electrically charged JP metric with a non-zero cosmological constant [39].

One can summarize it as:

1. If $\Lambda = 0$, we have the charged JP spacetime.
2. If $h(r,\theta) = 0$ we have the Kerr-Newman-(A)dS spacetime.
3. If $h(r,\theta) = \Lambda = 0$ we have the Kerr-Newman spacetime.
4. If $h(r,\theta) = Q = \Lambda = 0$ we have the Kerr spacetime.
5. If $h(r,\theta) = Q = \Lambda = a = 0$ we have the Schwarzschild metric.

One can detect that, excluding some global degrees of freedom change the properties of spacetime hence the parametric correction ϵ acts as a degree of freedom which violates the no-hair theorem [9]. Now, let us declare the JP spacetime by choosing $Q = 0$ & $\Lambda = 0$ and $M \neq 0, h(r,\theta) \neq 0$. The metric becomes:

$$\begin{aligned} ds^2 = -\Big[1 + h(r,\theta)\Big](1 - \frac{2Mr}{\Sigma(r,\theta)})dt^2 - 4a\frac{2Mr}{\Sigma(r,\theta)}\sin^2\theta[1 + h(r,\theta)]^2 dt d\phi \\ + \frac{\Sigma(r,\theta)[1 + h(r,\theta)]}{\Delta(r) + a^2\sin^2\theta h(r,\theta)}dr^2 + \Sigma(r,\theta)d\theta^2 \\ + \Bigg[\sin^2\theta\Big(r^2 + a^2 + a^2\sin^2\theta\frac{2Mr}{\Sigma(r,\theta)}\Big) \\ + h(r,\theta)a^2\sin^4\theta\frac{\Sigma(r,\theta) + 2Mr}{\Sigma(r,\theta)}\Bigg]d\phi^2. \quad (2.66) \end{aligned}$$

This summation on $h(r,\theta)$ can be truncated by observational data. If one sets $\epsilon_0 = \epsilon_1 = \epsilon_2 = 0$ and $\epsilon_i = 0 \iff i > 3$ we made a cut off in deviation [10]:

$$h(r,\theta) = \epsilon_3\frac{M^3 r}{\Sigma(r,\theta)^2}. \quad (2.67)$$

The horizon structure of JP spacetime can be evaluated numerically and the choice of the ϵ_3 parameter strictly changes the shape of the horizon. For rotating metrics, one can detect the null hypersurface as:

$$g_{t\phi}^2 - g_{tt}g_{\phi\phi} = 0. \quad (2.68)$$

This gives:

$$\begin{aligned} \Bigg[\frac{4aMr\sin^2\theta}{\Sigma(r,\theta)}\big(1 + \epsilon_3(\frac{M^3 r}{\Sigma(r,\theta)^2})\big]^2 \\ + \big[1 + \epsilon_3\frac{M^3 r}{\Sigma(r,\theta)^2}\big](1 - \frac{2Mr}{\Sigma(r,\theta)})\Bigg\{\sin^2\theta\Big[r^2 + a^2 + a^2\sin^2\theta\frac{2Mr}{\Sigma(r,\theta)})\Big] \\ + \big(1 + \epsilon_3\frac{M^3 r}{\Sigma(r,\theta)^2}a^2\sin^4\theta\frac{\Sigma(r,\theta) + 2Mr}{\Sigma(r,\theta)}\Bigg\} = 0, \quad (2.69) \end{aligned}$$

where the equation algebraically simplifies to

$$1 + \epsilon_3(\frac{M^3 r}{\Sigma(r,\theta)^2})\omega(r,\theta; M, \epsilon_3) = 0, \quad (2.70)$$

where ω is a function of $(r, \theta, M, \epsilon_3)$. This equation does not have any analytic algebraic solution. However, analyzing it when $\epsilon_3 = 0$, we have the null hypersurface of the Kerr black hole. This concludes at least the JP metric also has 2 horizons but it is deformed by the deviation parameter ϵ_3.

Black hole horizons in Einstein's gravity are null surfaces with compact cross-sections, hence the choice of ϵ_3 determines the types of orbits and finally their compactness. If $\epsilon_3 < 0$, it is not certain that we have a compact surface.

In the extremal limit, i.e. $a \simeq M$, the event horizon has a dumbbell shape at $\epsilon_3 = -1$ and non-compact at $\epsilon_3 = 1$. If $a = 0$, we have the modified Schwarzschild black hole and hence have a spherical horizon as usual at $r_s = 2M$. If $\epsilon_3 \geq -8$ one can show the horizon radius in terms of the deviation parameter as $r_h = (|\epsilon_3|)^{\frac{1}{3}} M$ at $\epsilon_3 = -8$ [9].

Now, let us examine the horizon more rigorously. Define a $C^\infty(\mathcal{M})$ function $f(r,\theta)$ on so-called null-surface. The normal on the null surface can be found as:

$$df(r,\theta) = g^{rr}\big(\frac{\partial f(r,\theta)}{\partial r}\big)^2 - 2g^{r\theta}\frac{\partial f(r,\theta)}{\partial \theta}\frac{\partial f(r,\theta)}{\partial r} + g^{\theta\theta}\big(\frac{\partial f(r,\theta)}{\partial \theta}\big)^2 = 0. \quad (2.71)$$

This partial differential equation should be solved to find the function on the surface. The metric structure of the JP allows us to simplify this equation. Since $g_{rr} = \frac{1}{g^{rr}}$,$g_{\theta\theta} = \frac{1}{g^{\theta\theta}}$ and $g^{r\theta} = 0$ (2.71) becomes:

$$g^{rr}\big(\frac{\partial f(r,\theta)}{\partial r}\big)^2 + g^{\theta\theta}\big(\frac{\partial f(r,\theta)}{\partial \theta}\big)^2 = 0. \quad (2.72)$$

Now let us choose an $f(r,\theta)$ such that it only depends on $r_h = H(\theta)$ and $\frac{\partial f(r,\theta)}{\partial r} = 0$ it becomes:

$$g^{rr} + g^{\theta\theta}\big(\frac{\partial H(\theta)}{\partial \theta}\big)^2 = 0. \quad (2.73)$$

This equation can be analyzed for different polar angles $\theta = (0, \pi, \frac{\pi}{2})$. At $\theta = 0, \pi$ we are at the poles and $\theta = \frac{\pi}{2}$ corresponds the equatorial plane. For all of those values, the event horizon equation becomes:

$$g^{rr} = 0\,, \quad (2.74)$$

but at the poles, this equation corresponds to $g^{rr} = \Delta(r) = 0$ which gives the Kerr radius $r = r_{Kerr}$. However, JP spacetime prevents some values of ϵ_3 at the poles

$$g^{rr} = \frac{\Delta(r) + a^2\sin^2\theta\frac{\epsilon_3 M^3 r}{\Sigma(r,\theta)^2}}{\Sigma(r,\theta)\big(1 + \frac{\epsilon_3 M^3 r}{\Sigma(r,\theta)^2}\big)}. \quad (2.75)$$

(2.75) gives the Kerr horizon radius iff $\epsilon_3 \neq \frac{-(2Mr_{Kerr})}{M^3 r_{Kerr}}$, where the denominator diverges. For $\theta = \frac{\pi}{2}$, equatorial plane:

$$g^{rr} = \frac{\Delta(r) + a^2\sin^2\theta\big(\frac{\epsilon_3 M^3 r}{\Sigma(r,\theta)^2}\big)}{\Sigma(r,\theta)\big(1 + \frac{\epsilon_3 M^3 r}{\Sigma(r,\theta)^2}\big)} \implies r^2 + a^2 - 2Mr + \frac{a^2\epsilon_3 M^3}{r^3} = 0, \quad (2.76)$$

the fifth-order algebraic equation in (2.75) becomes:

$$r^5 - 2Mr^4 + a^2r^3 + a^3\epsilon_3 M^3 = 0 \tag{2.77}$$

There is no exact solution for this equation. There is a subtle approach, one can treat the parametric deviation as a linear perturbation [9]. The linear perturbation parameter ϵ should affect the horizon radius. Let

$$g_{\mu\nu}^{(JP)} = g_{\mu\nu}^{(Kerr)} + h_{\mu\nu}^{(\epsilon)}, \tag{2.78}$$

such that

$$h_{tt} = -\frac{M^3 r(\Sigma(r,\theta) - 2Mr)}{\Sigma(r,\theta)}, \qquad h_{rr} = \frac{M^3 r(\Sigma(r,\theta) - 2Mr)}{\Sigma(r,\theta)\Delta(r)^2}, \tag{2.79}$$

$$h_{t\phi} = -\frac{2aM^4 r^2 \sin^2\theta}{\Sigma(r,\theta)^3}, \qquad h_{\phi\phi} = \frac{2a^2 M^4 r^2 \sin^4\theta}{\Sigma(r,\theta)^3}, \tag{2.80}$$

$$h_{\theta\theta} = 0. \tag{2.81}$$

The event horizon can be found by perturbation as [10]:

$$g_K^{rr}(1 - \epsilon g_K^{rr} h_{rr}) = 0, \tag{2.82}$$

such that the event horizon is where $g_K^{rr} = \frac{\Delta(r)}{\Sigma(r,\theta)}$ is the rr-component of Kerr metric.

$$r_{JP} = r_K(1 + \lambda\epsilon), \tag{2.83}$$

such that the deviation from the Kerr radius is:

$$\lambda = -\frac{a^2 M^3 \sin^2\theta}{2\sqrt{M^2 - a^2}(2Mr_K - a^2\sin^2\theta)}. \tag{2.84}$$

As we stated before, this metric is regular and CTC-free outside of the Killing horizon. At the equatorial plane of $g^{\phi\phi}$ component one can find the upper bound for ϵ

$$\epsilon^{CTC} = -\frac{r^3(r^2(a^2 + r^2) - 2a^2 Mr)}{a^2 M^3(2Mr + r^2)},$$

while $\epsilon^{Horizon}$ is:

$$\epsilon^{Horizon} = -\frac{r^3((a^2 + r^2) - 2Mr)}{a^2 M^3}, \tag{2.85}$$

obviously $\epsilon^{Horizon} > \epsilon^{CTC}$, this means all the CTCs are inside the horizon. We will not be bothered with them in astrophysical applications, since the outer horizon observer (OHO) will not be aware of the events inside the black hole.

CHAPTER 3

A DISCUSSION ON THE MEMBRANE PARADIGM

3.1 The Membrane Paradigm

Smooth geometries of spacetime generically have a boundary at infinity. According to Einstein-Hilbert action of General Relativity, the extra term coming from this boundary topology cannot change the field equations but the action must be augmented by a new boundary action which cancels the contribution coming from the variation of the original action at infinity [31]. This is needed for a valid calculation of the variation procedure.

These are surely boundary terms that constrain the topology of spacetime. However, black hole geometries induce a different boundary surface which is unique in its properties [22]. It is called the event horizon. The event horizon, as a null surface, is an unusual object. It is a surface of no return, where time becomes spatial and vice-versa.

It most commonly shows itself as a coordinate singularity in the usual forms of the metric. However, a true horizon can be detected by a global knowledge of the spacetime [46] and some curvature invariants, i.e. Kretschmann Scalar [47], which detects the change in curvature through the horizon radius.

An observer outside should see an observer on the horizon as infinitely red-shifted. It is a surface where the "Jacobian" is singular and the vectors on them are orthogonal to itself. Studies on black hole thermodynamics [48] constituted global statements on the macroscopic state while local statements govern the electrodynamics and the mechanics of the black hole [18].

Since the inside of the black hole is not available for an observer outside, it is understood that for a time-like observer outside of the horizon, there should be a surface that mimics the black hole as a "fluid bubble" [19] with electrical conductivity, shear & bulk viscosities through transport coefficients with the calculations on the surface gravity [17].

A careful calculation on the horizon led to Ohm's law, Joule's law and the non-relativistic Navier-Stokes equation [20]. Early works of Damour, followed by Thorne and Price [18] showed a way to mimic the horizon without its sickness. They constructed a fake horizon or *streched horizon* just outside of the *true horizon* with similar properties. A fiducial observer sees it as a time-like hypersurface, a.k.a a membrane, which has no pathologies in the kinds of the true horizon.[1] The *streched horizon*, as an arbitrarily close surface to the true horizon is indeed a 2+1 time-like surface. This surface can be linked through the true horizon by ingoing null congruence as an injection.

Since the membrane is not null by nature, it has a non-degenerate metric on it which brings an outstanding approximation to the behaviour of the true horizon. This approach to black hole dynamics was coined as *Membrane Paradigm of Black Holes* [49].

One can detect the gravitational and electromagnetic descriptions of a membrane through its boundary contribution. The main idea of this description relies on admitting the true nature of black holes as a causal sink where an outside observer is not aware of its dynamics inside [22]. Moreover, field equations for that observer are restricted to the complementary set of spacetime events outside of the horizon [49].

However, defining an extra topology, stretched horizon brings a new boundary term which must be supported and cancelled out by a chosen source on the membrane. It can be electrical and gravitational sources.[2]

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[1] This new bubble is in the analogy with one of the early works of electrodynamics the "method of images" [21].

[2] As always, a mathematical surface that brings topology to spacetime should be supported by topological, electrical or gravitational sources. They behave as external action surface terms which cancel the residual boundary term [50].

3.1.1 A Mathematical Construction of the Membrane Paradigm

Our discussions in previous sections prepared us to develop a clear mathematical understanding of the Membrane Paradigm.

Let H be the horizon, $\exists$ a null generator $l^\mu \in T_pM | \forall p \in M$ it is normalized with the time coordinate at infinity. One can define a fake horizon or stretched horizon H_s arbitrarily close to true horizon H. Let the metric on H_s be a time-like metric such that the lapse function α behaves as a parameter $\alpha << 1$ that pierces the foliations of spacetime and in the limit $\alpha \longrightarrow 0$, $H_s|_{\alpha \longrightarrow 0} = H$. This parametrization is needed such that in the limit, the stretched horizon and the true horizon coincide [20]. The lapse function α is a truly geometrical object hence it diverges together with some of the physical structures we will define on the surface. Since it acts as an arbitrary parameter, it can be factorized through renormalization such that algebraically α is also a regulator.

Parikh in his thesis claims that: "The stretched horizon, although the FIDO observers on it endure immerse acceleration and measurements diverge while $\alpha \longrightarrow 0$, is more fundamental than the true horizon. The phenomenological applicability of H_s allows for an observer to measure and report it back. If the length scale is smaller than the proper distance between H_s and H, it is natural for an observer to use H_s as if it is not a mimicking surface but a true measure of black hole properties" [20].

Let FIDO's have worldlines $U^\mu(\tau)$, H_s as a time-like surface has space-like unit normal n^μ such that

$$\alpha U^\mu \longrightarrow l^\mu, \qquad \alpha n^\mu \longrightarrow l^\mu, \tag{3.1}$$

while as $\alpha \longrightarrow 0$, $l^\mu l_\mu = 0$. Also, (M, g) is the total spacetime with dim$(M) = 4$. (H_s, h) is the stretched horizon with dim$(H_s) = 3$ with $h^\mu{}_\nu$ as a projector from $T_pM \longrightarrow T_p(H_s)$. Moreover, $(\Gamma(H_s), \gamma)$ is the spacelike cross-section of H_s with the 2-metric γ_{AB}. U^μ is normal to $\Gamma(T_pH_s)$. This identifies spacetime with a $2+1+1$ splitting as we discussed in the previous chapters.

As before, the connections on manifolds are; 4-covariant derivative ∇_μ, 3-covariant derivative $|_\mu$, 2-covariant derivative $||_\mu$ [22].

Let a vector $V^\mu \in T_p(M)$ then:

$$h^\gamma{}_\mu \nabla_\gamma V^\nu = V^\nu|_\mu - K^\gamma{}_\mu \omega_\gamma n^\nu, \tag{3.2}$$

where $K^\gamma{}_\mu = h^\gamma{}_\nu \nabla_\gamma n^\mu$ is the extrinsic curvature on H_s. The tools for Membrane Paradigm can be summarized as [22]:

$$\begin{aligned}
l^\mu l_\mu &= 0, & n^\mu n_\mu &= 1, \\
U^\mu U_\mu &= -1 & & \\
U^\mu &= (\frac{d}{d\tau})^\mu, & a^\mu &= n^\gamma \nabla_\gamma n^\mu = 0, \\
h^\mu{}_\nu &= g^\mu{}_\nu - n^\mu n_\nu, & \gamma^\mu{}_\nu = h^\mu{}_\nu + U^\mu U_\nu &= g^\mu{}_\nu - n^\mu n_\nu + U^\mu U_\nu, \\
K_{\mu\nu} n^\nu &= 0, & K_{\mu\nu} &= K_{\nu\mu}, \\
K^\gamma{}_\mu &= h^\gamma{}_\nu \nabla_\gamma n^\mu & & \\
\lim_{\alpha\to\infty} \alpha U^\mu &= l^\mu, & \lim_{\alpha\to\infty} \alpha n^\mu &= l^\mu.
\end{aligned}$$

3.1.2 An Action Formalism for the Membrane Paradigm

Generically one can find the field equations through the variation of the relevant action with respect to the metric or the relevant degrees of freedom. It is not always possible to find field equations directly, one should define boundary conditions. The Dirichlet boundary condition can be applied to the end of the spacetime as $\delta\varphi = 0, \forall\varphi \in M$. In our case, we do not have a globally smooth spacetime. We have a singular region, this region should obey the boundary condition

The action can be extremized through those boundaries, however, the stretched horizon is not a Dirichlet surface to be variated out. One should add external action to cancel it. It is not stationary. Hence the total action can be implicitly written as [51]:

$$S_{total} = (S_{in} - S_{surface}) + (S_{out} - S_{surface}). \tag{3.3}$$

Both the first and second parts of the total action are stationary on their own. Hence,

$$\delta S_{total} = \delta(S_{in} - S_{surface}) + \delta(S_{out} - S_{surface}), \tag{3.4}$$

where both parts of the addition are zero individually. This prevents new boundary terms from being added to the total action since it is stationary.

As we mentioned before, these surface terms are supportive sources to make the membrane stabilized outside of the true horizon. New topology can be fixed by different sources: Maxwell sources, and Gravitational sources. One should assume that the membrane is fictitious. An observer outside will observe these sources but will not see any while crossing the membrane [18].

3.1.2.1 The Maxwellian Membrane

Let (M, g) be a spacetime with a singularity. The membrane can be sourced by a Maxwell field with the action

$$S_{out}[A_\mu] = \int_{\mathcal{M}} d^4x\sqrt{-g}\Big\{-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + J^\mu A_\mu\Big\}, \tag{3.5}$$

$$\delta S_{out}[A_\mu] = \int_{\mathcal{M}} d^4x\sqrt{-g}\Big\{-\frac{1}{8\pi}F_{\mu\nu}\delta F^{\mu\nu} + J^\mu \delta A_\mu\Big\}. \tag{3.6}$$

This leads to the inhomogeneous Maxwell Equations:

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu. \tag{3.7}$$

and the homogenous ones follow from the definition $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. From the variation above, there is an extra term which corresponds to a boundary.

$$\begin{aligned} S_{surface}[A_\mu] &= \int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\mu\big(F^{\mu\nu}\delta A_\nu\big) \\ &= \int_{\partial\mathcal{M}} d^3x\sqrt{-h}\big(F^{\mu\nu}n_\mu\delta A_\nu\big), \end{aligned} \tag{3.8}$$

with n^μ is a space-like unit normal to the stretched horizon. As it seems, this term is not stationary on the boundary surface of stretched horizon. One should add a cancellation term [31]

$$\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\big(j_s^\mu A_\mu\big), \tag{3.9}$$

where $j_s^\mu = \frac{1}{4\pi}F^{\mu\nu}n_\nu$ where j_s^μ is a surface 4-current. The time component of the surface 4-current is the charge density on the membrane which cancels the orthogonal component of $\vec{E}$ while its spatial part cancels the tangential component of $\vec{B}$.

$$E_\perp = U_\mu F^{\mu\nu}n_\nu = 4\pi\sigma, \tag{3.10}$$

$$\vec{B}_\parallel = \epsilon^A{}_B\gamma^B{}_\mu F^{\mu\nu}n_\nu = 4\pi(\vec{j}_s \times \hat{n})^A. \tag{3.11}$$

The σ and j_s are local densities, the total charge can be found by integration on a constant universal time. One can arrive at the continuity equation of j_s on the surface membrane. Its 4-covariant derivative can be projected on 2-surface as:

$$\begin{aligned}\partial_\tau\sigma + (\gamma^A{}_\mu j_s^\mu)_{||A} &= \nabla_\mu(\frac{1}{4\pi}F^{\mu\nu}n_\nu) \\ &= -J^\nu n_\nu. \end{aligned} \tag{3.12}$$

The continuity equation on the surface makes the surface impenetrable to falling charges. They align on the surface to cancel fluxes through the stretched horizon [22].

Initial conditions outside the horizon require simplicity on the free-falling observers (FFOs). They should measure finite fields at the stretched horizon. In contrast to FFOs, FIDOs are infinitely accelerated at the membrane. Both frames can be transformed into each other by singular Lorentz boosts [22].

$$\vec{E}_{||}^{FIDO} = \hat{n}\times\vec{B}_{||}^{FIDO}, \tag{3.13}$$

shows that all the radiation is ingoing hence the black hole is assumed to be a perfect absorber.

$$B_{||}^A = \epsilon^A{}_B\gamma^B{}_\mu F^{\mu\nu}n_\nu = 4\pi(j_s\times\hat{n})^A. \tag{3.14}$$

The constraint equation becomes:

$$\vec{E}_{||} = \hat{n}\times 4\pi(\vec{j}_s\times\hat{n})^A = 4\pi\vec{j}_s - \cancel{\hat{n}(\hat{n}\cdot\vec{j}_s)} = 4\pi\vec{j}_s, \tag{3.15}$$

since $\vec{j}_s \in T_pH_s$ while $n_\mu V^\mu = 0 \iff V^\mu \in T_pH_s$.

(3.15) shows that the membrane mimicking the horizon obeys the Ohm's Law with a surface resistivity $\rho = 4\pi \simeq 377\Omega$. The Poynting flux through the membrane can be found by the non-relativistic equation [35].

$$\vec{S} = \frac{1}{4\pi}(\vec{E}\times\vec{B}) = \vec{j}_s^A\times 4\pi(\vec{j}_s\times\hat{n})_A = -4\pi(j_s)^2\hat{n} + \cancel{\vec{j}_s(\hat{n}\cdot\vec{j}_s)} \tag{3.16}$$

This can be integrated into FIDOs' time on the stretched horizon but, there might be some integration surfaces that time-slicing is not enough to choose synchronized clocks of FIDOs. If the square of the lapse function α is added to the integrand then for some universal time t, it is possible to synchronize all the FIDOs clocks on the surface. This position-dependent lapse function α is judiciously chosen to convert the locally measured energy flux to flux at infinity.

The power radiated into the black hole becomes a measure for the increasing irreducible mass [52] with respect to universal time slice t.

$$\frac{dM_{irr}}{dt} = -\int \alpha^2 \vec{S} \cdot d\vec{A}_{area} = \int \alpha^2 j_s^2 \rho \, dA_{area} \tag{3.17}$$

This is the *Joule Heating Law*. The black hole membrane acts as an ohmic resistor.[3]

3.1.2.2 The Gravitational Membrane

From the Lagrangian formulation of GR, we know that the EH action should be supported by the boundary end of spacetime via the action

$$S_{out} = \frac{1}{16\pi} \int_{\mathcal{M}} d^4x \sqrt{-g} R + \frac{1}{8\pi} \oint_{\partial\mathcal{M}} d^3x \sqrt{\pm h} K. \tag{3.18}$$

The integral is not over the stretched horizon. However, a time-like membrane which is wrapped in black hole geometry also has an interior boundary. As we have shown before [25]:

$$g^{\mu\nu} \delta R_{\mu\nu} = g^{\mu\gamma} \nabla_\gamma \big(g^{\lambda\nu} \nabla_\lambda \delta g_{\mu\nu} \big) - g^{\alpha\beta} \nabla_\mu \delta g_{\alpha\beta}, \tag{3.19}$$

where $\delta g_{\mu\nu}$ can be raised and lowered as an ordinary tensor. So:

$$\int_{\mathcal{M}} d^4x \sqrt{-g} \big\{ g^{\mu\nu} \delta R_{\mu\nu} \big\} = \int_{\partial\mathcal{M}} d^3x \sqrt{-h} n^\mu h^{\nu\alpha} \big\{ \nabla_\alpha \delta g_{\mu\nu} - \nabla_\mu \delta g_{\nu\alpha} \big\} \equiv I. \tag{3.20}$$

We choose the normal unit vector n^μ as outward-pointing. Applying the Leibniz rule to the integrand gives:

$$I = \int_{\partial\mathcal{M}} d^3x \sqrt{-h} h^{\mu\nu} \big\{ \delta g_{\mu\alpha} \nabla_\nu n^\alpha - \delta g_{\mu\nu} \nabla_\alpha n^\alpha \big\} + \int_{\partial\mathcal{M}} d^3x \sqrt{-h} h^{\mu\nu} \big\{ \nabla_\alpha (n^\alpha \delta g_{\mu\nu}) - \nabla_\nu (n^\alpha \delta g_{\alpha\mu}) \big\}. \tag{3.21}$$

The second integral vanishes. It can be proven by assuming variations only in the normal direction without loss of any generality [20].

$$I_1 = \int_{\partial\mathcal{M}} d^3x \sqrt{-h} h^{\beta\gamma} \big\{ \nabla_\alpha (n^\alpha \delta h_{\beta\gamma}) - \nabla_\gamma (n^\alpha \delta h_{\alpha\beta}) \big\} = 0, \tag{3.22}$$

[3] In this thesis we mostly look into the gravitational membrane.

where $g^{\perp}_{\alpha\beta} = h_{\alpha\beta}$. Proving this identity should require the Liebniz and 2+1+1 splitting rules:

$$I_1 = \int_{\partial\mathcal{M}} d^3x\sqrt{-h}\big\{\nabla_\alpha(h^{\beta\gamma}n^\alpha\delta h_{\alpha\gamma}) - \nabla_\alpha h^{\beta\gamma}n^\alpha\delta h_{\beta\gamma} - \nabla_\gamma(h^{\beta\gamma}n^\alpha\delta h_{\alpha\beta}) + (\nabla_\gamma h^{\beta\gamma})n^\alpha\delta h_{\alpha\beta}\big\} = 0 \quad (3.23)$$

One can use hypersurface covariant derivatives to find:

$$I_1 = \int_{\partial\mathcal{M}} d^3x\sqrt{-h}\big\{\nabla_\alpha(h^{\beta\gamma})n^\alpha\delta h_{\beta\gamma}) - (-\nabla_\alpha(n^\beta n^\gamma)) - (h^{\beta\gamma}n^\alpha\delta h_{\alpha\beta})_{|\gamma} - h^{\beta\gamma}n^\alpha a_\gamma\delta h_{\alpha\beta} - Kn^\beta n^\alpha\delta h_{\alpha\beta} - a^\beta n^\alpha\delta h_{\alpha\beta}\big\} = 0. \quad (3.24)$$

Since $a^\mu = n^\gamma\nabla_\gamma n^\mu = 0$ and $h^{\mu\nu}n_\nu = K^{\mu\nu}n_\nu = 0$.

$$I_1 = \int_{\partial\mathcal{M}} d^3x\sqrt{-h}\big\{\nabla_\alpha(h^{\beta\gamma}n^\alpha\delta h_{\beta\gamma}) - Kn^\beta n^\alpha\delta h_{\alpha\beta}\big\}. \quad (3.25)$$

One can multiply and divide by α and factor the variation out.

$$I_1 = \int_{\partial\mathcal{M}} d^3x\sqrt{-h}\Big\{\nabla_\alpha(h^{\beta\gamma}\frac{\alpha}{\alpha}n^\alpha\delta h_{\beta\gamma}) - K\big(\delta(n^\beta n^\alpha h_{\alpha\beta}) - n^\alpha h_{\alpha\beta}\delta n^\beta - n^\beta h_{\alpha\beta}\delta n^\alpha\big)\Big\}. \quad (3.26)$$

Since $h_{\alpha\beta}n^\beta = K_{\alpha\beta}n^\alpha = 0$ and $\alpha n^\mu \longrightarrow l^\mu$.

$$\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\big\{\nabla_\alpha(h^{\beta\gamma}\frac{1}{\alpha}l^\alpha\delta h_{\beta\gamma})\big\} = 0. \quad (3.27)$$

Let $h^{\beta\gamma}\frac{1}{\alpha}l^\alpha\delta h_{\beta\gamma} = \omega^\alpha$ then (3.27) becomes:

$$\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\big(\omega^\alpha{}_{|\alpha}\big) = 0. \quad (3.28)$$

This finalizes the variation of the boundary as:

$$\delta S = \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\big(Kh_{\mu\nu} - K_{\mu\nu}\big)\delta g^{\mu\nu}. \quad (3.29)$$

Let $t^{stretched}_{\mu\nu} = \frac{1}{8\pi}(Kh_{\mu\nu} - K_{\mu\nu}) \in T_p(H_s)\otimes T_p(H_s)$, since both of the tensors rely on the stretched horizon H_s, the normal component contractions will give zero. On the stretched horizon $g^{\mu\nu} = h^{\mu\nu}$. A surface term to add this variation to that makes the action stationary is:

$$\delta S_{surface} = \frac{-1}{2}\int d^3x\sqrt{-h}\big(t^{stretched}_{\mu\nu}\delta h^{\mu\nu}\big). \quad (3.30)$$

Similarly, for the electrodynamic cases where the surface charge induces a discontinuity on the surface, $t_{\mu\nu}^{stretched}$ induces a discontinuity in stretched horizon's extrinsic curvature $K_{\mu\nu}$. This discontinuity creates a junction on the surface which can be identified by the Israel Junction condition [53].

$$t_{\mu\nu}^{stretched} = \frac{1}{8\pi}([K]h_{\mu\nu} - [K]_{\mu\nu}) \tag{3.31}$$

where $[K] = K^+ - K^-$ such that $[K]$ is the difference between external universe embedding of H_s. We should identify $K^- = 0$ so that the stretched horizon interior to the black hole side is a flat embedding.

Einstein's equations are in Gauss-Codazzi type:

$$(t_{\mu\nu}^{stretched})_{|\nu} = -h^\mu{}_\lambda T^{\lambda\gamma} n_\gamma. \tag{3.32}$$

This hints on the gravitational membrane act like a fluid obeying Damour-Navier-Stokes equations. If one can write $K^\mu{}_\nu$ in terms of surface gravity κ and extrinsic curvature $k^A{}_B$ of 2-space-like section of H_s.

$$U^\gamma \nabla_\gamma n^\mu = \frac{\alpha}{\alpha} U^\gamma \nabla_\gamma (\frac{\alpha}{\alpha} n^\mu) = \frac{1}{\alpha^2} l^\gamma \nabla_\gamma l^\mu, \tag{3.33}$$

the surface gravity can be identified as:

$$\frac{1}{\alpha^2} l^\gamma \nabla_\gamma l^\mu = \frac{1}{\alpha^2} \kappa_r l^\mu. \tag{3.34}$$

Also,

$$k^\mu{}_\nu U^\nu U_\mu = -\kappa = -\frac{1}{\alpha}\kappa_r, \qquad \gamma^A{}_\alpha K^\alpha{}_\beta U^\beta = 0. \tag{3.35}$$

$\kappa_r = \alpha\kappa$ is called the renormalized surface gravity at the horizon. The extrinsic 2-space-like curvature can be identified as:

$$\begin{aligned} \gamma_A{}^\lambda \nabla_\lambda n^\mu = \frac{\alpha}{\alpha} \gamma_A{}^\lambda \nabla_\lambda n^\mu = \gamma_A{}^\lambda \frac{1}{\alpha} \nabla_\lambda l^\mu \\ \implies K^A{}_B = \gamma^\alpha{}_B K^\beta{}_\alpha \gamma^A{}_\beta = \frac{1}{\alpha} K^A{}_B, \end{aligned} \tag{3.36}$$

where K_{AB} is 2-space-like section of H_s extrinsic curvature.

Let us separate the trace and the traceless parts of K_{AB}.

$$K_{AB} = \sigma_{AB} + \frac{1}{2}\gamma_{AB}\Theta, \tag{3.37}$$

where σ_{AB} is the shear tensor. Stretched horizon stress tensor becomes [20]:

$$t^{AB}_{stretched} = \frac{1}{8\pi}\big\{ -\gamma_\alpha{}^A\gamma_\beta{}^B[K]^{\alpha\beta} + [K]\gamma^{AB}\big\}$$
$$= \frac{1}{8\pi}\big\{ -\sigma^{AB} - \frac{1}{2}\gamma^{AB}\Theta + \gamma^{AB}(\Theta+\kappa)\big\}$$
$$= \frac{1}{8\pi}\big\{ -\sigma^{AB} + \gamma^{AB}(\frac{1}{2}\Theta+\kappa)\big\}. \tag{3.38}$$

3.2 Appliacation of the Membrane Paradigm to Different Black Hole Geometries

To find the 2+1+1 decomposition of spacetime, we need to write the metric as:

$$ds^2 = -U_\mu U_\nu dx^\mu dx^\nu + n_\mu n_\nu dx^\mu dx^\nu + \gamma_{\mu\nu}dx^\mu dx^\nu \tag{3.39}$$

Let $r = r_H$ be the horizon radius such that it is a null hypersurface H. Null hypersurface has a null normal l^μ

The extrinsic curvature $K_{\mu\nu}$ can be found by:

$$K_{\mu\nu} = \nabla_\mu n_\nu \tag{3.40}$$

and its trace will be K. The stretched horizon stress tensor is given as $t^{stretched}_{\mu\nu} = \frac{1}{8\pi}(Kh_{\mu\nu} - K_{\mu\nu})$.

$$h_{\mu\nu} = \gamma_{\mu\nu} - U_\mu U_\nu \tag{3.41}$$

$\gamma_{\mu\nu}$ is the metric on 2D surface both orthogonal to U^μ and n^μ.

3.2.1 Construction of Gravitational Membrane: A static spacetime:

A Schwarzschild-type geometry can be identified as [54]:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_2^2, \tag{3.42}$$

where $f(r) = 1 - \frac{2M(r)}{r}$ as we mentioned before. Our job is to decompose the geometry as $ds^2 = -U_\mu U_\nu dx^\mu dx^\nu + n_\mu n_\nu dx^\mu dx^\nu + \gamma_{\mu\nu}dx^\mu dx^\nu$. We have to pick U_μ and n_μ in such a way that at $r = r_H$ should be null. We know that $M(r)$ can only have global charges $M(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^3}{6}$.

Even with the most general case, in spacetimes with the compact horizon, one can detect that there exists some r_H that:

$$f(r_H) = 1 - \frac{2M(r_H)}{r_H} = 0. \tag{3.43}$$

From (3.43), the decomposition is easier for static spacetimes.

$$U_\mu dx^\mu = f^{\frac{1}{2}} dt \implies U_\mu = f^{\frac{1}{2}} \delta^t{}_\mu, \tag{3.44}$$

$$n_\mu dx^\mu = f^{-\frac{1}{2}} dr \implies n_\mu = f^{-\frac{1}{2}} \delta^r{}_\mu, \tag{3.45}$$

$$\gamma_{\mu\nu} dx^\mu dx^\nu = r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{3.46}$$

In 2D surface coordinates,

$$\gamma_{AB} = \begin{pmatrix} r^2 & 0 \\ a & r^2 \sin^2\theta \end{pmatrix}, \tag{3.47}$$

where $\{A, B\} = \{\theta, \phi\}$. The dual basis can be identified as:

$$U^\mu = g^{\mu\nu} U_\nu = g^{\mu\nu} f^{\frac{1}{2}} \delta^t{}_\nu = -f^{-\frac{1}{2}} \delta^\mu{}_t. \tag{3.48}$$

Similarly for the normal vector

$$n^\mu = g^{\mu\nu} n_\nu = g^{\mu\nu} f^{-\frac{1}{2}} \delta^r{}_\nu = f^{\frac{1}{2}} \delta^\mu{}_r \tag{3.49}$$

such that $U_\mu U^\mu = -1$, $n_\mu n^\mu = 1$ while $U_\mu n^\mu = 0$. One can calculate the 3D extrinsic curvature tensor as:

$$K_{\mu\nu} = \nabla_\mu n_\nu = \partial_\mu [f^{-\frac{1}{2}} \delta^r{}_\nu] - \Gamma^\gamma_{\mu\nu} [f^{-\frac{1}{2}} \delta^r{}_\gamma], \tag{3.50}$$

(3.50) can be analyzed component by component:

$$
\begin{aligned}
\nabla_r n_r &= \partial_r(f^{-\frac{1}{2}}\delta^r{}_r)) - \Gamma^r_{rr}(f^{-\frac{1}{2}}\delta^r{}_r) \\
&= -\frac{1}{2}f^{-\frac{3}{2}}\partial_r f - (\partial_r f)f^{-1}f^{-\frac{1}{2}}(-\frac{1}{2}) = 0,
\end{aligned}
\tag{3.51}
$$

$$
\begin{aligned}
\nabla_t n_t &= -\Gamma^r_{tt}(f^{-\frac{1}{2}}\delta^r{}_r) \\
&= (\partial_r f)(f)f^{-\frac{1}{2}}(-\frac{1}{2}) \\
&= -\frac{1}{2}(\partial_r f)f^{\frac{1}{2}},
\end{aligned}
\tag{3.52}
$$

$$
\begin{aligned}
\nabla_\theta n_\theta &= -\Gamma^r_{\theta\theta}(f^{-\frac{1}{2}}\delta^r{}_r) \\
&= -rf^{\frac{1}{2}},
\end{aligned}
\tag{3.53}
$$

$$
\begin{aligned}
\nabla_\phi n_\phi &= -\Gamma^r_{\phi\phi}(f^{-\frac{1}{2}}\delta^r{}_r) \\
&= -r\sin^2\theta f^{\frac{1}{2}},
\end{aligned}
\tag{3.54}
$$

where it can be represented in matrix and vector form as:

$$
K_{\mu\nu} = -f^{\frac{1}{2}}\begin{pmatrix} \frac{1}{2}(\partial_r f) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix}, \tag{3.55}
$$

$$
= -\frac{1}{2}f^{-\frac{1}{2}}\partial_r f U_\mu U_\nu + \frac{1}{r}f^{\frac{1}{2}}\gamma_{\mu\nu}. \tag{3.56}
$$

One can easily find its contraction as a trace of $K_{\mu\nu}$.

$$
\begin{aligned}
g^{\mu\nu}K_{\mu\nu} &= K^\mu{}_\mu \\
&= \frac{1}{2}f^{-\frac{1}{2}}\partial_r f U_\mu U^\mu + \frac{f^{\frac{1}{2}}}{r}g^{\mu\nu}\gamma_{\mu\nu} \\
&= \frac{1}{2}f^{-\frac{1}{2}}\partial_r f + \frac{f^{\frac{1}{2}}}{r}(\frac{r^2}{r^2} + \frac{r^2\sin^2\theta}{r^2\sin^2\theta}) \\
&= \frac{1}{2}f^{-\frac{1}{2}}\partial_r f + \frac{2f^{\frac{1}{2}}}{r}
\end{aligned}
\tag{3.57}
$$

According to the 2+1+1 approach, extrinsic curvature on H_S can be identified by choosing $\alpha = f^{\frac{1}{2}}$ as a renormalization factor.

$$K_{\mu\nu} \longrightarrow \alpha^{-1} k_{\mu\nu} + \alpha^{-1} \kappa U_\mu U_\nu \tag{3.58}$$

where $k_{\mu\nu} = \gamma_{\mu A}\gamma_{\nu B} k^{AB}$ and κ is the surface gravity [54].

While $\alpha \longrightarrow 0$, extrinsic curvature of the stretched membrane converges to the horizon and $K_{\mu\nu}$ becomes the surface gravity. The trace of $K_{\mu\nu}$ diverges since f has a pole at $r = r_H$.

$$\begin{aligned} \lim_{\alpha \to 0} \kappa = \frac{1}{2} f^{-\frac{1}{2}} \partial_r f = \lim_{\alpha \to 0} \frac{1}{2}\frac{1}{\alpha} \partial_r(\alpha^2) \\ Tr(\alpha^{-1} k_{\mu\nu} - \alpha^{-1}\kappa U_\mu U_\nu)|_{(r=r_H)}. \end{aligned} \tag{3.59}$$

(3.58),(3.41),(3.37),(3.57) can be combined to find the stress tensor in terms of given parameters:

$$\begin{aligned} t^{stretched}_{\mu\nu} &= \frac{1}{8\pi\alpha}\Big\{(\Theta + \kappa)(\gamma_{\mu\nu} - U_\mu U_\nu) + \kappa U_\mu U_\nu - (\sigma_{\mu\nu} + \frac{1}{2}\Theta\gamma_{\mu\nu})\Big\} \\ &= \frac{1}{8\pi\alpha}\Big\{\gamma_{\mu\nu}(\Theta - \frac{1}{2}\Theta) - U_\mu U_\nu) - \Theta U_\mu U_\nu - U_\mu U_\nu(\kappa - \kappa) - \sigma_{\mu\nu}\Big\} \\ &= \frac{1}{8\pi\alpha}\Big\{(\frac{1}{2}\Theta + \kappa)\gamma_{\mu\nu} - \Theta U_\mu U_\nu - \sigma_{\mu\nu}\Big\}. \end{aligned} \tag{3.60}$$

One can compare this stretched stress tensor with a viscous fluid stress tensor, this brings up a correspondence between the dual fluid and membrane itself.

$$\begin{aligned} t^{viscous}_{\mu\nu} = \alpha^{-1}\rho U_\mu U_\nu + \alpha^{-1}\gamma_{\mu A}\gamma_{\nu B}\big(P\gamma^{AB} - 2\eta\sigma^{AB} - \zeta\Theta\gamma^{AB}\big) \\ + \pi^A(\gamma_{\mu A} U_\nu + \gamma_{\nu B} U_\nu), \end{aligned} \tag{3.61}$$

where ρ: energy density, ζ: bulk viscosity, P: pressure, η: shear viscosity, π^A: momentum density, σ^{AB}: shear tensor on H_s, Θ: null geodesic expansion near event horizon. If we identify (3.60) and (3.61)

$$\rho = -\frac{1}{8\pi}\Theta, \qquad 2\eta = \frac{1}{8\pi}, \tag{3.62}$$

$$P - \zeta\Theta = \frac{1}{8\pi}(\frac{1}{2}\Theta + \kappa), \qquad \pi^A = 0. \tag{3.63}$$

If we solve these algebraic equations:

$$\rho = -\frac{1}{8\pi}\Theta, \qquad \eta = \frac{1}{16\pi}, \tag{3.64}$$

$$P = \frac{\kappa}{8\pi}, \qquad \zeta = -\frac{1}{16\pi}, \tag{3.65}$$

$$\pi^A = 0, \tag{3.66}$$

one can find these scalars geometrically, too.

$$t_{\mu\nu}^{stretched} = \frac{1}{8\pi}\{(\frac{f^{-\frac{1}{2}}}{2}\partial_r f + \frac{f^{\frac{1}{2}}}{r})\gamma_{\mu\nu} - \frac{2f^{\frac{1}{2}}}{r}U_\mu U_\nu\}, \tag{3.67}$$

such that

$$\Theta = \frac{2}{r}f, \qquad \sigma_{AB} = 0, \tag{3.68}$$

$$\kappa = \frac{\partial_r f}{2}. \tag{3.69}$$

If $f = (1 - \frac{2M}{r})$ (Schwarzschild geometry[54]), we have:

$$\Theta = \frac{2}{r_H}f(r_H) = \frac{2}{2M}(1-1) = 0, \qquad \sigma_{AB} = 0, \tag{3.70}$$

$$\lim_{r\to r_H}\kappa = \frac{1}{2}\partial_r(1 - \frac{2M}{r})|_{r=r_H} \equiv \frac{1}{2}\frac{2M}{r^2}|_{r=2M} = \frac{1}{4M}. \tag{3.71}$$

Our aim is to generalize the static geometry as much as possible, if we restate the transport coefficients of generic static black holes, we can classify them by choosing f.

Surface gravity κ, energy density ρ, pressure P, expansion Θ will change for different choices of f. However, η, σ^{AB}, ζ will be a classification for spherical horizons and will not change their values.

3.2.2 A Classification for Black Holes

Transport coefficients are:[55]

$$\Theta = \frac{2}{r}(1 - \frac{2M(r)}{r}), \qquad \kappa = \partial_r\big[\frac{1}{2}(1 - \frac{2M(r)}{r})\big], \tag{3.72}$$

$$P = \frac{1}{16\pi}\partial_r\big[1 - \frac{2M(r)}{r}\big], \qquad \rho = \frac{1}{4\pi r}(1 - \frac{2M(r)}{r}), \tag{3.73}$$

where $M(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^3}{6}$.

3.2.2.1 Schwarzschild Black hole (Q=Λ=0)

$$\Theta = \frac{2}{r}(1 - \frac{2M}{r}), \qquad \kappa = \frac{M}{r^2}, \tag{3.74}$$

$$P = \frac{1}{2}(\frac{M}{4\pi r^2}), \qquad \rho = \frac{-1}{4\pi r^2}(r - 2M), \tag{3.75}$$

If $l_\mu = \alpha n_\mu = \alpha U_\mu$ when $\alpha \longrightarrow 0, r = r_H = 2M$

$$\Theta = \frac{2}{2M}(1 - \frac{2M}{2M}) = 0, \quad \lim_{r \to r_H} \kappa = \frac{1}{2}\partial_r(1 - \frac{2M}{r})|_{r=r_H} \equiv \frac{1}{2}\frac{2M}{r^2}|_{r=2M} = \frac{1}{4M}. \tag{3.76}$$

3.2.2.2 Rindler Modified Schwarzschild Black hole

Let us define a deviation from Schwarzschild spacetime's geodesics with a Rindler acceleration a [56],

$$M(r) = (M - ar^2). \tag{3.77}$$

The horizon of this deviation can be easily found from the vanishing condition of the inverse radial component of the metric:

$$g^{rr} = 0,$$
$$f(r) = 1 - \frac{2M}{r} + 2ar = 0,$$
$$r_h = r \to \frac{\pm\sqrt{16aM + 1} - 1}{4a}.$$

then the transport coefficients become:

$$\Theta = \frac{2}{r}\big[1 - \frac{2(M - ar^2)}{r}\big], \qquad \kappa = \partial_r\big[\frac{1}{2}(1 - \frac{2(M - ar^2)}{r})\big], \tag{3.78}$$

$$P = \frac{1}{16\pi}\partial_r\big[1 - \frac{2(M - ar^2)}{r}\big], \qquad \rho = \frac{1}{4\pi r}\big[1 - \frac{2(M - ar^2)}{r}\big], \tag{3.79}$$

while $r \longrightarrow r_h$ the surface gravity on the true horizon becomes:

$$\kappa_{|_{r=r_h}} = a + \frac{16a^2M}{\big[-1 + (1 + 16aM)^{\frac{1}{2}}\big]^2}. \tag{3.80}$$

In the limit $a \longrightarrow 0$, we again have the Schwarzschild surface gravity $\kappa_{Schwarzschild} = \frac{1}{4M}$.

3.2.2.3 Reissner-Nordström Black hole ($\Lambda = 0$)

For

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{3.81}$$

the horizon can be found at:

$$g^{rr} = 0 \implies r^2 - 2Mr + Q^2 = 0$$
$$\implies r_H^{1,2} = \frac{2M \pm \sqrt{4M^2 - 4Q^2}}{2} = +M \pm \sqrt{M^2 - Q^2}. \tag{3.82}$$

The transport coefficients become:

$$\Theta = \frac{2}{r}\big[1 - \frac{2M}{r} + \frac{Q^2}{r^2}\big], \qquad \kappa = \frac{M}{r^2} - \frac{Q^2}{r^3},$$
$$P = \frac{1}{8\pi}\big[\frac{M}{r^2} - \frac{Q^2}{r^3}\big] = \frac{1}{2}\Big(\frac{1}{4\pi r^2}\big[M - \frac{Q^2 r}{r^2}\big]\Big),$$
$$\rho = \frac{-2}{8\pi r}\big[1 - \frac{2M}{r} + \frac{Q^2}{r^2}\big]. \tag{3.83}$$

On the horizon, the transport coefficients read as:

$$\Theta(r \longrightarrow r_H) = \frac{2}{M + \sqrt{M^2 - Q^2}}(1 - 1) = 0,$$
$$\lim_{r \to r_H} \kappa = \frac{M}{r^2} - \frac{Q^2}{r^3},$$
$$= \frac{-Q^2}{(M + \sqrt{M^2 - Q^2})^3} + \frac{M}{(M + \sqrt{M^2 - Q^2})^2}. \tag{3.84}$$

One can see that surface gravity κ for RN black hole can be reduced to Schwarzschild by setting $Q = 0$.

$$\lim_{Q \to 0} \kappa = \frac{1}{4M}. \tag{3.85}$$

3.2.2.4 Schwarzschild-AdS Black Hole (Q=0)

Let $f(r) = (1 - \frac{2M}{r} - \frac{\Lambda r^2}{3})$ Since,

$$\Theta = \frac{2}{r}\big[1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}\big], \qquad \kappa = \frac{M}{r^2} - \frac{\Lambda r}{3},$$
$$P = \frac{1}{8\pi}\big[\frac{M}{r^2} - \frac{\Lambda r}{3}\big], \qquad \rho = -\frac{1}{4\pi r^2}\big[\frac{-\Lambda r^3}{3} + r^2 - 2M\big]. \tag{3.86}$$

The horizon radius can be found by:

$$g^{rr} = 0 \implies (1 - \frac{2M}{r} - \frac{\Lambda r^2}{3}) = 0. \tag{3.87}$$

The solution can be found by the algorithm:

$$x^3 + 3bx^2 + 6cx + 2d = 0,$$
$$p = 2c - b^2 \qquad q = 3bc - b^3 - d,$$
$$r = \sqrt{q^2 + p^3}. \tag{3.88}$$

Hence the solution is:

$$x = -b + (q - r)^{\frac{1}{3}} + (q + r)^{\frac{1}{3}}. \tag{3.89}$$

In our case, $b = 0$, $c = \frac{-1}{2\Lambda}$, $d = \frac{3M}{\Lambda}$. This concludes that $p = \frac{-1}{\Lambda}$, $q = \frac{-3m}{\Lambda}$, $r = \sqrt{\frac{9M^2\Lambda - 1}{\Lambda}}$

$$r_H = \Big[\frac{-3M}{\Lambda} + \sqrt{\frac{9M^2\Lambda - 1}{\Lambda^3}}\Big]^{\frac{1}{3}} + \Big[\frac{-3M}{\Lambda} - \sqrt{\frac{9M^2\Lambda - 1}{\Lambda^3}}\Big]^{\frac{1}{3}}. \tag{3.90}$$

$\Lambda < 0$ gives the real solutions to the equations.

3.2.2.5 A Stringy Solution is Possible: GGHS-Schwarzschild Solution:

The metric in the Schwarzschild coordinates are:[57]

$$ds^2 = -\frac{f(r)}{h(r)}dt^2 + \big(\frac{f(r)}{h(r)}\big)^{-1}dr^2 + h(r)r^2(d\theta^2 + sin^2\theta d\phi^2). \tag{3.91}$$

where $f(r) = 1 - \frac{r_1}{r}$, $h(r) = 1 - \frac{r_2}{r}$ and $r_1 = (2M - \frac{Q^2}{M})$,$r_2 = \frac{Q^2}{M}$. where M and Q are the mass and charge respectively. This solution represents a dilatonic black hole geometry [58]. One can apply the membrane paradigm with similar identifications to the Schwarzschild case. The horizon can be found by:

$$g^{rr} = (1 - \frac{r_1}{r})(1 + \frac{r_2}{r})^{-1} = 0 \implies r_H = r_1 = (2M - \frac{Q^2}{M}). \tag{3.92}$$

Hence, to exist, the dilaton black hole needs its charges. Otherwise, it collapses into Schwarzschild's black hole and loses dilatonic symmetry. In that sense, in the regime $Q \longrightarrow 0 \implies h \longrightarrow 1$ should give the static solution. Let us define a Parikh-Wilzchek-like 2+1+1 decomposition such that

$$U_\mu dx^\mu = \big(\frac{f}{h}\big)^{\frac{1}{2}}dt, \qquad n_\mu dx^\mu = \big(\frac{h}{f}\big)^{-\frac{1}{2}}dr, \tag{3.93}$$

$$\gamma_{\mu\nu} = h \begin{pmatrix} r^2 & 0 \\ 0 & r^2 sin^2\theta \end{pmatrix}, \tag{3.94}$$

where $U_\mu U^\mu = -1$ and $n^\mu n_\mu = 1$, $\{A, B\} = \{\theta, \varphi\}$. This means the 3-dimensional space-like surface with spherical topology is rescaled with the factor $h(r) = 1 - \frac{r_2}{r}$. Spacetime is 4D, horizon H is 3D, and embedded surface[20] with both U^μ and n^μ normal to is 2D. One can calculate the acceleration as follows:

$$a_\nu = n^\mu \nabla_\mu n_\nu = 0. \tag{3.95}$$

There is no acceleration as in the Schwarzschild case. The extrinsic curvature becomes:

$$K_{\mu\nu} = \nabla_\mu n_\nu$$
$$= \frac{1}{2}\sqrt{\frac{f}{h}} \begin{pmatrix} \frac{-h\partial_r f + f\partial_r h}{h^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r(\frac{2h+r\partial_r h}{2}) & 0 \\ 0 & 0 & 0 & -r\sin^2\theta(\frac{2h+r\partial_r h}{2}) \end{pmatrix}. \tag{3.96}$$

The scalar extrinsic curvature is the contraction:

$$K = g^{\mu\nu}K_{\mu\nu} = \frac{\sqrt{\frac{h}{f}}\left[rh\partial_r f + f(r\partial_r h + 4h)\right]}{2rh^2} \tag{3.97}$$

$$t^{stretched}_{\mu\nu} = -\Theta U_\mu U_\nu - \sigma_{\mu\nu} + (\frac{1}{2}\Theta + \kappa)\gamma_{\mu\nu}, \tag{3.98}$$

while the dual viscous fluid is:

$$t^{viscous}_{\mu\nu} = f^{-\frac{1}{2}}\rho U_\mu U_\nu + \alpha^{-1}\gamma_{\mu A}\gamma_{\nu B}\left(P\gamma^{AB} - 2\eta\sigma^{AB} - \zeta\Theta\gamma^{AB}\right) + \pi^A(\gamma_{\mu A}U_\nu + \gamma_{\nu B}U_\nu), \tag{3.99}$$

where transport coefficients can be identified as:[4]

$$\rho = -\frac{1}{8\pi}, \qquad \eta = \frac{1}{16\pi},$$
$$P = \frac{\kappa}{8\pi}, \qquad \zeta = -\frac{1}{16\pi},$$
$$\pi^A = 0, \tag{3.101}$$

Now:

$$t_{\mu\nu} = (Kh_{\mu\nu} - K_{\mu\nu}) = -K(\gamma_{\mu\nu} - U_\mu U_\nu) + K_{\mu\nu},$$

$$t_{\mu\nu} = \frac{1}{16\pi}\Big\{\gamma_{\mu\nu}\frac{h}{f}\frac{(2f + r\partial_r f)}{rh} - \frac{4h + 2\partial_r h}{rh(\frac{h}{f})^{\frac{1}{2}}}U_\mu U_\nu\Big\}. \tag{3.102}$$

If Dilatonic charge $Q = 0 \implies h = 1$

$$t^{str}_{\mu\nu|Q\longrightarrow 0} = \frac{1}{8\pi}\{(\frac{f^{-\frac{1}{2}}}{2}\partial_r f + \frac{f^{\frac{1}{2}}}{r})\gamma_{\mu\nu} - \frac{2f^{\frac{1}{2}}}{r}U_\mu U_\nu\}, \tag{3.103}$$

[4] One should not forget that for $D+1$ dimensional spacetime:

$$\zeta = -2\frac{(D-2)}{(D-1)}\eta \tag{3.100}$$

by comparing with $t^{stretched}$. In the current dilatonic black hole case:

$$\Theta = \frac{2h + \partial_r h}{rh\left(\frac{h}{f}\right)^{\frac{1}{2}}}, \qquad \sigma_{AB} = 0, \tag{3.104}$$

$$\kappa = \frac{rh\partial_r f - f\partial_r h}{2rfh\left(\frac{h}{f}\right)^{\frac{1}{2}}}. \tag{3.105}$$

again in $Q \longrightarrow 0$, these reduce to Schwarzschild's transport coefficients. For the true horizon the black hole is expectedly finite:

$$\Theta|_{r=r_H} = 0, \qquad \kappa = \frac{1}{2}\frac{M}{\sqrt{2}\sqrt{\frac{M^2}{2M^2-Q^2}}(2M^2 - Q^2)}, \tag{3.106}$$

which reduces to Schwarzschild surface gravity κ_{Sch} when $Q \longrightarrow 0$.

$$\kappa^{Q\longrightarrow 0} = \frac{1}{4M}. \tag{3.107}$$

3.2.2.6 A Five-Dimensional Metric: Schwarzschild-Tangherlini Spacetime

The metric in the Schwarzschild coordinates is [59]:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2[d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\varphi^2] \tag{3.108}$$

where $f(r) = 1 - \frac{2M(r)}{r}$, $M(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^2}{6}$. Let us define a Parikh-Wilzchek-like 3+1+1 decomposition such that

$$U_\mu dx^\mu = f^{\frac{1}{2}}dt, \qquad n_\mu dx^\mu = f^{-\frac{1}{2}}dr, \tag{3.109}$$

$$\gamma_{\mu\nu} = \begin{pmatrix} r^2 & 0 & 0 \\ 0 & r^2\sin^2\theta & 0 \\ 0 & 0 & r^2\cos^2\theta \end{pmatrix}, \tag{3.110}$$

where $U_\mu U^\mu = -1$ and $n^\mu n_\mu = 1$, $\{A, B\} = \{\theta, \varphi, \phi\}$.

Spacetime is 5D, horizon H is 4D, and the embedded surface with both U^μ and n^μ

normal to is 3D. One can calculate the acceleration as follows [60]:

$$
\begin{aligned}
a_\nu = n^\mu \nabla_\mu n_\nu &= n^\mu \big(\partial_\mu n_\nu - \Gamma^\gamma_{\mu\nu} n_\gamma\big) \\
&= f^{\frac{1}{2}} \delta^\mu{}_r \Big[\partial_\mu (f^{-\frac{1}{2}} \delta^r{}_\nu) - \Gamma^\gamma_{\mu\nu} (f^{-\frac{1}{2}} \delta^r{}_\gamma)\Big] \\
&= f^{\frac{1}{2}} \partial_r f^{-\frac{1}{2}} + \frac{\partial_r f}{2f} \\
&= \cancel{\partial_r (f^{\frac{1}{2}} f^{-\frac{1}{2}})} - \frac{\partial_r f^{\frac{1}{2}}}{f^{\frac{1}{2}}} + \frac{\partial_r f}{2f} \\
&= -\partial_r \ln(f^{\frac{1}{2}}) + \partial_r (\frac{1}{2} \ln(f)) = 0. \qquad (3.111)
\end{aligned}
$$

There is no acceleration as in the 4D case $a^\mu = 0$.

The extrinsic curvature of the 4D stress tensor embedded in the 5D surface is:

$$
K_{\mu\nu} = \nabla_\mu n_\nu = \partial_\mu (f^{-\frac{1}{2}} \delta^r{}_\nu) - \Gamma^\gamma_{\mu\nu} (f^{-\frac{1}{2}} \delta^r{}_\gamma). \qquad (3.112)
$$

The component-by-component analysis gives:

$$
\nabla_t n_t = 0 - \Gamma^r_{tt} (f^{-\frac{1}{2}}) = -\frac{1}{2} (f^{\frac{1}{2}}) \partial_r f,
$$

$$
\begin{aligned}
\nabla_r n_r &= 0, \\
\nabla_\theta n_\theta &= -\Gamma^r_{\theta\theta} (f^{-\frac{1}{2}}) = r (f^{\frac{1}{2}}),
\end{aligned}
$$

$$
\nabla_\phi n_\phi = -\Gamma^r_{\phi\phi} (f^{-\frac{1}{2}}) = r (f^{\frac{1}{2}}) \sin^2\theta,
$$

$$
\nabla_\varphi n_\varphi = -\Gamma^r_{\varphi\varphi} (f^{-\frac{1}{2}}) = r (f^{\frac{1}{2}}) \cos^2\theta,
$$

where $\Gamma^r_{tt} = \frac{1}{2} f \partial_r f$. The extrinsic curvature becomes:

$$
K_{\mu\nu} = -f^{\frac{1}{2}} \begin{pmatrix} \frac{1}{2}\partial_r f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 & 0 \\ 0 & 0 & 0 & -r\sin^2\theta & 0 \\ 0 & 0 & 0 & 0 & -r\cos^2\theta \end{pmatrix}, \qquad (3.113)
$$

$$
= -\frac{1}{2} f^{-\frac{1}{2}} \partial_r f U_\mu U_\nu + \frac{f^{\frac{1}{2}}}{r} \gamma_{\mu\nu}. \qquad (3.114)
$$

Even though $dim(H_s) = 4$ the structure did not change! The trace of $K_{\mu\nu}$ is:

$$\begin{aligned} g^{\mu\nu}K_{\mu\nu} &= g^{tt}K_{tt} + g^{rr}K_{rr} + g^{\theta\theta}K_{\theta\theta} + g^{\phi\phi}K_{\phi\phi} + g^{\varphi\varphi}K_{\varphi\varphi} \\ &= -f^{-1}(-f^{\frac{1}{2}}\partial_r f) + 0 + \frac{f^{\frac{1}{2}}r}{r^2} + \frac{f^{\frac{1}{2}}r\sin^2\theta}{r^2\sin^2\theta} + \frac{\frac{1}{2}r\cos^2\theta}{r\cos^2\theta}, \\ &= \frac{\partial_r f}{f^{\frac{1}{2}}} + \frac{3}{rf^{-\frac{1}{2}}}. \end{aligned} \tag{3.115}$$

Hence $K = \frac{\partial_r f}{2f^{\frac{1}{2}}} + \frac{3f^{\frac{1}{2}}}{r}$ will be the extrinsic curvature scalar.

$K_{\mu\nu}$ can be divided into 2 parts:

$$\begin{aligned} K_{\mu\nu} &= f^{-\frac{1}{2}}k_{\mu\nu} - f^{-\frac{1}{2}}\kappa U_\mu U_\nu, \\ K_{tt} &= -f^{\frac{1}{2}}\kappa U_t U_t \implies \kappa = \frac{1}{2}\partial_r f, \\ K_{rr} &= 0. \end{aligned} \tag{3.116}$$

Then the surface gravity on the 3D cross-section is:

$$k_{\theta\theta} = f^{\frac{1}{2}}K_{\theta\theta} = f^{\frac{1}{2}}(-r),$$

$$k_{\phi\phi} = f^{\frac{1}{2}}K_{\phi\phi} = f^{\frac{1}{2}}(-r\sin^2\theta),$$

$$k_{\varphi\varphi} = f^{\frac{1}{2}}K_{\phi\phi} = f^{\frac{1}{2}}(-r\cos^2\theta), \tag{3.117}$$

where k_{AB} is 3D extrinsic curvature! It can be divided into traceful and traceless parts:

$$k_{AB} = \sigma_{AB} + \frac{1}{3}\Theta\gamma_{AB}, \tag{3.118}$$

where $\gamma_{AB}\gamma^{AB} = 3$.

$$t^{stretched}_{\mu\nu} = -\Theta U_\mu U_\nu - \sigma_{\mu\nu} + (\frac{2}{3}\Theta + \kappa)\gamma_{\mu\nu}. \tag{3.119}$$

while the dual viscous fluid is:

$$\begin{aligned} t^{viscous}_{\mu\nu} = \alpha^{-1}\rho U_\mu U_\nu + \alpha^{-1}\gamma_{\mu A}\gamma_{\nu B}\big(P\gamma^{AB} - 2\eta\sigma^{AB} - \zeta\Theta\gamma^{AB}\big) \\ + \pi^A(\gamma_{\mu A}U_\nu + \gamma_{\nu B}U_\nu), \end{aligned} \tag{3.120}$$

where transport coefficients can be identified as:

$$\rho = -\frac{1}{8\pi}\Theta, \qquad \eta = \frac{1}{16\pi},$$
$$P = \frac{\kappa}{8\pi}, \qquad \zeta = -\frac{1}{12\pi},$$
$$\pi^A = 0. \tag{3.121}$$

Now:

$$t_{\mu\nu} = (Kh_{\mu\nu} - K_{\mu\nu}) = -K(\gamma_{\mu\nu} - U_\mu U_\nu) + K_{\mu\nu}$$
$$= K\gamma_{\mu\nu} - KU_\mu U_\nu + \frac{1}{2}f^{-\frac{1}{2}}\partial_r f U_\mu U_\nu - \frac{f^{\frac{1}{2}}}{r}\gamma_{\mu\nu},$$

$$t_{\mu\nu} = \gamma_{\mu\nu}\Big(\frac{\partial_r f}{2f^{\frac{1}{2}}} + \frac{2f^{\frac{1}{2}}}{r}U_\mu U_\nu\Big) - \frac{3f^{\frac{1}{2}}}{r}U_\mu U_\nu. \tag{3.122}$$

By comparing with $t_{\mu\nu}^{stretched}$[5].

$$\Theta = \frac{3f}{r}, \qquad \sigma_{AB} = 0,$$
$$\kappa = \frac{1}{2}\partial_r f. \tag{3.124}$$

3.2.2.7 A Lower Dimensional Solution is Possible: BTZ Black hole

For the solutions of 4D black holes, we considered 2+1+1 decomposition of space-time in the case of 3D black holes, we should have 1+1+1 decomposition. The metric can be written as:

$$ds^2 = -U_\mu U_\nu dx^\mu dx^\nu + n_\mu n_\nu dx^\mu dx^\nu + \gamma_\mu \gamma_\nu dx^\mu dx^\nu, \tag{3.125}$$

where γ_μ is a 1D spacelike cross-section of the stretched horizon H_s.

The extrinsic curvature becomes:

$$\nabla_\mu n_\nu \equiv K_{\mu\nu}$$

[5] n-dimensional spheres will not change the structure of the transport coefficients:

$$\Theta = \frac{d-2}{r}(1 - \frac{M}{r^{d-3}}), \qquad \kappa = \frac{d-3}{2}\frac{M}{r^{d-2}}. \tag{3.123}$$

Since for $(D+1)$-dim spacetimes with $(D-1)$-spacelike cross-section, the bulk viscosity and shear viscosity can be read as:

$$P = \frac{\kappa}{8\pi}, \qquad \eta = \frac{1}{16\pi},$$
$$\zeta = -\frac{D-2}{8\pi(D-1)}.$$

For 3D spacetime, horizon $\dim(H) = 2D$, cross-section is 1D. Hence,

$$P = \frac{\kappa}{8\pi}, \qquad \eta = \frac{1}{16\pi},$$
$$\zeta = 0.$$

Mainly, transport coefficients depend on the underlying theory however, the shear viscosity in Einstein's Gravity is universal. Also, according to Stoke's Hypothesis [61] black hole thermal states must be in thermal equilibrium if and only if $\zeta = 0$.

Now, let us write down our metric and use the Membrane Paradigm's full machinery.

$$ds^2 = -(-M - \Lambda r^2)dt^2 + (-M - \Lambda r^2)dr^2 + r^2 d\phi^2, \tag{3.126}$$

where $\Lambda < 0$ to have non-existing CTCs. The 1+1+1 split becomes:

$$\begin{aligned} U_\mu dx^\mu = f^{\frac{1}{2}} dt &\implies U_\mu = f^{\frac{1}{2}} \delta^t{}_\mu, \\ n_\mu dx^\mu = f^{-\frac{1}{2}} dr &\implies n_\mu = f^{-\frac{1}{2}} \delta^r{}_\mu, \\ \gamma_\mu dx^\mu &= r^2 d\phi^2, \end{aligned} \tag{3.127}$$

where $f = (-M - \Lambda r^2)$. The extrinsic curvature can be calculated easily:

$$K_{\mu\nu} = -f^{\frac{1}{2}} \begin{pmatrix} \frac{1}{2}\partial_r f & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -r \end{pmatrix}. \tag{3.128}$$

The extrinsic scalar curvature then becomes:

$$K = \frac{f^{\frac{1}{2}}}{r} + \frac{1}{2} f^{\frac{-1}{2}} \partial_r f. \tag{3.129}$$

The metric on the horizon is:

$$h_{\mu\nu} = -f dt^2 + r^2 d\phi^2. \tag{3.130}$$

Then, the stress tensor on the stretched horizon is:

$$t^{stretched}_{\mu\nu} = \gamma_\mu \gamma_\nu \Big(\frac{\partial_r f}{2 f^{\frac{1}{2}}}\Big) - \frac{f^{\frac{1}{2}}}{r} U_\mu U_\nu, \tag{3.131}$$

and from the viscous fluid description, one can check that

$$t_{\mu\nu} = \frac{f^{\frac{1}{2}}}{8\pi}\Big(- \Theta U_\mu U_\nu - \sigma_{\mu\nu} + \kappa\gamma_\mu\gamma_\nu\Big). \tag{3.132}$$

Then one can identify these as:

$$\pi^\mu = 0, \qquad \zeta = 0,$$
$$\sigma_{\mu\nu} = 0.$$

Hence the viscous fluid stress tensor reduces to:

$$t^{viscous}_{\mu\nu} = \alpha^{-1}\rho U_\mu U_\nu + \alpha^{-1}\gamma_\mu\gamma_A\gamma_\nu\gamma_B(P\gamma^A\gamma^B). \tag{3.133}$$

The identification reads:

$$\Theta = \frac{f}{r}, \qquad \kappa = \frac{\partial_r f}{2}. \tag{3.134}$$

3.2.3 Construction of Gravitational Membrane: A rotating spacetime:

Let us rewrite the metric for a generic rotating black hole in Kerr [62] form with Boyer-Lindquist coordinates [11]:

$$\begin{aligned} ds^2 = -f dt^2 + \frac{\Sigma(r,\theta)}{\Delta(r,\theta)} dr^2 + 2a(f-1)\sin^2\theta dt d\phi \\ + \Sigma(r,\theta) d\theta^2 + \big\{r^2 + a^2 - [1-f]a^2\sin^2\theta\big\}\sin^2\theta d\phi^2 \end{aligned} \tag{3.135}$$

where $f = 1 - \frac{2M(r)}{r}\big(\frac{r^2}{\Sigma}\big)$, $\Sigma(r,\theta) = r^2 + a^2\cos^2\theta$, $\Delta(r,\theta) = f\Sigma + a^2\sin^2\theta$ There is a more compact way to write this metric such that 2+1+1 formalism is applicable.[54]

$$F_t^2 = f, \qquad F_r^2 = \frac{\Sigma(r,\theta)}{\Delta(r,\theta)},$$

$$\begin{aligned} F_\phi^2 = \Big[r^2 + a^2 + (1-f)a^2 sin^2\theta\Big]\sin^2\theta, \\ \omega = -a(f-1)\sin^2\theta F_t^{-1}, \end{aligned} \tag{3.136}$$

with these definitions, the generic metric becomes:

$$ds^2 = -F_t^2 dt^2 - 2\omega F_t dt d\phi + F_\phi^2 d\phi^2 + F_r^2 dr^2 + \Sigma d\theta^2, \tag{3.137}$$

where (3.137) can be completed to square[43]

$$ds^2 = -(F_t dt^2 + \omega d\phi)^2 + F_\phi^2 d\phi^2 + \omega^2 d\phi^2 + F_r^2 dr^2 + \Sigma d\theta^2$$
$$= -(F_t dt^2 + \omega d\phi)^2 + F_r^2 dr^2 + \Sigma d\theta^2 + (F_\phi^2 + \omega) d\phi^2. \quad (3.138)$$

Now, we should identify this metric with a 2+1+1 dictionary:

$$ds^2 = (-U_\mu U_\nu + n_\mu n_\nu + \gamma_{AB} e^A{}_\mu e^B{}_\nu) dx^\mu dx^\nu. \quad (3.139)$$

Let $U_\mu dx^\mu = F_t dt + \omega d\phi$, $n_\mu dx^\mu = F_r dr$, $\gamma_{\mu\nu} dx^\mu dx^\nu = \Sigma d\theta^2 + (F_\phi^2 + \omega^2) d\phi^2$ one can easily show that their dual vectors are:

$$U^\mu \partial_\mu = F_t^{-1} \partial_t \implies U^\mu = F_t^{-1} \delta_t{}^\mu,$$
$$n^\mu \partial_\mu = F_r^{-1} \partial_r \implies n^\mu = F_r^{-1} \delta_r{}^\mu, \quad (3.140)$$

such that $U^\mu U_\mu$ is time-like, $n^\mu n_\mu$ is space-like, $U^\mu \partial_\mu \in T_p H_s$ while $U^\mu n^\mu = 0$.

It can be observed that, unlike the Schwarzschild-type black holes, n^μ vector field has non-trivial acceleration, it induces non-zero momentum on the membrane.

$$n^\mu \nabla_\mu n_\nu = (F_r^{-1} \delta_r{}^\mu) \nabla_\mu (F_r^{-1} \delta^r{}_\nu) = (F_r^{-1} \delta_r{}^\mu) \big\{ \partial_\mu (F_r^{-1} \delta^r{}_\nu) - \Gamma^\gamma_{\mu\nu} (F_r^{-1} \delta^r{}_\gamma \big\} \quad (3.141)$$

(3.141) is non-zero when $\mu = r, \gamma = r, \nu = r$

$$(F_r^{-1} \delta_r{}^r) \big\{ \partial_r (F_r^{-1} \delta_r{}^r) - \Gamma^r_{rr} (F_r^{-1} \delta_r{}^r) \big\}. \quad (3.142)$$

Hence, the equation for rr-component acceleration becomes:

$$\frac{\Delta}{\Sigma} \Big\{ \partial_r (\frac{\Sigma}{\Delta}) - \Gamma^r_{rr} (\frac{\Sigma}{\Delta}) \Big\} \neq 0. \quad (3.143)$$

Hence $n^\mu \nabla_\mu n_\nu = a_\nu \neq 0$. [6]

Let us calculate the extrinsic curvature and membrane-stress tensor:

$$K_{\mu\nu} = h_\mu{}^\gamma \nabla_\gamma n_\nu = h_{\sigma\mu} h^{\sigma\gamma} \nabla_\gamma n_\nu = (\gamma_{\mu\sigma} - U_\mu U_\sigma)(\gamma^{\gamma\sigma} - U^\gamma U^\sigma) \nabla_\gamma n_\nu,$$

$$K_{\mu\nu} = \gamma_{\mu\sigma} \gamma^{\gamma\sigma} \nabla_\gamma n_\nu - \gamma_{\mu\sigma} U^\sigma U^\gamma \nabla_\gamma n_\nu - U_\mu U_\sigma \gamma^{\sigma\gamma} \nabla_\gamma n_\nu - U_\mu U_\sigma U^\sigma U^\gamma \nabla_\gamma n_\nu,$$

$$K_{\mu\nu} = h_\mu{}^\gamma \nabla_\gamma n_\nu = h_\mu{}^\gamma (\partial_\gamma n_\nu - \Gamma^\sigma_{\gamma\nu} n_\sigma). \quad (3.144)$$

[6] The calculations through membrane paradigm, we do not need the explicit structure of Hajicek fields, however, non-zero momentum will completely change the membrane paradigm correspondence. So, they should be explicitly shown.

One can again find the components of the extrinsic curvature term by term

$$K_{t\phi} = h_t{}^{\gamma}(\partial_\gamma n_\phi - \Gamma^{\sigma}_{\gamma\phi} n_\sigma) = h_t{}^{t}(\partial_t n_\phi - \Gamma^{r}_{t\phi} n_r) = -\frac{\omega\partial_r F_t + F_t \partial_r \omega}{2F_r}$$

$$K_{\theta\theta} = h_\theta{}^{\gamma}(\partial_\theta n_\theta - \Gamma^{\sigma}_{\gamma\theta} n_\sigma) = h_\theta{}^{\gamma}(0 - \Gamma^{r}_{\theta\theta} n_r) = \frac{\partial_r \Sigma}{2F_r}$$

$$K_{\phi\phi} = h_\phi{}^{\phi}(\partial_\phi n_\phi - \Gamma^{r}_{\phi\phi} n_r) = \frac{\partial_r F_\phi^2}{2F_r}$$

$$K_{tt} = h_t{}^{t}(\partial_t n_t - \Gamma^{r}_{tt} n_r) = -\frac{\partial_r F_t^2}{2F_r}$$

Hence in matrix form:

$$K_{\mu\nu} = h_\mu{}^{\gamma} \nabla_\gamma n_\nu = \begin{pmatrix} -\partial_r F_t^2 & 0 & 0 & -\partial_r(\omega F_r) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_r \Sigma^2 & 0 \\ -\partial_r(\omega F_r) & 0 & 0 & \partial_r F_\phi^2 \end{pmatrix}. \tag{3.145}$$

The trace of 3D extrinsic curvature is:

$$\begin{aligned} h^{\mu\nu} K_{\mu\nu} &= h^{tt} K_{tt} + h^{rr} K_{rr} + h^{\theta\theta} K_{\theta\theta} + h^{\phi\phi} K_{\phi\phi} \\ &= \frac{1}{2F_r}\Big\{ F_t^{-2} \partial_r F_t^2 + 0 + \frac{\partial_r \Sigma^2}{\Sigma^2} + \frac{\partial_r F_\phi^2}{F_\phi^2 + \omega^2} \Big\}. \end{aligned} \tag{3.146}$$

since $\partial_r\big(\ln(F_r^2)\big) = \frac{\partial_r F_r^2}{F_r^{-2}}$. The trace can be written in a more compact form:

$$K = h^{\mu\nu} K_{\mu\nu} = \frac{1}{2F_r}\Big\{ \partial_r \big[\ln\big(\Sigma^2 F_t^2 (\omega^2 + F_\phi^2)\big) \big] \Big\} \tag{3.147}$$

As we did in a static solution, we need a lapse function α which acts as a renormalization factor which takes the time-like vectors and space-like vectors to null ones as $\alpha \longrightarrow 0$ Let $\alpha = F_r^{-1}$ such that $l^\mu = F_r^{-1} n^\mu$. This is not a random choice. The norm of the vector is:

$$l^\mu l_\mu = F_r^{-2} = \frac{\Delta}{\Sigma}, \tag{3.148}$$

where Δ has a $r = r_H$ value such that in the limit $r \longrightarrow r_H$, l^μ is null [54].

The 3D extrinsic curvature can be written in terms of 2D decomposition while the time-like vector U^μ is normal to the surface. This surface can be chosen from $\{\theta, \phi\}$ 2×2 block of $K_{\mu\nu}$. The block acts as a metric on the 2D surface which is both normal to n^μ and U^μ.

$$k_{AB} = \frac{1}{F_r^2}\begin{pmatrix} \partial_r\Sigma & 0 \\ 0 & \partial_r(\omega^2 + F_\phi^2) \end{pmatrix}. \tag{3.149}$$

Since every tensor can be separated into trace and traceless parts. $k_{AB} = \sigma_{AB} + \frac{1}{2}\Theta\gamma_{AB}$ where σ_{AB} is shear tensor, Θ is the expansion. $Tr(\sigma_{AB}) = 0$.

k_{AB} has 2 independent equations to solve and a traceless constraint on σ_{AB} that lowers degrees of freedom on the hypersurface.

$$k_{\theta\theta} = \sigma_{\theta\theta} + \frac{1}{2}\Theta\gamma_{\sigma\sigma}, \qquad k_{\phi\phi} = \sigma_{\theta\theta} + \frac{1}{2}\Theta\gamma_{\phi\phi}, \tag{3.150}$$

$$\gamma^{\theta\theta}\sigma_{\theta\theta} + \gamma^{\phi\phi}\sigma_{\phi\phi} = 0. \tag{3.151}$$

Logically, as in the case of 3D $K_{\mu\nu}$ the trace of k_{AB} will give:

$$\gamma^{AB}k_{AB} = \cancel{\gamma^{AB}\sigma_{AB}} + \frac{1}{2}\Theta\gamma^{AB}\gamma_{AB} = \frac{2}{2}\Theta = \Theta. \tag{3.152}$$

Hence,

$$\Theta = \frac{1}{F_r^2}\partial_r\Big[\ln(\Sigma(\omega^2 + F_\phi^2))\Big]. \tag{3.153}$$

The expansion Θ is the 2×2 block-trace of 4×4 K_{AB}. Since we know k_{AB} and Θ uniquely, we can use the traceless condition again to find σ_{AB}.

1. $\gamma^{\theta\theta}\sigma_{\theta\theta} = -\gamma^{\phi\phi}\sigma_{\phi\phi}$,
2. $k_{\theta\theta} - \frac{1}{2}\Theta\gamma_{\theta\theta} = \sigma_{\theta\theta}$,
3. $k_{\phi\phi} - \frac{1}{2}\Theta\gamma_{\phi\phi} = \sigma_{\phi\phi}$.

By using (1) and (2),

$$\sigma_{\theta\theta} = \frac{1}{2F_r^2}\partial_r\Sigma - \frac{1}{2}\Sigma\frac{1}{2F_r^2}\partial_r\big[\ln\big(\Sigma(\omega^2 + F_\phi^2)\big)\big]$$

$$\sigma_{\theta\theta} = \frac{1}{4F_r^2}\Big\{\frac{\partial_r\Sigma}{\Sigma}\Sigma - \Sigma\partial_r\ln(\Sigma(\omega^2 + F_\phi^2))\Big\}$$

$$\sigma_{\theta\theta} = \frac{\Sigma}{4F_r^2}\big\{\partial_r\ln(\Sigma) - \partial_r[\Sigma(\omega^2 + F_\phi^2)]\big\}$$

$$= \frac{\Sigma}{4F_r^2}\Big\{\partial_r\frac{\Sigma}{(\omega^2 + F_\phi^2)}\Big\}. \tag{3.154}$$

One can find $\sigma_{\phi\phi}$ with similar arguments:

$$\sigma_{\phi\phi} = k_{\phi\phi} - \frac{1}{2}\Theta\gamma_{\phi\phi}$$

$$\sigma_{\phi\phi} = \frac{1}{2F_r^2}\partial_r(\omega^2 + F_\phi^2) - \frac{1}{4F_r^2}\partial_r[\Sigma(\omega^2 + F_\phi^2)](F_\phi^2 + \omega^2)$$

$$= \frac{(\omega^2 + F_\phi^2)}{4F_r^2}\partial_r \ln\Big(\frac{\omega^2 + F_\phi^2}{\Sigma}\Big) \quad (3.155)$$

So to understand the transport coefficients of the dual fluid in terms of rotating spacetimes, we have to find membrane stress tensor $t_{\mu\nu}^{stretched}$. Einstein's gravity concludes:

$$t_{\mu\nu}^{stretched} = \frac{1}{8\pi}\big(Kh_{\mu\nu} - K_{\mu\nu}\big), \quad (3.156)$$

where $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ and for Kerr-type black holes we found that:

$$K = \frac{1}{2F_r}\partial_r \ln[\Sigma F_t^2(\omega^2 + F_\phi^2)], \quad (3.157)$$

$$K_{\mu\nu} = \frac{1}{2F_r}\begin{pmatrix} -\partial_r F_t^2 & 0 & 0 & -\partial_r(\omega F_r) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_r\Sigma^2 & 0 \\ -\partial_r(\omega F_r) & 0 & 0 & \partial_r F_\phi^2 \end{pmatrix}, \quad (3.158)$$

The stress tensor can be found component by component:

$$t_{tt}^{stretched} = \frac{1}{8\pi}(Kh_{tt} - K_{tt}) = \frac{1}{8\pi}(-KF_t^2 + \frac{1}{2F_r}\partial_r F_t^2),$$

$$t_{rr}^{stretched} = 0,$$

$$t_{\theta\theta}^{stretched} = \frac{1}{8\pi}(Kh_{\theta\theta} - K_{\theta\theta}), = \frac{1}{8\pi}\big[K\Sigma - \frac{1}{F_r}\partial_r\Sigma\big],$$

$$t_{\phi\phi}^{stretched} = \frac{1}{8\pi}(Kh_{\phi\phi} - K_{\phi\phi}) = \frac{1}{8\pi}\big(KF_\phi^2 - \frac{1}{2F_r}\partial_r F_\phi^2\big),$$

$$t_{\phi\theta}^{stretched} = \frac{1}{8\pi}(Kh_{\phi\theta} - K_{\phi\theta}) = \frac{1}{8\pi}[-K\omega F_t + \frac{1}{2F_r}\partial_r(\omega F_t)],$$

By following the same algorithm we followed for static spacetimes, the stress tensor for viscous fluid for rotating metrics is given in (3.61). Unlike static spacetimes, $\pi^A \neq 0$, this complicates the algebra a little bit but it is not unsolvable [54]. Now, taking the gravitational membrane as a viscous fluid, we should identify $t_{\mu\nu}^{stretched} \simeq t_{\mu\nu}^{fluid}$. This will allow us to find the fluid membrane properties of rotating black holes. Firstly, let us start with the momentum density that identifies with off-diagonal $\{\theta, \phi\}$ and tt-components.

$$
\begin{aligned}
t_{tt}^{fluid} = \frac{1}{\alpha}\rho U_t U_t = \frac{1}{8\pi}(-KF_t^2 + \frac{1}{2F_r}\partial_r F_t^2) \equiv t_{tt}^{stretched} \\
\rho = \frac{\alpha}{U_t U_t} - KF_t^2 + \frac{1}{2F_r}\partial_r F_t^2 \\
\rho = -\frac{K}{F_r} + \frac{1}{2F_r^2}\partial_r(\ln F_t^2)
\end{aligned} \tag{3.159}
$$

is the energy density of the fluid membrane. The momentum density can be considered as,

$$
t_{\mu\nu}^{fluid,momentum} = \pi^A(\gamma_{\mu A}U_\nu + \gamma_{\nu A}U_\mu). \tag{3.160}
$$

Since $U_\theta = 0 \implies \pi^\theta = 0$,

$$
t_{\phi\theta}^{fluid} = \pi^\phi\big(\gamma_{\phi\phi}U_t + \gamma_{t\phi}U_\phi\big) + \frac{1}{\alpha}\rho U_t U_\phi = \frac{1}{8\pi}\big(-K\omega F_t + \frac{1}{2F_r}\partial_r(\omega F_t)\big) \tag{3.161}
$$

$$
\begin{aligned}
\pi^\phi &= \frac{1}{8\pi\gamma_{\phi\phi}U_t}\big[-K\omega F_t + \frac{1}{F_r}\partial_r(\omega F_t)\big] - \frac{1}{\alpha\gamma_{\phi\phi}U_t}\rho U_t U_\phi \\
&= \frac{1}{8\pi}\frac{1}{F_t(\omega^2 + F_\phi^2)}\big[-K\omega F_t + \frac{1}{2F_r}\partial_r(\omega F_r) - \frac{F_r\omega}{(\omega^2 + F_\phi^2)}\rho\big].
\end{aligned} \tag{3.162}
$$

Since we have already found the energy density in (3.159), the ϕ-momentum reads

$$
\pi^\phi = \frac{1}{16\pi}\big[\frac{\omega}{F_r(\omega^2 + F_\phi^2)}\partial_r \ln(\frac{\omega}{F_t})\big]. \tag{3.163}
$$

So the momentum density on the 2D surface is:

$$
(\pi^\theta, \pi^\phi) = \Big(0, \frac{1}{16\pi}\frac{\omega}{F_r(\omega^2 + F_\phi^2)}\partial_r \ln(\frac{\omega}{F_t})\Big). \tag{3.164}
$$

We have 2 more equations to identify and factor out:

$$
t_{\theta\theta}^{stretched} \equiv t_{\theta\theta}^{fluid} = \frac{1}{8\pi}(K\Sigma - \frac{1}{2F_r}\partial_r\Sigma) = \frac{1}{\alpha}\gamma_{\theta\theta}\gamma_{\phi\phi}\Big[(P - \Theta\zeta)\gamma^{\theta\theta} - 2\eta\sigma^{\theta\theta}\Big] \tag{3.165}
$$

$$
t_{\phi\phi}^{stretched} \equiv t_{\phi\phi}^{fluid} = \frac{1}{8\pi}(KF_\phi^2 - \frac{1}{2F_r}\partial_r F_\phi^2)
$$

$$= \frac{1}{\alpha}\rho U_\phi U_\phi + 2\pi^\phi(\gamma_{\phi\phi}) \quad (3.166)$$
$$+\frac{1}{\alpha}\gamma_{\phi\phi}\gamma_{\phi\phi}\Big[(P-\eta\Theta)\gamma_{\theta\theta} - \frac{2\eta}{\alpha}\sigma^{\phi\phi}\Big]U_\phi.$$

If we take $(P-\Theta\zeta)$ parenthesis for $(\phi\phi)$ and $(\theta\theta)$ components from equation (3.2.3):

$$(P-\Theta\zeta) = \big(\alpha t^{fluid}_{\theta\theta} + 2\eta\sigma_{\theta\theta})\big)\frac{1}{\gamma_{\theta\theta}}$$
$$= \Big\{\alpha\big[t^{fluid}_{\phi\phi} - \frac{\rho}{\alpha}U_\phi U_\phi - 2\pi^\phi(\gamma_{\phi\phi}U_\phi)\big] + \frac{1}{\alpha}2\eta\sigma_{\phi\phi}\Big\}\frac{1}{\gamma_{\phi\phi}} \quad (3.167)$$

From now on, we know every function except η:

$$\eta = \frac{1}{16\pi}\big[1 + \frac{\partial_r\omega^2 - 2K\omega^2F_r - 2\omega^2F_r\partial_r\ln(\frac{\omega}{F_t})}{(\omega^2+F_\phi^2)\partial_r\ln(\frac{\Sigma}{\omega^2+F_\phi^2})}\big] - \frac{\rho\omega^2F_r^2}{(\omega^2+F_\phi^2)\partial_r\ln(\frac{\rho^2}{\omega^2+F_\phi^2})} \quad (3.168)$$

using (3.147), the nominator of 2^{nd} and 3^{rd} term in η becomes:

$$\partial_r\omega^2 - 2K\omega^2F_r - 2\omega^2F_r\partial_r\ln(\frac{\omega}{F_t}) + K\omega^2F_r - \frac{1}{2F_r^2}\partial_rF_t^2 = 0 \quad (3.169)$$

divide all with ω^2, then the whole nominator cancels with each other. Hence, for Kerr-type black holes and Schwarzschild-type black holes shear viscosity is a geometrized constant.

$$\eta = \frac{1}{16\pi} \quad (3.170)$$

Now, we should use the traceless condition (3.151) acting on t^{fluid}_{AB}, with σ^{AB}.

$$(P-\Theta\zeta) = -\frac{1}{2}\rho\gamma^{\phi\phi}U^\phi U_\phi - \alpha\pi^\phi U_\phi + \frac{\alpha}{2}\big(\gamma^{\theta\theta}t_{\theta\theta} + \gamma^{\phi\phi}t_{\phi\phi}\big). \quad (3.171)$$

$$P = -\zeta\Theta - \frac{1}{2}\rho\gamma^{\phi\phi}U^\phi U_\phi - \alpha\pi^\phi U_\phi + \frac{1}{2F_r}\big(\gamma^{\theta\theta}t_{\theta\theta} + \gamma^{\phi\phi}t_{\phi\phi}\big) \quad (3.172)$$

where the last term in (3.172) is:

$$\frac{1}{2F_r}\big(\gamma^{\theta\theta}t_{\theta\theta} + \gamma^{\phi\phi}t_{\phi\phi}\big) = \frac{1}{2F_r}\Big\{\frac{1}{\Sigma}\frac{1}{8\pi}[K\Sigma - \frac{1}{F_r}\partial_r\Sigma]$$
$$+\frac{1}{8\pi}\frac{1}{F_\phi^2+\omega^2}(KF_\phi^2 - \frac{1}{2F_r^2}\partial_rF_\phi^2)\Big\}. \quad (3.173)$$

(3.173) can be written in terms of expansion Θ, the 2^{nd} and 4^{th} term gives logarithms and by inserting $\cong \partial_r\omega^2$ terms to the equation allows us to complete the Leibniz rule.

$$\frac{1}{2F_r}\big(\gamma^{\theta\theta}t_{\theta\theta} + \gamma^{\phi\phi}t_{\phi\phi}\big) = -\frac{1}{16\pi}\Theta + \frac{1}{16\pi F_r}\Big(2K - \frac{K\omega^2}{(\omega^2+F_\phi^2)} + \frac{\partial_r\omega^2}{2F_r(\omega^2+F_\phi^2)}\Big) \quad (3.174)$$

$$P = \zeta\Theta - \frac{1}{16\pi}\Big(\frac{K}{F_r} + \frac{1}{2F_r^2}\partial_r{}_t^2 + \frac{1}{F_r^2}\frac{\omega^2}{(\omega^2+F_\phi^2)}\partial_r(\ln\omega - \ln F_t)\Big) - \frac{1}{16\pi}\Theta$$
$$+\frac{1}{16\pi F_r}\Big(2K - \frac{K\omega^2}{\omega^2+F_\phi^2} + \frac{\partial_r\omega^2}{2F_r(\omega^2+F_\phi^2)}\Big). \quad (3.175)$$

$2^{nd}, 3^{rd}, 6^{th}, 7^{th}$ terms are canceling each other and finally:

$$P = -\frac{K}{16\pi F_r} + \frac{2K}{16\pi F_r} + \zeta\Theta - \frac{1}{16\pi}\Theta$$
$$= \frac{1}{8\pi}\frac{K}{F_r} + (\zeta - \frac{1}{16\pi})\Theta. \quad (3.176)$$

We have already found the transport coefficients of dual static fluid, our solutions must induce the static case while angular momentum $a \longrightarrow 0$.

If $\lim_{a\to 0} P_{Kerr-type} = P_{Schwarzschild-type}$. We can identify ζ as a constant. From the previous section, we know that:

$$\zeta = -\frac{1}{16\pi}. \quad (3.177)$$

[7] Choosing $\zeta = -\frac{1}{16\pi}$ simplifies the pressure equation further.

$$P = \frac{1}{8\pi}\frac{K}{F_r} + (\zeta - \frac{1}{16\pi})\Theta = \frac{1}{8\pi}\Big(\frac{K}{F_r} - \Theta\Big)$$
$$= \frac{1}{8\pi}\Big\{\frac{1}{2F_r^2}\partial_r\ln[\Sigma F_t^2(\omega^2+F_\phi^2)] - \frac{1}{2F_r^2}\partial_r\ln(\Sigma(\omega^2+F_\phi^2))\Big\}$$

$$P = \frac{1}{16\pi F_r^2}\partial_r\ln F_t^2 = \frac{1}{8\pi F_r^2}\partial_r\ln F_t. \quad (3.178)$$

The calculations above led us to find Kerr-type geometries' viscous fluid dual on the stretched horizon. For Kerr metric functions are left generic one can choose any combination of $M(r) = M - \frac{Q^2}{2r} + \frac{\Lambda r^2}{6}$ to find different metrics' transport coefficients.

[7] Even in rotating geometries, the dual fluid bulk viscosity is negative. This negativity coincides with a decrease in viscosity thus a decrease in entropy.

3.2.3.1 Kerr Black Hole ($Q = 0, \Lambda = 0$)

The extrinsic curvature becomes:

$$K_{tt} = \frac{\sqrt{2}M(a^2\cos^2\theta - r^2)\sqrt{\frac{a^2+r(r-2M)}{a^2\cos 2\theta + a^2 + 2r^2}}}{(a^2\cos^2\theta + r^2)^2},$$

$$K_{t\phi} = \frac{\sqrt{2}aM\sin^2\theta(a^2\cos^2\theta - r^2)\sqrt{\frac{a^2+r(r-2M)}{a^2\cos 2\theta + a^2 + 2r^2}}}{(a^2\cos^2\theta + r^2)^2},$$

$$K_{\theta\theta} = \sqrt{2}r\sqrt{\frac{a^2 + r(r-2M)}{a^2\cos 2\theta + a^2 + 2r^2}},$$

$$K_{\phi\phi} = \frac{\sqrt{2}\sqrt{a^2 + r(r-2M)}(2a^2Mr\sin^4\theta + (a^2 + r^2)\sin^2\theta(a^2\cos^2\theta + r^2))}{(a^2\cos 2\theta + a^2 + 2r^2)^{3/2}}.$$

The extrinsic curvature scalar becomes:

$$K = \frac{2\sqrt{2}(a^2(M\sin^2\theta - M + r\cos^2\theta + r) + r^2(2r - 3M))}{\sqrt{a^2 + r(r-2M)}(a^2\cos 2\theta + a^2 + 2r^2)^{3/2}}.$$

The shear tensor reads as:

$$\sigma_{\theta\theta} = \frac{a^2\sin^2\theta(r - M)}{a^2\cos 2\theta + a^2 + 2r(r-2M)},$$

$$\sigma_{\phi\phi} = \frac{2a^2\sin^4\theta(M - r)(a^2 + r(r-2M))}{(a^2\cos 2\theta + a^2 + 2r(r-2M))^2}.$$

The momentum can be calculated as follows:

$$\begin{aligned}\pi_\phi = \frac{1}{A[r,\theta]}aM[&-3a^6 + a^4r(13M - 2r) + 4a^2r^2(-4M^2 - 3Mr + 3r^2)\\ &- 4a^2\cos 2\theta(a^4 + a^2r(r-4M) + r^2(4M^2 - 3Mr + r^2))\\ &- a^4\cos 4\theta(a^2 + r(2r - 3M)) + 8r^4(r - 2M)^2],\end{aligned}$$

where $A[r,\theta] = 4\pi[a^2 + r(r-2M)]^{3/2}(a^2\cos 2\theta + a^2 + 2r^2)^{7/2}\sqrt{2 - \frac{4Mr}{a^2\cos^2\theta + r^2}}$

$$\eta = \frac{1}{16\pi}, \qquad \zeta = -\frac{1}{16\pi}.$$

The pressure becomes:

$$P = -\frac{M[a^2 + r(r-2M)](a^2\cos^2\theta - r^2)}{\pi(a^2\cos 2\theta + a^2 + 2r^2)^2[a^2\cos 2\theta + a^2 + 2r(r-2M)]}.$$

The null expansion is:

$$\begin{aligned}\Theta = \frac{1}{B[r,\theta]}a^4\cos 4\theta(r - M) &+ a^4(M + 7r) + 4a^2r^2(5r - 11M)\\ &+ 4a^2r\cos 2\theta(2a^2 + r(3r - 5M)) + 16r^3(r - 2M)^2,\end{aligned}$$

where $B[r,\theta] = (a^2\cos 2\theta + a^2 + 2r^2)^2[a^2\cos 2\theta + a^2 + 2r(r - 2M]).$

3.2.3.2 Kerr-Newman Black Hole ($\Lambda = 0$)

The extrinsic curvature becomes:

$$K_{tt} = \frac{\sqrt{a^2 + r(r-2M)}[a^2 M \cos^2\theta + r(Q^2 - Mr)]}{(a^2\cos^2\theta + r^2)^{5/2}},$$

$$K_{t\phi} = \frac{a\sin^2\theta\sqrt{a^2 + r(r-2M)}[a^2 M \cos^2\theta + r(Q^2 - Mr)]}{(a^2\cos^2\theta + r^2)^{5/2}},$$

$$K_{\theta\theta} = r\sqrt{\frac{a^2 - 2Mr + r^2}{a^2\cos^2\theta + r^2}},$$

$$K_{\phi\phi} = \frac{\sqrt{a^2 + r(r-2M)}[(a^2 + r^2)\sin^2\theta(a^2\cos^2\theta + r^2) - a^2\sin^4\theta(Q^2 - 2Mr)]}{2(a^2\cos^2\theta + r^2)^{3/2}}.$$

The extrinsic curvature scalar becomes:

$$K = \frac{1}{Z[r,\theta]} 2\sqrt{2}\sqrt{a^2 + r(r-2M)}[a^2(M\sin^2\theta + r\cos^2\theta)$$
$$+ a^2(r - M) + r(r(2r - 3M) + Q^2)].$$

$$Z[r,\theta] = (a^2 + r(r-2M) + Q^2)[a^2\cos 2\theta + a^2 + 2r^2]^{3/2}$$

The shear tensor reads as:

$$\sigma_{\theta\theta} = -\frac{a^2\sin^2\theta(M - r)(a^2 + r(r-2M))}{(a^2 + r(r-2M) + Q^2)[a^2\cos 2\theta + a^2 + 2(-2Mr + Q^2 + r^2)]},$$

$$\sigma_{\phi\phi} = \frac{2a^2\sin^4\theta(M - r)(a^2 + r(r-2M))}{[a^2\cos 2\theta + a^2 + 2(-2Mr + Q^2 + r^2)]^2}.$$

The momentum can be calculated as follows:

$$\pi_\phi = \frac{1}{A[r,\theta]}\Big\{r\big[a^3(Q^2 r(7M - 3r) + 2Mr^2(2r - 3M) - 2Q^4)$$
$$+ ar(-2r^3(4M^2 + Q^2) + 8Mr^2(M^2 + Q^2) - Q^2 r(10M^2 + Q^2) + 3MQ^4 + 2Mr^4)\big]$$
$$+ a^5\cos^4\theta\big[-2a^2M + 6M^2r - M(Q^2 + 4r^2) + Q^2 r\big] - a^3\cos^2\theta\big[a^2(-2M^2r + M(Q^2 - 2r^2)$$
$$+ 3Q^2 r) + 8M^3r^2 - 6M^2r(Q^2 + r^2) + M(Q^4 + Q^2r^2 + 2r^4) - Q^4 r + Q^2 r^3\big]\Big\},$$

$$\pi_\theta = 0,$$

where,

$$A[r,\theta] = \sqrt{2}\pi\sqrt{a^2 + r(r-2M)}(a^2 + r(r-2M) + Q^2)[a^2\cos 2\theta$$
$$+ a^2 + 2r^2]^{7/2}\sqrt{\frac{Q^2 - 2Mr}{a^2\cos^2\theta + r^2} + 1},$$

such that,

$$\eta = \frac{1}{16\pi}, \qquad \zeta = -\frac{1}{16\pi}.$$

The pressure becomes:

$$P = -\frac{[a^2 + r(r-2M)][a^2 M \cos^2\theta + r(Q^2 - Mr)]}{8\pi(a^2\cos^2\theta + r^2)^2[a^2\cos^2\theta + r(r-2M) + Q^2]}.$$

The null expansion is:

$$\Theta = \frac{1}{B[r,\theta]}\Big\{[a^2+r(r-2M)]\big[a^4\cos 4\theta(r-M)+a^4(M+7r)+4a^2r\cos 2\theta(2(a^2+Q^2) - 5Mr + 3r^2) + 4a^2r(r(5r-11M)+6Q^2) + 16r(r(r-2M)+Q^2)^2\big]\Big\},$$

where

$$B[r,\theta] = (a^2 + r(r-2M) + Q^2)a^2\cos 2\theta + a^2 + 2r^2)^2[a^2\cos 2\theta + a^2 + 2(-2Mr + Q^2 + r^2)].$$

3.2.3.3 Kerr-(A)dS Black Hole ($Q = 0$)

The extrinsic curvature becomes:

$$K_{tt} = \frac{\sqrt{a^2 - 2Mr - \frac{\Lambda r^3}{3} + r^2}\big[3a^2\cos^2\theta(2M + \Lambda r^2) - 6Mr^2 + \Lambda r^4\big]}{6(a^2\cos^2\theta + r^2)^{5/2}},$$

$$K_{t\phi} = \frac{a\sin^2\theta\sqrt{a^2 - 2Mr - \frac{\Lambda r^3}{3} + r^2}\big[3a^2\cos^2\theta(2M + \Lambda r^2) - 6Mr^2 + \Lambda r^4\big]}{6(a^2\cos^2\theta + r^2)^{5/2}},$$

$$K_{\theta\theta} = \frac{r}{\sqrt{\frac{a^2\cos^2\theta + r^2}{a^2 - 2Mr - \frac{\Lambda r^3}{3} + r^2}}},$$

$$K_{\phi\phi} = \frac{\sqrt{a^2 - 2Mr - \frac{\Lambda r^3}{3} + r^2}\big[a^2 r\sin^4\theta(6M + \Lambda r^2) + 3(a^2 + r^2)\sin^2\theta(a^2\cos^2\theta + r^2)\big]}{6(a^2\cos^2\theta + r^2)^{3/2}}.$$

The extrinsic curvature scalar becomes:

$$K = \frac{1}{C[r,\theta]}\sqrt{\frac{2}{3}}\big\{3a^2[\sin^2\theta(2M + \Lambda r^2) - 2M + 2r\cos^2\theta + r(2 - \Lambda r)] + r^2[r(12 - 5\Lambda r) - 18M]\big\}.$$

where $C[r,\theta] = \sqrt{3a^2 - r(6M + r(\Lambda r - 3))}(a^2\cos 2\theta + a^2 + 2r^2)^{3/2}$. The shear tensor reads as:

$$\sigma_{\theta\theta} = -\frac{3a^2\sin^2\theta[2M + r(\Lambda r - 2)]}{6a^2\cos 2\theta + 6a^2 - 4r(6M + r(\Lambda r - 3))},$$
$$\sigma_{\phi\phi} = \frac{3a^2\sin^4\theta[2M + r(\Lambda r - 2)]\{3a^2 - r[6M + r(\Lambda r - 3)]\}}{\left[3a^2\cos 2\theta + 3a^2 - 2r(6M + r(\Lambda r - 3))\right]^2}.$$

The momentum can be calculated as follows:

$$\pi_\phi = \frac{1}{A[r,\theta]}\Big\{ar^3\big[-3a^2(108M^2 + 24Mr(\Lambda r - 3) + \Lambda^2 r^4) - 2r(\Lambda r^2 - 6M)(6M + r(\Lambda r - 3))^2\big] - 9a^5\cos^4\theta\big[6a^2(2M + \Lambda r^2) + 8\Lambda r^3(r - 3M) + 12Mr(2r - 3M) - 3\Lambda^2 r^5\big] + 3a^3 r\cos^2\theta\big[36M(a^2(M+r) - r(4M^2 - 3Mr + r^2)) + \Lambda^2 r^4(3a^2 + r(17r - 28M)) - 6\Lambda r^2(a^2(3r - 4M) + 5r(r - 2M)^2) - 2\Lambda^3 r^7\big]\Big\}, \quad (3.179)$$

$$\pi_\theta = 0, \quad (3.180)$$

where

$$A[r,\theta] = 6\pi\{3a^2 - r[6M + r(\Lambda r - 3)]\}^{3/2}(a^2\cos 2\theta + a^2 + 2r^2)^3\sqrt{6a^2\cos 2\theta + 6a^2 - 4r(6M + r(\Lambda r - 3))}.$$

$$\eta = \frac{1}{16\pi}, \qquad \zeta = -\frac{1}{16\pi}.$$

The pressure becomes:

$$P = \frac{[3a^2 - r(6M + r(\Lambda r - 3))]\{3a^2\cos^2\theta(2M + \Lambda r^2) - 6Mr^2 + \Lambda r^4\}}{48\pi(a^2\cos^2\theta + r^2)^2[r(6M + r(\Lambda r - 3)) - 3a^2\cos^2\theta]}.$$

The null expansion is:

$$\Theta = \frac{1}{B[r,\theta]}\Big[-9a^4\cos 4\theta[2M + r(\Lambda r - 2)] + 9a^4[2M + r(\Lambda r + 14)] - 36a^2r^2[22M + r(3\Lambda r - 10)] + 12a^2 r\cos 2\theta\big\{12a^2 + r[r(18 - 7\Lambda r) - 30M]\big\} + 32r^3[6M + r(\Lambda r - 3)]^2\Big],$$

where $B[r,\theta] = 6(a^2\cos 2\theta + a^2 + 2r^2)^2\left[3a^2\cos 2\theta + 3a^2 - 2r(6M + r(\Lambda r - 3))\right]$.

3.2.3.4 Kerr-Newman-(A)dS Black Hole

The extrinsic curvature becomes:

$$
\begin{aligned}
K_{tt} &= \frac{\sqrt{a^2-2Mr+Q^2-\frac{\Lambda r^3}{3}+r^2}\big[3a^2\cos^2\theta(2M+\Lambda r^2)+r(-6Mr+6Q^2+\Lambda r^3)\big]}{6(a^2\cos^2\theta+r^2)^{5/2}},\\
K_{t\phi} &= \frac{1}{6(a^2\cos^2\theta+r^2)^{5/2}}\Big\{a\sin^2\theta\sqrt{a^2-2Mr+Q^2-\frac{\Lambda r^3}{3}+r^2}\big[3a^2\cos^2\theta(2M+\Lambda r^2)\\
&\qquad +r(-6Mr+6Q^2+\Lambda r^3)\big]\Big\},\\
K_{\theta\theta} &= \frac{r}{\sqrt{\frac{a^2\cos^2\theta+r^2}{a^2-2Mr+Q^2-\frac{\Lambda r^3}{3}+r^2}}},\\
K_{\phi\phi} &= \frac{1}{\Phi[r,\theta]}\Big\{\sqrt{a^2-2Mr+Q^2-\frac{\Lambda r^3}{3}+r^2}\big[a^2\sin^4\theta(6Mr-3Q^2\\
&\qquad +\Lambda r^3)+3(a^2+r^2)\sin^2\theta(a^2\cos^2\theta+r^2)\big]\Big\},
\end{aligned}
$$

where $\Phi[r,\theta]=6(a^2\cos^2\theta+r^2)^{3/2}$ the extrinsic curvature scalar becomes:

$$
\begin{aligned}
K = \frac{1}{\Upsilon[r,\theta]}\sqrt{\frac{2}{3}}\Big\{&3a^2\big[\sin^2\theta(2M+\Lambda r^2)-2M\\
&+2r\cos^2\theta+r(2-\Lambda r)\big]+6r[r(2r-3M)+Q^2]-5\Lambda r^4\Big\},
\end{aligned}
$$

where

$$
\Upsilon[r,\theta]=(a^2\cos 2\theta+a^2+2r^2)^{3/2}\sqrt{3[a^2+r(r-2M)+Q^2]-\Lambda r^3}.
$$

The shear tensor reads as:

$$
\begin{aligned}
\sigma_{\theta\theta} &= -\frac{3a^2\sin^2\theta(2M+r(\Lambda r-2))}{6a^2\cos 2\theta+6a^2+12(-2Mr+Q^2+r^2)-4\Lambda r^3},\\
\sigma_{\phi\phi} &= \frac{3a^2\sin^4\theta[2M+r(\Lambda r-2)][3(a^2+r(r-2M)+Q^2)-\Lambda r^3]}{\big[3a^2\cos 2\theta+3a^2+6(-2Mr+Q^2+r^2)-2\Lambda r^3\big]^2}.
\end{aligned}
$$

the momentum can be calculated as follows:

$$\pi_\phi = \frac{1}{A[r,\theta]}\Bigg\{-3a^3r\Big\{r^2[108M^2+24Mr(\Lambda r-3)+\Lambda^2r^4]-3Q^2r[42M+r(5\Lambda r-18)] \\ +36Q^4\Big\}-9a^5\cos^4\theta\Big\{6\big[2a^2M-r(6M^2+Q^2)+3MQ^2+4Mr^2\big]+\Lambda r^2\big[6a^2+8r(r-3M) \\ +9Q^2\big]-3\Lambda^2r^5\Big\}+3a^3\cos^2\theta\Big\{18\big[Mr^2(2a^2-8M^2+Q^2)+r(a^2(2M^2-3Q^2) \\ +8M^2Q^2-2Q^4)-MQ^2(a^2+2Q^2)+r^3(6M^2-Q^2)-2Mr^4\big]+\Lambda^2r^5[3a^2+r(17r-28M) \\ +12Q^2]-3\Lambda r^2\big[a^2(-8Mr+3Q^2+6r^2)+Q^2r(13r-32M)+10r^2(r-2M)^2+6Q^4\big] \\ -2\Lambda^3r^8\Big\}-2ar(-6Mr+6Q^2+\Lambda r^3)\big[r(6M+r(\Lambda r-3))-3Q^2\big]^2\Bigg\}, \quad (3.181)$$

$$\pi_\theta = 0,$$

where

$$A[r,\theta] = 6\pi\big[a^2\cos 2\theta+a^2+2r^2\big]^3\big[3(a^2+r(r-2M)+Q^2)-\Lambda r^3\big]^{3/2}\sqrt{6a^2\cos 2\theta+6a^2+12(-2Mr+Q^2+r^2)-4\Lambda r^3},$$

$$\eta = \frac{1}{16\pi}, \qquad \zeta = -\frac{1}{16\pi},$$

The pressure becomes:

$$P = \frac{1}{R[r,\theta]}\big[3(a^2+r(r-2M)+Q^2)-\Lambda r^3\big]\Big\{3a^2\cos^2\theta(2M+\Lambda r^2) \\ +r(-6Mr+6Q^2+\Lambda r^3)\Big\}.$$

$$R[r,\theta] = 48\pi(a^2\cos^2\theta+r^2)^2(-3a^2\cos^2\theta+r(6M+r(\Lambda r-3))-3Q^2)$$

The null expansion is:

$$\Theta = \frac{1}{B[r,\theta]}\Bigg\{-9a^4\cos 4\theta[2M+r(\Lambda r-2)]+9a^4[2M+r(\Lambda r+14)] \\ +12a^2r\cos 2\theta\big[6(2(a^2+Q^2)-5Mr+3r^2)-7\Lambda r^3\big]-36a^2r\big[r(22M+r(3\Lambda r-10)) \\ -12Q^2\big]+32r\big[r(6M+r(\Lambda r-3))-3Q^2\big]^2\Bigg\}, \quad (3.182)$$

where

$$B[r,\theta] = 6[a^2\cos 2\theta+a^2+2r^2]^2\big\{3a^2\cos 2\theta+3a^2+6(-2Mr+Q^2+r^2)-2\Lambda r^3\big\}.$$

3.2.3.5 A Stringy Solution is Possible: Rotating Axion-Dilaton Black Hole

Heterotic string theory solutions of black holes in low energy limits will correspond to Einstein-Maxwell-Dilaton-Axion gravity. This gravitation theory has a spherically symmetric solution. The metric corresponding to this spacetime is Newman-Janis [23] applicable [57]. Its non-dynamic version is already analyzed under the static spacetime's membrane paradigm section.

Now, we will apply the membrane paradigm to this black hole and check its fluid membrane properties. First, we should give its metric according to [57]:

$$\begin{aligned} ds^2 &= -f dt^2 - 2a\sin^2\theta[f-1]dtd\phi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &+ \sin^2\theta\left\{-\left[a^2\sin^2\theta(f-1)\right] + a^2 + r\left(\frac{Q^2}{M}+r\right)\right\}d\phi^2 \end{aligned} \tag{3.183}$$

and write it in the equation of the form (3.138) by choosing the metric seeds as:

$$\begin{aligned} F_t &= \sqrt{f}, \\ F_r &= \sqrt{\frac{\Sigma}{\Delta}}, \\ F_\phi^2 &= \sqrt{\sin^2\theta(a^2\sin^2\theta(1-f) + (a^2 + r(\frac{Q^2}{M}+r)))}, \\ \omega &= -a\sin^2\theta(1-f)F_t^{-1}, \end{aligned}$$

where

$$\begin{aligned} \Delta &= a^2 + r[r - (2M - \frac{Q^2}{2M})], \\ \Sigma &= a^2\cos^2\theta + r[r - (2M - \frac{Q^2}{2M})] + 2M, \\ f &= 1 - \frac{2rM}{\Sigma} \end{aligned}$$

The structure of this metric can be put in the form of (3.137) such that one can directly start to use the algorithm and declare the important factors that underline the membrane paradigm of black holes:

$$K_{tt} = \frac{M(\Sigma - r\partial_r\Sigma)}{\Sigma^2\sqrt{\frac{\Sigma}{\Delta}}},$$

$$K_{rr} = 0,$$

$$K_{t\phi} = \frac{aM\sin^2\theta(\Sigma - r\partial_r\Sigma)}{\Sigma^2\sqrt{\frac{\Sigma}{\Delta}}} = K_{\phi t},$$

$$K_{\theta\theta} = \frac{\partial_r\Sigma}{2\sqrt{\frac{\Sigma}{\Delta}}},$$

$$K_{\phi\phi} = \frac{\sin^2\theta\left[\frac{2a^2Mr\sin^2\theta}{\Sigma} + a^2 + r(\frac{Q^2}{M} + r)\right]}{2\sqrt{\frac{\Sigma}{\Delta}}},$$

where its contraction with the metric becomes:

$$K = \frac{1}{A[r,\theta]}[M(a^2+r^2) + Q^2r]\partial_r\Sigma + (2Mr + Q^2)\Sigma - M\big[a^2M\cos 2\theta + a^2M + 6Mr^2 + 4Q^2r\big],$$

where

$$A[r,\theta] = 2\sqrt{\frac{\Sigma}{\Delta}}\Big\{\Sigma[a^2M + r(Mr+Q^2)] - Mr\big[a^2M\cos 2\theta + a^2M + 2r(Mr+Q^2)\big]\Big\}.$$

Also, by following the construction for a generic Kerr-like Membrane Paradigm algorithm one can find the other important transport coefficients. For instance, the shear tensor becomes:

$$\sigma_{\theta\theta} = \frac{1}{D[r,\theta]}\Big\{\Delta\big[2a^2M^3r^2\partial_r\Sigma + 2a^2M^3r^2\cos 2\theta\partial_r\Sigma - 2a^2M^2r\Sigma\partial_r\Sigma - 2a^2M^2r\cos 2\theta\Sigma\partial_r\Sigma + a^2M\Sigma^2\partial_r\Sigma - a^2M^2\Sigma^2 + a^2M^2\cos 2\theta\Sigma^2 + 4M^3r^4\partial_r\Sigma + 4M^2Q^2r^3\partial_r\Sigma - 4M^2r^3\Sigma\partial_r\Sigma - 4MQ^2r^2\Sigma\partial_r\Sigma + Mr^2\Sigma^2\partial_r\Sigma + Q^2r\Sigma^2\partial_r\Sigma - 8M^3r^3\Sigma - 4M^2Q^2r^2\Sigma + 8M^2r^2\Sigma^2 + 4MQ^2r\Sigma^2 - 2Mr\Sigma^3 - Q^2\Sigma^3\big]\Big\}.$$

$$\sigma_{\phi\phi} = \frac{1}{C[r,\theta]}\Big\{ \sin^2\theta\Delta\big[\partial_r\Sigma\big(-\Sigma^2(a^2M+r(Mr+Q^2))+2Mr\Sigma(a^2M\cos 2\theta+a^2M$$

$$+2r(Mr+Q^2))-2M^2r^2[a^2M\cos 2\theta+a^2M+2r(Mr+Q^2)]\big)+\Sigma(-M\Sigma\big[a^2M\cos 2\theta$$

$$-a^2M+4r(2Mr+Q^2)\big]+(2Mr+Q^2)\Sigma^2+4M^2r^2(2Mr+Q^2))\big]\Big\},$$

where $C[r,\theta] = 4M\Sigma^2(\Sigma - 2Mr)^2$ and

$$D[r,\theta] = 4\Sigma(2Mr-\Sigma)(-a^2M\Sigma - Mr^2\Sigma$$

$$-Q^2r\Sigma + a^2M^2r\cos 2\theta + a^2M^2r + 2M^2r^3 + 2MQ^2r^2).$$

And the non-zero component of the momentum will become:

$$\pi^\phi = \frac{1}{G[r,\theta]}\left[aM^2\sqrt{\frac{\Sigma}{\Delta}}\big\{(r\Sigma(2Mr-\Sigma)\partial_r\Delta+\Delta\big(r(3\Sigma-4Mr)\partial_r\Sigma+2\Sigma(Mr-\Sigma)\big)\big\}\right],$$

$$\pi^\theta = 0,$$

where

$$G[r,\theta] = \Sigma^3\sqrt{1-\frac{2Mr}{\Sigma}}\Big\{\Sigma[a^2M+r(Mr+Q^2)]-Mr[a^2M\cos 2\theta+a^2M+2r(Mr+Q^2)]\Big\}.$$

The pressure reads as follows:

$$P = \frac{M\Delta(\Sigma - r\partial_r\Sigma)}{\Sigma^2(2Mr-\Sigma)}.$$

The null-expansion of the charged axion-dilaton black hole is:

$$\Theta = \frac{1}{N[r,\theta]}\Big\{\Delta\big[\partial_r\Sigma(\Sigma(a^2M+r(Mr+Q^2))(\Sigma-4Mr)+2M^2r^2(a^2M\cos 2\theta+a^2M$$

$$+2r(Mr+Q^2))\big]+\Sigma\big(2M\Sigma\big[a^2M\sin^2\theta-2r(2Mr+Q^2)\big]+(2Mr+Q^2)\Sigma^2$$

$$+4M^2r^2(2Mr+Q^2)\big)\Big]\Bigg\},$$

where

$$N[r,\theta] = 2\Sigma^2(2Mr-\Sigma)(Mr(a^2M\cos 2\theta+a^2M+2r(Mr+Q^2))-\Sigma(a^2M+r(Mr+Q^2))).$$

Since the governing action is the same as before the other transport coefficients are the same.

3.2.3.6 A Lower Dimensional Solution is Possible: Rotating BTZ Black Hole

The rotating BTZ solution can be found by admitting NJA to a static BTZ solution [63]. The rotating BTZ solution can be understood extrinsically, it is basically Kerr solution at constant $\theta = \frac{\pi}{2}$ hypersurface. Now, let us write down the rotating BTZ black hole geometry:

$$ds^2 = -(-M - \Lambda r^2)dt^2 - 2a\big(1 - (-M - \Lambda r^2)\big)dtd\phi + \frac{r^2}{\Delta(r)}dr^2 + \big[(r^2 + a^2) + a^2(1 - (-M - \Lambda r^2))\big]d\phi^2$$

Let us call the metric seeds respectful to Arslaniev's notation.

$$F_t^2 = f, \qquad F_r^2 = \frac{r^2}{\Delta},$$
$$F_\phi^2 = \Big[(r^2 + a^2) + (1 - f)a^2\Big],$$
$$\omega = -a(f - 1)F_t^{-1},$$

where $f = (-M - \Lambda r^2)$,$\Delta(r) = r^2(-M - \Lambda r^2) + a^2$ while cosmological constant is $\Lambda < 0$ (AdS). with these definitions, the generic metric becomes square [43]:

$$ds^2 = -(F_t dt + \omega d\phi)^2 + F_r^2 dr^2 + (F_\phi^2 + \omega^2)d\phi^2.$$

Now, we should identify this metric with a 1+1+1 dictionary:

$$ds^2 = (-U_\mu U_\nu + n_\mu n_\nu + \gamma_\mu \gamma_\nu)dx^\mu dx^\nu.$$

Let $U_\mu dx^\mu = F_t dt + \omega d\phi$, $n_\mu dx^\mu = F_r dr$, $\gamma_\mu dx^\mu = r d\phi$ Now, since we constructed the geometry, one can wake the Membrane Paradigm up. Firstly, it is customary for rotating spacetimes to accelerate but for BTZ it is different.

$$a_\nu = n^\gamma \nabla_\gamma n_\nu = 0.$$

The extrinsic curvature can be found as:

$$K_{\mu\nu} = h_\mu^\gamma \nabla_\gamma n_\nu,$$

$$K_{\mu\nu} = \begin{pmatrix} -\frac{\sqrt{\Delta(r)}\partial_r f}{2r} & 0 & \frac{a\sqrt{\Delta(r)}\partial_r f}{2r} \\ 0 & 0 & 0 \\ \frac{a\sqrt{\Delta(r)}\partial_r f}{2r} & 0 & \frac{\sqrt{\Delta(r)}(2r - a^2\partial_r f)}{2r} \end{pmatrix},$$

The extrinsic curvature scalar becomes:

$$K = \frac{\sqrt{\Delta(r)}(r\partial_r f + 2f)}{2(a^2 + r^2 f)}.$$

The 1D cross-section of the extrinsic curvature can be calculated by taking the Lie derivative of γ^μ along null vector l^μ

$$k_\mu = \pounds_l \gamma = l^\mu \nabla_\mu \gamma_\nu + \nabla_\mu l_\nu \gamma^\mu,$$
$$k_\mu = \frac{\Delta(r)\sqrt{\frac{a^2}{f} + r^2}(2rf^2 - a^2\partial_r f)}{2r^2 f(a^2 + r^2 f)},$$

where $\gamma_\mu dx^\mu = \sqrt{\frac{a^2}{f} + r^2} d\phi$. By following the algorithm that is given in the Kerr metric the null expansion becomes:

$$\Theta = \gamma^\mu k_\mu,$$
$$\Theta = \frac{2rf^2\Delta(r) - a^2\Delta(r)\partial_r f}{2a^2r^2 f + 2r^4 f^2}.$$

Also, the by using the same algebraic relations given in rotating spacetime's generic solution one can find the vanishing shear "vector" as:

$$\sigma_\mu = k_\mu - \Theta\gamma_\mu = 0. \tag{3.184}$$

By using the extrinsic curvature and the horizon metric one can find the stress tensor of the stretched horizon as:

$$t_{\mu\nu}^{stretched} = \begin{pmatrix} \frac{\sqrt{\Delta(r)}(a^2\partial_r f - 2rf^2)}{2r(a^2+r^2 f)} & 0 & -\frac{a\sqrt{\Delta(r)}((a^2+r^2)\partial_r f - 2r(f-1)f)}{2r(a^2+r^2 f)} \\ 0 & 0 & 0 \\ -\frac{a\sqrt{\Delta(r)}((a^2+r^2)\partial_r f - 2r(f-1)f)}{2r(a^2+r^2 f)} & 0 & \frac{\sqrt{\Delta(r)}((a^2+r^2)^2\partial_r f - 2a^2 r(f-1)^2)}{2r(a^2+r^2 f)} \end{pmatrix},$$

Now, according to the membrane paradigm, the Newtonian Viscous Fluid description of the stress tensor should be identified with the stress tensor of the boundary fictitious horizon.t_{tt} component of both will give us the energy density.

$$\rho = \frac{\Delta(r)(a^2\partial_r f - 2rf^2)}{2r^2 f(a^2 + r^2 f)}. \tag{3.185}$$

As in the case of the Kerr black hole membrane paradigm non-dynamical limiting case should constrain the value of the bulk viscosity ζ. In the static case, bulk viscosity vanished due to thermal equilibrium correspondence. Hence

$$\zeta = 0. \tag{3.186}$$

However, the rotating black hole should have a momentum vector.

$$\pi^{\phi} = (\rho - F_r \rho F_t \omega(r)) \frac{1}{(\gamma_\phi)^2 F_t},$$
$$\pi^{\phi} = \frac{\sqrt{\Delta(r)}((a+1)f - a)(a^2 \partial_r f - 2rf^2)}{2r\sqrt{f}(a^2 + r^2 f)^2}.$$

Similarly, a common parenthesis for pressure from the viscous fluid stress can be found as:

$$P = \frac{\Delta(r)[((a^4 + a^3 + 2a^2r^2 + r^4)f - a^3)\partial_r f - 2ar(f-1)f[(a+1)f - a]]}{2r^2(a^2 + r^2 f)^2}.$$

To check whether this is correct, one should understand that in the limit $a \longrightarrow 0$ the surface gravity that is written as $P = \frac{\kappa}{4\pi}$ should correspond to static BTZ black hole:

$$\lim_{a \to 0} P = \frac{\partial_r f}{2}, \tag{3.187}$$

which gives the value of surface gravity.

3.2.4 Construction of Gravitational Membrane: A Parametrically Deviated Spacetime

3.2.4.1 Modified Static Black Hole Metric

The metric of parametrically deviated static black hole is given in (2.57). Let us restate the metric and put it into 2+1+1 form.

$$ds^2 = -f(1+h)dt^2 + f^{-1}(1+h)dr^2 + r^2 d\Omega_2^2, \tag{3.188}$$

where $f(r) = 1 - \frac{2M(r)}{r}$,$h(r) = \sum_{k=0}^{\infty} \epsilon_k \left(\frac{M}{r}\right)^k$ [64]. 2+1+1 decomposition of the membrane paradigm requires U^μ and n^μ with a regulator α such that in the limit $r \longrightarrow 0 \implies \alpha \longrightarrow 0$ while $U^\mu, n^\mu \longrightarrow l^u$ The horizon radius can be stated as:

$$g^{rr} = \frac{f}{1+h} = 0, \exists\, r = r_H \,|g^{rr} = 0. \tag{3.189}$$

Now let us rewrite the metric in a different form:

$$ds^2 = -F_t dt^2 + F_r dr^2 + r^2 d\Omega_2^2, \tag{3.190}$$

where $F_t = f(1+h)$ and $F_r = \frac{1+h}{f}$. Moreover, two components can be written in terms of each other as $F_t = f^2 F_r$. The metric is:

$$ds^2 = -f^2 F_r dt^2 + F_r dr^2 + r^2 d\Omega_2^2. \tag{3.191}$$

Then the 2+1+1 decomposition reads as:

$$\begin{aligned} U_\mu dx^\mu = f F_r^{\frac{1}{2}} dt \implies & \qquad\qquad U_\mu = f F_r^{\frac{1}{2}}, \\ n_\mu dx^\mu = F_r^{\frac{1}{2}} dr \implies & \qquad\qquad n_\mu = F_r^{\frac{1}{2}}, \end{aligned} \tag{3.192}$$

such that on the dual basis:

$$\begin{aligned} U^\mu = g^{\mu\nu} U_\nu = g^{\mu\nu}\big(f F_r^{\frac{1}{2}} \delta^t{}_\nu\big) = \frac{-1}{f^2 F_r}(f F_r^{\frac{1}{2}}) = \frac{-1}{f F_r^{\frac{1}{2}}} \\ n^\mu = g^{\mu\nu} n_\nu = g^{\mu\nu}\big(F_r^{\frac{1}{2}} \delta^r{}_\nu\big) = F_r^{-\frac{1}{2}} \end{aligned}$$

where the 2D cross-section of the black hole horizon is

$$\gamma_{\mu\nu} dx^\mu dx^\nu = r^2 d\Omega_2^2. \tag{3.193}$$

First of all, one needs to check the acceleration $a_\nu = n^\mu \nabla_\mu n_\nu$.

$$a_\nu = n^\mu\big(\partial_\mu n_\nu - \Gamma^\gamma_{\mu\nu} n_\gamma\big) F_r^{\frac{-1}{2}} \delta^\mu{}_r \Big[\partial_\mu (F_r^{\frac{1}{2}} \delta^r{}_\nu) - \Gamma^\gamma_{\mu\nu}(F_r^{\frac{1}{2}} \delta^r{}_\gamma))\Big] \tag{3.194}$$

The component-by-component analysis gives zero acceleration as expected from a static spacetime geometry.

$$a_r = F_r^{-\frac{1}{2}} \Big[\frac{1}{2} F_r^{\frac{-1}{2}} \partial_r (F_r) - \frac{\partial_r F_r}{2} (F_r^{-1} F_r^{\frac{1}{2}})\Big] = 0 \tag{3.195}$$

There is no acceleration. Now, let us calculate the extrinsic curvature tensor:

$$K_{\mu\nu} = \nabla_\mu n_\nu. \tag{3.196}$$

The component-by-component analysis gives,

$$K_{tt} = 0 - \Gamma^{r}_{tt} F_r^{\frac{1}{2}} = -\frac{f^2 \partial_r F_r}{2F_r} - f\partial_r f) F_r^{\frac{1}{2}},$$

$$K_{rr} = 0,$$

$$K_{\theta\theta} = 0 - \Gamma^{r}_{\theta\theta} F_r^{\frac{1}{2}} = -\frac{-r}{F_r} F_r^{\frac{1}{2}} = r F_r^{\frac{-1}{2}},$$

$$K_{\phi\phi} = 0 - \Gamma^{r}_{\phi\phi} F_r^{\frac{1}{2}} = -\frac{-r\sin^2\theta}{F_r} F_r^{\frac{1}{2}} = r\sin^2\theta F_r^{\frac{-1}{2}}.$$

Then in matrix form, the extrinsic curvature of the 3D stretched horizon is:

$$K_{\mu\nu} = -F_r^{\frac{-1}{2}} \begin{pmatrix} -(\frac{f^2\partial_r F_r)}{2} + f\partial_r f F_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin^2\theta \end{pmatrix}, \tag{3.197}$$

$$K_{\mu\nu} = \frac{1}{r} F_r^{-\frac{1}{2}} \gamma_{\mu\nu} - \Big(\frac{F_r^{\frac{-3}{2}} \partial_r F_r}{2} + \frac{F_r^{\frac{-1}{2}}}{f} \partial_r f\Big) U_\mu U_\nu. \tag{3.198}$$

To check this result, we have to remove the effects of the deformation and see whether the limiting case reduces to Static black hole case. $F_r = \frac{1+h}{f}$ where h is the deformation function on the manifold. If $h = 0$:

$$\begin{aligned} K_{\mu\nu}^{h\longrightarrow 0} &= \frac{f^{\frac{1}{2}}}{r} \gamma_{\mu\nu} - \Big[-\frac{f^{\frac{3}{2}} f^{-2} \partial_r f}{2} + f^{-\frac{1}{2}} \partial_r f \Big] U_\mu U_\nu \\ &= -(\frac{1}{2} f^{-\frac{1}{2}} \partial_r f) U_\mu U_\nu + \frac{1}{2} f^{\frac{1}{2}} \gamma_{\mu\nu} \equiv K_{\mu\nu}^{static} \end{aligned}$$

Hence, what we found for modified Static black hole geometry is correct. The trace equation can be found by using the properties of decomposed vector and 2-metric:

$$\begin{aligned} g^{\mu\nu} K_{\mu\nu} &= g^{\mu\nu} \Big(\frac{1}{r} F_r^{-\frac{1}{2}} \gamma_{\mu\nu} - \big(\frac{F_r^{\frac{-3}{2}} \partial_r F_r}{2} + \frac{F_r^{\frac{-1}{2}}}{f} \partial_r f\big) U_\mu U_\nu \Big) \\ &= \frac{F_r^{\frac{-3}{2}} \partial_r F_r}{2} + \frac{F_r^{\frac{-1}{2}}}{f} \partial_r f + \frac{2}{r} F_r^{-\frac{1}{2}}. \end{aligned} \tag{3.199}$$

The stretched horizon stress tensor is (3.60) where $\sigma_{\mu\nu} = 0$ in non-rotating spacetimes. Since:

$$t^{stretched}_{\mu\nu} = \frac{1}{8\pi}\big(\big(Kh_{\mu\nu} - K_{\mu\nu}\big) \tag{3.200}$$
$$= \frac{1}{8\pi}\big(K\gamma_{\mu\nu} - U_\mu U_\nu) - K_{\mu\nu}\big)$$
$$= \frac{1}{8\pi}\big(K\gamma_{\mu\nu} - KU_\mu U_\nu - K_{\mu\nu}\big)$$

$$t^{stretched}_{\mu\nu} = \frac{1}{8\pi}\Big\{(K - \frac{1}{r}F_r^{-\frac{1}{2}})\gamma_{\mu\nu} - \big(\frac{F_r^{-\frac{3}{2}}\partial - rF_r}{2} + F_r^{\frac{-1}{2}}\frac{\partial_r f}{f} + K\big)U_\mu U_\nu\Big\}$$
$$t^{stretched}_{\mu\nu} = \frac{1}{8\pi F_r^{\frac{1}{2}} f}\Big\{ - \frac{2}{r} f U_\mu U_\nu + f\big[\frac{1}{r} + (\frac{\partial_r f}{f} + \frac{1}{2}\frac{\partial_r F_r}{F_r})\big]\Big\}. \tag{3.201}$$

Now, as we did before, we can identify both of these equations with each other and viscous fluid description to extract the transport coefficients of parametrically deviated static black hole spacetime.

$$\rho = -\frac{1}{8\pi}\Theta, \qquad \eta = \frac{1}{16\pi}, \tag{3.202}$$
$$P = \frac{\kappa}{8\pi}, \qquad \zeta = -\frac{1}{16\pi}, \tag{3.203}$$
$$\pi^A = 0, \tag{3.204}$$

while surface gravity, shear tensor and expansion become:

$$\Theta = \frac{2}{r}f, \qquad \sigma_{AB} = 0, \tag{3.205}$$
$$\kappa = \frac{\partial_r f}{2} + \frac{\partial_r h f}{1+h}. \tag{3.206}$$

One can check the above relations with equations from (3.66) to (3.68). For a timelike observer, surface gravity changed by the factor of $\partial_r(\ln(1+h))f$ it is clear that if $h = 0$ (no deformation) the surface gravity is identical to static spacetime geometry. If $r \longrightarrow r_H$, $h \neq 0$ however, $f = 0$ hence on the horizon observer will find the same surface gravity as an ordinary static case. This happens because parametrical deviations of static black holes' horizon radius are exactly the same as the ordinary static black holes [64].

3.2.4.2 Johannsen-Psaltis Black Hole

We will use the metric given in (2.65),

$$
\begin{aligned}
ds^2 = -[1+h]f dt^2 + 2a\big(-\frac{2M(r)}{r}\frac{r^2}{\Sigma}\sin^2\theta[1+h]\big)dtd\phi + \frac{\Sigma[1+h]}{\Delta(r)+a^2\sin^2\theta h}dr^2 \\
+ \Sigma d\theta^2 + \Bigg[\sin^2\theta\Big(r^2+a^2+a^2\sin^2\theta\big(\frac{2M(r)}{r}\frac{r^2}{\Sigma}\big)\Big) \\
+ ha^2\sin^4\theta\big(1+\frac{2M(r)}{r}\big)\frac{r^2}{\Sigma}\Bigg]d\phi^2, \quad (3.207)
\end{aligned}
$$

and write it in the equation of the form (3.138) by choosing the metric seeds as:

$$
\begin{aligned}
F_t &= \sqrt{f}\sqrt{h+1}, \\
F_r &= \sqrt{\frac{(h+1)\Sigma}{a^2\sin^2\theta h+\Delta}}, \\
F_\phi^2 &= \sqrt{\sin^2\theta(a^2\sin^2\theta((2-f)h+(1-f))+(a^2+r^2))}, \\
\omega &= -a\sin^2\theta(1-f)(h+1)F_t^{-1},
\end{aligned}
$$

and the seed functions read:

$$
\begin{aligned}
\Sigma &= [a^2\cos^2\theta + r^2], \\
\Delta &= [f\Sigma + a^2\sin^2\theta], \\
f &= 1 - \frac{2r^2M(r)}{r\Sigma}, \\
h &= \frac{\epsilon_3(M^3r)}{\Sigma^2}.
\end{aligned}
$$

The structure of this metric can be put in the form of (3.137) such that one can directly start to use the algorithm and declare the important factors that underline the membrane paradigm of black holes:

$$K_{tt} = -\frac{(h+1)\partial_r f + f\partial_r h}{2\sqrt{\frac{(h+1)\Sigma}{a^2\sin^2\theta h+\Delta}}},$$

$$K_{rr} = 0,$$

$$K_{t\phi} = -\frac{a\sin^2\theta((h+1)\partial_r f + (f-1)\partial_r h)}{2\sqrt{\frac{(h+1)\Sigma}{a^2\sin^2\theta h+\Delta}}} = K_{\phi t},$$

$$K_{\theta\theta} = \frac{\partial_r\Sigma}{2\sqrt{\frac{(h+1)\Sigma}{a^2\sin^2\theta h+\Delta}}},$$

$$K_{\phi\phi} = \frac{\sin^2\theta(a^2\sin^2\theta(-f(h+1)+2h+1)+a^2+r^2)}{2\sqrt{\frac{(h+1)\Sigma}{a^2\sin^2\theta h+\Delta}}},$$

where its contraction with the metric becomes:

$$K = \frac{1}{A[r,\theta]}\Big[\Sigma((h+1)((a^2\cos 2\theta + a^2 + 2r^2)\partial_r f + 4rf) + \partial_r h(f(a^2\cos 2\theta + a^2 + 2r^2) + 4a^2\sin^2\theta(h+1))) + (h+1)\partial_r\Sigma(f(a^2\cos 2\theta + a^2 + 2r^2) + 2a^2\sin^2\theta(h+1)\Big], \quad (3.208)$$

where

$$A[r,\theta] = 4(h+1)\Sigma(a^2\sin^2\theta(-f+h+1) + (a^2+r^2)f)\sqrt{\frac{(h+1)\Sigma}{a^2\sin^2\theta h+\Delta}}.$$

Also, by following the construction for a generic Kerr-like Membrane Paradigm algorithm one can find the other important transport coefficients. For instance, the shear tensor becomes:

$$\sigma_{\theta\theta} = \frac{1}{B[r,\theta]}\Big[(a^2\sin^2\theta h + \Delta)(f\partial_r\Sigma(a^2\sin^2\theta(-f+h+1) + (a^2+r^2)f) + \Sigma(a^2\sin^2\theta(h+1)\partial_r f - f(a^2\sin^2\theta\partial_r h + 2rf)))\Big], \quad (3.209)$$

$$\sigma_{\phi\phi} = \frac{1}{C[r,\theta]} \sin^2\theta(a^2\sin^2\theta h + \Delta)(\Sigma(a^2\sin^2\theta f\partial_r h - a^2\sin^2\theta(h+1)\partial_r f$$
$$+ 2rf^2) - f\partial_r\Sigma(a^2\sin^2\theta(-f+h+1) + (a^2+r^2)f)), \quad (3.210)$$

where

$$B[r,\theta] = 2f(h+1)\Sigma(f(a^2\cos 2\theta + a^2 + 2r^2) + 2a^2\sin^2\theta(h+1)),$$

and

$C[r,\theta] = 4f^2(h+1)\Sigma^2$, and the non-zero momentum will become:

$$\pi^\phi = \frac{1}{G[r,\theta]}\Big[a\sqrt{\frac{(h+1)\Sigma}{a^2\sin^2\theta h + \Delta}}(\Sigma((f-1)f(a^2\sin^2\theta\partial_r h + \partial_r\Delta)$$
$$+ a^2\sin^2\theta(f+1)h\partial_r f + (f+1)\Delta\partial_r f) - (f-1)f\partial_r\Sigma(a^2\sin^2\theta h + \Delta))\Big],$$
$$\pi^\theta = 0, \quad (3.211)$$

where

$$G[r,\theta] = \sqrt{f}\sqrt{h+1}\Sigma^2(a^2\sin^2\theta(-f+h+1) + (a^2+r^2)f.).$$

The pressure reads as follows:

$$P = \frac{(a^2\sin^2\theta h + \Delta)((h+1)\partial_r f + f\partial_r h)}{16\pi f(h+1)^2\Sigma}. \quad (3.212)$$

The null-expansion of the Johannsen-Psaltis black hole is:

$$\Theta = -\frac{1}{N[r,\theta]}\Big[(a^2\sin^2\theta h + \Delta)(\partial_r\Sigma(\Sigma(\Sigma - 4Mr)(a^2\sin^2\theta h + a^2 + r^2)$$
$$+ 2M^2r^2(a^2\cos 2\theta + a^2 + 2r^2)) + \Sigma(\Sigma(a^2\sin^2\theta((\Sigma - 2Mr)\partial_r h$$
$$+ 2Mh) + 2r\Sigma + 2a^2M\sin^2\theta - 8Mr^2) + 8M^2r^3))\Big], \quad (3.213)$$

$$N[r,\theta] = 2(h+1)\Sigma^2(2Mr-\Sigma)(\Sigma(a^2\sin^2\theta h + a^2 + r^2) - Mr(a^2\cos 2\theta + a^2 + 2r^2))).$$

Other transport coefficients should be the same.

REFERENCES

[1] K. Schwarzschild, "On the gravitational field of a mass point according to Einstein's theory," *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*, vol. 1916, pp. 189–196, 1916.

[2] D. Finkelstein, "Past-Future Asymmetry of the Gravitational Field of a Point Particle," *Phys. Rev.*, vol. 110, pp. 965–967, 1958.

[3] J. Earman, "Cosmic censorship," *PSA: Proceedings of the Biennial Meeting of the Philosophy of Science Association*, vol. 1992, pp. 171–180, 1992.

[4] R. P. Kerr, "Gravitational field of a spinning mass as an example of algebraically special metrics," *Phys. Rev. Lett.*, vol. 11, pp. 237–238, Sep 1963.

[5] G. C. Debney, R. P. Kerr, and A. Schild, "Solutions of the Einstein and Einstein-Maxwell Equations," *J. Math. Phys.*, vol. 10, p. 1842, 1969.

[6] R.-R. Hsu, "The no hair theorem," *Chinese Journal of Physics - CHIN J PHYS*, 1992.

[7] N. P. Warner, "Lectures on Microstate Geometries," 12 2019.

[8] A. de la Cruz-Dombriz, A. Dobado, and A. L. Maroto, "Black Holes in f(R) theories," *Phys. Rev. D*, vol. 80, p. 124011, 2009. [Erratum: Phys.Rev.D 83, 029903 (2011)].

[9] T. Johannsen, "A metric for testing the nature of black holes," *Journal of Physics: Conference Series*, vol. 410, p. 012136, feb 2013.

[10] T. Johannsen and D. Psaltis, "A Metric for Rapidly Spinning Black Holes Suitable for Strong-Field Tests of the No-Hair Theorem," *Phys. Rev. D*, vol. 83, p. 124015, 2011.

[11] R. H. Boyer and R. W. Lindquist, "Maximal analytic extension of the Kerr metric," *J. Math. Phys.*, vol. 8, p. 265, 1967.

[12] O. Zelenka and G. Lukes-Gerakopoulos, “Chaotic motion in the Johannsen-Psaltis spacetime,” in *Workshop on Black Holes and Neutron Stars*, 11 2017.

[13] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” *Phys. Lett. B*, vol. 379, pp. 99–104, 1996.

[14] J. M. Maldacena, *Black holes in string theory*. PhD thesis, Princeton U., 1996.

[15] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” *Adv. Theor. Math. Phys.*, vol. 2, pp. 231–252, 1998.

[16] D. Marolf, “Black Holes, AdS, and CFTs,” *Gen. Rel. Grav.*, vol. 41, pp. 903–917, 2009.

[17] T. Damour, “Quelques propriétés mécaniques, électromagnétiques, thermodynamiques et quantiques des trous noirs (th‘ese de doctorat d’ ´etat),” ,(unpublished),1979.

[18] K. S. Thorne, K. S. Thorne, R. H. Price, and D. A. MacDonald, *Black holes: the Membrane Paradigm*. Yale university press, 1986.

[19] R. Ruffini, “2. marcel grossmann meeting on general relativity,” *Phys. Rev. Lett.*, vol. 26, pp. 587–608, Jul 1979.

[20] M. K. Parikh, “Membrane horizons: The Black hole’s new clothes,” other thesis, 10 1998.

[21] J. D. Jackson, *Classical Electrodynamics*. Wiley, third ed., 1998.

[22] M. K. Parikh and F. Wilczek, “An action for black hole membranes,” *Physical Review D*, vol. 58, aug 1998.

[23] E. T. Newman and A. I. Janis, “Note on the Kerr spinning particle metric,” *J. Math. Phys.*, vol. 6, pp. 915–917, 1965.

[24] T. Johannsen and D. Psaltis, “Metric for rapidly spinning black holes suitable for strong-field tests of the no-hair theorem,” *Phys. Rev. D*, vol. 83, p. 124015, Jun 2011.

[25] M. Nakahara, *Geometry, topology and physics*. 2003. Bristol, UK: Hilger (1990) 505 p. (Graduate student series in physics).

[26] B. Schutz, *A First Course in General Relativity*. Cambridge University Press, 2nd ed., June 2009.

[27] F. P. Schuller, "Lecture series at the we–heraeus institute of international winter school on gravity and light," 2015.

[28] R. M. Wald, *General Relativity*. Chicago, USA: Chicago Univ. Pr., 1984.

[29] E. Altaş and B. Tekin, "Advanced lectures in general relativity," 2023.

[30] T. P. Sotiriou and V. Faraoni, "f(R) Theories Of Gravity," *Rev. Mod. Phys.*, vol. 82, pp. 451–497, 2010.

[31] S. Chakraborty, *Boundary Terms of the Einstein–Hilbert Action*, pp. 43–59. Cham: Springer International Publishing, 2017.

[32] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation*. San Francisco: W. H. Freeman, 1973.

[33] N. DURRER, R; STRAUMANN, "Some applications of the 3+1 formalism of general relativity," *Helvetica Physica Acta*, 1988.

[34] S. Dengiz, "3+1 orthogonal and conformal decomposition of the einstein equation and the adm formalism for general relativity," *Thesis at METU*, 2011.

[35] R. H. Price and K. S. Thorne, "Membrane viewpoint on black holes: Properties and evolution of the stretched horizon," *Phys. Rev. D*, vol. 33, pp. 915–941, Feb 1986.

[36] E. Gourgoulhon, "3+1 formalism and bases of numerical relativity," 3 2007.

[37] T. Adamo and E. T. Newman, "The Kerr-Newman metric: A Review," *Scholarpedia*, vol. 9, p. 31791, 2014.

[38] T. Johannsen, "Regular Black Hole Metric with Three Constants of Motion," *Phys. Rev. D*, vol. 88, no. 4, p. 044002, 2013.

[39] Y.-C. Chou, "Extension Rules of Newman–Janis Algorithm for Rotation Metrics in General Relativity," *Physical Science International Journal*, vol. 24, p. 1–14, Jul. 2020.

[40] E. Newman and R. Penrose, "An Approach to gravitational radiation by a method of spin coefficients," *J. Math. Phys.*, vol. 3, pp. 566–578, 1962.

[41] H. Erbin, "Janis–Newman Algorithm: Generating Rotating and NUT Charged Black Holes," *Universe*, vol. 3, no. 1, p. 19, 2017.

[42] R. A. Konoplya and A. Zhidenko, "Shadows of parametrized axially symmetric black holes allowing for separation of variables," *Phys. Rev. D*, vol. 103, no. 10, p. 104033, 2021.

[43] M. Visser, "The Kerr spacetime: A Brief introduction," in *Kerr Fest: Black Holes in Astrophysics, General Relativity and Quantum Gravity*, 6 2007.

[44] D. G. Boulware, "Naked Singularities, Thin Shells, and the Reissner-Nordström Metric," *Phys. Rev. D*, vol. 8, no. 8, p. 2363, 1973.

[45] K. Gödel, "An example of a new type of cosmological solutions of einstein's field equations of gravitation," *Rev. Mod. Phys.*, vol. 21, pp. 447–450, Jul 1949.

[46] A. Tavlayan and B. Tekin, "Event horizon detecting invariants," *Phys. Rev. D*, vol. 101, no. 8, p. 084034, 2020.

[47] I. Gkigkitzis, I. Haranas, and O. Ragos, "Kretschmann Invariant and Relations Between Spacetime Singularities, Entropy and Information," *Physics International*, vol. 5, no. 1, pp. 103–111, 2014.

[48] S. W. Hawking, "Gravitational radiation from colliding black holes," *Phys. Rev. Lett.*, vol. 26, pp. 1344–1346, May 1971.

[49] N. Straumann, "The membrane model of black holes and applications," in *Black Holes: Theory and Observation* (F. W. Hehl, C. Kiefer, and R. J. Metzler, eds.), (Berlin, Heidelberg), pp. 111–156, Springer Berlin Heidelberg, 1998.

[50] I. Bah and P. Heidmann, "Topological Stars and Black Holes," *Phys. Rev. Lett.*, vol. 126, no. 15, p. 151101, 2021.

[51] R. H. Price and K. S. Thorne, "Membrane viewpoint on black holes: Properties and evolution of the stretched horizon," *Phys. Rev. D*, vol. 33, pp. 915–941, Feb 1986.

[52] J. P. S. Lemos and O. B. Zaslavskii, “Black hole mass formula in the membrane paradigm,” *Phys. Rev. D*, vol. 97, no. 6, p. 064008, 2018.

[53] D. Marolf and S. Yaida, “Energy conditions and junction conditions,” *Phys. Rev. D*, vol. 72, p. 044016, 2005.

[54] A. J. Nurmagambetov and A. M. Arslanaliev, “Kerr Black Holes within the Membrane Paradigm,” *LHEP*, vol. 2022, p. 328, 2022.

[55] D. Christodoulou, “Reversible and irreversible transformations in black-hole physics,” *Phys. Rev. Lett.*, vol. 25, pp. 1596–1597, Nov 1970.

[56] M. Halilsoy, O. Gurtug, and S. Habib Mazharimousavi, “Rindler modified Schwarzschild geodesics,” *Gen. Rel. Grav.*, vol. 45, pp. 2363–2381, 2013.

[57] S. Yazadjiev, “Newman-Janis method and rotating dilaton axion black hole,” *Gen. Rel. Grav.*, vol. 32, pp. 2345–2352, 2000.

[58] S. S. Yazadjiev, “Exact static solutions in Einstein-Maxwell dilaton gravity with arbitrary dilaton coupling parameter,” *Bulg. J. Phys.*, vol. 27N3, pp. 58–61, 2000.

[59] B. P. Singh and S. G. Ghosh, “Shadow of Schwarzschild–Tangherlini black holes,” *Annals Phys.*, vol. 395, pp. 127–137, 2018.

[60] R. Emparan and H. S. Reall, “Black Holes in Higher Dimensions,” *Living Rev. Rel.*, vol. 11, p. 6, 2008.

[61] M. Papalexandris, “On the applicability of stokes’ hypothesis to low–mach–number flows,” *Continuum Mechanics and Thermodynamics*, vol. 32, 05 2019.

[62] E. T. Newman and A. I. Janis, “Note on the Kerr spinning particle metric,” *J. Math. Phys.*, vol. 6, pp. 915–917, 1965.

[63] H. Kim, “Spinning btz black hole versus kerr black hole: A closer look,” *Phys. Rev. D*, vol. 59, p. 064002, Feb 1999.

[64] R. Rahim and K. Saifullah, “Charging the Johannsen–Psaltis spacetime,” *Annals Phys.*, vol. 405, pp. 220–233, 2019.

[65] T. Adamo, J. J. M. Carrasco, M. Carrillo-Gonzalez, M. Chiodaroli, H. Elvang, H. Johansson, D. O'Connell, R. Roiban, and O. Schlotterer, "Snowmass White Paper: the Double Copy and its Applications," in *Snowmass 2021*, 4 2022.

[66] M. Gurses and B. Tekin, "Classical Double Copy: Kerr-Schild-Kundt metrics from Yang-Mills Theory," *Phys. Rev. D*, vol. 98, no. 12, p. 126017, 2018.

[67] C.-Y. Chen and P. Chen, "Separability of the Klein-Gordon equation for rotating spacetimes obtained from Newman-Janis algorithm," *Phys. Rev. D*, vol. 100, no. 10, p. 104054, 2019.

[68] U. C. Çelik, "Generalized Kerr-Schild Metrics In Einstein's Gravity," *Thesis at METU*, 2020.

[69] D. Rajan and M. Visser, "Cartesian Kerr–Schild variation on the Newman–Janis trick," *Int. J. Mod. Phys. D*, vol. 26, no. 14, p. 1750167, 2017.

[70] I. Bah, R. Dempsey, and P. Weck, "Kerr-Schild Double Copy and Complex Worldlines," *JHEP*, vol. 02, p. 180, 2020.

[71] R. Monteiro, D. O'Connell, and C. D. White, "Black holes and the double copy," *JHEP*, vol. 12, p. 056, 2014.

[72] N. Deruelle, N. Merino, and R. Olea, "Einstein-Gauss-Bonnet theory of gravity: The Gauss-Bonnet-Katz boundary term," *Phys. Rev. D*, vol. 97, no. 10, p. 104009, 2018.

APPENDIX A

DOES PARAMETRICALLY DEVIATED SPACETIMES ACKNOWLEDGE DOUBLE COPY?

The double copy phenomenon is basically square rooting a known gravity solution in Kerr-Schild form to produce a gauge field of a corresponding self-dual Yang-Mills theory. It is abbreviated as $gravity = YM^2$. It is an application of Color-Kinematics Conjecture[65] which is tested multiple times for different classical black hole solutions. The Kerr-Schild form can also be extended to Kerr-Schild-Kundt metrics for underlying Yang-Mills theory [66]. This appendix aims to find single-copy and corresponding parametric corrections of a parametrically deviated metric proposed by Johannsen-Psaltis [24]. Johannsen-Psaltis spacetime is a parametric deformation to Kerr. It has a chaotic geodesic equation and non-separable Klein-Gordon equation [67] in massless scalar field background [12]. Still, it is intrinsically Hamilton-Jacobi separable hence its photon orbits are already classified by the founders of the metric[38]. One should consider that even with the linear order parametric deviation equations of highly non-linear and computability of curvature is really hard. Classical double-copy conjecture is already applied to Schwarzschild and Kerr solutions and they gave Coulomb Charge and a thin rotating disc of charge respectively.

Our approach starts with putting the modified Schwarzschild metric in the Kerr-Schild form. It is easy to see its single copy. A Coulomb's potential with a correction in $\mathcal{O}(\epsilon)$. After that, we turn the Newman-Janis procedure on to find the Johannsen-Psaltis metric in Kerr-Schild form. Similarly to the Kerr case, the single copy of the results corresponds to the gauge potential of the parametrically corrected thin rotating disc of charge. The schematics of the approach are given below:

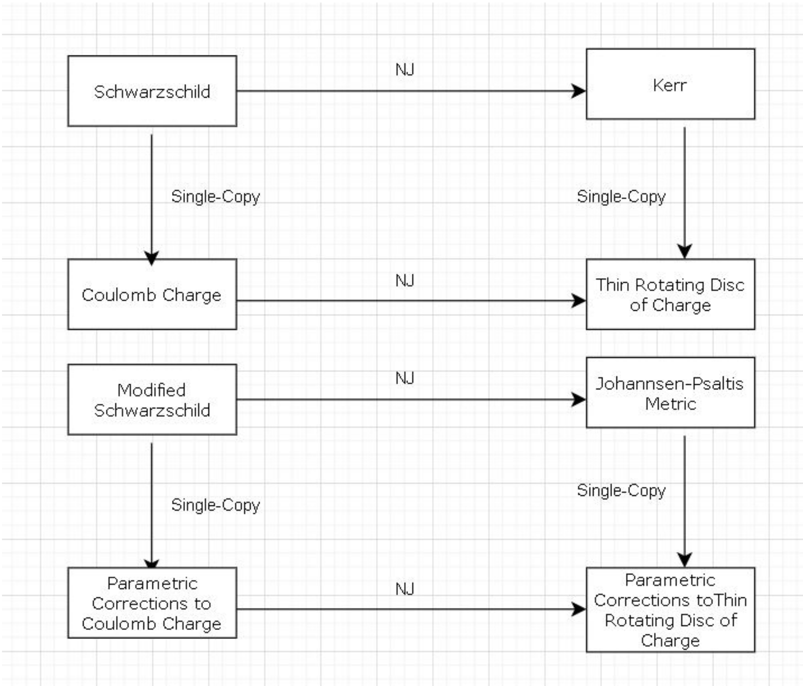

Figure A.1: Schematics of double copy procedure.

A.1 METHOD

A.1.1 Modified Schwarzschild Metric

Hereby we present our new look into JP double copy. If we start with the Modified Schwarzschild metric which is a parametric correction to non-rotating, spherically symmetric spacetime [24]

$$ds^2 = -f(1+h)dt^2 + f^{-1}(1+h)dr^2 + r^2 d\Omega^2{}_{S2}, \tag{A.1}$$

where $f = (1 - 2M/r)$, $h = \epsilon_3 M^3/r^3$ is the astrophysical correction suggested by Johannsen-Psaltis,$d\Omega^2{}_{S2}$ is the metric of 2-sphere.

Its Kerr-Schild form should be visible upon,

$$\begin{aligned} ds^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + 2\phi(r)k_\mu k_\nu dx^\mu dx^\nu \\ &= -dt^2 + dr^2 + r^2 d\Omega^2{}_{S2} + (1 - f(1+h))dt^2 + (-1 + f^{-1}(r)(1+h))dr^2 \\ &= ds^2{}_{Mink_{4x4}}|_{S2} + [1 - f(1+h)]\{dt^2 + \frac{(-f + (1+h)}{f(1+h)(1-f)}dr^2\}. \end{aligned} \tag{A.2}$$

From the mathematical perspective $\exists$ a Kerr-Schild form iff,

$$2\phi(r)k_\mu k_\nu dx^\mu dx^\nu = [1 - f(1+h)]\{dt^2 + \frac{(-f + (1+h)}{f(1+h)(1-f)}dr^2\}. \tag{A.3}$$

To show this existence let us make a coordinate transformation to stationary metric,

$$\begin{aligned} \tilde{t} &= t - A(r), \\ dt^2 &= d\tilde{t}^2 + (A'(r))^2 dr^2 + 2A'(r)d\tilde{t}dr. \end{aligned}$$

Now, let me analyse two parts of the metric

$$\begin{aligned} &1) ds^2{}_{Mink_{4x4}}|_{S2}, \\ &2) Rest. \end{aligned}$$

The first part is:

$$\begin{aligned} ds^2{}_{Mink_{4x4}}|_{S2} &= -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \\ &= -d\tilde{t}^2 + (1 - (A'(r))^2)dr^2 - 2A'(r)drd\tilde{t} + r^2 d\Omega^2{}_{S2}. \end{aligned}$$

The second part is:

$$[1 - f(1+h)]\{d\tilde{t}^2 + (A'(r))^2 dr^2 + A'(r)dr^2 d\tilde{t}^2 + (\frac{(1+h-f)}{f(1 - f(1+h))})dr^2\}.$$

Altogether we have,

$$\begin{aligned} ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \Big\{&(1 - f(1+h))d\tilde{t}^2 - 2A'(r)f(1+h)drd\tilde{t} \\ &+ [-f(1+h)A'(r) + \frac{(1+h-f)}{f(1-f)(1+h)}(1 - f(1+h))]dr^2\Big\} \end{aligned}$$

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \Big(1 - f(1+h)\Big)\Big\{d\tilde{t}^2 - 2A'(r)\frac{f(1+h)}{1-f(1+h)}drd\tilde{t}$$
$$+ [-\frac{f(1+h)}{1-f(1+h)}(A'(r))^2 + \frac{(1+h)-f)}{f(1-f)(1+h)}]dr^2\Big\}. \quad \text{(A.4)}$$

One can easily check that, if $h = 0$ we have Schwarzschild spacetime in Kerr-Schild form[68]. To find the complete square we have to find an $A(r)$ such that the term in the curly bracket becomes a null 2-form. The condition for completing the square is indicated below:

$$\Big(A'(r)\frac{f(1+h)}{1-f(1+h)}\Big)^2 = -(A'(r))^2\Big(\frac{f(1+h)}{1-f(1+h)}\Big) + \frac{(1+h)-f}{f(1-f(1+h))}, \quad \text{(A.5)}$$

$A'(r)$ can be isolated by the equation:

$$(A'(r))^2 = \frac{1}{1-f}\Big\{\frac{[1+h-f](1-f(1+h))^2}{(f(1+h))^2}\Big\}. \quad \text{(A.6)}$$

The integrability of $A(r)$ is enough to say that we can write it in Kerr-Schild form. Which is checked by Mathematica. Then, one can find the Schwarzschild metric in Kerr-Schild form:

$$\begin{aligned} g_{\mu\nu}dx^\mu dx^\nu &= \eta_{\mu\nu}dx^\mu dx^\nu + (1 - f(1+h))(d\tilde{t} \mp dr)^2, \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + 2(\frac{M}{r} - \frac{M^3\epsilon_3}{2r^3} + \frac{M^4\epsilon_3}{r^4})(d\tilde{t} \mp dr)^2, \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + 2\phi(r)k_\mu k_\nu dx^\mu dx^\nu. \end{aligned}$$

where $f = (1 - \frac{2M}{r})$ and $h = \sum_{k=0}^{\infty}\epsilon_k(\frac{M}{r})^k = 1$.Hence we can identify the null one-forms and the $\phi(r)$ as

$$\begin{aligned} \phi(r) &= (\frac{M}{r} - \frac{M^3\epsilon_3}{2r^3} + \frac{M^4\epsilon_3}{r^4}), \\ k_\mu dx^\mu &= (d\tilde{t} \mp dr). \end{aligned}$$

A.1.2 Newman-Janis Algorithm to Modified Schwarzschild Metric

Roy Kerr found a unique rotating black hole solution [4]. This solution is tedious to find. However, Newman-Janis[23] found a beautiful way to find these unique solutions upon complexification of radial spacetime parameters, it is still an active line of research to understand its deep mathematical and physical correspondence and mappings between different classes of spacetimes. One can realize that both Kerr and

Schwarzschild solutions are unique solutions, the mapping between them through the NJ algorithm is a valid approach hence it should be valid for the deviations of the spacetime. It was the original approach of Johannsen and Psaltis.

Generically this rotation through complexification is dangerously arbitrary. It becomes more and more complicated in different powers of $\mathcal{O}(\frac{1}{r})$ and higher dimensional cases [69]

$$\phi(r\bar{r}) = \Big(M(\frac{1}{r} + \frac{1}{\bar{r}}) + \frac{M^3\epsilon_3}{2}(\frac{1}{r} + \frac{1}{\bar{r}})\frac{1}{r\bar{r}} + \frac{M^4\epsilon_3}{(r\bar{r})^2})\Big), \tag{A.7}$$

where $r = \tilde{r} + ia\cos\theta$, $\bar{r} \equiv r^*$. This choice is specific in keeping the order of r the same as the Modified Schwarzschild metric.

$$\begin{aligned}\phi(r,\theta) &= \Big(\frac{2Mr}{r^2 + a^2\cos^2\theta} + \frac{M^3 r\epsilon_3}{(r^2 + a^2\cos^2\theta)^2} + \frac{M^4\epsilon_3}{(r^2 + a^2\cos^2\theta)^2})\Big),\\ \phi(r,\theta) &= \Big(\frac{2Mr}{r^2 + a^2\cos^2\theta} + \epsilon_3(\frac{M^3(M+r)}{(r^2 + a^2\cos^2\theta)^2}\Big). \end{aligned}\tag{A.8}$$

This should be a possible correction to the Kerr-Schild parameter since if we took the deviation $\epsilon_3 \longrightarrow 0$ it is clear that we find the Kerr-Schild form of Kerr spacetime.

A.1.3 The Dictionary of the Double Copy

Now the dictionary for the Classical Double Copy will be presented for simple cases. The corresponding gauge potentials will behave like Lienard-Wiechert single copy fields[70]

A.1.3.1 Modified Schwarzschild Single copy

From the Modified Schwarzschild metric, we found a simple form for the Kerr-Schild parameter $\phi(r)$

$$\phi(r) = (\frac{M}{r} - \frac{M^3\epsilon_3}{2r^3} + \frac{M^4\epsilon_3}{r^4}), \tag{A.9}$$

$$k_\mu dx^\mu = (d\tilde{t} \mp dr). \tag{A.10}$$

One should replace $M \equiv q$, $\kappa \equiv g$ such that the gravitational solution corresponds to an Abelian Yang-Mills solution of the underlying electromagnetic theory. With this dictionary gauge potential can be written as:

$$A_\mu^{ModSch} = \frac{g}{4\pi}\Big(\frac{q}{r} + \epsilon_3(-\frac{q^3}{2r^3} + \frac{q^4}{r^4})\Big)\Big\{1, -1, 0, 0\Big\}. \tag{A.11}$$

One can check that we have the Coulomb potential for $\epsilon_3 \longrightarrow 0$. Also, the time-like vector can be left untouched to find the current density.

$$F^{\mu\nu} = \partial^\mu A^\nu + \partial^\nu A^\mu,$$

$$\partial_\mu F^{\mu\nu} = J^\nu,$$

$$J^\mu = \frac{g}{4\pi}(-4\pi q\delta(r) + 4\pi\epsilon_3 \frac{3q^3(4q-r)}{r^6}). \tag{A.12}$$

A.1.3.2 Johannsen-Psaltis Single Copy

From the Johannsen-Psaltis metric, we found a simple form for the Kerr-Schild parameter $\phi(r, \theta)$

$$\phi(r,\theta) = \Big(\frac{2Mr}{r^2 + a^2\cos^2\theta} + \epsilon_3(\frac{M^3(M+r)}{(r^2+a^2\cos^2\theta)^2}\Big). \tag{A.13}$$

One should replace $M \equiv q$, $\kappa \equiv g$ such that the gravitational solution corresponds to an Abelian Yang-Mills solution of the underlying electromagnetic theory [71]. With this dictionary gauge potential can be written as:

$$A^\mu_{JP} = \frac{g}{4\pi}\Big(\frac{qr}{r^2 + a^2\cos^2\theta} + \epsilon_3(\frac{q^3(q+r)}{(r^2+a^2\cos^2\theta)^2)}\Big)\Big\{1, -1, 0, \frac{-a}{a^2+r^2}\Big\}, \tag{A.14}$$

one can check that for $\epsilon_3 \longrightarrow 0$ we have the thin rotating disc of charge's gauge potential.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The component-by-component analysis brings:

$$F^{tr} = -F^{rt} = \frac{1}{4\pi(h+1)\Sigma^4}\Big(gq(a^2\sin^2\theta h + \Delta)(\frac{\partial\Sigma}{\partial r}(r\Sigma + 2\epsilon_3(q+r)) - \Sigma(\Sigma+\epsilon_3))\Big)$$

$$F^{t\theta} = F^{\theta r} = -F^{r\theta} = -F^{\theta t} = \frac{gq}{4\pi\Sigma^3}\big[r\Sigma + 2\epsilon_3(q+r)\big]\partial_\theta \ln\Sigma, \tag{A.15}$$

$$F^{r\phi} = -F^{\phi r} = \frac{2ar\big[a^2\sin^2\theta h + \Delta\big]}{(a^2+r^2)^2(h+1)\Sigma}. \tag{A.16}$$

APPENDIX B

MEMBRANE PARADIGM OF EINSTEIN-HAWKING-HOROWITZ THEORY

B.1 An approach with Einstein-Hawking-Horowitz Theory through Schwarzschild-AdS $\&$ AdS

Let us start with a variation of Einstein-Hilbert Action:

$$\delta I = \frac{1}{16\pi}\int d^D x\sqrt{-g}\, G_{\mu\nu} - \frac{1}{8\pi}\oint d^{D-1}x\, \delta(\sqrt{h}K),$$
$$+\frac{1}{16\pi}\int d^{D-1}x\sqrt{h}\,(K_{\mu\nu} - h_{\mu\nu}K)\delta h^{\mu\nu} - \frac{1}{16\pi}\int d^{D-1}x\sqrt{h}\,(t^{stretched}_{\mu\nu})\delta h^{\mu\nu} = 0. \tag{B.1}$$

As we discussed earlier, the second integral cancels the extra contribution at infinity coming from the variation of the first integral. Moreover, the fourth integral cancels the extra contribution coming from the stretched horizon embedded arbitrarily close to the true horizon. One can choose a background spacetime while considering a background correction to the stretched horizon stress tensor.

$$I_{EHH} = I_E + I_{HH} = \frac{1}{16\pi}\int_{\mathcal{M}} d^D x\sqrt{-g}\, R + \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{h}\,(K - \bar{K}),$$

where $I_{HH} = \frac{1}{16\pi}\int_{\partial\mathcal{M}} d^{D-1}x\sqrt{h}\,(K - \bar{K})$ After some calculations given in (eqn 2.4) of [72], one can follow a different approach and define the found tensor as stretched horizon tensor.

$$I_{HH} = \frac{1}{16\pi}\int d^{D-1}x\sqrt{h}\,\delta h^{\mu\nu}(K_{\mu\nu} - h_{\mu\nu}(K - \bar{K})),$$

$$t^{stretched}_{\mu\nu} = \frac{1}{16\pi}(K_{\mu\nu} - h_{\mu\nu}(K - \bar{K})). \tag{B.2}$$

Let us choose the spacetime M as SAdS and the background spacetime $\bar{M}$ as AdS. This is a judicious choice because SAdS and AdS are massless and massive versions of each other in (t, r) submanifold and their (θ, ϕ) submanifold is exactly the same. The metrics read as:

$$ds^2_{SAdS} = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega_2^2,$$
$$ds^2_{AdS} = -g dt^2 + g^{-1} dr^2 + r^2 d\Omega_2^2, \tag{B.3}$$

where $g = 1 - \frac{\Lambda r^2}{3}$ and $f = 1 - \frac{\Lambda r^2}{3} - \frac{2M}{r}$ as we mentioned before. Our job is to decompose the geometry as $ds^2 = -U_\mu U_\nu dx^\mu dx^\nu + n_\mu n_\nu dx^\mu dx^\nu + \gamma_{\mu\nu} dx^\mu dx^\nu$. We have to pick U_μ and n_μ in such a way that at $r = r_H$ they should be null. For main spacetime,

$$U_\mu dx^\mu = f^{\frac{1}{2}} dt \implies U_\mu = f^{\frac{1}{2}} \delta^t{}_\mu,$$
$$n_\mu dx^\mu = f^{-\frac{1}{2}} dr \implies n_\mu = f^{-\frac{1}{2}} \delta^r{}_\mu,$$
$$\gamma_{\mu\nu} dx^\mu dx^\nu = r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{B.4}$$

for background spacetime

$$V_\mu dx^\mu = g^{\frac{1}{2}} dt \implies V_\mu = g^{\frac{1}{2}} \delta^t{}_\mu,$$
$$m_\mu dx^\mu = g^{-\frac{1}{2}} dr \implies m_\mu = g^{-\frac{1}{2}} \delta^r{}_\mu,$$
$$\gamma_{\mu\nu} dx^\mu dx^\nu = r^2 (d\theta^2 + \sin^2\theta d\phi^2). \tag{B.5}$$

In 2D hypersurface coordinates $\{A, B\}$,

$$\gamma_{AB} = \begin{pmatrix} r^2 & 0 \\ a & r^2 \sin^2\theta \end{pmatrix}, \tag{B.6}$$

where $\{A, B\} = \{\theta, \phi\}$. such that $U_\mu U^\mu = -1$, $n_\mu n^\mu = 1$ while $V_\mu V^\mu = -1$, $m_\mu m^\mu = 1$ One can now find 3D Extrinsic curvature, extrinsic curvature scalar and write down the stretched horizon tensor in terms of given spacetime vectors as:

$$K_{\mu\nu} = -\sqrt{f} \begin{pmatrix} \frac{1}{2}\partial_r f & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & r\sin^2\theta \end{pmatrix},$$

$$\bar{K}_{\mu\nu} = -\sqrt{g} \begin{pmatrix} \frac{1}{2}\partial_r g & 0 & 0 \\ 0 & -r & 0 \\ 0 & 0 & r\sin^2\theta \end{pmatrix}, \tag{B.7}$$

where the extrinsic curvature scalar of the main and background spacetime is:

$$K = \frac{r\partial_r f + 4f}{2r\sqrt{f}}, \tag{B.8}$$

$$\bar{K} = \frac{r\partial_r g + 4g}{2r\sqrt{g}}, \tag{B.9}$$

then the stretched horizon tensor can be found as:

$$t_{\mu\nu}^{stretched} = -\big(2\frac{f^{\frac{1}{2}}}{r} - \frac{4g + r\partial_r g}{rg^{\frac{1}{2}}}\big)U_\mu U_\nu + \frac{1}{2r}\big(4g^{\frac{1}{2}} - \frac{2f + r\partial_r f}{f^{\frac{1}{2}}} + \frac{r\partial_r g}{g^{\frac{1}{2}}}\big)\gamma_{\mu\nu}.$$

One can write this in a more fashionable manner:

$$t_{\mu\nu}^{stretched} = -\big(\frac{2f^{\frac{1}{2}}}{r} - \frac{4g^{\frac{1}{2}}}{r} - g^{\frac{-1}{2}}\partial_r g\big)U_\mu U_\nu + \big(\frac{2g^{\frac{1}{2}}}{r} - \frac{f^{\frac{1}{2}}}{r} - \frac{1}{2}f^{\frac{-1}{2}}\partial_r f + \frac{1}{2}g^{-\frac{1}{2}}\partial_r g\big)\gamma_{\mu\nu},$$

then the membrane paradigm gives:

$$\rho = -\frac{1}{8\pi}\Theta, \qquad \eta = \frac{1}{16\pi},$$

$$P = \frac{\kappa}{8\pi}, \qquad \zeta = -\frac{1}{16\pi},$$

$$\pi^A = 0, \tag{B.10}$$

$$\Theta = \frac{2f}{r} - \frac{4g^{\frac{1}{2}}f^{\frac{1}{2}}}{r} - f^{\frac{1}{2}}rg^{\frac{-1}{2}}\partial_r g,$$

$$\sigma_{AB} = 0,$$

$$\kappa = \frac{\partial_r f}{2}.$$

It seems for Einstein-Hawking-Horowitz's theory the surface gravity of background spacetime is directly affecting the expansion Θ and the energy density of the main spacetime while the surface gravity of the main spacetime does not feel the existence of the background metric.

Acceleration on the stretched horizon surface induced by the surface gravity of AdS relates to the expansion of null geodesics however, the surface gravity affects the FIDO observer dominated by the main spacetime.

One can check that if the background metric's effect on the black hole stretched horizon vanishes i.e $g \longrightarrow 0$:

$$\lim_{g\to 0}\Theta = \frac{2f(r)}{r}, \tag{B.11}$$

which is the well-known solution for expansion Θ in static spacetimes.

If we substitute the values for SAdS-AdS:

$$\Theta = \frac{2}{3r^2}\left[r\left(3-2\sqrt{3-\Lambda r^2}\sqrt{-\frac{6M}{r}-\Lambda r^2+3}\right)+\frac{\Lambda r^4\sqrt{-\frac{6M}{r}-\Lambda r^2+3}}{\sqrt{3-\Lambda r^2}}-6M-\Lambda r^3\right],$$

$$\sigma_{AB} = 0,$$

$$\kappa = \frac{1}{2}\left(\frac{2M}{r^2}-\frac{2\Lambda r}{3}\right).$$