TOROIDAL COMPACTIFICATION OF MODULI SPACES OF K3 SURFACES

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ABSTRACT

TOROIDAL COMPACTIFICATION OF MODULI SPACES OF K3 SURFACES

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The theory of moduli spaces of K3 surfaces, a frequently studied concept in Algebraic Geometry, is very interesting because of its connections in both mathematics and theoretical physics. The compactification of these fascinating spaces has also been the focus of attention of mathematicians for years. In this thesis, we will examine the toroidal compactification of the moduli space of K3 surfaces. After giving the necessary information and important examples about K3 surfaces and their moduli spaces, we introduce the concept of toric variety required for toroidal compactification. Then, we present the definitions in toroidal compactification through a specific example. Finally, we give our main theorem and apply it to the moduli spaces of K3 surfaces.

Keywords: K3 surfaces, Toric Varieties, Toroidal Compactification

ÖZ

K3 YÜZEYLERİNİN MODULİ UZAYLARININ TOROİDAL TIKIZLAŞTIRILMASI

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Cebirsel Geometride sıkça çalışılan bir kavram olan K3 yüzeylerinin modül uzaylarının teorisi, hem matematikte hem de teorik fizikte bağlantıları nedeniyle çok ilgiçttir. Bu büyüleyici yüzeylerin tıkızlaştırılması da yıllarca matematikçilerin ilgi odağı olmuştur. Bu tezde K3 yüzeylerinin modül uzayının toroidal tıkızlaştırılmasını inceleyeceğiz. K3 yüzeyleri ve onların modül uzayları hakkında gerekli bilgileri ve önemli örnekleri verdikten sonra toroidal tıkızlaştırma için gerekli olan toric değişken kavramını tanıtacağız. Ardından özel bir örnek üzerinden toroidal tıkızlaştırmadaki tanımları sunacağız. En son noktada da ana teoremimizi vereceğiz ve K3 yüzeylerinin modül uzaylarına uygulayacağız.

Anahtar Kelimeler: K3 yüzeyleri, toric değişkenler, toroidal tıkızlaştırılma

To my family

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INTRODUCTION

Moduli spaces play a central role in Algebraic Geometry, since they provide a framework for studying families of algebraic varieties and understanding their geometric properties. Within this wide working area of moduli spaces, especially the moduli space of K3 surfaces stands out as a particularly fascinating and interesting object of study. A K3 surface is a simply connected compact complex surface with trivial canonical bundle and vanishing first Chern class. Algebraic K3 surfaces were first studied by Andre Weil in the late 1950s, and as Weil said the name of K3 sufaces was given "in honor of Kummer, Kähler, Kodaira, and the beautiful mountain K2 in Kashmir". Soon they became one of the favorite spaces of mathematicians. Moreover, K3 surfaces can be defined as Calabi-Yau varieties of dimension 2. This perspective made K3 surfaces one of the favorite spaces of physicists since Calabi-Yau conjecturally exhibit a remarkable duality known as mirror symmetry.

One of the central goals in the study of K3 surfaces is to understand their moduli space, since it acts as a category for all K3 surfaces, allowing mathematicians to study the general properties and variations of these surfaces. The Torelli Theorems, named after an Italian mathematician Ruggiero Torelli, are gold for us to understand K3 surfaces and their moduli spaces. The Torelli Theorem states that the geometry of a K3 surface is completely determined by its Hodge structure and thus the Torelli Theorem provides a powerful tool for understanding the geometry and moduli space of K3 surfaces by giving a connection between the algebraic and geometric properties of K3 surfaces.

One of the most important things we expect from moduli spaces is that they are compact. However, in the classical sense the moduli space of K3 surfaces is not compact.

So, we want to find a "good" compactification which resolves this problem. To address this issue and to understand the moduli space fully, we can consider at the concept of toroidal compactification.

A toroidal compactification is a systematic method used in geometry and topology to compactify a space by adding extra points or boundary components to make it a compact space. Specifically, it involves compactifying a space by attaching tori. The resulting compactified space contains the original space as an open subset.

To understand toroidal compactification, we should have a solid understanding of toric varieties. Toric varieties can be obtained in a combinatorial way. The combinatorial data associated with toric varieties provides an explanation for toroidal compactification. Moreover, since toric varieties have well-understood singularities, we can obtain a compactified smooth space by resolving the singularities.

Once we review K3 surfaces and toric varieties, we will begin to discuss the toroidal compactification of moduli spaces of K3 surfaces.

In summary, in this thesis, we aim to shed light on the geometric and topological aspects of toroidal compactificatin of moduli spaces of K3 surfaces. By combining techniques from algebraic geometry, toric geometry, and related areas, we want to deepen our understanding of the moduli spaces of K3 surfaces, their toric structures, and their connections to various branches of mathematics.

We organize this thesis as follows:

First of all, in Chapter 2, we will review the information we need to know about K3 surfaces. In this chapter, we will go towards "Torelli Theorem", which is one of the most remarkable theorems in mathematics. In Chapter 3, we will focus on toric varieties. This chapter will consist of purely combinatorial definitions and examples. By the end of Chapter 3, we will reviewed the background material on toric varieties which is necessary to understand toroidal compactification. In the last chapter, we will start with the definitions required for toroidal compactification eand give the main theorem in [2]. Finally, we will discribe the toroidal compactification of the moduli space of K3 surfaces.

CHAPTER 2

K3 SURFACES

2.1 Definition and Examples

Definition 1. [13] A K3 surface X over $\mathbb{C}$ is a complete non-singular variety X of dimension two such that canonical bundle $\omega_{\mathrm{X}} = \bigwedge^2 \Omega_{\mathrm{X}} \simeq \mathcal{O}_{\mathrm{X}}$ is trivial and $H^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$.

Remark 2.1.1. [13] Over an arbitrary field, any smooth complete surface is projective. Thus, K3 surfaces are always projective.

Example 2.1.1. [13] Consider a smooth degree 4 surface X in $\mathbb{P}^3$,a "quartic surface" in $\mathbb{P}^3$. The short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathrm{X}} \longrightarrow 0,$$

induces a long exact sequence

$$\ldots \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}(-4)) \longrightarrow H^1(\mathbb{P}^3, \mathcal{O}) \longrightarrow H^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}})$$

$$\longrightarrow H^2(\mathbb{P}^3, \mathcal{O}(-4)) \longrightarrow \ldots$$

Since $H^1(\mathbb{P}^3, \mathcal{O}) = H^2(\mathbb{P}^3, \mathcal{O}(-4)) = 0$, we have $H^1(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$. On the other hand by adjunction formula, (see [10]), $\omega_{\mathrm{X}} = \mathcal{O}_{\mathrm{X}}$. Hence, a quartic surface is a K3 surface.

Example 2.1.2. [13] [17] Let $k = \mathbb{C}$, and A an abelian variety of dimension 2 over k. Let $i : A \to A$ be the automorphism of A defined by $i(x) = -x$. Observe that $i^2 = id_A$ and the group $G = \{id_A, i\}$ acts on A. Moreover, for a point $x \in A$ $i(x) = x$ if and only if the order of x in the abelian group $(A, +)$ is 1 or 2. By [19], § 6], there are exactly 16 fixed points on A. The quotient by this automorphism gives

only rational double point singularities, and by blowing up these point, they can be resolved. Also, we can first start with blowing up these fixed points, then take the quotient. So, we have the commutative diagram

$$\begin{array}{ccc} \tilde{A} & \longrightarrow & A \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ X & \longrightarrow & A/i, \end{array}$$

where the map $\tilde{A} \to A$ is blowing up. The canonical bundle formula for the blow up [[10], V.Prop.3.3] gives

$$\omega_{\tilde{A}} \simeq \mathcal{O}(\sum E_i),$$

and the canonical bundle formula for the branched covering π gives

$$\omega_{\tilde{A}} \simeq \pi^* \omega_X \otimes \mathcal{O}(\sum E_i),$$

where E_i are the exceptional divisors of blowing up, and their images $\overline{E}_i$) in X gives $\pi^*\mathcal{O}(\overline{E}_i) \simeq \mathcal{O}(2E_i)$. Thus, we have $\pi^*\omega_X \simeq \mathcal{O}_{\tilde{A}}$. Suppose the line bundle L is a square root of $\mathcal{O}(\overline{E}_i)$. Then, we get $\pi_*\mathcal{O}_{\tilde{A}} \simeq \mathcal{O}_X \oplus L^*$, and finally by the projection formula we have $\omega_X \simeq \mathcal{O}_X$. Observe that the image of the induced map $H^1(X, \mathcal{O}_X) \to H^1(\tilde{A}, \mathcal{O}_{\tilde{A}}) \simeq H^1(X, \mathcal{O}_X)$ is in the subspace invariant under the action of i. Hence, $H^1(X, \mathcal{O}_X) = 0$

Later, we will try to understand the integral cohomology structures of K3 surfaces. So, we need to focus on some results from Algebraic Geometry without proofs.

Theorem 2.1.1. *(Serre Duality [10])*

Let $\mathcal{F}$ be a locally free sheaf over an n-dimensional smooth projective variety X. Then for any $0 \leq k \leq n$, we have

$$H^k(X, \mathcal{F}) = H^{n-k}(X, \mathcal{F}^* \otimes \omega_X)^*,$$

where $\mathcal{F}^* = Hom(\mathcal{F}, \mathcal{O}_X)$.

As soon as we know Serre duality, we can directly deduce the next corollary,

Corollary 2.1.2. [13] Let X be a K3 Surface. Then, $dim(H^2(X, \mathcal{O}_X) = 1$

Proof. First note that since the canonical bundle is trivial, we have

$$\mathcal{O}_X^* \otimes \omega_X \simeq \mathcal{O}_X^* \otimes \mathcal{O}_X \simeq \mathcal{O}_X^* \simeq \mathcal{O}_X.$$

By Serre duality with $k = 2$ and $n = 2$, $H^2(X, \mathcal{O}_X) = H^0(X, \mathcal{O}_X) = \mathbb{C}$.

□

Definition 2. [10] Let $\mathcal{F}$ be a sheaf over a variety X of dimension n. The Euler Characteristic of $\mathcal{F}$ is

$$\mathcal{X}(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i dim(H^i(X, \mathcal{F})).$$

We now recall basics of the intersection pairing of divisors on smooth surfaces.

Definition 3. [13] Let X be an arbitrary non-singular complete surface over k and L_1 and L_2 be line bundles over X, i.e. $L_1, L_2 \in Pic(X)$. Then, define the intersection form

$$(L_1.L_2) := \mathcal{X}(\mathcal{O}_X) - \mathcal{X}(L_1^*) - \mathcal{X}(L_2^*) + \mathcal{X}(L_1^* \otimes L_2^*).$$

Theorem 2.1.3. *The Riemann-Roch Theorem for line bundles on surfaces [[13],2.1. iv)] Let X be a smooth surface and $L \in Pic(X)$. Then,*

$$\mathcal{X}(L) = \frac{L.(L \otimes \omega_X^*)}{2} + \mathcal{X}(\mathcal{O}_X).$$

Corollary 2.1.4. [13] In addition to the Riemann-Roch Theorem, if X is a K3 surface, then

$$\mathcal{X}(L) = \frac{1}{2} L.L + 2,$$

and $\mathcal{X}(\mathcal{O}_X) = 2$

Proof. Since X is a K3 surface $\omega_X^* \simeq \mathcal{O}_X^* \simeq \mathcal{O}_X$ and thus $\mathcal{X}(\mathcal{O}_X) = 2$, since $H^0(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X = \mathbb{C}$ and $H^1(X, \mathcal{O}_X) = 0$. □

we will try to understand the lattice structures of K3 surfaces in next sections. So, a few properties of the intersection form will come in handy for us. We start with a well-known theorem whose proof can be found in [[10], Chapter 5, Theorem 1.1]

Theorem 2.1.5. *[17] Let X be a smooth projective surface. Then the intersection form $(.) : Div(X) \times Div(X) \longrightarrow \mathbb{Z}$, satisfies the followings for any two divisors C,D:*
i) if C and D are non-singular curves meeting transversally, then $(C.D) = \#(C \cap D)$ which is the number of intersection points,
ii) the pairing is symmetric, i.e. $(C.D) = (D.C)$,
iii) the pairing is bilinear, i.e. if $C_1, C_2, D \in Div(X)$, then $((C_1 + C_2).D) = (C_1.D) + (C_2.D)$,
iv) the pairing depends only on the linear equivalence classes, i.e. if $C_1 \sim C_2$, then $(C_1.D) = (C_2.D)$.

Definition 4. [17] The Néron–Severi group of an algebraic surface $NS(X) := Pic(X)/Pic^0(X)$, where $Pic^0(X)$ is the subgroup of all line bundles that are algebraically equivalent to zero.

Definition 5. [13] A line bundle L is called numerically trivial if $(L.L') = 0$ for all line bundles L'. The subgroup of numerically trivial line bundle is denoted by $Pic^\tau(X)$ and

$$Num(X) := Pic(X)/Pic^\tau(X).$$

Proposition 2.1.6. [13] $NS(X)$ and $Num(X)$ are finitely generated groups. (The rank of $NS(X)$ is called the Picard number $\rho(X) = rk(NS(X))$).

Theorem 2.1.7. *(Hodge index theorem) [17] Let X be a surface and H be an ample divisor on X. Suppose D is a divisor which is not numerically trivial and such that $(D.H) = 0$. Then $(D.D) = D^2 < 0$.*

Proposition 2.1.8 (see [10], Chapter 5, Theorem 1.9)**.** The signature of the intersection form on $Num(X)$ is $(1, \rho(X) - 1)$.

Proposition 2.1.9. [13] Let X be a K3 surface. The natural surjections are isomorphisms:

$$Pic(X) \xrightarrow{\sim} NS(X) \xrightarrow{\sim} Num(X).$$

Moreover, the intersection form on $Pic(X)$ is non-degenerate, even and of signature $(1, \rho - 1)$

Proof. Let L be a non-trivial line bundle such that for all line bundle L', $(L.L') = 0$. Then also $(L^*.L')$. Note that if L is an ample line bundle and $C \subset X$ is a curve, then $(L.\mathcal{O}(C)) > 0$. Thus, L and L^* do not arise as bundles corresponding to effective divisors in X, so they cannot have nonzero global sections. So, $H^0(L) = H^0(L^*) = 0$. By Serre duality, $H^2(L) = H^0(L^*)^* = 0$. Thus, $\mathcal{X}(L) = -h^1(L) \leq 0$. By the Riemann-Roch formula for K3 surfaces

$$\mathcal{X}(L) = \frac{1}{2}(L)^2 + 2 \leq 0.$$

Since the intersection pairing gives an integer, $(L)^2 < 0$. So, L is not numerically trivial. Hence,

$$Pic(X) \xrightarrow{\sim} NS(X) \xrightarrow{\sim} Num(X).$$

The intersection form is non-degenerate on $Num(X)$, so it is non-degenerate on $Pic(X)$.

Finally, by Riemann-Roch formula $(L)^2 = 2(\mathcal{X}(L) - 4)$ and we have that the form is even.

□

Remark 2.1.2. We will give a general definition of K3 surfaces as complex manifolds which are not necessarily algebraic. By Serre's GAGA, any projective variety is algebraic, so the new ones are not projective. However, for non-projective complex K3 surfaces it can happen that $Pic(X)$ and $Num(X)$ are not isomorphic. Thus, the above proposition is not useful to find cohomology groups of non-projective complex K3 surfaces.

2.2 Topology of K3 surfaces

In this section, we try to calculate cohomology groups of K3 surfaces. Observe that the dual bundle Ω_X^* of the cotangent bundle is called the tangent bundle and is denoted by T_X. Thus, we have the natural alternating pairing

$$\Omega_X \times \Omega_X \longrightarrow \omega_X \simeq \mathcal{O}_X.$$

and induces a non-canonical isomorphism

$$T_X = \Omega_X^* = Hom(\Omega_X, \mathcal{O}_X) \simeq \Omega_X.$$

The Hodge numbers of a K3 surface X are

$$h^{p,q} := dim H^q(X, \Omega_X^p).$$

For a locally free sheaf (or an arbitrary coherent sheaf) F on a K3 surface X, Hirzebruch–Riemann–Roch formula gives

$$\mathcal{X}(F) = \int ch(F)td(X) = ch_2(F) + 2rk(F),$$

where $ch(F)$ is the exponential Chern character of F and $td(F)$ is the Todd class of F. Moreover, for $F = \mathcal{O}_X$ the first equality is the Noether formula, where $c_1(X) = -c_1(\omega_X) = 0$ by the definition

$$2 = \mathcal{X}(\mathcal{O}_X) = \frac{c_1^2(X) + c_2(X)}{12} = \frac{c_2(X)}{12}.$$

So, $c_2(X) = 24$. Recall that,

$$H^0(X, \Omega_X^0) = H^0(X, \Omega_X^2) = H^2(X, \Omega_X^0) = H^2(X, \Omega_X^2) = \mathbb{C},$$

$$H^0(X, \Omega_X^1) = 0$$

for any K3 surface.

By the Hirzebruch–Riemann–Roch formula,

$$2h^0(X, \Omega_X) - h^1(X, \Omega_X) = ch_2(\Omega_X) + 4 = 4 - c_2(\Omega_X) = -20.$$

Since $h^0(X, \Omega_X) = 0$, $h^1(X, \Omega_X) = 20$, the Hodge diamond looks like

$$\begin{array}{ccccc}
 & & h^{0,0} = 1 & & \\
 & h^{1,0} = 0 & & h^{0,1} = 0 & \\
h^{2,0} = 1 & & h^{1,1} = 20 & & h^{0,2} = 1 \\
 & h^{2,1} = 0 & & h^{1,2} = 0 & \\
 & & h^{2,2} = 1 & &
\end{array}$$

As we mentioned before, we will also study non-projective complex K3 surfaces as well since the Global Torelli Theorem always involves non-projective K3 surfaces.

Definition 6. [13] A complex K3 surface is a compact connected manifold X of dimension 2 such that $\Omega_X^2 \simeq \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$.

Proposition 2.2.1. [13] Let $k = \mathbb{C}$. If X is an algebraic K3 surface over k, then the associated complex space X^{an} is a complex K3 surface. Also, all complex K3 surfaces obtained in this way are projective.

The space X^{an} mentioned above is created by Serre's GAGA principle, and the underlying set of points of X^{an} is the set of all closed points of X (see [24]).

So, we have a proper full embedding of categories

$$\{\text{algebraic K3 surfaces over } \mathbb{C}\} \hookrightarrow \{\text{complex surfaces}\}.$$

Now since we have a complex manifold, the exponential sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^{\mathsf{x}} \to 0$$

gives the long cohomology exact sequence for K3 surfaces X

$$0 \to H^0(X, \mathbb{Z}) = \mathbb{Z} \to H^0(X, \mathcal{O}) = \mathbb{C} \to H^0(X, \mathcal{O}^{\mathsf{x}}) = \mathbb{C} \to H^1(X, \mathbb{Z}) \to 0 \to$$

$$\to H^1(X, \mathcal{O}^{\mathsf{x}}) \to H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}) = \mathbb{C} \to H^2(X, \mathcal{O}^{\mathsf{x}}) \to H^3(X, \mathbb{Z}) \to 0.$$

So, we have $H^1(X, \mathbb{Z}) = 0$. Since X is simply connected, $\pi_1(X) = H_1(X, \mathbb{Z}) = 0$. By Poincaré duality also $H^3(X, \mathbb{Z}) = 0$ up to torsion. Thus, the only non-trivial integral singular cohomology group of X is $H^2(X, \mathbb{Z})$ other than $H^0(X, \mathbb{Z}) \simeq H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$.

Since $H^1(X, \mathcal{O}\mathsf{x}) \simeq Pic(X)$, we obtain the new variant of the above exact sequence

$$0 \to Pic(X) \to H^2(X, \mathbb{Z}) \to \mathbb{C} \to \dots .$$

As both $Pic(X)$ and $\mathbb{C}$ are torsion free, $H^2(X, \mathbb{Z})$ is torsion free by exactness. By Poincare duality and the Universal Coefficient Theorem, we have $H^3(X, \mathbb{Z})$ is torsion free and indeed it is trivial. By The Riemann-Roch computations, we know that $c_2(X) = 24$ and for a complex surface it equals the topological Euler number

$$c_2(X) = \mathcal{X}(X) = \sum_{i=0}^{4} (-1)^i b_i(X) = \sum_{i=0}^{4} (-1)^i rank(H^i) = 24.$$

Since $b_1(X) = b_3(X) = 0$ and $b_0(X) = b_4(X) = 1$, $b_2(X) = 22$. Hence,

Proposition 2.2.2. The integral cohomology of a K3 surface X is

$$H^i(X,\mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } i=0 \\ 0 & \text{if } i=1 \\ \mathbb{Z}^{22} & \text{if } i=2 \\ 0 & \text{if } i=3 \\ \mathbb{Z} & \text{if } i=0 \end{cases}$$

2.3 The K3 Lattice

In this section we try to understand a special lattice Λ which is called the K3 lattice. But first we start with general definitions.

Definition 7. A lattice Λ is a free $\mathbb{Z}$-module of finite rank together with a symmetric bilinear form

$$(.) : \Lambda x \Lambda \to \mathbb{Z},$$

which we will always assume to be non-degenerate.

Remark 2.3.1. A lattice Λ and the $\mathbb{R}$-linear extension of its bilinear form (.) gives the real vector space $\Lambda_{\mathbb{R}} := \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ endowed with a symmetric bilinear form.

Definition 8. [7] Let V be a finite dimensional real vector space and $\Phi : V \times V \to \mathbb{R}$ be a (non-singular) symmetric bilinear form. Say V has a basis $\{x_1, ..., x_r, y_1, ..., y_s\}$, which always exists, such that $\Phi(x_i, y_j) = 0$, $\Phi(x_i, x_i) = 1$, $\Phi(x_i, x_j) = 0$ and $\Phi(y_j, y_j) = -1$, $\Phi(y_i, y_j) = 0$ for $i \neq j, 1 \leq i \leq r$, $1 \leq j \leq s$. (r, s) is called the signature of Λ and $r - s$ is called the index of Λ.

Definition 9. [4]

- The signature of Λ is the signature of the bilinear form on the vector space $\Lambda_{\mathbb{R}}$.
- Λ is indefinite if $r, s > 0$
- The bilinear form is called unimodular if the map induced by the bilinear form $\Lambda \to \Lambda^\vee = Hom(\Lambda, \mathbb{Z})$ is an isomorphism,

- A lattice is even if $(v.v)$ is even for all $v \in \Lambda$, otherwise the lattice is called old,
- $disc(U)$ is the determinant of the matrix which defines the lattice,
- A sublattice L of Λ is primitive if Λ/L' is torsion free,
- A sublattice L of Λ is isotropic if the restriction of the blinear form to $L \times L$ is zero.

Let's give some examples of even, unimodular lattices:

Example 2.3.1. The matrix $U := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ determines a lattice on $\mathbb{Z}^2$ of signature (1,1), i.e. $U \simeq e.\mathbb{Z} \oplus f.\mathbb{Z}$ with the quadratic form given by $(e)^2 = (f)^2 = 0$ and $(e.f)^2 = 1$. So, the lattice is even. Moreover, since $disc(U) = -1$ the lattice is unimodular.

Example 2.3.2. Consider the matrix

$$E_8 := \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & -1 & & & \\ & & -1 & 2 & 0 & & & \\ & & -1 & 0 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}$$

This matrix determines the E_8-lattice which is even and unimodular. For more details see [[13], Chapter 14].

The next definition and lemmas are from Algebraic Topology, see [[23]]:

Definition 10. [23] Let M be a simply-connected,closed orientable 4-manifold. Then its intersection form

$$\mathcal{Q}_M : H_2(M, \mathbb{Z}) \times H_2(M, \mathbb{Z}) \to \mathbb{Z},$$

is defined by $\mathcal{Q}_M(\alpha, \beta) = (S_\alpha.S_\beta)$, where S_α and S_β are any two surfaces representing the classes α and β.

Remark 2.3.2. [23] Since M is simply-connected, $H_2(M,\mathbb{Z})$ is a free $\mathbb{Z}$-module by the Universal Coefficient Theorem. But actually we do not need this assumption since by linearity the intersection form will always vanish on torsion classes for any closed, orientable 4-manifold M. Thus, we assume that $H_2(M,\mathbb{Z})$ is free.

Lemma 2.3.1. [23] The form $\mathcal{Q}_M(\alpha,\beta) = (S_\alpha . S_\beta)$ on $H_2(M,\mathbb{Z})$ coincides with the pairing $\mathcal{Q}_M(\alpha^*,\beta^*) = (\alpha^* \cup \beta^*)[M]$ on $H^2(M,\mathbb{Z})$.

Lemma 2.3.2. [23] The intersection form $\mathcal{Q}_M$ of a 4-manifold is unimodular. Moreover, by Poincaré duality, the intersection form on $H^2(M,\mathbb{Z})$ also defines a unimodular lattice.

Proof. Unimodularity is equivalent to the property that, for every $\mathbb{Z}$-linear function $f : H_2(M,\mathbb{Z}) \to \mathbb{Z}$, there exists a unique $\alpha \in H_2(M,\mathbb{Z})$ so that $f(x) = (\alpha.x)$. So, the intersection form is unimodular if and only if the map

$$\begin{aligned} \Psi : H_2(M,\mathbb{Z}) &\longrightarrow Hom(H_2(M,\mathbb{Z}),\mathbb{Z}) \\ \alpha &\mapsto (x \mapsto \alpha.x) \end{aligned}$$

is an isomorphism. We will show that the last map coincides with the Poincare duality morphism. Indeed, the Poincare duality is the isomorphism

$$\begin{aligned} H_2(M,\mathbb{Z}) &\longrightarrow H^2(M,\mathbb{Z}) \\ \alpha &\mapsto \alpha^* \end{aligned}$$

with $\alpha^* \cap [M] = \alpha$. Since $H_2(M,\mathbb{Z})$ is free, then by the Universal Coefficient Theorem we have

$$\begin{aligned} H^2(M,\mathbb{Z}) &\xrightarrow{\simeq} Hom(H_2(M,\mathbb{Z}),\mathbb{Z}) \\ \alpha^* &\mapsto (x \mapsto \alpha^*[x]). \end{aligned}$$

However, as argued in the last lemma, we have $(\alpha.x) = \alpha^*[x]$. Therefore, the above lemma coincides with the map Φ. Hence, the intersection form is unimodular. □

Hence, the intersection form (.) on $H^2(X,\mathbb{Z})$ for a K3 surface X defines a unimodular lattice, since X is a compact, oriented, real 4-dimensional manifold.

Before starting the our next proposition, we should mention the general classification of unimodular lattices:

Corollary 2.3.3. [25] Let Λ be an indefinite and a unimodular lattice of signature (n_+, n_-). Then,

(i) If Λ is even of index $\tau = n_+ - n_-$, then $\tau \equiv 0$ mod(8) and

$$\Lambda \simeq E_8^{\oplus \frac{\tau}{8}} \oplus U^{\oplus n_-}$$

or

$$\Lambda \simeq E_8(-1)^{\oplus \frac{-\tau}{8}} \oplus U^{\oplus n_+},$$

according to the sign of τ,

(ii) If Λ is odd, then

$$\Lambda \simeq \langle 1 \rangle^{\oplus n_+} \oplus \langle -1 \rangle^{\oplus n_-}.$$

Proposition 2.3.4. [13] The integral cohomology $H^2(X, \mathbb{Z})$ of a complex K3 surface X endowed with the intersection form (.) is isomorphic to the lattice

$$H^2(X, \mathbb{Z}) \simeq \Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3},$$

where U is the hyperbolic plane and $E_8(-1)$ is the E_8-lattice as in Example 2.3.2 with the quadratic form changed by a sign.

Proof. (Sketch) By the above classification, it is enough to show that $H^2(X, \mathbb{Z})$ is even of signature (3,19). Note that the intersection product of X is even if and only if its second Stiefel–Whitney class $\omega_2(X)$ is trivial ,see [18], and since $\omega_2(X) \equiv c_1(X)$ mod(2), the intersection form is even.

Moreover, by the Thom–Hirzebruch index theorem the index of X is -16. Since $b_2(X) = rank H^2(X, \mathbb{Z}) = 22$, the signature is (3,19). □

Before we go any further, we need to understand briefly the Hodge structures needed to describe the Global Torelli Theorem.

Let V be a free $\mathbb{Z}$-module of finite rank or a finite-dimensional $\mathbb{Q}$- vector space, and let $V_{\mathbb{C}} = V \otimes_{\mathbb{Z}} \mathbb{C}$ or $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$ be the complex vector space obtained by scalar

extension. Note that, under this construction $V_{\mathbb{C}}$ comes with a real structure. Indeed, we have an $\mathbb{R}$-linear isomorphism $V_{\mathbb{C}} \to V_{\mathbb{C}}$ defined as $v \mapsto \overline{v}$ (complex conjugation).

Definition 11. [13] A Hodge structure of weight $n \in \mathbb{Z}$ on V is a direct sum decomposition of $V_{\mathbb{C}}$:

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$.

Example 2.3.3. [17] The Tate Hodge structure is the Hodge structure of weight

-2, denoted by $\mathbb{Z}(1)$, given by the free $\mathbb{Z}$-module of rank one such that $\mathbb{Z}(1)^{-1,-1}$ is one-dimensional. Similarly, we can define the rational version, $\mathbb{Q}(1)$ and the twisted version $\mathbb{Z}$ and $\mathbb{Q}$.

Definition 12. [13] A Hodge structure V is of K3 type if $dimV^{2,0} = dimV^{0,2} = 1$, $dimV^{1,1} = 20$ and the others are zero.

Now, let C be the Weil operator, which acts on $V^{p,q}$ by multiplication with i^{p-q}. Clearly, C preserves the real part of the Hodge structure.

Definition 13. [13] A polarization of a rational Hodge structure V of weight n is a morphism of Hodge structures

$$\psi : V \otimes V \to \mathbb{Q}(-n)$$

such that its $\mathbb{R}$-linear extension yields a positive definite symmetric form

$$(v, w) \mapsto \psi(v, Cw)$$

on the real part of $V^{p,q} \oplus V^{q,p}$, and (V, ψ) is called polarized Hodge structure. A Hodge structure is called polarizable if it admits a polarization. Let (V_1, ψ_1) and (V_2, ψ_2) be two polarized Hodge structures. An isomorphism $V_1 \tilde{\to} V_2$ of Hodge structures that is compatible with given polarizations ψ_1, respectively ψ_2, is called a Hodge isometry.

Now we can move on to the main part of this chapter.

2.4 Period Map and Torelli Theorems

Let Λ be the K3 lattice, i.e. $\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. Let $\Lambda_{\mathbb{C}} = \Lambda \otimes \mathbb{C}$ be the $\mathbb{C}$-vector space endowed with the $\mathbb{C}$-linear extension of the form on Λ which corresponds to a homogenous quadratic polynomial. The latter defines a quadric in $\mathbb{P}(\Lambda_{\mathbb{C}})$, which is smooth because the bilinear form is nondegenerate.

Definition 14. [17] The open subset of the quadric:

$$D := \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) | (x)^2 = 0, (x, \overline{x}) > 0\} \subset \mathbb{P}(\Lambda_{\mathbb{C}})$$

is called the period domain.

Proposition 2.4.1. [17] There exists a natural bijection between D and the set of Hodge structures of K3 type on Λ such that any (2,0)-class σ satisfies:

1. $(\sigma)^2 = 0$

2. $(\sigma.\overline{\sigma}) > 0$

3. $\Lambda^{1,1} \perp \sigma$

Proof. Since the dimension of the (2,0)-part of any Hodge structure of K3 type on Λ is one, σ defines a line in $\Lambda_{\mathbb{C}}$. Thus, if 1. and 2. are satisfied, then the line defines a point in D. Conversely, if $l \in D$, then there exists a Hodge structure with l, and it satisfies 1. and 2. By adding the condition 3. we define the (1,1)-part uniquely. Hence the Hodge structure with above conditions is unique. □

Hence, an element $x \in D$ determines a Hodge decomposition:

$$\Lambda_{\mathbb{C}} = x \oplus (x \oplus \overline{x})^{\perp} \oplus \overline{x},$$

which is the same as the Hodge decomposition of the second cohomology of a K3 surface.

The period domain D associated with an arbitrary Λ of signature (n_+, n_-) has a nice relation with the Grassmanian. First recall that $\Lambda_{\mathbb{R}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is a real vector space with the $\mathbb{R}$-linear extension of $(.)$. Now consider the Grassmannian $Gr(2, \Lambda_{\mathbb{R}})$ of

planes in $\Lambda_{\mathbb{R}}$ which is a real manifold. Define $Gr^{p}(2,\Lambda_{\mathbb{R}}) \subset Gr(2,\Lambda_{\mathbb{R}})$ as the subset of all planes $P \subset \Lambda_{\mathbb{R}}$ for which $(.)|_P$ is positive definite. Also define $Gr^{po}(2,\Lambda_{\mathbb{R}})$ as the manifold of all such positive planes together with the choice of an orientation.

Theorem 2.4.2. *[13] There exist diffeomorphsims*

$$D\tilde{\to}Gr^{po}(2,\Lambda_{\mathbb{R}})\tilde{\to}O(n_+,n_-)/(SO(2)\times O(n_+-2,n_-)),$$

where $O(n_+,n_-)$ is the orthogonal group of $\mathbb{R}^{n_++n_-}$ endowed with the diagonal quadratic form $diag(1,...,1,-1,...,-1)$ of signature (n_+,n_-). Note that $O(n_+,n_-)$ is a Lie group.

Proof. (Sketch) Observe that there is a natural transitive action of $O(n_+,n_-)$ on $Gr^{po}(2,\Lambda_{\mathbb{R}})$ and the stabilizer of the plane spanned by the first two unit vectors (orientation is important here) v_1, v_2 is the subgroup

$$SO(2)\times O(<v_1,v_2>^{\perp}) \simeq SO(2)\times O(n_+-2,n_-).$$

So, this relation gives the second diffeomorphism. Now, consider the map

$$D \longrightarrow Gr^{po}(2,\Lambda_{\mathbb{R}})$$

$$x \longmapsto \mathbb{R}.Re(x)\oplus\mathbb{R}.Im(x)$$

One can easily see that $x \in D$, it satisfies $(x)^2 = 0$, $(x.\bar{x}) > 0$. Thus, $e_1 := Re(x)$, $e_2 := Im(x)$ are orthogonal to each other, and $(e_1)^2 = (e_2)^2 > 0$. Moreover, for any $\lambda \in \mathbb{C}^*$ λx also defines the same oriented plane. Conversely, any oriented positive plane P with a chosen orthonormal basis e_1, e_2 can be mapped to $x = e_1 + ie_2$. □

Remark 2.4.1. [16] [13]

i) When $n_+ > 2$, D is connected,

ii) When $n_+ = 2$, D has two connected components. Let's say $D = D^+ \sqcup D^-$, and the elements can be interchanged by complex conjugation, i.e. D consists of two connected components according to $Im(x)$ being positive or negative.

Now, we will see the relation between the period domain and tube domain. This relation gives a very important result which is essential for Chapter 3.

Definition 15. [13] Let $\Lambda_{\mathbb{R}} = U_{\mathbb{R}} \oplus W$ be an orthogonal decomposition, and e, f be the standard basis for $U_{\mathbb{R}}$ with $(e)^2 = (f)^2 = 0$ and $(e.f) = 1$. Then a point $x \in \mathbb{P}(\Lambda_{\mathbb{C}})$ can be written as $x = \alpha e + \beta f + z \in U_{\mathbb{C}} \oplus W_{\mathbb{C}}$ which can be denoted by $[\alpha : \beta : z]$ in homogeneous coordinates. The associated tube domain is

$$\mathcal{H} := \{z \in W_{\mathbb{C}} | (Im(z))^2 > 0\}.$$

Proposition 2.4.3. [13] Let $n_+ = 2$. Then

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\sim} & D \\ z & \longmapsto & [z : -(z)^2 : \sqrt{2}z] \end{array},$$

is a biholomorphic map.

Proof. First, let us show that $x = (1, -(z)^2, \sqrt{2}z)$ is in D. Recall that $x = \alpha e + \beta f + z$. Then, for all $Z \in \mathcal{H}$

$$(x)^2 = (e - (z)^2 f.e - (z)^2 f) + 2(z)^2 = e - (z)^2 f + 2(z)^2 = 0,$$

$$(x, \bar{x}) = (e - (z)^2 f.e - (\bar{z})^2 f) + 2(z.\bar{z}) = -2Re(z)^2 + 2(z.\bar{z}) = 4(Im(z)^2) > 0.$$

So, the map takes values in D.

Let $x = [\alpha : \beta : \gamma] \in D$. Observe that $\alpha \neq 0$. Then by quadratic equation of D, we have $2\beta + (\gamma)^2 = 0$ and since $(x.\bar{x}) > 0$, $(Im(\gamma)^2 > 0$.

□

To see the importance of the map more clearly, let $n_+ = 2$ and for simplicity $n_- = 1$, i.e. we may assume $W_{\mathbb{C}} = \mathbb{C}$ with the standard quadratic form. Then

$$\mathcal{H} = \mathbb{C} \setminus \{x - axis\} = \mathbb{H} \sqcup (-\mathbb{H}),$$

where $\mathbb{H}$ is the upper-half plane. As we discussed above D has two connected components, $D = D^+ \sqcup D^-$. Then the above biholomorphic map gives a map between the connected components of $\mathcal{H}$ and D, i.e.

$$\begin{array}{ccc} \mathbb{H} & \xrightarrow{\sim} & D^+ \\ z & \longmapsto & [z : -(z)^2 : \sqrt{2}z] \end{array}$$

A basic Complex Analysis fact tells us that $\mathbb{H}$ is biholomorphic to the unit disc $\triangle$ in $\mathbb{C}$. Hence, for $n_+ = 2$ the period domain D^+ is always a bounded symmetric domain of type IV which will be explained again in Chapter 3.

Definition 16. [16]

- Suppose that a topological group G acts on a topological space M continuously, i.e,

$$G \times M \longrightarrow M$$

$$(g, x) \longmapsto g \cdot x,$$

  is continuous.

- Assume both G, M are Hausdorff, and M is locally compact. Then the action is called proper if

$$G \times M \longrightarrow M \times M$$

$$(g, x) \longmapsto (g \cdot x, x),$$

  is proper,i.e. the inverse image of any compact set is compact.

- The action is called properly discontinuous if there is a finite number of $g \in G$ such that $g(K) \cap K \neq \emptyset$ for any compact set $K \subset M$.

Remark 2.4.2. [16] If G acts on M properly discontinuous, then the orbit of G is a discrete set in M and the stabilizer subgroup of G is finite. Conversely, when the action is proper, then Γ is a discrete subgroup of G if and only if Γ acts on M properly discontinuously.

Remark 2.4.3. [13] The period domain D associated with the lattice Λ comes with a natural action of the discrete group $O(\Lambda)$. However, for $n_+ > 2$ we do not expect that $O(\Lambda)$ acts properly discontinuously on D. Thus, the quotient $O(\Lambda) \setminus D$ is not Hausdorff for $n_+ > 0$. For details see [[4], Chapter VIII, §].

For $n_+ = 2$, since $SO(2) \times O(n_+ - 2, n_-) = SO(2) \times O(n_-)$ is compact, by the discussion in the above remark $O(\Lambda)$ acts on D properly discontinuously. Hence, $D \setminus O(\Lambda)$ is Hausdorff.

Let Λ be a non-degenerate lattice with its bilinear form (.) and (n_+, n_-) be its signature. Suppose $n_+ > 2$. Fix $l \in \Lambda$ with $(l)^2 > 2$. Then $l^\perp \subset \Lambda$ induces a linear embedding $\mathbb{P}(l^\perp_{\mathbb{C}}) \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ and

$$D_{l^\perp} = D \cap \mathbb{P}(l^\perp_{\mathbb{C}}),$$

where $D_{l^\perp}$ is the period domain associated with $l^\perp$.

Now, observe that if we take $\Lambda = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$ and $l = e_1 + de_2$, where e_1, e_2 is the standard basis of a copy of U, then

$$\Lambda_d := l^\perp \simeq E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \mathbb{Z}(-2d),$$

and $D_d \subset \mathbb{P}(\Lambda_{d\mathbb{C}}) \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ is the associated period domain. Indeed by [[13], Corollary 14.1.10], any primitive $l \in \Lambda$ with $(l)^2 = 2d$ as above, is of this form. If $O(\Lambda_d)$ is the group of orthogonal transformations of Λ_d which come from orthogonal transformations of Λ fixing l, then by the above remarks Λ_d acts on D_d properly continuously, and thus the quotient $D_d \backslash O(\lambda_d)$ is Hausdorff.

Now, let $f : X \to S$ be a smooth, proper family of K3 surfaces, $X_t := f^{-1}(t)$ where t is an element of a connected manifold S with a distinguished point $0 \in S$. The locally constant system $R_2 f_* \mathbb{Z}$ with the fiber $H^2(X_t, \mathbb{Z})$ at t gives a representation of $\pi_1(S)$ on $H^2(X_0, \mathbb{Z})$, and if S is simply connected, i.e, $\pi_1(S)$ is trivial, the locally constant system is canonically isomorphic to the constant system $\underline{H^2(X_0, \mathbb{Z})}$. Thus, all fibers are canonically isomorphic to Λ. For more details see [13]. Now, fix an isomorphism between K3 lattices $\varphi : H^2(X_0, \mathbb{Z}) \to \Lambda$, which is called a marking of X_0. If we take a simply connected S, this gives a canonical markings for all fibers.

Proposition 2.4.4. [13] The period map defined by

$$\begin{aligned} \mathcal{P} : S &\longrightarrow \mathbb{P}(\Lambda_{\mathbb{C}}) \\ t &\mapsto [\varphi(H^{2,0}(X_t))] \end{aligned}$$

is a holomorphic map that lands in the period domain $D \subset \mathbb{P}(\Lambda_{\mathbb{C}})$ and depends on the distinguished point $0 \in S$ and the marking φ.

Definition 17. [13] Let $S' \to S$ be a holomorphic map that sends $0' \in S'$ to $0 \in S$. Now consider the diagram

$$\begin{array}{ccc} X' := X \times_S S' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

Then the family $X \to S$ is complete if for any other smooth proper family $X' \to S'$ with $X_0 \simeq X'_0$, the latter family can be obtained from the first one as in the above diagram. If the tangent map of $S' \to S$ is unique, then $X \to S$ is called versal. And if the holomorphic map $S' \to S$ is unique then $X \to S$ is called the universal deformation. The universal deformation of X_0 is denoted by

$$X \to Def(X_0).$$

Theorem 2.4.5. *[13] If X_0 is a compact, complex manifold then X_0 has a versal deformation. Moreover, the tangent space of the versal deformation $T_0Def(X_0)$ is isomorphic to $H^1(X_0, T_{X_0})$. In addition, if $H^2(X_0, T_{X_0}) = 0$ then there is a smooth versal deformation. If $H^0(X_0, T_{X_0}) = 0$, then a universal deformation exists.*

So, if we take X_0 as a K3 surface, X_0 admits a smooth universal deformation $X \to Def(X_0)$ with $Def(X_0)$ smooth of dimension 20 since

$$H^0(X_0, T_{X_0}) = H^2(X_0, T_{X_0}) = 0$$

and $h_1(X_0, T_{X_0}) = 20$

Proposition 2.4.6. [13] (Local Torelli Theorem) Let $X \to S := Def(X_0)$ be the universal deformation of a complex K3 surface X_0. Then the period map

$$\mathcal{P} : S \longrightarrow D \subset \mathbb{P}(H^2(X, \mathbb{C}))$$

is a local isomorphism.

Our next goal is to show the case where the period map is surjective. This is one of the most important results about K3 surfaces. First, we need to construct the moduli space of marked K3 surfaces N. Let $N = \{\text{marked K3 surfaces}\}/_{\sim}$, as a set. Our aim is to put a complex structure on it. By the Local Torelli Theorem, the period map $Def(X_0) \to D$ is injective. Since the deformation $X \to Def(X_0)$ is universal,

roughly speaking the complex structures on $Def(X)$ can be glued into a complex structure on N. Similarly, the universal deformations can be glued together into a universal marked family. However, we should be careful here:

Remark 2.4.4. The moduli space N is a 20-dimensional complex manifold. However, it is not Hausdorff as we discussed above.

Using the universal marking of the universal family $X \to N$, we can define a global period map

$$\mathcal{P} : N \to D \subset \mathbb{P}(\Lambda_{\mathbb{C}}).$$

Theorem 2.4.7. *(Surjectivity of the period map) [13] The global period map*

$$\mathcal{P} : N \to D \subset \mathbb{P}(\Lambda_{\mathbb{C}})$$

is surjective and a local isomorphism.

We may want to take our results to the next level. We have two questions that immediately come to mind: 1) Can we build a moduli space of K3 spaces that is Hausdorff? 2) Can we add structures so that period map becomes injective? First, we need some extra definitons.

Definition 18. [17] A polarized K3 surface is a K3 surface with a fixed ample line bundle L. The K3 surface is primitively polarized if L is primitive, i.e, it is not a tensor power of another ample line bundle.

Now we can construct the moduli space N_d of primitively polarized marked K3 surfaces of degree $2d$ similar to the construction above. Let us start with a primitively polarized K3 surface (X, L) such that $(L)^2 = 2d$ and fix an isomorphism $\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$, so that Λ has a Hodge structure. Let points of N_d parametrize (X, L, φ). Then, by Lefschetz Theorem on (1,1)-classes, $Pic(X) \simeq H^2(X, \mathbb{Z}) \cap H^{1,1}(X)$, for a line bundle L we can find a vector l in the space of Hodge classes of Λ, i.e. $\varphi(L) = l$. So, we have the moduli space N_d consisting of triples (X, L, φ) where $\varphi(L) = v$. Moreover, this moduli space N_d is a Hausdorff complex manifold, as we discussed before.[see, [13] Chapter 3.3].

Now, as we did before the vector l in the space of Hodge classes of Λ which is the image of a line bundle L, gives a sublattice $l^{\perp}$ of Λ, denote it by $\Lambda_d := l^{\perp}$. And the

corresponding period domain is $D_d := D \cap \mathbb{P}(\Lambda_d)$. Observe that it is a 19-dimensional complex manifold. So, the new global period map is

$$\mathcal{P}_d : N_d \to D_d.$$

Moreover, let $O(\Lambda)$ be the orthogonal transformations of Λ, and $O(\Lambda_d)$ be the be the orthogonal transformations of Λ fixing the vector l. Then $O(\Lambda_d)$ acts on N_d, and the action $O(\Lambda_d) \times \Lambda_d \longrightarrow \Lambda_d$ defined by

$$g.(X, L, \varphi) = (X, L, g \circ \varphi),$$

for any $g \in O(\Lambda_d)$. The quotient $O(\Lambda_d)/N_d$ parametrizes all primitively polarized K3 surfaces of degree $2d$. Also, the global period map $\mathcal{P}_d$ is a local isomorphism and compatible with the action of $O(\Lambda_d)$.

Theorem 2.4.8. *(Baily-Borel) [see, [3]] If $\Gamma \subset O(\Lambda)$ is torsion free, then $\Gamma \backslash D$ is a smooth quasi-projective variety..*

By the above theorem, $O(\Lambda_d)\backslash D_d$ is a smooth quasi-projective variety. So, we have a new holomorphic map

$$\overline{\mathcal{P}_d} : O(\Lambda_d)\backslash N_d \to O(\Lambda_d)\backslash D_d.$$

Finally, we have the main theorem of this chapter:

Theorem 2.4.9. *(Global Torelli Theorem) [17] The period maps*

$$\mathcal{P}_d : N_d \to D_d,$$

and

$$\overline{\mathcal{P}_d} : O(\Lambda_d)\backslash N_d \to O(\Lambda_d)\backslash D_d,$$

are injective.

Corollary 2.4.10. (Global Torelli Theorem) [17] Let (X_1, L_1) and (X_2, L_2) be two polarized complex K3 surfaces. Then $(X_1, L_1) \simeq (X_2, L_2)$ if and only if there is a Hodge isometry $H^2(X_1, \mathbb{Z}) \simeq H^2(X_2, \mathbb{Z})$ mapping l_1 to l_2, where l_1 and l_2 are cohomology classes of L_1 and L_2, respectively

CHAPTER 3

TORIC VARIETIES

3.1 Lattices

Although we will start by giving the most general definition of a toric variety, we will try to understand how to construct these toric varieties in the following sections.

Definition 19. [8] A toric variety X is an irreducible variety such that

- a torus $T \simeq (\mathbb{C}^*)^n$ is a Zariski open subset of X, and
- the standard action of T on itself extends to an action of T on X.

Although this definition is the most general one, we will also give definitions of affine, projective or abstract toric varieties what follows.

Let us give the first examples that come to mind.

Example 3.1.1. [5] Clearly, $(\mathbb{C}^*)^n$ and $\mathbb{C}^n$ are toric, since they directly contain the torus $T \simeq (\mathbb{C}^*)^n$ and the action naturally extends. Now let us look at projective space $\mathbb{P}^n$. Let $x_0, ..., x_n$ be homogeneous coordinates. Then, by the map $(\mathbb{C}^*)^n \to \mathbb{P}^n$ defined by $(t_1, ..., t_n) \mapsto (1, t_1, ..., t_n)$ we see that $\mathbb{P}^n$ contains the torus

$$T_{\mathbb{P}^n} = \mathbb{P}^n \setminus V(x_0x_1...x_n) = \{(a_0, ..., a_n) \in \mathbb{P}^n | a_0 \dots a_n \neq 0\} =$$

$$\{(1, t_1, ..., t_n) \in \mathbb{P}^n | t_1, ..., t_n \in \mathbb{C}^*\} \simeq (\mathbb{C}^*)^n.$$

And by setting the action

$$(t_1, ..., t_n) \cdot (a_0, a_1, ..., a_n) = (a_0, t_1a_1, ..., t_na_n),$$

we see that $\mathbb{P}^n$ is a toric variety.

Now let us look at how one can construct affine toric varieties. As we explained in the previous chapter, a lattice N is a free Abelian group of finite rank. If we chose a $\mathbb{Z}$-basis for N, then $N \simeq \mathbb{Z}^n$. Moreover, we can define the dual lattice $M = Hom_{\mathbb{Z}}(N, \mathbb{Z})$, and the canonical pairing between N and M is denoted by $< m, u >$ for $m \in M$ and $u \in N$. The basis we choose for N gives an isomorphism $M \simeq \mathbb{Z}^n$ and the canonical pairing becomes dot product.

Let $T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^n$ be the torus of N. Every element $u \in N$ gives a map $\lambda^u : \mathbb{C}^* \to T_N$ defined by $\lambda^u(t) = (t^{a_1}, ..., t^{a_n})$ where $a_i \in \mathbb{Z}$, and this is the 1-parameter subgroup of T_N. Secondly, every element $m \in M$ gives a map

$$\chi^m : T_N \to \mathbb{C}^*,$$

defined by $\chi^m(t_1, ..., t_n) = t_1^{b_1} \dots t_n^{b_n}$, $b_i \in \mathbb{Z}$, which is called a character of T_N. So, M is called a character group. Indeed, since we have $M \simeq \mathbb{Z}^n \simeq N$, characters are Laurent monomials which lie in the ring $\mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$ of Laurent polynomials. That is why Laurent monomials are very important for us. At the end of this chapter, we will construct affine toric varieties by using these Laurent monomials. But first let us take a look at the next example:

Example 3.1.2. [5] Let $V = \mathbf{V}(xy - zw) \subset \mathbb{C}^4$. We can determine the torus by taking the intersection of the variety and the torus of $\mathbb{C}^4$, i.e.

$$V \cap (\mathbb{C}^*)^4 = \{(t_1, t_2, t_3, t_1 t_2 t_3^{-1}) | t_i \in \mathbb{C}^*\}.$$

Using the isomorphism defined as $(t_1, t_2, t_3) \mapsto ((t_1, t_2, t_3, t_1 t_2 t_3^{-1})$, we can see that $V \cap (\mathbb{C}^*)^4 \simeq (\mathbb{C}^*)^3$. As we discussed above, we need to find out which Laurent monomials extend to a function $V \to \mathbb{C}$. If $m = (a, b, c) \in M$, then the functions on V are of type $x^a y^b z^c$. So, actually we are asking that for the values of a, b, c for which this function extends. Suppose $c < 0$, $a + c \geq 0$ and $b + c \geq 0$. Since $xy = zw$ on V, we have $x^a y^b z^c = x^{a+c} y^{b+c} w^{-c}$, which extends to a function $V \to \mathbb{C}$. Now, the above inequalities suggest the definition of cones and fans.

3.2 Cones

Definition 20. [6] A convex polyhedral cone in $\mathbb{R}^n$ is

$$\sigma = Cone(S) = \{\sum_{v \in S} \lambda_v v | \lambda_v \geq 0\} \subset \mathbb{R}^n,$$

where $S \subset \mathbb{R}^n$, and set $Cone(\emptyset) = 0$. We say that σ is generated by S.

Apart from the cone definition, we can also construct a cone using polytopes.

Definition 21. [6] A polytope in $\mathbb{R}^n$ is

$$P = Conv(S) = \{\sum_{v \in S} \lambda_v v | \lambda_v \geq 0, \sum_{v \in S} \lambda_v = 1\} \subset \mathbb{R}^n,$$

where the set $S \in \mathbb{R}^n$ is finite. P is said to be the convex hull of S.

Now, if $P = Conv(S)$ is a polytope in $\mathbb{R}^n$ and $\sigma = \{\lambda \cdot (v, 1) \in \mathbb{R}^{n+1} | v \in P, \lambda \geq 0\}$, then σ is a convex polyhedral cone. Indeed, $\sigma = Cone(s \times 1)$.

We can also define the dimension $\dim(\sigma)$ of a convex polyhedral cone σ. The number $\dim(\sigma)$ is the dimension of the smallest subspace containing σ. Similarly the dimension of a polytope P is the dimension of the smallest affine space containing P. So, one can easily prove that $\dim(\sigma) = \dim P + 1$.

Let σ be a convex polyhedral cone in $\mathbb{R}^n$, and define its dual cone by

$$\sigma^\vee = \{u \in \mathbb{R}^{n*} | < u, v > \geq 0 \text{ for all } v \in \sigma\},$$

where $\mathbb{R}^{n*}$ is the dual space of $\mathbb{R}^n$ and $<, >$ is the natural pairing. Moreover, if $\sigma = Cone(S)$ then

$$\sigma^\vee = \{u \in R^{n*} | < u, v > \geq 0 \text{ for all } v \in \sigma\},$$

since S generates σ.

Definition 22. [6] If $u \neq 0$ in $\mathbb{R}^{n*}$, then the hyperplane defined bu u is

$$H_u = \{v \in \mathbb{R}^n | < u, v >= 0\}$$

and the associated closed half-space is

$$H_u^+ = \{v \in \mathbb{R}^n | < u, v > \geq 0\}.$$

A set $\tau = H_u \cap \sigma$ for some u is called a face of $\sigma \subset H_n^+$, denoted by $\tau \prec \sigma$.
Also, if $\dim(\tau) = \dim(\lambda) - 1$, i.e. codimension 1 face, then τ is called a facet.
An edge of a convex polyhedral cone is a 1-dimensional face.

Lemma 3.2.1. [6] Let σ be a convex polyhedral cone. Then,

- Every face of σ is a convex polyhedral cone.
- An intersection of two faces of σ is again a face of σ.
- A face of a face of σ is again a face of σ.

Definition 23. [6] A convex polyhedral cone $\sigma \subset \mathbb{R}^n$ is strongly convex if $\sigma \cap (-\sigma) = 0$. Equivalently, $\dim(\sigma^\vee) = n$.

The next proposition will be very useful in defining affine toric varieties.

Proposition 3.2.2. [5] Let σ be a strongly convex polyhedral cone. Then σ can be generated by its edges.

So far we dealt with cones in $\mathbb{R}^n$. But we can generalize this to any lattice. In particular, if N is a lattice with dual M, then $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R} \simeq \mathbb{R}^n$ and similarly, $M_\mathbb{R} \simeq \mathbb{R}^{n*}$.

Definition 24. [5] If $\sigma = Cone(S)$ for a finite set $S \subset N$, then $\sigma \in N_\mathbb{R}$ is a rational polyhedral cone.

Now we are ready to define affine toric varieties.

3.3 Affine Toric Varieties

Let $\sigma \subset N_\mathbb{R}$ be a rational polyhedral cone. We can define a semigroup $S_\sigma = \sigma^\vee \cap M$ under addition with identity $0 \in S_\sigma$. Moreover, Gordan's Lemma tells us that S_σ is finitely generated. Now, we can define its semigroup algebra $\mathbb{C}[S_\sigma]$. For every generator $m \in S_\sigma$, we can find a character χ^m, and these characters define a basis for $\mathbb{C}[S_\sigma]$. Thus, $\mathbb{C}[S_\sigma]$ is finitely generated. In general, an element of $\mathbb{C}[S_\sigma]$ is of the

form $\sum_{m\in S_\sigma} a_m\chi^m$, where only finitely many a_m are non-zero. One can check that if we replace χ^m with $t^m \in \mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}]$, then

$$\mathbb{C}[S_\sigma] \subset \mathbb{C}[t_1, t_1^{-1}, ..., t_n, t_n^{-1}].$$

Hence, $\mathbb{C}[S_\sigma]$ is a finitely generated integral domain as a $\mathbb{C}$-algebra.

Definition 25. [6] Let σ be a strongly convex polyhedral cone. The affine toric variety associated to σ is an irreducible affine variety defined by

$$V_\sigma = Spec(\mathbb{C}[S_\sigma]).$$

Example 3.3.1. [5] Let us start with some obvious examples. Let

$$\sigma = Cone(e_1, ..., e_n) \subset \mathbb{R}^n.$$

Observe that $\sigma^\vee \simeq \sigma$, and $S_\sigma = \sigma^\vee \cap M = \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}^n_{\geq 0}$. Hence $\mathbb{C}[S_\sigma] = \mathbb{C}[x_1^1, ..., x_n^1]$. Hence $V_\Sigma = \mathbb{C}^n$.
Similarly, let $\sigma = Cone(e_1, ..., e_d) \subset \mathbb{R}^n$, where $0 < d < n$. Then

$$\sigma^\vee = Cone(e_1', ..., e_d', \pm e_{d+1}', ..., \pm e_n'),$$

and

$$\mathbb{C}[S_\sigma] = \mathbb{C}[x_1, ..., x_d, x_{d+1}^{\pm 1}, ..., x_n^{\pm 1}].$$

Hence, $V_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$.

Theorem 3.3.1. *[6] $\sigma \subset N_\mathbb{R}$ is a strongly convex rational polyhedral cone if and only if V_σ is normal.*

For the proof see [[6], Theorem 1.3.5]. Moreover, the proof shows that the first definition and the combinatorial definiton of toric varieties are the same.

Moreover, an affine semigroup $S \subseteq M$ is called saturated if $\forall k \in \mathbb{N} \setminus 0$ and $m \in M$, $km \in S$ implies $m \in S$. Also, strongly convex polyhedral cone σ gives saturated semigroups S_σ. The converse is also true. Therefore, suppose S generates a cone, then we have

$$\mathbb{C}[S] \text{ is normal} \Leftrightarrow S \text{ is saturated}$$

$$\Leftrightarrow S = \sigma^\vee \cap \text{M for some rational polyhedral cone } \sigma \subset N_\mathbb{R}.$$

We can also define an affine toric variety corresponding to a face τ of a strongly convex rational polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$ in a similar way. Moreover, the relation between these toric varieties can be seen with the help of next proposition:

Proposition 3.3.2. [5] Let $\tau = H_m \cap \sigma$ for some $m \in \sigma^\vee \cap M$. The semigroup albegra $\mathbb{C}[S_\tau] = \mathbb{C}[\tau^\vee \cap M]$ is the localization of $\mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M]$ at $\mathcal{X}^m$, i.e.

$$\mathbb{C}[S_\tau] = \mathbb{C}[\tau^\vee \cap M] = \mathbb{C}[S_\sigma] = \mathbb{C}[\sigma^\vee \cap M]_{\mathcal{X}^m}.$$

The proof depends on the fact $S_\sigma + \mathbb{Z}(-m) = S_\tau$. Fora more general statement see [[6], Proposition 1.3.16].

Next we want to define maps between affine toric varieties that respect torus actions.

Definition 26. [6] Let $V_1 = Spec(\mathbb{C}[S_1])$ and $V_2 = Spec(\mathbb{C}[S_2])$ be affine toric varieties. A morphism $\phi : V_1 \to V_2$ is called toric if the map $\phi^* : \mathbb{C}[S_2] \to \mathbb{C}[S_1]$ is induced by a semigroup homomorphism $\hat{\phi} : S_2 \to S_1$.

The next proposition is very useful for understanding toric morphisims.

Proposition 3.3.3. [6] Let T_1, T_2 be the tori of $V_{\sigma_1}, V_{\sigma_2}$, respectively. Then $\phi : V_{\sigma_1} \to V_{\sigma_2}$ is a toric morphism if and only if $\phi(T_1) \subseteq T_2$ and $\phi|_{T_1} : T_1 \to T_2$ is a group homomorphism.

Proposition 3.3.4. [21] Let σ and τ be two strongly convex polyhedral cones. Then τ is a face of σ if and only if $\tau \subseteq \sigma$ and the induced map of affine toric varieties $\phi : V_\tau \to V_\sigma$ is an open immersion. Moreover, V_τ is a Zariski open subset of V_σ.

Proof. (Sketch)

$$\tau \prec \sigma \Leftrightarrow \tau^\vee \succ \sigma^\vee \Leftrightarrow \mathbb{C}[\tau^\vee \cap M] \supseteq \mathbb{C}[\sigma^\vee \cap M]$$

$$\begin{array}{ccccc} & & V_\tau & \to & V_\sigma \\ \Leftrightarrow & & \cup & & \cup \\ & & T_N & = & T_N \end{array}$$

□

3.4 Fans and Corresponding Toric Varieties

So far, we have affine toric varieties. We will now try to paste them into a more general toric variety. In general, we can create an abstract variety by gluing affine varieties. For example projective varieties can be formed as union of affine pieces. So, before describing abstract toric varieties, let us start with gluing together affine varieties in general. Suppose we have a collection of affine varieties and some maps

$$(\{V_\alpha\}_\alpha, \{V_{\alpha\beta}\}_{\alpha,\beta}, \{g_{\alpha\beta}\}_{\alpha,\beta}),$$

where $\{V_\alpha\}_\alpha$ is a finite collection of affine varieties, $V_{\alpha\beta} \subset V_\alpha$ is Zariski open, and $g_{\alpha\beta} : V_{\alpha\beta} \simeq V_{\beta\alpha}$ are isomorphism satisfying:

- $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ for all α, β,
- $g_{\beta\gamma}|_{V_{\beta\alpha}\cap V_{\beta\gamma}} \circ g_{\alpha\beta}|_{V_{\alpha\beta}\cap V_{\alpha\gamma}} = g_{\alpha\gamma}|_{V_{\alpha\beta}\cap V_{\alpha\gamma}}$ for every α, β, γ.

Then we define the abstract variety

$$X = \bigcup_\alpha V_\alpha / \sim,$$

and for $a \in V_\alpha$ and $b \in V_\beta$, $a \sim b$ if $a \in V_{\alpha\beta}$ and $g_{\alpha\beta}(a) = b$.

Definition 27. [6] A fan Σ is a finite collection of cones in $N_{\mathbb{R}}$ such that:

- Every $\sigma \in \Sigma$ is a strongly convex polyhedral cone.
- If $\sigma \in \Sigma$ and $\tau \prec \sigma$, then $\tau \in \Sigma$.
- If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of each cone and hence is in Σ.

The support of Σ is $|\Sigma| = \bigcup_{\alpha\in\Sigma} \sigma$.

Every cone $\sigma \in \Sigma$ gives an affine toric variety V_σ. Let $\sigma, \sigma' \in \Sigma$ be cones and $\sigma \cap \sigma'$ be their common face. As Proposition 3.3.4, we have open immersions

$$V_{\sigma\cap\sigma'} \to V_\sigma,$$

$$V_{\sigma\cap\sigma'} \to V_{\sigma'},$$

and denote the images of these maps by $V_{\sigma\sigma'}$ and $V_{\sigma'\sigma}$, respectively. Then we get

$$g_{\sigma\sigma'} : V_{\sigma\sigma'} \simeq V_{\sigma'\sigma}.$$

So, we construct the glueing data as above. Hence, we defined the X_Σ associated to the fan Σ.

Theorem 3.4.1. *[5] X_Σ is a normal separated toric variety.*

Let us give some important examples.

Example 3.4.1. Let Σ be a fan in $N_\mathbb{R} = \mathbb{R}^2$ in the figure below, where $N = \mathbb{Z}^2$ and e_1, e_2 be a standard basis for N. Σ has cones

$$\sigma_0 = Cone(e_1, e_2), \sigma_1 = Cone(-e_1 - e_2, e_2), \sigma_2 = Cone(e_1, -e_1 - e_2),$$

together with their rays.

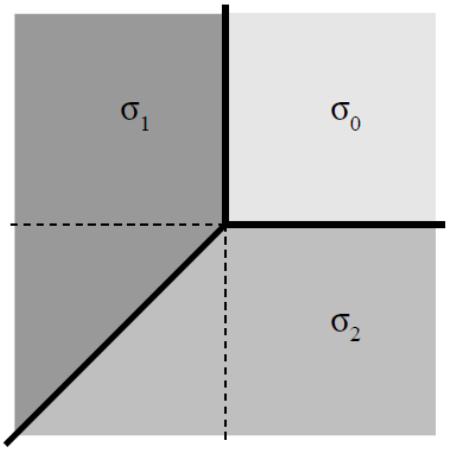

Figure 3.1: A fan Σ for $\mathbb{P}^2$, [6]

Then, the toric variety X_Σ is covered by the affine open sets

$$U_{\sigma_0} = Spec(\mathbb{C}[S_{\sigma_0}]) \simeq Spec(\mathbb{C}[x, y]),$$

$$U_{\sigma_1} = Spec(\mathbb{C}[S_{\sigma_1}]) \simeq Spec(\mathbb{C}[x^{-1}, x^{-1}y]),$$

$$U_{\sigma_2} = Spec(\mathbb{C}[S_{\sigma_2}]) \simeq Spec(\mathbb{C}[xy^{-1}, y^{-1}]).$$

And the gluing data on the coordinate rings

$$g_{10}^* : \mathbb{C}[x, y]_x \simeq \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}},$$

$$g_{20}^* : \mathbb{C}[x,y]_y \simeq \mathbb{C}[xy^{-1}, y^{-1}]_{y^{-1}},$$

$$g_{21}^* : \mathbb{C}[x^{-1}, x^{-1}y]_{x^{-1}y} \simeq \mathbb{C}[x, xy^{-1}]_{xy^{-1}},$$

Now, observe that if $(x_0 : x_1 : x_2)$ is the homogeneous coordinates on $\mathbb{P}^2$, then the maps $x \mapsto \frac{x_2}{x_0}$ and $y \mapsto \frac{x_2}{x_0}$ identify the open cover $U_i \subseteq \mathbb{P}^2$ with $U_{\sigma_i} \subseteq X_\Sigma$. Hence, $X_\Sigma = \mathbb{P}^2$.

Example 3.4.2. Let r be a non-negative integer, and Σ_r be a fan in $N_\mathbb{R} = \mathbb{R}^2$ consisting of the fan in the figure

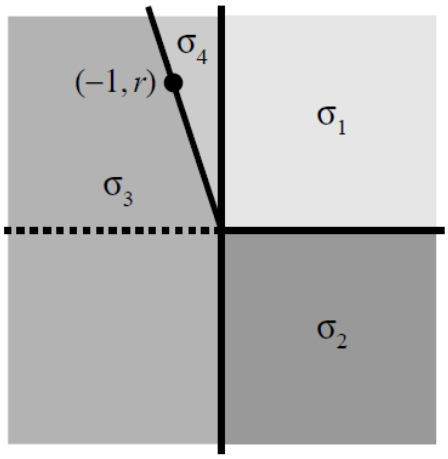

Figure 3.2: A fan Σ_r with $X_{\Sigma_r} \simeq \mathcal{H}_r$, [6]

Then, the toric variety X_{Σ_r} is covered by

$$U_{\sigma_1} = Spec(\mathbb{C}[x,y]) \simeq \mathbb{C}^2,$$

$$U_{\sigma_2} = Spec(\mathbb{C}[x,y^{-1}]) \simeq \mathbb{C}^2,$$

$$U_{\sigma_3} = Spec(\mathbb{C}[x^{-1}, x^{-r}y^{-1}]) \simeq \mathbb{C}^2,$$

$$U_{\sigma_4} = Spec(\mathbb{C}[x^{-1}, x^r y]) \simeq \mathbb{C}^2,$$

and glue them as we did in the previous example. The final toric variety $X_{\Sigma_r} \simeq \mathcal{H}_r$ is called the r^{th} Hirzebruch surface.

3.5 The Orbit-Cone Correspondence

By definition, there is an action of the torus T_N on the toric variety X_Σ. In this section, we want to figure out the orbits of this action. Let us try to look at the orbits in the example $\mathbb{P}^2 = X_\Sigma$. Consider the map $\lambda^u : \mathbb{C}^* \to \mathbb{P}^2$ where $u \in N$. Since $N \simeq \mathbb{Z}^2$, put $u = (a, b) \in \mathbb{Z}^2$. Then, $\lambda^u(t) = (1, t^a, t^b)$ and we have,

$$\lim_{t\to 0}(1, t^a, t^b) = \begin{cases} (1,0,0) & \text{if } a, b > 0 \\ (1,0,1) & \text{if } a > 0, b = 0 \\ (1,1,0) & \text{if } a = 0, b > 0 \\ (1,1,1) & \text{if } a = b = 0 \\ (0,0,1) & \text{if } a > b, b < 0 \\ (0,1,0) & \text{if } a < 0, a < b \\ (0,1,1) & \text{if } a < 0, a = b. \end{cases}$$

The conditions in the limit gives 7 disjoint regions in $\mathbb{R}^2$ which are exactly T_N-orbits in $\mathbb{P}^2$:

- The open sets $a, b > 0$, $a < 0, a < b$, $a > b, b < 0$ give the orbits

  $$O_1 = \{(x_0, x_1, x_2) | x_1 = x_2 = 0, x_0 \neq 0\},$$

  $$O_2 = \{(x_0, x_1, x_2) | x_0 = x_2 = 0, x_1 \neq 0\},$$

  $$O_3 = \{(x_0, x_1, x_2) | x_0 = x_1 = 0, x_2 \neq 0\},$$

  respectively

- The open rays $a > 0, b = 0$, $a = 0, b > 0$, $a < 0, a = b$ give the orbits

  $$O_4 = \{(x_0, x_1, x_2) | x_0 = 0, x_1, x_2 \neq 0\}$$

  $$O_5 = \{(x_0, x_1, x_2) | x_1 = 0, x_0, x_2 \neq 0\}$$

  $$O_6 = \{(x_0, x_1, x_2) | x_2 = 0, x_0, x_1 \neq 0\},$$

  respectively

- The point $a = b = 0$ gives the orbit

$$O_7 = \{(x_0, x_1, x_2) | x_i \neq 0, \forall i\}$$

So, for each orbit we have a unique limit point, which is called the distinguished point γ_σ. Hence, we have a bijective correspondence between cones in the fan and the T_N-orbits in $\mathbb{P}^2$.

The situation we see in this example can be generalized.

Theorem 3.5.1. *(Orbit-Cone Correspondance) [6] Let X_Σ be the toric variety of the fan $\Sigma \subset N_\mathbb{R}$.*

- *There is a bijective correspondance*

$$\{\sigma \in \Sigma\} \qquad \Leftrightarrow \qquad T_N - orbits$$

$$\sigma \qquad \Leftrightarrow \qquad O(\sigma) \simeq Hom_\mathbb{Z}(\sigma^\perp \cap M, \mathbb{C}^*)$$

- *If $dim N_\mathbb{R} = n$, then $dim(O(\sigma)) = n - \dim \sigma$.*

- *The affine open subset V_α is the is disjoint union of orbits, i.e.*

$$V_\sigma = \bigcup_{\tau \prec \sigma} O(\tau).$$

- *$\tau \prec \sigma$ if and only if $O(\sigma) \subseteq \overline{O(\tau)}$, and*

$$\overline{O(\tau)} = \bigcup_{\tau \prec \sigma} O(\sigma),$$

where $\overline{O(\tau)}$ is the closure in both the classical and Zariski topologies.

In the section 3.3, we defined toric morphisms for affine toric varieties. Now, let us define the abstract variety version of it.

Definition 28. [6] Let Σ_1, Σ_2 be fans in $\subset (N_1)_\mathbb{R}, (N_2)_\mathbb{R}$, respectively, and $X_{\Sigma_1}, X_{\Sigma_2}$ be the corresponding normal toric varieties. Then a morphism $\phi : X_{\Sigma_1} \to X_{\Sigma_2}$ is toric if $\phi(T_{N_1}) \subseteq T_{N_2}$ where $T_{N_1} \subseteq X_{\Sigma_1}, T_{N_2} \subseteq X_{\Sigma_2}$ and $\phi|_{T_{N_1}}$ is a group homomorphism.

One of the most important morphisms in Algebraic Geometry is the blow-up map. Now, we will look at its toric version. However, first we need some definitions.

Definition 29. [6] Let Σ be a fan in $N_{\mathbb{R}}$. Then we say a fan Σ' refines Σ if every cone of Σ' is contained in Σ and $|\Sigma'| = |\Sigma|$. One can observe that actually every cone in Σ is the union of cones of Σ'.

Definition 30. [6] Let Σ be a fan in $N_{\mathbb{R}}$, and $\sigma(e_1, ..., e_n) \in \Sigma$ be a smooth cone, so that $e_1, ..., e_n$ is a basis for N. Put $e_0 = e_1 + ... + e_n$. Let $\Sigma'(\sigma)$ be the set of all cones generated by the subsets of $e_0, ..., e_n$ not containing $e_1, ..., e_n$. Then the star subdivision of Σ along σ is

$$\Sigma^*(\sigma) = (\Sigma \setminus \sigma) \cup \Sigma'(\sigma),$$

which is a fan in $N_{\mathbb{R}}$.

Let us give an easy example.

Example 3.5.1. Let $\sigma = Cone(e_1, e_2, e_3)$ and $e_0 = e_1 + e_2 + e_3$. Then $\Sigma^*(\sigma)$ consists of the cones

$$\sigma_1 = Cone(e_0, e_1, e_2), \sigma_2 = Cone(e_0, e_1, e_2), \sigma_3 = Cone(e_0, e_2, e_3),$$

together with their faces.

Proposition 3.5.2. [6] $\Sigma^*(\sigma)$ is a refinement of Σ, and the induced toric morphisim

$$\phi : X_{\Sigma^*(\sigma)} \to X_{\Sigma}$$

makes $X_{\Sigma^*(\sigma)}$ the blowup of X_{Σ} at the distinguished point γ_σ corresponding to the cone σ.
The blowup X_{Σ} at γ_σ is denoted by $Bl_{\gamma_\sigma}(X_{\Sigma})$.

3.6 Weil And Cartier Divisors on Toric Varieties

Recall that toric varieties contain an algebraic torus as an open dense subset. Therefore, if X_{Σ_1} and X_{Σ_2} are two smooth, n-dimensional toric varieties, then they'll just look different on the boundary. Therefore, understanding the structure of divisors on toric varieties is essential in the study toric varieties.

Let us start with some general facts from Algebraic Geometry.

Remark 3.6.1. [26] [6]

Let X be an irreducible variety and $D_1, ..., D_r$ be irreducible subvarieties of codimension 1. Then $D = a_1 D_1 + .. + a_r D_r$ is called a Weil divisor on X, $a_1, ..., a_r \in \mathbb{Z}$

- If $a_i \geq 0$, then D is called effective.
- $D = 1.D_i$ is called a prime divisor.
- $Div(X)$ is the free abelian group of Weil Divisors on X, generated by prime divisors.

Let X be a normal variety and $f \in \mathbb{C}(X)^*$

- The divisor of $f \in \mathbb{C}(X)^*$ is the finite sum over prime divisors $D \subseteq X$,

  $$div(f) = \sum_D \nu_D(f) D,$$

  where ν_D is the valuation, and a divisor of the form $div(f)$ is called principal divisors.
- $Div_0(X) = \{div(f) | 0 \neq f \in \mathbb{C}(X)\}$ is the set of all principal divisors, which is a subgroup of $Div(X)$.
- $Cl(X) = Div(x)/Div_0(X)$ is called the class group of X.
- D_1 and D_2 is said to be linearly equivalent, $D_1 \sim D_2$, if they represent the same class in $Cl(X)$, i.e. $D_1 - D_2 = div(f)$ for some $f \in \mathbb{C}(X)$.
- A Weil divisor D on X is Cartier if it is locally principal, i.e. $D|_{U_i} = div(f_i)|_{U_i}$ for some $i \in I$ where $\{U_i\}_{i \in I}$ is an open cover for X.
- Cartier divisors form an Abelian group $CDiv(X)$, and

  $$Pic(X) = CDiv(X)/Div_0(X)$$

  is called the Picard Group of X

Now we will start with defining the class group of a normal toric variety X_Σ of a fan Σ in $N_\mathbb{R}$ with $dim N_\mathbb{R} = n$ using torus invariant prime divisors and characters.
By Orbit-Cone correspondence, the rays, which are the 1-dimensional cones, corresponds to the codimension 1 orbit closures in X_Σ. Let $\Sigma(1)$ be the set of all rays of Σ and let D_ρ be the irreducible torus-invariant divisor corresponding to $\rho \in \Sigma_1$. Therefore, we can describe the torus T_N of X_Σ with D_ρ:

$$T_N = X_\Sigma \setminus \bigcup_\rho D_\rho.$$

Also note that, the ray $\rho \in \Sigma(1)$ has a minimal generator $u_\rho \in \rho \cap N$.

Proposition 3.6.1. [6] If the ray $\rho \in \Sigma(1)$ has a minimal generator u_ρ and $\mathcal{X}^m$ is a character corresponding to $m \in M$, then

$$\nu_\rho(\mathcal{X}^m) = \langle m, u_\rho \rangle.$$

Proof. Extend u_ρ to a basis $e_1 = u_\rho, e_2, ..., e_n$ of N. Then $\rho = Cone(e_1) \subseteq \mathbb{R}^n$ and the corresponding affine variety is $V_\rho = Spec(\mathbb{C}[x_1, x_2^{\pm 1}, ..., x_n^{\pm 1}])$. So, $D_\rho \cap U_\rho$ is defined by $x_1 = 0$. Assume $f \in \mathbb{C}(x_1, ..., x_n)$ has valuation $\nu_\rho(f) = s$. then $f = x_1^s \frac{g}{h}$, where $g, h \in \mathbb{C}[x_1, ..., x_n] \setminus \langle x_1 \rangle$. Note that

$$\mathcal{X}^m = x_1^{<m,u_\rho>} \dots x_n^{<m,e_n>}.$$

Hence, $\nu_\rho(\mathcal{X}^m) = \langle m, u_\rho \rangle$. □

Proposition 3.6.2. [6]

$$div(\mathcal{X}^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho.$$

Now we can define the group of T_N-invariant Weil Divisors on X_Σ,

$$Div_{T_n}(X_\Sigma) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho \subseteq Div(X_\Sigma).$$

Theorem 3.6.3. *[6] We have a short exact sequence*

$$0 \longrightarrow M \longrightarrow Div_{T_N}(X_\Sigma) \longrightarrow Cl(X_\Sigma) \longrightarrow 0$$

$$m \longmapsto div(\mathcal{X}^m)$$

$$D_\rho \longmapsto \textit{its divisor class}$$

if and only if the set of minimal generators spans $N_{\mathbb{R}}$.
Actually the sequence is still exact except for the first map, even if the minimal generators do not span.

Example 3.6.1. Let $\mathscr{H}_r$ be the r^{th} Hirzebruch surface. Then, we have the ray generators $u_1 = -e_1 + re_2, u_2 = e_2, u_3 = e_3, u_4 = -e_2$. The class group is generated by the classes of 4 T_N-invariant divisors D_1, D_2, D_3, D_4 and

$$0 \sim div(\mathcal{X}^{e_1}) =< e_1, u_1 > D_1 + < e_1, u_2 > D_2 + < e_1, u_3 > D_3 + < e_1, u_4 > D_4$$

$$= -D_1 + D_3,$$

$$0 \sim div(\mathcal{X}^{e_2}) =< e_2, u_1 > D_1 + < e_2, u_2 > D_2 + < e_2, u_4 > D_4 = rD_1 + D_2 - D_4.$$

Hence, actually $Cl(\mathcal{H}_r)$ is a free abelian group generated by $[D_1]$ and $[D_2]$, since D_3 depends on D_1 and D_4 depends on D_1 and D_2.

Let $CDiv_{T_N}$ be the set of T_N-invariant Cartier divisors. For the Cartier divisors, we have a similar description, since $div(\mathcal{X}^m) \in CDiv_{T_N}$ for all $m \in M$.

Theorem 3.6.4. *[6] The sequence*

$$M \longrightarrow CDiv_{T_N}(X_\Sigma) \longrightarrow Pic(X_\Sigma) \longrightarrow 0 \ ,$$

is exact as above. Furthermore, if the set of minimal generator spans $N_{\mathbb{R}}$, then

$$0 \longrightarrow M \longrightarrow CDiv_{T_N}(X_\Sigma) \longrightarrow Pic(X_\Sigma) \longrightarrow 0 \ ,$$

is a short exact sequence.

CHAPTER 4

TOROIDAL COMPACTIFICATION

4.1 Toroidal Embeddings

Definition 31. (Toroidal Embedding) [14] Let $U \subset X$, where U is a Zariski open subset and X is an n-dimensional normal variety. Then, (X, U) is a toroidal embedding if for every closed $x \in X$, there is an affine toric varietry X_σ with n-dimensional torus $T \subset X_\sigma$, a point $t \in X_\sigma$ and an isomorphism of completed local rings

$$\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \hat{\mathcal{O}}_{X_\sigma,t} \ ,$$

such that the ideal in $\hat{\mathcal{O}}_{X,x}$ generated by the ideal X_U corresponding under this isomorphisim to the ideal in $\hat{\mathcal{O}}_{X_\sigma,t}$ generated by the ideal of $X_\sigma - T$. X_σ together with the above isomorphism is called a "local model" at x.

Here, the last condition is very important for us because in this way we will see that the stratification of the boundary divisors behaves like the stratification into orbits of a toric variety.

We can generalize this definition:

Definition 32. [2] Over $\mathbb{C}$ a pair $U \subset X$, where X is an analytic space and U is open in the complex topology, is called a toroidal embedding if for every $x \in X$, there is a small neighborhood $W_x \subset X$ of x such that $(W_x, W_x \cap U)$ is isomorphic to $(V_t, V_t \cap T)$ for some neighborhood $V_t \subset X_\sigma$ of some $t \in X_\sigma$, where X_σ is an affine toric variety with n-dimensional torus T.

When X, U are varieties as above, then these two definitions are the same. For now, let us consider only the first definition and try to understand it better.

Observe that $X_\sigma - T$ contains just boundary divisors, so $X_\sigma - T$ is purely one codimensional. Thus, $X - U$ is also purely one codimensional. So, we can write $X - U = \bigcup_{i \in I} E_i$ where E_i are boundary divisors.

Definition 33. [14] A "stratification" of a variety X is a finite collection of locally closed subsets $S_1, ..., S_n$, called "strata", such that every point of X is in exactly one stratum and the closure of a stratum is a union of strata. If Y is a stratum, define

$$StarY = \bigcup_{\substack{\text{strata } Z \text{ such that} \\ Y \subset \overline{Z}}} Z = X - \bigcup_{\substack{\text{strata } Z \text{ such that} \\ \overline{Z} \cap Y = \emptyset}} Z.$$

Note that $StarY$ is an open set.

We can also generalize this definition for analytic spaces as follows, i.e.

Definition 34. [2] W_x, which was the neighborhood of x in the general definition of toroidal embedding, has a canonical stratification $Y_{\alpha,x}$ into non-singular locally closed analytic strata with $\bar{Y}_{\alpha,x}$ normal and connected (second condition can be always satisfied if we shrink W_x): let E_i be the irreducible components of $W_x \setminus W_x \cap U$, and let for any subset J of I,

$$Y_{\alpha,x} = \bigcap_{i \in J} E_i \setminus \bigcup_{i \notin J} E_i.$$

Observe that these strata patch up on overlaps as x varies and thus we can stratify all of X into a connected, locally closed, smooth analytic strata Y_α. Moreover, $Y_\alpha \cap W_x$ is a union of the $Y_{\beta,x}$. If $Y_\alpha \cap W_x$ is always one stratum then we say (X, U) has no self-intersection, otherwise it has self-intersection. Studying (X, U)'s without self-intersection is much easier, but there is also a nice class of toroidal embeddings with self-intersection.

Suppose $Y_{\beta_1,x}$, $Y_{\beta_2,x}$ are parts of Y_α, and let us say $x = Y_{\beta_3,x}$ such that

$$\overline{Y}_{\beta_3,x} = \overline{Y}_{\beta_1,x} \cap \bar{Y}_{\beta_2,x}.$$

Let $\overline{Y}_{\beta_3,x}$ define a global stratum Y_γ. Then (X, U) is called "without monodromy" if Y_γ has a neighborhood W such that $Y_{\beta_1,x}$ and $\overline{Y}_{\beta_2,x}$ lie in different components of $Y_\alpha \cap W$.

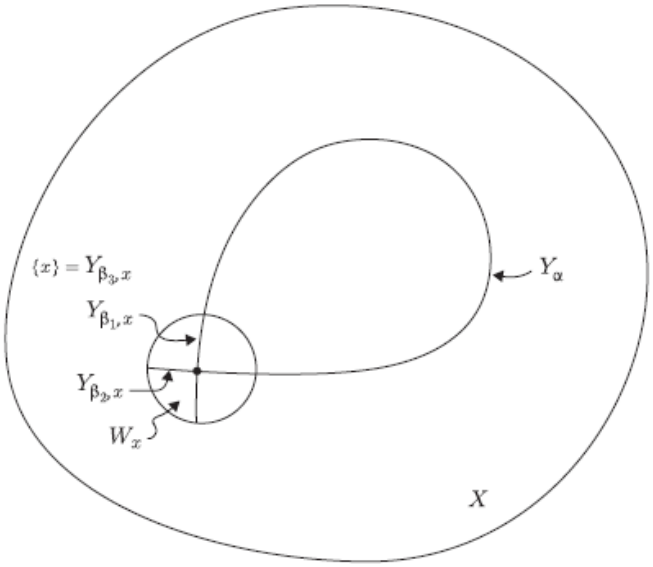

Figure 4.1: (X, U) has self-intersection [2]

If (X, U) is without monodromy, again we can define $Star_0(Y_\alpha)$ as the small complex neighborhood of Y_α, and all the local strara $Y_{\beta,x}$ remain distinct. Moreover, we have $Star_0(Y_\alpha) \simeq X_\sigma \times Y_\alpha$.

Now for simplicity, let us go back to the case where X is a variety, and let us consider the case of (X, U) without self-intersection. We assume that the E_i are normal.

Proposition 4.1.1. [14] Let (X, U) be without self-intersection. Then for any $J \subset I$,

i) $\bigcap_{i\in J} E_i$ is normal ,

ii) $\bigcap_{i\in J} E_i \setminus \bigcup_{i\notin J} E_i$ is non-singular.

Moreover, the components of the sets $\bigcap_{i\in J} E_i - \bigcup_{i\notin J} E_i$ define a stratification of X, as for the analytic case. If $x \in X$ and (X_σ, t) is a local model at x, then the closures $\bar{Z}$ of the strata Z, with $x \in \bar{Z}$, correspond formally to the closures of the orbits in X_σ. In particular, if x is a stratum Y, then Y corresponds formally to $\mathcal{O}(t)$ itself, where $\mathcal{O}(t)$ is the orbit of t.

Definition 35. [14] Let Y be a stratum of (X, U), then define

$$M^Y = \text{Cartier divisors on } Star_0(Y), \text{ supported on } Star_0(Y) \setminus U \cap Star_0(Y),$$

$$M^Y_{\mathbb{R}} = M^Y \otimes \mathbb{R},$$

$$M^Y_+ = \text{sub-semigroup of effective divisors},$$

$$N^Y = Hom(M^Y, \mathbb{Z}),$$

$$N^Y_{\mathbb{R}} = N^Y \otimes \mathbb{R} = Hom(M^Y_{\mathbb{R}}, \mathbb{R}),$$

$$\sigma^Y = \{x \in N^Y_{\mathbb{R}} | < D, x > \geq 0, \text{for all } D \in M^Y_+\} \subset N^Y_{\mathbb{R}}.$$

Corollary 4.1.2. There are canonical isomorphism

i) $M^Y \simeq M/\{r|\ r \equiv 0 \text{ on } \sigma\}$, where M is the character group of the affine toric variety,

ii) $N^Y_{\mathbb{R}} \simeq \{$subgroup spanned by σ, where N is the one parameter subgroup,$\}$

iii) $\sigma^Y \simeq \sigma$

Now, we have [[14],p.63]

$$\begin{aligned} \{\text{strata in Star}Y\} \quad &\cong \quad \{\text{orbits in} - X_\sigma\} \\ &\cong \quad \{\text{faces of} - \sigma\} \\ &\cong \quad \{\text{faces of} - \sigma^Y\} \end{aligned}$$

Definition 36. [14]

$$R.S.^U(X) = \{\text{set of morphisms } \lambda : Spec(k[[t]]) \to X \text{ such that } \lambda(y) \in U\ \}.$$

R.S. is short for "Riemann surface" as used by Zariski [[14],p. 64.] Now, we have a pairing defined by

$$<,>: R.S^U(StarY) \times M^Y \to \mathbb{Z},$$

defined by $<\ \lambda, D\ >= ord_0(\lambda^{-1}D)$, where ord_0 is the order of vanishing at the closed point 0 of a divisor, i.e. the multiplicity of a divisor at 0.
This pairing dualizes to a map

$$ord : R.S.^U(StarY) \to \sigma^Y \cap N^Y.$$

In fact, $\sigma^Y \cap N^Y = Im(ord)$ but it requires a proof.

We will end this section by gluing together all the polyhedral cones σ^Y into a big "conical polyhedral complex". But what is the relation between σ^Y and σ^Z when Z is a stratum in $StarY$?

Lemma 4.1.3. [2] For all strata Z_0 in $Star(Y)$, let Z be the stratum of X containing Z_0, then construct a map

$$\alpha_{Z_0} : M^Y \to M^Z,$$

by restricting a divisor on $Star(Y)$ to be the component of $Star(Y) \cap Star(Z)$ containing Z_0, and then extending it to $Star(Z)$. Next, construct the dual map of α_{Z_0}:

$$\beta_{Z_0} : N^Z_{\mathbb{R}} \to N^Y_{\mathbb{R}},$$

and its restriction is an isomorphism

$$\beta_{Z_0} : \sigma^Z \hat{\to} \sigma^Y,$$

For the proof see [[14], p. 68.].

Definition 37. [14] A conical (compact) polyhedral complex $\triangle$ is a topological space $\mid \triangle \mid$ with a finite number of closed subsets $\sigma_\alpha \subset \mid \triangle \mid$, called its cells, and a finite dimensional real vector space V_α of real valued continuous functions on σ_α such that:

i) if $dim V_\alpha = n_\alpha$, we get a homeomorphism

$$\phi_\alpha : \sigma_\alpha \tilde{\to} \sigma'_\alpha \subset \mathbb{R}^{n\alpha},$$

where σ'_α is a conical convex polyhedra in $\mathbb{R}^{n\alpha}$, not contained in a hyperplane,

ii) $\phi^{-1}(\text{faces of } \sigma'_\alpha) = (\text{faces of } \lambda_\alpha)$,

iii) $\mid \triangle \mid = \{\text{disjoint union of } Int(\sigma_\alpha)\}$,

iv) if $\sigma_\beta \succ \sigma_\alpha$, then $res_{\sigma_\beta} V_\alpha = V_\beta$.

Definition 38. [14] An integral structure on a conical polyhedral complex is a set of finitely generated abelian groups $L_\alpha \subset V_\alpha$ such that

i) $L_\alpha \otimes \mathbb{R} V_\alpha$

ii) if σ_β is a face of σ_α, then $res_{\sigma_\beta} L_\alpha = L_\beta$.

Finally, we have a conical polyhedral complex with integral structure whose cells are in one-to-one correspondence with the strata of X. To show this define

$$| \triangle | = \bigcup_Y \sigma^Y \backslash (\text{equivalence relation generated by the maps } \beta_{Z_0}) \cong \bigsqcup_Y Int\sigma^Y,$$

4.2 Bounded Symmetric Domains

Definition 39. [2] Let D be a connected Riemannian manifold. If for every point $x \in D$ there is an involutive automorphism s_x which has x as an isolated fixed point, then D is called a Riemannian symmetric space.
In addition, let D be a complex Hermitian manifold. If for every point $x \in D$ there is an involutive automorphism s_x which has x as an isolated fixed point, then D is called a Hermitian symmetric space. Note that s_x is also holomorphic.

Proposition 4.2.1. [11] [27] [2] If D is a hermitian symmetric space, then the Riemannian manifold D decomposes as

$$D = D_0 \times D_1 \times \ldots \times D_n,$$

where

- D_0 is the quotient of a complex vector space by a discrete group of translations, and it is called Euclidean type
- $D_i, i \neq 0$ is an irreducible and non-Euclidean hermitian symmetric space. If D_i, for some $i \neq 0$ are compact, then they are called compact type and are rational projective varieties. If D_i, for some $i \neq 0$ are non-compact, then they are of non-compact type, and is a bounded domain in $\mathbb{C}^n$.

Definition 40.

- If D does not contain a D_0 in the decomposition as defined above, D is called non-Euclidean.
- Let D be simply connected. If all of the D_i in the decomposition are of non-compact type, then D is called a symmetric domain.
- Let D be simply connected. If all of the D_i in the decomposition are of compact type, then D is called compact type.

Now, our discussion in Chapter 2 has become more meaningful with the above definition. As we mentioned in Chapter 2, for $n_+ = 2$, the connected components of the period domain D are bounded symmetric domain of type IV. The geometric interpretation of being type IV is Lie-sphere defined by $|z|^2 < \frac{1}{2}(1 + (z \cdot z)^2) < 1$ for any point z. One can see that the points of D^+ (or D^-) satisfy this condition, see [[16], Chapter 5].

Let Λ be the K3 lattice. Hence, if we fix once and for all a primitive vector $l \in \Lambda$ with $(l)^2 = 2d$, we have a new lattice $l^\perp = \Lambda_d =< -2d > \oplus U^{\oplus 2} \oplus E_8^{\oplus 2} \subset \Lambda$, and a new period domain $D_d = \{z \in \mathbb{P}(\Lambda_d \otimes \mathbb{C}) : (z)^2 = 0, (z, \bar{z}) > 0\}$ which has two connected components, say $D_d = D_d^+ \sqcup D_d^-$ where $D_d^+ \cong O(2, 19)/SO(2) \times O(19)$ is a bounded symmetric domain of type IV.

D is a non-Euclidean hermitian symmetric space and G is the identity component of its Lie group of automorphisms. The identity component means G is the connected component of its Lie group of automorphisms that contains the identity element of the group. Then, G acts transitively on D, and choosing a basepoint $o \in D$, $D = G/K$, where $K \subset G$ is a compact subgroup. If D is a hermitian symmetric domain, then G is semi-simple and adjoint, and K is its maximal compact subgroup. The above decomposition of D is called the Cartan decomposition with these assumptions.

4.3 Boundary Components

Toroidal compactification can generally be built for locally symmetric varieties. However, in order to write the theory in all its details, many different fields of mathematics are needed. In this part, we will try to elucidate many definitions and theorems through examples. So, let us start our discussion with an important example:

Example 4.3.1. [20] The Siegel upper half space of degree g is the set of $g \times g$ matrices defined over $\mathbb{C}$ with positive-definite imaginary part, i.e.

$$\mathscr{G}_g = \{\tau \in M_g(\mathbb{C}) : \tau^t = \tau, Im(\tau) > 0\}.$$

Definition 41. [21] The subgroup of $GL(2g,\mathbb{R})$ defined by the matrices of the form

$$G = \left\{ M \in M(2g,\mathbb{R}) : M^t \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} M = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} \right\}$$

$$= \left\{ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : A^tC = C^tA, B^tD = D^tB, A^tD - C^tB = I_g \right\}$$

$$= \left\{ M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : M^{-1} = \begin{bmatrix} D^t & -B^t \\ -C^t & A^t \end{bmatrix} \right\},$$

is called the real symplectic group of degree g and denoted by $Sp(2g,\mathbb{R})$.

Proposition 4.3.1. [20] $G \in Sp(2g,\mathbb{R})$ (after this point fix $G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$) acts on $\mathscr{G}_g$ biholomorphically a

$$Sp(2g,\mathbb{R}) \times \mathscr{G}_g \longrightarrow \mathscr{G}_g$$

$$(M,\tau) \longmapsto M \cdot \tau = (A\tau + B)(C\tau + D)^{-1}$$

.

Moreover, the action is transitive.

Remark 4.3.1. [20] All biholomorphic automorphisms of $\mathscr{G}_g$ can be expressed as above, i.e.

$$Aut(\mathscr{G}_g) = Sp(2g,\mathbb{R})/\pm 1,$$

where ± 1 is the center of $Sp(2g,\mathbb{R})$ and the quotient is a simple group.

Remark 4.3.2. [20]

$$Iso(iI_g) := \left\{ M \in Sp(g,\mathbb{R}) : M \cdot (iI_g) = iI_g \right\}$$

$$= \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in Sp(2g,\mathbb{R}) \tilde{\to} U(g) = \{A + iB\} \right\},$$

where $U(g)$ is the unitary group. This implies that $\mathscr{G}_g \cong Sp(2g,\mathbb{R})/U(g)$is a real analytic manifold. Realize that $Sp(2g,\mathbb{R})$ is a Lie group and $U(g)$ is a maximal compact subgroup unique up to conjugation.

Remark 4.3.3. [20] By the Cayley Transformation,

$$\begin{array}{lccc} c: & \mathscr{G}_g & \longrightarrow & D_g = \{Z \in M(g,\mathbb{C}) : Z^t = Z, Z^t\bar{Z} < I_g\} \\ & \cup & & \cup \\ & \tau & \longmapsto & Z = (\tau - iI_g)(\tau + iI_g)^{-1} \end{array}$$

$\mathscr{G}_g$ actually is a bounded domain.

Remark 4.3.4. [20] Let s be an involution ($s^2 = 1$) on $\mathscr{G}_g$ defined by

$$\begin{array}{lccc} s: & \mathscr{G}_g & \longrightarrow & \mathscr{G}_g \\ & \cup & & \cup \\ & \tau & \longmapsto & s \cdot \tau = \begin{bmatrix} 0 & I_g \\ -I_g & 0 \end{bmatrix} \cdot \tau = -\tau^{-1} \end{array}.$$

Observe that iI_g is an isolated fixed point of s. So, s is a symmetry at iI_g. Since we found a symmetry on the identity and $Sp(2g,\mathbb{R})$ acts transitively, the Siegel upper half space is a symmetric domain.

As we can see, the Siegel upper half space of degree g has very similar properties to moduli space of polarized K3 surfaces of degree $2d$. Therefore, thanks to the examples above, we can define many things on the moduli space of polarized K3 surfaces of degree $2d$ as well.

Remark 4.3.5. [20] $\mathscr{G}_g$ is a tube domain.

$$\mathscr{G}_g = \mathscr{H}_g + i\mathscr{H}_g^+,$$

where $\mathscr{H}_g = \{y \in M(g,\mathbb{R} : y^t = y\}$ is the vector space of symmetric matrices and $\mathscr{H}_g^+ = \{y \in \mathscr{H}_g : y > 0\}$ is the cone of positive definite matrices in $\mathcal{H}_g$.

Definition 42. [20] A subgroup Γ of $Sp(2g,\mathbb{R})$ is an arithmetic subgroup if $\Gamma \subset Sp(2g,\mathbb{Q})$ and for a faithful rational representation $p : Sp(2g,\mathbb{Q}) \to GL(n,\mathbb{Q})$, $p(\Gamma)$ is commensurable with $p(Sp(2g,\mathbb{R}) \cap GL(n,\mathbb{Z})$.

Remark 4.3.6. In our case, $\Gamma = Sp(2g,\mathbb{Z})$.

Corollary 4.3.2. [20] If Γ acts on $D_g = Sp(2g, \mathbb{R})$ without fixed points, then $\Gamma \backslash D_g$ is a manifold. Indeed, it is a quasi-projective algebraic variety. Moreover, $\Gamma \backslash D_g$ has only quotient singularities.

Let $\bar{D}_g$ be the closure of D_g in $\mathscr{H}_{g,\mathbb{C}}$ which is the vector space of $\mathbb{C}$- matrices of degree g. Define an equivalence relation for $p, q \in \bar{D}_g$:

$p \sim q$ if and only if $\exists$ holomorphic maps $\xi_i : \triangle = \{z \in \mathbb{C} : |z| < 1\} \to \bar{D}_g$, $i = 1, ..., m$, such that $\xi_1(0) = p$, $\xi_m(0) = q$, and $\xi_i(\triangle) \cap \xi_{i+1}(\triangle) \neq \emptyset$. This means that if $p \sim q$, then they are connected by a finite number of holomorphic curves.

Definition 43. [20] A maximal subset in $\bar{D}_g$ of mutually equivalent points is called a boundary component of D_g.

One of the good proporties about this definition is that the action of $Sp(2g, \mathbb{R})$ on D_g can be extended to $\bar{D}_g$.

Proposition 4.3.3. [20]

i) $\overline{D}_g = \{Z \in M(g, \mathbb{C}) : Z^t = Z, I_g - Z\bar{Z} \geq 0\}$.

ii) There is a 1-1 correspondence between the set of boundary components $\{F\}$ and the set of real subspaces of $\mathbb{R}^{2g}$ of dimension $g' \leq g$ $\{V\}$ defined by

$$F = F(V) =$$

$$\left\{ Z \in \overline{D}_g : \text{for} W := \text{the vector spaces on } \mathbb{C}^{2g} \text{ spanned by the columns of} \right.$$

$$\left. \begin{bmatrix} Z + I_g \\ i(Z - I_g) \end{bmatrix}, V_{\mathbb{C}} = W \cap \bar{W} \right\}.$$

iii) $\overline{D}_g = \bigcup_{0 \leq g' \leq g} G \cdot F_{g'}$.

Definition 44. [20] Let F be a boundary component of D_g. Then

i) $N(F) = \{h \in Sp(2g, \mathbb{R}) : hF = F\}$ is the stabilizer group associated with F. Note that it is a maximal parabolic subgroup of $Sp(2g, \mathbb{R})$

ii) $W(F) =$ the unipotent radical of $N(F)$,

iii) $U(F) =$ the center of $W(F)$.

Definition 45. [20] For $F = F_{g'}$,

$$G_h(F_{g'}) := \left\{ \begin{bmatrix} A' & 0 & B' & 0 \\ 0 & I_{g''} & 0 & 0 \\ C' & 0 & D' & 0 \\ 0 & 0 & 0 & I_{g''} \end{bmatrix} : \begin{bmatrix} A' & B' \\ C' & B' \end{bmatrix} \in Sp(g'', \mathbb{R}) \right\} \simeq Sp(g', \mathbb{R}).$$

$$G_l(F_{g'}) := \left\{ \begin{bmatrix} I_{g'} & 0 & 0 & 0 \\ 0 & u & 0 & 0 \\ 0 & 0 & I_{g'} & 0 \\ 0 & 0 & 0 & (u^{-1})^t \end{bmatrix} : u \in GL(g'', \mathbb{R}), g'' = g - g' \right\} \simeq GL(g'', \mathbb{R}).$$

Proposition 4.3.4. [20] For $g' < g$, let $F = F_{g'}$, $N_{g'} = N(F_{g'})$, $W(F_{g'}) = W_{g'}$, $U(F_{g'}) = U_{g'}$. Then, first we have $N_{g'} = (G_h \rtimes G_l)W_{g'}$.

Moreover, $W(F_{g'})$ is a metabelian group and we have a short exact sequence which splits

$$0 \to U_{g'} \to W_{g'} \to W_{g'}/U_{g'} \to 0.$$

Here $U_{g'}, W_{g'}/U_{g'}$ are vector groups.

The normalizer of $U_{g'}$ in $Sp(2g, \mathbb{R})$ is $N_{g'}$.

Let $w = [I_{g''}] \in U_{g'}$ for some $0 \leq g'' \leq g' < g$, and $C(F_{g'}) = C_{g'}$ be the $N_{g'}$-orbit of w in $\mathcal{U}_{g'}$, where the action is defined by conjugation. Then, we have isomorphisms

$$\begin{array}{ccc} C_{g'} & \xrightarrow{\sim} & \mathscr{H}^+_{g''} \\ \cup & & \cup \\ U_{g'} & \xrightarrow{\sim} & \mathscr{H}_{g''} \end{array}$$

Note that, $\mathscr{H}_{g''}$ has a quadratic form $(y_1, y_2) \to tr(y_1 y_2)$ by which $\mathscr{H}^+_{g''}$ is self-dual. Hence, we can see that $C_{g'}$ as a self-dual open cone in $U_{g'}$ since it is isomorphic to $\mathscr{H}^+_{g''}$.

We have epimorphisms

$$p_h : N_{g'} \longrightarrow G_h(F_{g'}) \xrightarrow{\sim} Aut(D_{g'}) = Sp(g', \mathbb{R})$$

$$p_l : N_{g'} \longrightarrow G_l(F_{g'}) \xrightarrow{\sim} Aut(U_{g'}, C_{g'}) = Aut(\mathscr{H}_{g''}, \mathscr{H}^+_{g''}) = GL(g'', \mathbb{R})$$

Proposition 4.3.5. [20] If $F = F(V), V \subset \mathbb{R}^{2g}$, then the closure of F in $\overline{D}_g$ is

$$\overline{F} = \left\{ Z \in \overline{D}_g : W := \text{vector space spanned by the columns of } \begin{bmatrix} Z + I \\ i(Z-1) \end{bmatrix}, \right.$$

$$\left. \text{then } V_{\mathbb{C}} = W \cap \overline{W} \right\}.$$

If F, F' are two boundary components of D, then denote $F < F'$ if $F \subset \overline{F}'$.

Proposition 4.3.6. [20] If F, F' are two boundary components with $F < F'$, then $\overline{C}(F') = \overline{C}(F) \cap U(F')$. Moreover, there is a one-to-one correspondence between the set of boundary components F' with $F < F'$ and the set of boundary components of $C(F)$, i.e.

$$\begin{array}{ccc} \{F', F < F'\} & \longrightarrow & \{\text{boundary components of } C(F)\} \\ \cup & & \cup \\ F' & \longrightarrow & C(F') \end{array}$$

Note that, here the relation is reversed.

Definition 46. [20] Let F be a boundary component of D_g. F is called rational if $N(F)$ is defined over $\mathbb{Q}$.

$$D^r = \bigsqcup_{F-rational} F \subset \overline{D}_g$$

is called the rational closure of D_g.

Now, define

$$D(F) = U(F)_{\mathbb{C}} D_g \subset D_g^c,$$

$$D'(F) = U(F)_{\mathbb{C}} \backslash D(F).$$

Theorem 4.3.7. *[20] The following diagram of holomorphic maps commutes,*

$$\begin{array}{ccc} D(F) & \xrightarrow{\sim} & F \times V(F) \times U(F)_{\mathbb{C}} \\ \downarrow & & \downarrow \\ D'(F) & \xrightarrow{\sim} & F \times V(F) \\ \downarrow & & \downarrow \\ F & \xrightarrow{=} & F \end{array}$$

where π_F's are projections and $V(F) = W(F)/U(F)$.

Theorem 4.3.8. *[20] ϕ is a real analytic map that maps D_g onto $C(F)$, i.e.*

$$\begin{array}{rccc} \phi: & D(F) & \longrightarrow & U(F) \\ & \cup & & \cup \\ & D_g & \longrightarrow & C(F), \end{array}$$

and $\phi^{-1}(C(F)) = D_g$.

Remark 4.3.7. Theorem actually says that D is a family of tube domain parametrized by $D'(F)$ which is a vector bundle over F.

According to above theorem for two boundary components F, F' with $F < F'$, there is a holomorphic epimorphism $\pi_{F,F'}$ between them such that the following diagram commutes

$$\begin{array}{ccc} & & F' \\ & \overset{\pi_{F'}}{\nearrow} & \downarrow \pi_{F,F'} \\ D_g & & \\ & \underset{\pi_F}{\searrow} & \\ & & F \end{array}$$

Now, we introduce a very important theorem which allow us to introduce our first compactification, so called Baily-Borel compactification:

Theorem 4.3.9. *[3] [20] Let $\Gamma \subset G_{\mathbb{Q}}$ be an arithmetic subgroup acting on D, and D^r be the rational closure of D. Then $(\Gamma\backslash D)^{BB} := \Gamma\backslash D^r$ is compact and contains*

$\Gamma\backslash D$ as an open dense sunset. Moreover, $(\Gamma\backslash D)^{BB}$ is a projective algebraic variety. $(\Gamma\backslash D)^{BB}$ is called Satake-Baily-Borel compactification.

Now let us apply the definitions in the specific example above to the K3 surfaces.

Definition 47. [15] Recall that $D_d := \{x \in \mathbb{P}(\Lambda_d \otimes \mathbb{C})|(x)^2 = 0, (x.\overline{x}) > 0\}$ is the period domain. Let $G_{\mathbb{R}}$ be the connected component of a linear algebraic group $O(\Lambda_d \otimes \mathbb{R})$, defined over $\mathbb{Q}$. Then the period domain can be represented by $D_d = G_{\mathbb{R}}/K$, where K is a maximal compact subgroup of K. $D_d^c := \{x \in \mathbb{P}(\Lambda_d \otimes \mathbb{C})|(x)^2 = 0\}$ is called the compact dual of D_d. It is a compact hermitian symmetric domain containing D_d as an open subdomain. Let us denote the topological closure of D_d in D_d^c by $\bar{D}_d$. A boundary component F of D_d is a maximal connected complex analytic subset in $\overline{D}_d \setminus D_d$. Then the stabilizer group $N(F) = \{g \in G|gF \subset F\}$ is a maximal parabolic group of $G_{\mathbb{R}}$. A boundary component F is rational if $N(F)$ is defined over $\mathbb{Q}$.

Proposition 4.3.10. [15] [22] The set of all rational boundary components of D_d corresponds to the set of all primitive totally isotopic sublattices of Λ_d. If E is a primitive totally isotropic sublattice of Λ_d, then the corresponding rational component F_E is defined by $\mathbb{P}(E \otimes \mathbb{C}) \cap \bar{D}_d$. Since the signature of $\Lambda_d = (2, 19)$, the dimension of a rational boundary component F_E is either 0 or 1. Hence observe that if $rank(E) = 2$, then $F \cong \mathbb{H}$,and if $rank(E) = 1$, then $F \cong \{point\}$.

Definition 48. [1] $D_d^r = D_d \bigsqcup_E F_E \subset D_d^c$ is called the rational closure of D_d.

Theorem 4.3.11. *[3] The quotient $\overline{F}_d^{BB} := D^r/\Gamma$ is compact and has a structure of a projective variety, and the projective coordinate ring is the ring of modular forms for Γ.*

Definition 49. [1] The 0- and 1-cusps of $\overline{F}_d^{BB}$ are the zero- and one-dimensional boundary components, respectively.

Now, we want see D_d as a Siegel domain of the third kind.

Proposition 4.3.12. First define $D_d(F) := U(F)_{\mathbb{C}} \cdot D_d \subset (\bar{D}_d)$. Then there exists a holomorphic isomorphism

$$D_d(F) \simeq F \times \mathbb{C}^m \times U(F)_{\mathbb{C}},$$

and

$$D_d = \{(\tau, W, z) \in F \times \mathbb{C}^m \times U(F)_{\mathbb{C}} | Im(z) - h_\tau(w, w) \in C(F)\}$$

where $C(F)$ is a self-dual homogeneous cone in $U(F)$ with a positive define bilinear form $\langle , \rangle$ on $U(F)$ defined over $\mathbb{Q}$, and $h_\tau : \mathbb{C} \times \mathbb{C} \to U(F)$ is a quasi-hermitian form depending on $\tau \in F$, see [[2] Chapter 3.4]. Note that, self-dual means

$$C(F) = \{x \in U(F) : \langle x, y \rangle > 0 \forall y \in \overline{C(F)} \setminus \{0\}\}.$$

This representation is called a Siegel domain of the third kind of D_d.

To see more explicit versions of these groups see [20]

4.4 Main Theorems of Toroidal Compactification

Now, since we have given the necessary definitions, we can move on to the main theorems of toroidal compactification. Let us start by recalling the definitions we discussed:

Remark 4.4.1. [20] Let D, in general be a hermitian bounded symmetric domain, and $G = Aut(D)$ which is a semisimple real algebraic group. Let $\Gamma \subset G$ be an arithmetic group.

$$\overline{D} = D \cup \bigcup_\alpha F_\alpha, \text{where } F_\alpha \text{ are the boundary components.}$$

$$D^r = D \cup \bigcup_\alpha F_\alpha, \text{where this time } F_\alpha \text{ are rational}$$

$$N(F) - \text{the parabolic subgroup associated with the boundary component } F$$

$$W(F) - \text{the unipotent radical of } N(F)$$

$$U(F) - \text{the centre of } W(F)$$

$$C(F) - \text{a self dual open cone}$$

Definition 50. [2]

$$\Gamma_F = \Gamma \cap N(F),$$

$$\overline{\Gamma_F} := Im(N(F) \cap \Gamma \to Aut(U(F)).$$

Definition 51. [2] A decomposition of C into a rational polyhedral cones $\Sigma = \{\sigma_\alpha\}$ is called a Γ-admissible polyhedral collection of C if

i) a face of a cone is again a cone,

ii) $\sigma_\alpha \cap \sigma_\beta$ is a common face of both σ_α and σ_β,

iii) for all $\gamma \in \Gamma$, $\gamma\sigma_\alpha \in \Sigma$,

iv) $C = \bigcup_\alpha (\sigma_\alpha \cap C)$.

As in [2], we construct a toroidal compactification partially, i.e. we put a toric variety on each cusp F separately. It is called a "partial compactification in the direction of F of $(D/U(F))_{\Sigma(F)}$. Consider the fiber bundle defined in the last section:

$$D(F) = U(F)_{\mathbb{C}} \cdot D \xrightarrow{\pi_F} D(F)' = D(F)/U(F)_{\mathbb{C}},$$

Then, $D(F)$ is a principal fiber bundle over $D(F)'$ with structure group $U(F)_{\mathbb{C}}$. Now put

$$T(F) = U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}.$$

This is an algebraic torus defined by the quotient fiber bundle π'_F, see [2].

$$D(F)/U(F)_{\mathbb{Z}}) \times^{T(F)} T(F)_{\Sigma(F)}$$

is the fiber bundle over $D(F)'$ with fiber $T(F)_{\Sigma(F)}$. Define

$$(D/U(F)_{\mathbb{Z}})_{\Sigma(F)} = \text{interior of closure of } D/U(F)_{\mathbb{Z}} \text{ in}$$

$$(D(F)/U(F)_{\mathbb{Z}}) \times_{T(F)} T(F)_{\Sigma(F)}.$$

Observe that Γ_F acts on $(D/U(F)_{\mathbb{Z}})_{\Sigma(F)}$, since $\Sigma(F)$ is $\overline{\Gamma}$-invariant collection.

Definition 52. [2] A Γ-admissible collection of fans

$$\Sigma := \{\Sigma_F = \{\sigma_\alpha^F\} : F - \text{rational boundary component}\}$$

is a collection of polyhedra

$$\{\sigma_\alpha(F)\} \subset \overline{C(F)},$$

one for every rational component F of D, which are $\overline{\Gamma}$-admissible and which satisfy the following properties:

i) If F_1, F_2 are two boundary components such that $F_1 = \gamma \cdot F_2$ for some $\gamma \in \Gamma$, then γ can be seen as a natural isomorphism, i.e.

$$\gamma : C(F_2) \xrightarrow{\sim} C(F_1) \ ,$$

and

$$\{\sigma_\alpha(F_1)\} = \{\gamma \cdot \sigma_\alpha(F_2),$$

ii) if $F_1 \subset \overline{F_2}$, then $\{\sigma_\alpha(F_2)\} = \{\sigma_\alpha(F_1)\} \cap \overline{C(F_2)}$

Now we can state the first main theorem of toroidal compactification.

Theorem 4.4.1. *(Main Theorem I) [2] Let $\mathscr{G}$ be a semi-simple algebraic group defined over $\mathbb{Q}$ such that G is the connected component of the automorphism group of a hermitian symmetric domain D. Let $\Gamma \subset G$ be an arithmetic group. Let $\Sigma(F)$ be a Γ-admissible collection of polyhedra. Then there exists a unique Hausdorff analytic variety $\overline{D/\Gamma}^{\Sigma}$ containing D/Γ as an open dense subset and such that, for every rational boundary component F of D, there are analytic morphisms π_F making the diagram commute:*

$$\begin{array}{ccc} D/U(F)_{\mathbb{Z}} & \hookrightarrow & (D/U(F)_{\mathbb{Z}})_{\Sigma(F)} \\ \downarrow & & \downarrow{\scriptstyle \pi_F} \\ D/\Gamma & \hookrightarrow & \overline{D/\Gamma} \end{array}$$

and such that every point of $\overline{D/\Gamma}^{\Sigma}$ is in the image of one of the maps π_F. Also, $\overline{D/\Gamma}^{\Sigma}$ is a complex algebraic space.

Proposition 4.4.2. [2] There exists a natural morphism from $\overline{D/\Gamma}^{\Sigma}$ to the Baily-Borel compactification $\overline{D/\Gamma}^{BB}$, inducing the identity morphism on D/Γ.

Proposition 4.4.3. [20] Let Σ and Σ' be Γ-admissible decompositions and $(\overline{D\backslash\Gamma})^{\Sigma}$, $(\overline{D\backslash\Gamma})^{\Sigma'}$ be the associated toroidal compactifications, respectively. If Σ' is a refinement of Σ, then there exists a holomorphic map

$$p : (\overline{\Gamma\backslash D})^{\Sigma'} \to (\overline{D\backslash\Gamma})^{\Sigma},$$

which is the identity on $D\backslash\Gamma$.

4.5 Toroidal Compactification of Moduli Spaces of K3 Surfaces

Let $\Gamma := \Gamma_d^+ \subset O^+(\Lambda)$ be an arithmethic subgroup, and F be a rational boundary component. As before, let $N(F) \subset O^+(\Lambda_d \otimes \mathbb{R})$ be the stabilizer of F, and $U(F) \subset N(F)$ be the center of its unipotent radical. Note that $U(F) \cong \mathbb{R}^k$ for some k, see [15]. Define,

$$N(F)_{\mathbb{Z}} = N(F) \cap \Gamma,$$

$$U(F)_{\mathbb{Z}} = U(F) \cap \Gamma,$$

$$\overline{\Gamma}_F = Im(N(F)_{\mathbb{Z}} \to Aut(U(F)).$$

We saw that D_d can be realized in $D(F) \simeq F \times \mathbb{C}^m \times U(F)_{\mathbb{C}}$ as a Siegel domain of the third kind. Define the torus $T(F) = U(F)_{\mathbb{C}}/U(F)_{\mathbb{Z}}$. Let Σ be an admissible polyhedral decomposition of $C(F) \subset U(F)$. We can assume that all cones are regular. The collection of cones gives a torus embedding $T(F) \subset X_{\sigma_F}$, which is a toric variety. Define $(D_d/U(F)_{\mathbb{Z}})_\Sigma$ as the interior of the closure of $D_d/U(F)_{\mathbb{Z}}$ in $(D(F)/U(F)_{\mathbb{Z}}) \times_{T(F)} X_{\Sigma_F}$, which is smooth since every cone is regular. Then by the Main Theorem I [2], we have a compact analytic space $\overline{(D_d/\Gamma)}^{\Sigma}$.

Let us be more clear. As we know, we can see all boundary components in Baily-Borel compactification, i.e. $F/N(F)_\Gamma \subset \overline{D/\Gamma}^{BB}$. By the Proposition 4.4.2, we have a map $\beta : \overline{D/\Gamma}^{\Sigma} \to \overline{D/\Gamma}^{BB}$. Then, for any boundary component F, $\beta^{-1}(F/N(F)_{\mathbb{Z}})$ is the quotient of partial compactification inside a toric variety. The fan Σ_F of the toric varity, and the support $|\Sigma|$ is the rational closure $C(F)^r$ of a cone $C(F) \subset U(F)$. Note that every fan in this description is $\overline{\Gamma}_F$-admissible polyhedral decomposition (or just $\overline{\Gamma}_F$-fans and lies in a Γ-admissible collection of polyhedra (or fans), i.e. $\Sigma := \{\Sigma_F | F \text{is a rational boundary component}\}$ is a Γ-admissible collection of polyhedra where Σ_F is Σ_F is a $\overline{\Gamma}_F$-admissible polyhedral decomposition in $U(F)$ with $|\Sigma_F| = C(F)^r$.

We know that in this case, the boundary components are of dimension 0 or dimension 1, and they correspond to a primitive isotropic plane E or an primitive isotropic line l in Λ_d. By [9] or [15], $U(E)$ is of dimension 1, and $U(l) \cong l^\perp/l$ is a hyperbolic lattice of signature $(1, 18)$. Moreover, if we assume that $2d$ is square free, then by [[22], Theorem 4.0.1], we have a unique primitive isotropic line $l \subset \Lambda_d$. So, we can identify

$U(l)$ with $M_d = E_8(-1)^{\oplus 2} \oplus U \oplus < -2d >$. Also, the corresponding cone $C(l)$ can be seen as a connected component C_d in $M_{d,\mathbb{R}} := M_d \otimes \mathbb{R}$, and $C_d^r = Conv(\overline{C_d} \cap M_d)$ is the rational closure. Finally, we have

Proposition 4.5.1. [12] A toroidal compactification of D/Γ for a square free $2d$ is determined by a $\overline{\Gamma}_F$-admissible polyhedral decomposition in $M_{d,\mathbb{R}}$, and for every fan Σ, $|\Sigma| = C_d^r$.

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