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LOW-DIMENSIONAL LINEAR REPRESENTATIONS OF MAPPING CLASS GROUPS

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Abstract. Recently, John Franks and Michael Handel proved that, for $g \geq 3$ and $n \leq 2g-4$, every homomorphism from the mapping class group of an orientable surface of genus g to $\mathrm{GL}(n,\mathbb{C})$ is trivial. We extend this result to $n \leq 2g-1$, also covering the case $g = 2$. As an application, we prove the corresponding result for nonorientable surfaces. Another application is on the triviality of homomorphisms from the mapping class group of a closed surface of genus g to $\mathrm{Aut}(F_n)$ or to $\mathrm{Out}(F_n)$ for $n \leq 2g-1$.

1. Introduction

Let S be a compact connected oriented surface of genus $g \geq 1$ with $q \geq 0$ boundary components and with $p \geq 0$ marked points in the interior. Let $\mathrm{Mod}(S)$ denote the mapping class group of S, the group of isotopy classes of orientation–preserving self–diffeomorphisms of S. Diffeomorphisms and isotopies are assumed to be the identity on the marked points and on the boundary.

There is the classical representation of $\mathrm{Mod}(S)$ onto the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ induced by the action of the mapping class group on the first homology of the closed surface of genus g. In [6], Question 1.2, Franks and Handel asked whether every homomorphism $\mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ is trivial for $n \leq 2g-1$. They proved that, in fact, this is the case for $g \geq 3$ and $n \leq 2g-4$, improving a result of Funar [7] who showed that every homomorphism from the mapping class group to $\mathrm{SL}(n,\mathbb{C})$ has finite image for $n \leq \sqrt{g+1}$.

The aim of this paper is to give a complete answer to the question of Franks and Handel, and then consider the corresponding problem for nonorientable surfaces. We first prove the following theorem.

Theorem 1. *Let $g \geq 1$ and let $n \leq 2g-1$. Let $\phi : \mathrm{Mod}(S) \to \mathrm{GL}(n,\mathbb{C})$ be a homomorphism. Then the image $\mathrm{Im}(\phi)$ of ϕ is*

(1) *trivial if $g \geq 3$,*
(2) *a quotient of the cyclic group $\mathbb{Z}_{10}$ of order 10 if $g = 2$, and*

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(3) *a quotient of* $\mathbb{Z}_{12}$ *if* $(g,q)=(1,0)$ *and of* $\mathbb{Z}^q$ *if* $g=1$ *and* $q\geq 1$.

I was informed by Bridson [3] that he conjectured the above theorem. He hints this in the page 2 of [2].

The first corollary to Theorem 1 is the following.

Corollary 2. *Let* $g\geq 2$, $n\leq 2g-1$, Γ *be a quotient of* $\mathrm{Mod}(S)$ *and let* $\varphi:\Gamma\to \mathrm{GL}(n,\mathbb{C})$ *be a homomorphism. Then* $\mathrm{Im}(\varphi)$ *is trivial if* $g\geq 3$, *and is isomorphic to a quotient of* $\mathbb{Z}_{10}$ *if* $g=2$.

Note that the groups $\mathrm{Sp}(2g,\mathbb{Z})$, $\mathrm{Sp}(2g,\mathbb{Z}_m)$, $\mathrm{PSp}(2g,\mathbb{Z})$, and $\mathrm{PSp}(2g,\mathbb{Z}_m)$ are quotients of $\mathrm{Mod}(S)$. Since $\mathrm{Mod}(S)$ is residually finite [9, 11], there are many finite quotients.

In the definition of the mapping class group, if we allow the diffeomorphisms of S to permute the marked points, then we get a group $\mathcal{M}(S)$, which contains $\mathrm{Mod}(S)$ as a subgroup of index $p!$.

Corollary 3. *Let* $g\geq 2$ *and let* $n\leq 2g-1$. *Let* $\phi:\mathcal{M}(S)\to \mathrm{GL}(n,\mathbb{C})$ *be a homomorphism. Then* $\mathrm{Im}(\phi)$ *is finite.*

For a nonorientable surface N of genus g with $p\geq 0$ marked points, we define the mapping class group $\mathrm{Mod}(N)$ of N to be the group of isotopy classes of diffeomorphisms preserving the set of marked points (isotopies are assumed to fix the marked points). The action of a mapping class on the first homology of the closed surface $\overline{N}$ obtained by forgetting the marked points give rise to an automorphism of $H_1(\overline{N};\mathbb{Z})$ preserving the associated $\mathbb{Z}_2$–valued intersection form. It was proved by McCarthy and Pinkall [19], and also by Gadgil and Pancholi [8], that, in fact, all automorphisms of $H_1(\overline{N};\mathbb{Z})$ preserving the $\mathbb{Z}_2$–valued intersection form are induced by diffeomorphisms. By dividing out the torsion subgroup of $H_1(\overline{N};\mathbb{Z})$, we get a representation $\mathrm{Mod}(N)\to \mathrm{GL}(g-1,\mathbb{C})$. It is now natural to ask the triviality of the lower dimensional representations of $\mathrm{Mod}(N)$. Since the mapping class group $\mathrm{Mod}(N)$ has nontrivial first homology, we cannot expect that every such homomorphism is trivial. Instead, one may ask the following question.

Question 1.1. *Let* $g\geq 3$, *and let* $n\leq g-2$. *Is the image of every homomorphism* $\phi:\mathrm{Mod}(N)\to \mathrm{GL}(n,\mathbb{C})$ *finite?*

As an application of Theorem 1, we answer this question leaving only one case open; the case g is even and $n=g-2$.

Theorem 4. *Let* $g\geq 3$, *and let* $n\leq g-2$ *if* g *is odd and* $n\leq g-3$ *if* g *is even. Let* N *be a nonorientable surface of genus* g *with* $p\geq 0$ *marked points and let* $\phi:\mathrm{Mod}(N)\to \mathrm{GL}(n,\mathbb{C})$ *be a homomorphism. Then* $\mathrm{Im}(\phi)$ *is finite.*

The mapping class group of a nonorientable surface with boundary components may also be considered, but we restrict ourself to surfaces with marked points only in order to make the proof simpler.

As another application of Theorem 1, we prove the following result on the homomorphisms from the mapping class group of a closed orientable surface to $\mathrm{Aut}(F_n)$ and to $\mathrm{Out}(F_n)$, where F_n is the free group of rank n. Compare Theorem 5 with Question 16 in [4].

Theorem 5. *Let $g \geq 2$ and let S be a closed orientable surface of genus g. Let n be a positive integer with $n \leq 2g-1$. Let H denote either of $\mathrm{Aut}(F_n)$ or $\mathrm{Out}(F_n)$ and let $\varphi : \mathrm{Mod}(S) \to H$ be a homomorphism. Then the image of φ is*

(1) *trivial if $g \geq 3$, and*
(2) *a quotient of $\mathbb{Z}_{10}$ if $g = 2$.*

In [6], the main theorem, Theorem 1.1, is proved by induction on g. It is first proved for the cases $g \geq 3$ and $n \leq 2$. The main improvement of this paper is that we can start the induction from the cases $g = 2$ and $n \leq 3$. The rest of the proof of Theorem 1 follows from the arguments of [6]. We give a slight modification of this proof. The proofs in cases $g = 2$ and $n \leq 2$ follow from the proof of Lemma 3.1 of [6] with the additional information that the commutator subgroup of $\mathrm{Mod}(S)$ is perfect. In the case $(g, n) = (2, 3)$ we need to treat all possible Jordan forms of the image of the Dehn twist about a nonseparating simple closed curve.

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2. Algebraic preliminaries

We state two properties of subgroups of $\mathrm{GL}(n, \mathbb{C})$. They are either well-known, or easy to prove. Therefore, we do not prove them. These properties will be used in the proof of Theorem 1.

Lemma 2.1. *Let $C = \begin{pmatrix} z & * & * \\ 0 & z & * \\ 0 & 0 & z \end{pmatrix}$ and $D = \begin{pmatrix} w & * & * \\ 0 & w & * \\ 0 & 0 & w \end{pmatrix}$ be two elements of $\mathrm{GL}(3, \mathbb{C})$. Then $CDC = DCD$ if and only if $C = D$.*

Lemma 2.2. *The subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *consisting of upper triangular matrices is solvable.*

We will also require the following lemma from [6].

Lemma 2.3. ([6], Lemma 2.2) *Let* G *be a perfect group, and* H *a solvable group. Then any homomorphism* $G \to H$ *is trivial.*

3. Mapping class groups and commutator subgroups

Let S be a compact oriented surface of genus g with $p \geq 0$ marked points and with $q \geq 0$ boundary components. In this section we give the results on mapping class groups required in the proof of Theorem 1. For further information on mapping class groups, the reader is referred to [12], or [5]. For a simple closed curve a on S we denote by t_a the (isotopy class of the) right Dehn twist about a.

Theorem 3.1. ([17], Theorem 1.2) *Let* $g \geq 1$. *Suppose that* a *and* b *are two nonseparating simple closed curves on* S. *Then there is a sequence*

$$a = a_0, a_1, a_2, \dots, a_k = b$$

of nonseparating simple closed curves such that a_{i-1} *intersects* a_i *at only one point*

Theorem 3.2. *Let* $g \geq 2$. *Then the mapping class group* $\mathrm{Mod}(S)$ *is generated Dehn twists about nonseparating simple closed curves on* S.

Theorem 3.3. ([17], Theorem 2.7) *Let* $g \geq 2$. *Let* a *and* b *be two nonseparating simple closed curves on* S *intersecting at one point. Then the commutator subgroup of* $\mathrm{Mod}(S)$ *is generated normally by* $t_a t_b^{-1}$.

Theorem 3.3 should be interpreted as follows: If a normal subgroup of $\mathrm{Mod}(S)$ contains $t_a t_b^{-1}$, then it contains the commutator subgroup of $\mathrm{Mod}(S)$, the (normal) subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$.

Theorem 3.4. ([17], Theorem 4.2) *Let* $g \geq 2$. *Then the commutator subgroup of* $\mathrm{Mod}(S)$ *is perfect.*

Note that, in [17], the group $\mathrm{Mod}(S)$ of this paper is denoted by $\mathcal{P}M_S$. In that paper, Theorems 3.3 and 3.4 above are proved for surfaces with marked points (=punctures), but the same proof apply to surfaces with boundary as well since $\mathrm{Mod}(S)$ is generated by Dehn twists about nonseparating simple closed curves.

We record the following well–known relations among Dehn twists.

Lemma 3.5. *Let* a *and* b *be two simple closed curves on* S, *and let* t_a *and* t_b *denote the right Dehn twists about them.*

(1) *If a and b are disjoint, then t_a and t_b commute.*
(2) *If a intersects b transversely at one point, then they satisfy the braid relation $t_a t_b t_a = t_b t_a t_b$.*

Recall that the first homology group $H_1(G;\mathbb{Z})$ of a group G is isomorphic to the abelianization G/G', where G' is the (normal) subgroup of G generated by all commutators $[g_1, g_2]$.

Theorem 3.6. ([15], Theorem 5.1) *Let $g \geq 1$. Then the first homology group $H_1(\mathrm{Mod}(S);\mathbb{Z})$ is*

(1) *trivial if $g \geq 3$,*
(2) *isomorphic to the cyclic group of order 10 if $g = 2$,*
(3) *isomorphic to the cyclic group of order 12 if $(g, q) = (1, 0)$, and*
(4) *isomorphic to $\mathbb{Z}^q$ if $g = 1$ and $q \geq 1$.*

Note that since any two Dehn twists about nonseparating simple closed curves are conjugate in $\mathrm{Mod}(S)$, their classes in $H_1(\mathrm{Mod}(S);\mathbb{Z})$ are equal. In particular, we conclude the next lemma.

Lemma 3.7. *Let $g \geq 1$, and let b and c be two nonseparating simple closed curves on S. If H is an abelian group and if $\phi : \mathrm{Mod}(S) \to H$ is a homomorphism, then $\phi(t_b) = \phi(t_c)$.*

4. Homomorphisms $\mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$

In this section we prove Theorem 1. So let $n \leq 2g - 1$ and let $\phi : \mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ be a homomorphism.

For the proof, we adopt the proof of Theorem 1.1 in [6]. Franks and Handel use the fact that when $g \geq 3$ the group $\mathrm{Mod}(S)$ is perfect. If this is rephrased as "the commutator subgroup of $\mathrm{Mod}(S)$ is perfect" then it is still true for the case $g = 2$ as well, and that is what we use below. The proof given in [6] for the case $g \geq 3$ and $n \leq 2$, also works for the case $g = 2$ and $n \leq 2$ with a slight modification. For the case $g = 2$ and $n = 3$, we need to analyze six possible Jordan forms of the image of the Dehn twist about a nonseparating simple closed curve. Once Theorem 1 is established for the cases $g = 2$ and $n \leq 3$, we then again follow a modified version of the idea of Franks and Handel to induct g.

Following [6], for a simple closed curve x on S we denote $\phi(t_x)$ by L_x. If λ is an eigenvalue of a linear operator L, the corresponding eigenspace is denoted by $E_\lambda(L)$. We write E^x_λ for $E_\lambda(L_x)$.

Proposition 4.1. *Let $g = 2$ and $n \leq 2$. Then $\mathrm{Im}(\phi)$ is a quotient of the cyclic group $\mathbb{Z}_{10}$.*

Proof. If $n = 1$ then $\mathrm{GL}(n, \mathbb{C}) = \mathbb{C}^*$ is abelian. Hence, ϕ factors through the first homology of $\mathrm{Mod}(S)$, which is isomorphic to $\mathbb{Z}_{10}$. So assume that $n = 2$. In this proof, we set $G = \mathrm{Mod}(S)$.

Let a, b and c be three nonseparating simple closed curves on S such that a is disjoint from $b \cup c$, and that b intersects c transversely at one point. Clearly, in order to complete the proof, it suffices to prove that $\phi(G') = \{I\}$, where G' is the commutator subgroup of G. There are three possibilities for the Jordan form of L_a.

(i). Suppose that L_a has two distinct eigenvalues λ_1 and λ_2, with corresponding eigenvectors v_1 and v_2; $L_a(v_i) = \lambda_i v_i$. With respect to the basis $\{v_1, v_2\}$, the matrix L_a is diagonal. Since L_b and L_c preserve each eigenspace of L_a, they are diagonal too. In particular, they commute. Now from the braid relation $L_bL_cL_b = L_cL_bL_c$, we get $L_b = L_c$, or $\phi(t_bt_c^{-1}) = I$. Since G' is generated normally by $t_bt_c^{-1}$ (c.f. Theorem 3.3), we conclude that $\phi(G')$ is trivial

(ii). If the matrix L_a has only one eigenvalue λ and if the Jordan form of L_a is λI, then $L_x = \lambda I$ for each nonseparating simple closed curve x on S. This is because L_x is conjugate to L_a. Since the group G is generated by Dehn twists about such curves, we have that $\phi(G)$ is cyclic.

(iii). Suppose finally that the Jordan form of L_a is not diagonal, so that the matrix of L_a is

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

in some fixed basis. Because L_b and L_c preserve the eigenspace of L_a, with respect to the same basis, the matrices L_b and L_c are upper triangular whose diagonal entries are λ. In particular, we have $L_bL_c = L_cL_b$. From the braid relation $L_bL_cL_b = L_cL_bL_c$ again, we get $L_b = L_c$, or $\phi(t_bt_c^{-1}) = I$. From this we conclude that $\phi(G')$ is trivial.

This completes the proof of the proposition. □

Lemma 4.2. *Let a, b, x, y be four nonseparating simple closed curves on S such that there is an orientation–preserving diffeomorphism f of S with $f(x) = a$ and $f(y) = b$. Let λ be an eigenvalue of $L_a = \phi(t_a)$. Then $E^a_\lambda = E^b_\lambda$ if and only if $E^x_\lambda = E^y_\lambda$.*

Proof. Let $F = \phi(f)$. The assumptions $f(x) = a$ and $f(y) = b$ imply that $ft_xf^{-1} = t_a$ and $ft_yf^{-1} = t_b$, and hence $FL_xF^{-1} = L_a$ and $FL_yF^{-1} = L_b$. Therefore,

$$E^a_\lambda = E_\lambda(L_a) = E_\lambda(FL_xF^{-1}) = F(E^x_\lambda)$$

and

$$E^b_\lambda = E_\lambda(L_b) = E_\lambda(FL_yF^{-1}) = F(E^y_\lambda).$$

The lemma now follows from these two. □

Proposition 4.3. *If $g = 2$ then the image of any homomorphism $\phi : \mathrm{Mod}(S) \to \mathrm{GL}(3, \mathbb{C})$ is a quotient of the cyclic group $\mathbb{Z}_{10}$*

Proof. Let $G = \mathrm{Mod}(S)$, and let G' denote the commutator subgroup of G. Since $H_1(G, \mathbb{Z})$ is isomorphic to $\mathbb{Z}_{10}$, it suffices to prove that $\phi(G)$ is abelian, or equivalently that $\phi(G')$ is trivial.

Since $\phi(G')$ is generated normally by a single element $t_x t_y^{-1}$ for any two nonseparating simple closed curves x and y intersecting at one point, it is also sufficient to find two such curves with $\phi(t_x t_y^{-1}) = I$.

Let a be a nonseparating simple closed curve on S. The Jordan form of L_a is one of the following six matrices:

$$\text{(i)} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \ \text{(ii)} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \ \text{(iii)} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 1 \\ 0 & 0 & \mu \end{pmatrix},$$
$$\text{(iv)} \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \ \text{(v)} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \ \text{(vi)} \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}.$$

Here, distinct notations represent distinct eigenvalues. In each case we fix a basis with respect to which the matrix L_a is in its Jordan form. Recall that the eigenspace of L_x corresponding to an eigenvalue λ is denoted by E^x_λ. We now analyze each case.

(i). In this case, each eigenspace of L_a is 1–dimensional. Let b_1 and b_2 be two nonseparating simple closed curves intersecting at one point such that each b_i is disjoint from a. Then each eigenspace $E^a_{\lambda_i}$ is invariant under each L_{b_i}, so that L_{b_i} are diagonal. In particular, L_{b_1} and L_{b_2} commute. Now the braid relation $L_{b_1} L_{b_2} L_{b_1} = L_{b_2} L_{b_1} L_{b_2}$ implies that $L_{b_1} = L_{b_2}$. Thus, we have $\phi(L_{b_1}) = \phi(L_{b_2})$, i.e. $\phi(t_{b_1} t_{b_2}^{-1}) = I$.

(ii). If b is a nonseparating simple closed curve on S, then L_b is conjugate to L_a, so that $L_b = \lambda I$. Since G is generated by all such Dehn twists, we get that $\phi(G)$ is cyclic.

(iii). Let b_1 and b_2 be two nonseparating simple closed curves intersecting at one point such that they are disjoint from a. The matrices L_{b_1} and L_{b_2} preserve each eigenspace E^a_λ and E^a_μ, so that they are of the form

$$L_{b_i} = \begin{pmatrix} x_i & 0 & w_i \\ 0 & y_i & u_i \\ 0 & 0 & z_i \end{pmatrix}.$$

The braid relation $L_{b_1} L_{b_2} L_{b_1} = L_{b_2} L_{b_1} L_{b_2}$ implies that $x = x_1 = x_2$, $y = y_1 = y_2$ and $z = z_1 = z_2$. The equality $L_{b_i} L_a = L_a L_{b_i}$ then gives

$w_i = 0$ and $y = z$. Hence, we have $x = \lambda$, $y = z = \mu$, and so

$$L_{b_i} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & u_i \\ 0 & 0 & \mu \end{pmatrix}.$$

But then we have $L_{b_1}L_{b_2} = L_{b_2}L_{b_1}$. The braid relation $L_{b_1}L_{b_2}L_{b_1} = L_{b_2}L_{b_1}L_{b_2}$ again implies that $L_{b_1} = L_{b_2}$, and hence $\phi(t_{b_1}t_{b_2}^{-1}) = I$.

(iv). Let b_1 and b_2 be two nonseparating simple closed curves intersecting at one point such that they are disjoint from a. In this case, $\ker(L_a - \lambda I) = E^a_\lambda$ is 1-dimensional, $\ker(L_a - \lambda I)^2$ is 2-dimensional, and they are L_{b_i}–invariant for $i = 1, 2$. It follows that

$$L_{b_i} = \begin{pmatrix} \lambda & * & * \\ 0 & \lambda & * \\ 0 & 0 & \lambda \end{pmatrix}.$$

Since there is the braid relation $L_{b_1}L_{b_2}L_{b_1} = L_{b_2}L_{b_1}L_{b_2}$, Lemma 2.1 implies that $L_{b_1} = L_{b_2}$. Hence, $\phi(t_{b_1}t_{b_2}^{-1}) = I$.

(v). The eigenspace E^a_λ is 2-dimensional in this case. Suppose first that $E^a_\lambda \neq E^b_\lambda$ for some (hence all) nonseparating simple closed curve b intersecting a at one point. Choose two nonseparating simple closed curves c_1 and c_2 disjoint from $a \cup b$ such that c_1 intersects c_2 at one point. Then $E^a_\lambda \cap E^b_\lambda$ and E^a_λ are L_{c_i}–invariant subspaces, so that with respect to a suitable basis

$$L_{c_i} = \begin{pmatrix} \lambda & * & * \\ 0 & \lambda & * \\ 0 & 0 & \lambda \end{pmatrix}.$$

Note that we have the braid relation $L_{c_1}L_{c_2}L_{c_1} = L_{c_2}L_{c_1}L_{c_2}$. We now use Lemma 2.1 to conclude that $L_{c_1} = L_{c_2}$, so that $\phi(t_{c_1}t_{c_2}^{-1}) = I$.

If $E^a_\lambda = E^b_\lambda$ for some (hence all) b intersecting a at one point, then $E^a_\lambda = E^x_\lambda$ for all nonseparating simple closed curves x. This is because, by Theorem 3.1 there is a sequence $a = a_0, a_1, a_2, \ldots, a_k = x$ of nonseparating simple closed curves such that a_{i-1} intersects a_i at one point for all $1 \leq i \leq k$, and $E^{a_{i-1}}_\lambda = E^{a_i}_\lambda$. Since G is generated by Dehn twists about nonseparating simple closed curves, E^a_λ is $\phi(G)$–invariant, so that ϕ induces a homomorphism $\bar{\phi} : G \to \mathrm{GL}(E^a_\lambda) = \mathrm{GL}(2, \mathbb{C})$. By Proposition 4.1, $\bar{\phi}(G)$ is cyclic, and hence $\bar{\phi}(f) = I$ all $f \in G'$. Thus the matrix of $\phi(f)$ is of the form

$$\begin{pmatrix} 1 & 0 & z_1 \\ 0 & 1 & z_2 \\ 0 & 0 & z_3 \end{pmatrix}.$$

Since the subgroup of $\mathrm{GL}(3,\mathbb{C})$ consisting of upper triangular matrices are solvable and since G' is perfect, $\phi(G')$ is trivial.

(vi). In this last case, the eigenspace E^a_λ is, again, 2-dimensional. If $E^a_\lambda = E^b_\lambda$ for some nonseparating simple closed curve b intersecting a at one point, then from Theorem 3.1 and Lemma 4.2 we obtain that $E^a_\lambda = E^x_\lambda$ for all nonseparating simple closed curves x. We conclude now as in the case (v) that $\phi(G')$ is trivial.

Suppose finally that $E^a_\lambda \neq E^b_\lambda$ for some (hence all) nonseparating simple closed curve b intersecting a at one point. By Lemma 4.2, $E^x_\lambda \neq E^y_\lambda$ for all nonseparating simple closed curves x and y intersecting once. Let $a = c_4$ and $b = c_5$. Choose three nonseparating simple closed curves c_1, c_2, c_3 such that

- c_i intersects c_j at one point if $|i-j| = 1$, and
- c_i is disjoint from c_j if $|i-j| \geq 2$.

Let $v_1 \in E^a_\lambda \cap E^b_\lambda$, $v_2 \in E^a_\lambda$ and $v_3 \in E^a_\mu$ so that $\{v_1, v_2, v_3\}$ is a basis. With respect to this basis, the matrix of L_a is its Jordan matrix. Since $E^a_\lambda \cap E^b_\lambda$, E^a_λ and E^a_μ are L_{c_i}–invariant for $i = 1, 2$, we have

$$L_{c_i} = \begin{pmatrix} x_i & w_i & 0 \\ 0 & y_i & 0 \\ 0 & 0 & z_i \end{pmatrix},$$

with $\{x_i, y_i, z_i\} = \{\lambda, \mu\}$. Then the braid relation

$$L_{c_1} L_{c_2} L_{c_1} = L_{c_2} L_{c_1} L_{c_2} \tag{1}$$

gives us $x_1 = x_2 = x, y_1 = y_2 = y$ and $z_1 = z_2 = z$.

If $z = \mu$ then $x = y = \lambda$, and hence $L_{c_1} L_{c_2} = L_{c_2} L_{c_1}$. Now the braid relation (1) implies again that $L_{c_1} = L_{c_2}$. It follows that $\phi(G')$ is trivial as above.

Suppose that $z = \lambda$. We will show that this case is not possible by arriving at a contradiction. If $x = \lambda$ then $y = \mu$. But then we have $E^{c_1}_\lambda = E^{c_2}_\lambda$, which is a contradiction. If $x = \mu$ then $y = \lambda$. Since $E^{c_1}_\mu$ is L_{c_3}–invariant, we have

$$L_{c_3} = \begin{pmatrix} u & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$

Now the braid relation $L_a L_{c_3} L_a = L_{c_3} L_a L_{c_3}$ gives $u = \lambda$, while the braid relation $L_{c_3} L_{c_2} L_{c_3} = L_{c_2} L_{c_3} L_{c_2}$ gives $u = \mu$. Since $\lambda \neq \mu$, we get a contradiction again.

This completes the proof of the proposition. □

Finally, we are in a position to prove Theorem 1.

Proof of Theorem 1. If $g = 1$ then $n = 1$, and $\phi : \mathrm{Mod}(S) \to \mathrm{GL}(1, \mathbb{C})$. Since $\mathrm{GL}(1, \mathbb{C})$ is abelian, ϕ factors through $H_1(\mathrm{Mod}(S); \mathbb{Z})$. The theorem now follows from Theorem 3.6. If $g = 2$ then $n \leq 3$. These cases are proved in Propositions 4.1 and 4.3. We assume that $g \geq 3$ and that the theorem holds true for all surfaces of genus $g - 1$. Since $\mathrm{GL}(k-1, \mathbb{C})$ is isomorphic to a subgroup of $\mathrm{GL}(k, \mathbb{C})$, it suffices to prove the theorem for $n = 2g - 1$.

In what follows R denotes a subsurface of S diffeomorphic to a compact connected surface of genus $g - 1$ with one boundary component. We embed $\mathrm{Mod}(R)$ into $\mathrm{Mod}(S)$ by extending self-diffeomorphisms of R to S by the identity. We set $G = \mathrm{Mod}(S)$ and $H_R = \mathrm{Mod}(R)$.

If, for some subsurface R, there exists a $\phi(H_R)$–invariant subspace V of dimension r with $2 \leq r \leq n-2$, then ϕ induces homomorphisms $\phi_1 : H_R \to \mathrm{GL}(V) = \mathrm{GL}(r, \mathbb{C})$ and $\phi_2 : H_R \to \mathrm{GL}(\mathbb{C}^n/V) = \mathrm{GL}(n-r, \mathbb{C})$. Note that $r \leq 2(g-1) - 1$ and $n - r \leq 2(g-1) - 1$. By assumption, the image of each ϕ_i is cyclic. In particular, if b and c are two simple closed curves on R intersecting at one point, then $\phi_i(t_b) = \phi_i(t_c)$ by Lemma 3.7. That is, $\phi_i(t_b t_c^{-1}) = I$. Since the commutator subgroup H'_R of H_R is generated normally by $t_b t_c^{-1}$ we get that $\phi_i(f) = I$ for all $f \in H'_R$. It follows that, with respect to some basis of $\mathbb{C}^n$,

$$\phi(f) = \begin{pmatrix} I_r & F \\ 0 & I_{n-r} \end{pmatrix}$$

for all $f \in H'_R$. Since the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of such matrices is abelian and since H'_R is perfect, we conclude that $\phi(H'_R)$ is trivial. In particular, we have $\phi(t_b t_c^{-1}) = I$. Since $G' = G$ is generated normally by $t_b t_c^{-1}$, $\phi(G)$ is trivial.

We now fix a subsurface R of genus $g - 1$ with one boundary component. Let a and b be two nonseparating simple closed curves on S intersecting at one point such that $a \cup b$ is disjoint from R.

CASE 1. Suppose that there is a subspace V of dimension r with $2 \leq r \leq n-2$ which is a direct sum of eigenspaces of L_a (Note that there exists such a subspace if L_a has at least three distinct eigenvalues.). Then V is $\phi(H_R)$–invariant. Hence, $\phi(G)$ is trivial.

CASE 2. Suppose that there is no subspace V as in CASE 1. In particular, L_a has at most two eigenvalues and each eigenspace of L_a is either 1–dimensional, or $(n-1)$–dimensional, or n–dimensional. The Jordan form of L_a is one of the following four matrices:

(i) λI_n, (ii) $\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$, (iii) $\begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 0 & 0 \\ 0 & 0 & \cdots & 0 & \lambda & 1 \\ 0 & 0 & \cdots & 0 & 0 & \lambda \end{pmatrix}$,

(iv) $\begin{pmatrix} \lambda I_{n-1} & 0 \\ 0 & \mu \end{pmatrix}$.

We fix a basis so that the matrix L_a is equal to its Jordan form.

In the case (i), if x is a nonseparating simple closed curve on S, then $L_x = \lambda I$ since it is conjugate to L_a. Since G is generated by Dehn twists about nonseparating simple closed curves, $\phi(G)$ is cyclic, and hence it is trivial.

In the case (ii), the subspace $\ker(L_a - \lambda I)^2$ is a $\phi(H_R)$–invariant subspace of dimension 2, so that $\phi(G)$ is trivial.

It remain to consider the cases (iii) and (iv). In these cases, the eigenspace E^a_λ is of dimension $n-1$. The eigenspace E^b_λ is also of dimension $n-1$. If $E^a_\lambda \neq E^b_\lambda$, then $E^a_\lambda \cap E^b_\lambda$ is a $\phi(H_R)$–invariant subspace of dimension $n-2(>1)$. Hence, $\phi(G)$ is trivial.

Suppose finally that $E^a_\lambda = E^b_\lambda$. It follows from Lemma 4.2 that $E^x_\lambda = E^y_\lambda$ for any two nonseparating simple closed curves x and y on S intersecting at one point. We then conclude from Theorem 3.1 that $E^a_\lambda = E^x_\lambda$ for all such nonseparating x on S. Since $\mathrm{Mod}(S)$ is generated by Dehn twists about nonseparating simple closed curves, it follows that for each $f \in G$, the matrix $\phi(f)$ is upper triangular. Hence, $\phi(G)$ is contained in the subgroup consisting of upper triangular matrices. But this subgroup of $\mathrm{GL}(n, \mathbb{C})$ is solvable. Since G is perfect, we conclude from Lemma 2.3 that $\phi(G)$ is trivial.

This completes the proof of Theorem 1. □

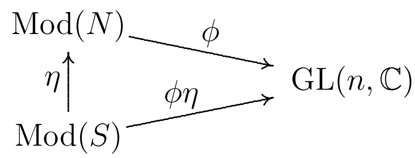
5. Nonorientable surfaces

The purpose of this section is to prove Theorem 4. So let $\phi : \mathrm{Mod}(N) \to \mathrm{GL}(n, \mathbb{C})$ be a homomorphism, where N is a nonorientable surface of genus $g \geq 3$ with $p \geq 0$ marked points, and $n \leq g-2$ for odd g and $n \leq g-3$ for even g.

Proof of Theorem 4. If $g = 3$ or $g = 4$ then $n = 1$, and $\mathrm{GL}(1, \mathbb{C})$ is abelian, so that ϕ factors through the first homology $H_1(\mathrm{Mod}(N); \mathbb{Z})$. Since $H_1(\mathrm{Mod}(N); \mathbb{Z})$ is finite (c.f. [13, 14]), the result follows. So we assume that $g \geq 5$.

Let T denote the subgroup of the mapping class group $\mathrm{Mod}(N)$. Write $g = 2r+1$ if g is odd and $g = 2r+2$ if g is even. Hence, we have $r \geq 2$ and $n \leq 2r-1$.

Let S be a compact orientable surface of genus r with one boundary component. Embed S into N, so that the boundary of S bounds a Möbius band (resp. Klein bottle with one boundary) on N with p marked points if g is odd (resp. even). Extending diffeomorphisms S to N by the identity induces a homomorphism $\eta : \mathrm{Mod}(S) \to \mathrm{Mod}(N)$. Then the composition $\phi\eta$ is a homomorphism from $\mathrm{Mod}(S)$ to $\mathrm{GL}(n, \mathbb{C})$.



If $r \geq 3$ ($g \geq 7$) then $\phi\eta$ is trivial by Theorem 1. It follows that $\phi(t_a) = I$ for any Dehn twist supported on S. If b is a two-sided nonseparating simple closed curve on N whose complement in nonorientable, it follows from Theorems 3.1 and 5.3 in [14] that a Dehn twist t_b about b is conjugate to a Dehn twist supported on S. Hence, $\phi(t_b)$ is trivial for all such b. Since T is generated by such Dehn twists (c.f. [14], proof of Theorem 5.12), we get that $\phi(T)$ is trivial. We also know that the index of T in $\mathrm{Mod}(N)$ is $p! \cdot 2^{p+1}$ ([14], Corollary 6.2). The conclusion of the theorem now follows.

If $r = 2$ ($g = 5$ or $g = 6$) then $\phi\eta$ is cyclic by Theorem 1. It follows that $\phi(t_a t_b^{-1}) = I$ for any two nonseparating simple closed curves on S. Let x and y be two two-sided nonseparating simple closed curves on N intersecting at one point such that the complement of each is nonorientable. It can easily be shown that there is a diffeomorphism f of N such that $f(x \cup y) \subset S$, so that $t_x t_y^{-1}$ can be conjugated to $t_a t_b^{-1}$, where a and b are on S. Hence, $\phi(t_x t_y^{-1}) = I$, or $\phi(t_x) = \phi(t_y)$. We now apply Theorem 3.1 in [14] to conclude that $\phi(t_x) = \phi(t_y)$ for all two-sided nonseparating simple closed curves whose complements are nonorientable. (Such simple closed curves are called essential in [14].) Since T is generated by such Dehn twists, we get that $\phi(T)$ is cyclic, so that $\phi(T') = \{I\}$, where T' is the commutator subgroup. Stukow proved that the index of T' in T is 2 (c.f. [20], Theorem 8.1). We conclude that $\phi(\mathrm{Mod}(N))$ is a finite group of order at most $p! \cdot 2^{p+2}$.

This finishes the proof of Theorem 4. □

Remark 5.1. If $g \geq 7$ and if N is closed, then the above proof shows that the image of ϕ in Theorem 4 is either trivial or is isomorphic to $\mathbb{Z}_2$.

In fact, it is easy to find a homomorphism whose image is $\mathbb{Z}_2$; send all Dehn twists to the identity and crosscap slides (=Y-homeomorphisms) to an element of order 2.

6. Homomorphisms to $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$

We prove Theorem 5 in this section. The proof in the case $n = 1$ is trivial. So we assume that $2 \leq n \leq 2g - 1$.

Let F_n denote the free group of rank n. The action of the automorphism group $\mathrm{Aut}(F_n)$ of F_n on the abelianization of F_n gives rise to a surjective homomorphism $\eta : \mathrm{Aut}(F_n) \to \mathrm{GL}(n, \mathbb{Z})$. The kernel of this map is usually denoted by IA_n. Let $\mathrm{Out}(F_n)$ denote the group of outer automorphisms of F_n, so that it is the quotient of $\mathrm{Aut}(F_n)$ with the (normal) subgroup $\mathrm{Inn}(F_n)$ of inner automorphisms.

Suppose that S is a closed surface of genus 2. Choose five pairwise nonisotopic nonseparating simple closed curves c_1, c_2, c_3, c_4, c_5 on S such that c_i is disjoint from c_j if $|i - j| \geq 2$ and that c_i intersects c_j at one point if $|i - j| = 1$. Let t_i denote the right Dehn twists about c_i. Then the group $\mathrm{Mod}(S)$ is generated by t_1, t_2, t_3, t_4, t_5. Let us set $\sigma = t_1t_2t_3t_4t_5$. It is well–known that σ is a torsion element of order six. One can easily show that

$$\sigma t_i \sigma^{-1} = t_{i+1} \tag{2}$$

for each $1 \leq i \leq 4$.

Lemma 6.1. *Let $g = 2$. The normal closure of σ^2 in $\mathrm{Mod}(S)$ is equal to the commutator subgroup of $\mathrm{Mod}(S)$.*

Proof. Let N denote the the normal closure of σ^2 in $\mathrm{Mod}(S)$, the intersection of all normal subgroups containing σ^2. Since any right Dehn twist in $\mathrm{Mod}(S)$ about a nonseparating simple closed curve maps to the generator of $H_1(\mathrm{Mod}(S), \mathbb{Z})$ under the natural homomorphism and since σ^2 is a product of 10 such Dehn twists, we see that σ^2 is contained in $[\mathrm{Mod}(S), \mathrm{Mod}(S)]$. Hence, $N \subset [\mathrm{Mod}(S), \mathrm{Mod}(S)]$.

Let $q : \mathrm{Mod}(S) \to \mathrm{Mod}(S)/N$ denote the quotient map. The equality (2) implies that $q(t_1) = q(t_3) = q(t_5)$ and $q(t_2) = q(t_4)$. On the other hand, from the braid relation we get

$$\begin{aligned} q(t_1)q(t_4)q(t_1) &= q(t_1)q(t_2)q(t_1) \\ &= q(t_1t_2t_1) \\ &= q(t_2t_1t_2) \\ &= q(t_2)q(t_1)q(t_2) \\ &= q(t_4)q(t_1)q(t_4). \end{aligned}$$

Since t_1 and t_4 commute, we obtain $q(t_1) = q(t_4)$. It follows that $\mathrm{Mod}(S)/N$ is cyclic. In particular, N contains $[\mathrm{Mod}(S), \mathrm{Mod}(S)]$, finishing the proof of the lemma. $\square$

Proof of Theorem 5. Suppose that $H = \mathrm{Aut}(F_n)$. Let ϕ be the composition of φ with η, so that we have a commutative diagram:

$$\begin{array}{ccccccccc} & & & & \mathrm{Mod}(S) & & & & \\ & & & \overset{\varphi}{\swarrow} & \downarrow \varphi & \overset{\phi}{\searrow} & & & \\ 1 & \longrightarrow & IA_n & \longrightarrow & \mathrm{Aut}(F_n) & \xrightarrow{\eta} & \mathrm{GL}(n, \mathbb{Z}) & \longrightarrow & 1. \end{array}$$

We now apply Theorem 1. If $g \geq 3$ then ϕ is trivial, implying that the image of φ is contained in IA_n. Since IA_n is torsion–free by a result of Baumslag-Taylor [1], all torsion elements in $\mathrm{Mod}(S)$ are contained in the kernel of φ. Since $\mathrm{Mod}(S)$ is is generated by torsion elements (cf. [18, 10, 16]), we conclude that φ is trivial.

Suppose now that $g = 2$. Theorem 1 gives us that the image of ϕ is cyclic, implying that the commutator subgroup N of $\mathrm{Mod}(S)$ is contained in the kernel of ϕ. Therefore, $\varphi(N)$ is contained in IA_n. Since IA_n is torsion–free and since $\sigma^2 \in N$ is a torsion element (of order 3), we have $\varphi(\sigma^2) = 1$. It follows from Lemma 6.1 that $\varphi(N)$ is trivial. Consequently, φ factors through the natural homomorphism $\mathrm{Mod}(S) \to H_1(\mathrm{Mod}(S), \mathbb{Z})$, and so the image of φ is a quotient of $\mathbb{Z}_{10}$.

This completes the proof of Theorem 5 for $\mathrm{Aut}(F_n)$.

The case $H = \mathrm{Out}(F_n)$ is completely similar and uses the fact that the subgroup $IA_n/\mathrm{Inn}(F_n)$ is torsion–free (c.f. [1]). $\square$

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