Non-linear periodic response analysis of mistuned bladed disk assemblies in modal domain

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1 ABSTRACT

Mistuning phenomenon has been studied extensively for almost half a century, especially for bladed disk assemblies. However, the studies hitherto focus on either linear models with distributed parameter mistuning or nonlinear models with point mistuning. The former method is not realistic under significant non-linear effects. Whereas the latter method lacks accuracy since discrete mistuning elements distort the mode shapes unless their numbers are large. Therefore a model which includes both nonlinearities and distributed parameter mistuning is required. In this study, a formulation for the analysis of mistuned bladed disk assemblies under periodic loads in the presence of distributed parameter mistuning and nonlinearity is given. The proposed method combines the component mode synthesis based reduced order modeling approach with non-linear forced response analysis technique in modal space. The calculations are carried out in modal domain which reduces the computational effort considerably, especially for large size finite element (FE) models. The mistuning is imposed on individual blade natural frequencies, which is more realistic compared to adding discrete mistuning elements. A case study is presented to demonstrate the application of the method and the effect of macro-slip friction type nonlinearity on the dynamic analysis of a mistuned bladed disk assembly. It is concluded that considering non-linear effects in the dynamic analysis of mistuned bladed disks is crucial when there exists significant non-linearity, such as gaps and friction dampers in the system. It is believed that this is the first study in which non-linear dynamic analysis of bladed disk assemblies is carried out with distributed parameter mistuning.

2 INTRODUCTION

Forced response predictions of bladed disk assemblies have been the main focus of researchers for the last few decades. Due to their cyclic symmetric structure, slight differences in structural properties and geometry of blades in bladed disk assemblies may cause significant increases in vibration amplitudes. This weakens the robustness and durability of the structure. In order to predict such responses, mistuning can be imposed on FE models. But these models usually have large numbers of degrees of freedom (DOF) requiring enormous amount of computational time for response predictions. In addition, FE analyses can no longer use a single sector of the assembly for the dynamic analysis of the mistuned assembly, since mistuning destroys the cyclic symmetric property.

Studies hitherto can be grouped by the complexity of the models used, as several DOF per sector models [1-4] and FE models [5-17]. Studies using FE approach aim to construct reduced order models which provide faster results. In some studies, frequency response function (FRF) coupling or updating techniques were employed on the modal analysis results of the tuned FE model. Yang and Griffin [14] suggested coupling the tuned disk and mistuned blades via FRF coupling. A transformation which expresses the blade-disk interface motion by a number of translational and rotational rigid body modes is applied to reduce the number of connection DOF. Yet, the FRF matrix for each DOF on the blade should be calculated for each point in the frequency range.

Petrov et al. [17] proposed a frequency domain approach which is based on FRF updating. The mistuned FRF is determined by applying linear mistuning such as stiffness and point mass as a modification on a number of DOF of the FE model of a tuned assembly. In that study, keeping the modal vector data for the DOF at which the modification is applied will be sufficient. Consequently, if it is desired to apply distributed mistuning throughout the blade, each DOF in the blade should be kept. Such an approach will be computationally expensive when fine meshes are used.

In another group of studies, reduced order models in modal domain were introduced [5-11]. Óttarsson et al. [6] developed a model in which the mistuned mode shapes are represented as a combination of disk mode shapes and cantilevered blade mode shapes. Method proves to yield accurate results when the cantilevered blade natural frequencies are tuned to compensate the stiffening caused by the fixed boundary conditions. Bladh et al. [7] extended the formulation for shrouded blade disk assemblies. On the other hand, Yang and Griffin [11, 12] suggested reducing the mistuned system by using a subset of nominal modes (SNM) of the tuned system. Mistuning is applied as linear modification matrices which are transformed into the modal domain using the subset of selected modes. They also developed a formulation for transient response calculations using the reduced order model [13]. Later, Bladh [10] developed a formulation which combines a similar approach as SNM method with fixed interface coupling method which is called Craig-Bampton method [22]. Using this formulation, mistuning is applied on the cantilevered natural frequencies of the blades.

Non-linear response of bladed disk assemblies has been studied extensively in frequency domain [e.g. see 24-28]. Recently, Petrov [18] developed formulations for typical non-linear elements and used numerical path following methods to trace the solution in frequency domain where multi-harmonics were included. The model is reduced using Craig-Bampton method and non-linear elements are connected between statically condensed DOF. This reduction aims to represent the contact interface motion more accurately under non-linear loads since it may not be accurate enough when expressed by a combination of nominal modes of the system. On the other hand, Cigeroglu et al. [19] implemented a new two-dimensional micro-slip friction model to a bladed disk.

The objective of this study is to address the question of how non-linearity and mistuning affect the displacement and stress patterns in bladed disk assemblies. Proposed technique avoids using point mistuning elements since in order to apply mistuning throughout the blade, a large number of DOF should be kept in the model. Instead, it is suggest using a reduced order model. The mode shapes of the mistuned bladed disk are calculated by Craig-Bampton method as formulated by Bladh [21]. The non-linear periodic response is calculated in modal domain, by using the method suggested by Kuran and Özgüven [20]. Numerical path following approach based on Newton method is applied in the solution. The purpose of the proposed approach is to calculate non-linear response of mistuned bladed disk assemblies with mistuning applied on the individual blade natural frequencies. Since the response is calculated in modal domain, the size of the problem is reduced to the number of modes retained irrespective of the complexity of the FE model used.

3 THEORY

3.1 Reduced Order Model

It is a well known fact that mode shapes of a mistuned bladed disk assembly are closely related to the natural frequencies of individual blades attached to the disk. It is the mistuning pattern, which is the set of deviations of natural frequencies of each blade from the ideally tuned case, that mainly affects the mode shapes of the mistuned bladed disks. Therefore a formulation which is capable of mistuning individual blade natural frequencies rather than adding point mistuning elements is to be preferred.

In this study a component mode synthesis method, namely the Craig-Bampton method [22], is used for model reduction. Bladh [21] employed the Craig-Bampton method for mistuned bladed disk assemblies and obtained a reduced order model in which it is possible to impose mistuning on cantilevered natural frequencies of each blade in the assembly. The mass and stiffness matrices, $\left[M^{cb}\right]$ and $\left[K^{cb}\right]$, respectively, of the reduced order model are given below using the original notation used in [21]:

$$\left[M^{cb}\right]=\begin{bmatrix} [I] & [\tilde{\mu}_{dc}] & [0] \\ [\tilde{\mu}_{dc}]^T & [\tilde{\mu}_{cc,d}]+[I]\otimes[\mu_{cc,b}] & [\hat{F}]^T\left[[I]\otimes[\mu_{bc}]^T\right] \\ [0] & \left[[I]\otimes[\mu_{bc}]\right][\hat{F}] & [I] \end{bmatrix} \tag{1}$$

$$\left[K^{cb}\right]=\begin{bmatrix} [\tilde{\Lambda}_d] & [0] & [0] \\ [0] & [\tilde{\kappa}_{cc,d}]+[I]\otimes[\kappa_{cc,b}] & [0] \\ [0] & [0] & [I]\otimes[\Lambda_b] \end{bmatrix} \tag{2}$$

Here $[I]$ is the identity matrix, $[0]$ is the zero matrix, $[\widetilde{\Lambda}_d]$ is the modal stiffness matrix of the disk, $[\Lambda_b]$ is the modal stiffness matrix of a single cantilevered blade, $[\mu]$ and $[\kappa]$ are the reduced mass and stiffness matrices, respectively, $[\hat{F}]$ is the transformation matrix from cyclic coordinates to physical coordinates, and $\otimes$ denotes the Kronecker product. Subscripts b, d and c represent the blade, the disk and the connection DOF, respectively.

Note that the lower-right element of the stiffness matrix is a diagonal matrix. It contains the modal stiffness values of each blade in the assembly, which makes modal stiffness mistuning possible. In order to impose mistuning, cantilevered blade modal stiffness values on the diagonal elements of $[I]\otimes[\Lambda_b]$ are perturbed as:

$$\underset{n=1,\ldots,N}{\mathrm{Bdiag}}\left[\underset{m=1,\ldots,m_b}{\mathrm{diag}}\left(1+\delta_n^k\right)\left[\Lambda_b\right]\right] \quad (3)$$

where $Bdiag[\bullet]$ denotes the block diagonal matrix, the argument being the n^{th} (n=1,2,…) diagonal block. The details of the formulation are given in reference [21].

3.2 Non-linear Forced Response in Modal Domain

The equation of motion of a non-linear system can be expressed for harmonic vibrations as:

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}+i[H]\{x\}+\{f_{NL}\}=\{f\} \quad (4)$$

where [M], [K], [C] and [H] are the mass, stiffness, viscous damping and structural damping matrices, $\{x\}$, $\{\dot{x}\}$ and $\{\ddot{x}\}$ are the vectors of physical displacements, velocities and accelerations, respectively. $\{f_{NL}\}$ is the vector of non-linear internal forces and $\{f\}$ is the vector of external forces. i is the unit imaginary number.

For a periodic excitation at fundamental frequency ω, the response can be assumed as a summation of harmonic responses at ω and its harmonics as:

$$\{x\}=\sum_{m=0}^{\infty}\{X\}_m e^{im\omega t} \quad (5)$$

Then the non-linear internal force due to non-linear elements can also be written as:

$$\{f_{NL}\}=\sum_{m=0}^{\infty}\{F_{NL}\}_m e^{im\omega t} \quad (6)$$

Knowing that the aerodynamic forces on blades may not be expressed by a single harmonic function only, the external forcing term can also be written as a summation of several harmonic functions as:

$$\{f\}=\sum_{m=0}^{\infty}\{F\}_m e^{im\omega t} \quad (7)$$

Substituting equations (5), (6) and (7) into the equation (4) we obtain:

$$[K]\{X\}_0 + \{F_{NL}\}_0 = \{F\}_0 \tag{8}$$

$$\left[-(m\omega)^2[M] + i(m\omega)[C] + [K] + i[H]\right]\{X\}_m + \{F_{NL}\}_m = \{F\}_m \text{ for m = 1, 2…} \tag{9}$$

Expressing the physical displacements as a linear combination of mode shapes as:

$$\{X\}_m = [\phi]\{p\}_m \tag{10}$$

and substituting equation (10) into equations (8) and (9) we obtain:

$$\left[\omega_r^2\right]\{p\}_0 + \{\overline{F}_{NL}\}_0 = \{\overline{F}\}_0 \tag{11}$$

$$\left[-(m\omega)^2[I] + i(m\omega)[\overline{C}] + \left[\omega_r^2\right] + i[\overline{H}]\right]\{p\}_m + \{\overline{F}_{NL}\}_m = \{\overline{F}\}_m \text{ for m = 1, 2…} \tag{12}$$

where

$$\{X\}_m = [\phi]\{p\}_m \,, \qquad [\overline{C}] = [\phi]^T[C][\phi], \quad [\overline{H}] = [\phi]^T[H][\phi],$$

$$\{\overline{F}_{NL}\}_m = [\phi]^T\{F_{NL}\}_m, \quad \{\overline{F}\}_m = [\phi]^T\{F\}_m$$

It should be noted that non-linear forces occur only at DOF where non-linear elements are connected to, which will be referred to as “non-linear DOF”. Therefore, in order to transform the non-linear forcing term $\{\overline{F}_{NL}\}_m$ into modal domain to obtain $\{F_{NL}\}_m$, only the mode shapes of the non-linear DOF are required. Thus, the internal non-linear forcing vector can be rewritten as:

$$\{\overline{F}_{NL}\}_m = [\phi_{NL}]^T\{F_{NL,NL}\}_m \tag{13}$$

where $\{F_{NL,NL}\}$ is the vector of internal non-linear forcing at non-linear DOF and $[\phi_{NL}]$ is the mode shape matrix which belongs to the non-linear DOF. In the original formulation of Kuran and Özgüven [20] the internal non-linear force vector is expressed as a matrix composed of describing functions representing nonlinearity, multiplied by the response vector. Then the response of the system is calculated by solving equations (11) and (12) iteratively. The details of the modal analysis of non-linear systems can be found in reference [20].

3.3 Numerical Solution with Path Following

In this study, the forced response of the system is calculated by using Newton solution procedure. Arranging the terms in the equations (11) and (12) we obtain:

$$\{R_0(\{p\}_0,\omega)\} = \left[\omega_r^2\right]\{p\}_0 + \{\overline{F}_{NL}\}_0 - \{\overline{F}\}_0 = \{0\} \tag{14}$$

$$\{R_m(\{p\}_m,\omega)\} = \left[-(m\omega)^2[I] + i(m\omega)[\overline{C}] + \left[\omega_r^2\right] + i[\overline{H}]\right]\{p\}_m + \{\overline{F}_{NL}\}_m - \{\overline{F}\}_m = \{0\} \tag{15}$$

Combining the modal displacement vectors into a single vector as:

$$\{p\} = \left[\{p\}_0^T \quad \{p\}_1^T \quad \{p\}_2^T \quad \cdots\right]^T \tag{16}$$

and collecting the functions into a single function, we obtain:

$$\{R(\{p\},\omega)\} = \left[\{R_0(\{p\}_0,\omega)\}^T \quad \{R_1(\{p\}_1,\omega)\}^T \quad \{R_2(\{p\}_2,\omega)\}^T \quad \cdots\right]^T \tag{17}$$

Then a Newton solution procedure is applied to solve for the modal displacements. For each frequency, ω_k, iteration is applied as follows:

$$\{p\}_k^{i+1} = \{p\}_k^i - \left[\frac{\partial\{R(\{p\},\omega)\}}{\partial\{p\}}\right]^{-1}\Bigg|_{\{p\}_k^i,\omega_k} \left\{R\left(\{p\}_k^i,\omega_k\right)\right\} \tag{18}$$

where $\{p\}_k^i$ is the modal solution vector at k^th^ frequency ω_k and at i^th^ iteration, $\left[\frac{\partial\{R(\{p\},\omega)\}}{\partial\{p\}}\right]$ is the Jacobian matrix, and $\left\{R\left(\{p\}_k^i,\omega_k\right)\right\}$ is the function evaluated at $\{p\}_k^i$ and ω_k.

For non-linear systems, there may be multiple solutions for a single *ω*. In such cases, there will be some points on the solution curve at which the Jacobian matrix is singular [23]. Then a numerical continuation procedure is required to trace the solution, which can be achieved by introducing a new equation

$$\{\Delta q\}_k^{i\,T}\{\Delta q\}_k^i = s^2 \tag{19}$$

that makes the Jacobian matrix non-singular where

$$\{\Delta q\}_k^i = \left\{\{q\}_k^i - \{q\}_{k-1}\right\} \tag{20}$$

and

$$\{q\} = \begin{Bmatrix} \{p\} \\ \omega \end{Bmatrix} \tag{21}$$

Now here k represents a solution point on the curve. Equation (19) guarantees that the k^th^ solution lies on a hyper-sphere centered at $\{q\}_{k-1}$ having a radius of *s*.

Arranging the terms in equation (19) to define a new function

$$\left\{g\left(\{p\}_k^i,\omega_k^i\right)\right\} = \{\Delta q\}_k^{i\,T}\{\Delta q\}_k^i - s^2 = \{0\} \tag{22}$$

The Newton iteration becomes:

$$\{q\}_k^{i+1} = \{q\}_k^i - \begin{bmatrix} \frac{\partial\{R(\{p\},\omega)\}}{\partial\{p\}} & \frac{\partial\{R(\{p\},\omega)\}}{\partial\omega} \\ \frac{\partial g(\{p\},\omega)}{\partial\{p\}} & \frac{\partial g(\{p\},\omega)}{\partial\omega} \end{bmatrix}^{-1}\Bigg|_{\{p\}_k^i,\omega_k^i} \begin{Bmatrix} \left\{R\left(\{p\}_k^i,\omega_k^i\right)\right\} \\ g\left(\{p\}_k^i,\omega_k^i\right) \end{Bmatrix} \tag{23}$$

where $\begin{bmatrix} \dfrac{\partial\{R(\{p\},\omega)\}}{\partial\{p\}} & \dfrac{\partial\{R(\{p\},\omega)\}}{\partial\omega} \\ \dfrac{\partial g(\{p\},\omega)}{\partial\{p\}} & \dfrac{\partial g(\{p\},\omega)}{\partial\omega} \end{bmatrix}$ is the new Jacobian matrix which is not singular.

During the solution procedure, first order estimators, which are calculated by using the Jacobian inverse found at the previously converged solution, are used. The details of the numerical continuation methods can be found in reference [23].

4 CASE STUDY

In this section a case study is presented to demonstrate the application of the analysis method proposed. A bladed disk assembly which has 24 sectors with an angle of 15° per sector is used. The isometric view of the FE model of a single sector is given in Figure 1. The model is formed by eight node brick elements. The blades are of mid-shroud type. The disk is constrained at the nodes on the inner rim.

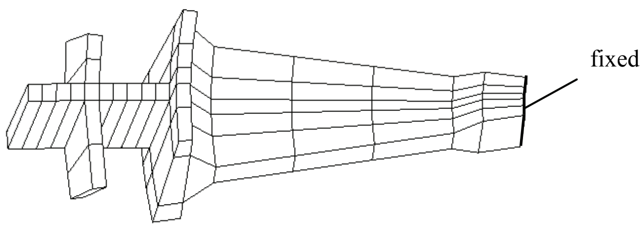

Fig 1 - FE model of a single sector

To construct the reduced order model, 20 modes for each blade and 260 modes (20 modes for each nodal diameter) for the disk are used. The resulting reduced order model has 1028 DOF. All finite element work is done in ANSYS.

As an example to non-linearity in the system, the friction between shrouds of adjacent blades is included into the analysis. The details of the formulation of the macro-slip friction element under constant normal load are given in reference [18]. Note that the dry friction between contact surfaces under changing normal loads can also be included into the analysis. The external forcing on the system is taken only in the axial direction of the disk. 3 engine-order (EO) single harmonic forcing is applied on all the nodes located at blade tips. The response is calculated using the first 50 modes.

The linear response of the tuned assembly in tangential direction at the tip of the first blade is given in Figure 2.

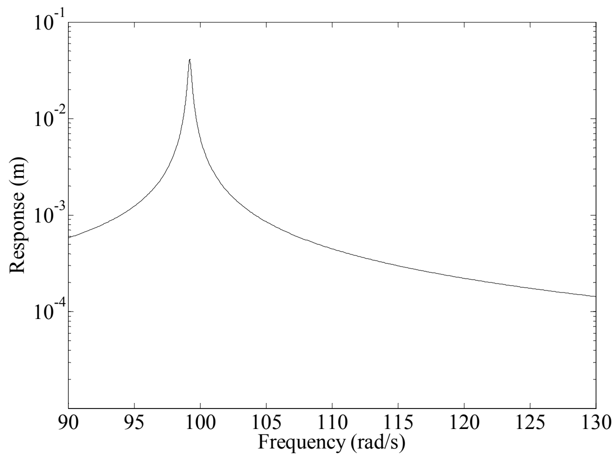

Fig 2 - Linear response of the tuned assembly in axial direction

To study the effect of mistuning, random deviations in the range of ± 5% are applied to the modal stiffness values of individual blades. The linear response of the mistuned assembly at the same DOF as above is given in Figure 3. Note that a number of new resonances have appeared whilst the fundamental resonance split into two. This is due to the fact that a 3EO excitation can excite only 3 nodal diameter (ND) modes of a tuned assembly, whereas in a mistuned case this is not so, since the whole structure will lose its cyclic symmetry. Therefore, a 3EO excitation can excite all the modes in a mistuned assembly.

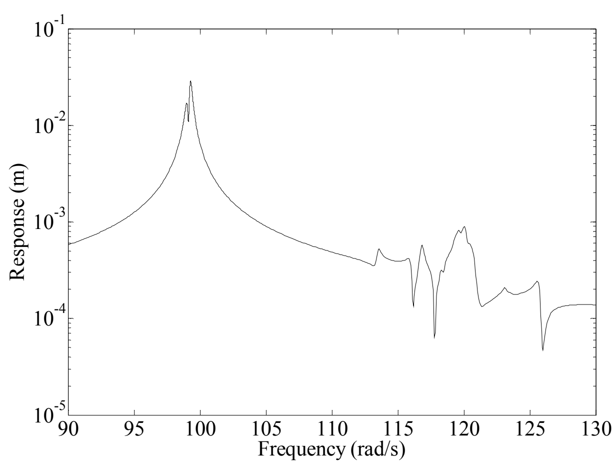

Fig 3 - Linear response of the mistuned assembly in axial direction

In order to consider the non-linearity in the system due to the friction between shrouds of adjacent blades, the relative motion between contact surfaces in normal direction is neglected and constant normal load assumption is made. Macro-slip friction elements with normal load values between 1N and 100N are used. Then both tuned and mistuned responses of the same blade are calculated. The mistuning pattern used in linear case is also employed here. The non-linear responses of tuned and mistuned bladed disk assemblies are given in Figures 4 and 5, respectively.

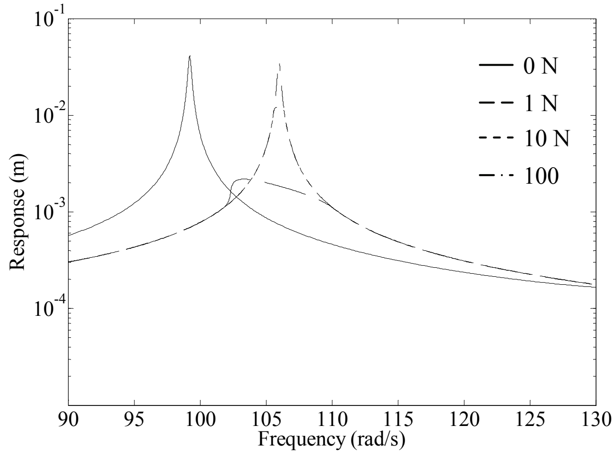

Fig 4 - Non-linear response of the tuned assembly in axial direction for various normal load values

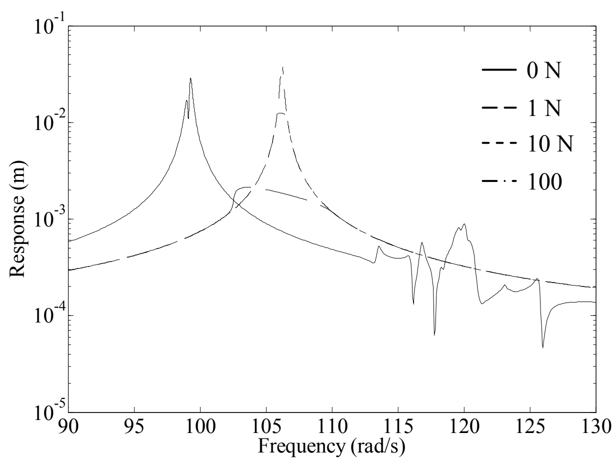

Fig 5 - Non-linear response of the mistuned assembly in axial direction for various normal load values

As demonstrated above, friction type nonlinearity can alter the forced response of both tuned and mistuned bladed disk assemblies significantly both in peak magnitudes and resonant frequencies.

5 CONCLUSION

In this study, a new approach is suggested for the dynamic analysis of bladed disk assemblies under periodic loads in the presence of distributed parameter mistuning and nonlinearity. To the best of our knowledge, this is the first study which uses distributed parameter mistuning in non-linear dynamic analysis. The method proposed combines the component mode synthesis based reduced order modeling approach with non-linear forced response analysis technique in modal space. Mistuning is imposed on individual blade natural frequencies of the reduced order model. Path following approach based on Newton method is employed in numerical solutions.

In the case study presented, a mistuned bladed disk assembly with friction type non-linearity between shrouds of adjacent blades is considered. Both linear and non-linear analyses of the system are carried out, and the forced responses of a mistuned blade under periodic loads are calculated. In both analyses several normal load values for macro-slip friction are employed. By comparing the linear and non-linear forced responses obtained in the case study, it is demonstrated that non-linearity can change the response of the system considerably.

Note that the current state-of-the-art suggests either adding point mistuning elements to FE model of the system or using modal synthesis and perturbing individual blade natural frequencies for the dynamic analysis of mistuned bladed disk assemblies. In the former method, in order to have realistic models, fine meshing is required and mistuning should be applied on a large number of DOF. The latter method has been applied only for linear systems. The method proposed in this study has the advantage of making non-linear analysis while keeping the benefits of using modal synthesis and thus applying distributed parameter mistuning. Making the non-linear solution in modal domain reduces computational time drastically.

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