DICHOTOMY THEOREMS AND FRUCHT THEOREM IN DESCRIPTIVE GRAPH COMBINATORICS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

ONUR BİLGE

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

AUGUST 2023

Approval of the thesis:

DICHOTOMY THEOREMS AND FRUCHT THEOREM IN DESCRIPTIVE GRAPH COMBINATORICS

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ABSTRACT

DICHOTOMY THEOREMS AND FRUCHT THEOREM IN DESCRIPTIVE GRAPH COMBINATORICS

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August 2023, 40 pages

Descriptive graph combinatorics studies graph-theoretic concepts under definable constraints. The systematic study of the field was started by Kechris, Solecki and Todorčević and the field has been mainly focused on Borel chromatic numbers and Borel matchings.

One of the major investigations in the field has been about finding conditions for a definable graph to have a specific Borel chromatic number. The $\mathbf{G}_0$ dichotomy theorem is one such theorem for graphs with uncountable Borel chromatic numbers. After the first proof of this dichotomy theorem, a classical proof was found by Ben Miller. Later, Carroy, Miller, Schrittesser and Vidnyánszky used this technique to prove the $\mathbf{L}_0$ dichotomy theorem, an analogue of the $\mathbf{G}_0$ dichotomy theorem for Borel chromatic number at least three. In the first part of this thesis, we provide a survey of these results.

In the second part, we will be concerned with definable automorphism groups of definable graphs. In classical graph theory, one of the most prominent theorems in the study of automorphism groups of graphs is Frucht theorem that states that any group can be realized as the automorphism group of a graph. We will prove that Frucht theorem generalizes to both topological and Borel measurable setting. More specifically, we shall show that every standard Borel group (respectively, Polish group) can be realized as the Borel (respectively, homeomorphic) automorphism group of a Borel graph on a standard Borel (respectively, Polish) space.

Keywords: Borel chromatic number, dichotomy, Borel graph automorphism, Frucht's theorem

ÖZ

BETİMSEL ÇİZGİ KOMBİNATORİĞİNDE DİKOTOMİ TEOREMLERİ VE FRUCHT TEOREMİ

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Ağustos 2023 , 40 sayfa

Betimsel çizge kombinatoriği, çizge teorisi kavramlarını tanımlanabilir kısıtlamalar altında inceler. Alanın sistematik çalışması Kechris, Solecki ve Todorčević tarafından başlatılmıştır ve bu alan, genellikle Borel kromatik sayılarına ve Borel eşleşmelerine odaklanmıştır.

Alandaki en büyük araştırmalardan birisi, tanımlanabilir bir çizgenin belirli bir Borel kromatik numarasına sahip olması için çeşitli koşulların bulunmasıyla ilgilidir. $\mathbf{G}_0$ dikotomi teoremi, sayılamayan Borel kromatik sayısına ait çizgeler için bu konuda bir teoremdir. Bu dikotomi teoreminin ilk ispatından sonra, Ben Miller tarafından klasik yaklaşımla yeni bir kanıt bulunmuştur. Daha sonra, Carroy, Miller, Schrittesser ve Vidnyánszky bu tekniği kullanarak $\mathbf{G}_0$ dikotomi teoreminin en az üç Borel kromatik sayısı için benzeri olan $\mathbf{L}_0$ dikotomi teoremini kanıtladı. Bu tezin ilk bölümünde, bu sonuçların bir özetini sunacağız.

Bu tezin ikinci bölümünde, tanımlanabilir çizgelerin tanımlanabilir otomorfizm grupları ile ilgileneceğiz. Klasik çizge teorisinde, çizgelerin otomorfizm grupları ile ilgili en öne çıkan teoremlerden birisi, her grubun bir çizgenin otomorfizm grubu olduğunu ifade eden Frucht teoremidir. Frucht teoreminin topolojik ve Borel ölçülebilir çevrede genelleştirmelerini kanıtlayacağız. Özellikle, her standard Borel grubun (sırasıyla, Polish grup) bir standard Borel (sırasıyla, Polish) uzay üzerindeki Borel çizgenin Borel (sırasıyla, homeomorfik) otomorfizm grubu olduğunu kanıtlayacağız.

Anahtar Kelimeler: Borel kromatik sayısı, dikotomi, Borel çizge otomorfizmi, Frucht teoremi

To them who brought me this far.

ACKNOWLEDGMENTS

I would like to thank my family and my dear Aylin for their unconditional support and love.

I would like to thank my advisor for his relentless efforts and valuable guidance.

I would also like to thank Ali Doğanaksoy for being a role model for me and for his support.

Some of the main results and some parts of this thesis will appear verbatim in a joint publication with Burak Kaya [1].

This work is supported by TÜBİTAK through 2210-E scholarship.

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CHAPTER 1

INTRODUCTION

1.1 Descriptive Graph Combinatorics and Preliminaries

Descriptive graph combinatorics is a recently developed field of mathematics that lies in the intersection of descriptive set theory and graph theory.

Descriptive set theory is the study of *Polish spaces* i.e., separable completely metrizable topological space, and their “definable" subsets such as Borel, analytic etc. sets. Recall that a subset of a Polish space is *Borel* if it is in the smallest σ-algebra generated by open sets; and a subset of a Polish space is called *analytic* if it is the continuous image of a Borel subset of another Polish space. For a general review of descriptive set theory, we refer the reader to [10] and [17].

The main purpose in descriptive graph combinatorics is to investigate how the behavior of graphs changes under definable constraints, i.e., how classical results in graph theory extend if one requires various graph-theoretic objects such as edge relations, colorings, matchings to be Borel, analytic etc. This was first systematically studied in [9], although one may find some isolated prior results on this theme. For a general review of the field that collects almost all current results, we refer the reader to [11].

In [9], it was shown that several fundamental results in classical graph theory cannot be extended to Borel measurable setting. This is mainly due to the fact that, when dealing with uncountable graphs, one has to use the axiom of choice to obtain certain colorings and matchings. Before proceeding further, we shall provide some preliminary definitions.

1.2 Preliminaries

Let us first recall some basic definitions from classical graph theory in the abstract setting. We suggest the reader [5] for further background in graph theory.

A *graph* $\mathbf{G}$ is a pair (X, G) where X is a set and G is an irreflexive, symmetric relation defined on $X \times X$. Here we call X the *vertex set* of the graph and G the *edge relation* of the graph. We call two vertices $x, y \in X$ *adjacent* if $(x, y) \in G$ and we call an edge $(x, y) \in G$ is *incident* to a vertex $z \in X$ if either $x = z$ or $y = z$.

Fix a graph $\mathbf{G} = (X, G)$. Let $n \in \mathbb{N}^+$. A *path of length* n in $\mathbf{G}$ is a sequence $(x_0, x_1, \ldots, x_n)$ of vertices in X such that $(x_i, x_{i+1}) \in G$ for all $0 \leq i < n$. A path $(x_0, x_1, \ldots, x_n)$ is called *simple* if $x_i \neq x_j$ for all $0 \leq i \neq j \leq n$. A path $(x_0, x_1, \ldots, x_n)$ is called a *cycle* if $x_0 = x_n$; and such a cycle is called *simple* if the path $(x_0, x_1, \ldots, x_{n-1})$ is simple. The graph $\mathbf{G}$ is called *acyclic* if there are no simple cycles of length $n \geq 3$.

Consider the equivalence relation E_G given by xE_Gy if and only if there is a path from x to y in $\mathbf{G}$. The equivalence classes of this equivalence relation are called the *connected components* of $\mathbf{G}$. The graph $\mathbf{G}$ is called *connected* if there is only one connected component.

The *degree* of a vertex is the number of vertices adjacent to it. $\Delta(\mathbf{G})$ denotes the least upper bound of degrees of vertices. A graph $\mathbf{G}$ is called to be *bounded degree* if $\Delta(\mathbf{G}) \leq n$ for some $n \in \mathbb{N}$, it is called *locally finite* if each degree is finite and it is called *locally countable* if $\Delta(\mathbf{G}) \leq \aleph_0$.

An *independent set* A in a graph $\mathbf{G} = (X, G)$ is a subset of X such that no two vertices in A are adjacent. A Y*-coloring* of a graph $\mathbf{G} = (X, G)$ is defined to be a function $c : X \to Y$ such that $(x, y) \in G \implies c(x) \neq c(y)$. Hence, c is a coloring of X such that for all $y \in Y$, $c^{-1}(y)$ is an independent set in $\mathbf{G}$. The *chromatic number* of $\mathbf{G}$, denoted by $\chi(\mathbf{G})$, is the smallest cardinality of a set Y such that there is a Y-coloring $c : X \to Y$.

We define the *line graph* of $\mathbf{G}$, denoted by $L(\mathbf{G}) = (\breve{X}, \breve{G})$, as follows:

- $\breve{X}$ is the set of edges of $\mathbf{G}$ seen as two element subsets of X. Notice that the pairs (x, y) and (y, x) represent the same edge, so $\{x, y\}$ represents a single vertex in $\breve{X}$.
- Two vertices in $\breve{X}$ are adjacent if and only if their corresponding edges in G are incident to a common vertex in X.

An *edge coloring* of $\mathbf{G}$ is a coloring of $L(\mathbf{G})$. By the *edge chromatic number* of $\mathbf{G}$, denoted by $\chi'(\mathbf{G})$, we mean the chromatic number of $L(\mathbf{G})$.

Given two graphs $\mathbf{G} = (X, G)$ and $\mathbf{H} = (Y, H)$, a *homomorphism* from $\mathbf{G}$ into $\mathbf{H}$ is a function $f : X \to Y$ such that if $(x_1, x_2) \in G$, then $(f(x_1), f(x_2)) \in H$. So, a graph homomorphism is a function between two graphs that preserves the adjacency relationships. An *automorphism* of $\mathbf{G}$ is a homomorphism from $\mathbf{G}$ into $\mathbf{G}$ which is a bijective map whose inverse is also a homomorphism.

The set of all automorphisms of the graph $\mathbf{G}$ forms a group under composition of functions. This group is called the *automorphism group* of $\mathbf{G}$ and is denoted by $\mathrm{Aut}(\mathbf{G})$.

Let us now provide the Borel measurable counterparts of these notions on a standard Borel space.

Recall that a *standard Borel space* is defined as a measurable space $(X, \mathcal{B})$ such that $\mathcal{B}$ is the Borel σ-algebra of some Polish topology on X. A *Borel graph* (respectively, analytic) on a standard Borel space $(X, \mathcal{B})$ is a graph $\mathbf{G} = (X, G)$ where the edge relation $G \subseteq X \times X$ is a Borel (analytic) subset of the product measurable space.

For the remaining definitions, assume that X and Y are standard Borel spaces. A coloring $c : X \to Y$ is called a *Borel n-coloring* if $|Y| = n$ and c is a Borel map. The *Borel chromatic number* of $\mathbf{G}$, denoted by $\chi_B(\mathbf{G})$, is defined as

$$\chi_B(\mathbf{G}) = \min\{|Y| : \text{ there exists a Borel coloring } c : X \to Y\}.$$

Observe that by Borel isomorphism theorem in [17, Theorem 3.3.13], if $|Y|$ is uncountable, then $|Y| = 2^{\aleph_0}$. So, the possible values for Borel chromatic number are in $\{1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$.

Recall that, given a topological space X, the *Effros Borel space* of X is the measurable space defined on the set $F(X)$ of closed subsets of X endowed with the σ-algebra generated by the collection

$$\{F \in F(X) : F \cap U \neq \emptyset\}$$

where U ranges over all open subsets of X. The Effros Borel space is a standard Borel space whenever X is Polish [10, Theorem 12.6].

Observe that, for a line graph $L(\mathbf{G}) = (\breve{X}, \breve{G})$ of a Borel graph $\mathbf{G}$, the set $\breve{X}$ can be endowed with a standard Borel structure induced from the Effros Borel space of X. We define a *Borel edge coloring* of $\mathbf{G}$ to be a *Borel coloring* of $L(\mathbf{G})$ and the *Borel edge chromatic number* of $\mathbf{G}$ to be the Borel chromatic number of $L(\mathbf{G})$, denoted by $\chi'_B(\mathbf{G})$.

The group of automorphisms of $\mathbf{G}$ consisting of all automorphisms that are Borel functions will be denoted by $\mathrm{Aut}_B(\mathbf{G})$. In the case that $\mathbf{G}$ is a Borel graph on a Polish space (X, τ), the group of automorphisms of $\mathbf{G}$ consisting of automorphisms that are homeomorphisms only, will be denoted by $\mathrm{Aut}_h(\mathbf{G})$.

1.3 Borel vs Non-Borel

It is an interesting phenomenon that, while some results in classical graph theory still hold when definable constraints are applied, some results do not generalize and indeed fail drastically. If analyzed, one sees that this is mostly due to uses of the axiom of choice and the ineligibility to choose a Borel transversal for a Borel equivalence relation.

We will now exemplify this phenomenon using one of the most simple folklore results regarding chromatic numbers.

PROPOSITION 1. *A graph $\mathbf{G} = (X, G)$ is 2-colorable if and only if it has no odd cycles.*

Proof. Assume that $\mathbf{G}$ is 2-colorable with a coloring function c, using colors 0 and 1 and assume that $\mathbf{G}$ has an odd cycle $x_1, x_2, \ldots, x_{2m+1}, x_1$. Assume without loss of generality that $c(x_1) = 0$. Then, we must have $c(x_k) = 0$ for every odd k and $c(x_l) = 1$ for every even l. Hence, $c(x_1) = x(x_{2m+1})$ which is a contradiction as these two vertices are adjacent. So, if there is a 2-coloring, then there cannot be any odd cycle.

Now, assume that the graph does not contain any odd cycle. Suppose initially that **G** is connected. Pick a vertex $x \in G$. Let A denote the set of vertices in X such that the shortest path from every vertex to x is of odd length. Let B denote the set of vertices in X such that the shortest path from every vertex to x is of even length. The shortest path is either even or odd so $A \cap B = \emptyset$ and also note that $x \in B$. Assume that there are two vertices $a_0, a_1 \in A$ such that $(a_0, a_1) \in G$. Then, combining the two shortest path from these two vertices to x, we obtain a path as follows: $(x, \ldots, a_0, a_1, \ldots, x)$ which is of odd length. Every graph that contains a closed path also contains an odd cycle. But then, **G** has odd cycle which is a contradiction. So, there are no adjacent vertices in A. By a similar argument there are no adjacent vertices in B. Hence, A and B are independent sets and there is a 2-coloring.

If the graph is disconnected i.e., there are more than one connected component, one can pick a vertex from each connected component and use the same method to find a 2-coloring on each connected component to find a 2-coloring of the graph. □

Note that in the case that there are uncountable many connected components, one has to use the axiom of choice to pick a vertex from each connected component to use this proof.

One can ask whether, if a Borel graph has no odd cycles, it has Borel chromatic number 2. In general, this turns out to be not the case. Consider the following example.

EXAMPLE 2 ([9]). *Let* $\mathbf{G} = (X, G)$ *where* $X = \mathbb{R}$ *and the edge relation* G *is defined by* $(x, y) \in G$ *if and only if* $|x - y| = 3^k$ *for some* $k \in \mathbb{Z}$.

This edge relation of this graph on $\mathbb{R}$ *can be defined as follows:*

$$G = \bigcup_{n \in \mathbb{Z}} \{(x, y) : x, y \in \mathbb{R}, |x - y| = 3^n\}$$

which is a countable union of closed sets. Hence, G *is a Borel (indeed, a* F_σ*) subset of* $\mathbb{R} \times \mathbb{R}$.

This graph contains no odd cycles. To show this, let $x_0, x_1, \ldots, x_n$ be a cycle of length n. For each $i = 1, 2, \ldots, n$, $x_i = x_{i-1} + \delta_i 3^{k_i}$ where $\delta_i = -1, 1$ and $k_i \in \mathbb{Z}$. This sequence of vertices construct a cycle so we have the sum $\sum_i \delta_i 3^{k_i} = 0$. For a sufficiently large N, we have $\sum_i \delta_i 3^{k_i+N} = 0$ and this sum is a sum of odd integers. Hence, there must be even number of terms which implies that the cycle has to be even.

PROPOSITION 3 ([9]). *$\chi(\mathbf{G}) = 2$ and $\chi_B(\mathbf{G}) = 2^{\aleph_0}$.*

Proof. Since the graph does not contain any odd cycles, by 1, $\chi(\mathbf{G}) = 2$.

Now, assume that the Borel chromatic number of $\mathbf{G}$ is countable and $c : \mathbb{R} \to \mathbb{N}$ is a Borel coloring. Then, $\mathbb{R} = A_1 \cup A_2 \cup \ldots$ where $A_i = c^{-1}(i)$ and for some $k \in \mathbb{N}$, A_k must have positive Lebesgue measure. A theorem of Steinhaus [18, Théorème VII] states that if $A \subseteq \mathbb{R}$ is of positive measure, then $A - A = \{a_1 - a_2 : a_1, a_2 \in A\}$ contains an open interval $(-\epsilon, \epsilon)$ around 0. We can find $N \in \mathbb{Z}$ such that $0 < \frac{1}{3^N} < \epsilon$. This implies that there exists $x, y \in A_k$ such that $\frac{1}{3^N} = x - y$ so $(x, y) \in G$ which is a contradiction as A_k needed to be independent. $\square$

Having seen that a classical result that does not generalize to Borel setting, let us now provide an example that does generalize to Borel setting. Consider the following fact.

PROPOSITION 4. *Let $\mathbf{G} = (X, G)$ be a locally countable graph, then $\chi'(\mathbf{G}) \leq \aleph_0$.*

Proof. In a locally countable graph, every connected component has countable many vertices and thus, countably many edges. Since there are countable many edges in each connected component, giving each edge in a connected component a unique color, we can obtain an $\aleph_0$-edge coloring. $\square$

This theorem still holds with Borel constraints which is proved in a non-trivial way:

PROPOSITION 5 ([9, Proposition 4.10]). *Let $\mathbf{G} = (X, G)$ be a locally countable Borel graph on a standard Borel space X. Then, $\chi'_B(\mathbf{G}) \leq \aleph_0$.*

Proof. Consider the connectedness relation E_G. Observe that each connected component is countable and consequently, the relation E_G is a countable Borel equivalence relation. By [17, Proposition 5.8.13], also known as the Feldman-Moore Theorem, there exists a countable group $\mathcal{G}$ and a Borel action of $\mathcal{G}$ on X such that E_G is the orbit equivalence relation of this Borel action. Moreover, the proof of this theorem reveals that one can choose a sequence $\{g_n\}_{n\in\mathbb{N}} \subseteq \mathcal{G}$ of involutions such that

$$xE_Gy \text{ if and only if there exists } n \in \mathbb{N} \text{ with } g_n \cdot x = y$$

Define $c : \check{X} \to \mathbb{N}$ to be the coloring of $L(\mathbf{G})$ given by

$$c(\{x, y\}) = \min\{n \in \mathbb{N} : g_n \cdot x = y\}$$

Then one can check that $c : L(\mathbf{G}) \to \mathbb{N}$ is a Borel coloring. Consequently, we have a Borel $\aleph_0$-edge coloring of $\mathbf{G}$.

□

1.4 Borel Chromatic Numbers

The Borel chromatic number of a graph is one of the most studied concepts in descriptive graph combinatorics. The article [9] is mainly about Borel chromatic numbers and provides numerous results: Interesting examples of Borel graphs for which chromatic number is strictly smaller than Borel chromatic number, Borel graphs generated by Borel functions and their Borel chromatic numbers, some results on Borel edge chromatic number and the $\mathbf{G}_0$ dichotomy.

Since then, the theory has been expanded greatly. Many concepts in classical graph theory such as Hedetniemi's Conjecture, Vizing's Theorem and Brook's Theorem have been studied under definable constraints, where some of the theorems hold in the Borel setting and some of them do not.

For example, it was found in [9, Theorem 5.1] that if a Borel graph is generated by a single function (meaning that two vertices are adjacent if the function takes one of these vertices to the other), then possible values for its Borel chromatic number are $\{1, 2, 3, \aleph_0\}$. It was an open problem in the same article that given a Borel graph generated by n Borel functions, is $2n + 1$ an upper bound for its Borel chromatic number? In his dissertation, with theorem [14, Theorem 2.1], Palamourdas answered the question by proving that if $\mathbf{G}$ is a Borel graph generated by k many commuting Borel functions, then $\chi_B(\mathbf{G}) \leq 2k+1$ or $\chi_B(\mathbf{G}) = \aleph_0$.

Another example comes from a well-known conjecture in classical graph theory. The product of two graphs $\mathbf{G} = (X, G)$ and $\mathbf{H} = (Y, H)$ is the graph $\mathbf{G} \times \mathbf{H}$ with the vertex set $X \times Y$ and $((x_0, y_0), (x_1, y_1)) \in G \times H$ if and only if $(x_0, x_1) \in G$ and $(y_0, y_1) \in H$. In classical graph theory, Hedetniemi's conjecture is a well-known conjecture that states if $\mathbf{G}$ and $\mathbf{H}$ are two finite graphs, then

$$\chi(\mathbf{G} \times \mathbf{H}) = \min\{\chi(\mathbf{G}), \chi(\mathbf{H})\}.$$

Let $C(k)$ be the statement that for finite graphs $\mathbf{G}$ and $\mathbf{H}$,

$$\chi(\mathbf{G}), \chi(\mathbf{H}) \geq k \implies \chi(\mathbf{G} \times \mathbf{H}) \geq k,$$

then Hedetniemi's conjecture is equivalent to the statement $C(k)$ holds for all $k \geq 2$. Hedetniemi's conjecture was recently disproven in [16]. Thus $C(k)$ does not hold for sufficiently large k.

The Borel version of Hedetniemi's conjecture is presented in [11, Problem 4.23]. For any $k \in \{1, 2, 3, \ldots, \aleph_0, 2^{\aleph_0}\}$, $C_B(k)$ is defined to be the statement: For any analytic graphs $\mathbf{G}$ and $\mathbf{H}$ we have

$$\chi_B(\mathbf{G}), \chi_B(\mathbf{H}) \geq k \implies \chi_B(\mathbf{G} \times \mathbf{H}) \geq k.$$

Using $\mathbf{G}_0$ dichotomy theorem (which will be presented in detail later), it is obvious to see that $C_B(2^{\aleph_0})$ holds. It was also proven in [11, Proposition 4.25] that $C_B(3)$ holds as well. However, it is not even known whether $C_B(4)$ or $C_B(5)$ holds.

A lot of work has been done on necessary and sufficient conditions for a graph to have different chromatic number and Borel chromatic number. In some cases, the reason for the graph to have this type of difference has been discovered as the graph to contain a specific graph in terms of continuous homomorphism.

In [9], a graph called $\mathbf{G}_0$ is constructed using a dense sequence in $2^{\aleph_0}$. The choice of the sequence is irrelevant as the obtained graph is unique. The importance of this graph is, there is a dichotomy theorem stating that any analytic graph $\mathbf{G}$ either has countable Borel chromatic number or there is a continuous homomorphism from $\mathbf{G}_0$

into the $\mathbf{G}$. So, the necessary and sufficient condition for an analytic graph to have uncountable Borel chromatic number is the existence of a continuous homomorphism from $\mathbf{G}_0$ into the graph.

The original proof of the $\mathbf{G}_0$ dichotomy theorem includes technical tools from effective descriptive set theory as stated in [9]. Later, Ben Miller found a classical proof of this result using classical methods from graph theory and descriptive set theory in [12] and gave a detailed proof in [13]. His idea can be summarized as follows:

> The "nice" parts of the graph under investigation which are Borel $\aleph_0$-colorable are removed by transfinite recursion iteratively. Along this removal process, one also constructs finite approximations to the graph $\mathbf{G}_0$. When the removal process stops, if the whole space is exhausted, then the graph is Borel $\aleph_0$-colorable. If not, then one is able to extend these approximations which allows to construct the necessary homomorphism.

Later on, using the same proof with some tweaks and observations on Borel 2-colorability, in [2], another graph $\mathbf{L}_0$ was found for a similar dichotomy separating analytic graphs with Borel chromatic number at most 2 and at least 3.

There have been negative results as well. For example, in [9], a problem was asked whether the shift graph on the Baire space could be used to prove the analogue of the $\mathbf{G}_0$ dichotomy for graphs with infinite Borel chromatic number. Conley and Miller showed the answer to this question is negative in [3]. This graph does not satisfy the analogue of the $\mathbf{G}_0$ dichotomy, in addition, in [19], it is shown that there is no graph (not even a countable set of graphs) satisfying such an analogue by proving that there is no Borel graph of chromatic number at least 4 which would admit a homomorphism to each graph with infinite Borel chromatic number. This also concludes that there is no analogue of $\mathbf{G}_0$ dichotomy for any $n \in \mathbb{N}$ where $n \geq 4$.

1.5 Borel Automorphism Groups

It is a natural direction in descriptive graph combinatorics to focus on automorphism groups and other concepts in graph theory related to graph automorphisms.

One of the most famous theorems about graph automorphisms is Frucht's theorem, which can be found in [6], that states that every finite group can be realized as the automorphism group of a graph. Later, both DeGroot in [4] and Sabidussi in [15] proved that this result can be extended to infinite groups as well. However, their construction on some groups of cardinality $2^{\aleph_0}$ leads to graphs with vertex set of cardinality greater than $2^{\aleph_0}$. It is natural to question what would be the descriptive set theory analogue of Frucht's theorem. We will provide two versions of this theorem, one being in the Borel setting and the other in the topological setting.

A triple $(\mathcal{G}, \cdot, \mathcal{B})$ is said to be a *standard Borel group* if $(\mathcal{G}, \mathcal{B})$ is a standard Borel space and $(\mathcal{G}, \cdot)$ is a group for which the multiplication $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the inversion $^{-1} : \mathcal{G} \to \mathcal{G}$ operations are Borel maps. A triple $(\mathcal{G}, \cdot, \tau)$ is said to be a *Polish group* if $(\mathcal{G}, \tau)$ is a Polish space and $(\mathcal{G}, \cdot)$ is a group for which $\cdot : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and $^{-1} : \mathcal{G} \to \mathcal{G}$ are continuous maps.

When Borel constraints are added, one can ask whether, for every standard Borel group, there exists a Borel graph such that the Borel automorphism group of this graph is isomorphic to the given group? DeGroot's and Sabidussi's construction builds non-Borel graphs using graphs with vertex sets of cardinality larger than $2^{\aleph_0}$. The construction needs to be modified so that we can code the necessary Borel graph in a Polish space.

The results in [4] and [15] seem to use Frucht's original idea in [6] which can be summarized as follows:

> Given a group $\mathcal{G}$ with a generating set S, consider the Cayley graph G with respect to S as a directed labeled graph. Then the group of automorphisms of G as a directed labeled graph is isomorphic to $\mathcal{G}$. Match each label with a connected undirected asymmetric graph. Systematically replace each directed labeled edge by the connected undirected asymmetric graph it is assigned with to obtain an undirected graph. Then the automorphism group of the resulting undirected graph is isomorphic to $\mathcal{G}$.

While this idea does not seem to invoke any non-explicit methods at first glance, such as the use of the axiom of choice that often results in non-measurable objects, it remains a non-trivial question to answer whether or not the "systematically replace" part of this idea can actually be done in a uniform way in Borel setting. Indeed, the arguments in [4] and [15] do not seem to produce Borel graphs. That said, the "forking" idea that will appear in our construction already appeared [4] in a different form. Nevertheless, the answer turns out to be affirmative as we shall see later.

1.6 Contributions of the Thesis

This thesis can be divided into two parts.

In the first part of the thesis, we will provide a survey of the technique discovered in [12], which is used in [13] and [2] to prove $\mathbf{G}_0$ and $\mathbf{L}_0$ dichotomies respectively. One aspect that will be different from these papers will be that the original proof given in [2] for the $\mathbf{L}_0$ dichotomy uses directed graphs that allows the authors to prove further results. However, using digraphs, the proof is slightly more complicated. Using (undirected) graphs instead of directed graphs, as it will be done in this thesis, allows one to simplify the proof.

In the second part of the thesis, the focus is on automorphism groups of graphs and the main contribution is to extend Frucht's theorem to Borel setting. Namely, we shall prove the following.

THEOREM 6. *For every standard Borel group $(\mathcal{G}, \cdot, \mathcal{B})$, there exists a Borel graph $\mathbf{G} = (X, G)$ on a standard Borel space $(X, \widehat{\mathcal{B}})$ such that $\mathcal{G}$ and $Aut_B(\mathbf{G})$ are isomorphic as abstract groups.*

A slight modification of our argument in the proof of Theorem 6 also gives the following variation in the topological setting.

THEOREM 7. *For every Polish group $(\mathcal{G}, \tau)$, there exists a $\mathbf{\Sigma^0_2}$-graph $\mathbf{G} = (X, G)$ on a Polish space $(X, \widehat{\tau})$ such that $\mathcal{G}$ and $Aut_h(\mathbf{G})$ are isomorphic. Moreover, this isomorphism can be taken to be a homeomorphism where $Aut_h(\mathbf{G}) \subseteq Homeo(X)$ is endowed with the subspace topology induced from the compact-open topology of $Homeo(X)$.*

1.7 The Outline of the Thesis

In chapter 2, we will first prove the $\mathbf{G}_0$ dichotomy following [13] and prove the $\mathbf{L}_0$ dichotomy following [2] with slight modifications. In Chapter 3, we will prove Theorem 6 and Theorem 7. In Chapter 4, we will conclude the thesis with further research directions and open questions.

1.8 Remarks on Notation

Throughout the thesis, $2^{\mathbb{N}}$ denotes the Cantor space, i.e., the Polish space consisting of binary sequences indexed by $\mathbb{N}$, R^* denotes the symmetrization of a relation R on a set, i.e., $R^* = R \cup R^{-1}$, Δ_X denotes the identity relation on a set X, $2^{<\mathbb{N}}$ denotes the set of finite binary sequences, 2^n denotes the set of binary sequences of length n and $\mathbb{N}_{\geq k}$ denotes the set of natural numbers greater than or equal to k. The notation $\leq_c$ between two graphs denote that there is a continuous homomorphism from the former to the latter.

As usual, we consider a sequence $(x_i)_{i\in I}$ with index set I over a set X as a function $\mathbf{x} : I \to X$, and hence, as a set consisting of ordered pairs. Consequently, given two sequences $\mathbf{a}, \mathbf{b}$ over a set, the subset inclusion $\mathbf{a} \subseteq \mathbf{b}$ implies that the sequence $\mathbf{b}$ extends the sequence $\mathbf{a}$ (meaning that these two sequences meet on every index on which $\mathbf{a}$ is defined), which we shall denote by $\mathbf{a} \sqsubseteq \mathbf{b}$. Observe that given a sequence $\{\mathbf{a_n}\}_{n\in\mathbb{N}}$ of sequences over a set X, each of which extends the previous one, the union $\bigcup\limits_{n\in\mathbb{N}} \mathbf{a_n}$ corresponds to the sequence whose index set is the union of the index sets of $\mathbf{a_n}$'s which extend all these sequences, that is, the sequence obtained by "gluing" all these sequence in order. Given two sequences $\mathbf{a}, \mathbf{b}$ where $\mathbf{a}$ is a finite sequence, we use the notation $\mathbf{a} \frown \mathbf{b}$ to denote their concatenation.

CHAPTER 2

G_0 AND L_0 DICHOTOMIES

2.1 G_0 Dichotomy

Recall that the $\mathbf{G}_0$ dichotomy provides a necessary and sufficient condition for an analytic graph to have uncountable Borel chromatic number. The first proof was given in [9] using tools from effective descriptive set theory. In this section, we shall prove the $\mathbf{G}_0$ dichotomy theorem following [13] mainly, but also with help from [9].

In order to construct $\mathbf{G}_0$, fix a sequence $\{\mathbf{g_n}\}$, where $\mathbf{g_n} \in 2^n$ for all $n \in \mathbb{N}$ and $\{\mathbf{g_n}\}$ sequence is dense, i.e. for each $\mathbf{a} \in 2^{<\mathbb{N}}$ there exists $n \in \mathbb{N}$ such that $\mathbf{a} \subseteq \mathbf{g_n}$.

Corresponding to the sequence $\{\mathbf{g_n}\}$ will be defined a graph $\mathbf{G}_0$ on the vertex set $2^{\mathbb{N}}$. Consider $\mathbf{G}_0 = (2^{\mathbb{N}}, R_0)$, where the edge relation R_0 is defined as follows:

$$(x, y) \in R_0 \text{ if and only if}$$

- there exists $n \in \mathbb{N}$ such that $\mathbf{g_n} \sqsubseteq x, y$ and $x(m) = y(m)$ for all $m < n$,
- $x(n) = 1 - y(n)$ and
- for all $k > n$, we have $x(k) = y(k)$.

Having defined the graph $\mathbf{G}_0$, we can state the $\mathbf{G}_0$-dichotomy theorem.

THEOREM 8 ([13, Theorem 2.2.1]). *For every analytic graph $\mathbf{G} = (X, R)$ on a Polish space X, exactly one of the following holds:*

1. $\chi_B(\mathbf{G}) \leq \aleph_0$
2. $\mathbf{G}_0 \leq_c \mathbf{G}$

A straightforward Baire category argument shows that $\mathbf{G}_0$ has uncountable Borel chromatic number:

PROPOSITION 9 ([9, Proposition 6.2]). $\chi_B(\mathbf{G}_0) > \aleph_0$.

Proof. Assume to the contrary that $\chi_B(\mathbf{G}_0) \leq \aleph_0$, say, $c : \mathbf{G}_0 \to \mathbb{N}$ is a Borel coloring. Then there is a countable Borel partition of the vertices of $\mathbf{G}_0$ into independent sets. More precisely, we have

$$2^{\mathbb{N}} = \bigcup_{i \in \mathbb{N}} A_i$$

where $A_i = c^{-1}(i)$. A countable union of meager sets is meager, so at least one of A_i must be non-meager, say, A_m is non-meager since $2^{\mathbb{N}}$ is non-meager. By Proposition 3.5.6 in [17], A_m is comeager in $N_{\mathbf{a}}$ where

$$N_{\mathbf{a}} = \{\mathbf{b} \in 2^{\mathbb{N}} : \mathbf{a} \subseteq \mathbf{b}\}$$

for some $\mathbf{a} \in 2^{<\mathbb{N}}$. Since $\mathbf{g_n}$ is a dense sequence, there exists $\mathbf{g_k}$ such that $\mathbf{a} \subseteq \mathbf{g_k}$. Let

$$f : N_{\mathbf{g_k} \frown \mathbf{0}} \to N_{\mathbf{g_k} \frown \mathbf{1}}$$

be the homeomorphism where

$$f(\mathbf{g_k} \frown \mathbf{0} \frown \mathbf{b}) = \mathbf{g_k} \frown \mathbf{1} \frown \mathbf{b}$$

Then A_m is comeager in $N_{\mathbf{g_k} \frown \mathbf{0}}$ and in $N_{\mathbf{g_k} \frown \mathbf{1}}$, which implies that both A_m and $f(A_m \cap N_{\mathbf{g_k} \frown \mathbf{0}})$ are comeager on $N_{\mathbf{g_k} \frown \mathbf{1}}$. Consequently, we have

$$A_m \cap f(A_m \cap N_{\mathbf{g_k} \frown \mathbf{0}}) \cap N_{\mathbf{g_k} \frown \mathbf{1}} \neq \emptyset$$

Now choose $\mathbf{c} \in A_m \cap f(A_m \cap N_{\mathbf{g_k} \frown \mathbf{0}}) \cap N_{\mathbf{g_k} \frown \mathbf{1}}$. Then, $(\mathbf{c}, f^{-1}(\mathbf{c})) \in R_0$ and this is a contradiction since $\mathbf{c}$ and $f^{-1}(\mathbf{c})$ are both in A_m so they share the same color. $\square$

It follows that (1) and (2) in the statement of Theorem 8 are mutually exclusive. Now, all there remains to show is that if an analytic graph has uncountable Borel chromatic number, then there is a continuous homomorphism from $\mathbf{G}_0$ into this graph.

Given an analytic graph $\mathbf{G} = (X, R)$, by [13, Proposition 1.4.8], there is a continuous surjection $\phi_R : \mathbb{N}^{\mathbb{N}} \to R$ and, by [13, Propositons 1.4.1, 1.4.4 and 1.4.8], there is a continuous function $\phi_X : \mathbb{N}^{\mathbb{N}} \to X$ such that the image of ϕ_X consists of the points which are in at least one projection of R. These two functions ϕ_R and ϕ_X will enable us to encode parts of the $\mathbf{G}_0$ inside $\mathbf{G}$.

Before we proceed, let us define the finite approximations to $\mathbf{G}_0$ that we shall use to construct the necessary homomorphism. An *n-approximation* is $a = (\phi^a, \psi^a)$, where $\phi^a : 2^n \to \mathbb{N}^n$ and $\psi^a : \bigsqcup_{m<n} 2^m \to \mathbb{N}^n$. An m-approximation $b = (\phi^b, \psi^b)$ is said to be a *one-step extension* of an n-approximation $a = (\phi^a, \psi^a)$ if

1. $m = n + 1$
2. for all $\mathbf{a} \in 2^n$, for all $\mathbf{b} \in 2^m$, if $\mathbf{a} \sqsubseteq \mathbf{b}$, then $\phi^a(\mathbf{a}) \sqsubseteq \phi^b(\mathbf{b})$
3. for all $\mathbf{a} \in \bigcup_{\hat{n}<n} 2^{\hat{n}}$, for all $\mathbf{b} \in \bigcup_{\hat{m}<m} 2^{\hat{m}}$ such that $|\mathbf{b}| = |\mathbf{a}| + 1$, if $\mathbf{a} \sqsubseteq \mathbf{b}$, then $\psi^a(\mathbf{a}) \sqsubseteq \psi^b(\mathbf{b})$

An *n-configuration* is $\gamma = (\phi^\gamma, \psi^\gamma)$, where $\phi^\gamma : 2^n \to \mathbb{N}^{\mathbb{N}}$ and $\psi^\gamma : \bigsqcup_{m<n} 2^m \to \mathbb{N}^{\mathbb{N}}$ such that for each $m < n$ and for each $\mathbf{s} \in 2^{n-(m+1)}$:

$$(\phi_R \circ \psi^\gamma)(\mathbf{s}) = ((\phi_X \circ \phi^\gamma(\mathbf{g_m} \frown (0) \frown \mathbf{s})), (\phi_X \circ \phi^\gamma(\mathbf{g_m} \frown (1) \frown \mathbf{s})))$$

An n-configuration $\gamma = (\phi^\gamma, \psi^\gamma)$ is *compatible* with an n-approximation $a = (\phi^a, \psi^a)$ if

1. $\phi^a(\mathbf{a}) \sqsubseteq \phi^\gamma(\mathbf{a})$ for all $\mathbf{a} \in 2^n$, and
2. $\psi^a(\mathbf{a}) \sqsubseteq \psi^\gamma(\mathbf{a})$ for all $\mathbf{a} \in \bigsqcup_{\hat{n}<n} 2^{\hat{n}}$.

An n-configuration $\gamma = (\phi^\gamma, \psi^\gamma)$ is *compatible* with a subset $Y \subseteq X$ if

$$(\phi_X \circ \phi^\gamma)(2^n) \subseteq Y$$

An n-approximation $a = (\phi^a, \psi^a)$ is called Y-terminal if there is no configuration which is both compatible with a one step extension of a and Y.

Also, define $A(a, Y)$ to be

$$\{\phi_X \circ \phi^\gamma(\mathbf{g_n}) | n\text{-configuration } \gamma \text{ is compatible with } a \text{ and } Y\}$$

which denotes the set of points $\phi_x \circ \phi^\gamma(\mathbf{g_n})$, where $\mathbf{g_n}$ is the element of the dense sequence used to construct $\mathbf{G}_0$ and ϕ^γ comes from any n-configuration γ that is compatible with both a and Y. Observe that the set $A(a, Y)$ is analytic.

We will recursively define a sequence of analytic subsets $(X^\alpha)_{\alpha<\omega_1}$ by throwing away $\mathbb{N}$-colorable subsets of X. Set the initial element of the sequence to be $X^0 = X$. For limit stages, set

$$X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$$

whenever λ is a limit ordinal. In order to define the successor stages, we will first make some observations:

LEMMA 10 ([13, Lemma 2.2.2]). *Let $Y \subseteq X$ and a is a Y-terminal n-approximation. Then, there is a Borel set $B(a, Y) \supseteq A(a, Y)$ such that $A(a, Y)$ and $B(a, Y)$ are both R-independent.*

Proof. Assume towards contradiction that $A(a, Y)$ is not R-independent, that there exists n-configurations γ_0 and γ_1 which are compatible with both a and Y with $((\phi_X \circ \phi^{\gamma_0})(\mathbf{g_n}), (\phi_X \circ \phi^{\gamma_1})(\mathbf{g_n})) \in R$. The aim is to show that if we put these two configurations together by joining them with the edge

$$((\phi_X \circ \phi^{\gamma_0})(\mathbf{g_n}), (\phi_X \circ \phi^{\gamma_1})(\mathbf{g_n})) = \phi_R(\mathbf{b})$$

for some $\mathbf{b} \in \mathbb{N}^\mathbb{N}$, we obtain a new configuration of a one step extension of a which still stays in Y. Let γ be the $n+1$-configuration with

$$\phi^\gamma(\mathbf{a} \frown \mathbf{0}) = \phi^{\gamma_0}(\mathbf{a}) \text{ and } \phi^\gamma(\mathbf{a} \frown \mathbf{1}) = \phi^{\gamma_1}(\mathbf{a})$$

for any $\mathbf{a} \in 2^n$ and

$$\psi^\gamma(\mathbf{a} \frown \mathbf{0}) = \psi^{\gamma_0}(\mathbf{a}) \text{ and } \psi^\gamma(\mathbf{a} \frown \mathbf{1}) = \psi^{\gamma_1}(\mathbf{a})$$

for all $\mathbf{a} \in \bigcup_{0<m<n} 2^m$ and $\psi^\gamma(\emptyset) = \mathbf{b}$. Then γ is a configuration of $\mathbf{G}_{0,n+1}$ because it preserves the edge relations as required. There is a unique $(n+1)$-approximation b which is compatible with this configuration and this approximation is a one-step extension of a which is a contradiction as a was assumed to be Y-terminal. Hence, $A(a, Y)$ is R-independent.

Since $A(a, Y)$ is R-independent,

$$\pi_1(R \cap (X \times A(a, Y)) \cap A(a, Y) = \emptyset$$

Both $\pi_1(R \cap (X \times A(a, Y))$ and $A(a, Y)$ are analytic sets so we can use the separation theorem [17, Theorem 4.4.1] to get a Borel set $\hat{A} \supseteq A(a, Y)$ such that

$$\hat{A} \cap \pi_1(R \cap (X \times A(a, Y)) = \emptyset$$

Then, we have $\pi_2(R \cap (\hat{A} \times X)) \cap A(a, Y) = \emptyset$ and again both of the sets of this intersection are analytic so use the separation theorem again to get $\hat{\hat{A}} \supseteq A(a, Y)$. Set $B(a, Y) = \hat{A} \cap \hat{\hat{A}}$. Then we have that

$$\pi_2(R \cap (B(a, Y) \times X)) \cap B(a, Y) \subseteq \pi_2(R \cap (\hat{A} \times X)) \cap \hat{\hat{A}} = \emptyset$$

which proves that $B(a, Y)$ is a Borel R-independent set containing $A(a, Y)$. $\square$

We now know that $B(a, Y)$ are Borel R-independent sets, so they can be of the same color. Since there are countable many approximations, the set

$$\bigcup_{a \text{ is } X^\alpha\text{-terminal}} B(a, X^\alpha)$$

is Borel $\aleph_0$-colorable. Define

$$X^{\alpha+1} = X^\alpha \setminus \bigcup_{a \text{ is } X^\alpha\text{-terminal}} B(a, X^\alpha)$$

In order to prove the theorem the following lemma is also required:

LEMMA 11 ([13, Lemma 2.2.3]). *Suppose $\alpha < \omega_1$ and a is an n-approximation that is not $X^{\alpha+1}$-terminal. Then there exists a one-step extension of a that is not X^α-terminal.*

Proof. Since a is not $X^{\alpha+1}$-terminal, there exist an $(n+1)$-approximation b and an $(n+1)$-configuration γ compatible with b and $X^{\alpha+1}$. It follows that we have $(\phi_X \circ \phi^\gamma)(\mathbf{a_{n+1}}) \in X^{\alpha+1}$. This subsequently implies that $B(b, X^\alpha) \cap X^{\alpha+1} \neq \emptyset$. Therefore b must be not X^α-terminal since otherwise the previous equality contradicts the definition of $X^{\alpha+1}$. $\square$

Since there are countable many approximations, we can set $\alpha < \omega_1$ such that X^α-terminal approximations and $X^{\alpha+1}$-terminal approximations are same.

Proof of Theorem 8. Let a_0 be the 0-approximation of the single vertex graph. Then, $A(a_0, Y) = Y$ for all $Y \subseteq X$. Suppose that a_0 is X^α-terminal. Then

$$X^{\alpha+1} \subseteq X^\alpha \setminus A(a_0, X^\alpha) = \emptyset$$

Since countably many Borel $\aleph_0$-colorable subsets of X are removed and we obtained the empty set after these removals, there exists a Borel $\aleph_0$-coloring of $\mathbf{G}$.

Now, assume that a_0 is not X^α-terminal. Then we can apply the last lemma iteratively to get one step extensions a_{n+1} of a_n for all $n \in \mathbb{N}$. Since we chose α so that X^α-terminal and $X^{\alpha+1}$-terminal approximations are the same, each of these one step extensions is not X^α-terminal. Set

$$\phi : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \text{ such that } \phi(\mathbf{a}) = \bigcup_{n\in\mathbb{N}} \phi^{a_n}(\mathbf{a} \upharpoonright n)$$

and

$$\psi : \bigsqcup_{m\in\mathbb{N}} 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \text{ such that } \psi_m : 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}} \text{ is defined as}$$
$$\psi_m(\mathbf{a}) = \bigcup_{k>m} \psi^{a_k}(\mathbf{a} \upharpoonright (k-(m+1))$$

Observe that the function ϕ is continuous. Since ϕ_X is also continuous, we have that $f = \phi_X \circ \phi$ is continuous. We now must show that f is a graph homomorphism from $\mathbf{G}_0$ to $\mathbf{G}$. It is sufficient to show the stronger condition that

$$(\phi_R \circ \psi_m)(\mathbf{a}) = ((\phi_X \circ \phi)(\mathbf{g_m} \frown 0 \frown \mathbf{a}), (\phi_X \circ \phi)(\mathbf{g_m} \frown 1 \frown \mathbf{a}))$$

for any $\mathbf{a} \in 2^{\mathbb{N}}$ and $m \in \mathbb{N}$. In order to do this, it is enough to show that for any open neighborhood U with $(\phi_R \circ \psi_m)(\mathbf{a}) \in U$ and any open neighborhood V with

$$((\phi_X \circ \phi)(\mathbf{a_m} \frown 0 \frown \mathbf{a}), (\phi_X \circ \phi)(\mathbf{a_m} \frown 1 \frown \mathbf{a})) \in V$$

we have $U \cap V \neq \emptyset$. Then there exists $k > m$ and there are basis elements of $\mathbb{N}^{\mathbb{N}}$: $N_{\phi^{a_k}(\mathbf{g_m}\frown 0\frown \mathbf{b})}$, $N_{\phi^{a_k}(\mathbf{g_m}\frown 1\frown \mathbf{b})}$ and $N_{\psi^{a_k,m}(\mathbf{b})}$ such that $\phi_R(N_{\psi^{a_k,m}(\mathbf{b})}) \subseteq U$ and

$$\phi_X(N_{\phi^{a_k}(\mathbf{g_m}\frown 0\frown \mathbf{b})}) \times \phi_X(N_{\phi^{a_k}(\mathbf{g_m}\frown 1\frown \mathbf{b})}) \subseteq V$$

where $\mathbf{b} = \mathbf{a} \upharpoonright (k-(m+1))$. We know that a_k is not an X^α-terminal approximation and thus, there is a configuration γ compatible with a_k. Hence, $(\phi_G \circ \psi^{(\gamma,n)})(\mathbf{b}) \in U$

and $((\phi_X \circ \phi^\gamma)(\mathbf{g_m} \frown 0 \frown \mathbf{b}), (\phi_X \circ \phi^\gamma)(\mathbf{g_m} \frown 1 \frown \mathbf{b})) \in V$ where these two elements are the same. Thus, $U \cap V \neq \emptyset$. This completes the proof that f is a graph homomorphism. $\square$

2.2 L_0 Dichotomy

Recall that $\mathbf{L}_0$ dichotomy theorem is an analogue of $\mathbf{G}_0$ dichotomy theorem that gives a necessary and sufficient condition for analytic graphs to have Borel chromatic number at least 3. In this section, we shall prove the $\mathbf{L}_0$ dichotomy theorem following [2], but we will modify the arguments for undirected graphs instead of directed graphs to simplify the proof further.

To define $\mathbf{L}_0$-type graphs, let $\mathbf{c} \in (2\mathbb{N}+1)^{\mathbb{N}}$. For each $n \in \mathbb{N}$, let $L_n = (X_n, R_n)$ denote the graph shaped like a line segment on the vertices $X_n = \{0, 1, \ldots, n\}$ given by

$$(a, b) \in R_n \text{ if and only if } |a - b| = 1$$

for $a, b \in X_n$. Let $\mathbf{l_n}$ be a sequence of finite sequences such that $\mathbf{l_0} = \mathbf{c}(0)$ and $\mathbf{l_n} = 0^n \frown 1$ whenever $n > 0$. Let $L_{\mathbf{c},n} = (X_{\mathbf{c},n}, R_{\mathbf{c},n})$, where

$$X_{\mathbf{c},n} = \bigcup_{m \leq n} \{0, 1, \ldots, \mathbf{c}(m)\} \times 2^{n-m}$$

with edge relations defined recursively as follows:

- $L_{\mathbf{c},0} = L_{\mathbf{c}(0)}$
- $L_{\mathbf{c},n+1}$ is the acyclic, connected graph with

  $$(\mathbf{v_0} \frown 0, \mathbf{v_1} \frown 0), (\mathbf{v_0} \frown 1, \mathbf{v_1} \frown 1) \in R_{\mathbf{c},n+1}$$

  whenever $(\mathbf{v_0}, \mathbf{v_1}) \in R_{\mathbf{c},n}$,

  $$((n), (m)) \in R_{\mathbf{c},n+1}$$

  whenever $|n - m| = 1$,

  $$((\mathbf{l_n}, 0), (0)) \in R_{\mathbf{c},n+1}$$

  and

  $$((\mathbf{c}(n+1)), (\mathbf{l_n}, 1)) \in R_{\mathbf{c},n+1}$$

Hence, each $L_{\mathbf{c},n+1}$ is a graph that consists of two copies of $L_{\mathbf{c},n}$ which are connected by their endpoints (chosen by $\mathbf{l_n}$) by the graph $L_{\mathbf{c}(n+1)}$.

Set $\mathbf{X_c} = \{(n,k,\mathbf{r}) \in \mathbb{N} \times \mathbb{N} \times 2^{\mathbb{N}} \mid k \leq \mathbf{c}(n)\}$. Now let $\mathbf{L_c} = (\mathbf{X_c}, \mathbf{R_c})$ and $\pi_{\mathbf{c},n} : \mathbf{X_c} \cap (\{0,\ldots,n\} \times \mathbb{N} \times 2^{\mathbb{N}}) \to X_{\mathbf{c},n}$ be the projection defined by

$$\pi_{\mathbf{c},n}(m,k,\mathbf{r}) = (k) \frown \mathbf{r} \restriction (n-m)$$

for all $n \in \mathbb{N}$. Then, define $\mathbf{L_c}$ to be the graph where $\mathbf{R_c}$ consists of edges of the form $((n_0,k_0,\mathbf{r}_0),(n_1,k_1,\mathbf{r}_1))$, where for all $n \geq \max(n_0,n_1)$, we have

$$(\pi_{\mathbf{c},n}(n_0,k_0,\mathbf{r}_0), \pi_{\mathbf{c},n}(n_1,k_1,\mathbf{r}_1)) \in L_{\mathbf{c},n}$$

Define $\mathbf{L}_0$ to be the graph $\mathbf{L_c}$, where $\mathbf{c}(0) = 1$ and $\mathbf{c}(n) = 2n-1$ for all $n > 0$.

Now, we can state the $\mathbf{L}_0$-dichotomy theorem:

THEOREM 12 ([2, Theorem 1.1]). *For every analytic graph $\mathbf{G} = (X,R)$ on a Polish space X, exactly one of the following holds:*

1. $\chi_B(\mathbf{G}) \leq 2$
2. $\mathbf{L}_0 \leq_c \mathbf{G}$

The proof of this theorem consists of two stages. In the first stage, for each sequence $\mathbf{c} \in (2\mathbb{N}+1)^{\mathbb{N}}$ of odd numbers, a Borel graph $\mathbf{L_c}$ is constructed; and the following weaker dichotomy is proven. (Note that this theorem is proved for graphs with Hausdorff vertex sets so it is a more general result).

THEOREM 13 ([2, Theorem 3.1]). *Let $\mathbf{G} = (X,R)$ be an analytic graph on a Hausdorff space X. Then, exactly one of the following holds:*

1. $\chi_B(\mathbf{G}) \leq 2$
2. *There exists sequence $\mathbf{c} \in (2\mathbb{N}+1)^{\mathbb{N}}$ such that there is a continuous homomorphism from $\mathbf{L_c}$ into $\mathbf{G}$.*

In the second stage, it is proven that these continuum many Borel graphs, which are called $\mathbf{L}_0$-type graphs, contains a homomorphic copy of the graph $\mathbf{L}_0$ that is a special example of these graphs.

The method that will be used in the proof of the first stage of this theorem is a slight modification of the proof of $\mathbf{G}_0$ dichotomy. However, some extra lemmas on Borel 2-colorability of graphs will be required. Since the methods in the second stage of the proof are not relevant to the approximation techniques used in the $\mathbf{G}_0$ dichotomy and requires different definitions, we chose not to include this construction in this thesis. Therefore, we will provide only the proof of the weaker dichotomy stated above. The omitted second stage can be found in [2].

A simple Baire category argument proves that $\mathbf{L_c} = 3$.

PROPOSITION 14. $\chi_B(\mathbf{L_c}) = 3$.

Proof. First, the Borel chromatic number of a locally finite Borel graph cannot exceed the degree of the graph plus one, as proven in Proposition 4.6 in [9]. Hence we have $\chi_B(\mathbf{L}_c) \leq 3$.

The proof that $\chi_B(\mathbf{L_c}) = 3$ will resemble that of Proposition 9. Assume to the contrary that $\chi_B(\mathbf{L_c}) \leq 2$. Then there exists a Borel coloring $c : X_\mathbf{c} \to \{0, 1\}$ and consequently, a Borel partition $X_\mathbf{c} = A_0 \cup A_1$ into independent sets, where $A_0 = c^{-1}(0)$ and $A_1 = c^{-1}(1)$. As before, one of these subsets has to be non-meager. Without loss of generality assume that A_0 is not meager. By Proposition 3.5.6 in [17], A_0 is co-meager in some open subset $[(n, k, \mathbf{t})] = \{(n, k, \mathbf{r}) \in X_\mathbf{c} : \mathbf{t} \sqsubseteq \mathbf{r}\}$. Then A_0 is also co-meager in $[(n, k, \mathbf{t} \frown (0))]$. The function

$$f : [(n, k, \mathbf{t} \frown (0))] \to [(n, k, \mathbf{t} \frown (1))]$$

given by $f((n, k, \mathbf{t} \frown (0))) = (n, k, \mathbf{t} \frown (1))$ is a homeomorphism. A_0 is co-meager in $[(n, k, \mathbf{t} \frown (1))]$ as well and $f(A_0 \cap [(n, k, \mathbf{t} \frown (0))])$ is also co-meager in $[(n, k, \mathbf{t} \frown (1))]$. Then,

$$f(A_0 \cap [(n, k, \mathbf{t} \frown (0))]) \cap [(n, k, \mathbf{t} \frown (1))] \cap A_0 \neq \emptyset$$

Hence, there exists an element $(n, k, \mathbf{t} \frown (1) \frown \mathbf{r})$ of this intersection such that

$$c((n, k, \mathbf{t} \frown (0) \frown \mathbf{r})) = c((n, k, \mathbf{t} \frown (1) \frown \mathbf{r})) = 0$$

However, we now obtain that the distance between the vertices $(n, k, \mathbf{t} \frown (0) \frown \mathbf{r})$ and $(n, k, \mathbf{t} \frown (1) \frown \mathbf{r})$ is odd and the graph is acyclic so they cannot share the same color which is a contradiction. $\square$

Since $\mathbf{G}$ is analytic there is a continuous surjection $\phi_R : \mathbb{N}^{\mathbb{N}} \to R$ and a continuous function $\phi_X : \mathbb{N}^{\mathbb{N}} \to X$ such that $\phi_X(\mathbb{N}^{\mathbb{N}})$ is the union of two projections of R to X. We will define a decreasing sequence $(X^\alpha)_{\alpha<\omega_1}$ of analytic subsets of X by throwing away $\mathbf{G}$-invariant sets that are Borel 2-colorable. Let $X^0 = \phi_X(\mathbb{N}^{\mathbb{N}})$, $X^\lambda = \bigcap_{\alpha<\lambda} X^\alpha$ when λ is a limit ordinal. To define the successor stage, we need the definitions:

A $(\mathbf{c}, n)$-approximation a is a pair of functions (ϕ^a, ψ^a) where $n \in \mathbb{N}$, $\mathbf{c} \in (2\mathbb{N}+1)^n$, $\phi^a : X_{\mathbf{c},n} \to \mathbb{N}^n$ and $\psi^a : L_{\mathbf{c},n} \to \mathbb{N}^n$. A $(\mathbf{c}', n+1)$-approximation a' is called a one-step extension of a $(\mathbf{c}, n)$-approximation a if:

1. $\mathbf{c} \sqsubset \mathbf{c}'$
2. For all $x \in dom(\pi_, \mathbf{c}, n, n+1)$, we have $\phi^a \circ \pi_{\mathbf{c},n,n+1}(x) \sqsubset \phi^{a'}(x)$, where $\pi_{\mathbf{c},n,n+1} : X_{\mathbf{c},n+1} \to X_{\mathbf{c},n}$ is a projection
3. $\forall x, y \in dom(\pi_{\mathbf{c},n,n+1})$, $(x, y) \in L_{\mathbf{c},n+1}$ we have
$$\psi^a(\pi_{\mathbf{c},n,n+1}(x), \pi_{\mathbf{c},n,n+1}(y)) \sqsubset \psi^{a'}(x, y)$$

A $(\mathbf{c}, n)$-configuration γ is a pair of functions $(\phi^\gamma, \psi^\gamma)$ where $n \in \mathbb{N}$, $\mathbf{c} \in (2\mathbb{N}+1)^n$, $\phi^\gamma : X_{\mathbf{c},n} \to \mathbb{N}^{\mathbb{N}}$, $\psi^\gamma : L_{\mathbf{c},n} \to \mathbb{N}^{\mathbb{N}}$ and $\forall (x, y) \in L_{\mathbf{c},n}$ we have
$$(\phi_R \circ \psi^\gamma)(x, y) = (\phi_X \circ \phi^\gamma(x), \phi_X \circ \phi^\gamma(y))$$

A $(\mathbf{c}, n)$-configuration γ is compatible with a $(\mathbf{c}, n)$-approximation a if

1. $\forall x \in X_{\mathbf{c},n}$, $\phi^a(x) \sqsubset \phi^\gamma(x)$
2. $\forall (x, y) \in L_{\mathbf{c},n}$, $\psi^\gamma(x, y) \sqsubset \psi^\gamma(x, y)$

A $(\mathbf{c}, n)$-configuration γ is compatible with a set $Y \subseteq X$ if $\phi_X \circ \phi^\gamma(X_{\mathbf{c},n}) \subseteq [Y]_{E_R}$. A $(\mathbf{c}, n)$-approximation a is Y-terminal if there is no configuration which is compatible with both Y and a one-step extension of a. Also, define
$$A(a, Y) = \{\phi_X \circ \phi^\gamma(s_n) | (\mathbf{c}, n)\text{-configuration } \gamma \text{ is compatible with } a \text{ and } Y\}$$

CLAIM 15 ([2, Claim 3.3]). *Assume that $A \subseteq X$ is analytic such that for every $x, y \in A$, every R-walk from x to y has even length. Then there exists an R-invariant Borel set $B \supseteq [A]_{E_R}$ such that $G \restriction B$ has a Borel 2-coloring.*

Proof. Define $A_0 \subseteq [A]_{E_R}$ such that $x \in A_0$ if and only if there is a walk of even length from x to some $y \in A$. Similarly, define $A_1 \subseteq [A]_{E_R}$ such that $x \in A_1$ if and only if there is a walk of odd length from x to some $y \in A$. A_0 and A_1 are analytic subsets. For each $x \in [A]_{E_R}$, either there is a walk of even length or there is a walk of odd length, from x to some $y \in A$, because every walk connecting two vertices in A has even length. Hence, $A_0 \cup A_1 = [A]_{E_R}$ and $A_0 \cap A_1 = \emptyset$. Using separation theorem for analytic sets, there are Borel sets B_1, B_2 such that $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$ and $B_0 \cap B_1 = \emptyset$. Let $c(x) = 0$ if $x \in B_0$ and $c(x) = 1$ if $x \in B_1$. Define

$$C = \{x \in X | c \text{ is a 2-coloring of } G \restriction [x]_{E_R}\}$$

Then, since c is a 2-coloring on $A_0 \cup A_1$, we have two analytic R-invariant sets $X \setminus C$ and $A_0 \cup A_1$ such that $(X \setminus C) \cap (A_0 \cup A_1) = \emptyset$. Then, by [7, Lemma 5.1], there is an R-invariant Borel set $B \supseteq A_0 \cup A_1$ and $B \cap (X \setminus C) = \emptyset$. Hence, $c \restriction B$ is a Borel 2-coloring of $\mathbf{G} \restriction B$. $\square$

LEMMA 16 ([2, Lemma 3.2]). *Assume that $Y \subseteq X$ is an analytic subset and a is a Y-terminal $(\mathbf{c}, n)$-approximation. Then, there exists an R-invariant Borel set $B(a, Y) \supseteq [A(a, Y)]_{E_R}$ so that $\mathbf{G} \restriction B(a, Y)$ has a Borel 2-coloring $c_{a,Y}$.*

Proof. By its definition $A(a, Y)$ is analytic. If for every $x, y \in A(a, Y)$, every R-walk from x to y has even length and the assumptions in the previous claim hold. Assume that between two vertices in $A(a, Y)$, there is an R-walk of odd length $t+2$. Our aim is to show a is not Y-terminal. So, we have two vertices $z_0, z_{t+2} \in A(a, Y)$ with an R-walk of odd distance $t + 2 \geq 3$ between them and there are two $(\mathbf{c}, n)$-configurations γ_0, γ_1 compatible with both a and Y such that $z_0 = \phi_X \circ \phi^{\gamma_0}(\mathbf{l_n})$ and $z_{t+2} = \phi_X \circ \phi^{\gamma_1}(\mathbf{l_n})$. Then we have these configurations and a path between z_0 and z_{t+2}: $(z_0, z_1, \ldots, z_{t+2})$ of length $t+2$, where t is odd. Choose $r_0, r_1, ..., r_{t+2} \in \mathbb{N}^{\mathbb{N}}$ and $e_0, e_1, \ldots, e_{t+1} \in \mathbb{N}^{\mathbb{N}}$ such that $r_0 = \phi^{\gamma_0}(\mathbf{l_n})$, $r_{t+2} = \phi^{\gamma_1}(\mathbf{l_n})$, $\forall i \leq t+2$, $\phi_X(r_i) = x_i$ and $\forall j < t+2$, $\phi_R(e_i) = (x_i, x_{i+1})$. Now, define a new $(\mathbf{c} \frown (t), n+1)$-configuration γ as: $\gamma = (\phi^\gamma, \psi^\gamma)$ where $\phi^\gamma : X_{\mathbf{c}\frown(t),n+1} \to \mathbb{N}^{\mathbb{N}}$, $\psi^\gamma : L_{\mathbf{c}\frown(t),n+1} \to \mathbb{N}^{\mathbb{N}}$ and

$$\phi^\gamma : \begin{cases} \phi^\gamma(x \frown (i)) = \phi^{\gamma_i}(x) & x \in X_{\mathbf{c},n}, i < 2 \\ \phi^\gamma((i)) = r_{i+1} & i \leq m \end{cases}$$

and

$$\psi^\gamma : \begin{cases} \psi^\gamma(x \frown (i), y \frown (j)) = \psi^{\gamma_i}(x,y) & (x,y) \in L_{\mathbf{c},n}, i < 2 \\ \psi^\gamma((\mathbf{s_n} \frown (0), (0))) = e_0 & \\ \psi^\gamma(((t), \mathbf{s_n} \frown (1))) = e_{t+1} & \\ \psi^\gamma((i, i+1)) = e_{i+1} & i \leq t-1 \end{cases}$$

Then, γ is a $(\mathbf{c} \frown (t), n+1)$-configuration and is compatible with both Y and the unique $(\mathbf{c} \frown (t), n+1)$-approximation a' that is a one-step extension of a which is a contradiction as a was supposed to be Y-terminal.

Hence, if a is Y-terminal then every R-walk between the vertices in $A(a,Y)$ has even length. Then, by the claim, there is a Borel R-invariant set $B(a,Y) \supseteq [A(a,Y)]_{E_R}$ that has a Borel 2-coloring on it. $\square$

Now, define the successor stage:

$$X^{\alpha+1} = X^\alpha \setminus \bigcup_{\alpha \text{ is } X^\alpha\text{-terminal}} B(a, X^\alpha)$$

There are countable many approximations and X^0 is an analytic set. Hence, X^α is analytic $\forall \alpha < \omega_1$. Also, since each $B(a, X^\alpha)$ is R-invariant, each X^α is also R-invariant.

LEMMA 17 ([2, Lemma 3.6]). *Assume that $\alpha < \omega_1$ and a is a $(\mathbf{c}, n)$-approximation that is not $X^{\alpha+1}$-terminal. Then a has a one-step extension that is not X^α-terminal.*

Proof. We know that a is not $X^{\alpha+1}$-terminal. Hence, it has a one-step extension $(\mathbf{c}', n+1)$-approximation a' and a $(\mathbf{c}', n+1)$-configuration γ such that γ is compatible with both a' and $X^{\alpha+1}$. We also know that $(\phi_X \circ \phi^\gamma)(X_{\mathbf{c},n+1}) \neq \emptyset$ and since $X^{\alpha+1}$ is R-invariant, $[X^{\alpha+1}]_{E_R} = X^{\alpha+1}$ so

$$\emptyset \neq (\phi_X \circ \phi^\gamma)(X_{\mathbf{c}',n+1}) \subseteq X^{\alpha+1}$$

If a' is X^α-terminal, then $[\phi_X \circ \phi^\gamma(X_{\mathbf{c}',n+1})]_{E_R} \subseteq [A(a', X^\alpha)]_{E_R} \subseteq B(a', X^\alpha)$. This implies that $X^{\alpha+1} \cap B(a', X^\alpha) \neq \emptyset$. However,

$$X^{\alpha+1} = X^\alpha \setminus \bigcup_{\alpha \text{ is } X^\alpha\text{-terminal}} B(a, X^\alpha)$$

which provides a contradiction as $B(a', X^\alpha)$ should've been extracted from X^α when obtaining $X^{\alpha+1}$. Hence, a' is not X^α-terminal. $\square$

Since there are countably many approximations, we can set $\alpha < \omega_1$ such that X^α-terminal approximations and $X^{\alpha+1}$-terminal approximations are same. From now on, we fix such an α.

LEMMA 18 ([2, Lemma 3.7]). *If every approximation is $X^{\alpha+1}$-terminal, then* $\mathbf{G}$ *has a Borel 2-coloring.*

Proof. Assume that for some $x, y \in X^{\alpha+1}$, $(x, y) \in R$. Then, there is a $((1), 1)$-configuration γ that is compatible with $\{x, y\}$ and a unique $((1), 1)$-approximation a such that γ is compatible with a. We know that a is $X^{\alpha+1}$-terminal so that we have $x, y \in [A(a, X^{\alpha+1})]_{E_R}$ but since we fixed α above, a is X^α-terminal as well which further implies that $x, y \in [A(a, X^\alpha)]_{E_R} \subseteq B(a, X^\alpha)$. However,

$$X^{\alpha+1} \cap B(a, X^\alpha) = \emptyset$$

which provides a contradiction. Hence, $X^{\alpha+1}$ is R-independent. It is also R-invariant and since X^0 consists of projections of R, we have that $X^{\alpha+1} = \emptyset$. Now we can define $e : \{(a, \beta) | a \text{ is } X^\beta\text{-terminal}, \beta \leq \alpha\} \to \mathbb{N}$ to be any injection. Let c_{a,X^β} be the Borel 2-coloring on $B(a, X^\beta)$ given be previous lemma, for $(a, \beta) \in dom(e)$. If $x \in X$, let

$$c(x) = \begin{cases} c_{a,X^\beta}(x) & e(a, \beta) \text{ is minimal such that } x \in B(a, X^\beta) \\ 0 & x \notin \bigcup\limits_{(a,\beta)\in dom(e)} B(a, X^\beta) \end{cases}$$

It is a Borel map and a 2-coloring as each $B(a, X^\beta)$ is R-invariant. $\square$

We are ready to prove the theorem of the first stage.

Proof of Theorem 13. Assume that $\chi_B(\mathbf{G}) > 2$. By the previous lemma, there exists an approximation that is not $X^{\alpha+1}$-terminal. One can find a (0)-approximation a_0 that is not $X^{\alpha+1}$-terminal. Applying the lemma recursively, there is a one-step extension $(c_{a_{n+1}}, n + 1)$-approximation a_{n+1} of (c_n, n)-approximation a_n that is not

X^{α}-terminal. Let $c = \bigcup_{n\in\mathbb{N}} c_n$. Define $\phi : X_c \to \mathbb{N}^{\mathbb{N}}$ by letting

$$\phi(m,k,\mathbf{r}) = \bigcup_{n\geq m} \phi^{a_n}(\pi_{c,n}(m,k,\mathbf{r}))$$

and $\psi : L_c \to \mathbb{N}^{\mathbb{N}}$ by

$$\psi(((m_0,k_0,\mathbf{r_0}),(m_1,k_1,\mathbf{r_1}))) = \bigcup_{n\geq m_0,m_1} \psi^{a_n}((\pi_{c,n}(m_0,k_0,\mathbf{r_0}),\pi_{c,n}(m_1,k_1,\mathbf{r_1})))$$

.

Aim is to show that $\phi_X \circ \phi$ is a continuous homomorphism from $\mathbf{L_c}$ to $\mathbf{G}$. To prove that it is a homomorphism, let $(x_0,x_1) \in \mathbf{L_c}$, $x_0 = (n_0,k_0,\mathbf{r_0})$, $x_1 = (n_1,k_1,\mathbf{r_1})$. If we show that $(\phi_R \circ \psi)(x_0,x_1) = ((\phi_X \circ \phi)(x_0),(\phi_X \circ \phi)(x_1))$ we are done. In order to show this it is enough to show that for any U,V open subsets containing the former and the latter respectively, $U \cap V \neq \emptyset$ (because X is Hausdorff). Using the definition of L_c, $(\pi_{\mathbf{c},n}(x_0)),(\pi_{\mathbf{c},n}(x_1)) \in L_{\mathbf{c},n}$ for all $n \geq max(n_0,n_1)$. Since ϕ,ψ,ϕ_R,ϕ_X are all continuous, there exists $n \geq max(n_0,n_1)$ with

$$\phi_R([\psi^{a_n}((\pi_{\mathbf{c},n}(x_0),\pi_{\mathbf{c},n}(x_1)))]) \subseteq U$$
$$\phi_X([\psi^{a_n} \circ \pi_{\mathbf{c},n}(x_0) \times \phi_X([\psi^{a_n} \circ \pi_{\mathbf{c},n}(x_1))]) \subseteq V$$

Let γ be a configuration compatible with a_n. By the definition of configuration we have

$$(\phi_R \circ \psi^{\gamma})((\pi_{\mathbf{c},n}(x_0),\pi_{\mathbf{c},n}(x,1))) = ((\phi_X \circ \phi^{\gamma})(\pi_{\mathbf{c},n}(x_0)),(\phi_X \circ \phi^{\gamma})(\pi_{\mathbf{c},n}(x_1)))$$

and since γ and a_n are compatible,

$$((\phi_X \circ \phi^{\gamma})(\pi_{\mathbf{c},n}(x_0)),(\phi_X \circ \phi^{\gamma})(\pi_{\mathbf{c},n}(x_1))) \in$$
$$\phi_X([\psi^{a_n} \circ \pi_{\mathbf{c},n}(x_0) \times \phi_X([\psi^{a_n} \circ \pi_{\mathbf{c},n}(x_1))]) \subseteq V$$

and

$$(\phi_R \circ \psi^{\gamma})((\pi_{\mathbf{c},n}(x_0),\pi_{\mathbf{c},n}(x,1))) \in \phi_R([\psi^{a_n}((\pi_{\mathbf{c},n}(x_0),\pi_{\mathbf{c},n}(x_1)))]) \subseteq U$$

so that $U \cap V \neq \emptyset$. Therefore $\phi_X \circ \phi$ is a homomorphism. $\square$

CHAPTER 3

BOREL AUTOMORPHISM GROUPS

3.1 Constructing the Graph

Recall that given a standard Borel group, we are trying to use Frucht's original idea in [6] to construct a Borel graph whose Borel automorphism group is isomorphic to the given group. First, let us show the construction of the graph. Let $(\mathcal{G}, \cdot, \mathcal{B})$ be a standard Borel group.

For each $\mathbf{a} \in 2^{\mathbb{N}}$, consider the graph $\mathbf{G_a} = (\mathbb{N}_{\geq 2}, R^*_{\mathbf{a}})$ where the edge relation is the symmetrization of the relation $R_{\mathbf{a}} = A_{\text{initial}} \cup A_{\text{fork}} \cup A_{\text{nofork}}$ with

$$\begin{aligned} A_{\text{initial}} &= \{(2,3),(3,4)\} \\ A_{\text{fork}} &= \left\{(n,n+1),(n,n+2) : \ n \in 2\mathbb{N}_{\geq 2}, \ \mathbf{a}\left(\frac{n-4}{2}\right) = 1\right\} \\ A_{\text{nofork}} &= \left\{(n,n+1),(n+1,n+2) : \ n \in 2\mathbb{N}_{\geq 2}, \ \mathbf{a}\left(\frac{n-4}{2}\right) = 0\right\} \end{aligned}$$

The placement of edges in $\mathbf{G_a}$ can be described as an iterative process as follows. Regardless of $\mathbf{a}$, we first put an edge between 2 and 3, and, 3 and 4. For each even integer $n \geq 4$, depending on whether $\mathbf{a}\left(\frac{n-4}{2}\right)$ is zero or one, we either create a fork at n using the next two vertices with odd vertex having degree one, or add an edge between successive vertices for the next two vertices. The following figure is an example:

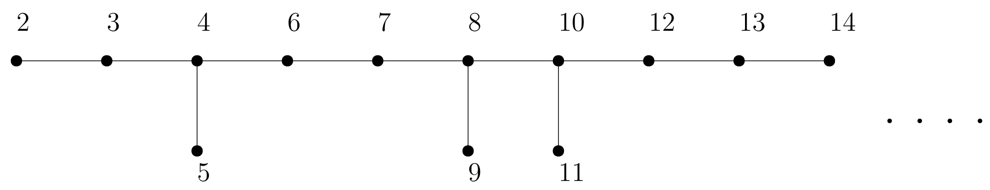

Figure 3.1: A diagrammatic representation of $\mathbf{G_a}$ with $\mathbf{a} = (1, 0, 1, 1, 0, \dots)$

We shall now argue that any such graph $\mathbf{G_a}$ is asymmetric, i.e., it has no non-trivial automorphisms. Let $\mathbf{a} \in 2^{\mathbb{N}}$ and $\varphi \in Aut(\mathbf{G_a})$. Observe that 2 is the only vertex of degree one that is adjacent to a vertex of degree two. Hence φ fixes 2 which immediately implies that 3 and 4 are fixed under φ as well. Let $n \geq 4$ be an even integer. Suppose that φ fixes all vertices $2 \leq k \leq n$. Then there are two possibilities:

- If $\mathbf{a}\left(\frac{n-4}{2}\right) = 1$, then $n+1$ is a vertex of degree one and $n+2$ is a vertex of degree at least two, in which case φ fixes both.
- If $\mathbf{a}\left(\frac{n-4}{2}\right) = 0$, then φ clearly fixes $n+1$ because n is fixed by φ and the other neighbors of n have already been fixed. But subsequently, φ must fix $n+2$ as well by a similar argument.

Therefore φ fixes all the vertices $2 \leq k \leq n+2$. By induction, φ fixes all vertices in $\mathbf{G_a}$. A similar inductive argument shows that $\mathbf{G_a}$ and $\mathbf{G_b}$ are not isomorphic whenever $\mathbf{a}$ and $\mathbf{b}$ are distinct elements of $2^{\mathbb{N}}$.

Next will be constructed the main graph associated to an uncountable standard Borel group. Fix an uncountable standard Borel group $(\mathcal{G}, \cdot, \mathcal{B})$. In order to implement Frucht's idea, we first need to find an appropriate Cayley graph for $(\mathcal{G}, \cdot, \mathcal{B})$. An obvious choice for a generating set is the Borel set $S = \mathcal{G} \setminus \{1_{\mathcal{G}}\}$. Suppose that we constructed the Cayley graph associated to this generating set. In this graph, there is a labeled directed edge from the first component to the second components of each element of $(\mathcal{G} \times \mathcal{G}) \setminus \Delta_{\mathcal{G}}$. We would like to replace each of these directed labeled edges by an appropriate asymmetric connected countable graph that we have already constructed. Consequently, for each element of $(\mathcal{G} \times \mathcal{G}) \setminus \Delta_{\mathcal{G}}$, we need to add countably many "new" vertices to "old" vertices. Therefore, it is natural to consider

$$X = \mathcal{G} \times \mathcal{G} \times \mathbb{N}$$

as the vertex set of the main (undirected) graph to be constructed. In this vertex set,

- the vertices of the form $(x, x, 0)$ where $x \in \mathcal{G}$ are supposed to represent the "old" vertices that are the group elements,

- the vertices of the form (x, y, k) where $x, y \in \mathcal{G}$ with $x \neq y$ and $k \in \mathbb{N}$ are the "new" vertices that are added after replacing the directed labeled edges, and

- the vertices of the form (x, x, k) where $x \in \mathcal{G}$ and $k \neq 0$ are "irrelevant" elements that will essentially serve no purpose. We could simply have taken these elements out of the vertex set, however, there is no harm in keeping them around. In order for these vertices to not create any additional symmetries, we will stick an infinite line formed by them to $(x, x, 0)$.

We shall next construct the main graph on the vertex set X. Recall that each directed labeled edge in the Cayley graph of $\mathcal{G}$ with respect to S, which corresponds to an element of S, is to be replaced by one of the continuum-many asymmetric graphs that we initially constructed. This supply of asymmetric graphs were parametrized by $2^{\mathbb{N}}$. Consequently, it suffices to parametrize $\mathcal{G}$ by $2^{\mathbb{N}}$. Since $(\mathcal{G}, \mathcal{B})$ is an uncountable standard Borel space, it follows from the Borel isomorphism theorem that there exists a Borel isomorphism $\Psi : \mathcal{G} \to 2^{\mathbb{N}}$.

Before we proceed, we would like to take a moment to let the reader know in advance that we will later require $\Psi : \mathcal{G} \to 2^{\mathbb{N}}$ to have other additional properties in the proof of Theorem 7. Indeed, as we shall see later, the Borel complexity of our graph, i.e., where it resides in the Borel hierarchy of the Polish space $(X \times X, \tau \times \tau)$, is completely determined by the Borel complexity of inverse images of the clopen basis elements of $2^{\mathbb{N}}$ under Ψ.

Consider the relation $G = G_{\text{irrelevant}} \cup G_{\text{blockbase}} \cup G_{\text{fork}} \cup G_{\text{nofork}}$ where

$$G_{\text{blockbase}} = \Big\{ \Big((x,x,0),(x,y,0) \Big), \Big((x,y,0),(x,y,1) \Big), \Big((x,y,0),(x,y,2) \Big), \Big((x,y,2),(x,y,3) \Big), \Big((x,y,3),(x,y,4) \Big), \Big((x,y,2),(y,y,0) \Big) : x,y \in \mathcal{G},\ x \neq y \Big\}$$

$$G_{\text{forks}} = \Big\{ \Big((x,y,n),(x,y,n+1) \Big), \Big((x,y,n),(x,y,n+2) \Big) : x,y \in \mathcal{G},\ x \neq y,\ n \in 2\mathbb{N}_{\geq 2},\ \Psi\left(x^{-1}y\right)\left(\frac{n-4}{2}\right) = 1 \Big\}$$

$$G_{\text{noforks}} = \Big\{ \Big((x,y,n),(x,y,n+1) \Big), \Big((x,y,n+1),(x,y,n+2) \Big) : x,y \in \mathcal{G},\ x \neq y,\ n \in 2\mathbb{N}_{\geq 2},\ \Psi\left(x^{-1}y\right)\left(\frac{n-4}{2}\right) = 0 \Big\}$$

$$G_{\text{irrelevant}} = \Big\{ \Big((x,x,n),(x,x,n+1) \Big) :\ x \in \mathcal{G},\ n \in \mathbb{N} \Big\}$$

An illustration of the edges in G^* for a pair of group elements x and y is given in Figure 3.2 as an undirected graph, where we assume for illustrative purposes that $\Psi\left(x^{-1}y\right) = (1,0,1,1,0,\dots)$ and $\Psi\left(y^{-1}x\right) = (1,1,1,1,0,\dots)$.

It is a routine verification to check that G is a Borel subset of $X \times X$. Here we will only show that G_{forks} is indeed Borel as a guiding example. Let $n \in 2\mathbb{N}_{\geq 2}$. Then the set

$$O = \left\{ \mathbf{a} \in 2^{\mathbb{N}} : \mathbf{a}\left(\frac{n-4}{2}\right) = 1 \right\}$$

is a clopen subset of $2^{\mathbb{N}} = \{0,1\}^{\mathbb{N}}$. Consider the map from $f : \mathcal{G} \times \mathcal{G} \to 2^{\mathbb{N}}$ given by $f(x,y) = \Psi\left(x^{-1}y\right)$. Since $(\mathcal{G}, \cdot, \mathcal{B})$ is a standard Borel group, f is a Borel map and hence $B = f^{-1}(O)$ is a Borel subset of $\mathcal{G} \times \mathcal{G}$. Set $A = B \setminus \Delta_{\mathcal{G}}$. It follows that $A \times \{n\}$, $A \times \{n+1\}$ and $A \times \{n+2\}$ are Borel subsets of X and hence, their pairwise cartesian products are Borel subsets of $X \times X$. Observe that the relation

$$D = \Big\{ \Big((x,y,i),(x,y,j) \Big) :\ x,y \in \mathcal{G},\ i,j \in \mathbb{N} \Big\}$$

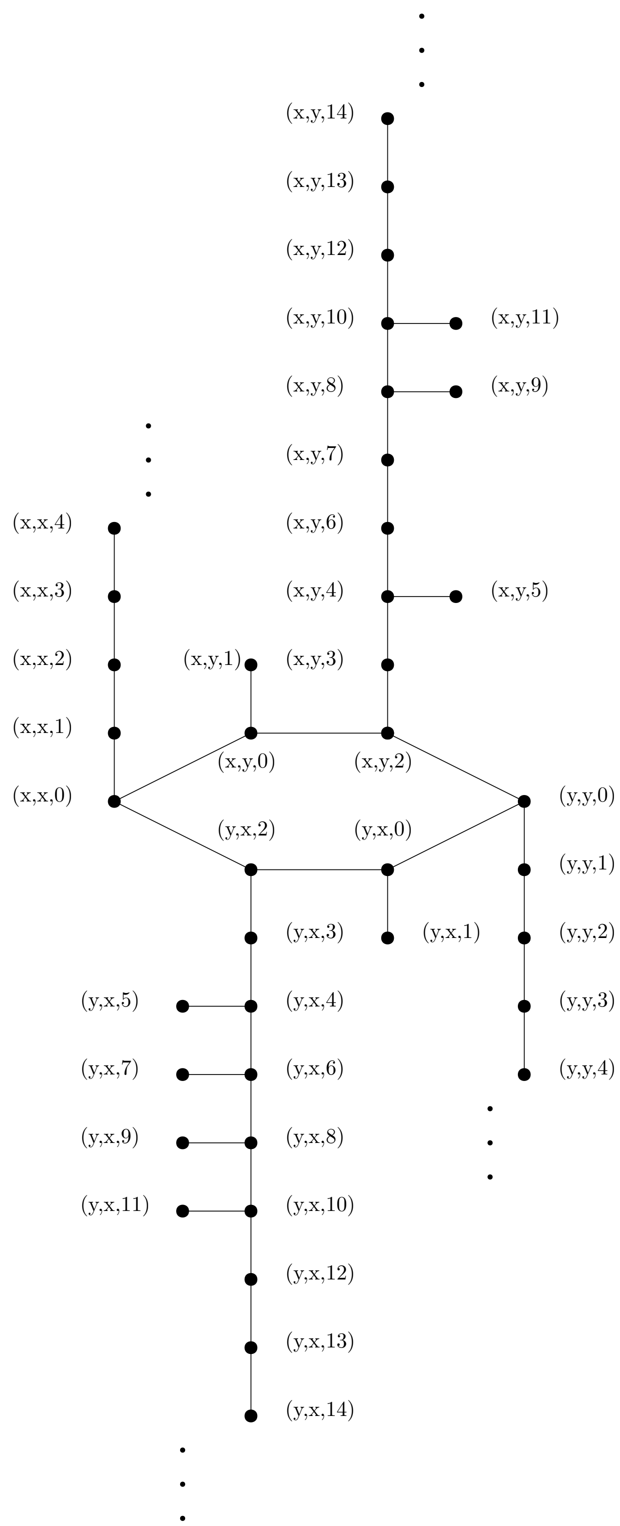

Figure 3.2: A representation of edges in G^* for a pair of group elements x and y

is a Borel subset of $X \times X$. Indeed, this is a closed subset of $X \times X$ once it is endowed with the product topology arising from the discrete topology on $\mathbb{N}$ and any topology turning $\mathcal{G}$ into a Polish space compatible with its Borel structure. But then

$$G_{\text{forks}} = D \cap \bigcup_{n \in 2\mathbb{N}_{\geq 2}} \big((A \times \{n\}) \times (A \times \{n+1\})\big) \cup \big((A \times \{n\}) \times (A \times \{n+2\})\big)$$

is a Borel subset of $X \times X$. That G_{noforks}, $G_{\text{blockbase}}$ and $G_{\text{irrelevant}}$ are Borel can be shown by similar arguments with appropriate modifications. Thus $\mathbf{G} = (X, G^*)$ is a Borel graph.

3.2 Proof of Theorem 6

Let $(\mathcal{G}, \cdot, \mathcal{B})$ be a standard Borel group. Suppose for the moment that $\mathcal{G}$ is uncountable. Set $\mathbf{G} = (X, G^*)$ to be the Borel graph constructed in Section 2 associated to $(\mathcal{G}, \cdot, \mathcal{B})$. We wish to show that $\mathcal{G}$ and $Aut_B(\mathbf{G})$ are isomorphic. For each $g \in \mathcal{G}$, consider the map $\varphi_g : X \to X$ given by

$$\varphi_g(x, y, k) = (gx, gy, k)$$

for all $x, y \in \mathcal{G}$ and $k \in \mathbb{N}$. Clearly φ_g is a bijective map. Observe that left-multiplying the first two components of each element of G by g leaves the sets $G_{\text{irrelevant}}$, $G_{\text{blockbase}}$, G_{fork} and G_{nofork} invariant. To see that G_{fork} and G_{nofork} are invariant under φ_g, observe that $x^{-1}y = (gx)^{-1}(gy)$. Thus φ_g is an automorphism. Since the group multiplication is Borel, so is φ_g. It follows that $\varphi_g \in Aut_B(\mathbf{G})$. Define the map $\Phi : \mathcal{G} \to Aut_B(\mathbf{G})$ by $\Phi(g) = \varphi_g$ for all $g \in \mathcal{G}$. Then clearly Φ is injective and moreover, we have

$$\Phi(gh) = \varphi_{gh} = \varphi_g \circ \varphi_h = \Phi(g) \circ \Phi(h)$$

Thus Φ is a group embedding. It remains to show that Φ is surjective.

Let $f \in Aut(\mathbf{G})$ be an arbitrary automorphism. Observe that the set of vertices which has one neighbor of infinite degree and another neighbor of degree one is precisely

$$\{(x, y, 0) : x, y \in \mathcal{G},\ x \neq y\}$$

Therefore, being an automorphism, f permutes this set. Let $x, y \in \mathcal{G}$ be distinct and set $(x', y', 0) = f(x, y, 0)$. Note that the only neighbors of $(x, y, 0)$ and $(x', y', 0)$

- of degree one is $(x, y, 1)$ and $(x', y', 1)$ respectively,
- of degree three is $(x, y, 2)$ and $(x', y', 2)$ respectively,
- of uncountable degree is $(x, x, 0)$ and $(x', x', 0)$ respectively.

Thus $f(x, y, 1) = (x', y', 1)$, $f(x, y, 2) = (x', y', 2)$ and $f(x, x, 0) = (x', x', 0)$. By a similar argument, since we already obtained $f(x, y, 2) = (x', y', 2)$, we must also have that $f(y, y, 0) = (y', y', 0)$. We would like to point out that the equalities $f(x, x, 0) = (x', x', 0)$ and $f(y, y, 0) = (y', y', 0)$ together show that x' only depends on x and y' only depends on y.

Recall that the graph $\mathbf{G}_{\Psi(x^{-1}y)} = \left(\mathbb{N}_{\geq 2}, R^*_{\Psi(x^{-1}y)}\right)$ constructed at the very beginning has no non-trivial automorphisms. Consequently, an inductive argument as was done in Section 2 shows that $f(x, y, n) = (x', y', n)$ for all $n \geq 2$.

This last conclusion immediately implies that $\Psi(x^{-1}y) = \Psi(x'^{-1}y')$. Since Ψ is injective, we have $x^{-1}y = x'^{-1}y'$ and hence $x'x^{-1} = y'y^{-1}$.

Set $g = x'x^{-1} \in \mathcal{G}$. Then we have $gx = x'$ and $gy = y'$. Therefore

$$f(x, y, n) = (x', y', n) = \varphi_g(x, y, n)$$

for all $n \in \mathbb{N}$. Recall that x' depends only on x and y' depends only on y. Consequently, if we used another $z \in \mathcal{G}$ instead of x or y, we still would have found the same group element g because in this case we would have $g = x'x^{-1} = y'y^{-1} = z'z^{-1}$. Therefore, we indeed have

$$f(x, y, n) = (x', y', n) = \varphi_g(x, y, n)$$

not only for the previously fixed x, y but for all distinct $x, y \in \mathcal{G}$ and $n \in \mathbb{N}$. Hence f agrees with φ_g on $X \setminus (\Delta_{\mathcal{G}} \times \mathbb{N})$. It also follows from $f(x, x, 0) = (x', x', 0) = (gx, gx, 0)$ via an inductive argument that $f(x, x, n) = (gx, gx, n)$ for all $n \in \mathbb{N}$. Thus f is identically φ_g on X. Hence Φ is an isomorphism and we indeed have $Aut(\mathbf{G}) = Aut_B(\mathbf{G})$.

Finally, suppose that $\mathcal{G}$ is a countable standard Borel group. In this case, we choose Ψ to be any (necessarily Borel) bijection from $\mathcal{G}$ to any (necessarily Borel) subset of $2^{\mathbb{N}}$ with cardinality $|\mathcal{G}|$ and implement the same construction. The exact same argument proving $Aut(\mathbf{G}) = Aut_B(\mathbf{G})$ in the uncountable case still goes through in the countable case, with appropriate modifications in the extreme case $|\mathcal{G}| = 1$.

3.3 Proof of Theorem 7

Let $(\mathcal{G}, \tau)$ be a Polish group. It is well-known [10, Theorem 4.14] that there exists a continuous injection $\gamma : \mathcal{G} \to [0,1]^{\mathbb{N}}$. Consider the map $\xi : [0,1]^{\mathbb{N}} \to 2^{\mathbb{N}\times\mathbb{N}}$ given by $\xi(\mathbf{x})(i,j) = 1$ if and only if the i-th digit of the binary expansion of x_j is equal to 1, where the binary expansions of dyadic rationals are taken to end in infinitely many repating 1's. It is straightforward to check that ξ is a $\mathbf{\Sigma^0_2}$-map, i.e., the inverse images of open sets are $\mathbf{\Sigma^0_2}$. Fix a homeomorphism $\zeta : 2^{\mathbb{N}\times\mathbb{N}} \to 2^{\mathbb{N}}$ and set $\widehat{\Psi} = \zeta \circ \xi \circ \gamma$.

We now carry out the same construction of $\mathbf{G} = (X, G^*)$ in Section 2 but we use the $\mathbf{\Sigma^0_2}$-injection $\widehat{\Psi} : \mathcal{G} \to 2^{\mathbb{N}}$ instead of the Borel bijection $\Psi : \mathcal{G} \to 2^{\mathbb{N}}$. Then the set A in the construction is $\mathbf{\Sigma^0_2}$. It follows that G_{forks} and G_{noforks} are $\mathbf{\Sigma^0_2}$. It is also easily seen that $G_{\text{blockbase}}$ and $G_{\text{irrelevant}}$ are $\mathbf{\Delta^0_2}$ and closed respectively. Therefore G^* is a $\mathbf{\Sigma^0_2}$-subset of $X \times X$.

We next execute the proof of Theorem 6 as it is. Observe that the automorphisms $\varphi_g : X \to X$ constructed in the proof are homeomorphisms. Moreover, $\widehat{\Psi}$ being injective suffices for the argument to go through. Thus we obtain that $\mathrm{Aut}(\mathbf{G}) = \mathrm{Aut}_h(\mathbf{G})$ and that $\Phi : \mathcal{G} \to \mathrm{Aut}_h(\mathbf{G})$ is an isomorphism.

We shall next prove that Φ is indeed a homeomorphism whenever the group $\mathrm{Aut}_h(\mathbf{G}) \subseteq \mathrm{Homeo}(X)$ is endowed with the subspace topology induced from the compact-open topology of $\mathrm{Homeo}(X)$. Let $\{O_\alpha\}_{\alpha\in I}$ be the usual basis for the product topology of X. Recall that the collection

$$\{\{f \in C(X,X) : \ f[K] \subseteq O_\alpha\} : \ K \subseteq X \text{ is compact}, \ \alpha \in I\}$$

is a subbase for the compact-open topology of $C(X,X)$. Let $U \subseteq \mathcal{G}$ be open. Then

$$\Phi(U) = \{\varphi_g \in \mathrm{Aut}_h(\mathbf{G}) : \ \varphi_g(1_{\mathcal{G}}, 1_{\mathcal{G}}, 1) \in U \times U \times \{1\}\}$$

Since the set $\{(1_{\mathcal{G}}, 1_{\mathcal{G}}, 1)\}$ is compact and $U \times U \times \{1\}$ is open in X, the set $\Phi(U)$ is open in the subspace topology of $\mathrm{Aut}_h(\mathbf{G})$. Hence Φ^{-1} is continuous.

Let $V_{K,O} \subseteq X$ be a subbasis element of the subspace topology of $\mathrm{Aut}_h(\mathbf{G})$ where $K \subseteq X$ is compact, $U_1 \times U_2 \times U_3 = O \subseteq X$ is a basis element with $U_1, U_2 \subseteq \mathcal{G}$ and $U_3 \subseteq \mathbb{N}$ open; and

$$V_{K,O} = \{\varphi \in \mathrm{Aut}_h(\mathbf{G}) : \varphi[K] \subseteq O\}$$

We wish to show that $\Phi^{-1}[V_{K,0}] = \{g \in \mathcal{G} : \varphi_g[K] \subseteq U_1 \times U_2 \times U_3\}$ is open. Observe that if $\pi_3[K] \nsubseteq U_3$, then $V_{K,O} = \emptyset$. So suppose that $\pi_3[K] \subseteq U_3$. Then we have

$$\Phi^{-1}[V_{K,0}] = \{g \in \mathcal{G} : g\pi_1[K] \subseteq U_1\} \cap \{g \in \mathcal{G} : g\pi_2[K] \subseteq U_2\}$$

We claim that both sets on the right hand side are open. To see this, let $g \in \mathcal{G}$ be such that $g\pi_i[K] \subseteq U_i$. For each $k \in \pi_i[K]$, since the multiplication on $\mathcal{G}$ is continuous and $gk \in U_i$, we can choose an open basis element $(g, k) \in V_k \times W_k$ of $\mathcal{G} \times \mathcal{G}$ such that $V_k \cdot W_k \subseteq U_i$. Since $\{W_k\}_{k \in K}$ is an open cover of the compact set $\pi_i[K]$, there exists a finite subcover $\{W_{k_j}\}_{j=1}^n$. Set $V = \bigcap_{j=1}^n V_{k_j}$. Then $g \in V$ and $V \cdot \pi_i[K] \subseteq U_i$. Thus $\{g \in \mathcal{G} : g\pi_i[K] \subseteq U_i\}$ is open. Hence Φ is continuous and so, is a homeomorphism.

CHAPTER 4

CONCLUSIONS

In the first part of the thesis, we surveyed the technique found by Ben Miller to prove the $\mathbf{G}_0$ dichotomy, using only classical methods. We also provided a proof of $\mathbf{L}_0$ dichotomy on undirected graphs using the same technique.

In the second part of the thesis, we provided a complete generalization of Frucht's theorem to Borel measurable and topological settings. However, due to the natural limitations of our coding technique, in topological setting, we were not able to obtain minimal complexity in Theorem 7. Therefore, we pose the following question.

Question. Is it true that for every Polish group $(\mathcal{G}, \cdot, \tau)$ there exists a closed or open graph $\mathbf{G} = (X, G)$ on a Polish space $(X, \widehat{\tau})$ such that $\mathcal{G}$ and $\mathrm{Aut}_h(\mathbf{G})$ are isomorphic as abstract or topological groups?

We strongly suspect that the answer is affirmative. Such a result may be obtained via a construction similar to ours that uses a continuous injection $\Psi : \mathcal{G} \to [0,1]^{\mathbb{N}}$ which we know exists for arbitrary second-countable metrizable spaces $\mathcal{G}$ [17, Theorem 2.1.32]. However, this would require one to construct continuum-many acyclic Borel graphs that code each element of the Hilbert cube $[0,1]^{\mathbb{N}}$ in such a way that each edge corresponds to an open or closed condition in $[0,1]^{\mathbb{N}}$. It is not clear to us how this can be done.

Observe that, due to the nature of the construction, the graphs that we obtained automatically ended up satisfying $\mathrm{Aut}(\mathbf{G}) = \mathrm{Aut}_B(\mathbf{G})$. However, it is trivial to observe via counting arguments that it is possible to have Borel graphs $\mathbf{G}$ such that $|\text{Aut}_B(\mathbf{G})| \leq 2^{\aleph_0} < 2^{2^{\aleph_0}} \leq |\text{Aut}(\mathbf{G})|$, for example, consider the complete graph $K_{\mathbb{R}}$. The next obvious question would be to ask whether it is possible to have $|\text{Aut}_B(\mathbf{G})| \leq \aleph_0 < 2^{\aleph_0} \leq |\text{Aut}(\mathbf{G})|$ for a Borel graph. Having corresponded with Andrew Marks, we learned that this question also has an affirmative answer. Here we briefly sketch his argument: Given a countable language $\mathcal{L}$, for any $\mathcal{L}$-structure $\mathcal{M}$ whose universe is a Polish space and whose functions and relations are Borel maps, one can construct a Borel graph $\mathbf{G}_{\mathcal{M}}$ on a Polish space such that $\text{Aut}_B(\mathcal{M}) \cong \text{Aut}_B(\mathbf{G}_{\mathcal{M}})$ and $\text{Aut}(\mathcal{M}) \cong \text{Aut}(\mathbf{G}_{\mathcal{M}})$. This can be achieved by appropriately modifying the argument which shows that arbitrary structures may be interpreted as graphs, e.g. see [8, Theorem 5.5.1]. Consequently, it suffices to find $\mathcal{M}$ such that $|\text{Aut}_B(\mathcal{M})| \leq \aleph_0 < 2^{\aleph_0} \leq |\text{Aut}(\mathcal{M})|$.

An example of such a structure would be $(\mathbb{R}, +, 1)$. Since any Borel measurable group automorphism of the Polish group $(\mathbb{R}, +)$ is automatically continuous [10, Theorem 9.10] and any continuous automorphism of $(\mathbb{R}, +)$ is precisely of the form $x \mapsto rx$, we have that $|\text{Aut}_B(\mathbb{R}, +, 1)| = 1$. On the other hand, since any permutation of a $\mathbb{Q}$-basis of $\mathbb{R}$ would induce a group automorphism and $\dim_{\mathbb{Q}}(\mathbb{R}) = 2^{\aleph_0}$, we have $|\text{Aut}(\mathbb{R}, +, 1)| = 2^{2^{\aleph_0}}$.

Having seen that the Borel and full automorphism groups of a Borel graph can be separated in cardinality, the following question seems to be the next step in our initial investigation.

Question. Given two standard Borel groups $\mathcal{H} \leq \mathcal{G}$, does there necessarily exist a Borel graph $\mathbf{G}$ such that $\text{Aut}_B(\mathbf{G}) \cong \mathcal{H}$ and $\text{Aut}(\mathbf{G}) \cong \mathcal{G}$, where the former isomorphism is the restriction of the latter?

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