Prospective Teachers' Instructional Responses based on Students' Functional Thinking: The Context of Pattern Generalization

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This qualitative case study aimed to investigate prospective middle school mathematics teachers' instructional responses based on the students' correct and incorrect functional thinking within the context of pattern generalization. The data were collected from thirty-two prospective teachers through a written task and semi-structured interviews and analyzed with open coding. The study's findings revealed that most prospective teachers could support the functional thinking of the student who had an incorrect solution. However, they could not extend the student's functional thinking of those who reached the correct solution. Instead, they asked the student to do another similar drill or provided a general response to the student with the correct solution without extending their functional thinking.

Introduction

Building mathematics instruction on students' thinking is important for effective mathematics teaching (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; National Council of Teachers of Mathematics, 2014). Instruction that builds on the students' thinking includes understanding students' existing knowledge and mathematical thinking by assessing their problem-solving processes and reaching appropriate instructional decisions (Carpenter et al., 1989). From this point of view, to provide instruction in which students' thinking is at the center, one of the key issues is the teachers' correct analysis of students' thinking (Carpenter et al., 1989; Jacobs, Franke, Carpenter, Levi, & Battey, 2007). To be aware of how students think, teachers should be able to identify students' understanding, explore their misconceptions and misunderstandings and make inferences about their mathematical understanding. In addition to understanding students' thinking, responding to students and making appropriate instructional decisions are the other key issues to constructive instruction in which students' thinking is at the center (Ball, Lubienski, & Mewborn, 2001). To understand students' solutions, analyze their understanding and propose appropriate instructions, teachers must have various competencies that reflect in the teaching practice (König, Blömeke & Kaiser, 2017). One of the teacher competencies to carry out the instructional foreground on students' thinking is teacher knowledge. Accordingly, mathematics education research has grown interested in transferring teacher knowledge into instruction (NCTM, 2014). A wealth of empirical studies in mathematics education focuses on investigating teacher knowledge and how to bring it into providing instruction (NCTM, 2014). Concerning this, teachers' attending to students' strategies, their interpretations of the students' mathematical understanding, and reflecting on these interpretations to decide the next teaching moves are regarded as the ways of enactment of teacher knowledge into the instruction (Adler & Davis, 2006). Ball and Bass (2009) affirmed the thoughts of Adler and Davis by attributing them as teaching responsibilities and acts dependent on the teacher's knowledge. The recent studies dealt with these responsibilities which focused on building the instruction concerning the students' thinking and referred to them as teacher noticing (Jacobs, Lamb, & Philipp, 2010; König, Blömeke, Klein, Suhl, Busse, & Kaiser, 2014; van Es & Sherin, 2002).

Teacher noticing involves the skill of analyzing the extent to which students' solutions are meaningful to learning mathematics conceptually and the nature of teachers' decisions while responding to students to support and extend their thinking (Hines & McMahon, 2005; Jacobs et al., 2010; van Es & Sherin, 2008). In general, teacher noticing is defined as noticing the classroom environment, teacher pedagogy, students' thinking, and behavior. However, in a more specialized sense, Jacobs et al. (2010) addressed the key issues of building instruction based on students' thinking within the framework of professional noticing of children's mathematical thinking. Therefore, teacher noticing includes three skills: identifying students' strategies, making sense of their understanding, and responding to students based on this understanding (Jacobs et al., 2010). Although all these skills have a critical role in giving opportunities to students to learn mathematics conceptually, they added that responding based on students' understanding is more effective in supporting and extending students' mathematics learning. This is because teachers need to provide an instructional response to students who produce incorrect and correct solutions. Thus, teachers can manage their students' misconceptions and develop their thinking.

In recent years, many research studies have been conducted to investigate teachers' responses to students with correct understanding or incorrect understanding (Casey, Lesseig, Monson, & Krupa, 2018; Doğan & Kılıç, 2019; Jacobs, Lamb, Philipp, & Schappelle, 2011; Land, Tyminski, & Drake, 2018; Son, 2013; Son & Sinclair, 2010). However, the number of research studies on teachers' responses to students having correct and incorrect understanding is limited. There is still room for investigation of the types of prospective teachers' instructional responses to elaborate and develop students' thinking. Therefore, in the present study, we aimed to investigate how prospective middle school mathematics teachers respond based on students' correct and incorrect functional thinking within the context of pattern generalization.

Theoretical Framework

Although many researchers agree on the importance of teacher noticing, they have used different frameworks to explore how teachers notice students' mathematical understanding. The two main frameworks that researchers have used are "Learning to Notice" and "Professional Noticing of Children's Mathematical Thinking." In the framework of Learning to Notice, van Es (2011) explored teacher noticing under two dimensions having four levels (baseline, mixed, focused, and extended): what teachers notice and how teachers notice (van Es, 2011). Van Es stated that there is a hierarchical order between the levels; that is, while the baseline level is a general level that deals with the classroom environment, the extended level is a more detailed level that focuses on student thinking. However, Jacobs, Lamb, and Philipp (2010) were interested in the extended level of the Learning to Notice framework. They added one more major dimension, which focused on teachers' in-the-moment decisions in their response to students. They claimed that building mathematics instruction based on students' thinking requires expertise in attending to their strategies, interpreting their thinking, and deciding how to respond to students based on their understanding. Jacobs et al. proposed the Professional Noticing of Children's Mathematical Thinking framework, which includes these three interrelated facets.

Professional Noticing of Children's Mathematical Thinking

Professional noticing of children's mathematical thinking is constructed on how and to what extent teachers notice children's mathematical thinking instead of what teachers notice (Jacobs et al., 2010). Specifically, it focuses on teachers' analysis of students' solutions in mathematical learning and their decisions on an instructional response (Jacobs et al., 2010). Within this context, teachers' expertise in professional noticing is conceptualized as "attending to children's strategies, interpreting children's understanding, and deciding how to respond based on children's understanding" (Jacobs et al., 2010, p. 172).

Attending, defined as highlighting or drawing attention to the mathematically meaningful and important points in the teaching environment (Jacobs et al., 2010), enables teachers to understand students' solutions (Carpenter, Fennema, Franke, Levi, & Empson, 1999). The skill of interpreting refers to making inferences about students' mathematical thinking by relating to students' solutions (Jacobs et al., 2010). The skill of deciding how to respond is related to how teachers change their in-the-moment decisions during instruction depending on students' mathematical thinking (Jacobs et al., 2010). Although attending and interpreting are important in revealing students' existing mathematical thinking and instructional decisions, they are not adequate for effective instructional response (Barnhart & van Es, 2015; Jacobs et al., 2010). Hence, teachers should understand that there is more to do than attending and interpreting. (Jacobs et al., 2010; Schoenfeld, 2011).

As Adler and Davis (2006) emphasized, the three skills of Professional Noticing of Children's Mathematical Thinking are the key to how teachers provide follow-up instruction. More specifically, to analyze the mathematically significant details embedded in students' strategies and making sense of students' understanding, specialized content knowledge (SCK) and knowledge of content and student (KCS) are essential teacher competencies (Ball et al., 2008). Additionally, teachers need to have knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC) to propose appropriate follow-up instructions. If the teachers have solid KCT, they know the most suitable instructions and teaching strategies for any mathematical topic (Ball et al., 2008). Moreover, they can build the link to the mathematical topics in the curriculum with the help of KCC (Ball et al., 2008). KCC enables them to give the most suitable response based on their understanding. Considering all this, SMK and PCK are essential for teachers to notice students' understanding.

Since the present study aims to explore prospective middle school mathematics teachers' instructional responses based on students' functional thinking, it is grounded on the third component of the Professional Noticing of Children's Mathematical Thinking framework, namely, *Deciding How to Respond based on Children's Mathematical Understandings.* Jacobs et al. (2010) argue that according to this component, there is no best response. It is important to understand to what extent the teachers' responses are suitable for students' mathematical thinking. Furthermore, while teachers respond to students, their role is not to tell students how they should think. Instead, the expected role of teachers in this component is to support and extend students' mathematical thinking by giving appropriate problems and asking questions while students are discussing solutions to problems (Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Jacobs et al., 2010). The teachers fulfill their role if they have SCK and KCS, which enables them to analyze and interpret students' mathematical thinking, KCT, and KCC allowing them to propose the most efficient teaching strategies (Ball et al., 2008). Deciding how to respond based on children's mathematical understanding requires having solid teacher knowledge (Barker, Lannin, Winsor, & Kirwan, 2019) and attending to and interpreting students' thinking (Jacobs et al., 2010). Therefore, it could be regarded as an important skill that extends students' correct functional thinking and helps students correct their incorrect functional thinking.

Functional Thinking

Mathematics consists of three main foci: things (e.g., numbers, shapes, and variables), relationships between things, and transformations of things (Scandura, 1971). Relationships and transformations lead to patterns and generalizations and are considered the power of mathematics (Johnassen, Beissner, & Yacci, 1993; Sfard, 1991). For this reason, the development of system justification and the generalization of patterns are the core of mathematics instruction (Sfard, 1991; Warren & Cooper, 2005). Functional thinking involving these three foci is defined as understanding relationships between variables or operations and generalizing (Blanton & Kaput, 2011; Warren & Cooper, 2005).

In national and international curricula, students are encouraged to think about functional relationships and transition linguistically from iconic and natural language to symbolic notation systems at early grades (Blanton & Kaput, 2011; MoNE, 2018; NCTM, 2014). Although this mathematical concept is important in the curriculum (NCTM, 2014; MoNE, 2018), students have difficulty developing functional thinking (Dede & Argün, 2003; Jurdak & Mouhayar, 2014). However, some researchers argue that students' functional thinking can be developed by providing the most appropriate instructional response depending on students' functional thinking (English & Warren, 1998; Jacobs et al., 2010). In other words, teachers could help students realize their mistakes/misconceptions in pattern generalization and make it easier to establish their functional thinking correctly. In addition, teachers could take students' functional thinking a step further by giving an appropriate response. For example, suppose students could build a functional relationship with algebraic expressions. In that case, teachers may ask a problem about the inverse process *-identifying the number of steps in a pattern by using inverse functional relationships-* to extend students' functional thinking. From this point of view, the prospective middle school mathematics teachers' instructional responses according to students' functional thinking are among the key factors in developing students' functional thinking.

Rationale of the Study

"Teaching is one of the most common—and also one of the most complicated—human activities" (Ball & Forzani, 2010, p.40). To teach effectively, teachers must learn to distinguish the noteworthy issues to pay attention to and to handle complicated events that happen in the classroom; therefore, teachers' noticing skill is a significant competency (Jacobs, Lamb, & Philipp, 2010; König et al., 2014; Sherin & Star, 2011; Star & Strickland, 2008; van Es & Sherin, 2002). For this reason, several research studies were conducted to explore teachers' professional noticing of students' mathematical thinking (Callejo & Zapatera, 2017; Jacobs et al., 2010; Kılıç & Doğan, 2021; LaRochelle et al., 2019; Llinares, 2013; Melhuish, Thanheiser, & Guyot, 2020; Osmanoğlu, Işıksal, & Koç, 2015; Schack et al., 2013). However, understanding and interpreting students' thinking are insufficient to advance their mathematical understanding (Jacobs & Ambrose, 2008; Jacobs & Empson, 2016). The instructional responses that support and extend students' mathematical thinking must focus on effective teaching (Ball et al., 2001; Jacobs & Ambrose, 2008; Jacobs & Empson, 2016). Despite this, research on prospective teachers' skill of responding according to students' mathematical thinking is limited. Some of these studies were conducted to construct a response rubric (Land et al., 2018; Milewski & Strickland, 2016), while others explored teachers' responses based on students' mathematical thinking (Casey et al., 2018; Crespo, 2002; Doğan & Kılıç, 2019; Rach, Ufer, & Heinze, 2013; Son, 2013; Son & Sinclair, 2010). Thus, it would be significant to investigate prospective teachers' professional noticing skills focusing on the types of their responses. Deciding how to respond is the most challenging dimension and requires expertise in attending students' strategies and interpreting their thinking (Jacobs et al., 2010). Considering this, prospective teachers' instructional responses would provide information on how they attend and interpret students' thinking. Therefore, investigating prospective teachers' instructional responses would contribute to the mathematics education literature.

Although many researchers focused on prospective teachers' responses to students who have incorrect solutions (Casey et al., 2018; Rach et al., 2013; Son, 2013; Son & Sinclair, 2010), there are few studies evaluating teachers' or prospective teachers' responses to students having both correct and incorrect solutions (Crespo, 2002). However, teachers' instructional responses to students' correct and incorrect solutions have a crucial role. This is especially important in extending students' correct understanding of new problems and supporting their misunderstandings through follow-up questions (Jacobs et al., 2010). With the help of PCK, teachers size up students' thinking and challenge them to think and explain their thinking (Ball et al., 2001; Ball et al., 2008). More specifically, teachers with a high level of SCK and KCS can easily assess students' solutions and interpret their thinking (Ball et al., 2008). After they analyze students' thinking, they can ask appropriate probing questions so that students can recognize their misconceptions or further their thinking through KCT and KCC (Ball et al., 2008). On the one hand, teachers should concentrate on how to take students' correct thinking forward. On the other hand, for those with incorrect solutions, teachers should focus on handling students' misconceptions by posing follow-up questions. Thus, it is essential to investigate teachers' instructional responses to students having correct and incorrect solutions. In addition, although functional thinking is one of the main foci for mathematics and mathematics teaching (Blanton & Kaput, 2011; Johnassen et al., 1993; Scandura, 1971; Sfard, 1991; Warren & Cooper, 2005), there is no study in the literature on teachers' instructional responses based on students' functional thinking.

Therefore, the following research question will guide this study.

1. What types of instructional responses do prospective middle school mathematics teachers give to students with correct and incorrect solutions based on students' functional thinking within the context of pattern generalization?

Method

Research Design

The qualitative case study design allows an in-depth understanding and exploration of the problem with one or more cases in a particular context (Creswell, 2007; Yin, 2009). Since the current study explores prospective middle school mathematics teachers' instructional responses in-depth, a qualitative case study design was preferred. To be more specific, since there is a single case, which is prospective middle school mathematics teachers, and one unit of analysis, which is the prospective teachers' responses based on students' functional thinking, the single-case holistic design was selected as the most appropriate design for this research study.

Context and Participants

The study context was the middle school mathematics teacher education program at one of the public universities in Ankara/Turkey. The middle school mathematics teacher education program is a four-year undergraduate program aiming to equip teachers with skills of problem-solving, critical thinking, and effective teaching of mathematics through technology. To achieve these objectives, the program includes content courses (e.g., Calculus for Functions of Several Variables and Introduction to Differential Equations), educational science courses (e.g., Introduction to Education and Guidance), and elementary mathematics education courses (e.g., Instructional Technology and Material Development, Methods of Teaching Mathematics I-II, and School Experience). After prospective teachers graduate from this program, they become mathematics teachers of 5^{th}-8^{th} grade students.

Thirty-two senior prospective middle school mathematics teachers studying at a public university in Ankara, Turkey, were selected as participants using the purposive sampling method depending on three criteria. The first criterion is that voluntary prospective teachers were preferred as participants. The second criterion for selecting participants is obtaining detailed information from participants. For this reason, prospective teachers who like to write and speak and can express their ideas clearly were selected. Thus, prospective teachers were expected to write detailed answers to open-ended questions and give detailed answers in the semi-structured interview. The final criterion is that those prospective teachers completed most of the courses in the program, including Methods of Teaching Mathematics I-II and School Experience courses. Thus, it was assumed that the prospective teachers in the study had sufficient knowledge about the methods of mathematics teaching for 5^{th}-8^{th} grade students and algebra and the ways of giving effective instructional responses to students based on their current functional thinking.

Data Collection

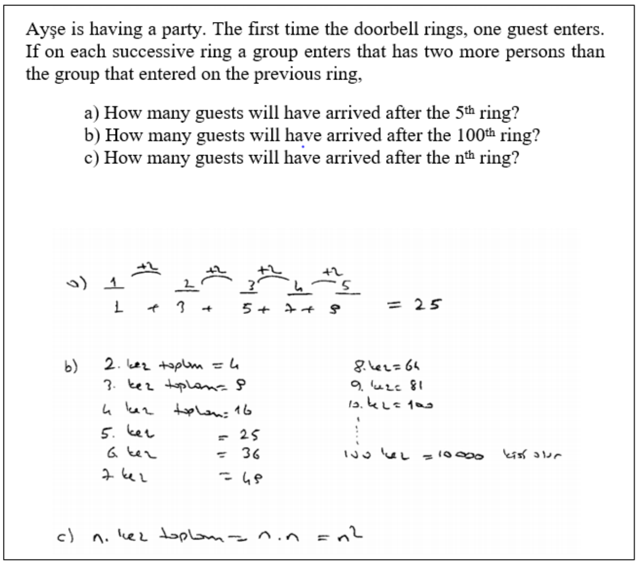
To achieve the aims of the study, data were collected in three phases via (1) a questionnaire to reveal 6^{th}-grade students' solutions, (2) a written task for prospective middle school mathematic teachers' instructional responses, and (3) semi-structured interviews with prospective teachers. In the first phase, a questionnaire including three problems about pattern generalization adapted from the study of Radford (2000), Kriegler (2008), and Meyer and Sallee (1983) was administered to 20 students. Then, one problem, which encourages students to think functionally, was chosen among these three problems to prepare for the written task. In the selected problem (Figure 1), while the students were expected to make near and far generalizations in sub-question *a* and sub-question *b,* respectively, they were asked to establish a functional relationship for the n^{th} term of the pattern to write the general rule of the pattern in sub-question *c*.

> Ayşe is having a party. The first time the doorbell rings, one guest enters. If on each successive ring a group enters that has two more persons than the group that entered on the previous ring,
>
> a. How many guests will have arrived after the 5^{th} ring?
>
> b. How many guests will have arrived after the 100^{th} ring?
>
> c. How many guests will have arrived after the n^{th} ring?

Figure 1. Problem (Meyer & Sallee, 1983)

Before collecting data from the prospective teachers, the written task was prepared using the 6^{th}-grade students' solutions to the selected problem (Figure 1). One of the reasons for using written student solutions instead of video or discursive context with students in class is that teachers with less experience focus on the students' behavior and classroom rather than students' thinking while watching the video (Berliner, Stein, Sabers, Clarridge, Cushing, & Pinnegar, 1988). Since participants of this study are prospective teachers, they could give importance to the classroom setting or environment rather than students' thinking. Another reason is that it is likely that prospective teachers would miss the details of students' strategies in the video or discursive context. For these reasons, written student solutions were preferred.

The written task consisted of one correct (Figure 2) and one incorrect student solution (Figure 3). This is because correct solutions give a chance to evaluate how teachers extend the understanding of the student and incorrect solutions provide an opportunity to evaluate how teachers support students' understanding (Jacobs et al., 2010). This way, prospective teachers' instructional responses to students having correct and incorrect solutions were asked. The student solutions are as follows:


Ayşe is having a party. The first time the doorbell rings, one guest enters. If on each successive ring a group enters that has two more persons than the group that entered on the previous ring,

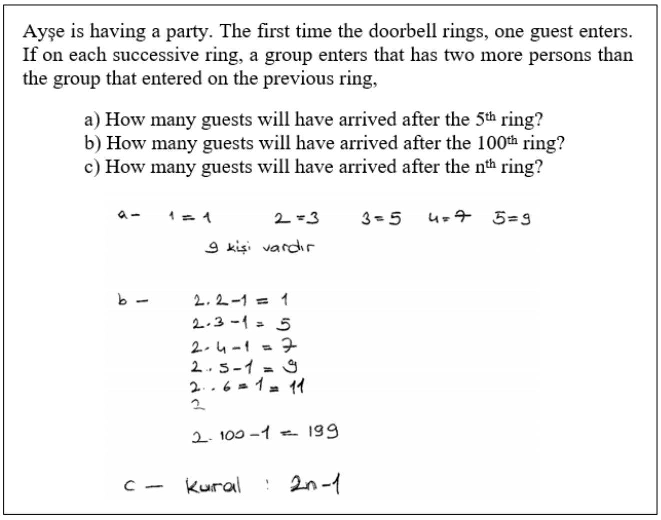
a) How many guests will have arrived after the 5^{th} ring?
b) How many guests will have arrived after the 100^{th} ring?
c) How many guests will have arrived after the n^{th} ring?

a) 1 +2 2 +2 3 +2 4 +2 5
1 + 3 + 5 + 7 + 9 = 25

b) 2. kez toplam = 4
3. kez toplam = 9
4 kez toplam = 16
5. kez = 25
6 kez = 36
7 kez = 49
8. kez = 64
9. kez = 81
10. kez = 100
⋮
100 kez = 10000 kişi olur

c) n. kez toplam = n.n = n^2

Figure 2. Correct student solution


Ayşe is having a party. The first time the doorbell rings, one guest enters. If on each successive ring, a group enters that has two more persons than the group that entered on the previous ring,

a) How many guests will have arrived after the 5^{th} ring?
b) How many guests will have arrived after the 100^{th} ring?
c) How many guests will have arrived after the n^{th} ring?

a – 1 = 1 2 = 3 3 = 5 4 = 7 5 = 9
9 kişi vardır

b – 2.2 – 1 = 1
2.3 – 1 = 5
2.4 – 1 = 7
2.5 – 1 = 9
2.6 = 1 = 11
2
2.100 – 1 = 199

c – Kural : 2n-1

Figure 3. Incorrect student solution

The correct solution shows that the student could establish the relationship in the pattern and calculate the number of guests who arrived after the 5^{th} ring and 100^{th} ring. Afterward, the student wrote the functional relationship between the number of rings and the number of arriving guests. Thus, a student with this correct solution has done all the important steps regarding pattern generalization, such as near generalization, far generalization, and writing the rule of pattern. Since it is a matter of curiosity how the teachers would give an instructional response to a student with such a solution, the solution in figure 2 was chosen as the correct solution for the study. On the other hand, the incorrect solution demonstrates that the student misunderstood the relationship in the pattern. Thus, the student miscalculated the number of guests who arrived at the 5^{th} and 100^{th} ring. For this reason, the student established the functional relationship between the number of rings and the number of arriving guests incorrectly. Students with such a misunderstanding and solution are a common occurrence. Therefore, prospective teachers will likely encounter such an incorrect solution in their future careers. For this reason, this incorrect solution was included in the study to investigate how the participants would provide instructions to a student with such an incorrect solution.

This written task was administered to thirty-two prospective middle school mathematics teachers in the second phase of the data collection process. To gain insight into prospective teachers' expertise in responding to students based on students' functional thinking, prospective teachers were asked to respond to the following prompt based on students' correct and incorrect solutions: "Pretend that you are the teacher of these children. What problem or problems might you pose next?" (Jacobs et al., 2010, pp. 178-179). In the third phase, semi-structured interviews were conducted with eight of the thirty-two prospective teachers to clarify prospective teachers' written answers to the question and further explore their instructional responses to students. Eight prospective teachers from whom the most detailed data could be obtained were selected for the semi-structured interview. Moreover, since these prospective teachers' written responses involved a variety of instructional responses, they were found to be the most suitable to interview among the 32 teachers.

Regarding ethical concerns, approval was obtained from the Human Subjects and Ethics Committee at the institution where the study was conducted. Furthermore, before collecting the data, information about the study was given, consent forms were completed, and the prospective teachers confirmed that they were voluntary participants. Moreover, researchers guaranteed that participants' details, responses, and video recordings would not be shared. Finally, a suitable classroom environment was established so that participants could answer the questionnaire and conduct the interview comfortably.

Data Analysis

The data was analyzed through open coding, in which researchers conceptualized all related data to review all the concepts (Charmaz, 2006). According to the data analysis, the prospective teachers' instructional responses were grouped under three categories: *extending/supporting students' functional thinking, asking students to do another drill, and providing a general response.* When the prospective teachers guided the student with the correct solution by posing a challenging problem that would take students' functional thinking further, their instructional responses were categorized as extending students' functional thinking. Instructional responses in which the prospective teachers helped the students recognize their errors using follow-up questions refer to supporting students' functional thinking. In responses categorized as extending/supporting students' functional thinking, teachers used PCK effectively. They understood students' strategies, analyzed their thinking correctly with the help of SCK and KCS, and proposed appropriate instructional acts or effective teaching methods with the help of KCT and KCC. When the prospective teachers asked students to do another similar drill without extending/supporting their functional thinking, these responses were categorized as asking students to do another similar drill as a response. When the prospective teachers' responses did not reflect the student's functional thinking and included general comments, these responses were categorized as providing a general response based on students' functional thinking. In responses classified as asking students to do another similar drill and providing a general response, teachers could not use or reflect their KCT and KCC in their instructional responses.

To ensure inter-rater reliability, a doctoral student in Mathematics Education coded each prospective teacher's written answer and transcribed each interview separately. Since the co-coder has knowledge and experience in the construct of teacher noticing and is familiar with the framework of professional noticing of children's mathematical thinking, she was chosen as a co-coder. The researchers' and co-coder's coding were compared to see commonalities and differences. Inter-rater reliability was calculated and established at 95% using the formula Miles and Huberman (1994) suggested. After assessing the inconsistencies, the raters reached a consensus, and the necessary changes were made.

Findings

Deciding how to respond based on children's mathematical thinking is the ability to make an in-the-moment instructional decision depending on students' existing mathematical understanding (Jacobs et al., 2010). Accordingly, prospective teachers' instructional responses were divided into three categories; *extending/supporting students' functional thinking, asking students to do another similar drill as a response, and providing a general response*. The characteristics of each category for correct and incorrect solutions are shown in Table 1.

Table 1
The Details of Teachers' Instructional Responses based on Students' Functional Thinking

	TEACHERS' INSTRUCTIONAL RESPONSES ON THE BASIS OF STUDENTS' FUNCTIONAL THINKING
Students Correct Solution	*Extending/Supporting Students' Functional Thinking* Extending students' existing functional thinking by posing new challenging questions
	Asking Students to do Another Similar Drill as a Response Asking a similar question to students to reinforce their knowledge without extending their functional thinking
	Providing a General Response Asking a question that is independent of student's functional thinking.
Student' s Incorrect Solution	*Extending/Supporting Students' Functional Thinking* Supporting students' existing functional thinking by asking a question to make them recognize their misunderstanding
	Asking Students to do Another Similar *Drill as a Response* Asking a similar question to students without supporting their functional thinking
	Providing a General Response Asking a question that is independent of the student's functional thinking.

In the present study, the percentages of prospective teachers' responses to students with both correct and incorrect solutions are summarized in Table 2.

Table 2
The Percentage of Prospective Teachers' Responses for Each Category

	Student's correct solution	**Student's incorrect solution**
Extending/Supporting Students' Functional Thinking	1 3.13%	20 62.5%
Asking Students to do Another Similar Drill as a Response	11 34.38%	4 12.50%
Providing a General Response	18 56.25%	7 21.88%
No answer	2 6.21%	1 3.13%

N=32

Extending/Supporting Students' Functional Thinking

When prospective teachers extend students' correct functional thinking or support their incorrect functional thinking through appropriate instructional responses, their responses were coded as extending/supporting students' functional thinking.

According to the data analysis, 20 (62.5%) prospective teachers supported the functional thinking of the student with an incorrect solution. However, only one of them could extend a student's correct functional thinking. The details are as follows:

PT3 suggested the following response and rationale behind his response to an incorrect answer.

> Response:
> PT3: I would ask this student the question for the solution. Guests go to Ayşe's house every day. On the 1st day, one guest comes. On each subsequent day, two more guests come to the house compared to the previous day. How many guests came to Ayşe's house on the 100th day?
> *(PT3, an excerpt from the interview)*
>
> Rationale*:*
> I would ask this question because I want him to see the difference between the original question and this question, and to question why the solutions are the same. Thanks to this question, the student can understand where he made a mistake and can correct it. I also didn't want to ask an easier or a more difficult question, as I thought the student had difficulty only in understanding the question. Seeing his error may help him solve the problem correctly.
> *(PT3, an excerpt from the questionnaire)*

As seen in the incorrect solution, the student misunderstood the problem. The student found the number of guests who came in the 5^{th}, 100^{th}, and n^{th} ring, although the total number of guests who arrived after the 5^{th}, 100^{th}, and n^{th} ring was required in the problem. For example, since PT3 realized that the student misunderstood the problem and gave a wrong answer, PT3 planned to help the student recognize his mistake by asking a problem in which the student had to calculate the number of guests who came in the 5^{th}, 100^{th}, and n^{th} ring. Thus, PT3 expected the student to solve the new problem, the solution to which is the same as the solution already found by the student. Also, the student was expected to realize the difference between the meanings of the number of guests who came in the 5^{th}, $100^{th,}$ and n^{th} ring and the total number of guests who came after the 5^{th}, $100^{th,}$ and n^{th} ring. Hence, PT3 planned to encourage the student to correct his solution upon realizing his mistake. It can be said that PT3 could support the student's functional thinking with an appropriate instructional response.

Asking Students to Do Another Similar *Drill as a Response*

When the prospective teachers asked a similar question to the students with the correct or incorrect solution without extending or supporting students' functional thinking, their responses were coded as asking students to do another similar drill as a response. While 11 (34.38 %) prospective teachers asked students to do another similar drill in response to the student with the correct solution, 4 (12.50%) prospective teachers asked the student with incorrect solutions to do another similar drill. PT3's response to the student who solved the problem correctly and her rationale for her response are below:

> Response:
> Instead of saying "two more people are coming home compared to the previous day", I would ask the question like "At the next ring, twice as many people come home as the number of people coming home at the previous ring".
> *(PT3, an excerpt from the questionnaire)*

> Rationale:
> I think the student understood the subject well. For this reason, I wanted to ask a more difficult question since I wanted to check whether the student learned to find a rule and (whether they) can make a generalization or not. This question will be a more complex question for the student.
> *(PT3, an excerpt from the questionnaire)*

As PT3 noticed that the student understood how to write the rule of pattern and make a generalization, she focused on the student's understanding, and her rationale was based on the interpretation of the student's mathematical understanding. Although PT3 emphasized the student's mathematical understanding in her rationale, she planned to ask a more difficult problem instead of a more sophisticated problem. PT3 asked a similar problem in which only the number of guests who came at each successive ring was changed. In the problem that the student solved, it was stated that two more people came compared to the people who entered on the previous ring. In the problem that PT3 asked, it is stated that twice as many people came home as the number of people coming home on the previous ring. Since the problem that PT3 proposed allowed the student to practice using only his current knowledge rather than extending the student's functional thinking, PT3's response was coded as asking students to do another similar drill.

PT4's response to the student with a correct solution, which was categorized as asking students to do another similar drill, and her rationale behind her response is below:

> Response:
> I would write a new question, based on the same question, changing the amount and way of the increase. For example, what happens if 3 times more guests come compared to the previous one each time?
>
> 1 3 9 27…3^{n-1}
>
> *(PT4, an excerpt from the questionnaire)*
>
> Rationale:
> The multiplication relationship may seem a little more difficult or scary than the addition and subtraction relationship. The student needs to form an association and realize that the exponent is one out of the number of steps.
> *(PT4, an excerpt from the questionnaire)*

According to PT4's rationale, while responding to the student with the correct solution, she focused on the characteristics of the problem. However, PT4 did not consider the student's mathematical understanding. In other words, PT4 did not ask a problem that would allow the student to extend his functional thinking; instead, PT4 asked a problem as further practice. Like PT3's response to the student with the correct solution, PT4 changed the problem by increasing the number of guests. Therefore, PT4's response to the student with the correct solution was coded under the category of asking students to do another similar drill as a response.

Providing a General Response

Some prospective teachers neither extended/supported students' functional thinking nor asked them to do another similar drill as a response. Instead, they asked the student to explain their solutions, mentioned the type of problem they would ask, or made a general comment. Such responses were categorized as providing a general response. While 18 (56.25%) prospective teachers provided a general response to the student with a correct solution, only 7 (21.88%) prospective teachers provided a general response to a student whose answer was incorrect. PT2's response to a student with an incorrect solution and the rationale behind his response follows:

> Response:
> I would ask him to read the question again and check whether his answer fits that question or not. I would ask different pattern problems.
> *(PT2, an excerpt from the interview)*
>
> Rationale:
> Here, I thought that the student understood the subject, but did not understand the question because when we look at the solution, it seems logical. He was able to find 2n-1 and show the steps, even if he made some minor errors at each step.
> *(PT2, an excerpt from the interview)*

According to PT2's rationale behind his response, PT2 considered the student's mathematical understanding and thought that he understood the concept, yet he did not understand the problem. However, PT2 did not aim to support the student's functional thinking. Instead, PT2 suggested and told the type of problem he would ask. For this reason, PT2's response was coded under the category of providing a general response.

PT1's response to a student with the correct solution and her rationale for her response follows:

> Response:
> How did you find that there were 10,000 people at the 100^{th} ring? Can you explain? Why do n^2 people arrive at the n^{th} ring?
> *(PT1, an excerpt from the questionnaire)*
>
> Rationale:
> The student did the operations correctly and reached the correct solutions. However, he did not write anything in the explanation part. So I would ask questions like "Why?, How did you find it?, Can you explain?" to understand whether he really learned it or to make sure he did not copy the solution from his friend.
> *(PT1, an excerpt from the questionnaire)*

PT1 decided on a response depending on the correctness of the student's answer rather than the student's mathematical understanding. Thus, PT1 asked the student to explain how she reached the solution. However, PT1 did not guide the student to extend her functional thinking or ask her to do another similar drill. For these reasons, PT1's response was categorized as providing a general response.

Consequently, prospective teachers who participated in this study had difficulty extending the student's functional thinking after the correct solution was given. However, most could support the student's functional thinking with the incorrect solution. In addition to that, prospective teachers tended to ask a drill to respond to students with the correct and incorrect solution. Finally, most of them provided a general response to a student with a correct solution instead of extending their mathematical understanding.

Discussion

In this study, prospective teachers' instructional responses to the students with correct and incorrect solutions based on their functional thinking within the context of pattern generalization were investigated.

The study's most striking finding is that the prospective teachers offered instructional responses to the student with an incorrect solution by supporting their functional thinking with further relevant questions, making them question their answers. However, when the student reached the correct solution, they sometimes asked the student to do another similar drill and provided a general response, which was insufficient. To be more specific, the prospective teachers tended to focus on the conceptual aspects of students' incorrect solutions, such as their misunderstanding. In other words, when the teachers responded, they asked follow-up questions and helped students realize their misunderstanding, meaning that they helped students correct their thinking. On the other hand, while the prospective teachers were responding to the student with the correct solution, they had difficulty extending student's functional thinking.

Previous studies revealed that teachers had difficulty responding to students who had errors since they focused on the procedural aspects of student solutions instead of students' conceptual understanding (Crespo, 2002; Son & Crespo, 2009). For this reason, teachers responded to the students with incorrect answers by showing the procedures or stating the correct answers rather than explaining their incorrect thinking (Crespo, 2002; Milewski & Strickland, 2016). In this study, the reason for prospective teachers' effective responses to the student with an incorrect solution might be asking the student to reflect on how their strategy relates to key mathematical ideas or relationships (Jacobs & Ambrose, 2008; Milewski & Strickland, 2016). To ask questions that help students recognize their misunderstandings, teachers need to have knowledge of students' misunderstandings and know how to overcome them. Therefore, this study indicates that they might have knowledge of content and students (KCS) and knowledge of content and teaching (KCT) about students' misunderstandings (Milewski & Strickland, 2016).

Additionally, since the prospective teachers in the present study took the Methods of Teaching Mathematics I/II courses, it is not surprising that many supported students' incorrect functional thinking. These courses covered details of each mathematics topic, students' possible thinking and misconceptions, and effective teaching strategies for these topics. For this reason, prospective teachers have the opportunity to improve their knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). In this way, with the Methods of Teaching Mathematics I/II courses, prospective teachers learn many methods about how to deal with students' misconceptions about each mathematics topic. Therefore, taking these courses may have contributed to developing prospective teachers' effective responses to students with an incorrect solution. Finally, thanks to the School Experience course, prospective teachers may gain experience with students' misconceptions in a real classroom environment and observe how their mentor teachers overcome them. This experience and observation might also have enhanced their ability to respond based on students' incorrect thinking.

However, similar to the findings of the previous studies, since the prospective teachers could not extend the existing functional thinking of the students with the correct solution, their responses to such students were insufficient (Crespo, 2002; Milewski & Strickland, 2016; Taylan, 2018). This finding might have resulted from the prospective teachers believing that the aim was met successfully when the students solved the problem correctly. Thus, they did not need to guide students to extend their existing knowledge; instead, they asked them to do another similar drill as a practice or provided a general response. Moreover, teachers might consider that praise is a sufficient response to correct solutions (Crespo, 2002; Milewski & Strickland, 2016) without extending students' mathematical understanding. Surprisingly, it was concluded that the correctness of the students' solutions affects the quality of prospective teachers' responses.

The findings of this study could have been affected by the context of the study (i.e., pattern generalization) since the expertise in professional noticing is domain-specific (Jacobs & Empson, 2016; Nickerson, Lamb, & LaRochelle, 2017; Walkoe, 2015). In this study, the question was developed based on the stages of pattern generalization: near generalization, far generalization, and writing the rule of the pattern consisting of a step-by-step solution (Radford, 2008). To offer high-quality responses to students, prospective teachers need to consider students' functional thinking in each step of pattern generalization. This nature of pattern generalization might make it easier to recognize and overcome students' misunderstandings in each step. In addition to incorrect solutions, correct solutions about pattern generalization include many steps completed correctly one by one.

For this reason, to extend the student's correct functional thinking, the prospective teachers needed to ask a challenging question involving all steps of the pattern generalization process. Posing questions including all steps of pattern generalization rather than posing questions including one or two steps of pattern generalization might be difficult for teachers. This nature of pattern generalization might make it easier for prospective teachers to respond to students with incorrect solutions while making it challenging to respond to students with correct solutions. Finally, prospective teachers' subject matter knowledge (SMK) about the process of pattern generalization and pedagogical content knowledge (PCK) about the understanding of students' functional thinking and teaching methods of pattern generalization may affect their responses.

A further interesting outcome from the analysis of the study was that the nature of their response does not determine prospective teachers' rationale. For example, PT3 justified her responses according to the student's mathematical understanding, and PT4 focused on the characteristic of the problem in the questions she asked. Despite the differences in their rationale, both prospective teachers asked the student to do another similar drill. Another example that supports this finding is related to the category of providing a general response. Although PT2 focused on students' mathematical understanding and PT1 was interested in the correctness of students' answers while justifying their responses, both provided a general response. Accordingly, it can be said that within the process, they asked students to do the given drill or give a general response, but the rationale they provided did not affect the nature of their responses.

One more remarkable aspect of this study is that although it was conducted on Jacobs et al.'s (2010) definition of professional noticing of students' mathematical thinking, we specifically focused on the third component of professional noticing. Moreover, we explored prospective teachers' responses to the students with correct and incorrect solutions based on students' functional thinking within the context of pattern generalization. In Jacobs et al.'s (2010) framework, prospective teachers' expertise in deciding how to respond based on children's understanding was divided into three categories: robust evidence, limited evidence, and lack of evidence. Their categorization provides a general perspective on teachers' decisions on how to respond. Jacobs et al. (2010) stated that "we were not seeking a particular next problem or rationale but were instead interested in the extent to which participants based their decisions on what they had learned about the children's understandings from the specific situation and how consistent their reasoning was with the research on children's mathematical development" (p.188). However, in this study, the prospective teachers provided specific types of instructional responses to students based on their functional thinking within the context of pattern generalization. Thus, we focused on the nature of prospective teachers' instructional responses to students having both correct and incorrect solutions.

Consequently, although the prospective teachers could support the student's incorrect thinking with follow-up questions, they were deficient in extending the student's correct thinking. The prospective teachers asked the student to do another similar drill or provided a general response to the students who reached the correct solution. Hence, it can be said that the correctness of solutions was a significant factor in prospective teachers' instructional responses. Moreover, since the expertise in professional noticing is domain-specific (Jacobs & Empson, 2016; Nickerson et al., 2017; Walkoe, 2015), teachers' competency in giving instructional responses might vary depending on the context. For this reason, the nature of prospective teachers' instructional responses to students with correct and incorrect solutions may differ due to the nature of the context, which is pattern generalization.

Based on the link between teachers' instructional responses and their knowledge, future studies which aim to examine how teachers' pedagogical content knowledge (PCK) and content knowledge (SMK) affect their instructional responses to students could be carried out. Moreover, future studies may examine teachers' responses to students with correct and incorrect solutions within other mathematical contexts using our categorizations. Additionally, how teachers' or prospective teachers' responses differ depending on their different types of rationale can be investigated. Finally, the same study might be conducted in an international context. The studies to be conducted in the future may show how culture affects prospective teachers' instructional responses to students who reach correct or incorrect solutions within the context of pattern generalization.

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