# CODES ON SUBGROUPS OF WEIGHTED PROJECTIVE TORI 

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#### Abstract

We obtain certain algebraic invariants relevant in studying codes on subgroups of weighted projective tori inside an $n$-dimensional weighted projective space. As application, we compute all the main parameters of generalized toric codes on these subgroups of tori lying inside a weighted projective plane of the form $\mathbb{P}(1,1, a)$.


## 1. Introduction

Let $\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$ be the weighted projective space over an algebraic closure $\overline{\mathbb{F}}_{q}$ of a finite field $\mathbb{F}_{q}$, defined by some positive integers $w_{0}, \ldots, w_{n}$. Without loosing generality, we assume that $n$ of these numbers have no common divisor. It is well known that the $\overline{\mathbb{F}}_{q}$-rational points of the weighted projective space $\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$ can be represented by the Geometric Invariant Theory quotient $\left(\overline{\mathbb{F}}_{q}^{n+1} \backslash\{0\}\right) / G$, where the group $G=\left\{\left(\lambda^{w_{0}}, \ldots, \lambda^{w_{n}}\right): \lambda \in \overline{\mathbb{F}}_{q}^{*}\right\}$. Therefore, a point is an orbit of the form $\left[p_{0}: \ldots: p_{n}\right]=\left\{\left(\lambda^{w_{0}} p_{0}, \ldots, \lambda^{w_{n}} p_{n}\right): \lambda \in \overline{\mathbb{F}}_{q}^{*}\right\}$ known as its homogeneous coordinates as in the classical projective case. Every $\mathbb{F}_{q}$-rational point has a representative from the set $\mathbb{F}_{q}^{n+1}$ in this correspondence.

For a thorough introduction to and a fairly good account on general properties of these spaces, see $[1,3,8,18]$. It is known that $X=\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$ is smooth if and only if it is the usual projective space $\mathbb{P}^{n}$, i.e., $w_{0}=\cdots=w_{n}=1$.

The coordinate ring $S=\mathbb{F}_{q}\left[x_{0}, \ldots, x_{n}\right]$ over the field $\mathbb{F}_{q}$ of a weighted projective space $\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$ is graded naturally by the numerical semigroup $\mathbb{N} \beta$ generated by $\operatorname{deg}\left(x_{i}\right)=w_{i}$, for $i=0, \ldots, n$, where $\mathbb{N}$ denotes the set of natural numbers with 0 . Thus, we have the following decomposition:
$S=\bigoplus_{\alpha \in \mathbb{N} \beta} S_{\alpha}$, where $S_{\alpha}$ is the vector space spanned by the monomials of degree $\alpha$.
For any $\alpha \in \mathbb{N} \beta$ and any subset $Y=\left\{P_{1}, \ldots, P_{N}\right\}$ of $\mathbb{F}_{q}$-rational points of $\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$, we have the following evaluation map:

$$
\begin{equation*}
\mathrm{ev}_{Y}: S_{\alpha} \rightarrow \mathbb{F}_{q}^{N}, \quad F \mapsto\left(F\left(P_{1}\right), \ldots, F\left(P_{N}\right)\right) \tag{1.1}
\end{equation*}
$$

The image $\mathcal{C}_{\alpha, Y}=\operatorname{ev}_{Y}\left(S_{\alpha}\right)$ is a linear code. The three basic parameters of $\mathcal{C}_{\alpha, Y}$ are block-length which is $N$, the dimension which is $K=\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathcal{C}_{\alpha, Y}\right)$, and the minimum distance $\delta=\delta\left(\mathcal{C}_{\alpha, Y}\right)$ which is the minimum of the number of nonzero components of vectors in $\mathcal{C}_{\alpha, Y} \backslash\{0\}$. When $Y$ is the full set of $\mathbb{F}_{q}$-rational points of $\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$, the code is known as the weighted Reed-Muller code. These codes are special cases of what is called generalized toric codes, see Section 2 for details.

[^0]Toric codes are introduced by Hansen in [10] for the set $Y=T_{X}\left(\mathbb{F}_{q}\right)$ of $\mathbb{F}_{q^{-}}$ rational points of the dense torus $T_{X}$ of a toric variety $X=X_{\Sigma}$ and examined further in e.g. [11, 19, 2, 22, 12, 4] producing some codes having the best known parameters. The vanishing ideal $I(Y)$ of $Y$ which is generated by homogeneous polynomials vanishing on $Y$ is a key in studying the parameters of $\mathcal{C}_{\alpha, Y}$. This is because, the kernel of $\mathrm{ev}_{Y}$ is nothing but the subspace $I_{\alpha}(Y):=I(Y) \cap S_{\alpha}$, and hence the code $\mathcal{C}_{\alpha, Y}$ is isomorphic to the vector space $S_{\alpha} / I_{\alpha}(Y)$. Therefore, the dimension $K=\operatorname{dim}_{\mathbb{F}_{q}}\left(\mathcal{C}_{\alpha, Y}\right)$ is the value $H_{Y}(\alpha)=\operatorname{dim}_{\mathbb{F}_{q}} S_{\alpha}-\operatorname{dim}_{\mathbb{F}_{q}} I_{\alpha}(Y)$ of the multigraded Hilbert function $H_{Y}$ of $Y$, see [26]. Most recently, Nardi developed combinatorial methods for studying codes on the full set $Y=X\left(\mathbb{F}_{q}\right)$ of $\mathbb{F}_{q}$-rational points of a toric variety, see $[16,17]$.

In literature, there are a few papers computing the main parameters of codes on weighted projective spaces. The main parameters of some weighted Reed-Muller codes are given explicitly for the set $Y=X\left(\mathbb{F}_{q}\right)$ of $\mathbb{F}_{q}$-rational points of the weighted projective planes $X=\mathbb{P}\left(1, w_{1}, w_{2}\right)$ when $\alpha$ is a multiple of the $\operatorname{lcm}\left(w_{1}, w_{2}\right)$. The main parameters have the most beautiful formulas in the special case of the plane $X=\mathbb{P}(1,1, a)$, see $[1]$.

If $Y=T_{X}\left(\mathbb{F}_{q}\right)=\left\{\left[1: t_{1}: \ldots: t_{n}\right] \mid t_{i} \in \mathbb{F}_{q}^{*}\right.$, for all $\left.i \in[n]:=\{1, \ldots, n\}\right\}$ is the set of $\mathbb{F}_{q}$-rational points of the torus $T_{X}$ in $X=\mathbb{P}^{n}$ and $\alpha \geq 1$, then the main parameters are given in [21]. On the other hand, [20] studied the degenerate tori

$$
Y_{Q}=\left\{\left[1: t_{1}^{a_{1}}: \ldots: t_{n}^{a_{n}}\right] \mid t_{i} \in \mathbb{F}_{q}^{*}, \text { for all } i \in[n]:=\{1, \ldots, n\}\right\}
$$

lying in the classical projective space $X=\mathbb{P}^{n}$, generalizing [21]. This is because, $Y_{Q}$ becomes the set of $\mathbb{F}_{q}$-rational points of the projective torus in $\mathbb{P}^{n}$, once $a_{i}=1$, for all $i \in[n]$. The results in [20] show that $I\left(Y_{Q}\right)$ is a complete intersection of the binomials $x_{i}^{s_{i}}-x_{0}^{s_{i}}$, for $i \in[n]$, its degree is $\left|Y_{Q}\right|=s_{1} \cdots s_{n}$ and $a$-invariant is $a_{Y}=s_{1}+\cdots+s_{n}-n-1$, where $s_{i}=(q-1) / \operatorname{gcd}\left(q-1, a_{i}\right)$ for all $i \in[n]$. Some nice formulas are given for the other parameters as well.

The present paper considers the analogue of the same parametrization $Y_{Q}$ but in the weighted projective space $X=\mathbb{P}\left(1, w_{1}, \ldots, w_{n}\right)$ with $a_{i}=w_{i}$ for all $i$. When $w_{i}=1$, for all $i$, our $Y_{Q}$ becomes the $\mathbb{F}_{q}$-rational points of the projective torus studied in [21], as well. In the next section, we review basic terminology and theory needed in the sequel. We prove that $I\left(Y_{Q}\right)$ is a complete intersection ideal in Proposition 3.3. We give a formula for the Hilbert function $H_{Y_{Q}}$ and compute the $a$-invariant of $Y_{Q}$ in Proposition 3.4. Theorem 4.1 gives formulas for the length and dimension of the code $\mathcal{C}_{\alpha, Y_{Q}}$. The final section displays more explicit formulas for the dimension and minimum distance of the codes coming from the weighted projective plane $\mathbb{P}(1,1, a)$, see Theorem 5.1.

## 2. Preliminaries

Let $\Sigma \subseteq \mathbb{R}^{n}$ be a complete simplicial fan with rays generated by the lattice vectors $\mathbf{v}_{1}, \ldots, \mathbf{v}_{r}$. Each cone $\sigma \in \Sigma$, defines an affine toric variety $U_{\sigma}=\operatorname{Spec}(\mathbb{K}[\check{\sigma} \cap$ $\left.\mathbb{Z}^{n}\right]$ ) over an algebraically closed field $\mathbb{K}$. Gluing these affine pieces, we obtain the toric variety $X_{\Sigma}$ as an abstract variety over $\mathbb{K}$. There is a nice correspondence between polytopes in real $n$-space and projective toric varieties. Namely, every lattice polytope $\mathcal{P}$ gives rise to a so called normal fan $\Sigma_{\mathcal{P}}$ whose rays are spanned by the inner normal vectors of $\mathcal{P}$. Assuming $X_{\Sigma}$ has a free class group, the ray
generator yields the following short exact sequence:

$$
\mathfrak{P}: 0 \longrightarrow \mathbb{Z}^{n} \xrightarrow{\phi} \mathbb{Z}^{r} \xrightarrow{\beta} \mathbb{Z}^{d} \longrightarrow 0
$$

where $\phi$ is the matrix $\left[\mathbf{v}_{1} \cdots \mathbf{v}_{r}\right]^{T}$ and $d=r-n$ is the rank of the class group $\mathrm{Cl} X_{\Sigma} \cong \mathbb{Z}^{d}$. There is an important lattice $L_{\beta}$ in $\mathbb{Z}^{r}$ that is isomorphic to $\mathbb{Z}^{n}$ via $\phi$, and is spanned by the columns $\mathbf{u}_{1}, \ldots, \mathbf{u}_{n}$ of $\phi$.

Applying $\operatorname{Hom}\left(-, \mathbb{K}^{*}\right)$ functor to $\mathfrak{P}$ gives the following dual short exact sequence:

$$
\mathfrak{P}^{*}: 1 \longrightarrow G \xrightarrow{i}\left(\mathbb{K}^{*}\right)^{r} \xrightarrow{\pi}\left(\mathbb{K}^{*}\right)^{n} \longrightarrow 1
$$

where $\pi(P)=\left(\mathbf{x}^{\mathbf{u}_{1}}(P), \ldots, \mathbf{x}^{\mathbf{u}_{n}}(P)\right)$ and $\mathbf{x}^{\mathbf{a}}(P)=p_{1}^{a_{1}} \cdots p_{r}^{a_{r}}$ for $P=\left(p_{1}, \ldots, p_{r}\right) \in$ $\left(\mathbb{K}^{*}\right)^{r}$ and $\mathbf{a}=\left(a_{1}, \ldots, a_{r}\right) \in \mathbb{Z}^{r}$.

As proved by Cox in [6], the set $X(\mathbb{K})$ of $\mathbb{K}$-rational points of the toric variety $X:=X_{\Sigma}$ is identified with the geometric quotient $\left[\mathbb{K}^{r} \backslash V(B)\right] / G$, where $B$ is the monomial ideal in $\mathbb{K}\left[x_{1}, \ldots, x_{r}\right]$ generated by the monomials $\mathbf{x}^{\hat{\sigma}}=\Pi_{\rho_{i} \notin \sigma} x_{i}$ corresponding to cones $\sigma \in \Sigma$. Hence, points of $X(\mathbb{K})$ are orbits $[P]:=G \cdot P$, for $P \in \mathbb{K}^{r} \backslash V(B)$. When $\mathbb{K}=\overline{\mathbb{F}}_{q}$ is an algebraic closure of a finite field $\mathbb{F}_{q}$, the $\mathbb{F}_{q}$-rational points $[P]$ are represented by points $P$ from the set $\mathbb{F}_{q}^{r} \backslash V(B)$.

The coordinate ring $S=\mathbb{F}_{q}\left[x_{1}, \ldots, x_{r}\right]$ of $X$ is graded via the columns of the matrix $\beta$, i.e. $\operatorname{deg}_{\beta}\left(x_{j}\right)=\beta_{j}$, for $j=1, \ldots, r$. There is a nice correspondence between subgroups of the torus $T_{X}\left(\mathbb{F}_{q}\right) \cong\left(\mathbb{F}_{q}^{*}\right)^{r} / G$ and $\beta$-graded lattice ideals in $S$, defined by:

$$
I_{L}=\left\langle\mathbf{x}^{\mathbf{m}^{+}}-\mathbf{x}^{\mathbf{m}^{-}} \mid \mathbf{m}=\mathbf{m}^{+}-\mathbf{m}^{-} \in L\right\rangle
$$

where $L$ is a sublattice of $L_{\beta}$, see [25]. In the case of the weighted projective space $\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$, we have the row matrix $\beta=\left[w_{0} \cdots w_{n}\right]$.


Figure 1. The polygon $\mathcal{P}$


Figure 2. The fan $\Sigma_{\mathcal{P}}$

Example 2.1. Let $X=\mathbb{P}(1,2,3)$ be the weighted projective space over $\overline{\mathbb{F}}_{3}$, which corresponds to the normal fan $\Sigma_{\mathcal{P}}$ depicted in Figure 2 of the polygon $\mathcal{P}$ depicted in Figure 1. Then, the first sequence above becomes:

$$
\mathfrak{P}: 0 \longrightarrow \mathbb{Z}^{2} \xrightarrow{\phi} \mathbb{Z}^{3} \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0,
$$

where

$$
\phi=\left[\begin{array}{lll}
-2 & 1 & 0 \\
-3 & 0 & 1
\end{array}\right]^{T} \quad \text { and } \quad \beta=\left[\begin{array}{lll}
1 & 2 & 3
\end{array}\right]
$$

The coordinate ring $S=\mathbb{F}_{3}[x, y, z]$ is multigraded via

$$
\operatorname{deg}_{\beta}(x)=1, \operatorname{deg}_{\beta}(y)=2 \text { and } \operatorname{deg}_{\beta}(z)=3
$$

Since $B=\langle x, y, z\rangle$, we remove the set $V(B)=V(x, y, z)=\{0\}$ and therefore obtain the quotient representation $X\left(\mathbb{F}_{3}\right)=\left(\mathbb{F}_{3}^{3} \backslash 0\right) / G$, where

$$
G=\left\{(x, y, z) \in\left(\overline{\mathbb{F}}_{3}^{*}\right)^{3} \mid x^{-2} y=x^{-3} z=1\right\}=\left\{\left(\lambda, \lambda^{2}, \lambda^{3}\right) \mid \lambda \in \overline{\mathbb{F}}_{3}^{*}\right\}
$$

is the zero locus in $\left(\overline{\mathbb{F}}_{3}^{*}\right)^{3}$ of the toric ideal:

$$
I_{L_{\beta}}:=\left\langle\mathbf{x}^{\mathbf{u}}-\mathbf{x}^{\mathbf{v}}: \mathbf{u}, \mathbf{v} \in \mathbb{N}^{r} \text { and } \beta \mathbf{u}=\beta \mathbf{v}\right\rangle=\left\langle x^{2}-y, x^{3}-z\right\rangle
$$

One needs to be careful about the field over which the group $G$ is considered. Even though we use representative from the affine space $\mathbb{F}_{3}^{3}$ recall that the equivalence of points in an orbit is determined via the subgroup $G$ of $\left(\overline{\mathbb{F}}_{3}^{*}\right)^{3}$. For instance, the points $[0: 0: 1]$ and $[0: 0: 2]$ are the same as $\mathbb{F}_{3}$-rational points, since there is $\lambda \in \overline{\mathbb{F}}_{3}^{*}$ such that $\lambda^{2}=2$ and thus we have $(0,0,2)=\left(\lambda, \lambda, \lambda^{2}\right) \cdot(0,0,1)$. But, these two points would be different if we considered equivalence with respect to the existence of $\lambda \in \mathbb{F}_{3}^{*}$ such that $\lambda^{2}=2$, since $\lambda^{2}=1$ for all $\lambda \in \mathbb{F}_{3}^{*}=\{1,2\}$.

Let us recall basics of linear codes. Our alphabet is the finite field $\mathbb{F}_{q}$ with $q$ elements. A linear code is a subspace $\mathcal{C} \subset \mathbb{F}_{q}^{N}$ whose elements are referred to as the codewords.

Definition 2.2. The parameters of a linear code $\mathcal{C} \subset \mathbb{F}_{q}^{N}$ are as follows:

- $N$ is the length of $\mathcal{C}$,
- $K=\operatorname{dim}_{\mathbb{F}_{q}} \mathcal{C}$ is the dimension of $\mathcal{C}$ as a subspace (a measure of efficiency),
- $\delta$ is the minimum distance of $\mathcal{C}$ (a measure of reliability), which is the minimum of all Hamming distances between different codewords in $\mathcal{C}$, where the Hamming distance between two codewords $c_{1}$ and $c_{2}$ is

$$
\operatorname{dist}\left(c_{1}, c_{2}\right):=\# \text { of non-zero entries in } c_{1}-c_{2} .
$$

So,

$$
\delta(\mathcal{C})=\min _{c \in \mathcal{C} \backslash\{0\}}(\# \text { of non-zero entries in } c)
$$

As in Equation (1.1), we get the so called generalized toric codes by evaluating homogeneous polynomials $F \in S_{\alpha}$ of degree $\alpha$ at some subset $Y$ of $\mathbb{F}_{q}$-rational points in a toric variety $X$.
Definition 2.3. Let $Y \subseteq X$ be a subset of a toric variety $X$. Its vanishing ideal $I(Y)$ is the (homogeneous) ideal in $S$ generated by homogeneous polynomials vanishing on $Y$. The multigraded Hilbert function of $Y$ is

$$
H_{Y}(\alpha):=\operatorname{dim}_{\mathbb{K}} S_{\alpha}-\operatorname{dim}_{\mathbb{K}} I_{\alpha}(Y)
$$

Since, the kernel of the evaluation map in Equation (1.1) consists of the homogeneous polynomials of degree $\alpha$ whose image is the point $(0, \ldots, 0) \in \mathbb{F}_{q}^{N}$, it follows that the dimension of the code $\mathcal{C}_{\alpha, Y}$ equals the value $H_{Y}(\alpha)$ of the Hilbert function of $Y$. When $Y$ lies in the torus $T_{X}$, the variables $x_{i}$ are all non zero-divisors in
the quotient ring $S / I(Y)$, and thus the Hilbert function does not decrease as we state in the following result. Below we use the partial ordering $\preceq$, where $\alpha \preceq \alpha^{\prime}$ if $\alpha^{\prime}-\alpha \in \mathbb{N} \beta$. Notice that this is the usual ordering in $\mathbb{N}$ for $X:=\mathbb{P}\left(1, w_{1} \ldots, w_{n}\right)$ as $\mathbb{N} \beta=\mathbb{N}$ in this case.

Proposition 2.4. [26, Corollary 3.18] Let $Y \subset T_{X}$. The dimension $H_{Y}(\alpha)$ of $\mathcal{C}_{\alpha, Y}$ is non-decreasing in the sense that $H_{Y}(\alpha) \leq H_{Y}\left(\alpha^{\prime}\right)$ for all $\alpha \preceq \alpha^{\prime}$.

On the other hand, the minimum distance behaves the opposite way as the following points out:

Proposition 2.5. [24, Proposition 2.22] Let $Y \subset T_{X}$. The minimum distance of $\mathcal{C}_{\alpha, Y}$ is non-increasing in the sense that $\delta\left(\mathcal{C}_{\alpha, Y}\right) \geq \delta\left(\mathcal{C}_{\alpha^{\prime}, Y}\right)$ for all $\alpha \preceq \alpha^{\prime}$.

These two results are not that surprising as we have the following well known relation between these two parameters given by the Singleton's bound:

$$
\delta\left(\mathcal{C}_{\alpha, Y}\right)+K\left(\mathcal{C}_{\alpha, Y}\right) \leq N\left(\mathcal{C}_{\alpha, Y}\right)+1
$$

There is an algebro-geometric invariant of the zero-dimensional subvariety $Y \subset$ $X\left(\mathbb{F}_{q}\right)$ used to eliminate trivial codes which we introduce now.
Definition 2.6. The multigraded regularity of $Y$, denoted $\operatorname{reg}(Y)$, is the set of $\alpha \in \mathbb{N} \beta$ for which $H_{Y}(\alpha)=|Y|$, the length of $\mathcal{C}_{\alpha, Y}$.
Proposition 2.7. If $\alpha \in \operatorname{reg}(Y)$ then $\delta\left(\mathcal{C}_{\alpha, Y}\right)=1$.
Proof. Let $\alpha \in \operatorname{reg}(Y)$. Then, the dimension of the code is nothing but the length. So, the claim follows from the Singleton bound, as we always have $\delta\left(\mathcal{C}_{\alpha, Y}\right) \geq 1$.

The multigraded regularity set is determined by a number also known as the $a$-invariant in the case of a weighted projective space. In order to state the precise result, we first recall some relevant concepts.

When $I$ is a weighted graded ideal, the quotient ring $S / I$ inherits this grading as well and has a decomposition $S / I=\bigoplus_{\alpha \in \mathcal{A}}(S / I)_{\alpha}$, where $(S / I)_{\alpha}=S_{\alpha} / I_{\alpha}$ is a finite dimensional vector space spanned by monomials of degree $\alpha$ in the numerical semigroup $\mathbb{N} \beta=\mathbb{N}\left\{w_{0}, \ldots, w_{n}\right\}$, which do not belong to $I$. This gives rise to the weighted Hilbert function and series defined respectively by

$$
\begin{gathered}
H_{S / I}(\alpha):=\operatorname{dim}_{\mathbb{K}}(S / I)_{\alpha}=\operatorname{dim}_{\mathbb{K}} S_{\alpha}-\operatorname{dim}_{\mathbb{K}} I_{\alpha} \\
\text { and } \quad H S_{S / I}(t):=\sum_{\alpha \in \mathbb{N} \beta} H_{S / I}(\alpha) t^{\alpha}
\end{gathered}
$$

Furthermore, the weighted Hilbert series has a rational function representation, that is, we have

$$
\begin{equation*}
H S_{S / I}(t)=\frac{p_{S / I}(t)}{\left(1-t^{w_{0}}\right) \cdots\left(1-t^{w_{n}}\right)} \tag{2.1}
\end{equation*}
$$

for a unique polynomial $p_{S / I}(t)$ with integer coefficients, see [14, Chapter 8].
Proposition 2.8. [26, Proposition 3.12] Let $Y \subset T_{X}$ for $X=\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$ with $w_{0}=1$. Then, the integer $a_{Y}=\operatorname{deg}\left(p_{S / I(Y)}(t)\right)-w_{0}-\cdots-w_{n}$ satisfies $\operatorname{reg}(Y)=1+a_{Y}+\mathbb{N}$.

A nice formula for the $a$-invariant is given for the $\mathbb{F}_{q}$-rational points of the torus $T_{X}$ when $X$ is a weighted projective space.
Proposition 2.9. [7, Corollary 3.9] If $Y=T_{X}\left(\mathbb{F}_{q}\right)$ for $X=\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$ and $g(\mathbb{N} \beta)$ is the Frobenius number of the numerical semigroup $\mathbb{N} \beta=\mathbb{N}\left\{w_{0}, \ldots, w_{n}\right\}$, then

$$
a_{Y}=(q-2)\left[w_{0}+\cdots+w_{n}+g(\mathbb{N} \beta)\right]+g(\mathbb{N} \beta) .
$$

There are subgroups of the torus $T_{X}$ referred to as degenerate tori which we briefly discuss now.

Definition 2.10. The following subgroup $Y_{A}=\left\{\left[t_{1}^{a_{1}}: \ldots: t_{r}^{a_{r}}\right]: t_{i} \in \mathbb{F}_{q}^{*}\right\}$ of the torus $T_{X}$ is called a degenerate torus, lying inside a toric variety $X_{\Sigma}$, for any positive integers $a_{1}, \ldots, a_{r}$, where $r$ is the number of rays in the fan $\Sigma$.

If $\mathbb{F}_{q}^{*}=\langle\eta\rangle$, every $t_{i} \in \mathbb{F}_{q}^{*}$ is of the form $t_{i}=\eta^{k_{i}}$, for some $0 \leq k_{i} \leq q-2$. Let $d_{i}=\left|\eta^{a_{i}}\right|$ and $D=\operatorname{diag}\left(d_{1}, \ldots, d_{r}\right)$.
Proposition 2.11. [13, Corollary 3.13 (ii)] If $Y=Y_{A}$ is a complete intersection in $X=\mathbb{P}^{r-1}$ and $g:=\operatorname{gcd}\left(d_{1}, \ldots, d_{r}\right)$ so that $d_{1}^{\prime}=d_{1} / g, \ldots, d_{r}^{\prime}=d_{r} / g$ generate $a$ numerical semigroup $\mathbb{N} D^{\prime}$ with the Frobenius number $g\left(\mathbb{N} D^{\prime}\right)$, then

$$
1+a_{Y}=g \cdot g\left(\mathbb{N} D^{\prime}\right)+d_{1}+\cdots+d_{r}-(r-1)
$$

Notice that when $a_{i}=1$ and $w_{j}=1$, for all $i$ and $j$, we have $d_{i}=q-1$, and so $d_{i}^{\prime}=1$. The greatest integer not belonging to the numerical semigroup $\mathbb{N} \beta=\mathbb{N} D^{\prime}=\mathbb{N}$ is $g(\mathbb{N} \beta)=g\left(\mathbb{N} D^{\prime}\right)=-1$ so both formulas in Proposition 2.9 and Proposition 2.11 yield $a_{Y}=n(q-2)-1$, for the torus $Y=T_{X}\left(\mathbb{F}_{q}\right)$ in the projective space $X=\mathbb{P}^{n}$.

Definition 2.12. A binomial is a polynomial of the form $\mathbf{x}^{\mathbf{a}}-\mathbf{x}^{\mathbf{b}}$, and $J$ is called a binomial ideal if it is generated by binomials. $J$ is called a complete intersection if it is generated by height $(J)$ many binomials.
Definition 2.13. For a lattice $L \subset \mathbb{Z}^{r}$, the lattice ideal $I_{L}$ is the binomial ideal generated by binomials $\mathbf{x}^{\mathbf{a}}-\mathbf{x}^{\mathbf{b}}$ for all $\mathbf{a}-\mathbf{b} \in L$. That is,

$$
I_{L}=\left\langle\mathbf{x}^{\mathbf{a}}-\mathbf{x}^{\mathbf{b}} \mid \mathbf{a}-\mathbf{b} \in L\right\rangle \subset S
$$

Theorem 2.14. [23, Theorem 4.5] If $Y=Y_{A}$ then $I(Y)=I_{L}$ for $L=D\left(L_{\beta D}\right)$.
If $a_{i}=1$, for all $i$, then $Y_{A}=T_{X}\left(\mathbb{F}_{q}\right)$ and $d_{i}=q-1$, for all $i$, so that the matrix $D$ is just $q-1$ times the identity matrix yielding the following:

Corollary 2.15. [23, Corollary 4.14 (ii)] If $Y=T_{X}\left(\mathbb{F}_{q}\right)$ then $I(Y)=I_{L}$ for $L=(q-1) L_{\beta}$.

Proposition 2.16. [23, Proposition 4.12] A generating system of binomials for $I\left(Y_{A}\right)$ is obtained from that of $I_{L_{\beta D}}$ by replacing $x_{i}$ with $x_{i}^{d_{i}} . I\left(Y_{A}\right)$ is a complete intersection if and only if so is the toric ideal $I_{L_{\beta D}}$. In this case, a minimal generating system is obtained from a minimal generating system of $I_{L_{\beta D}}$ this way.

## 3. Degenerate Tori on Weighted Projective Spaces

In this section, we explore properties of some degenerate tori on a weighted projective space. To start with, we prove that they are complete intersections of special type of binomial hypersurfaces.

We focus on a weighted projective space $X=\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$ and use the notation $S=\mathbb{F}_{q}\left[x_{0}, \ldots, x_{n}\right]$ for the Cox ring of $X$. Set

$$
\tilde{w}_{i}:=\frac{w_{i}}{\operatorname{gcd}\left(q-1, w_{i}\right)} \text { and } d_{i}:=\frac{q-1}{\operatorname{gcd}\left(q-1, w_{i}\right)} \text { for } i=0,1, \ldots, n
$$

The following concept is very helpful in determining when a lattice ideal is a complete intersection.
Definition 3.1. If each column of a matrix has both a positive and a negative entry we say that the matrix is mixed. Moreover, if the matrix does not have a square mixed submatrix, then it is called dominating.

Theorem 3.2. [15, Theorem 3.9] Let $L \subseteq \mathbb{Z}^{r}$ be a lattice with the property that $L \cap \mathbb{N}^{r}=0$. Then, $I_{L}$ is a complete intersection if and only if $L$ has a basis $\mathbf{m}_{1}, \ldots, \mathbf{m}_{k}$ such that the matrix $\left[\mathbf{m}_{1} \cdots \mathbf{m}_{k}\right]$ is mixed dominating. If $I_{L}$ is a complete intersection, then we have

$$
I_{L}=\left\langle x^{m_{1}^{+}}-x^{m_{1}^{-}}, \ldots, x^{m_{k}^{+}}-x^{m_{k}^{-}}\right\rangle
$$

Proposition 3.3. Let $Q=\operatorname{diag}\left(w_{0}, \ldots, w_{n}\right)$ and $Y_{Q}=\left\{\left[t_{0} w_{0}: \ldots: t_{n}{ }^{w_{n}}\right] \mid t_{i} \in \mathbb{F}_{q}^{*}\right\}$ be the corresponding subgroup of $T_{X}$ for $X=\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$. If $w_{0} \mid q-1$ and $F_{i}=x_{i}^{d_{i}}-x_{0}^{d_{0} \tilde{w}_{i}}, i=1,2, \ldots, n$, then, the vanishing ideal of $Y_{Q}$ is the following complete intersection lattice ideal:

$$
I\left(Y_{Q}\right)=\left\langle F_{1}, F_{2}, \ldots, F_{n}\right\rangle
$$

Proof. Since $D=\operatorname{diag}\left(d_{0}, \ldots, d_{n}\right)$ and $\beta=\left[w_{0} \cdots w_{n}\right]$, it follows that their product is $\beta D=\left[w_{0} d_{0} \cdots w_{n} d_{n}\right]$. It is clear that $\tilde{w}_{i}(q-1)=w_{i} d_{i}$, and so

$$
\operatorname{gcd}\left(w_{0} d_{0}, \ldots, w_{n} d_{n}\right)=(q-1) \operatorname{gcd}\left(\tilde{w}_{0}, \ldots, \tilde{w}_{n}\right)
$$

Therefore, we have the equality of the lattices $L_{\beta D}=L_{\tilde{W}}$, where $\tilde{W}$ is the matrix with columns $\tilde{w}_{i}$, for $i=0, \ldots, n$.

When $w_{0} \mid q-1$, we have $\tilde{w}_{0}=1$ and thus the lattice $L_{\tilde{W}}$ has the following basis

$$
\left\{\left(-\tilde{w}_{1}, \mathbf{e}_{1}\right), \ldots,\left(-\tilde{w}_{n}, \mathbf{e}_{n}\right)\right\}
$$

where $\mathbf{e}_{i}$ form the standard basis for $\mathbb{Z}^{n}$. Consider the matrix $M$ whose columns are the basis vectors of $L_{\tilde{W}}$ given above. Since the matrix $M$ is mixed-dominating, it follows from Theorem 3.2 that the lattice ideal of $L_{\tilde{W}}$ is a complete intersection generated by the binomials $x_{i}-x_{0}^{\tilde{w}_{i}}, i=1,2, \ldots, n$.

By Theorem 2.14, the vanishing ideal $I\left(Y_{Q}\right)$ is the binomial ideal $I_{L}$ for the lattice $L=D\left(L_{\beta D}\right)$, whose generators are obtained substituting $x_{i}^{d_{i}}$ for $x_{i}$ in the binomials above generating the lattice ideal of $L_{\tilde{W}}$, by Proposition 2.16. Therefore, the vanishing ideal $I\left(Y_{Q}\right)$ is a complete intersection generated by the binomials $F_{1}, F_{2}, \ldots, F_{n}$.
Proposition 3.4. Let $Q=\operatorname{diag}\left(w_{0}, \ldots, w_{n}\right)$ and $Y_{Q}=\left\{\left[t_{0} w_{0}: \ldots: t_{n}{ }^{w_{n}}\right] \mid t_{i} \in \mathbb{F}_{q}^{*}\right\}$ be the corresponding subgroup of $T_{X}$ for $X=\mathbb{P}\left(w_{0}, \ldots, w_{n}\right)$. If $w_{0} \mid q-1$ then, for any $\alpha \in \mathbb{N} \beta$ we have

$$
H_{Y_{Q}}(\alpha)=\sum_{s=0}^{n}(-1)^{s} \sum_{I \subseteq[n],|I|=s} \operatorname{dim}_{\mathbb{K}} S_{\alpha-\alpha_{I}}
$$

where $\alpha_{I}=\sum_{i \in I} \alpha_{i}$. Moreover, the a-invariant of $Y_{Q}$ is given by the formula $a_{Y_{Q}}=\left(d_{1}-1\right) w_{1}+\cdots+\left(d_{n}-1\right) w_{n}-w_{0}$.

Proof. Notice that $I\left(Y_{Q}\right)$ is a complete intersection by Proposition 3.3 generated by binomials of degrees $\alpha_{1}=d_{1} w_{1}, \ldots, \alpha_{n}=d_{n} w_{n}$. Thus, its minimal free resolution is given by the Koszul complex. As in the proof of [26, Proposition 3.13] we have the following exact sequence

$$
0 \rightarrow W_{n} \rightarrow \cdots \rightarrow W_{s} \rightarrow \cdots \rightarrow W_{1} \rightarrow S_{\alpha} \rightarrow\left(S / I\left(Y_{Q}\right)\right)_{\alpha} \rightarrow 0
$$

where, for every $s=1, \ldots, n$, the vector space $W_{s}$ is given by

$$
W_{s}=\bigoplus_{I \subseteq[n],|I|=s} S\left(-\alpha_{I}\right)_{\alpha}=\bigoplus_{I \subseteq[n],|I|=s} S_{\alpha-\alpha_{I}}
$$

Therefore, we obtain:

$$
\begin{align*}
H_{Y_{Q}}(\alpha) & =\operatorname{dim}_{\mathbb{K}} S_{\alpha}+\sum_{s=1}^{n}(-1)^{s} \operatorname{dim}_{\mathbb{K}} W_{s} \\
& =\sum_{s=0}^{n}(-1)^{s} \sum_{I \subseteq[n],|I|=s} \operatorname{dim}_{\mathbb{K}} S_{\alpha-\alpha_{I}}, \tag{3.1}
\end{align*}
$$

where $\alpha_{I}=\sum_{i \in I} \alpha_{i}$. By Proposition 8.23 in [14], the numerator of the Hilbert series in Equation 2.1 is as follows:

$$
p_{S / I\left(Y_{Q}\right)}=\sum_{s=0}^{n}(-1)^{s} \sum_{I \subseteq[n],|I|=s} t^{\alpha_{I}} .
$$

Hence, $p_{S / I\left(Y_{Q}\right)}$ has degree $\alpha_{1}+\cdots+\alpha_{n}=d_{1} w_{1}+\cdots+d_{n} w_{n}$, and thus

$$
a_{Y_{Q}}=\left(d_{1}-1\right) w_{1}+\cdots+\left(d_{n}-1\right) w_{n}-w_{0}
$$

by Proposition 2.8.
Example 3.5. Let $X=\mathbb{P}(1,1,2)$. Consider the matrix $Q=\operatorname{diag}(1,1,2)$ and $Y_{Q}=\left\{\left[t_{0}: t_{1}: t_{2}^{2}\right] \mid t_{0}, t_{1}, t_{2} \in \mathbb{F}_{q}^{*}\right\}$. Assume that $q$ is odd. So, we have

$$
\left(d_{0}, d_{1}, d_{2}\right)=(q-1, q-1,(q-1) / 2) \text { and }\left(\tilde{w}_{0}, \tilde{w}_{1}, \tilde{w}_{2}\right)=(1,1,1)
$$

Thus, $I\left(Y_{Q}\right)=\left\langle F_{1}, F_{2}\right\rangle=\left\langle x_{1}^{q-1}-x_{0}^{q-1}, x_{2}^{(q-1) / 2}-x_{0}^{q-1}\right\rangle$. As the degrees of the generators are $\alpha_{1}=q-1$ and $\alpha_{2}=q-1$, a graded minimal free resolution of $I\left(Y_{Q}\right)$ is given by:

$$
0 \rightarrow S_{\alpha-\alpha_{1}-\alpha_{2}} \xrightarrow{\left[-F_{2} F_{1}\right]^{T}} S_{\alpha-\alpha_{1}} \oplus S_{\alpha-\alpha_{2}} \xrightarrow{\left[F_{1} F_{2}\right]} S_{\alpha} \rightarrow\left(S / I\left(Y_{Q}\right)\right)_{\alpha} \rightarrow 0 .
$$

Therefore, the Hilbert function is computed to be

$$
\begin{aligned}
H_{Y_{Q}}(\alpha) & =\operatorname{dim}_{\mathbb{K}} S_{\alpha}-\operatorname{dim}_{\mathbb{K}} S_{\alpha-\alpha_{1}}-\operatorname{dim}_{\mathbb{K}} S_{\alpha-\alpha_{2}}+\operatorname{dim}_{\mathbb{K}} S_{\alpha-\alpha_{1}-\alpha_{2}} \\
& =\operatorname{dim}_{\mathbb{K}} S_{\alpha}-2 \operatorname{dim}_{\mathbb{K}} S_{\alpha-(q-1)}+\operatorname{dim}_{\mathbb{K}} S_{\alpha-2(q-1)}
\end{aligned}
$$

We first notice the following

$$
\operatorname{dim}_{\mathbb{K}} S_{\alpha}= \begin{cases}\left(\alpha_{0}+1\right)^{2} & \text { if } \alpha=2 \alpha_{0} \\ \left(\alpha_{0}+1\right)\left(\alpha_{0}+2\right) & \text { if } \alpha=2 \alpha_{0}+1\end{cases}
$$

Thus, if $0 \leq \alpha \leq q-2$, then $\operatorname{dim}_{\mathbb{K}} S_{\alpha-(q-1)}=\operatorname{dim}_{\mathbb{K}} S_{\alpha-2(q-1)}=0$. Hence,

$$
H_{Y_{Q}}(\alpha)= \begin{cases}\left(\alpha_{0}+1\right)^{2} & \text { if } \alpha=2 \alpha_{0} \\ \left(\alpha_{0}+1\right)\left(\alpha_{0}+2\right) & \text { if } \alpha=2 \alpha_{0}+1\end{cases}
$$

When, $q-1 \leq \alpha<2(q-1)$, we have $\operatorname{dim}_{\mathbb{K}} S_{\alpha-2(q-1)}=0$. It is easy to see that

$$
\operatorname{dim}_{\mathbb{K}} S_{\alpha-(q-1)}= \begin{cases}\left(\alpha_{0}+1-(q-1) / 2\right)^{2} & \text { if } \alpha=2 \alpha_{0} \\ \left(\alpha_{0}+1-(q-1) / 2\right)\left(\alpha_{0}+2-(q-1) / 2\right) & \text { if } \alpha=2 \alpha_{0}+1\end{cases}
$$

Hence, we have the following formula for $H_{Y_{Q}}(\alpha)$ :
$\begin{cases}\left(\alpha_{0}+1\right)^{2}-2\left(\alpha_{0}+1-(q-1) / 2\right)^{2} & \text { if } \alpha=2 \alpha_{0} \\ \left(\alpha_{0}+1\right)\left(\alpha_{0}+2\right)-2\left(\alpha_{0}+1-(q-1) / 2\right)\left(\alpha_{0}+2-(q-1) / 2\right) & \text { if } \alpha=2 \alpha_{0}+1 .\end{cases}$
Finally, when $\alpha \geq 2(q-1)$, we get

$$
\operatorname{dim}_{\mathbb{K}} S_{\alpha-2(q-1)}= \begin{cases}\left(\alpha_{0}+1-(q-1)\right)^{2} & \text { if } \alpha=2 \alpha_{0} \\ \left(\alpha_{0}+1-(q-1)\right)\left(\alpha_{0}+2-(q-1)\right) & \text { if } \alpha=2 \alpha_{0}+1\end{cases}
$$

Therefore, we have $H_{Y_{Q}}(\alpha)=(q-1)^{2} / 2=\left|Y_{Q}\right|$ which is not surprising as we have $\alpha>a_{Y_{Q}}$ in this case.

## 4. Length and Dimension when $X=\mathbb{P}\left(1, w_{1}, \ldots, w_{n}\right)$

Let $\mathbb{F}_{q}^{*}=\langle\eta\rangle$, then the order of $\eta_{i}:=\eta^{w_{i}}$ is

$$
d_{i}=\frac{q-1}{\operatorname{gcd}\left(q-1, w_{i}\right)} \quad i=1, \ldots, n
$$

By using $I\left(Y_{Q}\right)$, the length and the dimension of $\mathcal{C}_{\alpha, Y_{Q}}$ are computed as follows.
Theorem 4.1. Let $X=\mathbb{P}\left(1, w_{1}, \ldots, w_{n}\right)$ be a weighted projective space over the field $\overline{\mathbb{F}}_{q}$. Consider $Q=\operatorname{diag}\left(1, w_{1}, \ldots, w_{n}\right)$ and the subgroup it defines in $T_{X}\left(\mathbb{F}_{q}\right)$ :

$$
Y_{Q}=\left\{\left[t_{0}: t_{1}^{w_{1}}: \ldots: t_{n}^{w_{n}}\right] \mid t_{i} \in \mathbb{F}_{q}^{*}, \text { for all } i=0, \ldots, n\right\}
$$

Then, the length of $\mathcal{C}_{\alpha, Y_{Q}}$ is $\left|Y_{Q}\right|=d_{1} \cdots d_{n}$ and the dimension is
$\operatorname{dim}\left(\mathcal{C}_{\alpha, Y_{Q}}\right)=\sum_{m_{n}=0}^{\min \left\{\left\lfloor\frac{\alpha}{w_{n}}\right\rfloor, d_{n}-1\right\} \min \left\{\left\lfloor\frac{\alpha-m_{n} w_{n}}{w_{n}}\right\rfloor, d_{n-1}-1\right\}} \sum_{m_{n-1}=0}^{\min \left\{\left\lfloor\frac{\alpha-m_{n} w_{n}-\cdots-m_{2} w_{2}}{w_{1}}\right\rfloor, d_{1}-1\right\}} \sum_{m_{1}=0} 1$
Moreover, the a-invariant is given by

$$
a_{Y_{Q}}=\left(d_{1}-1\right) w_{1}+\cdots+\left(d_{n}-1\right) w_{n}-1
$$

Proof. We first prove that

$$
\begin{equation*}
Y_{Q}=\left\langle\left[1: \eta_{1}: 1: \ldots: 1\right]\right\rangle \times \cdots \times\left\langle\left[1: \ldots: 1: \eta_{n}\right]\right\rangle . \tag{4.1}
\end{equation*}
$$

Multiplying by $\left[\lambda: \lambda^{w_{1}}: \ldots: \lambda^{w_{n}}\right]$ does not change an equivalence class for every $\lambda \in \mathbb{F}_{q}^{*}$. So, we have the equality of the following points:

$$
\left[t_{0}: t_{1}^{w_{1}}: \ldots: t_{n}^{w_{n}}\right]=\left[1:\left(t_{1} / t_{0}\right)^{w_{1}}: \ldots:\left(t_{n} / t_{0}\right)^{w_{n}}\right] .
$$

Hence, we have

$$
Y_{Q}=\left\{\left[1: s_{1}^{w_{1}}: \ldots: s_{n}^{w_{n}}\right] \mid s_{i} \in \mathbb{F}_{q}^{*}, \text { for all } i=1, \ldots, n\right\} .
$$

Since $s_{i}=\eta^{k_{i}}$, for some $k_{i} \in \mathbb{N}$, it is clear that $s_{i}^{w_{i}}=\eta_{i}^{k_{i}}$ and thus

$$
Y_{Q}=\left\{\left[1: \eta_{1}^{i_{1}}: \ldots: \eta_{n}^{i_{n}}\right] \mid 0 \leq i_{1} \leq d_{1}, \ldots, 0 \leq i_{n} \leq d_{n}\right\}
$$

from which the claim in (4.1) is deduced, and thus $\left|Y_{Q}\right|=d_{1} \cdots d_{n}$.
If $w_{0}=1$, then $d_{0}=q-1$ and so the vanishing ideal of $Y_{Q}$ is generated by the binomials $F_{i}=x_{i}^{d_{i}}-x_{0}^{d_{i} w_{i}}$, for $i=1,2, \ldots, n$. With respect to any term order for which $x_{0}$ is the smallest variable, the leading monomial of $F_{i}$ is clearly $x_{i}^{d_{i}}$. Since the
monomials $x_{i}^{d_{i}}$ and $x_{j}^{d_{j}}$ are relatively prime for different $i$ and $j$, it readily follows that the binomials $F_{1}, \ldots, F_{n}$ form a Groebner basis for the vanishing ideal $I\left(Y_{Q}\right)$. It is well-known ( $[5, \mathrm{p} .232]$ ) then that a basis for the vector space $S_{\alpha} / I_{\alpha}\left(Y_{Q}\right)$ is given by the monomials $\mathbf{x}^{\mathbf{m}}=x_{0}^{m_{0}} x_{1}^{m_{1}} \cdots x_{n}^{m_{n}}$ of degree $\alpha$ that can not be divided by the leading monomials $x_{i}^{d_{i}}$ of $F_{i}$, for all $i=1,2, \ldots, n$ and for

$$
\alpha=m_{0}+m_{1} w_{1}+\cdots+m_{n} w_{n} \in \mathbb{N}=\left\langle 1, w_{1}, \ldots, w_{n}\right\rangle
$$

Therefore, a basis for $S_{\alpha} / I_{\alpha}\left(Y_{Q}\right)$ corresponds to the set of tuples ( $m_{0}, m_{1}, \ldots, m_{n}$ ) satisfying $\alpha=m_{0}+m_{1} w_{1}+\cdots+m_{n} w_{n}$ and $m_{i} \leq d_{i}-1$, for all $i=1,2, \ldots, n$. The elements of this set can be identified step by step as we explain now. We start first by choosing an integer $m_{n}$ between 0 and $\min \left\{\left\lfloor\frac{\alpha}{w_{n}}\right\rfloor, d_{n}-1\right\}$ and observe that the elements of the set in question can be partitioned into subsets for every choice of $m_{n}$ in the aforementioned range. More precisely, for each fixed $m_{n}$, we have a subset consisting of tuples $\left(m_{0}, m_{1}, \ldots, m_{n}\right)$ satisfying
$m_{0}+m_{1} w_{1}+\cdots+m_{n-1} w_{n-1}=\alpha-m_{n} w_{n}$ and $m_{i} \leq d_{i}-1$, for all $i=1,2, \ldots, n-1$.
As a second step, we fix $m_{n-1}$ between 0 and $\min \left\{\left\lfloor\frac{\alpha-m_{n} w_{n}}{w_{n-1}}\right\rfloor, d_{n-1}-1\right\}$, and look for the solutions ( $m_{0}, m_{1}, \ldots, m_{n-2}$ ) satisfying

$$
m_{0}+m_{1} w_{1}+\cdots+m_{n-2} w_{n-2}=\alpha-m_{n} w_{n}-m_{n-1} w_{n-1} \text { and } m_{i} \leq d_{i}-1
$$

for all $i=1,2, \ldots, n-2$. Continuing inductively, we end up with a unique $m_{0}$ satisfying

$$
m_{0}=\alpha-m_{n} w_{n}-m_{n-1} w_{n-1}-\cdots-m_{1} w_{1}
$$

Hence, the dimension of the code, which is nothing but the dimension of the vector space $S_{\alpha} / I_{\alpha}\left(Y_{Q}\right)$, is exactly the sum given by the formula
$\operatorname{dim}\left(\mathcal{C}_{\alpha, Y_{Q}}\right)=\sum_{m_{n}=0}^{\min \left\{\left\lfloor\frac{\alpha}{w_{n}}\right\rfloor, d_{n}-1\right\}} \sum_{m_{n-1}=0}^{\min \left\{\left\lfloor\frac{\alpha-m_{n} w_{n}}{w_{n}-1}\right\rfloor, d_{n-1}-1\right\}} \cdots \sum_{m_{1}=0}^{\min \left\{\left\lfloor\frac{\alpha-m_{n} w_{n}-\cdots-m_{2} w_{2}}{w_{1}}\right\rfloor, d_{1}-1\right\}} \ldots$
The $a$-invariant can be obtained from Proposition 3.4 , by substiting $w_{0}=1$.

## 5. Codes on $Y_{Q} \subset \mathbb{P}(1,1, a)$

For any positive integer $a$, we compute the basic parameters of the code $\mathcal{C}_{\alpha, Y_{Q}}$, for the subgroup $Y_{Q}=\left\{\left[t_{0}: t_{1}: t_{2}^{a}\right] \mid t_{0}, t_{1}, t_{2} \in \mathbb{F}_{q}^{*}\right\}$ of $T_{X}\left(\mathbb{F}_{q}\right)$ for the weighted projective space $X=\mathbb{P}(1,1, a)$.

Theorem 5.1. Let $d_{2}=\frac{q-1}{\operatorname{gcd}(a, q-1)}, k=\left\lfloor\frac{\alpha-(q-2)}{a}\right\rfloor$ and $\mu_{2}=\min \left\{\left\lfloor\frac{\alpha}{a}\right\rfloor, d_{2}-1\right\}$. Then, the length of $\mathcal{C}_{\alpha, Y_{Q}}$ is $N=\left|Y_{Q}\right|=(q-1) d_{2}$. Its dimension $K\left(\mathcal{C}_{\alpha, Y_{Q}}\right)$ is

$$
\begin{array}{ll}
\left(\mu_{2}+1\right)\left(\alpha+1-\mu_{2} a / 2\right), & \text { if } 0 \leq \alpha \leq q-2 \\
(q-1)(k+1)+\left(\mu_{2}-k\right)\left[\alpha+1-\left(\mu_{2}+k+1\right) a / 2\right], & \text { if } 0<\alpha-(q-2)<\left(d_{2}-1\right) a \\
N & \text { otherwise. }
\end{array}
$$

and the minimum distance of $\mathcal{C}_{\alpha, Y_{Q}}$ is:

$$
\delta\left(\mathcal{C}_{\alpha, Y_{Q}}\right)= \begin{cases}d_{2}(q-1-\alpha) & \text { if } 0 \leq \alpha \leq q-2 \\ d_{2}-k & \text { if } q-2 \leq \alpha<(q-2)+\left(d_{2}-1\right) a \\ 1 & \text { otherwise } .\end{cases}
$$

Proof. Since $w_{1}=1$, we have $d_{1}=q-1$. It follows from Equation 4.1 that

$$
Y_{Q}=\left\{\left[1: \eta_{1}^{i_{1}}: \eta_{2}^{i_{2}}\right] \mid 0 \leq i_{1} \leq d_{1} \text { and } 0 \leq i_{2} \leq d_{2}\right\}
$$

so the length of the code is $d_{1} d_{2}=(q-1) d_{2}$.
When $0 \leq \alpha \leq q-2$, the dimension formula in Theorem 4.1 specializes to

$$
\begin{aligned}
\operatorname{dim}\left(\mathcal{C}_{\alpha, Y_{Q}}\right) & =\sum_{m_{2}=0}^{\mu_{2}} \sum_{m_{1}=0}^{\min \left\{\alpha-m_{2} a, q-2\right\}} 1=\sum_{m_{2}=0}^{\mu_{2}} \sum_{m_{1}=0}^{\alpha-m_{2} a} 1 \\
& =\sum_{m_{2}=0}^{\mu_{2}}\left(\alpha-m_{2} a+1\right)=\left(\mu_{2}+1\right)(\alpha+1)-a \sum_{m_{2}=0}^{\mu_{2}} m_{2} \\
& =\left(\mu_{2}+1\right)(\alpha+1)-a \frac{\mu_{2}\left(\mu_{2}+1\right)}{2}
\end{aligned}
$$

If $q-2<\alpha<(q-2)+\left(d_{2}-1\right) a$, then using the formula in Theorem 4.1 again, we get

$$
\begin{aligned}
\operatorname{dim}\left(\mathcal{C}_{\alpha, Y_{Q}}\right) & =\sum_{m_{2}=0}^{\mu_{2}} \sum_{m_{1}=0}^{\min \left\{\alpha-m_{2} a, q-2\right\}} 1 \\
& =\sum_{m_{2}=0}^{k} \sum_{m_{1}=0}^{q-2} 1+\sum_{m_{2}=k+1}^{\mu_{2}} \sum_{m_{1}=0}^{\alpha-m_{2} a} 1 \\
& =(q-1)(k+1)+\sum_{m_{2}=k+1}^{\mu_{2}}\left(\alpha-m_{2} a+1\right) \\
& =(q-1)(k+1)+\left(\mu_{2}-k\right)(\alpha+1)-a \sum_{m_{2}=k+1}^{\mu_{2}} m_{2} \\
& =(q-1)(k+1)+\left(\mu_{2}-k\right)(\alpha+1)-a \frac{\mu_{2}\left(\mu_{2}+1\right)-k(k+1)}{2} .
\end{aligned}
$$

Notice that these dimensions are the number of lattice points of the polygons depicted below.


Figure 3. $\alpha \leq q-2$


Figure 4. $\alpha>q-2$

As for the minimum distance, we first give an upper bound on the number $\left|V_{Y_{Q}}(F)\right|$ of zeroes on $Y_{Q}$ of a homogeneous polynomial $F$ of degree $\alpha$ and then
demonstrate a specific polynomial attaining that bound. Let $\left[d_{2}\right]$ denote the set of non-negative integers smaller than $d_{2}$, and set

$$
J_{F}:=\left\{j \in\left[d_{2}\right] \mid x_{2}-\eta_{2}^{j} x_{0}^{w_{2}} \text { divides } F\right\}
$$

We claim that

$$
\begin{equation*}
\left|V_{Y_{Q}}(F)\right| \leq d_{1}\left|J_{F}\right|+\left(d_{2}-\left|J_{F}\right|\right) \operatorname{deg}_{x_{1}}(F) \tag{5.1}
\end{equation*}
$$

where $\operatorname{deg}_{x_{1}}(F)$ is the usual degree of $F$ in the variable $x_{1}$. The polynomial $f_{j}\left(x_{1}\right):=F\left(1, x_{1}, \eta_{2}^{j}\right) \in \mathbb{F}_{q}\left[x_{1}\right]$ vanishes at the points $\left[1: \eta_{1}^{i}: \eta_{2}^{j}\right]$, for every $i \in\left[d_{1}\right]$, when $j \in J_{F}$. Thus, there are $d_{1}\left|J_{F}\right|$ such roots of $F$. On the other hand, $f_{j}$ is not a zero polynomial when $j \notin J_{F}$, and in this case it can have at most its degree many zeroes, giving rise to $\left(d_{2}-\left|J_{F}\right|\right) \operatorname{deg}_{x_{1}}(F)$ many roots of $F$, completing the proof of the claim.

Since we always have

$$
F=\prod_{j=1}^{\left|J_{F}\right|}\left(x_{2}-\eta_{2}^{j} x_{0}^{w_{2}}\right) F^{\prime}
$$

it follows that $\operatorname{deg}_{x_{1}}(F)=\operatorname{deg}_{x_{1}}\left(F^{\prime}\right) \leq \alpha-\left|J_{F}\right| w_{2}$. Thus, we have

$$
\begin{align*}
\left|V_{Y_{Q}}(F)\right| & \leq d_{1}\left|J_{F}\right|+\left(d_{2}-\left|J_{F}\right|\right)\left(\alpha-\left|J_{F}\right| w_{2}\right) \\
& \leq d_{2} \alpha+\left|J_{F}\right|\left(d_{1}-\alpha-w_{2}\left(d_{2}-\left|J_{F}\right|\right)\right) \tag{5.2}
\end{align*}
$$

Notice that the number in the parenthesis above is
$d_{1}-\alpha-w_{2}\left(d_{2}-\left|J_{F}\right|\right)=d_{1}-\alpha-w_{2} d_{2}+w_{2}\left|J_{F}\right|=d_{1}-(q-1) \tilde{w}_{2}-\alpha+w_{2}\left|J_{F}\right|$
which is non-positive since $d_{1} \leq q-1 \leq(q-1) \tilde{w}_{2}$ and $\left|J_{F}\right| w_{2} \leq \operatorname{deg}(F)=\alpha$. Hence, altogether, we have the upper bound

$$
\begin{equation*}
\left|V_{Y_{Q}}(F)\right| \leq d_{2} \alpha \tag{5.3}
\end{equation*}
$$

Consider now the following polynomial:

$$
F_{0}=\prod_{i=1}^{\alpha}\left(x_{1}-\eta_{1}^{i} x_{0}\right)
$$

which vanishes at the points $\left[1: \eta_{1}^{i}: \eta_{2}^{j}\right]$, for every $i \in[\alpha]$ and $j \in\left[d_{2}\right]$, implying that $\left|V_{Y_{Q}}\left(F_{0}\right)\right|=d_{2} \alpha$. As the weight of the codeword $\mathrm{ev}_{Y_{Q}}\left(F_{0}\right)$ is clearly

$$
\left|Y_{Q}\right|-\left|V_{Y_{Q}}\left(F_{0}\right)\right|=d_{2}(q-1)-d_{2} \alpha
$$

and that of a general codeword $\mathrm{ev}_{Y_{Q}}(F)$ is

$$
\left|Y_{Q}\right|-\left|V_{Y_{Q}}(F)\right| \geq d_{2}(q-1)-d_{2} \alpha
$$

it follows that the minimum distance of the code is $d_{2}(q-1-\alpha)$, when $\alpha<q-1$.
When $\alpha \geq a_{Y}+1=(q-2)+\left(d_{2}-1\right) a$, the code is trivial, so $\delta\left(\mathcal{C}_{\alpha, Y_{Q}}\right)=1$.
From now on, assume that $q-2 \leq \alpha<a_{Y}+1=(q-2)+\left(d_{2}-1\right) a$. Let $k$ be the quotient and $r_{0}$ be the remainder of the division of $\alpha-(q-2)$ by $w_{2}=a$, i.e.
$\alpha-(q-2)=k a+r_{0}$ where $0 \leq k:=\left\lfloor\frac{\alpha-(q-2)}{a}\right\rfloor \leq d_{2}-2$ and $0 \leq r_{0} \leq a-1$.
When $\left|J_{F}\right|=d_{2}, F$ vanishes on $Y_{Q}$, so $F$ gives a codeword with zero weight. Thus, we suppose $\left|J_{F}\right| \leq d_{2}-1$.

If $\left|J_{F}\right| \leq k$ also, then by $\operatorname{deg}_{x_{1}}(F) \leq d_{1}-1=q-2$ and (5.1) we have

$$
\begin{aligned}
\left|V_{Y_{Q}}(F)\right| & \leq(q-1)\left|J_{F}\right|+\left(d_{2}-\left|J_{F}\right|\right)(q-2)=(q-2) d_{2}+\left|J_{F}\right| \\
& \leq(q-2) d_{2}+k
\end{aligned}
$$

If $\left|J_{F}\right|>k$, we let $\left|J_{F}\right|=k+j_{0}$ with $j_{0} \geq 1$. As $\alpha=q-2+k a+r$ with $r \leq a-1$, we have $\alpha-\left|J_{F}\right| a=\alpha-(k+1) a-a\left(\left|J_{F}\right|-k-1\right) \leq q-2-1-a\left(j_{0}-1\right)$. As $\operatorname{deg}_{x_{1}}(F) \leq \alpha-\left|J_{F}\right| a$, it follows from (5.1) that we have,

$$
\begin{aligned}
\left|V_{Y_{Q}}(F)\right| & \leq(q-1)\left|J_{F}\right|+\left(d_{2}-\left|J_{F}\right|\right)\left(\alpha-\left|J_{F}\right| a\right) \\
& \leq(q-1)\left|J_{F}\right|+\left(d_{2}-\left|J_{F}\right|\right)\left(q-2-1-a\left(j_{0}-1\right)\right) \\
& \leq(q-2) d_{2}+\left|J_{F}\right|+\left(d_{2}-\left|J_{F}\right|\right)\left(-1-a\left(j_{0}-1\right)\right)
\end{aligned}
$$

Since $d_{2}-\left|J_{F}\right| \geq 1$ and $a \geq 1$, we have

$$
\begin{aligned}
\left|V_{Y_{Q}}(F)\right| & \leq(q-2) d_{2}+\left|J_{F}\right|+\left(d_{2}-\left|J_{F}\right|\right)\left(-1-a\left(j_{0}-1\right)\right) \\
& \leq(q-2) d_{2}+\left|J_{F}\right|-1-\left(j_{0}-1\right)=(q-2) d_{2}+k
\end{aligned}
$$

We consider the following homogeneous polynomial of degree $\alpha$ now:

$$
G_{0}=x_{0}^{r_{0}} \prod_{i=1}^{q-2}\left(x_{1}-\eta_{1}^{i} x_{0}\right) \prod_{j=1}^{k}\left(x_{2}-\eta_{2}^{j} x_{0}^{a}\right)
$$

which vanish at the points $\left[1: \eta_{1}^{i}: \eta_{2}^{j}\right]$, for every $i \in[q-2]$ and $j \in\left[d_{2}\right]$, together with the points $\left[1: \eta_{1}^{i}: \eta_{2}^{j}\right]$, for $i=q-1$ and $j \in[k]$. Therefore, the number of roots is $\left|V_{Y_{Q}}\left(G_{0}\right)\right|=(q-2) d_{2}+k$. It readily follows that the minimum distance $\delta\left(\mathcal{C}_{\alpha, Y_{Q}}\right)$ of the code is the weight $(q-1) d_{2}-(q-2) d_{2}-k=d_{2}-k$ of the codeword corresponding to $G_{0}$.

Remark 5.2. It is very difficult to give a closed formula for some parameters of the code $\mathcal{C}_{\alpha, Y_{Q}}$, for the subgroup $Y_{Q}=\left\{\left[t_{0}: t_{1}^{w_{1}}: t_{2}^{w_{2}}\right] \mid t_{0}, t_{1}, t_{2} \in \mathbb{F}_{q}^{*}\right\}$ of $T_{X}\left(\mathbb{F}_{q}\right)$ in the more general case of the weighted projective plane $X=\mathbb{P}\left(1, w_{1}, w_{2}\right)$. This is mainly because of the formulas involving a division by the integer $w_{1}$. In this case one has to use the floor function when the ratio is not integer, which we explain in more details below. There are 4 cases to consider:

$$
\begin{aligned}
& \text { Case 1: }\left\lfloor\frac{\alpha}{w_{1}}\right\rfloor \leq d_{1}-1 \text { and }\left\lfloor\frac{\alpha}{w_{2}}\right\rfloor \leq d_{2}-1 \\
& \text { Case } 2:\left\lfloor\frac{\alpha}{w_{1}}\right\rfloor>d_{1}-1 \text { and }\left\lfloor\frac{\alpha}{w_{2}}\right\rfloor \leq d_{2}-1 \\
& \text { Case } 3:\left\lfloor\frac{\alpha}{w_{1}}\right\rfloor \leq d_{1}-1 \text { and }\left\lfloor\frac{\alpha}{w_{2}}\right\rfloor>d_{2}-1 \\
& \text { Case } 4:\left\lfloor\frac{\alpha}{w_{1}}\right\rfloor>d_{1}-1 \text { and }\left\lfloor\frac{\alpha}{w_{2}}\right\rfloor>d_{2}-1 .
\end{aligned}
$$

For instance, the dimension formula in Theorem 4.1 specializes to

$$
\operatorname{dim}\left(\mathcal{C}_{\alpha, Y_{Q}}\right)=\sum_{m_{2}=0}^{\mu_{2}} \sum_{m_{1}=0}^{\min \left\{\left\lfloor\frac{\alpha-m_{2} w_{2}}{w_{1}}\right\rfloor, d_{1}-1\right\}} 1
$$

where $\mu_{2}=\min \left\{\left\lfloor\frac{\alpha}{w_{2}}\right\rfloor, d_{2}-1\right\}$.

We first consider Cases 2 and 4. In these cases, as we have $\alpha \geq w_{1}\left(d_{1}-1\right)$, we choose

$$
\mu_{2}^{\prime}:=\left\lfloor\frac{\alpha-w_{1}\left(d_{1}-1\right)}{w_{2}}\right\rfloor
$$

so that $\min \left\{\left\lfloor\frac{\alpha-m_{2} w_{2}}{w_{1}}\right\rfloor, d_{1}-1\right\}=d_{1}-1$ for all $m_{2} \leq \mu_{2}^{\prime}$. Hence, we have

$$
\begin{aligned}
\operatorname{dim}\left(\mathcal{C}_{\alpha, Y_{Q}}\right) & =\sum_{m_{2}=0}^{\mu_{2}^{\prime}} \sum_{m_{1}=0}^{d_{1}-1} 1+\sum_{m_{2}=\mu_{2}^{\prime}+1}^{\mu_{2}}\left\lfloor\frac{\alpha-m_{2} w_{2}}{\sum_{1}}\right\rfloor \\
& =\left(\mu_{2}^{\prime}+1\right) d_{1}+\sum_{m_{2}=\mu_{2}^{\prime}+1}^{\mu_{2}}\left(\left\lfloor\frac{\alpha-m_{2} w_{2}}{w_{1}}\right\rfloor+1\right) \\
& =\mu_{2}^{\prime}\left(d_{1}-1\right)+\mu_{2}+d_{1}+\left\lfloor\frac{\alpha-\left(\mu_{2}^{\prime}+1\right) w_{2}}{w_{1}}\right\rfloor+\cdots+\left\lfloor\frac{\alpha-\mu_{2} w_{2}}{w_{1}}\right\rfloor
\end{aligned}
$$

A similar formula for $\operatorname{dim}\left(\mathcal{C}_{\alpha, Y_{Q}}\right)$ can be obtained in Cases 1 and 3 . In any case, we conclude that a closed formula for the dimension is difficult to get.

As for the minimum distance, one can generalize Theorem 5.1 as follows. Let $U(x, y)$ be a polynomial defined as

$$
U(x, y):=d_{1} y+\left(d_{2}-y\right) x \text { for } 0 \leq x \leq \min \left\{\left\lfloor\frac{\alpha-y w_{2}}{w_{1}}\right\rfloor, d_{1}-1\right\}, 0 \leq y \leq \mu_{2}
$$

Then, the upper bound in (5.1) becomes

$$
\left|V_{Y_{Q}}(F)\right| \leq U\left(\operatorname{deg}_{x_{1}}(F),\left|J_{F}\right|\right)
$$

Since we have

$$
x \leq \min \left\{\left\lfloor\frac{\alpha-y w_{2}}{w_{1}}\right\rfloor, d_{1}-1\right\}= \begin{cases}d_{1}-1 & \text { if } 0 \leq y \leq \mu_{2}^{\prime} \\ \left\lfloor\frac{\alpha-y w_{2}}{w_{1}}\right\rfloor & \text { if } \mu_{2}^{\prime}<y \leq \mu_{2}\end{cases}
$$

it follows that

$$
U(x, y) \leq u(y):=d_{1} y+\left(d_{2}-y\right) \min \left\{\left\lfloor\frac{\alpha-y w_{2}}{w_{1}}\right\rfloor, d_{1}-1\right\}
$$

Therefore, we get

$$
\begin{aligned}
& U(x, y) \leq d_{1} y+\left(d_{2}-y\right)\left(d_{1}-1\right)=d_{2}\left(d_{1}-1\right)+y \text { for all } y \in\left[0, \mu_{2}^{\prime}\right] \\
& \quad \text { and } U(x, y)<d_{1} y+\left(d_{2}-y\right)\left(d_{1}-1\right) \text { for all } y \in\left[\mu_{2}^{\prime}+1, \mu_{2}\right]
\end{aligned}
$$

Clearly, the polynomial $U(x, y)$ attains the maximum value at $\left(d_{1}-1, \mu_{2}^{\prime}\right)$, which is

$$
u\left(\mu_{2}^{\prime}\right)=d_{2}\left(d_{1}-1\right)+\mu_{2}^{\prime}=d_{1} d_{2}-\left(d_{2}-\mu_{2}^{\prime}\right)
$$

Thus, the minimum distance of the code $\mathcal{C}_{\alpha, Y_{Q}}$ will be $d_{2}-\mu_{2}^{\prime}$. This means that the proof of the second part of the Theorem 5.1 can be generalized very easily via replacing $q-1$ (resp. $q-2$ ) by $d_{1}$ (resp. $d_{1}-1$ ) to the Cases 2 and 4, namely for the values of $\alpha$ satisfying $w_{1}\left(d_{1}-1\right) \leq \alpha<w_{1}\left(d_{1}-1\right)+w_{2}\left(d_{2}-1\right)$.

When $w_{1}=1$, as in the proof of the first part of the Theorem 5.1, the ratio $\frac{\alpha-y w_{2}}{w_{1}}$ was an integer yielding $x=\left\lfloor\frac{\alpha-y w_{2}}{w_{1}}\right\rfloor=\alpha-y w_{2}$, and so the upper bound was

$$
u(y)=d_{1} y+\left(d_{2}-y\right)\left(\alpha-y w_{2}\right)=w_{2} y^{2}+\left(d_{1}-\alpha-d_{2} w_{2}\right) y+d_{2} \alpha
$$

As the quadratic polynomial $u(y)$ is concave up, it was clear that the absolute maximum is attained at the boundary points of the interval $[0,\lfloor\alpha / a\rfloor]$ and we were able to prove that $u(0)$ was the maximum value of $U$. However, the proof of the first case does not generalize to the Cases 1 and 3 as the maximum values are sometimes attained at interior points.

For instance, consider the case $q=31, w_{1}=8, w_{2}=9$ and $\alpha=34$. Then, we have $d_{1}=15, d_{2}=10,\left\lfloor\frac{\alpha}{w_{1}}\right\rfloor=4$ and $\left\lfloor\frac{\alpha}{w_{2}}\right\rfloor=3$. The function

$$
U(x, y):=15 y+(10-y) x \text { for } 0 \leq x \leq\left\lfloor\frac{34-9 y}{8}\right\rfloor, 0 \leq y \leq 3
$$

has the upper bound given by

$$
u(y)=15 y+(10-y)\left\lfloor\frac{34-9 y}{8}\right\rfloor, 0 \leq y \leq 3
$$

Notice that $[u(0), u(1), u(2), u(3)]=[40,42,46,45]$. Therefore, the maximum value 46 is attained at the interior point $y=2$.

As the value $u(0)=\left\lfloor\frac{\alpha}{w_{1}}\right\rfloor$ gives rise to an upper bound on the minimum distance in Cases 1 and 3, we have the following:

Theorem 5.3. Let $d_{1}=\frac{q-1}{\operatorname{gcd}\left(w_{1}, q-1\right)}, d_{2}=\frac{q-1}{\operatorname{gcd}\left(w_{2}, q-1\right)}$ and $k=\left\lfloor\frac{\alpha-w_{1}\left(d_{1}-1\right)}{w_{2}}\right\rfloor$. Then, the length of $\mathcal{C}_{\alpha, Y_{Q}}$ is $N=\left|Y_{Q}\right|=d_{1} d_{2}$. The minimum distance of $\mathcal{C}_{\alpha, Y_{Q}}$ satisfies

$$
\begin{array}{ll}
\delta\left(\mathcal{C}_{\alpha, Y_{Q}}\right) \leq d_{2}\left(d_{1}-\left\lfloor\frac{\alpha}{w_{1}}\right\rfloor\right) & \text { if } 0 \leq \alpha \leq w_{1}\left(d_{1}-1\right) \\
\delta\left(\mathcal{C}_{\alpha, Y_{Q}}\right)=d_{2}-k & \text { if } w_{1}\left(d_{1}-1\right) \leq \alpha<w_{1}\left(d_{1}-1\right)+w_{2}\left(d_{2}-1\right) \\
\delta\left(\mathcal{C}_{\alpha, Y_{Q}}\right)=1 & \text { otherwise. }
\end{array}
$$

We conclude the paper by showcasing an example with codes having good parameters obtained by our construction.

Example 5.4. Take $a=2, q=5$. So, $d_{2}=2$ and length is $d_{2}(q-1)=8$. Table 1 exhibits the main parameters of the code $\mathcal{C}_{\alpha, Y_{Q}}$ for $\alpha$ in the first column. According to Markus Grassl's Code Tables [9] a best-possible code with $N=8$ has $K+\delta=8$ or $K+\delta=9$ (MDS codes). This example provides us with 3 best possible codes whose parameters satisfy $K+\delta=8$ together with an MDS code [8, 7, 2].

Table 1. $\mathrm{a}=2$ and $\mathrm{q}=5$

| $\alpha$ | $[N, K, \delta]$ |
| :---: | :---: |
| 0 | $[8,1,8]$ |
| 1 | $[8,2,6]$ |
| 2 | $[8,4,4]$ |
| 3 | $[8,6,2]$ |
| 4 | $[8,7,2]$ |
| 5 | $[8,1,8]$ |

Example 5.5. Similarly, we take $a=3$ and $q=5$ so that $d_{2}=4$ and length is $d_{2}(q-1)=4 \cdot 4=16$. Table 2 gives the main parameters of the corresponding codes.

TABLE 2. $\mathrm{a}=3$ and $\mathrm{q}=5$

| $\alpha$ | $[N, K, \delta]$ |
| :---: | :---: |
| 0 | $[16,1,16]$ |
| 1 | $[16,2,12]$ |
| 2 | $[16,3,8]$ |
| 3 | $[16,5,4]$ |
| 4 | $[16,6,4]$ |
| 5 | $[16,7,4]$ |
| 6 | $[16,9,3]$ |
| 7 | $[16,10,3]$ |
| 8 | $[16,11,3]$ |
| 9 | $[16,13,2]$ |
| 10 | $[16,14,2]$ |
| 11 | $[16,15,2]$ |

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## References

[1] Yves Aubry, Wouter Castryck, Sudhir R. Ghorpade, Gilles Lachaud, Michael E. O'Sullivan, and Samrith Ram. Hypersurfaces in weighted projective spaces over finite fields with applications to coding theory. In Algebraic geometry for coding theory and cryptography, volume 9 of Assoc. Women Math. Ser., pages 25-61. Springer, Cham, 2017.
[2] Peter Beelen and Diego Ruano. The order bound for toric codes. In Applied algebra, algebraic algorithms, and error-correcting codes, volume 5527 of Lecture Notes in Comput. Sci., pages 1-10. Springer, Berlin, 2009.
[3] Mauro Beltrametti and Lorenzo Robbiano. Introduction to the theory of weighted projective spaces. Exposition. Math., 4(2):111-162, 1986.
[4] Gavin Brown and Alexander M. Kasprzyk. Seven new champion linear codes. LMS J. Comput. Math., 16:109-117, 2013.
[5] David Cox, John Little, and Donal O'Shea. Ideals, varieties, and algorithms. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2007. An introduction to computational algebraic geometry and commutative algebra.
[6] David A. Cox. The homogeneous coordinate ring of a toric variety. J. Algebraic Geom., 4(1):17-50, 1995.
[7] Eduardo Dias and Jorge Neves. Codes over a weighted torus. Finite Fields Appl., 33:66-79, 2015.
[8] Igor Dolgachev. Weighted projective varieties. In Group actions and vector fields (Vancouver, B.C., 1981), volume 956 of Lecture Notes in Math., pages 34-71. Springer, Berlin, 1982.
[9] Markus Grassl. Bounds on the minimum distance of linear codes and quantum codes. Online available at http://www.codetables.de, 2007. Accessed on 2023-01-17.
[10] Johan P. Hansen. Toric varieties Hirzebruch surfaces and error-correcting codes. Appl. Algebra Engrg. Comm. Comput., 13(4):289-300, 2002.
[11] David Joyner. Toric codes over finite fields. Appl. Algebra Engrg. Comm. Comput., 15(1):6379, 2004.
[12] John B. Little. Remarks on generalized toric codes. Finite Fields Appl., 24:1-14, 2013.
[13] Hiram H. López, Rafael H. Villarreal, and Leticia Zárate. Complete intersection vanishing ideals on degenerate tori over finite fields. Arab. J. Math. (Springer), 2(2):189-197, 2013.
[14] Ezra Miller and Bernd Sturmfels. Combinatorial Commutative Algebra. Cambridge Studies in Advanced Mathematics. Springer-Verlag New York, 2005.
[15] Marcel Morales and Apostolos Thoma. Complete intersection lattice ideals. J. Algebra, 284(2):755-770, 2005.
[16] Jade Nardi. Algebraic geometric codes on minimal Hirzebruch surfaces. J. Algebra, 535:556597, 2019.
[17] Jade Nardi. Projective toric codes. Int. J. Number Theory, 18(1):179-204, 2022.
[18] M. Rossi and L. Terracini. Linear algebra and toric data of weighted projective spaces. Rend. Semin. Mat. Univ. Politec. Torino, 70(4):469-495, 2012.
[19] Diego Ruano. On the structure of generalized toric codes. J. Symbolic Comput., 44(5):499506, 2009.
[20] Manuel González Sarabia, Carlos Rentería Márquez, and Antonio J. Sánchez Hernández. Minimum distance of some evaluation codes. Appl. Algebra Engrg. Comm. Comput., 24(2):95106, 2013.
[21] Eliseo Sarmiento, Maria Vaz Pinto, and Rafael H. Villarreal. The minimum distance of parameterized codes on projective tori. Appl. Algebra Engrg. Comm. Comput., 22(4):249-264, 2011.
[22] Ivan Soprunov. Toric complete intersection codes. J. Symbolic Comput., 50:374-385, 2013.
[23] Mesut Şahin. Toric codes and lattice ideals. Finite Fields Appl., 52:243-260, 2018.
[24] Mesut Şahin. Lattice ideals, semigroups and Toric codes. In Numerical semigroups, volume 40 of Springer INdAM Ser., pages 285-302. Springer, Cham, [2020] © 2020.
[25] Mesut Şahin. Rational points of lattice ideals on a toric variety and toric codes. Finite Fields Appl., 90:102226, 2023.
[26] Mesut Şahin and Ivan Soprunov. Multigraded Hilbert functions and toric complete intersection codes. J. Algebra, 459:446-467, 2016.

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