CODES ON SUBGROUPS OF WEIGHTED PROJECTIVE TORI

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ABSTRACT. We obtain certain algebraic invariants relevant in studying codes on subgroups of weighted projective tori inside an *n*-dimensional weighted projective space. As application, we compute all the main parameters of generalized toric codes on these subgroups of tori lying inside a weighted projective plane of the form $\mathbb{P}(1, 1, a)$.

1. INTRODUCTION

Let $\mathbb{P}(w_0, \ldots, w_n)$ be the weighted projective space over an algebraic closure $\overline{\mathbb{F}}_q$ of a finite field \mathbb{F}_q , defined by some positive integers w_0, \ldots, w_n . Without loosing generality, we assume that n of these numbers have no common divisor. It is well known that the $\overline{\mathbb{F}}_q$ -rational points of the weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$ can be represented by the Geometric Invariant Theory quotient $(\overline{\mathbb{F}}_q^{n+1} \setminus \{0\})/G$, where the group $G = \{(\lambda^{w_0}, \ldots, \lambda^{w_n}) : \lambda \in \overline{\mathbb{F}}_q^*\}$. Therefore, a point is an orbit of the form $[p_0 : \ldots : p_n] = \{(\lambda^{w_0} p_0, \ldots, \lambda^{w_n} p_n) : \lambda \in \overline{\mathbb{F}}_q^*\}$ known as its homogeneous coordinates as in the classical projective case. Every \mathbb{F}_q -rational point has a representative from the set \mathbb{F}_q^{n+1} in this correspondence.

For a thorough introduction to and a fairly good account on general properties of these spaces, see [1, 3, 8, 18]. It is known that $X = \mathbb{P}(w_0, \ldots, w_n)$ is smooth if and only if it is the usual projective space \mathbb{P}^n , i.e., $w_0 = \cdots = w_n = 1$.

The coordinate ring $S = \mathbb{F}_q[x_0, \ldots, x_n]$ over the field \mathbb{F}_q of a weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$ is graded naturally by the numerical semigroup $\mathbb{N}\beta$ generated by $\deg(x_i) = w_i$, for $i = 0, \ldots, n$, where \mathbb{N} denotes the set of natural numbers with 0. Thus, we have the following decomposition:

 $S = \bigoplus_{\alpha \in \mathbb{N}\beta} S_{\alpha}$, where S_{α} is the vector space spanned by the monomials of degree α .

For any $\alpha \in \mathbb{N}\beta$ and any subset $Y = \{P_1, \ldots, P_N\}$ of \mathbb{F}_q -rational points of $\mathbb{P}(w_0, \ldots, w_n)$, we have the following *evaluation map*:

(1.1)
$$\operatorname{ev}_Y : S_\alpha \to \mathbb{F}_q^N, \quad F \mapsto (F(P_1), \dots, F(P_N)).$$

The image $C_{\alpha,Y} = \operatorname{ev}_Y(S_\alpha)$ is a linear code. The three basic parameters of $C_{\alpha,Y}$ are block-length which is N, the dimension which is $K = \dim_{\mathbb{F}_q}(C_{\alpha,Y})$, and the minimum distance $\delta = \delta(C_{\alpha,Y})$ which is the minimum of the number of nonzero components of vectors in $C_{\alpha,Y} \setminus \{0\}$. When Y is the full set of \mathbb{F}_q -rational points of $\mathbb{P}(w_0, \ldots, w_n)$, the code is known as the *weighted Reed-Muller code*. These codes are special cases of what is called generalized toric codes, see Section 2 for details.

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Toric codes are introduced by Hansen in [10] for the set $Y = T_X(\mathbb{F}_q)$ of \mathbb{F}_q rational points of the dense torus T_X of a toric variety $X = X_{\Sigma}$ and examined further in e.g. [11, 19, 2, 22, 12, 4] producing some codes having the best known parameters. The vanishing ideal I(Y) of Y which is generated by homogeneous polynomials vanishing on Y is a key in studying the parameters of $\mathcal{C}_{\alpha,Y}$. This is because, the kernel of ev_Y is nothing but the subspace $I_{\alpha}(Y) := I(Y) \cap S_{\alpha}$, and hence the code $\mathcal{C}_{\alpha,Y}$ is isomorphic to the vector space $S_{\alpha}/I_{\alpha}(Y)$. Therefore, the dimension $K = \dim_{\mathbb{F}_q}(\mathcal{C}_{\alpha,Y})$ is the value $H_Y(\alpha) = \dim_{\mathbb{F}_q}S_{\alpha} - \dim_{\mathbb{F}_q}I_{\alpha}(Y)$ of the multigraded Hilbert function H_Y of Y, see [26]. Most recently, Nardi developed combinatorial methods for studying codes on the full set $Y = X(\mathbb{F}_q)$ of \mathbb{F}_q -rational points of a toric variety, see [16, 17].

In literature, there are a few papers computing the main parameters of codes on weighted projective spaces. The main parameters of some weighted Reed-Muller codes are given explicitly for the set $Y = X(\mathbb{F}_q)$ of \mathbb{F}_q -rational points of the weighted projective planes $X = \mathbb{P}(1, w_1, w_2)$ when α is a multiple of the lcm (w_1, w_2) . The main parameters have the most beautiful formulas in the special case of the plane $X = \mathbb{P}(1, 1, a)$, see [1].

If $Y = T_X(\mathbb{F}_q) = \{ [1 : t_1 : \ldots : t_n] \mid t_i \in \mathbb{F}_q^*, \text{ for all } i \in [n] := \{1, \ldots, n\} \}$ is the set of \mathbb{F}_q -rational points of the torus T_X in $X = \mathbb{P}^n$ and $\alpha \ge 1$, then the main parameters are given in [21]. On the other hand, [20] studied the degenerate tori

 $Y_Q = \{ [1: t_1^{a_1}: \ldots: t_n^{a_n}] \mid t_i \in \mathbb{F}_q^*, \text{ for all } i \in [n] := \{1, \ldots, n\} \}$

lying in the classical projective space $X = \mathbb{P}^n$, generalizing [21]. This is because, Y_Q becomes the set of \mathbb{F}_q -rational points of the projective torus in \mathbb{P}^n , once $a_i = 1$, for all $i \in [n]$. The results in [20] show that $I(Y_Q)$ is a complete intersection of the binomials $x_i^{s_i} - x_0^{s_i}$, for $i \in [n]$, its degree is $|Y_Q| = s_1 \cdots s_n$ and *a*-invariant is $a_Y = s_1 + \cdots + s_n - n - 1$, where $s_i = (q - 1)/\gcd(q - 1, a_i)$ for all $i \in [n]$. Some nice formulas are given for the other parameters as well.

The present paper considers the analogue of the same parametrization Y_Q but in the weighted projective space $X = \mathbb{P}(1, w_1, \ldots, w_n)$ with $a_i = w_i$ for all *i*. When $w_i = 1$, for all *i*, our Y_Q becomes the \mathbb{F}_q -rational points of the projective torus studied in [21], as well. In the next section, we review basic terminology and theory needed in the sequel. We prove that $I(Y_Q)$ is a complete intersection ideal in Proposition 3.3. We give a formula for the Hilbert function H_{Y_Q} and compute the *a*-invariant of Y_Q in Proposition 3.4. Theorem 4.1 gives formulas for the length and dimension of the code \mathcal{C}_{α,Y_Q} . The final section displays more explicit formulas for the dimension and minimum distance of the codes coming from the weighted projective plane $\mathbb{P}(1, 1, a)$, see Theorem 5.1.

2. Preliminaries

Let $\Sigma \subseteq \mathbb{R}^n$ be a complete simplicial fan with rays generated by the lattice vectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$. Each cone $\sigma \in \Sigma$, defines an affine toric variety $U_{\sigma} = \operatorname{Spec}(\mathbb{K}[\check{\sigma} \cap \mathbb{Z}^n])$ over an algebraically closed field \mathbb{K} . Gluing these affine pieces, we obtain the toric variety X_{Σ} as an abstract variety over \mathbb{K} . There is a nice correspondence between polytopes in real *n*-space and projective toric varieties. Namely, every lattice polytope \mathcal{P} gives rise to a so called normal fan $\Sigma_{\mathcal{P}}$ whose rays are spanned by the inner normal vectors of \mathcal{P} . Assuming X_{Σ} has a free class group, the ray generator yields the following short exact sequence:

$$\mathfrak{P}: 0 \longrightarrow \mathbb{Z}^n \xrightarrow{\phi} \mathbb{Z}^r \xrightarrow{\beta} \mathbb{Z}^d \longrightarrow 0 \ ,$$

where ϕ is the matrix $[\mathbf{v}_1 \cdots \mathbf{v}_r]^T$ and d = r - n is the rank of the class group $\operatorname{Cl} X_{\Sigma} \cong \mathbb{Z}^d$. There is an important lattice L_{β} in \mathbb{Z}^r that is isomorphic to \mathbb{Z}^n via ϕ , and is spanned by the columns $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of ϕ .

Applying Hom $(-, \mathbb{K}^*)$ functor to \mathfrak{P} gives the following dual short exact sequence:

$$\mathfrak{P}^*: 1 \longrightarrow G \xrightarrow{i} (\mathbb{K}^*)^r \xrightarrow{\pi} (\mathbb{K}^*)^n \longrightarrow 1$$

where $\pi(P) = (\mathbf{x}^{\mathbf{u}_1}(P), \dots, \mathbf{x}^{\mathbf{u}_n}(P))$ and $\mathbf{x}^{\mathbf{a}}(P) = p_1^{a_1} \cdots p_r^{a_r}$ for $P = (p_1, \dots, p_r) \in (\mathbb{K}^*)^r$ and $\mathbf{a} = (a_1, \dots, a_r) \in \mathbb{Z}^r$.

As proved by Cox in [6], the set $X(\mathbb{K})$ of \mathbb{K} -rational points of the toric variety $X := X_{\Sigma}$ is identified with the geometric quotient $[\mathbb{K}^r \setminus V(B)]/G$, where B is the monomial ideal in $\mathbb{K}[x_1, \ldots, x_r]$ generated by the monomials $\mathbf{x}^{\hat{\sigma}} = \prod_{\rho_i \notin \sigma} x_i$ corresponding to cones $\sigma \in \Sigma$. Hence, points of $X(\mathbb{K})$ are orbits $[P] := G \cdot P$, for $P \in \mathbb{K}^r \setminus V(B)$. When $\mathbb{K} = \overline{\mathbb{F}}_q$ is an algebraic closure of a finite field \mathbb{F}_q , the \mathbb{F}_q -rational points [P] are represented by points P from the set $\mathbb{F}_q^r \setminus V(B)$.

The coordinate ring $S = \mathbb{F}_q[x_1, \ldots, x_r]$ of X is graded via the columns of the matrix β , i.e. $\deg_{\beta}(x_j) = \beta_j$, for $j = 1, \ldots, r$. There is a nice correspondence between subgroups of the torus $T_X(\mathbb{F}_q) \cong (\mathbb{F}_q^*)^r/G$ and β -graded *lattice ideals* in S, defined by:

$$I_L = \langle \mathbf{x}^{\mathbf{m}^+} - \mathbf{x}^{\mathbf{m}^-} \mid \mathbf{m} = \mathbf{m}^+ - \mathbf{m}^- \in L \rangle,$$

where L is a sublattice of L_{β} , see [25]. In the case of the weighted projective space $\mathbb{P}(w_0, \ldots, w_n)$, we have the row matrix $\beta = [w_0 \cdots w_n]$.



FIGURE 1. The polygon \mathcal{P}

FIGURE 2. The fan $\Sigma_{\mathcal{P}}$

Example 2.1. Let $X = \mathbb{P}(1, 2, 3)$ be the weighted projective space over $\overline{\mathbb{F}}_3$, which corresponds to the normal fan $\Sigma_{\mathcal{P}}$ depicted in Figure 2 of the polygon \mathcal{P} depicted in Figure 1. Then, the first sequence above becomes:

$$\mathfrak{P}: 0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\phi} \mathbb{Z}^3 \xrightarrow{\beta} \mathbb{Z} \longrightarrow 0 ,$$

where

$$\phi = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}^T \quad and \quad \beta = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$

The coordinate ring $S = \mathbb{F}_3[x, y, z]$ is multigraded via

 $\deg_{\beta}(x) = 1$, $\deg_{\beta}(y) = 2$ and $\deg_{\beta}(z) = 3$.

Since $B = \langle x, y, z \rangle$, we remove the set $V(B) = V(x, y, z) = \{0\}$ and therefore obtain the quotient representation $X(\mathbb{F}_3) = (\mathbb{F}_3^3 \setminus 0)/G$, where

$$G = \{ (x, y, z) \in (\overline{\mathbb{F}}_3^*)^3 \mid x^{-2}y = x^{-3}z = 1 \} = \{ (\lambda, \lambda^2, \lambda^3) \mid \lambda \in \overline{\mathbb{F}}_3^* \}$$

is the zero locus in $(\overline{\mathbb{F}}_3^*)^3$ of the toric ideal:

$$I_{L_{\beta}} := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} : \mathbf{u}, \mathbf{v} \in \mathbb{N}^{r} \text{ and } \beta \mathbf{u} = \beta \mathbf{v} \rangle = \langle x^{2} - y, x^{3} - z \rangle.$$

One needs to be careful about the field over which the group G is considered. Even though we use representative from the affine space \mathbb{F}_3^3 recall that the equivalence of points in an orbit is determined via the subgroup G of $(\overline{\mathbb{F}}_3^*)^3$. For instance, the points [0:0:1] and [0:0:2] are the same as \mathbb{F}_3 -rational points, since there is $\lambda \in \overline{\mathbb{F}}_3^*$ such that $\lambda^2 = 2$ and thus we have $(0,0,2) = (\lambda,\lambda,\lambda^2) \cdot (0,0,1)$. But, these two points would be different if we considered equivalence with respect to the existence of $\lambda \in \mathbb{F}_3^*$ such that $\lambda^2 = 2$, since $\lambda^2 = 1$ for all $\lambda \in \mathbb{F}_3^* = \{1,2\}$.

Let us recall basics of linear codes. Our alphabet is the finite field \mathbb{F}_q with q elements. A *linear code* is a subspace $\mathcal{C} \subset \mathbb{F}_q^N$ whose elements are referred to as the *codewords*.

Definition 2.2. The parameters of a linear code $\mathcal{C} \subset \mathbb{F}_q^N$ are as follows:

- N is the length of C,
- $K = \dim_{\mathbb{F}_q} \mathcal{C}$ is the dimension of \mathcal{C} as a subspace (a measure of efficiency),
- δ is the minimum distance of C (a measure of reliability), which is the minimum of all Hamming distances between different codewords in C, where the Hamming distance between two codewords c_1 and c_2 is

 $dist(c_1, c_2) := \#of \text{ non-zero entries in } c_1 - c_2.$

So,

$$\delta(\mathcal{C}) = \min_{c \in \mathcal{C} \setminus \{0\}} (\#of \text{ non-zero entries in } c).$$

As in Equation (1.1), we get the so called *generalized toric codes* by evaluating homogeneous polynomials $F \in S_{\alpha}$ of degree α at some subset Y of \mathbb{F}_q -rational points in a toric variety X.

Definition 2.3. Let $Y \subseteq X$ be a subset of a toric variety X. Its vanishing ideal I(Y) is the (homogeneous) ideal in S generated by homogeneous polynomials vanishing on Y. The multigraded Hilbert function of Y is

$$H_Y(\alpha) := \dim_{\mathbb{K}} S_\alpha - \dim_{\mathbb{K}} I_\alpha(Y).$$

Since, the kernel of the evaluation map in Equation (1.1) consists of the homogeneous polynomials of degree α whose image is the point $(0, \ldots, 0) \in \mathbb{F}_q^N$, it follows that the dimension of the code $\mathcal{C}_{\alpha,Y}$ equals the value $H_Y(\alpha)$ of the Hilbert function of Y. When Y lies in the torus T_X , the variables x_i are all non zero-divisors in

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the quotient ring S/I(Y), and thus the Hilbert function does not decrease as we state in the following result. Below we use the partial ordering \leq , where $\alpha \leq \alpha'$ if $\alpha' - \alpha \in \mathbb{N}\beta$. Notice that this is the usual ordering in \mathbb{N} for $X := \mathbb{P}(1, w_1 \dots, w_n)$ as $\mathbb{N}\beta = \mathbb{N}$ in this case.

Proposition 2.4. [26, Corollary 3.18] Let $Y \subset T_X$. The dimension $H_Y(\alpha)$ of $\mathcal{C}_{\alpha,Y}$ is non-decreasing in the sense that $H_Y(\alpha) \leq H_Y(\alpha')$ for all $\alpha \preceq \alpha'$.

On the other hand, the minimum distance behaves the opposite way as the following points out:

Proposition 2.5. [24, Proposition 2.22] Let $Y \subset T_X$. The minimum distance of $\mathcal{C}_{\alpha,Y}$ is non-increasing in the sense that $\delta(\mathcal{C}_{\alpha,Y}) \geq \delta(\mathcal{C}_{\alpha',Y})$ for all $\alpha \leq \alpha'$.

These two results are not that surprising as we have the following well known relation between these two parameters given by the Singleton's bound:

$$\delta(\mathcal{C}_{\alpha,Y}) + K(\mathcal{C}_{\alpha,Y}) \le N(\mathcal{C}_{\alpha,Y}) + 1.$$

There is an algebro-geometric invariant of the zero-dimensional subvariety $Y \subset X(\mathbb{F}_q)$ used to eliminate trivial codes which we introduce now.

Definition 2.6. The multigraded regularity of Y, denoted reg(Y), is the set of $\alpha \in \mathbb{N}\beta$ for which $H_Y(\alpha) = |Y|$, the length of $\mathcal{C}_{\alpha,Y}$.

Proposition 2.7. If $\alpha \in \operatorname{reg}(Y)$ then $\delta(\mathcal{C}_{\alpha,Y}) = 1$.

Proof. Let $\alpha \in \operatorname{reg}(Y)$. Then, the dimension of the code is nothing but the length. So, the claim follows from the Singleton bound, as we always have $\delta(\mathcal{C}_{\alpha,Y}) \geq 1$. \Box

The multigraded regularity set is determined by a number also known as the a-invariant in the case of a weighted projective space. In order to state the precise result, we first recall some relevant concepts.

When I is a weighted graded ideal, the quotient ring S/I inherits this grading as well and has a decomposition $S/I = \bigoplus_{\alpha \in \mathcal{A}} (S/I)_{\alpha}$, where $(S/I)_{\alpha} = S_{\alpha}/I_{\alpha}$ is a finite dimensional vector space spanned by monomials of degree α in the numerical

semigroup $\mathbb{N}\beta = \mathbb{N}\{w_0, \ldots, w_n\}$, which do not belong to *I*. This gives rise to the weighted Hilbert function and series defined respectively by

$$\begin{aligned} H_{S/I}(\alpha) &:= \dim_{\mathbb{K}} (S/I)_{\alpha} = \dim_{\mathbb{K}} S_{\alpha} - \dim_{\mathbb{K}} I_{\alpha} \\ \text{and} \quad HS_{S/I}(t) &:= \sum_{\alpha \in \mathbb{N}\beta} H_{S/I}(\alpha) t^{\alpha}. \end{aligned}$$

Furthermore, the weighted Hilbert series has a rational function representation, that is, we have

(2.1)
$$HS_{S/I}(t) = \frac{p_{S/I}(t)}{(1 - t^{w_0}) \cdots (1 - t^{w_n})}$$

for a unique polynomial $p_{S/I}(t)$ with integer coefficients, see [14, Chapter 8].

Proposition 2.8. [26, Proposition 3.12] Let $Y \subset T_X$ for $X = \mathbb{P}(w_0, \ldots, w_n)$ with $w_0 = 1$. Then, the integer $a_Y = \deg(p_{S/I(Y)}(t)) - w_0 - \cdots - w_n$ satisfies $\operatorname{reg}(Y) = 1 + a_Y + \mathbb{N}$. A nice formula for the *a*-invariant is given for the \mathbb{F}_q -rational points of the torus T_X when X is a weighted projective space.

Proposition 2.9. [7, Corollary 3.9] If $Y = T_X(\mathbb{F}_q)$ for $X = \mathbb{P}(w_0, \ldots, w_n)$ and $g(\mathbb{N}\beta)$ is the Frobenius number of the numerical semigroup $\mathbb{N}\beta = \mathbb{N}\{w_0, \ldots, w_n\}$, then

 $a_Y = (q-2)[w_0 + \dots + w_n + g(\mathbb{N}\beta)] + g(\mathbb{N}\beta).$

There are subgroups of the torus T_X referred to as degenerate tori which we briefly discuss now.

Definition 2.10. The following subgroup $Y_A = \{[t_1^{a_1} : \ldots : t_r^{a_r}] : t_i \in \mathbb{F}_q^*\}$ of the torus T_X is called a degenerate torus, lying inside a toric variety X_{Σ} , for any positive integers a_1, \ldots, a_r , where r is the number of rays in the fan Σ .

If $\mathbb{F}_q^* = \langle \eta \rangle$, every $t_i \in \mathbb{F}_q^*$ is of the form $t_i = \eta^{k_i}$, for some $0 \leq k_i \leq q-2$. Let $d_i = |\eta^{a_i}|$ and $D = diag(d_1, \ldots, d_r)$.

Proposition 2.11. [13, Corollary 3.13 (ii)] If $Y = Y_A$ is a complete intersection in $X = \mathbb{P}^{r-1}$ and $g := \gcd(d_1, \ldots, d_r)$ so that $d'_1 = d_1/g, \ldots, d'_r = d_r/g$ generate a numerical semigroup $\mathbb{N}D'$ with the Frobenius number $g(\mathbb{N}D')$, then

$$1 + a_Y = g \cdot g(\mathbb{N}D') + d_1 + \dots + d_r - (r-1).$$

Notice that when $a_i = 1$ and $w_j = 1$, for all i and j, we have $d_i = q - 1$, and so $d'_i = 1$. The greatest integer not belonging to the numerical semigroup $\mathbb{N}\beta = \mathbb{N}D' = \mathbb{N}$ is $g(\mathbb{N}\beta) = g(\mathbb{N}D') = -1$ so both formulas in Proposition 2.9 and Proposition 2.11 yield $a_Y = n(q-2) - 1$, for the torus $Y = T_X(\mathbb{F}_q)$ in the projective space $X = \mathbb{P}^n$.

Definition 2.12. A binomial is a polynomial of the form $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}$, and J is called a binomial ideal if it is generated by binomials. J is called a complete intersection if it is generated by height(J) many binomials.

Definition 2.13. For a lattice $L \subset \mathbb{Z}^r$, the lattice ideal I_L is the binomial ideal generated by binomials $\mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}}$ for all $\mathbf{a} - \mathbf{b} \in L$. That is,

$$I_L = \langle \mathbf{x}^{\mathbf{a}} - \mathbf{x}^{\mathbf{b}} \mid \mathbf{a} - \mathbf{b} \in L \rangle \subset S.$$

Theorem 2.14. [23, Theorem 4.5] If $Y = Y_A$ then $I(Y) = I_L$ for $L = D(L_{\beta D})$.

If $a_i = 1$, for all *i*, then $Y_A = T_X(\mathbb{F}_q)$ and $d_i = q - 1$, for all *i*, so that the matrix *D* is just q - 1 times the identity matrix yielding the following:

Corollary 2.15. [23, Corollary 4.14 (ii)] If $Y = T_X(\mathbb{F}_q)$ then $I(Y) = I_L$ for $L = (q-1)L_\beta$.

Proposition 2.16. [23, Proposition 4.12] A generating system of binomials for $I(Y_A)$ is obtained from that of $I_{L_{\beta D}}$ by replacing x_i with $x_i^{d_i}$. $I(Y_A)$ is a complete intersection if and only if so is the toric ideal $I_{L_{\beta D}}$. In this case, a minimal generating system is obtained from a minimal generating system of $I_{L_{\beta D}}$ this way.

3. Degenerate Tori on Weighted Projective Spaces

In this section, we explore properties of some degenerate tori on a weighted projective space. To start with, we prove that they are complete intersections of special type of binomial hypersurfaces. We focus on a weighted projective space $X = \mathbb{P}(w_0, \ldots, w_n)$ and use the notation $S = \mathbb{F}_q[x_0, \ldots, x_n]$ for the Cox ring of X. Set

$$\tilde{w}_i := \frac{w_i}{\gcd(q-1, w_i)}$$
 and $d_i := \frac{q-1}{\gcd(q-1, w_i)}$ for $i = 0, 1, \dots, n$.

The following concept is very helpful in determining when a lattice ideal is a complete intersection.

Definition 3.1. If each column of a matrix has both a positive and a negative entry we say that the matrix is mixed. Moreover, if the matrix does not have a square mixed submatrix, then it is called dominating.

Theorem 3.2. [15, Theorem 3.9] Let $L \subseteq \mathbb{Z}^r$ be a lattice with the property that $L \cap \mathbb{N}^r = 0$. Then, I_L is a complete intersection if and only if L has a basis $\mathbf{m}_1, \ldots, \mathbf{m}_k$ such that the matrix $[\mathbf{m}_1 \cdots \mathbf{m}_k]$ is mixed dominating. If I_L is a complete intersection, then we have

$$I_L = \langle \boldsymbol{x}^{\boldsymbol{m}_1^+} - \boldsymbol{x}^{\boldsymbol{m}_1^-}, \dots, \boldsymbol{x}^{\boldsymbol{m}_k^+} - \boldsymbol{x}^{\boldsymbol{m}_k^-}
angle$$

Proposition 3.3. Let $Q = diag(w_0, \ldots, w_n)$ and $Y_Q = \{[t_0^{w_0}: \ldots: t_n^{w_n}]|t_i \in \mathbb{F}_q^*\}$ be the corresponding subgroup of T_X for $X = \mathbb{P}(w_0, \ldots, w_n)$. If $w_0 \mid q - 1$ and $F_i = x_i^{d_i} - x_0^{d_0 \tilde{w}_i}$, $i = 1, 2, \ldots, n$, then, the vanishing ideal of Y_Q is the following complete intersection lattice ideal:

$$I(Y_Q) = \langle F_1, F_2, \dots, F_n \rangle.$$

Proof. Since $D = diag(d_0, \ldots, d_n)$ and $\beta = [w_0 \cdots w_n]$, it follows that their product is $\beta D = [w_0 d_0 \cdots w_n d_n]$. It is clear that $\tilde{w}_i(q-1) = w_i d_i$, and so

$$gcd(w_0d_0,\ldots,w_nd_n) = (q-1)gcd(\tilde{w}_0,\ldots,\tilde{w}_n).$$

Therefore, we have the equality of the lattices $L_{\beta D} = L_{\tilde{W}}$, where \tilde{W} is the matrix with columns \tilde{w}_i , for $i = 0, \ldots, n$.

When $w_0 \mid q-1$, we have $\tilde{w}_0 = 1$ and thus the lattice $L_{\tilde{W}}$ has the following basis

$$\{(-\tilde{w}_1,\mathbf{e}_1),\ldots,(-\tilde{w}_n,\mathbf{e}_n)\}$$

where \mathbf{e}_i form the standard basis for \mathbb{Z}^n . Consider the matrix M whose columns are the basis vectors of $L_{\tilde{W}}$ given above. Since the matrix M is mixed-dominating, it follows from Theorem 3.2 that the lattice ideal of $L_{\tilde{W}}$ is a complete intersection generated by the binomials $x_i - x_0^{\tilde{w}_i}$, $i = 1, 2, \ldots, n$.

By Theorem 2.14, the vanishing ideal $I(Y_Q)$ is the binomial ideal I_L for the lattice $L = D(L_{\beta D})$, whose generators are obtained substituting $x_i^{d_i}$ for x_i in the binomials above generating the lattice ideal of $L_{\tilde{W}}$, by Proposition 2.16. Therefore, the vanishing ideal $I(Y_Q)$ is a complete intersection generated by the binomials F_1, F_2, \ldots, F_n .

Proposition 3.4. Let $Q = diag(w_0, \ldots, w_n)$ and $Y_Q = \{[t_0^{w_0} : \ldots : t_n^{w_n}] | t_i \in \mathbb{F}_q^*\}$ be the corresponding subgroup of T_X for $X = \mathbb{P}(w_0, \ldots, w_n)$. If $w_0 \mid q-1$ then, for any $\alpha \in \mathbb{N}\beta$ we have

$$H_{Y_Q}(\alpha) = \sum_{s=0}^{n} (-1)^s \sum_{I \subseteq [n], |I|=s} \dim_{\mathbb{K}} S_{\alpha - \alpha_I}$$

where $\alpha_I = \sum_{i \in I} \alpha_i$. Moreover, the a-invariant of Y_Q is given by the formula $a_{Y_Q} = (d_1 - 1)w_1 + \dots + (d_n - 1)w_n - w_0$.

Proof. Notice that $I(Y_Q)$ is a complete intersection by Proposition 3.3 generated by binomials of degrees $\alpha_1 = d_1 w_1, \ldots, \alpha_n = d_n w_n$. Thus, its minimal free resolution is given by the Koszul complex. As in the proof of [26, Proposition 3.13] we have the following exact sequence

$$0 \to W_n \to \cdots \to W_s \to \cdots \to W_1 \to S_\alpha \to (S/I(Y_Q))_\alpha \to 0,$$

where, for every s = 1, ..., n, the vector space W_s is given by

$$W_s = \bigoplus_{I \subseteq [n], |I|=s} S(-\alpha_I)_{\alpha} = \bigoplus_{I \subseteq [n], |I|=s} S_{\alpha - \alpha_I}.$$

Therefore, we obtain:

(3.1)
$$H_{Y_Q}(\alpha) = \dim_{\mathbb{K}} S_{\alpha} + \sum_{s=1}^{n} (-1)^s \dim_{\mathbb{K}} W_s$$
$$= \sum_{s=0}^{n} (-1)^s \sum_{I \subseteq [n], |I|=s} \dim_{\mathbb{K}} S_{\alpha-\alpha_I}$$

where $\alpha_I = \sum_{i \in I} \alpha_i$. By Proposition 8.23 in [14], the numerator of the Hilbert series in Equation 2.1 is as follows:

$$p_{S/I(Y_Q)} = \sum_{s=0}^{n} (-1)^s \sum_{I \subseteq [n], |I|=s} t^{\alpha_I}$$

Hence, $p_{S/I(Y_Q)}$ has degree $\alpha_1 + \cdots + \alpha_n = d_1w_1 + \cdots + d_nw_n$, and thus

$$a_{Y_Q} = (d_1 - 1)w_1 + \dots + (d_n - 1)w_n - w_0$$

by Proposition 2.8.

Example 3.5. Let $X = \mathbb{P}(1, 1, 2)$. Consider the matrix Q = diag(1, 1, 2) and $Y_Q = \{ [t_0 : t_1 : t_2^2] \mid t_0, t_1, t_2 \in \mathbb{F}_q^* \}$. Assume that q is odd. So, we have

$$(d_0, d_1, d_2) = (q - 1, q - 1, (q - 1)/2)$$
 and $(\tilde{w}_0, \tilde{w}_1, \tilde{w}_2) = (1, 1, 1).$

Thus, $I(Y_Q) = \langle F_1, F_2 \rangle = \langle x_1^{q-1} - x_0^{q-1}, x_2^{(q-1)/2} - x_0^{q-1} \rangle$. As the degrees of the generators are $\alpha_1 = q-1$ and $\alpha_2 = q-1$, a graded minimal free resolution of $I(Y_Q)$ is given by:

$$0 \to S_{\alpha-\alpha_1-\alpha_2} \xrightarrow{[-F_2 F_1]^T} S_{\alpha-\alpha_1} \oplus S_{\alpha-\alpha_2} \xrightarrow{[F_1 F_2]} S_{\alpha} \to (S/I(Y_Q))_{\alpha} \to 0.$$

Therefore, the Hilbert function is computed to be

$$H_{Y_Q}(\alpha) = \dim_{\mathbb{K}} S_{\alpha} - \dim_{\mathbb{K}} S_{\alpha-\alpha_1} - \dim_{\mathbb{K}} S_{\alpha-\alpha_2} + \dim_{\mathbb{K}} S_{\alpha-\alpha_1-\alpha_2}$$

= $\dim_{\mathbb{K}} S_{\alpha} - 2 \dim_{\mathbb{K}} S_{\alpha-(q-1)} + \dim_{\mathbb{K}} S_{\alpha-2(q-1)}.$

We first notice the following

$$\dim_{\mathbb{K}} S_{\alpha} = \begin{cases} (\alpha_0 + 1)^2 & \text{if } \alpha = 2\alpha_0\\ (\alpha_0 + 1)(\alpha_0 + 2) & \text{if } \alpha = 2\alpha_0 + 1. \end{cases}$$

Thus, if $0 \leq \alpha \leq q-2$, then $\dim_{\mathbb{K}} S_{\alpha-(q-1)} = \dim_{\mathbb{K}} S_{\alpha-2(q-1)} = 0$. Hence,

$$H_{Y_Q}(\alpha) = \begin{cases} (\alpha_0 + 1)^2 & \text{if } \alpha = 2\alpha_0\\ (\alpha_0 + 1)(\alpha_0 + 2) & \text{if } \alpha = 2\alpha_0 + 1. \end{cases}$$

When, $q-1 \leq \alpha < 2(q-1)$, we have $\dim_{\mathbb{K}} S_{\alpha-2(q-1)} = 0$. It is easy to see that

$$\dim_{\mathbb{K}} S_{\alpha-(q-1)} = \begin{cases} (\alpha_0 + 1 - (q-1)/2)^2 & \text{if } \alpha = 2\alpha_0\\ (\alpha_0 + 1 - (q-1)/2)(\alpha_0 + 2 - (q-1)/2) & \text{if } \alpha = 2\alpha_0 + 1. \end{cases}$$

Hence, we have the following formula for $H_{Y_Q}(\alpha)$:

 $\begin{cases} (\alpha_0+1)^2 - 2(\alpha_0+1 - (q-1)/2)^2 & \text{if } \alpha = 2\alpha_0\\ (\alpha_0+1)(\alpha_0+2) - 2(\alpha_0+1 - (q-1)/2)(\alpha_0+2 - (q-1)/2) & \text{if } \alpha = 2\alpha_0+1. \end{cases}$ Finally, when $\alpha \ge 2(q-1)$, we get

Finally, when
$$\alpha \geq 2(q-1)$$
, we get

$$\dim_{\mathbb{K}} S_{\alpha-2(q-1)} = \begin{cases} (\alpha_0 + 1 - (q-1))^2 & \text{if } \alpha = 2\alpha_0\\ (\alpha_0 + 1 - (q-1))(\alpha_0 + 2 - (q-1)) & \text{if } \alpha = 2\alpha_0 + 1. \end{cases}$$

Therefore, we have $H_{Y_Q}(\alpha) = (q-1)^2/2 = |Y_Q|$ which is not surprising as we have $\alpha > a_{Y_Q}$ in this case.

4. LENGTH AND DIMENSION WHEN $X = \mathbb{P}(1, w_1, \ldots, w_n)$

Let $\mathbb{F}_q^* = \langle \eta \rangle$, then the order of $\eta_i := \eta^{w_i}$ is

$$d_i = \frac{q-1}{\gcd(q-1,w_i)} \quad i = 1,\dots,n.$$

By using $I(Y_Q)$, the length and the dimension of \mathcal{C}_{α,Y_Q} are computed as follows.

Theorem 4.1. Let $X = \mathbb{P}(1, w_1, \ldots, w_n)$ be a weighted projective space over the field $\overline{\mathbb{F}}_q$. Consider $Q = diag(1, w_1, \ldots, w_n)$ and the subgroup it defines in $T_X(\mathbb{F}_q)$:

$$Y_Q = \{ [t_0 : t_1^{w_1} : \ldots : t_n^{w_n}] \mid t_i \in \mathbb{F}_q^*, \text{ for all } i = 0, \ldots, n \}$$

Then, the length of \mathcal{C}_{α,Y_Q} is $|Y_Q| = d_1 \cdots d_n$ and the dimension is

$$\dim(\mathcal{C}_{\alpha,Y_Q}) = \sum_{m_n=0}^{\min\{\lfloor\frac{\alpha}{w_n}\rfloor, d_n-1\}} \sum_{m_{n-1}=0}^{\min\{\lfloor\frac{\alpha-m_nw_n}{w_{n-1}}\rfloor, d_{n-1}-1\}} \cdots \sum_{m_1=0}^{\min\{\lfloor\frac{\alpha-m_nw_n-\cdots-m_2w_2}{w_1}\rfloor, d_1-1\}} 1.$$

Moreover, the a-invariant is given by

$$a_{Y_Q} = (d_1 - 1)w_1 + \dots + (d_n - 1)w_n - 1.$$

Proof. We first prove that

(4.1)
$$Y_Q = \langle [1:\eta_1:1:\ldots:1] \rangle \times \cdots \times \langle [1:\ldots:1:\eta_n] \rangle.$$

Multiplying by $[\lambda : \lambda^{w_1} : \ldots : \lambda^{w_n}]$ does not change an equivalence class for every $\lambda \in \mathbb{F}_a^*$. So, we have the equality of the following points:

$$[t_0: t_1^{w_1}: \ldots: t_n^{w_n}] = [1: (t_1/t_0)^{w_1}: \ldots: (t_n/t_0)^{w_n}].$$

Hence, we have

$$Y_Q = \{ [1: s_1^{w_1}: \ldots: s_n^{w_n}] \mid s_i \in \mathbb{F}_q^*, \text{ for all } i = 1, \ldots, n \}.$$

Since $s_i = \eta^{k_i}$, for some $k_i \in \mathbb{N}$, it is clear that $s_i^{w_i} = \eta_i^{k_i}$ and thus

$$Y_Q = \{ [1:\eta_1^{i_1}:\ldots:\eta_n^{i_n}] \mid 0 \le i_1 \le d_1,\ldots,0 \le i_n \le d_n \},\$$

from which the claim in (4.1) is deduced, and thus $|Y_Q| = d_1 \cdots d_n$.

If $w_0 = 1$, then $d_0 = q - 1$ and so the vanishing ideal of Y_Q is generated by the binomials $F_i = x_i^{d_i} - x_0^{d_i w_i}$, for i = 1, 2, ..., n. With respect to any term order for which x_0 is the smallest variable, the leading monomial of F_i is clearly $x_i^{d_i}$. Since the

monomials $x_i^{d_i}$ and $x_j^{d_j}$ are relatively prime for different *i* and *j*, it readily follows that the binomials F_1, \ldots, F_n form a Groebner basis for the vanishing ideal $I(Y_Q)$. It is well-known ([5, p.232]) then that a basis for the vector space $S_{\alpha}/I_{\alpha}(Y_Q)$ is given by the monomials $\mathbf{x}^{\mathbf{m}} = x_0^{m_0} x_1^{m_1} \cdots x_n^{m_n}$ of degree α that can not be divided by the leading monomials $x_i^{d_i}$ of F_i , for all $i = 1, 2, \ldots, n$ and for

$$\alpha = m_0 + m_1 w_1 + \dots + m_n w_n \in \mathbb{N} = \langle 1, w_1, \dots, w_n \rangle.$$

Therefore, a basis for $S_{\alpha}/I_{\alpha}(Y_Q)$ corresponds to the set of tuples (m_0, m_1, \ldots, m_n) satisfying $\alpha = m_0 + m_1 w_1 + \cdots + m_n w_n$ and $m_i \leq d_i - 1$, for all $i = 1, 2, \ldots, n$. The elements of this set can be identified step by step as we explain now. We start first by choosing an integer m_n between 0 and $\min\{\lfloor\frac{\alpha}{w_n}\rfloor, d_n - 1\}$ and observe that the elements of the set in question can be partitioned into subsets for every choice of m_n in the aforementioned range. More precisely, for each fixed m_n , we have a subset consisting of tuples (m_0, m_1, \ldots, m_n) satisfying

$$m_0 + m_1 w_1 + \dots + m_{n-1} w_{n-1} = \alpha - m_n w_n$$
 and $m_i \le d_i - 1$, for all $i = 1, 2, \dots, n-1$

As a second step, we fix m_{n-1} between 0 and $\min\{\lfloor \frac{\alpha - m_n w_n}{w_{n-1}} \rfloor, d_{n-1} - 1\}$, and look for the solutions $(m_0, m_1, \ldots, m_{n-2})$ satisfying

$$m_0 + m_1 w_1 + \dots + m_{n-2} w_{n-2} = \alpha - m_n w_n - m_{n-1} w_{n-1}$$
 and $m_i \le d_i - 1$,

for all i = 1, 2, ..., n - 2. Continuing inductively, we end up with a unique m_0 satisfying

$$m_0 = \alpha - m_n w_n - m_{n-1} w_{n-1} - \dots - m_1 w_1$$

Hence, the dimension of the code, which is nothing but the dimension of the vector space $S_{\alpha}/I_{\alpha}(Y_Q)$, is exactly the sum given by the formula

$$\dim(\mathcal{C}_{\alpha,Y_Q}) = \sum_{m_n=0}^{\min\{\lfloor\frac{\alpha}{w_n}\rfloor, d_n-1\}} \sum_{m_n=1}^{\min\{\lfloor\frac{\alpha}{w_n}\rfloor, d_n-1-1\}} \sum_{m_n=1}^{\min\{\lfloor\frac{\alpha}{w_n}\rfloor, d_n-1-1\}} \cdots \sum_{m_1=0}^{\min\{\lfloor\frac{\alpha}{w_n}\rfloor, d_n-1-1\}} 1$$

The *a*-invariant can be obtained from Proposition 3.4, by substituting $w_0 = 1$. \Box

5. Codes on $Y_Q \subset \mathbb{P}(1, 1, a)$

For any positive integer a, we compute the basic parameters of the code \mathcal{C}_{α,Y_Q} , for the subgroup $Y_Q = \{[t_0 : t_1 : t_2^a] \mid t_0, t_1, t_2 \in \mathbb{F}_q^*\}$ of $T_X(\mathbb{F}_q)$ for the weighted projective space $X = \mathbb{P}(1, 1, a)$.

Theorem 5.1. Let $d_2 = \frac{q-1}{\gcd(a,q-1)}$, $k = \lfloor \frac{\alpha - (q-2)}{a} \rfloor$ and $\mu_2 = \min\{\lfloor \frac{\alpha}{a} \rfloor, d_2 - 1\}$. Then, the length of \mathcal{C}_{α,Y_Q} is $N = |Y_Q| = (q-1)d_2$. Its dimension $K(\mathcal{C}_{\alpha,Y_Q})$ is

$$\begin{array}{ll} (\mu_2+1)(\alpha+1-\mu_2a/2), & \mbox{if } 0 \leq \alpha \leq q-2 \\ (q-1)(k+1)+(\mu_2-k)[\alpha+1-(\mu_2+k+1)a/2], & \mbox{if } 0 < \alpha-(q-2) < (d_2-1)a \\ N & \mbox{otherwise.} \end{array}$$

and the minimum distance of \mathcal{C}_{α,Y_Q} is:

$$\delta(\mathcal{C}_{\alpha,Y_Q}) = \begin{cases} d_2(q-1-\alpha) & \text{if } 0 \le \alpha \le q-2\\ d_2-k & \text{if } q-2 \le \alpha < (q-2) + (d_2-1)a\\ 1 & \text{otherwise.} \end{cases}$$

Proof. Since $w_1 = 1$, we have $d_1 = q - 1$. It follows from Equation 4.1 that

$$Y_Q = \{ [1:\eta_1^{i_1}:\eta_2^{i_2}] \mid 0 \le i_1 \le d_1 \text{ and } 0 \le i_2 \le d_2 \}$$

so the length of the code is $d_1d_2 = (q-1)d_2$.

When $0 \le \alpha \le q - 2$, the dimension formula in Theorem 4.1 specializes to

$$\dim(\mathcal{C}_{\alpha,Y_Q}) = \sum_{m_2=0}^{\mu_2} \sum_{m_1=0}^{\min\{\alpha-m_2a,q-2\}} 1 = \sum_{m_2=0}^{\mu_2} \sum_{m_1=0}^{\alpha-m_2a} 1$$
$$= \sum_{m_2=0}^{\mu_2} (\alpha - m_2a + 1) = (\mu_2 + 1)(\alpha + 1) - a \sum_{m_2=0}^{\mu_2} m_2$$
$$= (\mu_2 + 1)(\alpha + 1) - a \frac{\mu_2(\mu_2 + 1)}{2}.$$

If $q-2 < \alpha < (q-2) + (d_2-1)a$, then using the formula in Theorem 4.1 again, we get

$$\dim(\mathcal{C}_{\alpha,Y_Q}) = \sum_{m_2=0}^{\mu_2} \sum_{m_1=0}^{\min\{\alpha-m_2a,q-2\}} 1$$

= $\sum_{m_2=0}^k \sum_{m_1=0}^{q-2} 1 + \sum_{m_2=k+1}^{\mu_2} \sum_{m_1=0}^{\alpha-m_2a} 1$
= $(q-1)(k+1) + \sum_{m_2=k+1}^{\mu_2} (\alpha-m_2a+1)$
= $(q-1)(k+1) + (\mu_2 - k)(\alpha+1) - a \sum_{m_2=k+1}^{\mu_2} m_2$
= $(q-1)(k+1) + (\mu_2 - k)(\alpha+1) - a \frac{\mu_2(\mu_2+1) - k(k+1)}{2}$

Notice that these dimensions are the number of lattice points of the polygons depicted below.



As for the minimum distance, we first give an upper bound on the number $|V_{Y_Q}(F)|$ of zeroes on Y_Q of a homogeneous polynomial F of degree α and then

demonstrate a specific polynomial attaining that bound. Let $[d_2]$ denote the set of non-negative integers smaller than d_2 , and set

$$J_F := \{ j \in [d_2] \mid x_2 - \eta_2^j x_0^{w_2} \text{ divides } F \}.$$

We claim that

(5.1)
$$|V_{Y_Q}(F)| \le d_1|J_F| + (d_2 - |J_F|) \deg_{x_1}(F),$$

where $\deg_{x_1}(F)$ is the usual degree of F in the variable x_1 . The polynomial $f_j(x_1) := F(1, x_1, \eta_2^j) \in \mathbb{F}_q[x_1]$ vanishes at the points $[1 : \eta_1^i : \eta_2^j]$, for every $i \in [d_1]$, when $j \in J_F$. Thus, there are $d_1|J_F|$ such roots of F. On the other hand, f_j is not a zero polynomial when $j \notin J_F$, and in this case it can have at most its degree many zeroes, giving rise to $(d_2 - |J_F|) \deg_{x_1}(F)$ many roots of F, completing the proof of the claim.

Since we always have

$$F = \prod_{j=1}^{|J_F|} (x_2 - \eta_2^j x_0^{w_2}) F'$$

it follows that $\deg_{x_1}(F) = \deg_{x_1}(F') \leq \alpha - |J_F|w_2$. Thus, we have

(5.2)
$$\begin{aligned} |V_{Y_Q}(F)| &\leq d_1 |J_F| + (d_2 - |J_F|)(\alpha - |J_F|w_2) \\ &\leq d_2 \alpha + |J_F|(d_1 - \alpha - w_2(d_2 - |J_F|)) \end{aligned}$$

Notice that the number in the parenthesis above is

$$d_1 - \alpha - w_2(d_2 - |J_F|) = d_1 - \alpha - w_2d_2 + w_2|J_F| = d_1 - (q - 1)\tilde{w}_2 - \alpha + w_2|J_F|$$

which is non-positive since $d_1 \leq q - 1 \leq (q - 1)\tilde{w}_2$ and $|J_F|w_2 \leq \deg(F) = \alpha$
Hence, altogether, we have the upper bound

$$(5.3) |V_{Y_O}(F)| \le d_2 \alpha$$

Consider now the following polynomial:

$$F_0 = \prod_{i=1}^{\alpha} (x_1 - \eta_1^i x_0)$$

which vanishes at the points $[1 : \eta_1^i : \eta_2^j]$, for every $i \in [\alpha]$ and $j \in [d_2]$, implying that $|V_{Y_Q}(F_0)| = d_2\alpha$. As the weight of the codeword $ev_{Y_Q}(F_0)$ is clearly

$$|Y_Q| - |V_{Y_Q}(F_0)| = d_2(q-1) - d_2\alpha$$

and that of a general codeword $ev_{Y_Q}(F)$ is

$$|Y_Q| - |V_{Y_Q}(F)| \ge d_2(q-1) - d_2\alpha,$$

it follows that the minimum distance of the code is $d_2(q-1-\alpha)$, when $\alpha < q-1$.

When $\alpha \ge a_Y + 1 = (q-2) + (d_2 - 1)a$, the code is trivial, so $\delta(\mathcal{C}_{\alpha, Y_Q}) = 1$.

From now on, assume that $q - 2 \le \alpha < a_Y + 1 = (q - 2) + (d_2 - 1)a$. Let k be the quotient and r_0 be the remainder of the division of $\alpha - (q - 2)$ by $w_2 = a$, i.e.

$$\alpha - (q-2) = ka + r_0$$
 where $0 \le k := \left\lfloor \frac{\alpha - (q-2)}{a} \right\rfloor \le d_2 - 2$ and $0 \le r_0 \le a - 1$.

When $|J_F| = d_2$, F vanishes on Y_Q , so F gives a codeword with zero weight. Thus, we suppose $|J_F| \le d_2 - 1$.

If $|J_F| \leq k$ also, then by $\deg_{x_1}(F) \leq d_1 - 1 = q - 2$ and (5.1) we have

$$|V_{Y_Q}(F)| \leq (q-1)|J_F| + (d_2 - |J_F|)(q-2) = (q-2)d_2 + |J_F| \leq (q-2)d_2 + k.$$

If $|J_F| > k$, we let $|J_F| = k + j_0$ with $j_0 \ge 1$. As $\alpha = q - 2 + ka + r$ with $r \le a - 1$, we have $\alpha - |J_F|a = \alpha - (k+1)a - a(|J_F| - k - 1) \le q - 2 - 1 - a(j_0 - 1)$. As $\deg_{x_1}(F) \le \alpha - |J_F|a$, it follows from (5.1) that we have,

$$\begin{aligned} |V_{Y_Q}(F)| &\leq (q-1)|J_F| + (d_2 - |J_F|)(\alpha - |J_F|a) \\ &\leq (q-1)|J_F| + (d_2 - |J_F|)(q-2 - 1 - a(j_0 - 1)) \\ &\leq (q-2)d_2 + |J_F| + (d_2 - |J_F|)(-1 - a(j_0 - 1)) \end{aligned}$$

Since $d_2 - |J_F| \ge 1$ and $a \ge 1$, we have

$$V_{Y_Q}(F)| \leq (q-2)d_2 + |J_F| + (d_2 - |J_F|)(-1 - a(j_0 - 1))$$

$$\leq (q-2)d_2 + |J_F| - 1 - (j_0 - 1) = (q-2)d_2 + k$$

We consider the following homogeneous polynomial of degree α now:

$$G_0 = x_0^{r_0} \prod_{i=1}^{q-2} (x_1 - \eta_1^i x_0) \prod_{j=1}^k (x_2 - \eta_2^j x_0^a)$$

which vanish at the points $[1 : \eta_1^i : \eta_2^j]$, for every $i \in [q-2]$ and $j \in [d_2]$, together with the points $[1 : \eta_1^i : \eta_2^j]$, for i = q-1 and $j \in [k]$. Therefore, the number of roots is $|V_{Y_Q}(G_0)| = (q-2)d_2 + k$. It readily follows that the minimum distance $\delta(\mathcal{C}_{\alpha,Y_Q})$ of the code is the weight $(q-1)d_2 - (q-2)d_2 - k = d_2 - k$ of the codeword corresponding to G_0 .

Remark 5.2. It is very difficult to give a closed formula for some parameters of the code C_{α,Y_Q} , for the subgroup $Y_Q = \{[t_0 : t_1^{w_1} : t_2^{w_2}] \mid t_0, t_1, t_2 \in \mathbb{F}_q^*\}$ of $T_X(\mathbb{F}_q)$ in the more general case of the weighted projective plane $X = \mathbb{P}(1, w_1, w_2)$. This is mainly because of the formulas involving a division by the integer w_1 . In this case one has to use the floor function when the ratio is not integer, which we explain in more details below. There are 4 cases to consider:

$$Case \ 1: \left\lfloor \frac{\alpha}{w_1} \right\rfloor \le d_1 - 1 \ and \ \left\lfloor \frac{\alpha}{w_2} \right\rfloor \le d_2 - 1$$

$$Case \ 2: \left\lfloor \frac{\alpha}{w_1} \right\rfloor > d_1 - 1 \ and \ \left\lfloor \frac{\alpha}{w_2} \right\rfloor \le d_2 - 1$$

$$Case \ 3: \left\lfloor \frac{\alpha}{w_1} \right\rfloor \le d_1 - 1 \ and \ \left\lfloor \frac{\alpha}{w_2} \right\rfloor > d_2 - 1$$

$$Case \ 4: \left\lfloor \frac{\alpha}{w_1} \right\rfloor > d_1 - 1 \ and \ \left\lfloor \frac{\alpha}{w_2} \right\rfloor > d_2 - 1.$$

For instance, the dimension formula in Theorem 4.1 specializes to

$$\dim(\mathcal{C}_{\alpha,Y_Q}) = \sum_{m_2=0}^{\mu_2} \sum_{m_1=0}^{\min\left\{ \left\lfloor \frac{\alpha - m_2 w_2}{w_1} \right\rfloor, d_1 - 1 \right\}} 1,$$

where $\mu_2 = \min\left\{\left\lfloor \frac{\alpha}{w_2} \right\rfloor, d_2 - 1\right\}.$

We first consider Cases 2 and 4. In these cases, as we have $\alpha \ge w_1(d_1-1)$, we choose

$$\mu_2' := \left\lfloor \frac{\alpha - w_1(d_1 - 1)}{w_2} \right\rfloor$$

so that $\min\left\{\left\lfloor\frac{\alpha-m_2w_2}{w_1}\right\rfloor, d_1-1\right\} = d_1-1$ for all $m_2 \leq \mu'_2$. Hence, we have

$$\dim(\mathcal{C}_{\alpha,Y_Q}) = \sum_{m_2=0}^{\mu'_2} \sum_{m_1=0}^{d_1-1} 1 + \sum_{m_2=\mu'_2+1}^{\mu_2} \sum_{m_1=0}^{\left\lfloor\frac{\alpha-m_2w_2}{w_1}\right\rfloor} 1$$
$$= (\mu'_2+1)d_1 + \sum_{m_2=\mu'_2+1}^{\mu_2} \left(\left\lfloor \frac{\alpha-m_2w_2}{w_1} \right\rfloor + 1 \right)$$
$$= \mu'_2(d_1-1) + \mu_2 + d_1 + \left\lfloor \frac{\alpha-(\mu'_2+1)w_2}{w_1} \right\rfloor + \dots + \left\lfloor \frac{\alpha-\mu_2w_2}{w_1} \right\rfloor$$

A similar formula for dim $(\mathcal{C}_{\alpha,Y_Q})$ can be obtained in Cases 1 and 3. In any case, we conclude that a closed formula for the dimension is difficult to get.

As for the minimum distance, one can generalize Theorem 5.1 as follows. Let U(x, y) be a polynomial defined as

$$U(x,y) := d_1 y + (d_2 - y)x \text{ for } 0 \le x \le \min\left\{ \left\lfloor \frac{\alpha - yw_2}{w_1} \right\rfloor, d_1 - 1 \right\}, 0 \le y \le \mu_2.$$

Then, the upper bound in (5.1) becomes

$$|V_{Y_Q}(F)| \le U(\deg_{x_1}(F), |J_F|).$$

Since we have

$$x \le \min\left\{ \left\lfloor \frac{\alpha - yw_2}{w_1} \right\rfloor, d_1 - 1 \right\} = \begin{cases} d_1 - 1 & \text{if } 0 \le y \le \mu'_2 \\ \left\lfloor \frac{\alpha - yw_2}{w_1} \right\rfloor & \text{if } \mu'_2 < y \le \mu_2, \end{cases}$$

it follows that

$$U(x,y) \le u(y) := d_1 y + (d_2 - y) \min\left\{ \left\lfloor \frac{\alpha - yw_2}{w_1} \right\rfloor, d_1 - 1 \right\}.$$

Therefore, we get

$$U(x,y) \le d_1y + (d_2 - y)(d_1 - 1) = d_2(d_1 - 1) + y \text{ for all } y \in [0, \mu'_2]$$

and
$$U(x,y) < d_1y + (d_2 - y)(d_1 - 1)$$
 for all $y \in [\mu'_2 + 1, \mu_2]$.

Clearly, the polynomial U(x,y) attains the maximum value at (d_1-1,μ'_2) , which is

$$u(\mu'_2) = d_2(d_1 - 1) + \mu'_2 = d_1d_2 - (d_2 - \mu'_2).$$

Thus, the minimum distance of the code C_{α,Y_Q} will be $d_2 - \mu'_2$. This means that the proof of the second part of the Theorem 5.1 can be generalized very easily via replacing q-1 (resp. q-2) by d_1 (resp. d_1-1) to the Cases 2 and 4, namely for the values of α satisfying $w_1(d_1-1) \leq \alpha < w_1(d_1-1) + w_2(d_2-1)$.

When $w_1 = 1$, as in the proof of the first part of the Theorem 5.1, the ratio $\frac{\alpha - yw_2}{w_1}$ was an integer yielding $x = \left\lfloor \frac{\alpha - yw_2}{w_1} \right\rfloor = \alpha - yw_2$, and so the upper bound was

$$u(y) = d_1 y + (d_2 - y)(\alpha - yw_2) = w_2 y^2 + (d_1 - \alpha - d_2 w_2)y + d_2 \alpha.$$

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As the quadratic polynomial u(y) is concave up, it was clear that the absolute maximum is attained at the boundary points of the interval $[0, \lfloor \alpha/a \rfloor]$ and we were able to prove that u(0) was the maximum value of U. However, the proof of the first case does not generalize to the Cases 1 and 3 as the maximum values are sometimes attained at interior points.

For instance, consider the case q = 31, $w_1 = 8$, $w_2 = 9$ and $\alpha = 34$. Then, we have $d_1 = 15$, $d_2 = 10$, $\left|\frac{\alpha}{w_1}\right| = 4$ and $\left|\frac{\alpha}{w_2}\right| = 3$. The function

$$U(x,y) := 15y + (10 - y)x \text{ for } 0 \le x \le \left\lfloor \frac{34 - 9y}{8} \right\rfloor, 0 \le y \le 3$$

has the upper bound given by

$$u(y) = 15y + (10 - y) \left\lfloor \frac{34 - 9y}{8} \right\rfloor, 0 \le y \le 3.$$

Notice that [u(0), u(1), u(2), u(3)] = [40, 42, 46, 45]. Therefore, the maximum value 46 is attained at the interior point y = 2.

As the value $u(0) = \left\lfloor \frac{\alpha}{w_1} \right\rfloor$ gives rise to an upper bound on the minimum distance in Cases 1 and 3, we have the following:

Theorem 5.3. Let $d_1 = \frac{q-1}{\gcd(w_1,q-1)}$, $d_2 = \frac{q-1}{\gcd(w_2,q-1)}$ and $k = \left\lfloor \frac{\alpha - w_1(d_1-1)}{w_2} \right\rfloor$. Then, the length of \mathcal{C}_{α,Y_Q} is $N = |Y_Q| = d_1d_2$. The minimum distance of \mathcal{C}_{α,Y_Q} satisfies

$$\begin{aligned} \delta(\mathcal{C}_{\alpha,Y_Q}) &\leq d_2(d_1 - \left\lfloor \frac{\alpha}{w_1} \right\rfloor) & \text{if } 0 \leq \alpha \leq w_1(d_1 - 1) \\ \delta(\mathcal{C}_{\alpha,Y_Q}) &= d_2 - k & \text{if } w_1(d_1 - 1) \leq \alpha < w_1(d_1 - 1) + w_2(d_2 - 1) \\ \delta(\mathcal{C}_{\alpha,Y_Q}) &= 1 & \text{otherwise.} \end{aligned}$$

We conclude the paper by showcasing an example with codes having good parameters obtained by our construction.

Example 5.4. Take a = 2, q = 5. So, $d_2 = 2$ and length is $d_2(q-1) = 8$. Table 1 exhibits the main parameters of the code C_{α,Y_Q} for α in the first column. According to Markus Grassl's Code Tables [9] a best-possible code with N = 8 has $K + \delta = 8$ or $K + \delta = 9$ (MDS codes). This example provides us with 3 best possible codes whose parameters satisfy $K + \delta = 8$ together with an MDS code [8, 7, 2].

TABLE 1. a=2 and q=5

α	$[N,K,\delta]$
0	[8, 1, 8]
1	[8, 2, 6]
2	[8, 4, 4]
3	[8, 6, 2]
4	[8, 7, 2]
5	[8, 1, 8]

Example 5.5. Similarly, we take a = 3 and q = 5 so that $d_2 = 4$ and length is $d_2(q-1) = 4 \cdot 4 = 16$. Table 2 gives the main parameters of the corresponding codes.

TABLE 2. a=3 and q=5

α	$[N, K, \delta]$
0	[16, 1, 16]
1	[16, 2, 12]
2	[16, 3, 8]
3	[16, 5, 4]
4	[16, 6, 4]
5	[16, 7, 4]
6	[16, 9, 3]
7	[16, 10, 3]
8	[16, 11, 3]
9	[16, 13, 2]
10	[16, 14, 2]
11	[16, 15, 2]

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