Combinatorics on Heavy $(3, 2n+1)$- Multinets

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Abstract

A (k,d)-multinet is a certain configuration of lines and points with multiplicities in $\mathbb{CP}^2$. If there is at least one multiple line in a class of a (k,d)-multinet, it is called *heavy*. By using the main result proved by Yuzvinsky in [4] and the results of the article [1] , we conclude that if a multinet is heavy, the only k value is 3. Therefore, each heavy multinet is of the form $(3,d)$. A heavy $(3,2n)$-multinet is constructed for $n>1$ by Bartz in [2]. We discuss the possibilities for combinatorics of lines and points inside a heavy $(3,2n+1)$-multinet and we have showed that there exists neither a heavy $(3,3)$ nor a heavy $(3,5)$-multinet. Moreover, we have discovered several numerical results of a heavy $(3,2n+1)$-multinet containing a multiple line consisting of three points from $\mathcal{X}$.

Introduction

Multinets are certain configurations of lines and points with multiplicities in the complex projective plane $\mathbb{CP}^2$. They occur in the study of resonance varieties of complement of complex hyperplane arrangements and homology of Milnor fibers. There are some restrictions on the structure of heavy multinets discovered by M. Falk and S. Yuzvinsky, but some open problems remain. A multinet with at least one multiple line is called a *heavy multinet*. We will work over these multinets with odd degree to construct more numerical restrictions on them. The way that we have attempted is using certain rules of complex projective geometry to find these restrictions. We show that there cannot be any heavy $(3,3)$ and $(3,5)$-multinets in $\mathbb{CP}^2$. For heavy multinets of degrees 7 and 9, we have discovered some numerical conditions on them. It is easier to deal with a heavy $(3,2n+1)$-multinet containing a multiple line consisting of three points from $\mathcal{X}$ because we can use combinatorics extensively on these small degrees.

Preliminaries

Let $\mathcal{A}$ be an arrangement of lines in the complex projective plane. Let $m : \mathcal{A} \to \mathbb{Z}_{>0}$ be a function which assigns to each $l \in \mathcal{A}$ a positive integer $m(l)$ called the *multiplicity of the line*. The pair $(\mathcal{A}, m)$ is called a *multi-arrangement*.

Definition. *A (k,d)-multinet on a multi-arrangement $(\mathcal{A}, m)$ is a pair $(\mathcal{N}, \mathcal{X})$ where $\mathcal{N}$ is a partition of $\mathcal{A}$ into $k \geqslant 3$ classes $\mathcal{A}_1, \dots, \mathcal{A}_k$, and $\mathcal{X}$ is a set of points called base locus such that:*

(i) $\sum_{l\in\mathcal{A}_i} m(l) = d$*, independent of* i

(ii) for every $l \in \mathcal{A}_i$ *and* $l' \in \mathcal{A}_j$*, with* $i \neq j$*, the point* $l \cap l'$ *is in* $\mathcal{X}$

(iii) for each $p \in \mathcal{X}$, $\sum_{l\in\mathcal{A}_i, p\in l} m(l)$ *is constant, independent of* i

(iv) for $1 \le i \le k$*, for any* $l, l' \in \mathcal{A}_i$*, there is a sequence of* $l = l_0, l_1, \dots, l_r = l'$ *such that* $l_{j-1} \cap l_j \notin \mathcal{X}$ *for* $1 \le j \le r$*.*

Definition. *A (k,d)-multinet with $n_p = 1$ for all $p \in \mathcal{X}$ is called a (k,d)-net, i.e. $|\mathcal{X}| = d^2$.*

By using the main result proved by Yuzvinsky in [4] and the article [1], we immediately conclude that the following for a given (k,d)-multinet: The k value is either 3 or 4. If $k=4$, then $|\mathcal{X}| = d^2$, hence it is a net. It is visible from the definitions that all nets are multinets. However, there are multinets which are not net. These multinets are called *proper multinets*. Then, if a (k,d)-multinet is proper, the only k value is 3. Therefore, each proper multinet is of the form $(3,d)$.

Definition. *If each class of a (k,d)-multinet has no multiple line, the multinet is said to be light. If there is at least one multiple line in a class of a (k,d)-multinet, it is called heavy.*

We can say that every heavy (k,d)-multinet is a proper multinet immediately. It is known that there exist examples of a heavy $(3,2n)$-multinet, a light proper $(3,2n)$-multinet and a light proper $(3,2n+1)$-multinet constructed for $n>1$ by Bartz in [2].

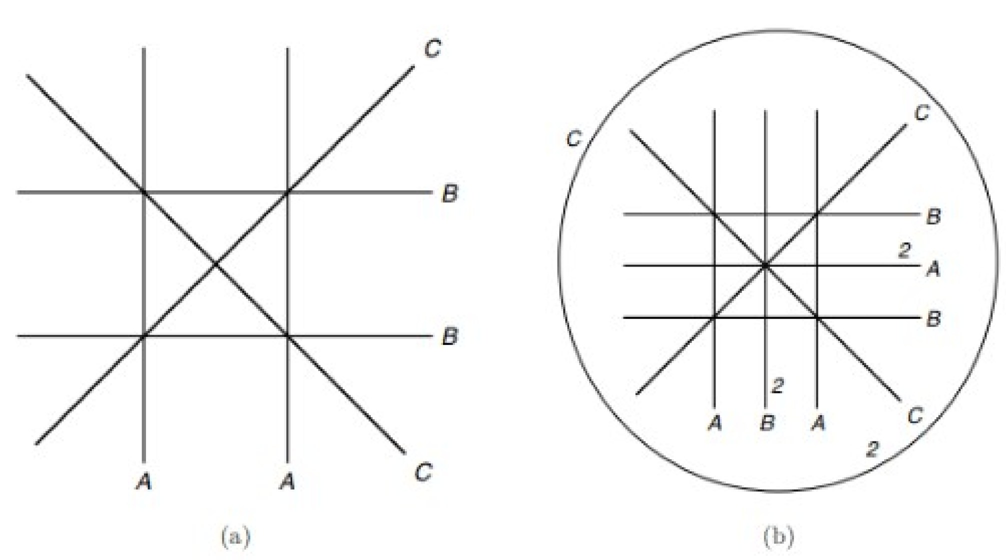

Figure 1: (a) A (3,2)-net (b) A heavy (3,4)-multinet

Results

We discuss the possibilities for combinatorics of lines and points inside a heavy $(3,2n+1)$-multinet and have showed that there exists neither a heavy $(3,3)$ nor a heavy $(3,5)$-multinet. We have found the following numerical conditions which lead to restrictions on the line and point multiplicities in heavy multinets for $d=7$ and 9.

Result 1. *A heavy $(3,2n+1)$-multinet $\mathcal{A} = \cup_{i=1}^{3} \mathcal{A}_i$ does not have a multiple line containing exactly two points in $\mathcal{X}$.*

Result 2. *If a heavy $(3,7)$-multinet exists, the only possible base locus $\mathcal{X}$ is the set $\{p_1, p_2, p_3, q_1, ..., q_{32}\}$ or the set $\{p_1, p_2, p_3, r_1, r_2, q_1, ..., q_{24}\}$ or the set$\{p_1, p_2, p_3, r_1, r_2, r_3, r_4, q_1, ..., q_{16}\}$ with $n_{p_1} = 3$, $n_{p_2} = n_{p_3} = n_{r_i} = 2$ and $n_{q_i} = 1$ for each i.*

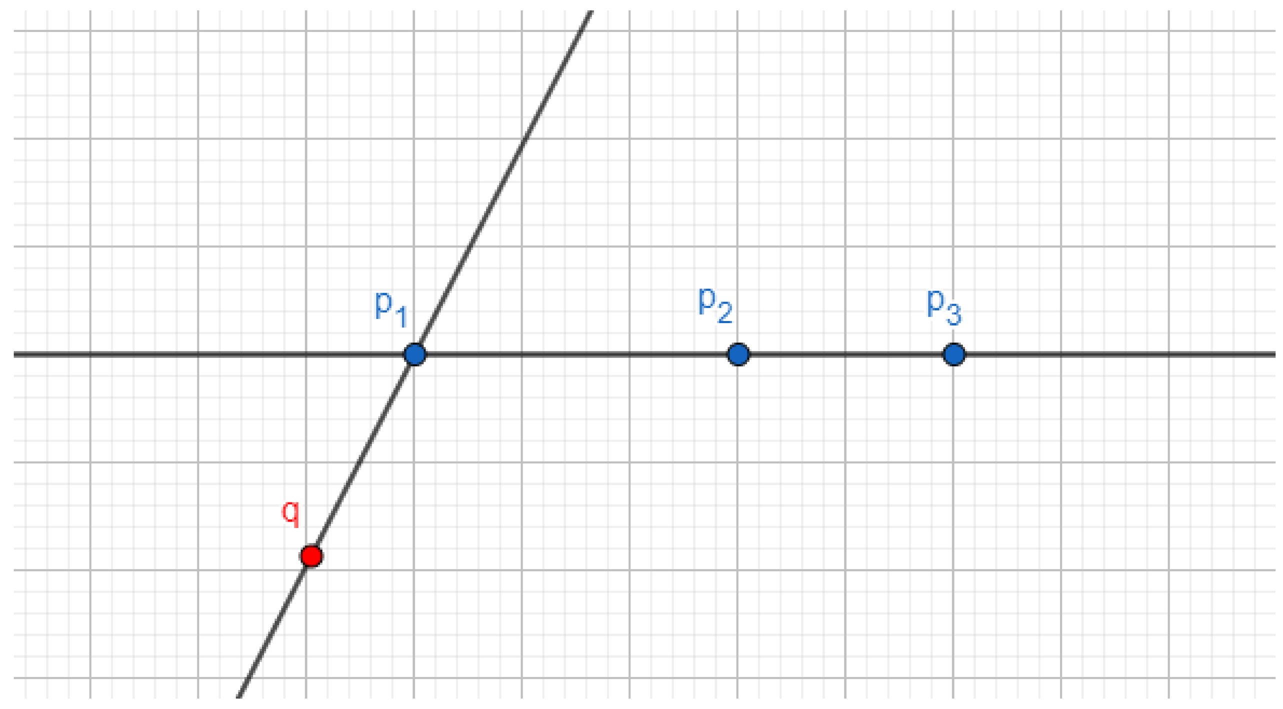

Figure 2: A class with a double line containing 3 multiple points in X

Result 3. *If a heavy (3,9)-multinet with a double line l_1 consisting of three triple points p_1, p_2 and p_3 from $\mathcal{X}$ exists, then $\mathcal{X} = \{p_1, p_2, p_3, q_1, ..., q_{54}\}$ where $n_{q_i} = 1$ for each i.*

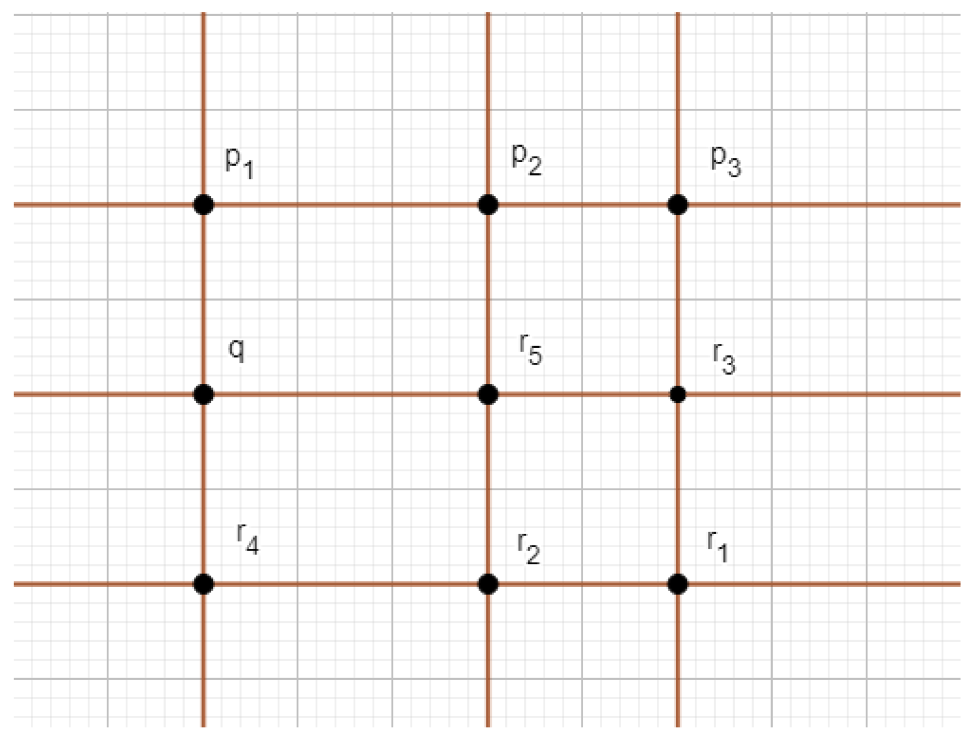

Figure 3: 3 double lines with 9 triple points in one class, *a grid*

Result 4. *If a heavy (3,9)-multinet with a triple line l_1 consisting of three triple points p_1, p_2 and p_3 from $\mathcal{X}$ exists, then $\mathcal{X} = \{p_1, p_2, p_3, q_1, ..., q_{54}\}$ where $n_{q_i} = 1$ for each i.*

Result 5. *If a heavy (3,9)-multinet with a multiple line l_1 consisting of 4 points p_1, p_2, p_3 and p_4 with $n_{p_1} = 3, n_{p_2} = n_{p_3} = n_{p_4} = 2$ from $\mathcal{X}$ exists, then $\mathcal{X} = \{p_1, p_2, p_3, p_4, q_1, ..., q_n\}$ where $n_{q_i} = 1$ or 2 for each i. Moreover, the line passing through p_1 in $\mathcal{A}_1$ has 6 simple points.*

Moreover, we have discovered the following results of a heavy $(3,2n+1)$-multinet containing a multiple line consisting of three points from $\mathcal{X}$.

Theorem 1. *A heavy $(3,2n+1)$-multinet $\mathcal{A}$ with a multiple line l_1 consisting of three points p_1, p_2 and p_3 from $\mathcal{X}$ with $n_{p_1} \ge n_{p_2} \ge n_{p_3}$ exists, then either $n_{p_1} = n_{p_2} = n_{p_3}$ or $n_{p_1} > n_{p_2} = n_{p_3} = m(l_1)$.*

Theorem 2. *A heavy $(3,2n+1)$-multinet $\mathcal{A}$ with a multiple line consisting of three points p_1, p_2 and p_3 from $\mathcal{X}$ with $n_{p_3} = 2$ does not exist for $n > 3$.*

Conclusions

- A heavy $(3,2n+1)$-multinet $\mathcal{A}$ with a multiple line l_1 consisting of three points p_1, p_2 and p_3 from $\mathcal{X}$ with $n_{p_1} \ge n_{p_2} \ge n_{p_3}$ exists, then either $n_{p_1} = n_{p_2} = n_{p_3}$ or $n_{p_1} > n_{p_2} = n_{p_3} = m(l_1)$.
- A heavy $(3,2n+1)$-multinet $\mathcal{A}$ with a multiple line consisting of three points p_1, p_2 and p_3 from $\mathcal{X}$ with $n_{p_3} = 2$ does not exist for $n > 3$.

Forthcoming Research

Now we will focus on finding more numerical restrictions on the line and point multiplicities in heavy $(3,2n+1)$-multinets and try to manipulate these restrictions to the calculation of signature of an algebraic complex surface constructed from a cyclic cover of the projective plane blown-up at the base locus X of the heavy $(3,2n+1)$-multinet as in the article [1] in order to detect whether there is a heavy $(3,2n+1)$-multinet in $\mathbb{CP}^2$ for each n.

References

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Acknowledgements

I would like to thank my advisor Ali Ulaş Özgür Kişisel for motivation, inspiration, and his helpful comments. I am indebted to him for sharing with me his knowledge and insight.I would like to thank the Department of Mathematics of Middle East Technical University for great support.