GEOMETRIC MODEL ERROR REDUCTION IN INVERSE PROBLEM OF ELECTROCARDIOGRAPHY

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ABSTRACT

GEOMETRIC MODEL ERROR REDUCTION IN INVERSE PROBLEM OF ELECTROCARDIOGRAPHY

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Electrocardiographic Imaging (ECGI) is a clinical tool for visualizing the electrical activity of the heart, detecting arrhythmias and mapping arrhythmic substrates. It has an inverse problem: obtaining electrical activity of the heart from body surface potential measurements, using a patient-specific model of the torso. This problem is an ill-posed one and does not have an exact solution.

The ill-posed nature of the inverse problem could be handled with statistical constraints on the solution. These are based on prior statistical information about the solution. Several estimation methods exist to provide this regularization. This study focuses on the Bayesian MAP estimation. In most studies, MAP only handles the measurement noise and neglects the errors caused by discrepancies in the geometric modeling of the heart and the torso.

This study, using measured electrogram data from the University of Utah, focuses on inspecting the effects of measurement noise and geometric model errors on the solution of the inverse problem. Further, some methods are proposed to compensate for these geometric model errors and improve the MAP solution. Methods are compared
with respect to numeric metrics. There are also visualizations that compare these compensations, showing results on the heart map.

Methods proposed in the thesis improve the performance of Bayesian MAP in estimating original epicardial potentials and activation times. Taking geometric model errors into account results in more accurate estimations of EP and pacing locations. This reduces the number of times the hearts of patients need to be modeled, which is an invasive procedure.

Keywords: Electrocardiographic Imaging, Inverse problem, Maximum a posteriori estimation (MAP), Geometric model errors
ÖZ

ELEKTROKARDİYOGRAFİ İÇİN TERS PROBLEMDEKİ GEOMETRİK MODEL HATALARININ AZALTILMASI

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Bu çalışma Utah Üniversitesinde kaydedilen elektrogram ölçümlerini kullanarak, elektrokardiyografide ters problemin çözümündeki ölçüm hatalarını ve geometrik model hatalarının ayrı ayrı ve birlikte oluşturdukları etkilerin incelemesine odaklanır. Da-
hası, bu geometrik model hatalarının etkilerinin aza indirgenmesi ve MAP çözümünün geliştirilmesi için metodlar sunar. Metodlar sayısal ölçürlar bazında kıyaslanır. Bu metodlar, kalp haritası üzerinde sunulan birtakım görsellerle de kıyaslanır.

Bu tezde sunular metodlar; Bayesian MAP kestirmesinin orijinal kalp potansiyellerini ve aktivasyon zamanlarını tahmin etmedeki performansını iyileştirmiştir. Geometrik model hatalarını hesaba katmak, gerçeğe daha yakın kalp potansiyeli ve öncü elektrot konumu tahminleri ile sonuçlanmıştır. Bu da klinik hastaların, müdahaleli bir prosedür olan, kalplerini modellemelerinin sayısında azalmaya yol açacaktır.

Anahtar Kelimeler: Elektrokardiyografik Görüntüleme, Ters Problem, MAP, Geometrik model hataları
To humorous minds
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# TABLE OF CONTENTS

**ABSTRACT** ................................................................. v

**ÖZ** ................................................................. vii

**ACKNOWLEDGMENTS** ................................................. x

**TABLE OF CONTENTS** ................................................. xi

**LIST OF FIGURES** ...................................................... xv

**LIST OF ABBREVIATIONS** ........................................... xxiii

**CHAPTERS**

1 **INTRODUCTION** ..................................................... 1

1.1 Motivation of the Thesis ........................................ 1

1.2 Scope of the Thesis ................................................ 2

1.3 Contribution of the Thesis ....................................... 2

1.4 Organization of the Thesis ....................................... 3

2 **BACKGROUND** ....................................................... 5

2.1 Anatomy of the Heart ............................................ 5

2.2 Action Potential Generation .................................... 7

2.3 Electrical Activity of the Heart ............................... 8

2.4 QRS Complex ...................................................... 9

2.5 12-Lead Electrocardiogram .................................... 10
APPENDICES

A. BAD LEAD REMOVAL .................................................. 83

B. METRICS FOR SHIFTING ACROSS Y AND Z AXES ............. 87
   B.1 Effect of Geometric Model Errors .............................. 87
   B.2 Compensation of Geometric Errors: Individual Epsilon ... 88
   B.3 Compensation of Geometric Errors: Categorical Epsilon .... 90
   B.4 Compensation of Geometric Errors: Unified Epsilon ....... 92
LIST OF FIGURES

FIGURES

Figure 2.1 Layers of the heart. ................................................. 6
Figure 2.2 Chambers and valves of the heart. ......................... 7
Figure 2.3 Diagram of individual subcomponents of the cardiac conduction system, cropped from the original. ............. 9
Figure 2.4 P wave, QRS complex, T wave, and their intervals. ........ 10
Figure 2.5 Chest precordial electrode placements in the 12-lead ECG. .... 11

Figure 3.1 Scaling error. Left: visualization of the torso. Right: visualization of the hearts. Black heart is the original heart model. Blue is a scaled up version with a scaling factor of 1.3, whereas red is a scaled down version with a scaling factor of 0.7. .......... 25
Figure 3.2 Shift error in the z-axis. Left: visualization of the torso. Right: visualization of the hearts. Black heart is the original heart model. Blue is shifted negative 15 mm, and red is shifted positive 15 mm, both along the z-axis. ................. 26
Figure 3.3 Rotational error around the x-axis. Left: visualization of the torso. Right: visualization of the hearts. Black heart is the original heart model. Blue is rotated negative 20 degrees, and red is rotated positive 20 degrees, both around the x-axis. ................. 26
Figure 3.4 The heart model, from the positive side of each axis. ........ 30
Figure 3.5  The torso model, from the positive side of each axis. Green dots are the electrode locations.

Figure 3.6  Example EP map and EGM. Left: an example EP map from the data-set. Right: an example EGM from the data-set. The EGM belongs to the lead that’s marked with a black dot on the map. The map belongs to the time instance that’s indicated with the black dashed line in the EGM. Green dashed line corresponds to the provided QRS beginning time, whereas red dashed line corresponds to the provided QRS end time.

Figure 4.1  Effect of SNR on tCC, tRE, sCC, and sRE for both Bayesian MAP and Tikhonov estimations.

Figure 4.2  Effect of introducing scaling errors on tCC, tRE, sCC, and sRE, for both Bayesian MAP and Tikhonov estimations.

Figure 4.3  Effect of introducing shifting errors across the x-axis on tCC, tRE, sCC, and sRE, for both Bayesian MAP and Tikhonov estimations.

Figure 4.4  Effect of introducing rotational errors around the x-axis on tCC, tRE, sCC, and sRE, for both Bayesian MAP and Tikhonov estimations.

Figure 4.5  Effect of introducing rotational errors around the y-axis on tCC, tRE, sCC, and sRE, for both Bayesian MAP and Tikhonov estimations.

Figure 4.6  Comparison of no compensation, compensation with IID, and compensation with CORR with respect to scaling errors, when the training composition scenario is the individual epsilon.

Figure 4.7  Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the x-axis, when the training composition scenario is the individual epsilon.
| Figure 4.8 | Comparison of no compensation, compensation with IID, and compensation with CORR with respect to rotational errors introduced around the x-axis, when the training composition scenario is the individual epsilon. | 45 |
| Figure 4.9 | Comparison of no compensation, compensation with IID, and compensation with CORR with respect to rotational errors introduced around the y-axis, when the training composition scenario is the individual epsilon. | 46 |
| Figure 4.10 | Comparison of no compensation, compensation with IID, and compensation with CORR with respect to scaling errors, when the training composition scenario is the categorical epsilon. | 47 |
| Figure 4.11 | Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the x-axis, when the training composition scenario is the categorical epsilon. | 48 |
| Figure 4.12 | Comparison of no compensation, compensation with IID, and compensation with CORR with respect to rotational errors introduced around the x-axis, when the training composition scenario is the categorical epsilon. | 48 |
| Figure 4.13 | Comparison of no compensation, compensation with IID, and compensation with CORR with respect to rotational errors introduced around the y-axis, when the training composition scenario is the categorical epsilon. | 49 |
| Figure 4.14 | Comparison of no compensation, compensation with IID, and compensation with CORR with respect to scaling errors, when the training composition scenario is the unified epsilon. | 50 |
Figure 4.22  Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), in terms of activation time correlation coefficients. Top left: AT CC for all error types. Top right: AT CC for only scaling errors. Bottom left: AT CC for only shifting errors. Bottom right: AT CC for only rotational errors...

Figure 4.23  Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), in terms of localization errors. Top left: LE for all error types. Top right: LE for only scaling errors. Bottom left: LE for only shifting errors. Bottom right: LE for only rotational errors...

Figure 4.24  Original and reconstructed EGMs with estimation methods of Tikhonov and Bayesian MAP with compensation methods of IID, CORR and nocomp. Left: EP map at t = 60 ms. Right: Ground truth and reconstructed EGMs. The blue dot on the map represents the lead, to which the EGMs on the right belong. On the right, the dashed vertical lines correspond to the estimated activation times. Estimated activation times of the original EGM and CORR EGM are exactly the same, so only one is shown. A moving average is applied on the reconstructions with a window of 5 ms, for easier visual inspection. The introduced error is shifting of -15 mm along the x-axis. The training composition scenario is the individual epsilon...

Figure 4.25  Maps of ground truth and reconstructed epicardial potentials, comparing no compensation with IID and CORR. The introduced error is shifting of -15 mm along the x-axis. The training composition scenario is the individual epsilon...

Figure 4.26  Maps of tCC, comparing reconstruction accuracies of no compensation with IID and CORR. The introduced error is shifting of -15 mm along the x-axis. The training composition scenario is the individual epsilon...
Figure 4.27  Maps of AT, comparing results of no compensation with IID and CORR. Red dots indicate the pacing locations (lead with smallest AT), whereas black dots show the original pacing location. The introduced error is shifting of -15 mm along the x-axis. The training composition scenario is the individual epsilon.  

Figure 4.28  Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), with stacked forward matrices, for all error types.  

Figure 4.29  Comparison of training composition scenarios for the epsilon covariance matrix (categorical, unified) and compensation methods (no comp, IID, CORR), in terms of activation time correlation coefficients, with stacked forward matrices, for all error types. 

Figure 4.30  Comparison of training composition scenarios for the epsilon covariance matrix (categorical, unified) and compensation methods (no comp, IID, CORR), in terms of localization errors, with stacked forward matrices, for all error types.  

Figure 4.31  Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), for all error types (scaling, shifting, rotational). Contrary to all other figures in this chapter, this one shows results for the experimental data-set. 

Figure A.1  Histogram of maximum potentials of all leads in the test set.  

Figure A.2  Histogram of maximum potentials of all leads in the test set, |V| < 5.  

Figure A.3  Histogram of maximum potentials of all leads in the test set, |V| < 1.
Figure A.4  Histogram of maximum potentials of provided bad leads in the test set. 85

Figure A.5  Histogram of maximum potentials of bad leads in the test set, $|V| < 1$. 86

Figure B.1  Effect of introducing shifting errors across the y-axis on $t_{CC}$, $t_{RE}$, $s_{CC}$, and $s_{RE}$, for both Bayesian MAP and Tikhonov estimations. 87

Figure B.2  Effect of introducing shifting errors across the z-axis on $t_{CC}$, $t_{RE}$, $s_{CC}$, and $s_{RE}$, for both Bayesian MAP and Tikhonov estimations. 87

Figure B.3  Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the y-axis, when the training composition scenario is the individual epsilon. 88

Figure B.4  Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the z-axis, when the training composition scenario is the individual epsilon. 89

Figure B.5  Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the y-axis, when the training composition scenario is the categorical epsilon. 90

Figure B.6  Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the z-axis, when the training composition scenario is the categorical epsilon. 91

Figure B.7  Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the y-axis, when the training composition scenario is the unified epsilon. 92
Figure B.8 Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the z-axis, when the training composition scenario is the unified epsilon.
LIST OF ABBREVIATIONS

AO          Aorta
AT          Activation Time
AV          Atrioventricular
BEM         Boundary Element Method
BSP         Body Surface Potential
BSPM        Body Surface Potential Measurements
CC          Correlation Coefficient
CG          Conjugate Gradient
CT          Computational Tomography
CORR        Independent with Correlated Leads
CVD         Cardiovascular Disease
ECG         Electrocardiogram
ECGI        Electrocardiographic Imaging
EGM         Electrogram
EP          Epicardial Potential
FEM         Finite Element Method
GMRES       Generalized Minimal Residual
IID         Independent and Identically Distributed
LV          Left Ventricular
LE          Localization Error
MAP         Maximum A Posteriori
MINRES      Minimal Residual
MRI         Magnetic Resonance Imaging
PDE         Partial Differential Equation
<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Full Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>RE</td>
<td>Relative Error</td>
</tr>
<tr>
<td>RMSE</td>
<td>Root Mean Squared Error</td>
</tr>
<tr>
<td>RV</td>
<td>Right Ventricular</td>
</tr>
<tr>
<td>SA</td>
<td>Sinoatrial</td>
</tr>
<tr>
<td>sCC</td>
<td>Spatial Correlation Coefficient</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal to Noise (Ratio)</td>
</tr>
<tr>
<td>sRE</td>
<td>Spatial Relative Error</td>
</tr>
<tr>
<td>tCC</td>
<td>Temporal Correlation Coefficient</td>
</tr>
<tr>
<td>tRE</td>
<td>Temporal Relative Error</td>
</tr>
<tr>
<td>TSVD</td>
<td>Truncated Singular Value Decomposition</td>
</tr>
<tr>
<td>UQ</td>
<td>Uncertainty Quantification</td>
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</tbody>
</table>
CHAPTER 1

INTRODUCTION

1.1 Motivation of the Thesis

Cardiovascular diseases (CVD) are the leading cause of death globally \[1, 2, 3, 4\]. This makes it crucial to detect abnormalities in the heart (i.e. arrhythmias). The de facto standard for the assessment of the heart’s electrical activity, and hence the detection of heart diseases and abnormalities is currently the 12-lead electrocardiogram (ECG) \[5\]. Although it is useful in the said tasks, the 12-lead electrocardiogram suffers from being limited in detecting the precise location of arrhythmias, lowering its clinical value in terms of sensitivity, specificity and accuracy \[6\].

Electrocardiographic imaging (ECGI) is an alternative to the 12-lead ECG. In ECGI, activation times or cardiac sources such as epicardial potentials (EP) are reconstructed from electrocardiographic body surface potentials (BSP) noninvasively \[7, 8, 9\]. The inverse problem of ECGI that solves the epicardial potentials when given the body surface potentials is ill-posed due to attenuation and smoothing of the cardiac signals within the thorax \[10, 11\]. This ill-posed nature of the inverse problem of ECGI has successfully been handled by several regularization methods, in which prior statistical knowledge about the solution is known \[12, 13, 14, 15\]. These methods include Bayesian Maximum a Priori (MAP) estimation, Tikhonov regularization, and Kalman filters, among others.

These regularization methods overcome the ill-posed nature of the inverse problem and handle the measurement noise to an extent. They have successfully been applied to the inverse problem of ECGI. However, the relevant studies, by incorporating these regularization methods, focus only on the measurement noise, while ignoring errors
caused by inaccuracies in the geometric model. To come up with a method whose solution is robust not only against measurement noises, but also against geometric model errors, there needs to be modifications in one of these methods that take into account the impact and the nature of geometric model errors. In this study, we first observe the impact of measurement noise and geometric model errors on the inverse solution of Bayesian MAP estimation and Tikhonov regularization, then we modify Bayesian MAP estimation to compensate for the forward model errors.

1.2 Scope of the Thesis

- This thesis focuses on the mathematical formulation of the inverse problem of electrocardiography, and explores various solution methods, along with observing the effect of measurement noise and geometric model errors on the estimation of the inverse problem solution.

- The methods that are investigated include Tikhonov regularization, Bayesian maximum a posteriori (MAP) estimation, and Bayesian MAP with geometric model error compensation. A method of combining matrices with geometric model errors by stacking them on top of each other is also investigated.

- The methods are applied to heart beats from two separate data-sets. First one is a simulated data-set, in which EP values are measurements, and BSP values are obtained via simulations. Second data-set is an experimental data-set, whose values of both EP and BSP are measurements.

- Using temporal and spatial cross correlation (CC) and relative error (RE), activation times (AT), and by visualizing the solutions, the different solution approaches are compared and evaluated for their ability to handle measurement noise and geometric errors.

1.3 Contribution of the Thesis

Part of the work in this thesis deals with incorporating model uncertainty to Bayesian MAP estimation to improve its performance in reconstructing EP from BSP. Using
geometric errors of heart position translations and heart size scaling, the noise correlation matrix for Bayesian MAP estimation is modified. There are two distinct types of modifications done. First one (IID) adds a diagonal square matrix to the noise correlation matrix, whereas the second one (CORR) adds a non-diagonal square matrix to the noise correlation matrix. These square matrices are themselves correlation matrices of a new noise term that represents the geometric model error. These two methods are compared with regular Bayesian MAP estimation with no compensation for model errors, in terms of spatial and temporal correlation coefficients (CC). It is observed that even in a simple form such as the IID, reconstructions of MAP improve when compared with the case of no compensation. Additionally, these compensations (IID & CORR) also improve the estimations of activation times (AT) and localization errors (LE).

This particular body of work has resulted in the following publication:


1.4 Organization of the Thesis

This thesis is divided into six chapters:

- Chapter 1 (this chapter) introduces the motivation, scope, and organization of the thesis.

- Chapter 2 touches on the literature review on the forward and inverse problems of electrocardiography, along with the electrical nature of the heart and its anatomy.

- Chapter 3 focuses on the various methods used for solving the inverse problem of electrocardiography and compensating for the geometric model errors. It also identifies the data-sets used in the work involved in this thesis.
• Chapter 4 is where the results of the methods are evaluated and compared, accompanied by discussion of the results.

• Chapter 5 concludes the thesis and lays ground for future work.
CHAPTER 2

BACKGROUND

In this chapter, the anatomy of the heart, action potential generation, and the electrical activity of the heart are explained. Then backgrounds on the 12-lead electrogram (ECG) and electrocardiographic imaging (ECGI) are introduced. Later, the forward problem of ECGI and the place of finite/boundary element methods in the forward problem are laid out before moving on to some background on the inverse problem. Lastly, geometric models of the heart and the torso and potential errors that can arise from those models are mentioned.

2.1 Anatomy of the Heart

The heart is a cone-shaped, muscular organ that plays a crucial role in the circulatory system [16, 17]. It is responsible for pumping blood throughout the body (hence keeping its host alive), supplying oxygen and nutrients to tissues and organs, and removing waste products. It is located in the thoracic cavity between the lungs, and it is protected by the rib cage. On average, an adult’s heart weighs 250 to 300 grams [18].

Figure 2.1 shows that the heart is separated from its surrounding tissues by a layer called the pericardium, inside which resides the heart wall, which consists of the following layers of its own [20]:

- **Epicardium** is the outer layer.
- **Myocardium** is the middle layer that makes up the bulk of the heart [21]. Myocardium is responsible for the heart’s contraction.
Figure 2.1: Layers of the heart [19].

- **Endocardium** is the inner layer that is responsible for the protection of the heart’s chambers and valves.

The heart consists of four main chambers, which work in the following order as they perform their duties in the cardiac cycle:

- **Right atrium** receives deoxygenated blood from the body and sends it to the right ventricle.
- **Right ventricle** pumps deoxygenated blood from the right atrium to the lungs.
- **Left atrium** receives oxygenated blood from the lungs and sends it to the left ventricle.
- **Left ventricle** pumps oxygenated blood from the left atrium to the body.

The right and left atria are the upper, thin-walled chambers of the heart, whereas ventricles are the lower, thick-walled chambers.

Ensuring the correct direction of blood flow in the heart is the responsibility of four valves, shown in Figure 2.2:

- **Tricuspid valve** handles blood flow from the right atrium to the right ventricle [22].
Mitral valve handles blood flow from the left atrium to the left ventricle [23].

Pulmonary valve ensures no blood is pumped back from the lungs to the left ventricle.

Aortic valve ensures no blood is pumped back from the body to the left ventricle [24].

Tricuspid and mitral valves are also called atrioventricular valves [25], since they are between an atrium and a ventricle. Pulmonary and aortic valves are semilunar valves [26].

2.2 Action Potential Generation

An action potential is a rapid electrical signal that is generated and conducted by the cells of the heart. It is a transient change in the electrical charge of a cell, which is initiated by the opening of voltage-gated ion channels [27].
The action potential has a characteristic shape that is defined by several phases, including the rising phase (depolarization), the peak (plateau), and the falling phase (repolarization). The shape of the action potential is determined by the balance between the movement of charged ions into and out of the cell, which creates a change in the electrical charge of the cell. This process is essential for the proper functioning of the heart.

2.3 Electrical Activity of the Heart

The electrical activity of the heart is responsible for coordinating the contraction and relaxation of its muscles, leading to the pumping of blood throughout the body. The heart’s electrical activity originates from a group of specialized cells known as the sinoatrial (SA) node [28, 29], located in the right atrium of the heart [30]. It can be seen in Figure 2.3 along with other components of the cardiac conduction system.

The SA node generates an electrical signal, which spreads through the atria, causing them to contract and pump blood into the ventricles. The electrical signal then passes through the atrioventricular (AV) node [31, 32], which acts as a gateway between the atria and the ventricles, slowing down the signal to allow for complete filling of the ventricles before they contract.

After passing through the AV node, the electrical signal travels down the bundle of His, a pathway of specialized conducting fibers that splits into two branches - the left and right bundle branches. These branches conduct the electrical signal to the ventricles, causing them to contract in a coordinated manner, starting from the bottom of the heart and moving upwards.

The electrical signal causes the muscle fibers of the ventricles to contract, leading to the ejection of blood from the heart into the circulatory system. The entire process of electrical conduction and contraction is repeated with each heartbeat, allowing the heart to function as an efficient pump.
2.4 QRS Complex

The QRS complex \cite{34} (see Figure 2.4) refers to the three distinct waves in an electrocardiogram (ECG) trace that correspond to the electrical activity of the heart during ventricular depolarization and contraction. The QRS complex is composed of three waves:

- **Q wave**: The first wave in the QRS complex, which is a small upward deflection in the ECG trace that represents the electrical depolarization of the interventricular septum.

- **R wave**: The tallest wave in the QRS complex, which is a large upward deflection in the ECG trace that represents the rapid depolarization of the ventricles.

- **S wave**: The final wave in the QRS complex, which is a downward deflection in the ECG trace that represents the slower repolarization of the ventricles.
2.5 12-Lead Electrocardiogram

Clinical applications of the electrocardiogram (ECG) can be traced back to the beginning of the 20th century when Willem Einthoven recorded an ECG in a clinically reproducible fashion, using a string galvanometer [36]. Einthoven’s three limb leads, combined with six chest leads and three augmented limb leads—introduced by Emanuel Goldberger—laid the grounds for the 12-lead electrogram later in the century. In the early days, the ECG was recorded using a complicated and time-consuming process involving multiple electrodes placed on the body. The introduction of the 12-lead ECG simplified the process by using only ten electrodes, which were placed on standardized locations on the body. Figure 2.5 shows the locations for the 6 precordial electrodes. The remaining 4 electrodes are on the wrists and ankles (one on each).

As a noninvasive clinical tool, the 12-lead ECG has successfully been used to capture the electrical activity of the heart and detect cardiovascular diseases. Nevertheless,
Figure 2.5: Chest precordial placements in the 12-lead ECG [37].

- V1 - Fourth intercostal space at the right border of the sternum
- V2 - Fourth intercostal space at the left border of the sternum
- V3 - Midway between placement of V2 and V4
- V4 - Fifth intercostal space at the midclavicular line
- V5 - Anterior axillary line on the same horizontal level as V4
- V6 - Mid-axillary line on the same horizontal level as V4 and V5
due to its limited capabilities, it can fall short of accurately detecting some abnormalities. Some examples for these are myocardial infarction (heart attack), transient myocardial ischemia, Wolff-Parkinson-White syndrome, and ventricular ectopy [38]. Moreover, the 12-lead ECG also fails to provide precise information on regional electrical activity in the heart, or the sequences of activation during arrhythmias [39].

2.6 Electrocardiographic Imaging

Electrocardiographic imaging (ECGI) is another noninvasive technique to measure and visualize the electrical activity of the heart. The technique involves recording electrical signals (potentials) from multiple electrodes placed on the surface of the body. The recorded signals (body surface potential measurements, or BSPM [40]) are used to create a map of the heart’s electrical activity, which can provide important information for the diagnosis and treatment of various cardiovascular diseases.

By visualizing the electrical activity of the heart, ECGI can help to identify the source of an arrhythmia and determine the most appropriate treatment. Additionally, ECGI can be used to assess the effectiveness of treatments for arrhythmias, such as ablation procedures, and to monitor patients with heart disease over time.

The spatial resolution is higher in ECGI than it is in the 12-lead ECG. This enables health practitioners to observe the electrical activity of the heart in greater detail. Consequently, ECGI is more able to help detect subtler abnormalities in the heart of a patient. Capabilities of ECGI in arrhythmia detection, cardiac modeling, risk stratification, and myocardial infarction (heart attack) assessment acquire the focus of researchers.

ECGI has a forward problem and an inverse problem. The forward problem of ECGI involves the simulation, or calculation, of the body surface potentials, given a distribution of cardiac electrical activity. The inverse problem, on the other hand, involves estimating the distribution of cardiac electrical activity on the heart, based on the recorded body surface potentials.
2.7 Forward Problem

The forward problem of ECGI is the computation of body surface potentials (BSP) from the epicardial potentials (EP). Assuming no active bioelectric sources between the heart surface and the body surface lets us take advantage of the following Laplace’s equation when formulating the forward problem [41, 42]:

$$\nabla \cdot \sigma \nabla \phi(p) = 0 \quad \forall p \in V,$$

(2.1)

where $\phi(p)$ is the potential at coordinate $p$, and $\sigma$ is the conductivity of $V$, which is the volume (region) between the heart and the body surfaces.

When solving Equation 2.1, Neumann boundary condition is assumed [43], as follows:

$$\nabla \phi(p) \cdot \hat{a} = 0 \quad \forall p \in S_B,$$

(2.2)

where $\hat{a}$ is the unit vector pointing outwards from $S_B$, the body surface.

Along with Neumann boundary condition, Dirichlet boundary condition is also assumed. It is given by the following equation:

$$\phi = \phi_H \quad \forall p \in S_H,$$

(2.3)

where $\phi_H$ is the source voltage distribution on the heart and $S_H$ is the heart surface.

Finite Element Method (FEM) and Boundary Element Method (BEM) are computational techniques used to solve these equations [44, 45, 46]. These methods are used to model the electrical activity of the heart and the body’s electrical conductive system, which is essential for understanding the origin and progression of cardiovascular diseases, such as arrhythmias.

Finite Element Method is a numerical method for solving partial differential equations (PDEs) that divides a volume into smaller volumes [47, 48, 49]. The PDEs are
then solved in each volume, and the solutions are combined to find the solution for the entire volume. FEM is particularly well-suited for solving problems with complex geometries or boundary conditions, and it can be easily adapted to handle multi-physics problems, such as the interaction between the electrical and mechanical systems of the heart.

Boundary Element Method is a surface mesh based, numerical method for solving PDEs that is well-suited for problems with boundary conditions, such as the electrical potentials recorded on the surface of the body in BSPM [50, 51, 52, 53]. BEM models the electrical potentials as a boundary condition on the surface of a volume, rather than solving the PDEs in the entire volume. This makes BEM computationally efficient, especially for problems with complex geometries, and it can be easily combined with other numerical methods to solve multi-physics problems.

With the given Laplace’s equation and the boundary conditions (Equations 2.1, 2.2, and 2.3), and using BEM/FEM to compute the forward transfer matrix $A$, forward problem of ECGI takes the following form:

$$ y = Ax, $$

where $y \in \mathbb{R}^M$ and $x \in \mathbb{R}^N$ are the body surface potentials (BSP) and electrode potentials (EP), respectively, and $A \in \mathbb{R}^{M \times N}$ is the forward transfer matrix, which reflects the properties of the true torso-heart geometry.

### 2.8 Inverse Problem

In the field of cardiology, the inverse problem involves the estimation of the epicardial potentials (EP), using body surface potential (BSP) measurements and a patient-specific mathematical model of the torso. This is a challenging task because the measurement of the body surface potentials is subject to noise and other sources of uncertainty. Due to smoothing and attenuation of the cardiac signals within the torso, this inverse problem is an ill-posed problem, and regularization is needed to obtain an accurate solution.
The inverse problem of ECGI is given by the following equation:

\[ y = Ax + n, \]  

(2.5)

where \( y \in \mathbb{R}^M \) and \( x \in \mathbb{R}^N \) are the BSP measurements and EGM, respectively, \( n \in \mathbb{R}^M \) is the measurement noise, and \( A \in \mathbb{R}^{M \times N} \) is the forward transfer matrix. Before moving forward with some regularization methods that handle the ill-posedness of this inverse problem, a description for well-posedness of a problem is due.

A well-posed problem \([54]\) can be summarized with the following criteria:

- A well-posed problem has a solution.
- Its solution is unique.
- With respect to the initial conditions, the solution varies continuously.

Contrary to Equation (2.4) of the forward problem, here in the inverse problem the known variables are \( y \) (BSP) and \( A \) (the forward matrix). While compensating for the measurement noise \( n \), the goal is to find a solution for the electrode potentials (EP), \( x \). Generally, \( A \) is not an invertible matrix. Even after compensating for the measurement noise, the inverse problem of ECGI, like most other inverse problems being researched, is ill-posed. To make sure small changes in the input data do not impact the solution greatly, a smoothing method must be applied on the basis of some assumptions on the solution. These methods are called regularization methods.

Tikhonov regularization \([55, 56, 57]\), Bayesian Maximum A Posteriori (MAP) estimation \([13, 14]\), Kalman filter \([15, 58, 59]\), truncated singular value decomposition (TSVD) \([60]\), conjugate gradient (CG) method \([61]\), minimal residual method (MIN-RES) \([62]\), and generalized minimal residual method (GMRES) \([63]\) are methods that can be used to address the ill-posed nature of this inverse problem and estimate the epicardial potentials in a stable and reliable manner.

Tikhonov regularization adds a penalty term to the optimization problem to discourage the optimization algorithm from fitting the observed data too closely, which can lead to overfitting and poor generalization performance.
Bayesian MAP estimation uses Bayes’ theorem \([64, 65]\) to calculate the maximum a posteriori probability of the unknown variable given the observed data and prior information. This approach allows the integration of prior information into the estimation process, which can help to stabilize the solution.

Kalman filter is a recursive method that is used to estimate the state of a system over time based on measurements and a model of the system dynamics. In the context of the inverse problem of ECGI, Kalman filter can be used to estimate the epicardial potentials by using a combination of measurements and a model of the underlying system.

TSVD aims to reduce the dimensionality of the problem by focusing on the most significant singular values and corresponding singular vectors of a matrix. This way, contributions of the smaller singular values are eliminated and a smoother solution is obtained.

CG is an iterative method that forces constraints on amplitude and first and second order spatial gradients. It is particularly advantageous when used on sparse systems. MINRES and GMRES are used when solving the inverse equation in more general cases. Specifically, CG deals with symmetric positive-definite matrices; MINRES deals with symmetric matrices; and GMRES can be used on non-symmetric matrices.

By using various models and optimization techniques, these methods can provide different approaches to finding a stable and reliable solution to the inverse problem, even in the presence of measurement noise and other sources of uncertainty.

All of these methods are successful, to some extent, in smoothing the inverse problem’s solution. They all have their advantages and disadvantages. Tikhonov regularization, while providing a stable solution that does not depend on prior information, can result in oversmoothing. Bayesian MAP estimation effectively takes advantage of prior information of EP and BSP values to come up with an accurate solution, whereas the need for a training phase to calculate covariance matrices can become computationally heavy. Kalman filter, due to its iterative nature, adapts well to changing conditions, even though that makes it sensitive to model errors. TSVD reduces the dimensionality of the problem, hence reducing computational costs, all the while potentially amplify-
CG is known for its rapid convergence; however, its performance is highly dependent on problem characteristics. MINRES is a low memory solution, but it requires a symmetric matrix; on the other hand, GMRES can be used on non-symmetric matrices, whereas its memory demand is high.

This thesis, while providing results of Tikhonov regularization for comparison purposes, focuses on Bayesian MAP estimation and how it can be modified to better accommodate geometric model errors. It is a well proven, accurate method, and its computation and time demands during its training phase are not blockers for this study. Additionally, the data-sets that are used in the study can be properly divided into training and test sets.

2.9 Geometric Model/Errors

As detailed in Section 2.7, the forward transfer matrix in ECGI is usually obtained from the heart and torso geometries, along with the conductivities of the organs inside the torso. To acquire these geometric models, medical imaging methods like magnetic resonance imaging (MRI) or computational tomography (CT) are used. The resulting geometric models can sometimes have errors caused by the following reasons: segmentation errors of the said medical images; discrepancies in the torso tissue conductivities; changes in the heart size, location, and rotation due to contraction and relaxation of the heart during measurements. Consequently, these geometric model errors increase the errors of solutions of the inverse problem of ECGI.

Studies have been done to investigate the effects of geometric model errors on the solutions of the inverse problem [66, 67, 68, 69, 70, 71, 72]. Some focus on the effects of torso inhomogeneities [69, 70]; some inspect using a tailored geometry [71]; some compare the effects of signal error, material property error, and geometric errors [72]. Most studies in the literature that inspect geometric model errors do not attempt to modify their inverse problem solving methods to accommodate for these errors.

A study that does accommodate for geometric model errors is by Jiang et al. [73]. They incorporate heart’s motion in their inverse problem solution, which represents
introducing shifting errors to the heart’s model. They simulate this motion in the T period of the activation potential and compare this compensation for zero order Tikhonov regularization and GMRES. Their study concludes that Tikhonov regularization’s estimations become more correlated with the ground truth, when compensation for the cardiac motion is included in the solution approach, whereas GMRES does not show much improvement.

Another study is done by Aydin et al. [15], attempting to compensate for geometric model errors with a modified version of the Kalman filter. They focus on scaling and shifting errors in the heart model. The study shows that introducing geometric model errors to the problem without compensating for them reduces the performance of the Kalman filter. Nevertheless, when the Kalman filter is modified to handle these errors, estimations via the Kalman filter improve in terms of relative difference measurement star (RDMS) and correlation coefficient (CC).

The compensation methods proposed in this thesis are similar to the method proposed in Aydin et al.’s study. Like in the mentioned study, modifications for compensating for geometric model errors include coming up with a Gaussian noise model for geometric model errors and adding that to the Gaussian noise model for the measurement noise. Following are the differences (contributions of this thesis).

- Multiple variations of the noise model for geometric errors are proposed.
- Introduced errors are not only in the forms of scaling and shifting, but also rotational.
- Instead of the Kalman filter, Bayesian MAP estimation is modified.
- Along with CC, following metrics are used to compare methods: relative error (RE), activation times (AT), localization error (LE). CC and RE are given in two variations: temporal and spatial.
- Methods are applied to two data-sets, one simulated and one experimental. Aydin et al. use only a simulated data-set.

Another noteworthy study that compensates for geometric model errors is carried out by Bergquist et al. [74]. It incorporates uncertainty quantification (UQ) to handle
changing heart positions (again, this represents shifting errors). Bergquist et al. show that as the shifting error is increased beyond a magnitude of 10 mm, the performance of second order Tikhonov regularization degrades. Nonetheless, accommodating for these errors by using UQ, the authors have successfully achieved reasonable solutions for the inverse problem of ECGI, in terms of root mean squared error (RMSE), spatial CC, and temporal CC, even at large shifting errors of 40 mm. The method proposed in the mentioned work is studied in this thesis, both with and without further compensation with our proposed methods.
CHAPTER 3

METHODS

In this chapter, Tikhonov regularization and Bayesian MAP estimation, two of the regularization methods for the inverse problem of ECGI, are explained. Then the methods that are used to generate/simulate geometric model errors are given. Afterwards, some modifications on Bayesian MAP estimation are introduced. The result of how these modifications help with the compensation of geometric model errors resides in the next chapter. This chapter is concluded with the metrics that are used to evaluate our findings.

3.1 Tikhonov Regularization

Tikhonov regularization is a widely used method for solving inverse problems in general, including the inverse problem of electrocardiography. The basic idea of Tikhonov regularization is to add a regularization term to the objective function that is minimized in order to find the solution. This regularization term is a measure of the smoothness of the estimated solution, which encourages solutions that are smooth and continuous, and discourages solutions that are highly oscillatory or noisy.

The Tikhonov regularization method seeks to minimize the following objective function:

\[ F(x) = ||Ax - y||^2 + \lambda||Lx||^2, \]  

(3.1)

where \( x \) is the vector of unknowns (i.e., the electrical activity of the heart), \( A \) is the forward matrix that maps the unknowns to the measurements, \( y \) is the vector...
of measured data, \( L \) is a regularization matrix that measures the smoothness of the estimated solution, and \( \lambda \) is the regularization parameter that controls the trade-off between fitting the data and promoting smoothness in the solution. The first term \( \|Ax - y\|^2 \) measures the misfit between the predicted data \( (Ax) \) and the measured data \( (y) \), and the second term \( \lambda\|Lx\|^2 \) measures the smoothness of the solution. The norm \( \|.\|^2 \) represents the squared Euclidean norm.

The regularization matrix \( L \) is typically chosen to be an operator that measures the local variation of the solution. For example, the Laplacian operator can be used to measure the second-order spatial derivatives of the solution, which corresponds to the local curvature of the electrical activity. In this thesis, zero-order Tikhonov is used, which corresponds to \( L \) being the identity matrix: \( L = I \). The regularization parameter \( \lambda \) determines the amount of smoothing in the solution. A larger value of \( \lambda \) promotes smoother solutions, while a smaller value of \( \lambda \) allows more oscillatory or noisy solutions.

### 3.2 Bayesian MAP Estimation

Bayesian MAP estimation maximizes the following posterior pdf of \( x \), which can be expressed in terms of the likelihood function \( p(y|x) \) and an a priori pdf \( p(x) \) of EGMs:

\[
p(x|y) = \frac{p(y|x)p(x)}{p(y)}
\]

(3.2)

If we assume that \( x \) and \( y \) are jointly Gaussian, \( x \sim N(\bar{x}, C_x) \), \( n \sim N(0, C_n) \), then the MAP estimate of \( x \), that solves Equation 2.5, becomes:

\[
\hat{x} = (A^T C_n^{-1} A + C_x^{-1})^{-1} (A^T C_n^{-1} y + C_x^{-1} \bar{x})
\]

(3.3)

To compute \( C_x \), all training beats are merged together. Each training beat has shape \( M \times T_i \) where \( M \) is the number of electrodes, and \( T_i \) is the duration of the beat’s QRS segment. Merging all \( N \) training beats along the time axis results in \( X_{tr} \), with shape \( M \times T \), where
\[ T = \sum_{i=1}^{N} T_i. \]  

\[ C_x \] can then be computed as follows:

\[ C_x = \frac{(X_{tr} - X_{mean})(X_{tr} - X_{mean})^T}{T}, \]  

where \( X_{mean} \) has repeating columns, each of which is the mean of \( X_{tr} \) along the time axis.

### 3.3 Geometric Model Error Compensation

The work explained in this section is published in Computing in Cardiology 2021 [10], where it is applied only on scaling and shifting errors, leaving out rotational errors.

The original relationship between the EP and BSP in ECGI is given in Equation 2.5. The geometric model errors that we aim to handle are in the forward matrix, \( A \). Taking this error into account, the relationship can be rewritten by the following equation:

\[ y = A x + n = (\tilde{A} + A_{err}) x + n, \]  

where \( \tilde{A} \) is the forward matrix corresponding to the incorrect geometry, and \( A_{err} \) is its deviation from the true matrix \( A \). Eqn. [3.6] can be modified to include a new geometric error term \( \epsilon = A_{err} x \) as:

\[ y = \tilde{A} x + \epsilon + n, \]  

which allows the introduction of a more general noise term:

\[ \tilde{n} = \epsilon + n. \]
Note that $\epsilon$ is correlated with the EPs, $x$. In this study, nonetheless, we ignore this correlation for simplicity of the model. We assume that $\epsilon$ and $n$ are uncorrelated with $x$ and with each other. The measurement noise is taken to be independent and identically distributed (iid) with $n \sim N(0, \sigma_n^2 I)$ and $\epsilon \sim N(0, C_\epsilon)$, resulting in $C_n = C_\epsilon + \sigma_n^2 I$.

For each geometric model error, $C_\epsilon$ is estimated using error training data:

$$E_{tr} = A_{err} X_{tr},$$  \hspace{1cm} (3.9)\)

where $A_{err}$ is the error in the forward model and $X_{tr}$ is the entire training data. Each of these $E_{tr}$ matrices is then used to define two different $C_\epsilon$ estimates:

1. **IID**: $\epsilon$ is independent and identically distributed.

   $$C_\epsilon = \sigma_\epsilon^2 I,$$  \hspace{1cm} (3.10)\)

   where scalar $\sigma_\epsilon^2$ is the variance of $E_{tr}$.

2. **CORR**: there is correlation across leads; $C_\epsilon$ is a non-diagonal matrix.

   $$C_\epsilon = \frac{E_{tr} E_{tr}^T}{N_\epsilon},$$  \hspace{1cm} (3.11)\)

   where $N_\epsilon$ is the number of columns of $E_{tr}$.

Additionally, we solve the inverse problem without compensating for these geometric errors (No comp) for comparison. For this case, $\epsilon$ is assumed to be zero, and hence $C_\epsilon = 0$.

We introduce the geometric error as shift, scaling, and rotational errors in the heart geometry [15]. The heart position is shifted by -15, -10, -5, 5, 10, and 15 mm along each axis (separately). For the scaling scenario, the size of the heart is modified by a scaling factor of 0.7, 0.8, 0.9, 1.1, 1.2, and 1.3. Rotations introduced are -20, -10, 10, and 20 degrees around the x-axis, as well as -20 and -10 degrees around the y-axis. 0 degree rotation, 0 mm shift and 1.0 scaling factor correspond to no geometric error.
Figure 3.1: Scaling error. Left: visualization of the torso. Right: visualization of the hearts. Black heart is the original heart model. Blue is a scaled up version with a scaling factor of 1.3, whereas red is a scaled down version with a scaling factor of 0.7.

For each geometric error scenario, a new forward matrix is computed using the BEM approach for a homogeneous torso, which is used to solve the inverse problem for the test beat.

Figures 3.1, 3.2, and 3.3 are displays of several modifications that are done on the heart model to introduce geometric model errors. In each figure, to the left is the model of the torso with the hearts, and to the right is the original heart model (black), along with the modified heart models (blue and red).

There are three different scenarios in which $C_e$ is constructed. The following list describes each scenario.

- **Individual Epsilon Covariance Matrix:** All geometric errors have their own sets of $C_e$ matrices.

  For example, for compensation of the geometric model error of scaling with a scaling factor of 0.7, $C_e$ is constructed only with the errored forward matrix that corresponds to a scaling factor of 0.7. Likewise for shifting and rotational errors, $C_e$ is constructed only with a single errored forward matrix:
Figure 3.2: Shift error in the z-axis. Left: visualization of the torso. Right: visualization of the hearts. Black heart is the original heart model. Blue is shifted negative 15 mm, and red is shifted positive 15 mm, both along the z-axis.

Figure 3.3: Rotational error around the x-axis. Left: visualization of the torso. Right: visualization of the hearts. Black heart is the original heart model. Blue is rotated negative 20 degrees, and red is rotated positive 20 degrees, both around the x-axis.
\[ A_{err} = A - \bar{A}. \] (3.12)

Note that this \( A_{err} \) is the matrix used to compute \( E_t \) (See Eqn. (3.9)).

- **Categorical Epsilon Covariance Matrix:** All geometric error types have their own sets of \( C_\epsilon \) matrices.

For instance, for compensation of the geometric model error of scaling with a scaling factor of 0.7, \( C_\epsilon \) is constructed with all errored forward matrices that correspond to all scaling errors, even 1.0 which corresponds to no error. Likewise, during construction of \( C_\epsilon \) for shifting and rotational errors, all errors from the respective error type (scaling, shifting, rotational) are used. Following equation displays how \( A_{err} \) for scaling errors is constructed in this epsilon scenario:

\[
A_{err,\text{scaling}} = 
\begin{bmatrix}
A_{err,scl,0.7} \\
A_{err,scl,0.8} \\
A_{err,scl,0.9} \\
A_{err,scl,1.0} \\
A_{err,scl,1.1} \\
A_{err,scl,1.2} \\
A_{err,scl,1.3}
\end{bmatrix}, \quad A_{err,scl,1.0} = 0.
\] (3.13)

Construction of \( A_{err} \) is similar for shifting and rotational errors. In \( A_{err,\text{shifting}} \), \( A_{err} \) for all shifting errors along all axes are stacked, as in Eqn. (3.13). Likewise for \( A_{err,\text{rotational}} \), all rotational errors are included, regardless of the axis around which the rotation is applied. This scenario results in three sets of \( C_\epsilon \) matrices, for scaling, shifting, and rotational errors.

- **Unified Epsilon Covariance Matrix:** There is only one set of \( C_\epsilon \) matrices.

In this scenario, \( C_\epsilon \) is constructed with all errors. Regardless of the error type or error size, compensation uses this \( C_\epsilon \). Following equation shows how \( A_{err} \) is constructed:
\[ A_{err} = \begin{bmatrix} A_{err,scaling} \\ A_{err,shifting} \\ A_{err,rotational} \end{bmatrix}, \quad (3.14) \]

where \( A_{err,scaling} \), \( A_{err,shifting} \), and \( A_{err,rotational} \) are the \( A_{err} \) matrices from the categorical epsilon scenario. Repetitions of \( A_{err,scl,1.0} \), \( A_{err,shft,0.0} \), and \( A_{err,rot,0.0} \) are removed. The resulting \( A_{err} \) matrix is a result of stacking 31 error matrices: 6 for scaling errors, 18 for shifting errors, 6 for rotational errors, and a no error matrix.

Note that a set refers to three \( C_\epsilon \) matrices: one for IID, CORR, and No comp.

### 3.3.1 Stacked Error Matrices

Bergquist et al. \[74\] propose a compensation method for Tikhonov regularization, where instead of the regular forward \( A \) matrix, they use \( \tilde{A}_{stack} \) which consists of \( n \) amounts of errored \( A \) matrices (\( \tilde{A}_i \)) stacked on top of each other:

\[ \tilde{A}_{stack} = \begin{bmatrix} \tilde{A}_1 \\ \tilde{A}_2 \\ \cdots \\ \tilde{A}_n \end{bmatrix}. \quad (3.15) \]

For consistency in size, they also use a modified version of \( y \) in their inverse computations:

\[ y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdots \\ y_n \end{bmatrix}, \quad (3.16) \]
where \( y_i = y \) for all \( i \in [1, n] \).

To observe how well this method accommodates for geometric model errors on its own, it is applied with no additional compensation (No comp, \( C_e = 0 \)) at first. For Tikhonov regularization and Bayesian MAP estimation computations, \( y \) and the forward matrix \( A \) are replaced with their respective stacked counterparts.

Additionally, we modify this approach to use it as an additional Bayesian MAP compensation method like so: in epsilon covariance matrix computations, we use \( \tilde{A}_{stack} \). Note that \( \tilde{A}_{stack} \) does not include \( A \) without errors. This is done for both IID and CORR, for both categorical and unified epsilons. Individual epsilon scenario is forgone, as it deals with single errors: stacking is not appropriate in that case.

3.4 Data

This section details the data-sets used in the thesis. There are two data-sets. Simulated data only have EP measurements, so BSP values need to be simulated. Experimental data have measurements of both EP and BSP.

3.4.1 Simulated Data

The simulated data used in this study are from 7 different experiments carried out at the University of Utah, Nora Eccles Harrison Cardiovascular Research and Training Institute (CVRTI) [75]. In these experiments, an excised dog heart, perfused by a support dog’s circulatory system, was suspended in an adolescent torso-shaped tank. The heart was ventricularly paced from the epicardial surface, and EPs were recorded via 490-lead sock electrodes (1000 Hz sampling rate). BSPs on the body surface were simulated by multiplying the EPs on the heart surface by a forward matrix. This forward matrix was computed by using the boundary element method (BEM) with lungs included in the geometric model. The fine torso geometry consisted of 771 nodes, whereas BSPS are simulated onto 192 leads on the surface of the body.

Figure 3.4 shows the heart model in the data-set, seen from the positive directions of x, y, and z axes. Figure 3.5 shows the torso model in the data-set, seen from the
Figure 3.4: The heart model, from the positive side of each axis.

Figure 3.5: The torso model, from the positive side of each axis. Green dots are the electrode locations.
Figure 3.6: Example EP map and EGM. Left: an example EP map from the data-set. Right: an example EGM from the data-set. The EGM belongs to the lead that’s marked with a black dot on the map. The map belongs to the time instance that’s indicated with the black dashed line in the EGM. Green dashed line corresponds to the provided QRS beginning time, whereas red dashed line corresponds to the provided QRS end time.

positive directions of x, y, and z axes. Figure 3.6 displays an example EP map and an example beat with vertical lines on the QRS beginning and end times that are provided in the data-set.

There are a total of 326 beats in the data-set, from the 7 experiments; we used 309 of these (3 experiments) in our training set and 17 of these (4 experiments) in our test set. None of the experiments in the training set and the test set overlap.

The test data-set helpfully includes the information of what leads are made up of unusable values (bad leads). Bad leads are the leads with inaccurate recordings. APPENDIX A includes information on how we went about further removal of bad leads. In the rest of this thesis, all analysis is done after forgoing these bad leads.

3.4.2 Experimental Data

To evaluate the methods on a data-set in which both EP and BSP are measured values as opposed to simulated, the data-set named Ischemia torso tank with sock and

31
needles (Utah-10-03-02) from EDGAR (Experimental data and geometric analysis repository) [76] Time Signal Catalog is used.

The experimental setup and the method of perfusion are similar with the simulated data. 48 transmural plunge needles, 247 sock electrodes, and 192 torso tank electrodes are used to acquire the data. The sampling rate is 1000 Hz. The data-set is made up of eight separate interventions with approximate durations of 13 minutes each.

In this thesis, four of these interventions are used:

- **Intervention 1:** Intervention is done with the supply ischemia protocol. Pacing rate is 350 ms, while blood flow rate goes from 20 ml/min to 5 ml/min. This intervention consists of 44 beats.

- **Intervention 2:** Intervention is done with the demand ischemia protocol. Blood flow rate is 35 ml/min, while pacing rate goes from 350 ms to 275 ms. This intervention consists of 44 beats.

- **Intervention 3:** Intervention is done with the supply ischemia protocol. Pacing rate is 325 ms, while blood flow rate goes from 20 ml/min to 5 ml/min. This intervention consists of 44 beats.

- **Intervention 4:** Intervention is done with the demand ischemia protocol. Blood flow rate is 25 ml/min, while pacing rate goes from 350 ms to 275 ms. This intervention consists of 39 beats.

During evaluations of the methods proposed in this thesis, data from Intervention 1, Intervention 3, and Intervention 4 are used in the training set, while data from Intervention 2 are used in the test set. This results in 127 training beats and 44 test beats.

### 3.4.3 Noise Modeling

In section [3.3] the noise in the inverse problem is described as follows:
\[ n \sim N(0, \sigma_n^2 I). \] (3.17)

Note that this is before any compensation method is introduced. Approaches to computing \( \sigma_n^2 \) are different for the simulated and experimental data-sets.

In the simulated data-set, the noise is simulated, and we use that simulated noise in our \( \sigma_n^2 \) computations. Each EP/BSP pair has its own noise matrix. Those separate noise matrices are used when computing \( \sigma_n^2 \), the variance of the noise.

In the experimental data-set, however, both EP and BSP are measured values, and we don’t have immediate knowledge of the noise. We start from the following equation:

\[ n = y - Ax. \] (3.18)

During the training phase, we generate the following augmented noise matrix:

\[ N_{tr} = Y_{tr} - AX_{tr} = \begin{bmatrix} n_1 & n_2 & \cdots & n_k \end{bmatrix}, \] (3.19)

where \( k \) is the number of training beats.

Initially, like in the simulated data-set, assumptions of 0 mean and a diagonal covariance matrix for \( N_{tr} \) were used to compute the noise characteristics. However, that proved to be inaccurate (mean absolute value of \( N_{tr} \) is 0.28) and resulted in improper reconstructions. Following is a more proper representation of the final Gaussian form of the noise:

\[ n \sim N(\bar{N}_{tr}, C_n), \] (3.20)

where \( \bar{N}_{tr} \) has repeating columns, holding values of the mean values of \( N_{tr} \)'s rows. Additionally, \( C_n = (N_{tr} - \bar{N}_{tr})(N_{tr} - \bar{N}_{tr})^T/K \), where \( K \) is the number of columns of \( N_{tr} \).
3.5 Evaluation Metrics

To compare our methods and evaluate their performances, we take advantage of several quantitative metrics.

After we go backwards with the inverse problem (from BSP to EP), we evaluate our accuracy by comparing against the recorded ground truth EP values. In evaluating this accuracy, we turn to correlation coefficient (CC) and relative error (RE).

3.5.1 Correlation Coefficient (CC)

The Pearson CC [77, 78], which is one of the most widely used CC methods, is often just called CC. It measures the linear relationship between two normally distributed variables. It is given by the following formula:

\[
CC = \frac{\text{cov}(x, \hat{x})}{\sigma_x \sigma_{\hat{x}}},
\]

where \(\text{cov}(x, \hat{x})\) is the covariance matrix of the estimation, \(\hat{x}\), and the ground truth, \(x\). \(\sigma\)s are the respective standard deviations.

The resulting CC value is bounded by -1 and 1. -1 indicates a negative linear correlation between the two random variables; 0 indicates no linear correlation, and 1 is for a total positive linear correlation. The closer the absolute value of CC to 1, the more linearly correlated the random variables are.

Since our EPs expend both in the time domain and the space domain, we have temporal and spatial variations of CC. Temporal CC (tCC) compares an estimation with the ground truth across the time axis; as a result, there is a tCC value for each lead on the heart. Spatial CC (sCC) on the other hand compares them across the leads; hence, there is a sCC value for each time instance.

3.5.2 Relative Error (RE)

The relative error is simply the deviation of the approximation from the actual value:
\[ RE = \frac{\| \hat{x} - x \|}{\| x \|}, \]  

(3.22)

where \( \hat{x} \) is an approximation for \( x \).

The RE value has no upper boundary. The lower it is, the more accurate the approximation is; 0 is a perfect match.

Like CC, RE has temporal and spatial variations in the context of this thesis. Temporal RE (tRE) is the error across the time axis, whereas spatial RE (sRE) is the error across leads. There is a tRE value for each lead and a sRE value for each time instance.

### 3.5.3 Activation Time & Pacing Location

Activation time (AT) is the time of a node when that node’s voltage undergoes the sharpest change in value \([79, 80]\). This is accepted to be the instance when that node joins the action potential propagation. Due to the relation of AT with the change in potential, some methods of computing AT involve finding the minimum of the first derivative of the EGM. This might result in inaccuracies caused by the noise and smoothing involved in the inverse problem. In this thesis, ATs are computed with the spatio-temporal method proposed in \([81]\).

Pacing location is the node that has the smallest activation time: it’s the node where the propagation of the action potential, hence the action, begins. Localization error (LE) is the distance between the estimated pacing location and the original pacing location, in millimeters for our case.
In this chapter, first the effect of measurement noise on the performance of Tikhonov regularization and Bayesian MAP estimation is observed. Then the effect of various geometric model errors on Bayesian MAP estimation’s performance is inspected, after which the results of introducing compensation for the model errors are laid out. Further in the chapter, results of using stacked forward matrices (See subsection 3.3.1) and results of compensation on the experimental data-set are given.

This chapter includes both quantitative and qualitative results. Quantitative results show the performances of estimations with and without compensation, in terms of temporal/spatial correlation coefficients (CC) and relative errors (RE), activation times (AT), and localization errors (LE). Qualitative inspections of the results are heart maps of AT, AT CC, and tCC comparing various compensation methods with no compensation, along with inspection of an action potential with a plot of ground truth and reconstructed electrograms (EGM).

In some plots of relative errors that are clipped between values of 0 and 1, it’s possible to see that some boxes in the boxplot are outside the y axis range. Though not so visually appealing, this is intended, as it’s more important to observe what’s happening between 0 and 1. A relative error value of more than 1 underperforms an estimator that guesses its output 0 every time, thus it does not have considerable contribution.

All results in this chapter are for the simulated data-set, with the exception of Section 4.6 whose results are for the experimental data-set.
4.1 Effect of Measurement Noise

An inspection is done on the effect of signal to noise (SNR) ratio of the introduced measurement noise on the solution of Bayesian MAP estimation and Tikhonov estimation alike.

To construct the a priori information for the Bayesian MAP, all training beats in the simulated data-set are used (See Section 3.2 and Subsection 3.4.3). Then the evaluation metrics of the solutions for all test beats in the simulated data-set are combined to produce boxplots, showing the median and quartile distributions. Outliers are removed from the boxplots, and the y-axis of each boxplot is clipped between the values of 0 and 1 for easier visual inspection.
Figure 4.1 shows how the temporal CC, temporal RE, spatial CC, and spatial RE vary with respect to the SNR. It is noteworthy that higher values of CC and lower values of RE indicate better performance. With smaller SNR values, the noise is too much for both Bayesian MAP and Tikhonov regularization to yield adequate results. After around 30 to 40 dB, increasing the SNR also hurts the performances of both MAP and Tikhonov. It can be inferred from Figure 4.1 that Bayesian MAP estimation generally performs better than Tikhonov regularization, in terms of temporal/spatial CC/RE, with the exception of an SNR value of 50 dB. Furthermore, as the SNR value decreases, MAP’s improvement over Tikhonov increases.

4.2 Effect of Geometric Model Errors

In this section, following error types are inspected: scaling, shifting, and rotational. Like in Section 4.1, this section’s figures are all boxplots of metrics (combined metrics of all test beats in the simulated data-set) for Bayesian MAP estimation and Tikhonov regularization.

4.2.1 Scaling error

We observe the impact of introducing scaling errors to the heart model, by comparing the temporal CC, temporal RE, spatial CC, and spatial RE values with respect to the scaling error introduced. Scaling errors introduced result in hearts with sizes that are 0.7, 0.8, 0.9, 1.1, 1.2, 1.3 times the original heart size. Note that a scaling factor of 1.0 times the original heart size corresponds to no error.

Figure 4.2 shows that temporal and spatial correlation coefficients are not impacted heavily from introducing scaling errors; their median values remain similar, whereas the stability of the solutions decreases. One particular example to this is the sCC values of Tikhonov regularization solutions on scaled down heart models: 25% of the sCC values are lower than 0.08 when the heart model is scaled down with a scaling factor of 0.7. Nonetheless, we can see that as the size of the heart deviates from its original scale (1.0), REs of both Bayesian MAP and Tikhonov estimations increase. This is stronger when the heart is smaller, rather than larger. Additionally, although
the difference is small, MAP generally performs better than Tikhonov when there are scaling errors. This difference increases as the heart size decreases: in the smallest hearts, MAP’s improvement over Tikhonov becomes more apparent.

### 4.2.2 Shift error

The impact of introducing shifting errors along the x-axis to the heart model is observed, using tCC, tRE, sCC, and sRE. Shifting errors introduced are -15, -10, -5, 5, 10, 15 mm along the x-axis. Note that a shifting error of 0 mm corresponds to no error.

Figure 4.3 shows that, as the introduced error increases, the performance of each estimator (MAP and Tikhonov) gets worse in terms of both CC and RE. Furthermore, this is seen in both variations of the metrics: temporal and spatial. Moreover, like with scaling errors, MAP generally performs better than Tikhonov when there are shifting errors.
4.2.3 Rotational error

Rotational errors are introduced around the x-axis and the y-axis. Around the x-axis, rotations of -20, -10, 10, and 20 degrees are applied. Around the y-axis, rotations of -20 and -10 degrees are applied. A rotation of 0 degrees stands for the unmodified heart model.

Figure 4.4 shows the results of both estimators with respect to rotational errors around the x-axis. Although not as drastically as when the introduced error is in the form of shifting, $tCC$, $tRE$, $sCC$, and $sRE$ get worse with increased rotational errors around the x-axis. The same can be said for rotational errors around the y-axis, results of which are shown in Figure 4.5. Like with scaling and shifting errors, when the introduced error is rotational, MAP performs slightly better than Tikhonov. This is true when the
Figure 4.4: Effect of introducing rotational errors around the x-axis on tCC, tRE, sCC, and sRE, for both Bayesian MAP and Tikhonov estimations.

Figure 4.5: Effect of introducing rotational errors around the y-axis on tCC, tRE, sCC, and sRE, for both Bayesian MAP and Tikhonov estimations.
rotation is applied either around the x-axis or around the y-axis.

4.2.4 Summary

As anticipated, introducing geometric model errors to the heart model decreases the performances of both Bayesian MAP estimation and Tikhonov regularization. In the case of scaling errors, this impact is more clear in temporal and spatial variations of RE, rather than CC. In the case of shifting errors, both temporal and spatial variations of RE and CC are impacted as the introduced error increases. This is also observed in the case of rotational errors, with reduced impact. For most of the geometric model errors introduced, Bayesian MAP estimation has proved to be more robust than Tikhonov regularization.

4.3 Compensation of Geometric Errors in Bayesian MAP Estimation

In this section, results of compensation are inspected under three different epsilon construction methods (training composition scenarios): individual epsilon, categorical epsilon, and unified epsilon. tCC, tRE, sCC, and sRE boxplots are given, comparing IID and CORR’s performances against no compensation. See Section 3.3 for a detailed distinction between the training composition scenarios.

4.3.1 Individual Epsilon Covariance Matrix

This subsection focuses on the results of individual epsilon scenario: $C_\epsilon$ for the Bayesian MAP estimation is constructed with a single errored forward matrix.

Figure 4.6 shows that compensating for scaling errors using IID and CORR methods helps greatly in the cases of scaling down: the tRE and sRE metrics of the solution are improved. When it comes to the other errors and the other metrics, we see that compensation does not help much; however, it also does not harm the solution by great margins. Furthermore, generally, temporal and spatial REs improve more when the compensation method is CORR, rather than IID.
Figure 4.6: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to scaling errors, when the training composition scenario is the individual epsilon.

Figure 4.7: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the x-axis, when the training composition scenario is the individual epsilon.
Figure 4.8: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to rotational errors introduced around the x-axis, when the training composition scenario is the individual epsilon.

Figure 4.7 shows the performance of Bayesian MAP estimation, comparing compensating for shifting errors with IID and CORR methods against no compensation. The shifting errors are introduced along the x-axis. Results for shifting errors along the y-axis and along the z-axis can be found in APPENDIX B.2. In general, both IID and CORR outperform no compensation in all metrics, when shifting errors are introduced. CORR outperforms IID in all metrics except tCC. Increasing the shifting error results in increased performance boost with compensation.

Figures 4.8 and 4.9 show the performance of Bayesian MAP estimation, comparing compensating for rotational errors with IID and CORR methods against no compensation. Figure 4.8 is for rotational errors introduced around the x-axis, whereas Figure 4.9 is for rotational errors introduced around the y-axis. From these figures, it is observed that IID does not improve the performance of Bayesian MAP estimation when there are rotational errors, while CORR shows slight improvements that increase with increased error, in all metrics, for rotations around both x and y axes.

In short, when the introduced errors are in the form of scaling, both IID and CORR
outperform no compensation, in terms of temporal and spatial REs. When the introduced errors are in the form of shifting, both IID and CORR outperform no compensation, in terms of temporal and spatial REs, as well as CCs. In rotational errors, there is not much improvement in the metrics for IID, whereas CORR outperforms no compensation slightly. In general, CORR has proved to be a better compensation method than IID when the training composition scenario is the individual epsilon.

### 4.3.2 Categorical Epsilon Covariance Matrix

This subsection focuses on the results of categorical epsilon scenario: $C_\epsilon$ for the Bayesian MAP estimation is constructed with all errored forward matrices from a single error type (scaling, shifting, or rotational).

Figure 4.10 shows that, like in the case of individual epsilon scenario, compensating for scaling errors using IID and CORR methods helps greatly in the cases of scaling down: the tRE and sRE metrics of the solution are improved. Again, when it comes to
Figure 4.10: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to scaling errors, when the training composition scenario is the categorical epsilon.

the other errors and the other metrics, we see that compensation does not help much; however, it also does not harm the solution by great margins. We see once again that CORR outperforms IID in terms of temporal and spatial REs.

Figure 4.11 shows the performance of compensating for the shifting errors instead of ignoring them for errors along the x-axis. Results for shifting errors along the y-axis and along the z-axis can be found in APPENDIX B.3. Both IID and CORR outperform no compensation in tCC, tRE, sCC, and sRE, when shifting errors are introduced with a training composition scenario of categorical epsilon. This time, IID outperforms CORR in terms of tCC, sCC, and sRE. They are comparable in terms of their improvements in tRE.

Figures 4.12 and 4.13 show the performance of compensating for the rotational errors instead of ignoring them for errors of rotation around the x-axis and the y-axis, respectively. There is not much improvement in the metrics when Bayesian MAP estimation is equipped with the IID compensation method, although CORR shows a slight improvement, even less than its improvement in the individual epsilon case.
Figure 4.11: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the x-axis, when the training composition scenario is the categorical epsilon.

Figure 4.12: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to rotational errors introduced around the x-axis, when the training composition scenario is the categorical epsilon.
Figure 4.13: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to rotational errors introduced around the y-axis, when the training composition scenario is the categorical epsilon.

To recall, when the introduced errors are in the form of scaling, both IID and CORR outperform no compensation, in terms of temporal and spatial REs. When the introduced errors are in the form of shifting, both IID and CORR outperform no compensation, in terms of temporal and spatial REs, as well as CCs. In rotational errors, no apparent improvement is observed in the metrics for IID, and slight improvement is observed for CORR. In the results for the training composition scenario of individual epsilon, it was shown that CORR outperforms IID. This time, however, it is outperformed by IID when the introduced error is in the form of shifting.

4.3.3 Unified Epsilon Covariance Matrix

This subsection focuses on the results of unified epsilon scenario: $C_\epsilon$ for the Bayesian MAP estimation is constructed with all errored forward matrices from all error types (scaling, shifting, or rotational).

Figure 4.14 shows that, like in the cases of individual and categorical epsilon scenar-
Figure 4.14: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to scaling errors, when the training composition scenario is the unified epsilon.

ios, compensating for the scaling errors using IID and CORR methods helps greatly in the cases of scaling down: the tRE and sRE metrics of the solution are improved. When it comes to the other errors and the other metrics, we see that compensation does not help much; however, it also does not harm the solution by great margins, as in the other training composition scenarios (individual and categorical). Again, when there are improvements in temporal and spatial REs for scaled down heart models, CORR’s improvements surpass those of IID.

Figure 4.15 shows the performance of compensating for the shifting errors instead of ignoring them for errors along the x-axis. Its counterparts (boxplots of metrics for shifting along the y-axis and along the z-axis) can be found in APPENDIX [B.4]. Both IID and CORR outperform no compensation when the introduced errors are in the form of shifting, for temporal and spatial REs. CORR does not show improvement in temporal or spatial CC, when compared to no compensation. IID, on the other hand, can produce estimations that are both temporally and spatially more correlated with the ground truth, when compared to no compensation.
Figure 4.15: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the x-axis, when the training composition scenario is the unified epsilon.

Figure 4.16: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to rotational errors introduced around the x-axis, when the training composition scenario is the unified epsilon.
4.3.4 Comparison of Training Compositions for the Epsilon Covariance Matrix

Previous three subsections, 4.3.1, 4.3.2, and 4.3.3, are observations of how Bayesian MAP with IID and CORR compensations perform against Bayesian MAP with no compensation.
Figure 4.18: Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), for all error types (scaling, shifting, rotational).

compensation, for all three training composition scenarios for the epsilon covariance matrix, namely the individual epsilon, the categorical epsilon, and the unified epsilon, in the order they appear as subsections.

In this subsection, for each training composition scenario, metrics for all geometric errors, for all test beats in the simulated data-set, are combined together to compare how well the compensation generally performs under each training composition scenario. To avoid repetition, metrics of heart models with no errors are not included. Results of compensation are investigated for all error types, for scaling errors, for shifting errors, and for rotational errors, in that order.

Figure 4.18 has seven x ticks for each four of its boxplots of metrics (tCC, tRE, sCC, and sRE):

- No comp: Metrics for no compensation.
- Ind IID: Metrics for IID compensation, with the training composition scenario of individual epsilon.
Figure 4.19: Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), for scaling errors.

- Cat IID: Metrics for IID compensation, with the training composition scenario of categorical epsilon.
- Uni IID: Metrics for IID compensation, with the training composition scenario of unified epsilon.
- Ind CORR: Metrics for CORR compensation, with the training composition scenario of individual epsilon.
- Cat CORR: Metrics for CORR compensation, with the training composition scenario of categorical epsilon.
- Uni CORR: Metrics for CORR compensation, with the training composition scenario of unified epsilon.

Figures 4.19, 4.20, and 4.21 have the same format, but they’re only for scaling, shifting, and rotational errors, respectively.

From Figures 4.18, 4.19, 4.20, and 4.21, we can see that median values of the
Figure 4.20: Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), for shifting errors.

Figure 4.21: Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), for rotational errors.
metrics undergo small improvements for scaling and rotational errors, whereas with shifting errors, compensation helps reconstruct the original epicardial potentials more accurately, in terms of temporal & spatial correlation coefficients and relative errors. Also, even in cases where compensation does not improve the median by great values, it results in smaller interquartile ranges for the metrics, which shows that results are less scattered and more robust. This is more apparent in boxplots of relative errors (tRE and sRE) than it is in boxplots of correlation coefficients (tCC and sCC).

Specifically, for scaling errors, CORR outperforms IID in terms of temporal and spatial REs, which itself outperforms no compensation in those metrics. For shifting errors, IID is the most performant compensation method in terms of tCC, whereas CORR outshines IID and no compensation in terms of tRE, sCC, and sRE. In all these metrics, both IID and CORR outperform no compensation. In rotational errors, IID does not improve the results, whereas CORR introduces slight improvements that get diminished as the epsilon scenario goes from the individual epsilon to the categorical epsilon and then to the unified epsilon. When the metrics from all error types are combined (Figure 4.18), we see that IID outperforms no compensation in all metrics and is the most performant in terms of tCC, whereas CORR is the most performant in terms of all other metrics, falling short of the performance of no compensation in terms of tCC.

In general, individual epsilon scenario performs better than its counterparts, as is expected: individual epsilon is a more overfit version of categorical epsilon, which itself is a more overfit version of unified epsilon.

### 4.3.4.1 Comparison of Training Compositions in terms of Activation Time CCs

Recall that AT is the time instance when a node’s voltage value undergoes the largest change, and that node’s activation starts.

ATs of original and estimated values of all test beats are computed and the corresponding boxplots can be seen in Figure 4.22, which compares correlation coefficients (CC) of these ATs with respect to the epsilon training composition and the compensation method that are used. There are four boxplots corresponding to all
Compensation does not improve estimation of activation times in considerable amounts, when the introduced error is in the form of a rotation. In both scaling errors and shifting errors, modifying Bayesian MAP with IID compensation method as opposed to CORR or no compensation results in more accurate estimations of activation times. All three forms of IID (individual, categorical, unified) prove to be better AT estimators than no compensation. Surprisingly, overall, in the existence of scaling errors, unified IID outperforms even individual and categorical IIDs, in terms of AT CCs.
4.3.4.2 Comparison of Training Compositions in terms of Localization Errors

Figure 4.23: Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), in terms of localization errors. Top left: LE for all error types. Top right: LE for only scaling errors. Bottom left: LE for only shifting errors. Bottom right: LE for only rotational errors.

Reminder: pacing location is the node that has the smallest AT, and localization error is the distance between the actual and estimated pacing locations.

Figure 4.23 shows LE metrics for various training composition scenarios (individual, categorical, unified) and compensation methods (No comp, IID, CORR), in four box-plots corresponding to all error types, scaling errors, shifting errors, and rotational errors. We see that LE is improved (smaller) around 30 to 40 percent with IID, regardless of the error type. In terms of estimating pacing locations, compensating with CORR method does not help much, when compared with no compensation.
4.4 Qualitative Inspections

In this section, results of integrating compensation in Bayesian MAP estimation is observed qualitatively. A random test beat and a random lead from the simulated data-set are selected for this purpose, ensuring that the corresponding ground truth EGM has a nicely observable action potential curve. The introduced error that is investigated in this section is arbitrarily chosen as a shifting error of -15 mm along the x-axis. The training composition scenario is the individual epsilon.

Reconstructions with and without compensation are observed by comparing EGM reconstructions and heart maps of EP, tCC, and AT.

4.4.1 Comparison of Original and Reconstructed EGMs

Reconstructed EGMs (No comp, IID, CORR, Tikhonov) are compared against the original EGM in Figure 4.24 for one test beat. In particular, lead 387 (arbitrary, with a proper ground truth action potential curve) is taken under observation. For each EGM, a corresponding vertical line is drawn to indicate the computed AT. The original AT line is not shown, as it is exactly the same as the AT line from CORR. A moving average is applied on the reconstructions with a window of 5 ms, for easier visual inspection.

In both shape and value, both IID and CORR result in EGM reconstructions with better fidelity to the ground truth than either No comp Bayesian MAP estimation or Tikhonov regularization. Even with slight smoothing (moving average) to make the comparison easier on the eye, EGM estimates of No comp and Tikhonov are noisy, which in turn makes their AT computations less accurate. CORR EGM is also somewhat noisy, but it is up there with IID in accurately estimating the original AT of the lead. Of all reconstructions, IID EGM resembles the original the most, with differences in amplitudes.
Figure 4.24: Original and reconstructed EGMs with estimation methods of Tikhonov and Bayesian MAP with compensation methods of IID, CORR and nocomp. Left: EP map at $t = 60$ ms. Right: Ground truth and reconstructed EGMs. The blue dot on the map represents the lead, to which the EGMs on the right belong. On the right, the dashed vertical lines correspond to the estimated activation times. Estimated activation times of the original EGM and CORR EGM are exactly the same, so only one is shown. A moving average is applied on the reconstructions with a window of 5 ms, for easier visual inspection. The introduced error is shifting of -15 mm along the $x$-axis. The training composition scenario is the individual epsilon.
Figure 4.25: Maps of ground truth and reconstructed epicardial potentials, comparing no compensation with IID and CORR. The introduced error is shifting of -15 mm along the x-axis. The training composition scenario is the individual epsilon.
4.4.2 Heart Maps

Figure 4.25 shows original and reconstructed EPs mapped on the heart model. The potential values belong to t = 63 ms (arbitrary), when the ground truth heart map is observed to be smooth in its transitions between higher and lower potential values (t = 0 is when the QRS complex begins). We see that EPs are more smoothly reconstructed when there is compensation in MAP estimation. That can also be inferred from the 29.17% increase in sCC for IID and 43.75% for CORR.

Temporal CC values are mapped on the heart model in Figure 4.26. Leads being more red than blue indicates more correlation with the original time signal on that lead. Better correlation with the ground truth epicardial potentials is observed when Bayesian MAP estimation is supported with a compensation method. The training composition is the individual epsilon.

Figure 4.27 shows maps of ATs for the ground truth, the no compensation Bayesian MAP estimation, and IID & CORR compensated Bayesian MAP estimation. sCC improves 54.72% for CORR and 69.81% for IID. Visually, the AT distribution of IID resembles the ground truth ATs more than its counterparts. All estimated AT maps have a red dot showing the estimated pacing location: the lead with the smallest AT. All maps have a black dot indicating the ground truth pacing location. Compensation helps predict a more accurate pacing site, with IID being the most accurate.
Figure 4.27: Maps of AT, comparing results of no compensation with IID and CORR. Red dots indicate the pacing locations (lead with smallest AT), whereas black dots show the original pacing location. The introduced error is shifting of -15 mm along the x-axis. The training composition scenario is the individual epsilon.
4.5 Compensation with Stacked Forward Matrices

Figure 4.28 shows how compensation improves Bayesian MAP estimation when the stacked forward matrix (See subsection 3.3.1) method is used. Results are for all error types, from all test beats in the simulated data-set. tCC, sCC, tRE, and sRE values all improve with IID compensation, regardless of the epsilon training composition. When errors are introduced to the estimator in this way, Bayesian MAP estimation without compensation fails to produce adequate results. Recall that no compensation means using the stacked error matrices in place of $A$ for the inverse problem. IID and CORR adds further compensation by also modifying the noise covariance matrix in the solution approach. Though a less complex method than CORR, IID here takes the cake as the savior of MAP estimation, bringing its results to the vicinity of those of
Tikhonov’s. Notably, performances of both categorical and unified IID surpass those of Tikhonov’s, in terms of tCC, sCC, tRE, and sRE.

![Activation Time CC (all errors)](image)

Figure 4.29: Comparison of training composition scenarios for the epsilon covariance matrix (categorical, unified) and compensation methods (no comp, IID, CORR), in terms of activation time correlation coefficients, with stacked forward matrices, for all error types.

Figures 4.29 and 4.30 show AT CC and LE performances of various stacked error methods. Results are again for all error types, from all test beats. We see once again that using the stacked error matrices without further compensation (No comp), Bayesian MAP estimation falls behind of Tikhonov. Nonetheless, IID compensation with unified epsilon reaches the performance of Tikhonov regularization, whereas IID compensation with categorical epsilon exceeds the performance of both. Categorical CORR improves MAP’s performance in terms of activation time correlations and localization errors, but its performance falls short of those of Tikhonov and MAP with IID compensation.
4.6 Compensation on Experimental Data

Figure 4.31 shows the results of compensation on the experimental data-set mentioned in Section 3.4.2. Compensating with IID improves MAP estimation in terms of tRE, sCC, and sRE when compared to no compensation. CORR, on the other hand, outperforms both IID and no compensation in all metrics. For both IID and CORR, performance improvements are reduced going from individual epsilon to categorical epsilon and from categorical epsilon to unified epsilon.
Figure 4.31: Comparison of training composition scenarios for the epsilon covariance matrix (individual, categorical, unified) and compensation methods (no comp, IID, CORR), for all error types (scaling, shifting, rotational). Contrary to all other figures in this chapter, this one shows results for the experimental data-set.
CHAPTER 5

CONCLUSION

This study tackles the inverse problem of ECGI with the Bayesian MAP estimation method. It focuses on how the performance of the Bayesian MAP depends on measurement noise and process noise, in particular noise from geometric model errors that can arise while measuring BSPs.

The simulated data are from the University of Utah and include the following:

- 309 training beats
- 17 test beats

The epicardial potentials from these beats are used to simulate body surface potentials, using the forward model with the heart and torso models.

The experimental data are also from the University of Utah and reside in EDGAR (Experimental Data and Geometric Analysis Repository). It has the following numbers of training and test beats:

- 127 training beats
- 44 test beats

Before any process noise (geometric model error) compensation, Bayesian MAP estimation and Tikhonov regularization performances are tested against varying measurement noise and process noise. The geometric model errors are introduced in forms of scaling, shifting, and rotational errors.

The evaluation of the solutions are done with the following metrics:
• Temporal Correlation Coefficient (tCC)
• Temporal Relative Error (tRE)
• Spatial Correlation Coefficient (sCC)
• Spatial Relative Error (sRE)
• Correlation Coefficient of Activation Times
• Localization Errors

5.1 Effects of Measurement Noise and Geometric Model Errors

It is observed that Bayesian MAP estimation and Tikhonov regularization perform the best when the measurement noise has an SNR of around 30 to 40 dB. Furthermore, MAP is more robust to measurement noise changes than Tikhonov. In geometric model error cases, the SNR ratio of the measurement noise is kept at 30 dB. Section 4.2 shows that tRE and sRE get worse with larger scaling, shifting, and rotational errors. tCC and sCC get worse with larger errors of shifting and rotation, whereas when we modify the scale of the heart, tCC and sCC do not show drastic changes in their median values.

5.2 Effects of Compensation on Simulated Data-set

Further in the Results chapter we see that activation time estimations, hence the localization error estimations get more accurate when Bayesian MAP estimation is compensated with IID method. Additionally, IID and CORR help improve Bayesian MAP estimation’s performance in terms of tCC, tRE, sCC, and sRE, for various geometric error types (scaling, shifting, rotational). Combining our compensation method with the stacked matrices from Bergquist et al. [74] results in improvements in Bayesian MAP estimation’s performance, again mostly with IID method.

Some qualitative analyses are done on the simulated data-set to observe the effects of compensation. These are in forms of heart maps of AT and tCC. They all show improvements in Bayesian MAP estimation, when compensation is introduced.
5.3 Effects of Compensation on Experimental Data-set

Results of compensation on experimental data-set show that introducing IID compensation to MAP improves tRE, sCC, and sRE when compared to no compensation. Moreover, introducing CORR to MAP performs better than both IID and no compensation in all of tCC, tRE, sCC, and sRE.

5.4 Takeaway

We show that using prior information of the epicardial potentials, Bayesian MAP estimation solution can be improved in cases of translational and rotational geometric heart model errors, along with errors in the size of the heart. Even in a simple form such as the IID, inclusion of geometric model errors in the formulation of Bayesian MAP beats ignoring them. Taking into account not only the measurement noise, but also the geometric model errors results in more accurate estimations of EP and pacing locations. To obtain the geometric model of the heart, clinical patients’ hearts need to be scanned via medical imaging methods like MRI or CT. These methods can sometimes be invasive procedures. Being able to compensate for geometric model errors with the methods proposed in this thesis potentially means that the number of times clinical patients’ hearts need to be scanned can be reduced.

5.5 Limitations

Following are some of the limitations in the study:

- The geometric model errors are assumed to be independent of EGMs. This is an oversimplification of the solution.

- The training set beats and the test set beats belong to the same dog heart. This might result in an overfit in the evaluations.

- All of EP, BSP, measurement noise, and geometric model errors are represented via Gaussian models. This is an idealistic model that might not fully represent
the variables.

5.6 Future Work

The following future work can increase the impact and value of the study:

- Unlike IID and CORR, a compensation method that includes the correlation between the EGM \((x)\) and the geometric model error term \((\epsilon = A_{err,x})\) can be tested against various geometric model errors.

- Compensation can be tested against geometric model errors that are the results of combinations of errors (scaling, shifting, rotation).
REFERENCES


[37] “12-lead ecg placement guide with illustrations.”

https://www.cablesandsensors.eu/pages/


Of the 8330 (490x17) leads in the test set, 124 leads are marked as bad leads. To see if there are any additional bad leads of which we need to be aware, we inspected the histogram of the maximum magnitude of potential value reached by every lead.

Figure A.1 and A.2 together show that there is an abnormality with some leads whose potential value magnitude never exceed 1.

Limiting this histogram to maximum potential magnitudes of 1 yields Figure A.3 which reveals to us that there are no leads whose maximum potential magnitudes are between 0.3 and 1.

Moving on, we want to compare these findings with the histogram of the provided bad leads.

Notice that Figure A.3 and Figure A.5 are almost identical. Of the 124 provided bad leads, 119 leads never exceed a potential value magnitude of 0.3. The total amount of leads whose potential value magnitudes never exceed 0.3 is 125. With the help of this inspection, we safely mark 6 more leads as bad leads.
Figure A.1: Histogram of maximum potentials of all leads in the test set.

Figure A.2: Histogram of maximum potentials of all leads in the test set, |V| < 5.
Figure A.3: Histogram of maximum potentials of all leads in the test set, $|V| < 1$.

Figure A.4: Histogram of maximum potentials of provided bad leads in the test set.
Figure A.5: Histogram of maximum potentials of bad leads in the test set, $|V| < 1$. 
APPENDIX B

METRICS FOR SHIFTING ACROSS Y AND Z AXES

B.1 Effect of Geometric Model Errors

Figure B.1: Effect of introducing shifting errors across the y-axis on tCC, tRE, sCC, and sRE, for both Bayesian MAP and Tikhonov estimations.

Figure B.2: Effect of introducing shifting errors across the z-axis on tCC, tRE, sCC, and sRE, for both Bayesian MAP and Tikhonov estimations.
B.2 Compensation of Geometric Errors: Individual Epsilon

Figure B.3: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the y-axis, when the training composition scenario is the individual epsilon.
Figure B.4: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the z-axis, when the training composition scenario is the individual epsilon.
B.3 Compensation of Geometric Errors: Categorical Epsilon

Figure B.5: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the y-axis, when the training composition scenario is the categorical epsilon.
Figure B.6: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the z-axis, when the training composition scenario is the categorical epsilon.
B.4 Compensation of Geometric Errors: Unified Epsilon

Figure B.7: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the y-axis, when the training composition scenario is the unified epsilon.
Figure B.8: Comparison of no compensation, compensation with IID, and compensation with CORR with respect to shifting errors introduced along the z-axis, when the training composition scenario is the unified epsilon.