REAL ZEROS OF RANDOM MODULAR FORMS

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ABSTRACT

REAL ZEROS OF RANDOM MODULAR FORMS

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Modular forms have been a highly important area of interest in many fields such as Algebraic Geometry, Number Theory and Applied Cryptography. These special functions possess very important and interesting arithmetic and geometric properties through which several applications occur in modern mathematics and geometry. Calculating the number of zeros of modular forms on a fundamental domain and finding their distribution behaviour are considered as major problems among them. In this study, the main focus will be on attacking this problem with a probabilistic approach by using standard normal variables and the basis elements of cusp forms through which one can define a so-called *random modular form*. For this purpose we first give basic definitions and fundamental properties of modular forms, then introduce cusp forms, which are defined as modular forms vanishing as $\operatorname{Im} z \to \infty$, which form a very crucial subspace of the finite dimensional vector space of modular forms. Afterwards, by using the basis elements of the vector space of cusp forms of weight k and independently identically distributed (i.i.d) real random variables, we construct *random modular forms*. Then we adapt the *Crofton's formula* for random modular forms to obtain the expected number of real zeros which are the zeros defined on some specific geodesic segments on a fundamental domain. In the end we obtain a

formula for the expected number of real zeros of a random modular form of weight k and give an upper bound for the infimum of this number.

Keywords: Elliptic curves, Cusp forms, Random modular forms, Real zeros

ÖZ

RASTGELE MODÜLER FORMLARIN GERÇEL SIFIRLARI

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Modüler formlar Cebirsel Geometri, Sayılar Teorisi ve Uygulamalı Kriptografi gibi birçok alanda oldukça önemli bir ilgi alanı olmuştur. Bu özel fonksiyonlar, modern matematik ve geometride çeşitli uygulamaların ortaya çıktığı çok önemli ve ilginç aritmetik ve geometrik özelliklere sahiptir. Modüler formların bir temel bölge üzerindeki sıfırlarının sayısını hesaplamak ve dağılım davranışlarını bulmak, modüler formların tüm özellikleri arasında en önemlileri olarak durmaktadır. Bu çalışmanın temel odağı, bu probleme, standart normal değişkenler ve sözde *rastgele modüler formlar*ı tanımlamakta kullanılan uç formların taban elemanları kullanılarak, olasılıksal bir yaklaşımla saldırmak olacaktır. Bu amaçla, önce modüler formların temel tanımlarını ve özelliklerini verdikten sonra, z'nin sanal kısmı sonsuza giderken sıfıra eşit olan modüler formlar olarak tanımlanan ve sonlu boyutlu modüler formların bir altuzayını oluşturan uç formları tanıtacağız. Sonrasında, k ağırlığındaki uç formlar vektör uzayının taban elemanlarını ve bağımsız özdeş dağılımlı reel rastgele değişkenleri kullanarak rastgele bir modüler form oluşturuyoruz. Daha sonra ise, rastgele modüler formlar için *Crofton formülü*'nü uyarlayarak, temel bir bölge üzerindeki bazı belirli jeodezik parçalar üzerindeki sıfırlar olarak tanımlanan reel sıfırların beklenen

sayısını etmeye çalışyoruz. Sonunda k ağırlığındaki rastgele bir modüler formun beklenen reel sıfır sayısı için bir formül elde edip bu sayının infimumu için bir üst sınır veriyoruz.

Anahtar Kelimeler: Eliptik eğriler, Uç formlar, Rastgele modüler formlar, Gerçel sıfırlar

To Derya and my family

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I saved my beloved wife to the last since thanking to her is the hardest one for me. I am sure that there are not enough words to use to express my gratitude and thanks for her, so the only thing to say about her is that because of her I am the luckiest man alive.

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INTRODUCTION

1.1 Motivation

Modular Forms are considered central objects in many areas of mathematics, specifically in Number Theory, Complex Analysis and Algebraic Geometry. They play an important role both in applicable and theoretical levels of mathematics and have been very useful in solving various significant problems, such as Fermat's last theorem, Diophantine equations, congruent number problem, construction of Ramanujan graphs and generating functions for partitions ([1], [2]). In recent years, Maryna Viazovska, Fields medalist for the work of cracking the Sphere-Packing Problem in dimension 8, has also used modular forms extensively ([3]). Besides all of these challenging problems with which mathematicians have been struggling for years, modular forms appear to be very handy in many applications of cryptography in which elliptic curves over finite fields have been used, especially they are the cornerstone of Elliptic Curve Cryptography ([4]).

Among all the interesting arithmetic properties of modular forms, which are roughly complex-valued functions on the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ satisfying some holomorphy and invariance conditions, finding the locations, asymptotic distributions and the number of the zeros of these specific functions constitute an important part of the field of study. There have been many interesting and nice results in this area, for instance in [5], it has been proved that the zeros of the Eisenstein series of weight $k > 2$,

$$G_k(z) = \frac{1}{2} \sum_{\substack{gcd(c,d)=1 \\ (c,d)\in\mathbb{Z}^*\times\mathbb{Z}^*}} (cz+d)^{-k}$$

are all on the arc $\{\|z\| = 1 \mid z \in \mathbb{Z}\}$ in the fundamental domain $SL_2(\mathbb{Z})\backslash\mathbb{H}$ (which will be depicted later) and uniformly distributed as $k \to \infty$.

In this thesis, we will be mostly interested in the case of "random modular forms" which are constructed by using i.i.d. (independently identically distributed) real random variables and basis elements of cusp forms which are a very special subset of modular forms, yet in order to get to this point, first we need basic definitions, facts and some preliminary information to construct such objects.

1.2 Elliptic Curves, Lattices and the Full Modular Group

Definition 1.2.1. An **Elliptic Curve** $E(K)$, over a field K, is a genus one non-singular projective algebraic plane curve with a specified base point $\mathcal{O}$ which serves as an identity element of the group structure (see §5 of the Chapter "Introduction to Rational Points on Plane Curves " in [6]) on the curve.

It is a well-known fact that if $char(K) \neq 2, 3$, then an elliptic curve $E(K)$ can be brought into the following standard form by a projective linear transformation

$$E(K) = \{(x, y) \in K \times K : y^2 = x^3 + Ax + B\} \cup \{\mathcal{O}\}$$

where the discriminant $\Delta = 4A^3 + 27B^2 \neq 0$. (See pg. 45 in [7])

Above, the equation of the curve in the affine plane is given and $\{\mathcal{O}\}$ is its unique point at infinity. Now in order to see every complex elliptic curve $E(\mathbb{C})$ as a complex torus, one needs to define *lattices* of rank 2 on the complex plane and construct an isomorphism between them.

Definition 1.2.2. A **lattice** of rank $\Lambda = \{n_1 z_1 + n_2 z_2 \mid n_1, n_2 \in \mathbb{Z}\} = \langle z_1, z_2 \rangle$ of rank 2 generated by $\mathbb{R}$-linearly independent complex numbers $z_1, z_2 \in \mathbb{C}$ is an additive subgroup of $\mathbb{C}$.

Proposition 1.2.3. $\Lambda = \langle z_1, z_2 \rangle$ and $\Lambda' = \langle z_1', z_2' \rangle$ generate the same lattice if and only if there exist a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $a, b, c, d \in \mathbb{Z}$ and $ad - bc = \mp 1$. (See Theorem 1.2 in [8])

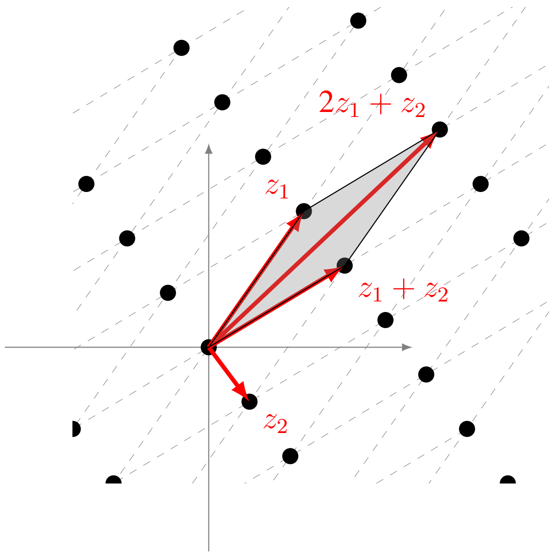

Figure 1.1: Lattice with generators z_1 and z_2

Definition 1.2.4. Let Λ_1 and Λ_2 be two lattices over $\mathbb{C}$. Then Λ_1 and Λ_2 are called *homothetic* if there is some $\lambda \neq 0$ such that $\Lambda_1 = \lambda\Lambda_2$.

It is straightforward to see that *homothety* is an equivalence relation. In the complex plane $\mathbb{C}$, as a consequence of *the uniformization theorem*, to every elliptic curve $E(\mathbb{C})$ there is an associated lattice unique up to homothety and an isomorphic map between $E(\mathbb{C})$ and the complex torus $\mathbb{C}/\Lambda$ (See VI.5 in [7]). Thanks to the isomorphism $E(\mathbb{C}) \cong \mathbb{C}/\Lambda$, an elliptic curve over $\mathbb{C}$ can also be seen as a complex torus.

It is also possible to find a standard representation for a complex elliptic curve by means of the following properties.

Proposition 1.2.5.

(i) $\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2$ if and only if Λ_1 and Λ_2 are homothetic, i.e., $\Lambda_1 = \lambda\Lambda_2$ for some $\lambda \in \mathbb{C}^*$.

(ii) $\langle z, 1\rangle$ and $\langle z', 1\rangle$ are homothetic if and only if $\gamma z = z'$ for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ where $\gamma z = \frac{az+b}{cz+d}$.

Proof.

(i) see VI.4 in [7].

(ii) See Remark 12.2 in [7]. $\square$

Now by using (i) above, a torus represented by $\mathbb{C}/\langle z_1, z_2\rangle$ is biholomorphic to another torus $\mathbb{C}/\langle \frac{z_1}{z_2}, 1\rangle$. This will pave the way to enable rather a more useful characterization which is indeed finding a *moduli space* for elliptic curves. Still there are some complications to resolve in order to obtain a direct way to represent every elliptic curve with respect to the lattices $\Lambda = \langle z, 1\rangle$ with the fixed generator 1.

Let E be an elliptic curve over $\mathbb{C}$ and $E \cong \mathbb{C}/\Lambda$ for some lattice Λ. Then by choosing an oriented basis $\langle z_1, z_2\rangle$ for Λ so that $\mathrm{Im}(\frac{z_1}{z_2}) > 0$ and using $\lambda = \frac{1}{z_2}$, we get the isomorphism $E \cong \mathbb{C}/\Lambda \cong \mathbb{C}/\langle \frac{z_1}{z_2}, 1\rangle$. As a final step to achieve the desired representation for elliptic curves, we use (ii) above and therefore have $E \cong \mathbb{C}/\langle \gamma(\frac{z_1}{z_2}), 1\rangle$ for each $\gamma \in SL_2(\mathbb{Z})$, where $SL_2(\mathbb{Z}) = \{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) | a, b, c, d \in \mathbb{Z}, ad - bc = 1\}$.

Finally, there exists a bijection between $SL_2(\mathbb{Z})\backslash\mathbb{H}$ and *isomorphism classes of elliptic curves*. This bijection also implies that $SL_2(\mathbb{Z})\backslash\mathbb{H}$ acts as a *moduli space* of elliptic curves on $\mathbb{C}$, in other words, $SL_2(\mathbb{Z})\backslash\mathbb{H}$ serves as a parameter space for the elliptic curves which are unique up to homothety. For that reason, it is crucial to understand the structure of this space, but before this let us discuss the group $SL_2(\mathbb{Z})$ in more details.

$SL_2(\mathbb{Z})$ is classically defined as the group of all 2×2 matrices with entries in $\mathbb{Z}$ whose determinant is 1. This group acts on the complex upper half plane in the way that $\gamma \circ z = \frac{az+b}{cz+d}$ where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. Note that $\gamma \circ z$ and $-\gamma \circ z$ give the same result under this action, i.e., $-\gamma z = \frac{-az-b}{-cz-d} = \frac{az+b}{cz+d} = \gamma z$. Therefore it will be wiser to use $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$ as the group acting on $\mathbb{Z}$. From now on, Γ will always denote $SL_2(\mathbb{Z})/\{\pm I\}$ which is also called the *full modular group*.

The full modular group Γ can be generated by using only two matrices. This results in the presentation given below.

Proposition 1.2.6. ([9]) $\Gamma = \langle S, T \mid S^2 = (ST)^3 = I\rangle$ where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right)$. In other words, Γ is the free product of the cyclic groups of order 2 and 3 generated by the matrices S and ST, respectively.

1.3 A Fundamental Domain

As explained in the previous section, $\Gamma\backslash\mathbb{H}$ serves as a *moduli space* for the isomorphism classes of complex elliptic curves. Thus, it is crucial to comprehend the geometric structure of this space. Each orbit of the action of Γ represents an elliptic curve which is isomorphic to a quotient of $\mathbb{C}$ by some lattice $\Gamma_z = \langle z, 1\rangle$ such that z corresponds a representative element of this equivalence class. We now describe a subset of $\mathbb{H}$ which contains one element from each orbit. For this purpose, firstly define the set

$$\mathcal{D} = \{z \in \mathbb{H} \mid -1/2 \leq \mathrm{Re}(z) \leq 1/2, \|z\| \geq 1\}$$

which is pictured below.

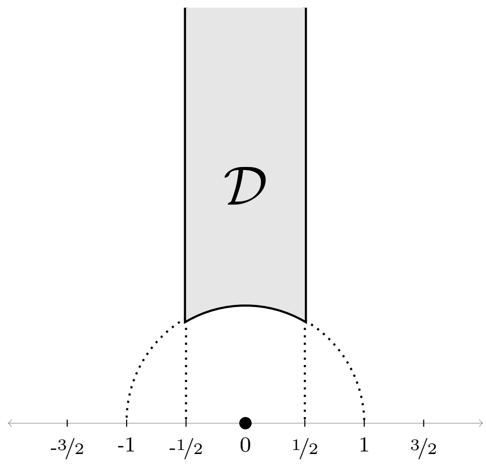

Figure 1.2: A Fundamental Domain

Theorem 1.3.1. Let the set $\mathcal{D}$ be described as above. Then

(i) For every $z \in \mathbb{H}$, $\gamma z \in \mathcal{D}$ for some $\gamma \in \Gamma$

(ii) Let z_1, $z_2 \in \mathcal{D}$ be congruent modulo Γ. Then, either $\mathrm{Re}(z_1) = \mp 1/2$ and $z_1 = z_2 \pm + - 1$ or $\|z_1\| = 1$ and $z_2 = -1/z_1$.

Proof. (i) Let $z = x + iy \in \mathbb{H}$ and $\epsilon > 0$. It is straightforward to see that $\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}\, z}{|cz+d|^2}$ for all $z \in \mathbb{H}$ and $\gamma \in \Gamma$. One can also easily see that the set

$\{(c,d) \in \mathbb{Z} \times \mathbb{Z} \mid |cz+d| < \epsilon\}$ is finite since $|cz+d|^2 = (cx+d)^2 + c^2y^2 = c^2(x^2+y^2) + d^2 < \epsilon$ for only finite number of integers c and d. This allows us to say that there can be found some $\gamma \in \Gamma$ such that $\mathfrak{Im}(\gamma z)$ is maximum.

Note that $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ translates every element 1 unit to the right while T^{-1} translates them 1 unit to the left. Thus $-1 \leq \mathrm{Re}(T^n\gamma z) \leq 1$ for some $n \in \mathbb{Z}$.

Now let us assume that $|T^n\gamma z| < 1$ and apply $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $\mathrm{Im}(ST^n\gamma z) = \frac{\mathrm{Im}(T^n\gamma z)}{|T^n\gamma Z|^2} = \frac{\mathrm{Im}(\gamma z)}{|T^n\gamma z|^2}$ will imply that $\mathrm{Im}(ST^n\gamma z) > \mathrm{Im}(\gamma z)$ which is not possible because of the maximality of $\mathrm{Im}(\gamma z)$. Therefore $|T^n\gamma z| \geq 1$.

(ii) Let $z_1, z_2 \in \mathcal{D}$ and without loss of generality assume that $\mathrm{Im}\, z_2 \geq \mathrm{Im}(z_1)$. By the assumption that z_1 and z_2 are congruent, $z_2 = \gamma z_1$ for some $\gamma \in \Gamma$. Since $\mathrm{Im}(\gamma z_1) = \frac{\mathrm{Im}(z_1)}{|cz+d|^2} \geq \mathrm{Im}(z_1)$ we obtain $|cz_1 + d| \leq 1$. Thus, c can only take the values 0, 1 or -1.

When $c = 0$, $d = \pm 1$. So $\gamma = \left(\begin{smallmatrix} 1 & b \\ 0 & \pm 1 \end{smallmatrix}\right)$. But since $\frac{-1}{2} \leq \mathrm{Re}(z_1), \mathrm{Re}(\gamma z_1) \leq \frac{1}{2}$ and $z_1 \neq z_2$, we get $b = \pm 1$ which gives $\mathrm{Re}(z_1) = \frac{-1}{2}$ and $\mathrm{Re}(\gamma z_1) = \frac{1}{2}$ or the other way around.

When $c = 1$ we get $|z_1 + d| \leq 1$. If $z_1 = e^{\frac{2\pi}{3}}$ then $d = 0$ or 1.

$d = 0$ implies $|z_1| = 1$ and $\gamma = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$. So $\gamma z_1 = \frac{az_1 - 1}{z} = a - \frac{1}{z_1}$. Since $|\frac{-1}{z_1}| = 1$ we have $a = 0$.

$d = 1$ implies $\gamma = \left(\begin{smallmatrix} a & b \\ 1 & 1 \end{smallmatrix}\right)$ where $a - b = 1$. So, $\gamma z_1 = \frac{az_1+b}{z_1+1} = \frac{a(z_1+1)-1}{z_1+1} = a - \frac{1}{z_1+1} = a + z_1$. Therefore we have $a = 0$ or 1. Now we check the case if $z_1 \neq \rho$. Then $d = 0$ since $|z_1 + d| \leq 1$. So $\gamma = \left(\begin{smallmatrix} a & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Therefore $\gamma z_1 = \frac{az_1 - 1}{z_1} = a - \frac{1}{z_1}$ which implies that $a = 0$.

When $c = -1$ all we have to do is to change the signs of a, b, c and d which does not change the action of γ on z_1.

□

Note that on the set $\mathcal{D}$ there are special points called *elliptic points* which by definition have nontrivial stabilizer subgroup of Γ (also called *isotropy subgroups*). Recall that *the stabilizer subgroup* of $x \in X$ in G where the group G acts on the set X is $\{g \in G \mid gx = x\}$.

Remark 1.3.2. Let $\rho = e^{\frac{\pi i}{3}}$ so that $\rho - 1 = \rho^2 = e^{\frac{2\pi i}{3}}$. Elliptic points in the domain $\mathcal{D}$ are i, ρ and ρ^2 whose stabilizer subgroups are generated by the elements S, ST and

TS, respectively. Therefore the order of the stabilizer subgroup of i is 2 while orders of the other subgroups are 3. (Recall Proposition 1.2.6.)

Notice also that under the action of the matrix T^n where $n \in \mathbb{Z}^+$, $T^n z \in \mathbb{H}$ just represents the n-times translation of z to right, while $T^n z$ represents n-times translation to the left in $\mathbb{H}$ if $n \in \mathbb{Z}^-$. On the other hand, under the action of the matrix S, the orbit Sz represents the symmetry of z with respect to the y-axis but whose norm is factored by $\frac{1}{|z|}$.

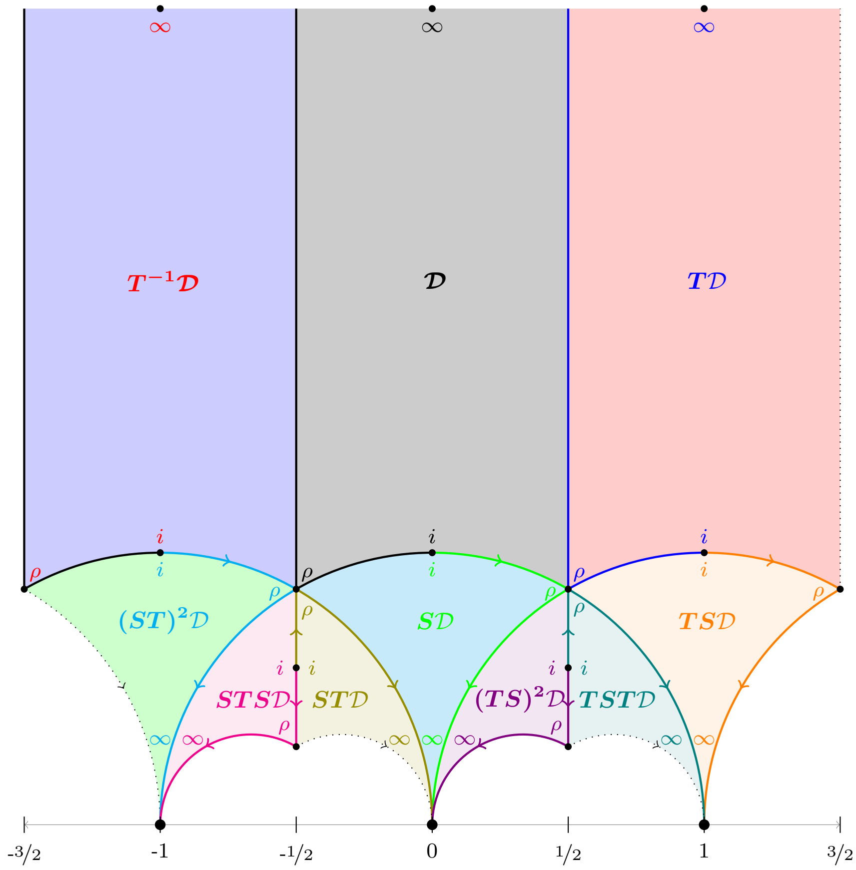

Figure 1.3: Γ-translates of $\mathcal{D}$

Finally with the help of theorem 1.3.1 and considering Γ-action on the set $\mathcal{D}$, one can define **a fundamental domain**

$$\widetilde{\mathcal{D}} = \{z \in \mathcal{D} \mid -1/2 < \mathrm{Re}(z) \text{ and } \|z\| > 1 \text{ if } \mathrm{Re}(z) < 0\}$$

which has exactly one element from each Γ-orbit.

So far we have treated the set $\mathcal{D}$ only as a set, but to be able to grasp the full geometric picture with all the essential properties, one needs to put a topology on it.
First of all, it is very natural to consider the quotient space $\Gamma\backslash\mathbb{H}$ which gives us the space $\mathcal{D}$ except the points either on the right side or left side of the boundary, i.e., $\{z \in \mathcal{D} \mid \mathrm{Re}(z) < 1/2$ and $\|z\| > 1$ if $\mathrm{Re}(z) > 0\}$ or $\{z \in \mathcal{D} \mid -1/2 < \mathrm{Re}(z)$ and $\|z\| > 1$ if $\mathrm{Re}(z) < 0\}$, respectively. On the other hand, in order to get a useful topology on $\mathcal{D}$ we identify the opposite sides of the boundary, i.e., the line $\mathrm{Re}(z) = -1/2$ with the other line $\mathrm{Re}(z) = 1/2$ in $\mathcal{D}$ and half-arc $\{z \in \mathcal{D} \mid \|z\| = 1, -1/2 \leq \mathrm{Re}(z) \leq 0\}$ with the other half-arc $\{z \in \mathcal{D} \mid \|z\| = 1, 0 \leq \mathrm{Re}(z) \leq 1\}$. Finally, by also adding an extra point $i\infty$ to the identification set $\mathcal{D}$, which will be denoted as $\overline{\mathcal{D}}$, it could also be given a natural structure under which $\overline{\mathcal{D}}$ is a compact Riemann surface of genus 0. (See Theorem 3 in [10])

CHAPTER 2

MODULAR FORMS

2.1 Definitions and Elementary Examples

In the previous chapter, it has first been shown how the Γ-action on the set $\mathbb{H}$ forms a moduli space for elliptic curves on $\mathbb{C}$. Then with the help of doing certain identifications on the boundary of the domain $\mathcal{D}$ and adding the point $i\infty$ to this very special set, a compact Riemann surface $\overline{\mathcal{D}}$ of genus 0 has been obtained. This is just the beginning of the road. Now, the next step would be to consider (holomorphic) sections of a line bundle over this domain $\overline{\mathcal{D}}$ and see what these will look like. But before constructing such sections with the necessary conditions, it would be wise to see them also in terms of *elliptic curves and lattices*.

Definition 2.1.1. Let E_1 and E_2 be two distinct elliptic curves with the base points $\mathcal{O}_1$ and $\mathcal{O}_2$ serving as identity elements, respectively. Then a non-constant morphism (of elliptic curves) $\varphi : E_1 \to E_2$ is called an **isogeny** if $\varphi(\mathcal{O}_1) = \mathcal{O}_2$.

If there is an isogeny between E_1 and E_2, then E_1 and E_2 are called $isogenous$. Furthermore if there exists two isogenies $\varphi_1 : E_1 \to E_2$ and $\varphi_2 : E_2 \to E_1$ so that $\varphi_1 \circ \varphi_2$ and $\varphi_2 \circ \varphi_1$ are equal to corresponding identity maps, then E_1 and E_2 will be isomorphic.

Corollary 2.1.2. Let E_1 and E_2 be two complex elliptic curves such that $E_1 \cong \mathbb{C}/\Lambda_1$ and $E_2 \cong \mathbb{C}/\Lambda_2$ where Λ_1 and Λ_2 are the corresponding lattices. Then E_1 and E_2 are isomorphic if and only if Λ_1 and Λ_2 are homothetic.

Proof. See §VI Corollarry 4.1.1 in [7]. $\square$

Remark 2.1.3. $\overline{\mathcal{D}}$ is a compact Riemann surface (of genus 0), so there exists no non-constant holomorphic function on this domain.

Because of the reason above, one should consider meromorphic functions. But the problem is that this class of functions is way too general and has no flexibility in terms of doing interesting arithmetic. Therefore one needs to try something different through which meromorphic functions could also be represented. In order to achieve this we will put a "*weight*" to those functions and change the transformation property a bit which enables us to consider holomorphic functions safely. However to grasp the idea behind this process, let us recall a well-known example from the projective plane and functions on it.

Recall that the complex projective plane $\mathbb{P}(\mathbb{C})$ is defined as the quotient space

$$(\mathbb{C} \times \mathbb{C} \backslash \{(0,0)\})/\mathbb{C}^*$$

that is $\lambda \sim \lambda'$ if $\lambda = k\lambda'$ for some $k \in \mathbb{C}^*$ where $\lambda, \lambda' \in \mathbb{C}$. Therefore all functions on this projective plane $\mathbb{P}(\mathbb{C})$ has to satisfy the condition $f(\lambda X, \lambda Y) = f(X, Y)$ for all $\lambda \in \mathbb{C}^*$. In order to obtain rational functions $f(X,Y) = \frac{g(X,Y)}{h(X,Y)}$ on $\mathbb{P}(\mathbb{C})$, we need the function $(x,y) \mapsto f(x,y)$ to satisfy the same rule. If g, $h \in \mathbb{C}[X,Y]$ are homogeneous polynomials of the same degree, i.e., for all $\lambda \in \mathbb{C}^*$ $g(\lambda X, \lambda Y) = \lambda^d g(X,Y)$ and $h(\lambda X, \lambda Y) = \lambda^d h(X,Y)$ for some degree d, then f will be a rational function on $\mathbb{P}(\mathbb{C})$.

When considering the setup above for the *"modular functions"*, one needs to define a new function that works like homogeneous polynomials as in the above case. With this motivation, let us take a function F which is from defined from the set of lattices to $\mathbb{C}$ and change the translation property from $F(\lambda\Lambda) = F(\Lambda)$ for all $\lambda \in \mathbb{C}^*$ to $F(\lambda\Lambda) = \lambda^{-k}F(\Lambda)$ such that k will be called $weight$ of the function F. Now let us investigate how this change translates into the language of functions f from $\mathbb{H}$ to $\mathbb{C}$. For this, let us first construct a bijection from the set of functions on the set of lattice functions to the set of functions on $\mathbb{H}$. Let

$S = \{f : \mathbb{H} \to \mathbb{C} \mid f(\gamma z) = f(z) \ \forall \gamma \in \Gamma\}$
$S' = \{F : \{\Lambda \mid \Lambda \text{ is a lattice in } \mathbb{C}\} \to \mathbb{C} \mid F(\lambda\Lambda) = F(\Lambda) \ \forall \lambda \in \mathbb{C}^*\}.$

Then by using the map

$$S \to S' \ , \ f(z) \mapsto F(\Lambda) = F(\langle w_1, w_2 \rangle) := f(w_1/w_2)$$

one could obtain a bijection.

With the help of the correspondence between f and F one may easily deduce

$$\begin{aligned} f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = F\left(\langle \gamma z, 1\rangle\right) = F\left(\left\langle \frac{az+b}{cz+d}, 1 \right\rangle\right) \\ = (cz+d)^k F(\langle az+b, cz+d\rangle) \end{aligned}$$

where λ is taken as $\frac{1}{cz+d}$ and $\gamma \in \Gamma$.
By also using proposition 1.2.3, we also know that $\langle az+b, cz+d\rangle = \langle z, 1\rangle$ through which we get $f(\gamma z) = (cz+d)^k f(z)$.
Note also that the converse is also true, namely, the weight condition

$$f(\gamma z) = (cz+d)^k f(z)$$

implies a similar weight condition

$$F(\lambda\Lambda) = \lambda^{-k} F(\Lambda)$$

Let $f(\gamma z) = (cz+d)^k f(z)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and F be the lattice function defined by $F(\Lambda) = F(\langle z_1, z_2\rangle) := z_2^{-k} f(\frac{z_1}{z_2})$. Then

$$F(\lambda\Lambda) = F(\lambda\langle z_1, z_2\rangle) = (\lambda z_2)^{-k} f(\frac{z_1}{z_2}) = \lambda^{-k} f(\frac{z_1}{z_2}) z_2^{-k} = \lambda^{-k} F(\Lambda)$$

Therefore F satisfies some sort of homogeneity condition with weight $-k$.

With the aid of the above setup, one can represent meromorphic functions on $\widetilde{\mathcal{D}}$ as the quotients of holomorphic functions on $\mathbb{H}$ satisfying the translation property $f(\gamma z) = (cz+d)^k f(z)$ where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $z \in \mathbb{H}$. Thus one can immediately define a *modular form* as below.

Definition 2.1.4. A complex-valued function $f : \mathbb{H} \to \mathbb{C}$ is called a **modular form of weight** k where $k \in \mathbb{Z}$ if the following conditions are satisfied

(i) f is holomorphic on $\mathbb{H}$

(ii) $f(\gamma z) = (cz+d)^k f(z)$ for every $z \in \mathbb{H}$ and $\gamma \in \Gamma$ where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

(iii) f is holomorphic at $i\infty$

One can immediately notice that there exists no nonzero modular form of odd weight since

$$f(z) = f\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)z\right) = (-1)^k f(z)$$

. Note also that a modular form f of weight 0 can only be a constant function otherwise f would be Γ-invariant because of the translation property and it has been earlier mentioned that any Γ-invariant holomorphic function on the compactified space $\overline{\mathcal{D}}$ must be constant.

At the end of this section, it will be shown that there exists no nonzero modular form of any negative weight and weight 2. Therefore only nontrivial examples of modular forms start from weight 4. Now let us give some nontrivial examples of modular forms. First example one could think of would be the famous *Eisenstein series* which is defined below:

Definition 2.1.5.

$$G_k(z) = \sum_{(c,d)\in(\mathbb{Z}^2)^*} \frac{1}{(cz+d)^k}$$

is called the *Eisenstein series of weight* k, where $k > 2$ and $(\mathbb{Z}^2)^* = \mathbb{Z}^2 - \{(0,0)\}$.

When one encounters some infinite series, the first question to be asked would be whether that series is convergent or not, therefore we give the following lemma and corollary.

Lemma 2.1.6.

(i) The following series

$$\sum_{(c,d)\in(\mathbb{Z}^2)^*} \frac{1}{(\sup\{|c|,|d|\})^k}$$

converges for integers $k \geq 3$. ([11])

(ii) Let $\beta_1, \beta_2 \in \mathbb{R}^+$ and

$$\mathcal{F} = \{z \in \mathbb{H} : |\operatorname{Re}(z)| \leq \beta_1,\ |\operatorname{Im}(z)| \geq \beta_2\}.$$

Then for all $\delta \in \mathbb{R}$, $z \in \mathbb{H}$, $|z+\delta| > K\sup\{1,|\delta|\}$ for some positive real number K. ([11])

Proof.

(i) Consider the partial sums

$$S_n = \sum_{(n_1,n_2)} \frac{1}{(\sup\{|n_1|,|n_2|\})^k} \qquad \text{where} \quad (n_1,n_2) \in \{(\mathbb{Z}^2)^* : |n_1|,|n_2| \leq n\}.$$

Note that sum is taken over the sets $\{(n_1,n_2) \in (\mathbb{Z}^2)^* : |n_1|,|n_2| \leq n\}$ which cover all $(\mathbb{Z}^2)^*$ as $n \to \infty$.

Observe that

$$S_n = 8\sum_{i=1}^{n} \frac{1}{i^{k-1}}$$

which immediately implies

$$\sum_{(c,d)\in(\mathbb{Z}^2)^*} \frac{1}{(\sup\{|c|,|d|\})^k} = \lim_{n\to\infty} S_n = \sum_{i=1}^{\infty} \frac{8}{i^{k-1}}$$

Notice that the series is convergent when $k \geq 3$.

(ii) See [11, Chapter 1]. ☐

Two important results are given below by using this lemma.

Corollary 2.1.7. Let $\mathcal{F}$ be a domain in $\mathbb{H}$ as above with some $\beta_1, \beta_2 \in \mathbb{R}^+$. Then,

(i) $G_k(z)$ converges absolutely and also uniformly on $\mathcal{F}$.

(ii) $G_k(z)$ is bounded on $\mathcal{F}$.

Proof.

(i)

$$\begin{aligned} G_k(z) &= \sum_{(c,d)\in(\mathbb{Z}^2)^*} \frac{1}{(cz+d)^k} = \sum_{d\in\mathbb{Z}\setminus\{0\}} \frac{1}{d^k} + \sum_{\substack{c\neq 0\\ d\in\mathbb{Z}}} \frac{1}{(cz+d)^k} \\ &= 2\zeta(k) + \sum_{\substack{c\neq 0\\ d\in\mathbb{Z}}} \frac{1}{(cz+d)^k} \end{aligned}$$

where $\zeta(k)$ is the Riemann zeta function. Then

$$\sum_{(c,d)\in(\mathbb{Z}^2)^*} \left|\frac{1}{(cz+d)^k}\right| \leq 2\,|\zeta(k)| + \sum_{\substack{c\neq 0\\ d\in\mathbb{Z}}} \frac{1}{|cz+d|^k} .$$

Since $\zeta(k)$ is bounded for $k \geq 3$ and when $c \neq 0$,

$$|cz + d| = |c||z + d/c| > |c|.C \sup\{1, |d/c|\}$$

by (ii) of corollary above, one can deduce that

$$\begin{aligned}\sum_{(c,d)\in(\mathbb{Z}^2)^*} \left|\frac{1}{(cz+d)^k}\right| &< 2|\zeta(k)| + \sum_{\substack{c\neq 0\\ d\in\mathbb{Z}}} \frac{1}{C^k(\sup\{1, |d/c|\})^k} \\ &= 2|\zeta(k)| + \sum_{\substack{c\neq 0\\ d\in\mathbb{Z}}} \frac{1}{C^k(\sup\{|c|, |d|\})^k}\end{aligned}$$

From (i) of lemma above, we obtain the absolute convergence of $G_k(z)$ on $\mathcal{F}$. Also, notice that the estimates above do not depend on the point $z \in \mathcal{F}$, hence the convergence is uniform.

(ii) Since

$$|G_k(z)| = \left|\sum_{(c,d)\in(\mathbb{Z}^2)^*} \frac{1}{(cz+d)^k}\right| \leq \sum_{(c,d)\in(\mathbb{Z}^2)^*} \left|\frac{1}{(cz+d)^k}\right|$$

and by using the calculations above, it is easily implied that G_k is bounded on $\mathcal{F}$.

□

Remark 2.1.8.

(i) The map

$$(\mathbb{Z}^2)^* \to (\mathbb{Z}^2)^*\ , \quad (c', d') \to \gamma(c', d') = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(c', d') = (ac' + bd', cc' + dd')$$

is a bijection.

(ii) Any given compact subset in $\mathbb{H}$ can be placed into a domain $\mathcal{F}$ defined in (ii) of Lemma 2.1.6 for some $\beta_1, \beta_2 > 0$. Therefore with the aid of (i) from the same lemma, G_k is holomorphic on $\mathbb{H}$.

(iii) G_k is $\mathbb{Z}$-invariant on $\mathbb{H} \cup \{\infty\}$, i.e., $G_k(z+1) = G_k(z)$ for any $z \in \mathbb{H} \cup \{\infty\}$, because of the bijectivity given in (i) of this remark. Therefore any point z, including ∞, could be moved to a suitable $\mathcal{F}$ which implies that G_k is bounded as $\mathrm{Im}(z) \to \infty$.

After this remark, there is only one criterion left, namely the translation property, to prove that G_k is a modular form of weight k.

$$\begin{aligned}G_k(\gamma z) &= \sum_{(c',d')\in(\mathbb{Z}^2)^*} \frac{1}{(c'\gamma z + d')^k} = \sum_{(c',d')\in(\mathbb{Z}^2)^*} \frac{1}{(c'(\frac{az+b}{cz+d}) + d')^k} \\ &= (cz+d)^k \sum_{(c',d')\in(\mathbb{Z}^2)^*} \frac{1}{((c'a+cd')z + (c'b+dd'))^k} \\ &= (cz+d)^k G_k(z)\end{aligned}$$

since the multiplication map $(c', d') \mapsto \gamma(c', d')$ where $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ is a bijection from $(\mathbb{Z}^2)^*$ to itself (Remark 2.1.8).

What makes *Eisenstein series* so important and useful in terms of modular forms (hence for number theory and algebraic geometry) is that $G_4(z)$ and $G_6(z)$ algebraically generate all modular forms. But before giving this theorem, let us make some crucial remarks about the set of all modular forms of some fixed weight k which will be denoted by M_k.

Remark 2.1.9.

(i) M_k is a $\mathbb{C}$-vector space.

(ii) If $f \in M_{k_1}$ and $g \in M_{k_2}$, then $fg \in M_{k_1+k_2}$ and $f/g \in M_{k_1-k_2}$, where g has no zeros on $\mathbb{H} \cup \{i\infty\}$ which yields $M = \bigoplus_{k\in\mathbb{Z}} M_k$ being a graded ring.

Proof.

(i) It is a straightforward proof that the set M_k satisfies all the vector space properties.

(ii) Product and quotient of two holomorphic functions on $\mathbb{H} \cup \{i\infty\}$ are holomorhic on the same domain as well and since

$$(fg)(\gamma z) = f(\gamma z)g(\gamma z) = (cz+d)^{k_1} f(z)(cz+d)^{k_2} g(z) = (cz+d)^{k_1+k_2}(fg)(z)$$
$$\left(\frac{f}{g}\right)(\gamma z) = \frac{f(\gamma z)}{g(\gamma z)} = \frac{(cz+d)^{k_1} f(z)}{((cz+d)^{k_2} g(z)} = (cz+d)^{k_1-k_2}\left(\frac{f}{g}\right)(z)$$

the result is immediate. $\square$

Now one may give two very important results determining the structure of the complex vector space M_k.

Theorem 2.1.10.

(i) If $f \in M_k$, then f can be written as a linear combination of the monomials in the Eisenstein series G_4 and G_6 of weights 4 and 6, respectively, that is

$$f = \sum_{4\alpha+6\beta=k} c_{\alpha,\beta} G_4^{\alpha} G_6^{\beta}$$

where α, β are non-negative integers and $c_{\alpha,\beta} \in \mathbb{C}$.

(ii)

$$\dim M_k = \begin{cases} \lfloor k/12 \rfloor & \text{if } k \equiv 2 (\bmod 12) \\ \lfloor k/12 \rfloor + 1 & \text{if } k \not\equiv 2 (\bmod 12) \end{cases}$$

Proof. (See [8, Chapter 6]) □

One could instantly obtain some direct results for weights $k < 12$ by using the theorem above. It has been mentioned earlier that there is no nonzero modular form of odd weight which could be confirmed by using this theorem as well. It might also be deduced immediately that there is no non-constant weight 0 modular forms. Also since there are no non-negative integers such that $4\alpha + 6\beta = 2$, there does not exist weight 2 modular forms other than the zero function. For weights $k = 4, 6, 8$ and 10, a modular form f is just a multiple of the monomials G_4, G_6, G_4^2 and $G_4 G_6$, respectively. So one might say that for $k < 12$ there is not much interesting stuff in the vector spaces M_k. The fist interesting vector space of modular forms to investigate would be M_{12} which has dimension 2. If $f \in M_{12}$, then f will be a $\mathbb{C}$-linear combination of $(G_4)^3$ and $(G_6)^2$. There is yet a special modular form Δ-function, also called the *discriminant function*, of weight 12 which will be explicitly constructed in the next section. What makes this Δ-function so special is that $\lim \Delta(z) = 0$ as $\mathrm{Im}(z) \to i\infty$. But before going into further details, let us make a new characterization of modular forms that enables one to express them as Fourier series.

Recall that a modular form f of weight k is holomorphic on $\mathbb{H}$ and at $i\infty$. Therefore these special functions might be written in terms of Fourier series near the points of

$\mathbb{H} \cup i\{\infty\}$. For this purpose let us define the two maps below.

$$q : \mathbb{H} \to \{z \in \mathbb{C}^* : \|z\| < 1\}, \ z \mapsto q(z) = e^{2\pi i z}$$
$$g : \{z \in \mathbb{C}^* : \|z\| < 1\} \to \mathbb{C}, \ z \mapsto f\left(\frac{\log z}{2\pi i}\right)$$

where $f(z)$ is a modular form of weight k. Note that $f = g \circ q$ through which we will be able to express f as a Fourier series near the origin.
Let $z = re^{i(\theta+2\pi n)}$ and $z' = r'e^{i(\theta'+2\pi m)}$. Then $z = z'$ implies that $z' = re^{i(\theta+2\pi k)}$. Furthermore one obtains that

$$\begin{aligned} g(z') = f\left(\frac{\log z'}{2\pi i}\right) &= f\left(\frac{\log r + i(\theta + 2\pi k)}{2\pi i}\right) = f(\frac{\log r + i\theta}{2\pi i + k}\right) = f\left(\frac{\log r + i\theta}{2\pi i}\right) \\ &= f\left(\frac{\log z}{2\pi i}\right) = g(z) \end{aligned}$$

since the modular form $f(z)$ is a $\mathbb{Z}$-periodic function, i.e., $f(z + k) = f(z)$ where $k \in \mathbb{Z}$. Thus the second map g is well-defined. Also note that both q and g are holomorphic maps and holomorphicity at the point $i\infty$ is equivalent to the holomorphicity at $q = 0$ since $q \to 0$ as $\mathrm{Im}(z) \to \infty$. Finally one obtains the result that every modular form f of weight k can be written as a Fourier series

$$f(z) = \sum_{n=0}^{\infty} c_n q^n$$

where $q = e^{2\pi i z}$. This expression is called the q-expansion of f.

2.2 Cusp Forms and the Discriminant Function Δ

In this section the goal is to introduce *cusp forms* of weight k which will constitute a highly important subspace of the vector space M_k of modular forms of weight k. To pave the way we will present the *discriminant function* which is the first nontrivial cusp form.
Previously the *discriminant function* has been briefly mentioned which will be denoted by Δ (also called Δ-function). We will now give the definition of this function and express it in terms of its q-expansion. So as to give a proper definition and an explicit construction for this special function one needs some useful identities.

Proposition 2.2.1. (see [11, Chapter 1])

(i)
$$\frac{1}{z}+\sum_{d=1}^{\infty}\left(\frac{1}{z-d}+\frac{1}{z+d}\right)=\pi\cot(\pi z)=\pi i\left(1-2\sum_{m=0}^{\infty}q^m\right)$$
where $q=e^{2\pi i z}$.

(ii)
$$G_k(z)=2\zeta(k)+2\frac{(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$
where $\sigma_k(n)=\sum\limits_{\substack{m|n\\ m>0}} m^k$ and $\zeta(k)=\sum\limits_{d=1}^{\infty}\frac{1}{d^k}$ is the Riemann zeta function.

Proof.

(i) Recall that
$$\sin\pi z=\pi z\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})$$
Then
$$\begin{aligned}\pi\cot\pi z&=\frac{\mathrm{d}}{\mathrm{d}z}(\log(\sin\pi z))=\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\left(\pi z\prod_{n=1}^{\infty}(1-\frac{z^2}{n^2})\right)\right)\\&=\frac{\mathrm{d}}{\mathrm{d}z}\left(\log(\pi z)+\sum_{n=1}^{\infty}\log\left(\frac{n^2-z^2}{n^2}\right)\right)\\&=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}\end{aligned}$$
gives the desired result for the first identity.
For the second identity one can simply use
$$\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2}\quad\text{and}\quad\sin(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2i}$$

(ii) In order to prove this identity one could differentiate (i) $(k-1)$ times and rearrange the terms. □

Recall from (i) in Theorem 2.1.7 that every modular form can be expressed in a unique form by using G_4 and G_6. Now let us define
$$g_2(z)=60G_4(z)\text{ and }g_3(z)=140G_6(z)$$
through which we give the definition below.

Definition 2.2.2. The *discriminant function* is defined as $\Delta(z) = g_2^3(z) - 27g_3^2(z)$

Corollary 2.2.3.

(i) $\Delta(z)$ is a modular form of weight 12.

(ii) The first coefficient in the q-expansion of $\Delta(z)$ equals to 0, i.e., $\lim\limits_{\mathrm{Im}(z)\to\infty} \Delta(z) = 0$.

Proof.

(i) Since $G_4(z)$ and $G_6(z)$ are modular forms of weight 4 and 6, respectively, and $G_4^3(z), G_6^2(z) \in M_{12}$

$$\begin{aligned}\Delta(\gamma z) &= g_2^3(\gamma z) - 27g_3^2(\gamma z) = (60G_4(\gamma z))^3 - 27(140G_6(\gamma z))^2 \\ &= 60^3((cz+d)^4G_4(z))^3 - 27.(140^2)((cz+d)^6G_6(z))^2 \\ &= (cz+d)^{12}(60^3G_4^3(z) - 27140^2G_6^2(z)) = (cz+d)^{12}(g_2^3(z) - 27g_4^2(z)) \\ &= (cz+d)^{12}\Delta(z).\end{aligned}$$

So translation property is satisfied.
It follows immediately that $\Delta(z)$ is holomorphic on $\mathbb{H}$ and at $i\infty$ since both G_4 and G_6 are holomorphic on on $\mathbb{H}$ and at $i\infty$.
Therefore $\Delta(z)$ is a modular form of weight 12.

(ii) In order to investigate the limiting behaviour of g_2 and g_4 as $\mathrm{Im}(z) \to \infty$, using first part of the last proposition yields immediately $g_2 \to 120\ \zeta(4)$ and $g_3 \to 280\ \zeta(6)$. Also since $\zeta(4) = \frac{\pi^4}{90}$ and $\zeta(6) = \frac{\pi^6}{945}$ it is easy to see that

$$g_2 \to \frac{4}{3}\pi^4 \text{ and } g_4 \to \frac{8}{27}\pi^6 \quad \text{as } \mathrm{Im}(z) \to \infty$$

which quickly follows $\Delta(z) = g_2^3(z) - 27g_3^2(z) \to 0$ as $\mathrm{Im}(z) \to \infty$. □

Next theorem explicitly gives the $q-$expansion of $\Delta(a)$ in product form.

Theorem 2.2.4. (see §2.4 in [12])

$$\Delta(z) = (2\pi)^{12}\sum_{n=1}^{\infty}\tau(n)q^n = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

where $\tau(n)$ is defined as *Ramanujan's tau* function.

It is again notable to mention that the Δ-function is the first non-trivial example of a modular form belonging to a vector space of dimension greater than 1 and having 0 as the first coefficient in its q-expansion. Now it is time to bring up one of the key subjects of this thesis, namely *cusp forms*.

Definition 2.2.5. Let $f(z)$ be a modular form of weight k such that first coefficient in its $q-$ expansion is 0, that is

$$f(z) = \sum_{n=1}^{\infty} c_n q^n$$

where $q = e^{2\pi iz}$. Then $f(z)$ is called a *cusp form* of weight k.

Note that $c_0 = 0$ is equivalent to the condition that $f(z) \to 0$ as $\mathrm{Im}(z) \to \infty$ and $\Delta(z)$ is a cusp form.

The set of all cusp forms of weight k which will be denoted by S_k forms a subspace in the vector space M_k and it has 1 less dimension than M_k has.

Proposition 2.2.6.

(i) $S_k \subseteq M_k$ is a subspace.

(ii) $\dim S_k = 0$ for $k = 2, 4, 6, 8, 10$ and $\dim S_k = \dim M_k - 1$ for $k \geq 12$, i.e.,

$$\dim S_k = \begin{cases} \lfloor k/12 \rfloor - 1 & \text{if } k \equiv 2 (\mathrm{mod} 12) \\ \lfloor k/12 \rfloor & \text{if } k \not\equiv 2 (\mathrm{mod} 12) \end{cases}$$

Proof.

(i) For $k = 2$ there is nothing to show since $M_2 = 0$. For other weights $k < 12$ and $k = 14$, recall from (ii) of Theorem 2.1.7 that M_4, M_6, M_8, M_{10} and M_{14} are of dimension 1. Besides from (i) of the same theorem it is known that $M_4, M_6, M_8, M_{10}, M_{14}$ are generated by the Eisenstein series $G_4, G_6, G_4^2, G_4G_6, G_4^2G_6$, respectively. Finally, since Eisenstein series are non-cusp forms, i.e., first coefficients in their q-expansions are $2\zeta(k)$ which are non-zero where $\zeta(k)$ is the Riemann zeta function, 0 function is the only cusp form for these weights.

For other weights, $S_k \neq \emptyset$ since 0 function is always a cusp form of all weights. Also it is obvious that addition of two cusp forms of the same weight is a cusp form and any multiple of a cusp form is again a cusp form. However there is rather a more elegant way to prove this and it also gives an exact formula for the dimension. Let

$$\phi : M_k \to \mathbb{C}, \quad f \mapsto f(i\infty)$$

It is not difficult to see that ϕ is a linear map since

$$\phi(f + f') = \phi(\sum_{n=0}^{\infty} c_n q^n + \sum_{n=0}^{\infty} c'_n q^n) = c_0 + c'_0 = \phi(f) + \phi(f').$$

where $f,\ f' \in M_k$ with q-expansions $\sum\limits_{n=0}^{\infty} c_n q^n$ and $\sum\limits_{n=0}^{\infty} c'_n q^n$, respectively. For this linear map note that kernel is exactly S_k, therefore S_k is subspace.

(ii) By using the linear map ϕ above it is trivial to see that

$$\dim M_k = \dim \ker \phi + \dim \operatorname{Image} \phi = \dim S_k + 1$$

where $k \geq 12$ and $k \neq 14$ and $\dim S_k = 0$ for $k = 2, 4, 6, 8, 10$. □

Recall from (i) of Theorem 2.1.7 any modular form of weight k can be expressed in a unique form by using G_4 and G_6. Yet one can give another practical characterization by using the subspace S_k of cusp forms. Moreover, any cusp form of weight k may be expressed as a linear combination of Eisentein series and the Δ function.

Lemma 2.2.7.

(i) For $k \geq 2$, $M_k = \langle G_k \rangle \bigoplus S_k$ where $\langle G_k \rangle$ is the subspace generated by G_k.

(ii) Let us assume that $f \in M_k$ with $k \geq 0$ and $G_0 = 1$ (recall that weight 0 modular forms are only constant functions). Then f has a unique representation which is composed of Eisentein series of weight k and Δ function, explicitly

$$f = \sum_{\substack{n=0 \\ k-12n \neq 2}}^{\lfloor \frac{k}{12} \rfloor} b_n \, G_{k-12n} \, \Delta^n$$

More specifically if $f \in S_k$ then $b_0 = 0$.

Proof.

(i) It is not difficult to see that $S_k \cap \langle G_k \rangle = 0$ for a fixed $k \geq 2$ since first coefficient in the q-expansion of an Eisenstein series of weight k is non-zero.
For $k < 12$ it is obvious since there is no cusp forms of weight $k < 12$. For $k \geq 12$, we have two subspaces S_k and $\langle G_k \rangle$ of M_k such that their intersection contains only 0 function and their dimensions are $(\dim M_k) - 1$ and 1, respectively. Therefore the result is immediate.

(ii) See §6.4 in [8]. □

Till now we have mostly been interested in the structure of the vector space of modular forms (respectively, cusp forms) of a given weight k, its dimension, expression of a modular form in terms of Eisenstein series and Δ function and the q-expansion of a modular form. What we are going to do as a next step is now to investigate their zeros. For this we have a powerful theorem below which gives an exact formula for the total number of zeros (counted with their multiplicities) that is dependent on weight k and says that the set of all zeros on the fundamental domain $\mathcal{D}$ is finite. But before that let us recall what the *order of vanishing* of a function at a point is.

Definition 2.2.8. Let $f : \mathbb{H} \to \mathbb{C}$ be a meromorphic function not identically zero and $z_0 \in \mathbb{H}$. The unique integer n such that $\frac{f(z)}{(z-z_0)^n}$ is holomorphic and non-vanishing at z_0 is called the *order of vanishing* of f at the point $z = z_0$ which will be denoted by $\nu_{z_0}(f)$. If $n > 0$, then z_0 will be a *zero* and if $n < 0$, then z_0 will be a pole of the function f.

Also note that since zeros and poles in $\mathbb{C}$ are isolated the above definition also applies for a disk around the point z_0. For the cusp point $i\infty$ we will denote the order of vanishing by $\nu_\infty(f)$.

Lemma 2.2.9.

(i) Let $f \in M_k \backslash S_k$. Then $\nu_\infty(f) = 0$. Specifically if $f \in S_k$ then $\nu_\infty(f) \geq 1$.

(ii) If $z_0 \in \mathbb{H}$ and $f \in M_k$ then $\nu_{z_0}(f) = \nu_{\gamma z_0}(f)$ for all $\gamma \in \Gamma$, i.e., investigating the order of vanishing $\nu_{z_0}(f)$ of f at the points $z_0 \in \overline{\mathcal{D}}$ will suffice.

Proof.

(i) Let $f \in M_k \backslash S_k$. Then f can be written as

$$f = \sum_{n=0}^{\infty} c_n q^n$$

where $c_0 \neq 0$ which automatically implies $\nu_\infty(f) = 0$ since $\operatorname{Im} z \to i\infty$ is equivalent to $q \to 0$.

If $f \in S_k$ then the q-expansion of f starts from at least 1 which completes the proof.

(ii) Let $\nu_{z_0}(f) = n$ where $z_0 \mathbb{H}$ and f is a modular form of weight k. Then $\frac{f(z)}{(z-z_0)^n}$ is holomorphic and nonzero at z_0 which also implies $\lim_{z \to z_0} \frac{f(z)}{(z-z_0)^n} \neq 0$. Also a simple calculation shows that $\gamma z - \gamma z_0 = (\frac{z-z_0}{(cz+d)(cz_0+d)})^n$ where $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$. Then

$$\begin{aligned}
\lim_{z \to \gamma z_0} \frac{f(z)}{(z - \gamma z_0)^n} &= \lim_{z \to z_0} \frac{f(\gamma z)}{(\gamma z - \gamma z_0)^n} \\
&= \lim_{z \to z_0} \frac{1}{(\gamma z - \gamma z_0)^n} f(z)(cz+d)^k \ \ (\text{since} f(\gamma z) = (cz+d)^k f(z)) \\
&= \lim_{z \to z_0} f(z)(cz+d)^k \frac{(cz+d)^n (cz_0+d)^n}{(z - z_0)^n} \\
&= \lim_{z \to z_0} \frac{f(z)}{(z - z_0)^n} (cz_0 + d)^{2n+k}
\end{aligned}$$

which is nonzero since $z_0 \neq \frac{-d}{c}$, i.e., z_0 is not a cusp point.

(Recall $f(z) = (cz+d)^{-k} f(\gamma z)$ implies all the cuspidal points are in $\mathbb{Q} \cup \{\infty\}$, where $k > 0$).

Holomorphicity is clear. $\square$

Now let us give the following powerful theorem which is also known as *valence formula*.

Theorem 2.2.10. Let f be a nozero modular form of weight k. Then

$$\nu_\infty(f) + \frac{1}{3}\nu_\rho(f) + \frac{1}{2}\nu_i(f) + \sum_{\substack{z_0 \in \Gamma/\mathbb{H} \\ z_0 \neq \rho, i}} \nu_{z_0}(f) = \frac{k}{12}$$

where $\rho = e^{\frac{2\pi i}{3}}$.

Proof. See §1.3 in [12]. $\square$

In the previous theorem note that the numbers $\nu_\rho(f)$ and $\nu_i(f)$ are weighted with the coefficients $\frac{1}{3}$ and $\frac{1}{2}$, respectively, while the others are not. This is because all the

points in $\overline{\mathcal{D}}$ have trivial stabilizer subgroups except ρ and i whose stabilizer subgroups have order 3 and 2, respectively (Recall Remark 1.3.2).
Recall that all modular forms of weight k are holomorphic on $\mathbb{H} \cup \{i\infty\}$ which yields $\nu_{z_0}(f) \geq 0$ for all $z_0 \in \Gamma\backslash\mathbb{H}$. So one may notice that it is a direct consequence of the valence formula that there exists no nonzero modular forms of negative weight. Furthermore one can also say that there is no modular form of weight 2 since $\frac{n}{3} + \frac{m}{2} \neq \frac{1}{6}$ for all $n, m \in \mathbb{Z}^+$.

2.3 Petersson Inner Product and Hecke Operators

In the previous section it has been mentioned that the vector space M_k of modular forms of weight k is finite dimensional and even has a specific dimension formula for each k. Furthermore it has been given that $M = \bigoplus_{k\in\mathbb{Z}} M_k$ forms a graded ring structure. Evidently the subspace $S_k \subset M_k$ which consists of all cusp forms of weight k possesses a very similar structure. In this part we will try to put a well-founded measure and an inner product on this subspace. After then we will define certain operators, namely *Hecke operators*, through which one obtains a basis for S_k in the end. Now let us start with a well-known measure which is called *hyperbolic measure*.

Definition 2.3.1. Let $d\mu(z) = \frac{dxdy}{y^2}$ where $z = x + iy$. Then define *hyperbolic measure* as

$$\mu(A) = \int_A d\mu(z)$$

where $A \subset \mathbb{H}$ is a measurable set.

The main reason why one uses this measure is that it is invariant under the action of Γ on $\mathbb{H}$, even more generally, it is invariant under $GL_2^+(\mathbb{R})$.

Remark 2.3.2. $\mu(\gamma A) = \mu(A)$ where $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma$ and A is a measurable subset of $\mathbb{H}$.

Proof. Let $\gamma z = \frac{az+b}{cz+d} = u + iv$. Since $\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$

$$\mu(\gamma A) = \int\limits_{\gamma A} \frac{dudv}{v^2} = \int\limits_{A} \frac{|cz+d|^4}{y^2} \frac{\partial(u,v)}{\partial(x,y)} dxdy$$

Recall that $\frac{\partial(u,v)}{\partial(x,y)} = u_x v_y - u_y v_x$. But using Cauchy-Riemann equations gives $\frac{\partial(u,v)}{\partial(x,y)} = (v_x)^2 + (v_y)^2$. Then simple calculations show that $\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{|cz+d|^4}$ which completes the proof of the hyperbolic measure μ defined above being Γ-invariant. $\square$

Now let us give the definition of a specific inner product, namely *Petersson inner product*, through which S_k will gain an orthonormal basis.

Definition 2.3.3. Let $f(z), g(z) \in S_k$. Then *Petersson inner product* is defined as

$$\langle f(z)|g(z)\rangle = \int\limits_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} y^k d\mu(z) = \int\limits_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)} y^k \frac{dxdy}{y^2}$$

It is clear that Petersson inner product is linear in its first argument and conjugate-symmetric which automatically implies the conjugate-linearity in its second argument. Positive-definiteness is also clear.
Note that this special inner product is very much like the one that we use for the vector space of complex-valued functions but with a difference of the factor y^k. The reason is that one would want it to be Γ-invariant. Moreover there is a reason why Petersson inner product is defined for cusp forms, and not for modular forms in general. But before explaining this let us give the proof of Petersson inner product being Γ-invariant.

Remark 2.3.4. Petersson inner product is Γ-invariant, i.e., for any $\gamma \in \Gamma$

$$\langle f(z)|g(z)\rangle = \langle f(\gamma z)|g(\gamma z)\rangle$$

Proof.

$$\begin{aligned}\langle f(\gamma z)|g(\gamma z)\rangle &= \int\limits_{\gamma(\Gamma\backslash\mathbb{H})} f(\gamma z)\overline{g(\gamma z)}(\mathrm{Im}\,\gamma z)^k d\mu(\gamma z) \\ &= \int\limits_{\Gamma\backslash\mathbb{H}} (cz+d)^k (\overline{cz+d})^k f(z)\overline{g(z)} \frac{y^k}{|cz+d|^{2k}} d\mu(z)\end{aligned}$$

since $\mathrm{Im}(\gamma z) = \frac{y}{|cz+d|^k}$ and $d\mu(z)$ is Γ-invariant. Then the result is immediate.

□

Proposition 2.3.5. Let $f(z), g(z) \in M_k$ such that at least one of them is a cusp form. Then

$$\int\limits_{\Gamma\backslash\mathbb{H}} f(z)\overline{g(z)}(\mathrm{Im}(z))^k \, d\mu(z)$$

is convergent.

Proof. Let us express both f and g as Fourier series, i.e.,

$$f(z) = \sum_{n=0}^{\infty} c_n q^n \qquad \text{and} \qquad g(z) = \sum_{m=0}^{\infty} d_m q^m$$

where $q = e^{2\pi i z}$.

Without loss of generality, say $f(z)$ is a cusp form and $g(z)$ is not. Then the series of the product $f(z)g(z)$ starts from at least $n = 1$, in other words it is a cusp form. Then

$$\begin{aligned}
|f(z)\overline{g(z)}| = \left|\sum_{n=1}^{\infty} c_n q^n \left(\overline{\sum_{m=0}^{\infty} d_m q^m}\right)\right| &= \left|\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}(c_n\overline{d_m})q^n(\overline{q})^m\right| \\
&\leq \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} |c_n\overline{d_m}|e^{-2\pi y(n+m)} \\
&= \sum_{r=1}^{\infty} s_r e^{-2\pi y r}, \qquad \text{where } s_r \geq 0
\end{aligned}$$

since $|q| = |\overline{q}| = e^{-2\pi y}$. Take the domain $\widehat{D} = \{z \in \widetilde{D} : \mathrm{Im}\, y > 1\}$. Then

$$\left|\int\limits_{\widehat{D}} f(z)\overline{g(z)}y^k \, d\mu(z)\right| \leq \int\limits_{\widehat{D}} \sum_{r=1}^{\infty} s_r e^{-2\pi y r} y^{k-2} \, dx\, dy = \int\limits_{1}^{\infty} \left(\sum_{r=1}^{\infty} s_r e^{-2\pi y r} y^{k-2}\right) \, dy$$

since $\frac{-1}{2} \leq x < \frac{1}{2}$. Then one can also find for some $r \geq 1$

$$\begin{aligned}
\int\limits_{1}^{\infty} e^{-2\pi y r} y^{k-2} \, dy = e^{-2\pi r} &\int\limits_{1}^{\infty} e^{-2\pi r(y-1)} y^{k-2} \, dy \\
&\leq e^{-2\pi r} \int\limits_{1}^{\infty} e^{-2\pi(y-1)} y^{k-2} \, dy \\
&= Ce^{-2\pi r}
\end{aligned}$$

since the last integral converges to some constant C. So this will automatically imply that

$$\left| \int_{\hat{D}} f(z)\overline{g(z)}y^k \right| \leq \sum_{r=1}^{\infty} s_r C e^{-2\pi r}$$

Note that last series is convergent since it is exactly $C \sum_{r=1}^{\infty} s_r e^{-2\pi y r}$ when $y = 1$ which converges for all $y \geq 1$.

Therefore $\int_{\widehat{D}} f(z)\overline{g(z)}(\mathrm{Im}(z))^k \, d\mu(z)$ is convergent. Also for the rest of the domain, namely $\widetilde{D} \backslash \widehat{D}$, convergence of the integral is immediate since $\widetilde{D} \backslash \widehat{D}$ is compact and $f(z)\overline{g(z)}(\mathrm{Im}(z))^k$ is a continuous function.

□

Furthermore there is a more general case which basically says integrating a continuous and bounded function from the upper half plane to $\mathbb{C}$ composed with a $\gamma \in \Gamma$ with respect to hyperbolic measure over $\overline{D}$ is also convergent. It is given by the remark below.

Remark 2.3.6. Let $\widetilde{D}$ be the usual fundamental domain. Then

(i)

$$\mathrm{Vol}(\widetilde{D}) = \int_{\widetilde{D}} d\mu(z) = \int_{\mathbb{H}} d\mu(z) = \frac{\pi}{3}$$

(ii) Let $\psi : \mathbb{H} \to \mathbb{C}$ be a bounded and continuous function. Then

$$\int_{\overline{D}} \psi(\gamma z) \, d\mu(z)$$

converges for any $\gamma \in \Gamma$.

Proof. Recall that $\widetilde{D}$ is the fundamental domain defined in the first chapter and $\overline{D}$ is the identification of this space compactified by adding $i\infty$.

(i) Using the Γ-invariance of the hyperbolic measure

$$\int_{\widetilde{D}=\Gamma\backslash\mathbb{H}} d\mu(z) = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{dy\,dx}{2} = \int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{3}$$

(ii) See [11, Chapter 5] $\square$

Finally, it is clearer now why one defines Petersson inner product specifically on $S_k \times S_k$.

Now let us define the so-called *Hecke operators* which basically send a modular form f of weight k to another modular form of the same weight through a specific summation which will be shown below. These operators provide us very important results in regard to the vector space structures of both M_k and S_k when combined with the Petersson inner product. Before giving the formulation of *Hecke operators* in terms of modular forms, let us express what their correspondence is in the view of lattice functions F on the set of lattices of rank 2.

Recall that a lattice function F of rank 2 is defined on the set of all lattices in $\mathbb{C}$ such that F assigns a lattice Λ to a complex number. A sub-lattice Λ' of index m of a lattice Λ is defined as a subgroup of the group Λ with finite index m. A basic example is that the lattice $\langle i, 1\rangle$ has 3 subgroups of index 2 which are $\langle 2i, 1\rangle$, $\langle i, 2\rangle$ and $\langle i+1, 2\rangle$. Now for a fixed index m, let T_m denote the operator so that

$$\mathrm{T_m}\, F(\Lambda) = \sum_{\Lambda' \leq \Lambda} F(\Lambda')$$

where summation is taken over all the sub-lattices of index m of Λ.

It is clear that $\mathrm{T_m}$ defines an operator on the set of lattice functions S_3 which was defined at the beginning of §2.1, in other words, $\mathrm{T_m}$ assigns a lattice function F to another lattice function such that translation property is preserved for any weight k, i.e., $\mathrm{T_m}\, F(\lambda\Lambda) = \lambda^{-k}\, \mathrm{T_m}\, F(\Lambda)$. It is also not hard to show that $\mathrm{T_m}$ is a linear operator on S_3.

In order to translate the above operator into the language of modular forms, we recall that in the beginning of §2.1, the sets S_2 and S_3 were defined and a bijective map $F \mapsto f(z) := F(\Lambda_z)$, where $\Lambda_z = \langle z, 1\rangle$, was built between them. Also with the help of lattice theory, one defines a *Hecke operator* $\mathrm{T_m}$ for $m \geq 1$ and a fixed weight k as the translation of the operator above into the language of modular forms. In other words,

$$\mathrm{T_m}\, f(z) = m^{k-1} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \backslash M_m(\mathbb{Z})} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

where $M_m = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Z}) : ad - bc = m\}$.
Here one notes that the full modular group Γ acts on the group $M_m(\mathbb{Z})$. Also, notice that m^{k-1} is put for the purpose of normalization so that a modular form $f(z)$ having integer Fourier coefficients is sent to another modular form again having integer Fourier coefficients.

This formula maybe be transformed to a more useful form below (See [12, §4.1]).

Definition 2.3.7. Let $m \in \mathbb{Z}^+$ and $f \in M_k$. Then a *Hecke operator* on the vector space M_k of modular forms with weight k is defined as

$$\mathrm{T_m}\, f(z) = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} \frac{1}{d^k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right)$$

For the purposes of doing some arithmetics, sometimes the above form could also be written below

$$\mathrm{T_m}\, f(z) = \frac{1}{m} \sum_{\substack{ad=m \\ 0 \le b < d \\ a,d>0}} a^k f\left(\frac{az+b}{d}\right)$$

Notice that for primes p,

$$\mathrm{T_p} f(z) = p^{k-1} f(pz) + \frac{1}{p} \sum_{b=0}^{p-1} f\left(\frac{z+b}{p}\right)$$

which contains only two sums.

Hecke operators possess various useful properties through which one understands much more about the structure of the vector spaces M_k and S_k, and one can construct a basis for S_k. But first thing to be checked would be that they are indeed operators on both M_k and S_k.

Theorem 2.3.8. (See [8, Theorem 6.11]) For a modular form f of weight k, $\mathrm{T_m}\, f$ is also a modular form of the same weight, where $m \in \mathbb{Z}^+$. This property is also satisfied for the cusp forms, i.e, if $g \in S_k$, then $\mathrm{T_m}\, g \in S_k$.

Let $f \in M_k$ and $m \in \mathbb{Z}^+$. Then since $\mathrm{T_m} f \in M_k$ it must have a q-expansion whose Fourier coefficients are related with the Fourier coefficients of f by the below theorem.

Theorem 2.3.9. Let f be a modular form of weight k such that it has a q-expansion

$$f(z) = \sum_{n=0}^{\infty} c_n q^n$$

where $q = e^{2\pi i z}$. Then $\mathrm{T_m} f$ will have a q-expansion

$$\mathrm{T_m} f(z) = \sum_{n=0}^{\infty} \left(\sum_{d \,|\, gcd(n,m)} d^{k-1} c_{mn/d^2} \right) q^n$$

Proof. Let $f \in M_k$ and $\mathrm{T_m}$ be the Hecke operator corresponding to $m \in \mathbb{Z}^+$. Then

$$\begin{aligned}
\mathrm{T_m} f(z) &= m^{k-1} \sum_{ad=m} \frac{1}{d^k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right) \\
&= \sum_{d|m} \left(\frac{m}{d}\right)^{k-1} \sum_{b=0}^{d-1} \frac{1}{d} \sum_{n=0}^{\infty} c_n e^{2\pi i n(\frac{az+b}{d})} \\
&= \sum_{n=0}^{\infty} \sum_{d|m} \left(\frac{m}{d}\right)^{k-1} \frac{1}{d} c_n e^{2\pi i \frac{az}{d} n} \sum_{b=0}^{d-1} e^{2\pi i \frac{b}{d} n}
\end{aligned}$$

It is clear that the finite sum $\sum\limits_{b=0}^{d-1} e^{2\pi i \frac{b}{d} n} = d$ if d divides n.

In the case of $d \nmid n$, it is easy to see that $\sum\limits_{b=0}^{d-1} \left(e^{2\pi i \frac{n}{d}}\right)^b = \frac{1-\left(e^{2\pi i \frac{n}{d}}\right)^d}{1-e^{2\pi i \frac{n}{d}}} = 0$. So

$$\mathrm{T_m} f(z) = \sum_{n=0}^{\infty} \sum_{\substack{d|m \\ d|n}} \left(\frac{m}{d}\right)^{k-1} c_n e^{2\pi i a z \frac{n}{d}}.$$

Since $d \mid n$, say $n = dd'$ for some $d' \in \mathbb{Z}^+$. Also since $ad = m$, put $a = \frac{m}{d}$. Then we obtain

$$\mathrm{T_m} f(z) = \sum_{d'=0}^{\infty} \sum_{d|m} \left(\frac{m}{d}\right)^{k-1} c_{dd'} e^{2\pi i z d' \frac{m}{d}}$$

Since $d \mid m$, d can be replaced by $\frac{m}{d}$ which yields

$$\begin{aligned}
\mathrm{T_m} f(z) &= \sum_{d'=0}^{\infty} \sum_{d|m} d^{k-1} c_{md'/d} e^{2\pi i z d' d} \\
&= \sum_{n=0}^{\infty} \sum_{\substack{d|m \\ d|n}} d^{k-1} c_{mn/d^2} q^n
\end{aligned}$$

So we obtain the result.

$\square$

2.4 Simultaneous Hecke Eigenforms and a Basis for S_k

In this section our goal is to find a proper basis for the vector space S_k of cusp forms of weight k. In order to achieve this goal, we will basically take advantage of the fact that Hecke operators are Hermitian with respect to the Petersson inner product and commute with each other, which will be given in more details, and make use of eigenvalues and eigenfunctions of the Hecke operators $\mathrm{T_m}$ for $m \in \mathbb{Z}^+$. After that, we will define so-called *random modular forms* in the next chapter.

Theorem 2.4.1. (See [8, §6.10])

Let $\mathrm{T_m}$ and $\mathrm{T_n}$ be two Hecke operators defined on the vector space M_k of modular forms of weight k. Then

$$\mathrm{T_m}\,\mathrm{T_n} = \sum_{d|gcd(m,n)} d^{k-1}\mathrm{T}_{\frac{\mathrm{mn}}{\mathrm{d}^2}} .$$

This theorem has two very important consequences given in the corollary below.

Corollary 2.4.2.

(i) Hecke operators commute with each other, i.e.,

$$\mathrm{T_m}\,\mathrm{T_n} = \mathrm{T_n}\,\mathrm{T_m}$$

for any $m, n \in \mathbb{Z}^+$.

(ii) $\mathrm{T_m}(\mathrm{T_n}) = \mathrm{T_{mn}}$, for any coprime $m, n \in \mathbb{Z}^+$.

Proof.

(i) By using the formula above, it is easy to see the commutativity of Hecke operators.

(ii) Suppose that m and n are coprimes, so that $d = 1$ for $d \mid gcd(m, n)$.
Therefore the proof follows immediately. $\square$

Earlier it has been mentioned that Hecke operators are well-defined on the vector space S_k of cusp forms of weight k as well. In the previous corollary we said that Hecke operators are commutative on the space M_k of modular forms of weight k, so it applies also for the space S_k. Yet there is another nice property about Hecke operators on S_k.

Theorem 2.4.3. Let $\mathrm{T_m}$ be a Hecke operator on S_k for a fixed $m \in \mathbb{Z}^+$. Then it is self-adjoint with respect to the Petersson inner product, namely,

$$\langle \mathrm{T_m}\, f, g\rangle = \langle f, \mathrm{T_m}\, g\rangle$$

for any cusp forms $f, g \in S_k$.

Proof. Recall that $\mathrm{T_m}$ is defined as

$$(\mathrm{T_m}\, f)(z) = m^{k-1} \sum_{\substack{ad=m \\ a,d>0}} \frac{1}{d^k} \sum_{b=0}^{d-1} f\left(\frac{az+b}{d}\right).$$

So for $m = 1$, the proof is obvious since $\mathrm{T_1}$ is the identity operator.
For $m \geq 2$, see [13, §3.4]. □

Now since we know that Hecke operators on the space S_k of modular forms of weight k are self-adjoint with respect to the Petersson inner product, due to the spectral theorem, one can easily say that S_k has an orthonormal basis whose elements are eigenvectors of some Hecke operators, which are also called *eigenfuctions*. From now on we will refer to these eigenfunctions as *Hecke eigenforms*. Moreover self-adjointness implies that all eigenvalues of Hecke operators are real.

Finally, for the vector space S_k we have an orthonormal basis composed of Hecke eigenforms. However there is more to say about those eigenforms since using (i) of corollary 2.4.2 yields also that one can find a set of eigenforms which simultaneously diagonalize S_k. In other words, a set of simultaneous Hecke eigenforms, which are by definition common eigenvectors for all Hecke operators, exist such that they form an orthonormal basis for S_k.

An immediate example for simultaneous Hecke eigenforms could be given for 1-dimensional spaces of modular forms. For instance, Eisenstein series G_4 is a common eigenform for all Hecke operators $\mathrm{T_m}$ for $m \in \mathbb{Z}^+$, since $\mathrm{T_m}\, f$ is again a modular form of weight 4 by theorem 2.3.8 and all modular forms of weight 4 is just multiples of G_4 because of theorem 2.1.10. This example could be extended to all spaces M_k and S_k with dimension 1.

As of now, we will heavily use simultaneous Hecke eigenforms, therefore it would be great to develop some useful properties about these special eigenforms. So, recall that

$$\mathrm{T_m}\, f = \sum_{n=0}^{\infty} \left(\sum_{d|gcd(m,n)} d^{k-1} c_{mn/d^2} \right) q^n$$

where $f \in M_k$ with $f = \sum\limits_{n=0}^{\infty} c_n q^n$ and $\mathrm{T_m}$ is the Hecke operator for $m \in \mathbb{Z}^+$.
Now let us investigate first two coefficients of $\mathrm{T_m}\, f$.
If $n = 0$, then

$$\sum_{d|m} d^{k-1} c_0 = \sigma_{k-1} c_0 \,.$$

where $\sigma_k = \sum\limits_{d|k} d^k$ is the usual divisor function.
If $n = 1$, we obtain

$$\sum_{d|gcd(m,n)} d^{k-1} c_{mn/d^2} = c_m \,.$$

One can obtain an interesting result by using the last identity.

Lemma 2.4.4. (see [8, §6.15] Let $f \in S_k$ be a nonzero cusp form of weight k with the q-expansion

$$f = \sum_{n=1}^{\infty} c_n q^n \,.$$

Then, f is a simultaneous Hecke eigenform for all Hecke operators $\mathrm{T_m}$ if and only if c_m is an eigenvalue for $\mathrm{T_m}$ where $m \in \mathbb{Z}^+$.

Proof. Let f be a simultaneous Hecke eigenform, i.e., $\mathrm{T_m}\, f = \lambda_m f$ for some λ where $m \in \mathbb{Z}^+$. Then, by using the last identity above we obtain c_m as the second Fourier coefficient in the q-expansion of $\mathrm{T_m}\, f$. Since f is a simultaneous Hecke eigenform we have

$$\mathrm{T_m}\, f = \lambda_m f = \lambda_m \sum_{n=1}^{\infty} c_n q^n = \lambda_m c_1 q + \sum_{n=2}^{\infty} \lambda_m c_n q^n \,.$$

Therefore we obtain $\lambda_m c_1 = c_m$.
In the case of $c_1 = 0$ one gets all $c_m = 0$ for $m \geq 1$ which implies $f = 0$, but we have assumed that f is an eigenform. Therefore $\lambda_m = \frac{c_m}{c_1}$ which implies that c_m is an eigenvalue for all $m \geq 1$.
Now let us assume that c_m is an eigenvalue for all Hecke operator $\mathrm{T_m}$ where $m \geq 1$. Let us fix m and say f' is the corresponding eigenform for $\mathrm{T_m}$, namely $\mathrm{T_m}\, f' = c_m f'$. Let

$$f' = \sum_{n=1}^{\infty} c'_n q^n$$

be its q-expansion. Therefore

$$\mathrm{T_m}\, f' = \sum_{n=1}^{\infty} \left(\sum_{d|gcd(n,m)} d^{k-1} c'_{mn/d^2} \right) q^n = c_m f' = \sum_{n=1}^{\infty} c_m c'_n q^n \, .$$

First coefficient in the q-expansion of $\mathrm{T_m}\, f'$ will be c'_m which implies $c'_m = c_m c'_1$. Since this applies for all $m \geq 1$, we easily obtain

$$f' = \sum_{n=1}^{\infty} c'_n q^n = \sum_{n=1}^{\infty} c_n c'_1 q^n = c'_1 f \, .$$

Again with the same reasons above, $c'_1 \neq 0$, so $f = \frac{f'}{c'_1}$ which completes the proof. □

Notice that in the proof above one finds out also that the first coefficient c_1 in the q-expansion of a simultaneous Hecke eigenform $f = \sum\limits_{n=1}^{\infty} c_n q^n$ is nonzero. This can also be easily proven for any modular form of weight $k \geq 4$ by using the same technique above.
From now on, we will always normalize simultaneous Hecke eigenforms by dividing them with the first Fourier coefficient so that $c_1 = 1$.

Corollary 2.4.5. $f \in S_k$ is a normalized simultaneous Hecke eigenform provided that the identity

$$c_m c_n = \sum_{d|gcd(m,n)} d^{k-1} c_{mn/d^2}$$

is satisfied for the Fourier coefficients of f where $m, n \in \mathbb{Z}^+$.

Proof. Proof immediately follows from the above lemma. □

CHAPTER 3

REAL ZEROS OF A RANDOM MODULAR FORM

3.1 Introduction to Random Modular Forms and Real Zeros

In the previous chapter definitions and some important examples of modular forms and cusp forms were given. Afterwards, Petersson inner product was introduced for the vector spaces of modular forms and cusp forms so that one would have a better understanding of the vector space structure of these special functions. Finally, Hecke operators were defined on the space of modular forms through which the vector space S_k of cusp forms of some weight k gained an orthonormal basis because of the Hermitian property of Hecke operators with respect to the Petersson inner product on S_k. Recall that elements of this orthonormal basis are composed of eigenfunctions of all Hecke operators since they commute with each other. Such eigenfunctions were called *simultaneous Hecke eigenforms*.

After this nice and proper setup, one of the first questions which pops into one's mind would be that how we could take advantage of this orthonormal basis to determine significant results for the cusp forms $f \in S_k$. Investigating the number of zeros of generic functions as well as locations and distribution behaviours of them have been one of the major problems throughout the history of mathematics and algebraic geometry. Therefore our purpose will be first to find out the number of zeros of a *random modular form* and then how they distribute on the fundamental domain $\widetilde{D}$. To accomplish this, we will take an orthonormal basis of S_k whose existence is by the previous chapter, then we are going to "randomize" these basis elements by using i.i.d. (independently identically distributed) real random variables.

Before constructing random modular forms, let us fix soem notation. From now on F_j^k

will denote basis elements, which are simultaneous Hecke eigenforms as mentioned earlier, of the vector space S_k of weight k where $1 \leq j \leq \dim S_k$.
Recall that

$$\dim S_k = \begin{cases} \lfloor k/12 \rfloor - 1 & \text{if } k \equiv 2(\mathrm{mod} 12) \\ \lfloor k/12 \rfloor & \text{if } k \not\equiv 2(\mathrm{mod} 12) \end{cases}$$

where $k \geq 12$. Also remember that there is no non-zero cusp form for positive weights $k < 12$.

Definition 3.1.1. A **random modular form of weight** k is

$$f_k(z) = \sum_{j=1}^{\dim S_k} a_j F_j^k$$

where a_j's are i.i.d (independently identically distributed) real random variables and F_j^k's are simultaneous Hecke eigenforms which form an orthonormal basis for the vector space S_k.

Since we now have a definition of a random modular form, our task will be to look into the number zeros of these functions. However there is yet a special subset of the domain $\mathcal{D}$ such that all cusp forms, and hence all random modular forms, will be real-valued on this subset. Before giving the definition of this subset and reasoning of this phenomenon we need some tools.

Remark 3.1.2. Due to a classical result from linear algebra. all Hecke eigenforms are simultaneous eigenforms because of the commutativity of all Hecke operators. Therefore one could use the terms 'simultaneous Hecke eigenforms' and 'Hecke eigenforms' interchangeably.

Corollary 3.1.3. Let $f \in S_k$ be a Hecke eigenform such that its q-expansion is

$$f = \sum_{n=1}^{\infty} c_n q^n$$

where $q = e^{2\pi i z}$.
Then all Fourier coefficients c_n are real.

Proof. Let f be a normalized (first Fourier coefficient is 1) Hecke eigenform and $\mathrm{T_m}$ be a Hecke operator where $m \in \mathbb{Z}+$. Recall that the first coefficient of the cusp form $\mathrm{T_m}\, f$ is c_m which comes from

$$\mathrm{T_m}\, f = \sum_{n=0}^{\infty} \left(\sum_{d|gcd(m,n)} d^{k-1} c_{mn/d^2} \right) q^n \qquad \text{(see theorem 2.3.9)}$$

Let us denote the Fourier coefficients of $\mathrm{T_m}\, f$ as c'_n. Then $c'_1 = c_m$. By assumption f is a Hecke eigenform that is $\mathrm{T_m}\, f = \lambda_m f = \sum_{n=1}^{\infty} \lambda_m c_n q^n$ for some nonzero eigenvalue λ_m which implies $\lambda_m = \lambda_m c_1 = c'_1 = c_m$. Recall that all eigenvalues of Hecke eigenforms are real, thus c_m is real. The same proof can be done for all Hecke operators where $m \in \mathbb{Z}^+$ which completes the proof that all Fourier coefficients of a (normalized) Hecke eigenform are real. □

Proposition 3.1.4. ([14]) Let $f(z)$ be a Hecke eigenform. Then the following identities are satisfied

$$f(-\overline{z}) = f(1-\overline{z}) = \overline{f(z)} \ \text{ and } \ f\left(\frac{1}{\overline{z}}\right) = (\overline{z})^k \overline{f(z)}\,.$$

Proof. It is clear that if $z \in \mathbb{H}$ then $-\overline{z} \in \mathbb{H}$. It is also clear that $f(-\overline{z}) = f(1-\overline{z})$ since $f\left(T(-\overline{z})\right) = f(-\overline{z})$ for $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma$.
One can easily notice that if $z \mapsto q$ then $q \mapsto \overline{q}$ since

$$\begin{aligned} e^{2\pi i z} &= e^{-2\pi y}\left(\cos 2\pi x + i \sin 2\pi x\right) \\ e^{-2\pi i \overline{z}} &= e^{-2\pi i y}\left(\cos 2\pi x - i \sin 2\pi x\right) = \overline{\left(e^{2\pi i z}\right)} \end{aligned}$$

where $z = x + iy$.
Recall from corollary 3.1.3 that all Fourier coefficients of a Hecke eigenform are real, in other words $c_n = \overline{c_n}$ where $f = \sum_{n=1}^{\infty} c_n q^n$ is a Hecke eigenform. So one can immeadiately deduce that

$$f(-\overline{z}) = \sum_{n=1}^{\infty} c_n (\overline{q})^n = \sum_{n=1}^{\infty} \overline{c_n} (\overline{q})^n = \overline{f(z)}$$

The second identity easily follows

$$f\left(\frac{1}{\overline{z}}\right) = f\left(\frac{-1}{-\overline{z}}\right) = (-\overline{z})^k f(-\overline{z}) = (\overline{z})^k \overline{f(z)}$$

since $f\left(\frac{-1}{-\overline{z}}\right) = f\left(S(-\overline{z})\right)$ where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \Gamma$, k is even and $f(-\overline{z}) = \overline{f(z)}$. □

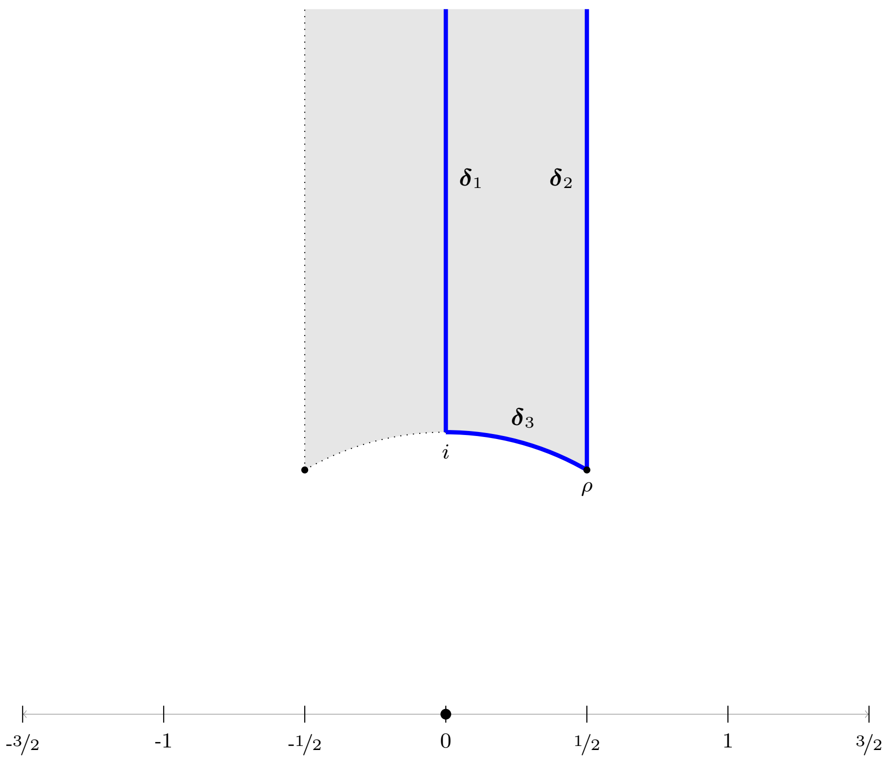

Figure 3.1: $\delta^* = \delta_1 \cup \delta_2 \cup \delta_3$

Let us define a subset $\delta^* = \delta_1 \cup \delta_2 \cup \delta_3$ of the fundamental domain $\widetilde{D}$ where

$$\delta_1 = \{z \in \widetilde{D} : \mathrm{Re}(z) = 0\,,\ \mathrm{Im}(z) \geq 1\}$$

$$\delta_2 = \{z \in \widetilde{D} : \mathrm{Re}(z) = \frac{1}{2}\,,\ \mathrm{Im}(z) \geq \frac{\sqrt{3}}{2}\}$$

$$\delta_3 = \{z \in \widetilde{D} : 0 \leq \mathrm{Re}(z) \leq \frac{1}{2}\,,\ |z| = 1\}$$

Now let us give a crucial corollary about Hecke eigenforms on δ^* through which we will be able to define 'real zeros' of a random modular form.

Corollary 3.1.5. ([15]) Let $f \in S_k$ be a Hecke eigenform. Then

(i) $f(z)$ is real-valued on $\delta_1 \cup \delta_2$

(ii) $z^{k/2} f(z)$ is real-valued on δ_3

Proof. Let $f(z)$ be a Hecke eigenform with the q-expansion

$$f(z) = \sum_{n=1}^{\infty} c_n q^n\,.$$

(i) One can easily see that on δ_1

$$q^n = e^{2\pi izn} = e^{-2\pi yn}$$

where $y \geq 1$.

Recall also that all Fourier coefficients c_n of a Hecke eigenform are real. Then $f(z)$ is a real-valued function on δ_1.

Similarly

$$q^n = e^{2\pi izn} = e^{2\pi i(\frac{1}{2}+iy)n} = (-1)^n e^{-2\pi yn}$$

where $y \geq \frac{\sqrt{3}}{2}$. Then $f(z)$ is real-valued on δ_2 as well.

(ii) First note that $\frac{1}{\overline{z}} = z$ on δ_3 which implies, on δ_3, $f(z) = f\left(\frac{1}{\overline{z}}\right) = (\overline{z})^k \overline{f(z)}$ because of the proposition 3.1.4. Then

$$\begin{aligned}\overline{(z^{k/2} f(z))} &= (\overline{z})^{k/2}\overline{f(z)} = (\overline{z})^{k/2}\frac{f(z)}{(\overline{z})^k} = \frac{1}{(\overline{z})^{k/2}} f(z)\frac{z^{k/2}}{z^{k/2}} = \frac{z^{k/2} f(z)}{|z|^k} \\ &= z^{k/2} f(z)\end{aligned}$$

since $|z| = 1$ on δ_3. So the result is immediate. □

Now we are ready to define "*real zeros of a random modular form*".

Definition 3.1.6. Let

$$f_k(z) = \sum_{j=1}^{\dim S_k} a_j F_j^k$$

be a random modular form of weight k where a_j are i.i.d. real random variables and F_j^k are Hecke eigenforms which form an orthonormal basis for S_k. Then zeros of $f_k(z)$ on δ^* will be called **real zeros**.

3.2 The Expected Number of Real Zeros of a Random Polynomial of Degree n

There have been many studies in the past few decades concerning the problems of locating all zeros of Hecke eigenforms and finding the number of zeros of these special functions on a fundamental domain. Rankin and Swinnerton-Dyer (see [5]) managed to find locations of all zeros of Eisenstein series which are on the arc of unit circle in the domain $\mathcal{D}$. However due to the structure of $\mathcal{D}$ with the identifications on the boundary, namely $\widetilde{\mathcal{D}}$, one can also directly say that all zeros of Eisenstein series lie on δ_3 which is depicted in the figure 3.1.

Now let us consider simultaneous Hecke eigenforms and try to investigate zeros of these functions. For this purpose first recall that $M_k = \langle G_k \rangle \bigoplus S_k$ and give an important proposition below.

Proposition 3.2.1. Let $f \in M_k \backslash S_k$, namely a modular form of weight k which is not a cusp form of the same weight. Also suppose that f is a simultaneous Hecke eigenform. Then f is a constant multiple of G_k.

Proof. Let us give a sketch of proof that basically uses a result from [8, §6.13] which is stated as

$$f(z) = \frac{(2k-1)!}{2(2\pi i)^k} G_k(z)$$

if and only if $f \in M_k \backslash S_k$ is a normalized simultaneous Hecke eigenform.
This result is due to the relation of Fourier coefficients with Hecke eigenvalues, that is

$$\mathrm{T_m}\, f = \sum_{n=0}^{\infty} \left(\sum_{d|gcd(m,n)} d^{k-1} c_{mn/d^2} \right) q^n = \lambda_m f = \sum_{n=0}^{\infty} c_n q^n \, .$$

and q-expansion of G_k

$$G_k(z) = \frac{(2k-1)!}{(2\pi i)^k} \zeta(k) + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

$\square$

Another way of interpreting this proposition is that all simultaneous Hecke eigenforms are either in a subspace generated by an Eisenstien series G_k or cusp forms. One can also recall that Hecke eigenforms in S_k coincide with simultaneous Hecke eigenforms. Since it is known that all zeros of an Eisenstein series G_k lie on δ_3, our focus will be on finding zeros of Hecke eigenforms in the space of S_k, especially real zeros of them.

We recall that all cusp forms can be written as a linear combination of Hecke eigenforms F_j^k. By assigning i.i.d real random variables on F_j^k, we have defined random modular forms. A natural candidate for the real random variables a_j are real Gaussian random variables where

$$f_k(z) = \sum_{j=1}^{\dim S_k} a_j F_j^k \,.$$

From now on we will be focusing on finding the "expected number" of real zeros of a random modular form. In order to solve this problem we will recall a solution of a similar problem given in the case of a "random polynomial of degree n".

Let f be **a random polynomial of degree** n with independent standard Gaussian random variables, namely,

$$f = \sum_{i=0}^{n} a_i x^i$$

where $a_i \sim \mathcal{N}(0, 1)$. Then an argument from the paper [16] suggests that the expected number of real zeros of a random polynomial of degree n with independent Gaussian random variables is exactly $\frac{1}{\pi}$ multiple of the length of the curve

$$\vec{\mathcal{R}} = \frac{\vec{r}(t)}{\|\vec{r}(t)\|}$$

where $\vec{r}(t) = [1, t, t^2, \ldots, t^n]^\top$ is the so-called **moment curve**. In other words, the expected number of real roots of a random polynomial f with independent standard normals is

$$\mathrm{E}[N_{real}(f(x))] = \frac{1}{\pi} L(\mathcal{R})$$

where $N_{real}(f(x))$ denotes the number of real roots of a random polynomial $f(x)$ of degree n and $L(\mathcal{R})$ is the length of the curve $\mathcal{R}$.

One may immediately notice that the moment curve is built from the canonical basis elements $[1, x, x^2, \ldots, x^n]$ of the function space of polynomials of degree n. It is also notable that the vectors $[a_0, a_1, a_2, \ldots, a_n]$ and $[1, t, t^2, \ldots, t^n]^\top$ are perpendicular to each other if and only if $x = t$ is a zero of the random polynomial $f(x)$. Indeed these two basic facts are key elements in the proof of the above formula (see [16]).

Remark 3.2.2.

(i) Let $\vec{a} = [a_0, a_1, a_2, \ldots, a_n]$ such that $a_i \sim \mathcal{N}(0, 1)$, namely a_i's are independent standard normals. Then $\frac{\vec{a}}{\|\vec{a}\|}$ on the n-dimensional sphere will be uniformly distributed

(ii) The formula $\mathrm{E}[N_{real}(f(x))] = \frac{1}{\pi}L(\mathcal{R})$ is applicable for any uniformly distributed random variables on the n-dimensional sphere.

Proof.

(i) Let $a_i \sim \mathcal{N}(0,1)$ and be independent. Then their joint probability density function is

$$\frac{1}{(\sqrt{2\pi})^n}e^{-\frac{1}{2}\|\vec{x}\|^2} = \frac{1}{(\sqrt{2\pi})^n}e^{-\frac{1}{2}(x_0^2+x_1^2+\cdots+x_n^2)}$$

where $x_i \in a_i$ for $0 \leq i \leq n$. Note that when restricted to the n-sphere S^n, this density function will be depending only on the radius by using the spherical coordinates. Therefore on S^n, $\frac{\vec{a}}{\|\vec{a}\|}$ is uniformly distributed.

(ii) See [16]. □

When working with the case of random polynomials of a fixed degree n, it is relatively easy to find a formula for the length of the curve $\vec{\mathcal{R}} = \frac{\vec{r}(t)}{\|\vec{r}(t)\|}$ which gives us the expected number of real zeros multiplied by π since we use only standard calculus and some algebra. We give the formula below.

Theorem 3.2.3. (See [17, theorem 2.1]) Let $f(x)$ be random polynomial of degree n with the independent standard normal variables, i.e, $f(x) = \sum_{i=0}^{n} a_i x^i$ where $a_i \sim \mathcal{N}(0,1)$ are independent. Then

$$\mathrm{E}[N_{real}(f(x))] = \frac{1}{\pi}\int_{-\infty}^{\infty}\sqrt{\frac{1}{(t^2-1)^2} - \frac{(n+1)^2t^{2n}}{(t^{2n+2}-1)^2}}\,dt$$

3.3 Moment Curve and the Expected Number of Real Zeros of a Random Modular Form

The ideas about finding the real zeros of a random polynomial from the previous section may show us the way to compute the expected number of real zeros of a random modular form. We recall that a random modular form of weight k

$$f_k(z) = \sum_{j=1}^{\dim S_k} a_j F_j^k$$

where a_j's are i.i.d. real random variables and F_j^k are (simultaneous) Hecke eigenforms so that they compose an orthonormal basis for S_k. Up to now we have not said anything specific about random variables a_j, however using standard normals, namely $a_j \sim N(0,1)$ with mean 0 and variance 1, would be a good point to start. Therefore from now on we will assume i.i.d. real random variables are distributed as standard normals. Also one may immediately notice that, when restricted to the sphere S^{dimS_k}, these random variables become uniformly distributed exactly as in remark 3.2.2. Thus the problem of finding the expected number of real zeros of a random modular form $f_k(z)$, which land on $\delta^* = \delta_1 \cup \delta_2 \cup \delta_3$ (see §3.1), reduces down to the problem of finding the arc-length of the normalized moment curve since

$$\mathrm{E}\left[N_{real}(f(x))\right] = \frac{1}{\pi} L(R)\,.$$

Proposition 3.3.1. Let $r(t) : \mathbb{R} \to \mathbb{R}^{n+1}$ be any differentiable curve so that $R(t) = \frac{r(t)}{\|r(t)\|}$ be a normalized curve on the sphere S^n. Then

$$\|R'(t)\|^2 = \frac{\partial^2}{\partial \tilde{y} \partial y} \log\left[\vec{r}(y) \cdot \vec{r}(\tilde{y})\right]\Big|_{y=\tilde{y}=t}$$

Proof. See [16, theorem 2.1]. □

Recall that $f(z)$ is real-valued on $\delta_1 \cup \delta_2$ while $z^{k/2} f(z)$ is real-valued on δ_3 for any Hecke eigenform $f \in S_k$. Therefore one can use the above formula for the moment curve

$$\vec{r}(z) = [F_1^k(z), F_2^k(z), \ldots, F_{\dim S_k}^k(z)]^\top$$

where F_j^k are Hecke eigenforms composing a basis for a random modular form $f_k(z)$ of weight k.

Our purpose is to compute $\frac{\partial^2}{\partial \tilde{z} \partial z} \log\left[\vec{r}(z) \cdot \vec{r}(\tilde{z})\right]\Big|_{z=\tilde{z}=t}$. For now let us restrict the computations to the segment δ_1. Let

$$F_j^k(z) = \sum_{n=1}^{\infty} c_n^j q^n \qquad \text{and} \qquad F_j^k(\tilde{z}) = \sum_{n=1}^{\infty} c_n^j (\tilde{q})^n$$

and where $q = e^{2\pi i z}$, $\tilde{q} = e^{2\pi i \tilde{z}}$ and $1 \leq j \leq \dim S_k$. Then

$$\log[\vec{r}(z) \cdot \vec{r}(\tilde{z})] = \log\left[\sum_{j=1}^{\dim S_k} \left(\sum_{n=1}^{\infty} c_n^j q^n\right)\left(\sum_{n=1}^{\infty} c_n^j (\tilde{q})^n\right)\right]$$

Since we work on δ_1 one can replace z with iy where $y > 1$ through which $q = e^{-2\pi y}$ and $\tilde{q} = e^{-2\pi\tilde{y}}$. We therefore notice that $\frac{\mathrm{d}}{\mathrm{d}y}q = -2\pi q$ and $\frac{\mathrm{d}}{\mathrm{d}\tilde{y}}\tilde{q} = -2\pi\tilde{q}$. Then

$$\frac{\partial}{\partial y}\log[\vec{r}(y)\cdot\vec{r}(\tilde{y})] = -2\pi\frac{\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j nq^n\right)\left(\sum_{n=1}^{\infty}c_n^j(\tilde{q})^n\right)}{\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j q^n\right)\left(\sum_{n=1}^{\infty}c_n^j(\tilde{q})^n\right)}$$

from which follows that the numerator of $\frac{\partial}{\partial\tilde{y}}\left(\frac{\partial}{\partial y}\log[\vec{r}(y)\cdot\vec{r}(\tilde{y})]\right)$ is

$$\begin{aligned}&(4\pi^2)\left[\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j nq^n\right)\left(\sum_{n=1}^{\infty}c_n^j n(\tilde{q})^n\right)\right]\left[\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j q^n\right)\left(\sum_{n=1}^{\infty}c_n^j(\tilde{q})^n\right)\right]\\&-(4\pi^2)\left[\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j nq^n\right)\left(\sum_{n=1}^{\infty}c_n^j(\tilde{q})^n\right)\right]\left[\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j q^n\right)\left(\sum_{n=1}^{\infty}c_n^j n(\tilde{q})^n\right)\right]\end{aligned}$$

while the denominator is

$$\left[\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j q^n\right)\left(\sum_{n=1}^{\infty}c_n^j(\tilde{q})^n\right)\right]^2$$

Put $y = \tilde{y} = t$ which implies $q^n = (\tilde{q})^n = e^{-2\pi tn}$.
For simplicity let

$$\sum_{n=1}^{\infty}c_n^j nq^n = a_j \qquad \text{and} \qquad \sum_{n=1}^{\infty}c_n^j q^n = b_j\,.$$

Then

$$\frac{\partial^2}{\partial\tilde{y}\partial y}[\log(\vec{r}(y)\cdot\vec{r}(\tilde{y}))]\Big|_{y=\tilde{y}=t} = (4\pi^2)\frac{\left(\sum_{j=1}^{\dim S_k}a_j^2\right)\left(\sum_{j=1}^{\dim S_k}b_j^2\right)-\left(\sum_{j=1}^{\dim S_k}a_jb_j\right)^2}{\left(\sum_{j=1}^{\dim S_k}b_j^2\right)^2}\,.$$

Let $\dim S_k = r$. Then the above formula will be

$$
(4\pi^2)\frac{\begin{pmatrix}
a_1^2b_1^2 + a_1^2b_2^2 + a_1^2b_3^2 + & \dots & +a_1^2b_r^2 + \\
a_2^2b_1^2 + a_2^2b_2^2 + a_2^2b_3^2 + & \dots & +a_2^2b_r^2 + \\
+ & \ddots & + \\
a_r^2b_1^2 + a_r^2b_2^2 + a_r^2b_3^2 + & \cdots & +a_r^2b_r^2 + \\
-(a_1b_1)^2 - a_1b_1a_2b_2 - a_1b_1a_3b_3 - & \dots & -a_1b_1a_rb_r \\
-a_2b_2a_1b_1 - (a_2b_2)^2 - a_2b_2a_3b_3 - & \dots & -a_2b_2a_rb_r \\
- & \ddots & - \\
-a_rb_ra_1b_1 - a_rb_ra_2b_2 - a_rb_ra_3b_3 - & \dots & -(a_rb_r)^2
\end{pmatrix}}{\left(\sum\limits_{j=1}^{r} b_j^2\right)^2}
$$

After cancellation and rearranging terms, we obtain

$$
(4\pi^2)\frac{\begin{pmatrix}
(a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + & \dots & +(a_1b_r - a_rb_1)^2 + \\
+(a_2b_3 - a_3b_2)^2 + & \dots & +(a_1b_r - a_rb_1)^2 + \\
+ & \ddots & + \\
+ & \dots & +(a_{r-1}b_r - a_{r-1}b_r)^2
\end{pmatrix}}{\left(\sum\limits_{j=1}^{r} b_j^2\right)^2}
$$

which can be formulated as

$$
(4\pi^2)\frac{\left[\sum\limits_{i=1}^{r-1}\sum\limits_{j=i+1}^{r}(a_ib_j - a_jb_i)^2\right]}{\left(\sum\limits_{j=1}^{r} b_j^2\right)^2}
$$

Now put the terms $\sum\limits_{n=1}^{\infty} c_n^j n q^n = a_j$ and $\sum\limits_{n=1}^{\infty} c_n^j q^n = b_j$ back and obtain that $\|R'(t)\|^2$ is

$$
(4\pi^2)\frac{\sum\limits_{i=1}^{r-1}\sum\limits_{j=i+1}^{r}\left(\sum\limits_{n=1}^{\infty} c_n^i n q^n \sum\limits_{n=1}^{\infty} c_n^j q^n - \sum\limits_{n=1}^{\infty} c_n^j n q^n \sum\limits_{n=1}^{\infty} c_n^i q^n\right)^2}{\left(\sum\limits_{j=1}^{r}\left(\sum\limits_{n=1}^{\infty} c_n^j q^n\right)^2\right)^2} \qquad (*)
$$

where $t \geq 1$, $q = e^{-2\pi t}$, $r = \dim S_k$ and $F_j^k = \sum\limits_{n=1}^{\infty} c_n^j q^n$

Now let us see what happens on δ_2. Similar to the previous case, one can replace z with $\frac{1}{2}+iy$ through which $q=(-1)e^{-2\pi y}$ and $\tilde{q}=(-1)e^{-2\pi\tilde{y}}$. Then we again obtain $\frac{\mathrm{d}}{\mathrm{d}y}q=-2\pi q$ and $\frac{\mathrm{d}}{\mathrm{d}\tilde{y}}\tilde{q}=-2\pi\tilde{q}$. So same process applies for the rest and we obtain (*) with the differences $t\geq\frac{\sqrt{3}}{2}$ and $q=(-1)e^{-2\pi t}$. Furthermore, since $q=(-1)e^{-2\pi t}$ on δ_2, (*) becomes

$$(4\pi^2)\frac{\sum\limits_{i=1}^{r-1}\sum\limits_{j=i+1}^{r}\left(\sum\limits_{n=1}^{\infty}(-1)^n c_n^i n q^n\sum\limits_{n=1}^{\infty}(-1)^n c_n^j q^n-\sum\limits_{n=1}^{\infty}(-1)^n c_n^j n q^n\sum\limits_{n=1}^{\infty}(-1)^n c_n^i q^n\right)^2}{\left(\sum\limits_{j=1}^{r}\left(\sum\limits_{n=1}^{\infty}(-1)^n c_n^j q^n\right)^2\right)^2} \tag{**}$$

where $t\geq\frac{\sqrt{3}}{2}$ and $q=e^{-2\pi t}$ as in the previous case.

Finally let us do the computations considering we work on δ_3 and therefore recall that

$$z^{\frac{k}{2}}F_j^k(z)=z^{\frac{k}{2}}\sum_{n=1}^{\infty}c_n^j q^n \qquad \text{and} \qquad (\tilde{z})^{\frac{k}{2}}F_j^k(\tilde{z})=(\tilde{z})^{\frac{k}{2}}\sum_{n=1}^{\infty}c_n^j(\tilde{q})^n$$

are real-valued where $q=e^{2\pi iz}$, $\tilde{q}=e^{2\pi i\tilde{z}}$ and $1\leq j\leq\dim S_k$.
Note that $z^{\frac{k}{2}}F_j^k(z)$ compose a an orthonormal basis for the function $z^{\frac{k}{2}}f_k(z)$ on δ_3 which therefore is also real-valued. So moment curve is for this function will be

$$z^{\frac{k}{2}}\vec{r}(z)=z^{\frac{k}{2}}[F_1^k(z),F_2^k(z),\ldots,F_{\dim S_k}^k(z)]^\top.$$

Then one obtains

$$\begin{aligned}\log[z^{\frac{k}{2}}(\tilde{z})^{\frac{k}{2}}\vec{r}(z)\cdot\vec{r}(\tilde{z})]&=\log\left[z^{\frac{k}{2}}(\tilde{z})^{\frac{k}{2}}\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j q^n\right)\left(\sum_{n=1}^{\infty}c_n^j(\tilde{q})^n\right)\right]\\&=\frac{k}{2}(\log z+\log\tilde{z})+\log\left[\sum_{j=1}^{\dim S_k}\left(\sum_{n=1}^{\infty}c_n^j q^n\right)\left(\sum_{n=1}^{\infty}c_n^j(\tilde{q})^n\right)\right]\end{aligned}$$

After taking the derivative with respect to y and $\tilde{y}$ consecutively, the term $\frac{k}{2}(\log y+\log\tilde{y})$ will be killed. Therefore a similar formula we have found for $\|R'(t)\|^2$ appears on δ_3 as well.

On δ_3 one could replace z with $e^{i\theta}$ where $\frac{\pi}{3}\leq\theta\leq\frac{\pi}{2}$ through which $q=e^{2\pi ie^{i\theta}}$ and $\tilde{q}=e^{2\pi ie^{i\tilde{\theta}}}$. This time we obtain $\frac{\mathrm{d}}{\mathrm{d}\theta}q=-2\pi e^{i\theta}q$ and $\frac{\mathrm{d}}{\mathrm{d}\tilde{\theta}}\tilde{q}=-2\pi e^{i\tilde{\theta}}\tilde{q}$. Then similar computations result the formulation on δ_3 as

$$(4\pi^2 e^{2i\theta})\frac{\sum\limits_{i=1}^{r-1}\sum\limits_{j=i+1}^{r}\left(\sum\limits_{n=1}^{\infty} c_n^i n q^n \sum\limits_{n=1}^{\infty} c_n^j q^n - \sum\limits_{n=1}^{\infty} c_n^j n q^n \sum\limits_{n=1}^{\infty} c_n^i q^n\right)^2}{\left(\sum\limits_{j=1}^{r}\left(\sum\limits_{n=1}^{\infty} c_n^j q^n\right)^2\right)^2} \qquad (***)$$

where $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$ and $q = e^{2\pi i e^{i\theta}}$.

However this leads us to the expected number of real zeros of the function $z^{\frac{k}{2}} f_k(z)$. But it is obvious that a point $z \in \delta_3$ is a zero of $f_k(z)$ is equivalent to z is also a zero of $z^{\frac{k}{2}} f_k(z)$. Therefore they have the same expected number of real zeros.

Finally the expected number of real zeros of a random modular form f_k of weight k is

$$E[N_{real}(f_k)] = \frac{1}{\pi}\left(\int\limits_{1}^{\infty} \|R_1'(t)\|\, dt + \int\limits_{\frac{\sqrt{3}}{2}}^{\infty} \|R_2'(t)\|\, dt + \int\limits_{\frac{\pi}{3}}^{\frac{\pi}{2}} \|R_3'(\theta)\|\, d\theta\right)$$

where $\|R_1'(t)\|^2 =$ (*) , $\|R_2'(t)\|^2 =$ (**) and $\|R_3'(\theta)\|^2 =$ (***).

CHAPTER 4

ON THE ESTIMATION OF FOURIER COEFFICIENTS

4.1 Estimates on the Bounds of the Fourier Coefficients of a Cusp Form of Weight k

In this chapter the main goal will be to try to find an efficient bound for the expected number $E[N_{real}(f_k)]$ of real zeros of a random modular form because the formulas found in the previous section are very lengthy and not quite computable. But it is still manageable since Fourier coefficients of a cusp form has an efficient upper bound and the variable $|q^n| = e^{2\pi iz}$ decays very fast when $\mathrm{Im}(z) \to \infty$.

Proposition 4.1.1. Let $f \in M_k \backslash S_k$ with the Fourier expansion

$$f = \sum_{n=0}^{\infty} c_n q^n \, .$$

In other words, it is a modular form of weight k with the first Fourier coefficient c_0 being non-zero. Then

$$c_n = O(n^{k-1})$$

Proof. See [9, §7.4.3]. □

We work with the random modular forms which can be written as a linear combination of some Hecke eigenforms which are basically cusp forms with being joint eigenfuctions for all Hecke operators. Therefore we will consider another asymptotic characterization which is more efficient for the Fourier coefficients of a cusp form.

Theorem 4.1.2. Let $f = \sum\limits_{n=1}^{\infty} c_n q^n \in S_k$. Then

$$c_n = O(n^{k/2}) \, .$$

Proof. See [8, theorem 6.17] . □

One can immediately deduce that there exists some $M > 0$ such that

$$|c_n| \leq M n^{k/2}\,, \qquad \text{as } k \to \infty\ .$$

There have been many attempts to improve the above inequality, and in 1974 P. Deligne, as a consequence of the Weil conjectures, accomplished to find $|c_p| \leq 2p^{\frac{k-1}{2}}$ where p is prime (see [18]) which results in

$$|c_n| \leq d(n) n^{\frac{k-1}{2}} \qquad \text{(called "Deligne's bound")}$$

where $d(n)$ is the divisor function.

4.2 An Upper Bound for the Infimum of the Density of the Expected Number of Real Zeros

Recall from the previous chapter that a random modular f_k of weight k is defined as

$$f_k(z) = \sum_{j=1}^{\dim S_k} a_j F_j^k$$

where a_j are i.i.d. real random variables and F_j^k are Hecke eigenforms which form an orthonormal basis for S_k. Then we considered $a_j \sim N(0,1)$ and computed the expected number of real zeros, namely zeros on δ^*, of a random modular form. Through these computations we have obtained three formulae for the densities

$$\|R_1'(t)\|^2\ ,\ \|R_2'(t)\|^2 \text{ and } \|R_3'(\theta)\|^2 \qquad \text{(see (*), (**) and (***))}$$

such that

$$E[N_{real}(f_k)] = \frac{1}{\pi}\left(\int_1^\infty \|R_1'(t)\|\,dt + \int_{\frac{\sqrt{3}}{2}}^\infty \|R_2'(t)\|\,dt + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \|R_3'(\theta)\|\,d\theta\right)$$

These three formulas include Fourier coefficients c_n^j of Hecke eigenforms F_j^k where $1 \leq j \leq \dim S_k$, namely, $F_j^k = \sum_{n=1}^{\infty} c_n^j q^n$. Since F_j^k are cusp forms as well, one can use Deligne's bound to estimate an upper bound.

Lemma 4.2.1.

(i)

$$\|R_1'(t)\|^2 \leq 4\pi^2 \left(\frac{\sum\limits_{j=1}^{r} \left(\sum\limits_{n=1}^{\infty} c_n^j n q^n \right)^2}{\sum\limits_{j=1}^{r} \left(\sum\limits_{n=1}^{\infty} c_n^j q^n \right)^2} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \frac{\left| \sum\limits_{n=1}^{\infty} c_n^i n q^n \sum\limits_{n=1}^{\infty} c_n^j n q^n \right|}{\left| \sum\limits_{n=1}^{\infty} c_n^i q^n \sum\limits_{n=1}^{\infty} c_n^j q^n \right|} \right)$$

where $q = e^{-2\pi t}$ and $t > 1$.

(ii)

$$\|R_2'(t)\|^2 \leq 4\pi^2 \frac{\sum\limits_{j=1}^{r} \left(\sum\limits_{n=1}^{\infty} (-1)^n c_n^j n q^n \right)^2}{\sum\limits_{j=1}^{r} \left(\sum\limits_{n=1}^{\infty} (-1)^n c_n^j q^n \right)^2} +$$

$$4\pi^2 \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \frac{\left| \sum\limits_{n=1}^{\infty} (-1)^n c_n^i n q^n \sum\limits_{n=1}^{\infty} (-1)^n c_n^j n q^n \right|}{\left| \sum\limits_{n=1}^{\infty} (-1)^n c_n^i q^n \sum\limits_{n=1}^{\infty} (-1)^n c_n^j q^n \right|}$$

where $q = e^{-2\pi t}$ and $t > \sqrt{3}/2$.

(iii)

$$\|R_3'(t)\|^2 \leq 4\pi^2 e^{2i\theta} \left(\frac{\sum\limits_{j=1}^{r} \left| \sum\limits_{n=1}^{\infty} c_n^j n q^n \right|^2}{\sum\limits_{j=1}^{r} \left| \sum\limits_{n=1}^{\infty} c_n^j q^n \right|^2} + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \frac{\left| \sum\limits_{n=1}^{\infty} c_n^i n q^n \sum\limits_{n=1}^{\infty} c_n^j n q^n \right|}{\left| \sum\limits_{n=1}^{\infty} c_n^i q^n \sum\limits_{n=1}^{\infty} c_n^j q^n \right|} \right)$$

where $q = e^{2\pi e^{i\theta}}$ and $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$.

Proof.

(i) Recall the notation $r = \dim S_k$, $\sum\limits_{n=1}^{\infty} c_n^j n q^n = a_j$ and $\sum\limits_{n=1}^{\infty} c_n^j q^n = b_j$. Then

$$\|R_1'(t)\|^2 = (*) = (4\pi^2) \frac{\left[\sum\limits_{i=1}^{r-1} \sum\limits_{j=i+1}^{r} (a_i b_j - a_j b_i)^2 \right]}{\left(\sum\limits_{j=1}^{r} b_j^2 \right)^2} .$$

Now expand $\left|\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}(a_ib_j - a_jb_i)^2\right|$ and rearrange its terms as

$$|a_1^2(b_2^2+b_3^2+\cdots+b_r^2)+a_2^2(b_1^2+b_3^2+\cdots+b_r^2)+\cdots+a_r^2(b_1^2+\cdots+b_{r-1}^2|$$
$$-2(a_1b_2a_2b_1+\cdots+a_{r-1}b_ra_rb_{r-1})]$$

Then note that

$$\frac{|b_1^2+b_2^2+\cdots+b_{i-1}^2+b_{i+1}^2+\cdots+b_r^2|}{\left|\sum_{j=1}^{r}b_j^2\right|}\leq 1$$

for all $i=1,\dots,r$. One can also recall that Hecke eigenforms are real-valued functions on δ_1 and δ_2 which implies thus both b_j and a_j are real-valued on δ_1 and δ_2 as well. So,

$$\begin{aligned}
(*) &\leq 4\pi^2\frac{\sum_{j=1}^{r}|a_j|^2}{\left|\sum_{j=1}^{r}b_j^2\right|}+8\pi^2\frac{|a_1a_2b_1b_2+\cdots+a_{r-1}a_rb_{r-1}b_r|}{\left|\sum_{j=1}^{r}b_j^2\right|^2}\\
&\leq 4\pi^2\frac{\sum_{j=1}^{r}|a_j|^2}{\left|\sum_{j=1}^{r}b_j^2\right|}+8\pi^2\frac{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}|a_ia_j||b_ib_j|}{\left|\sum_{j=1}^{r}b_j^4+2\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}b_i^2b_j^2\right|}\\
&\leq 4\pi^2\frac{\sum_{j=1}^{r}|a_j|^2}{\left|\sum_{j=1}^{r}b_j^2\right|}+8\pi^2\frac{\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}|a_ia_j||b_ib_j|}{2\left|\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}b_i^2b_j^2\right|}\\
&\leq 4\pi^2\frac{\sum_{j=1}^{r}a_j^2}{\sum_{j=1}^{r}b_j^2}+4\pi^2\sum_{i=1}^{r-1}\sum_{j=i+1}^{r}\frac{|a_ia_j|}{|b_i||b_j|}
\end{aligned}$$

Therefore the result is immediate.

(ii) It is very similar to the above case.

(iii) When working on δ_3, a little more attention is required since $f(z)$ may be complex valued where $f(z)$ is a Hecke eigenform, however one can recall that $z^{k/2}f(z)$ is real-valued (corollary 3.1.5).

One can write $\frac{z^{k/2}b_j}{z^{k/2}}$ and $\frac{z^{k/2}a_j}{z^{k/2}}$ in place of b_j and a_j, respectively, then a very similar result is obtained bu using the same techniques. $\square$

Lemma 4.2.2. Let $F_j^k = \sum_{n=1}^{\infty} c_j^n q^n$ be cusp forms of weight k for $j = 1, 2, \ldots, r$ and $\mathrm{Im}(z) = y$. Then

(i)
$$\left(\sum_{n=1}^{\infty} c_n^j n q^n\right)^2 \leq 4 \sum_{n=1}^{\infty} n^{k+3} e^{-2\pi(n+1)y}$$

(ii)
$$\sum_{n=N+1}^{\infty} n^{k+3} e^{-2\pi(n+1)y} \leq \frac{e^{-2\pi yN}}{2\pi y}(k+4)N^{k+3}$$

where $N = \lceil \frac{k+2}{2\pi y} \rceil$.

Proof.

(i) By using Cauchy product for infinite series, we note that

$$\begin{aligned}\left(\sum_{n=1}^{\infty} c_n^j n q^n\right)^2 &= \sum_{n=1}^{\infty} \sum_{m=1}^{n} c_m^j m q^m c_{n+1-m}^j (n+1-m) q^{n+1-m} \\ &= \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n} c_m^j c_{n+1-m}^j m(n+1-m)\right) q^{n+1}\end{aligned}$$

Notice that for all $n = 1, 2, \ldots$

$$\sum_{m=1}^{n} c_m^j c_{n+1-m}^j m(n+1-m) \leq \sum_{m=1}^{n} d(m) m^{\frac{k+1}{2}} d(n+1-m)(n+1-m)^{\frac{k+1}{2}}$$

because of Deligne's bound.

Then with the inequalities $d(n) \leq 2\sqrt{n}$, $m \leq n$ and $n+1-m \leq n$ one obtains

$$\sum_{m=1}^{n} c_m^j c_{n+1-m}^j m(n+1-m) \leq 4n^{k+3}$$

Therefore one can find an upper bound

$$\left(\sum_{n=1}^{\infty} c_n^j n q^n\right)^2 \leq 4 \sum_{n=1}^{\infty} n^{k+3} e^{-2\pi(n+1)y} = 4 \sum_{n=1}^{\infty} n^{k+3} e^{-2\pi(n+1)y}$$

since $|q|^{n+1} = e^{-2\pi y(n+1)}$, where $y = \mathrm{Im}(z)$.

(ii) One may easily notice that $f(x) = x^{k+3}e^{-2\pi(x+1)y}$ is a positive, continuous and non-increasing function on the interval $[\frac{k+3}{2\pi y}, \infty)$. Let $N = \lceil \frac{k+2}{2\pi y} \rceil$. Then

$$\sum_{N+1}^{\infty} n^{k+3}e^{-2\pi(n+1)y} \leq \int_{N}^{\infty} x^{k+3}e^{-2\pi(x+1)y}\,dx$$

$$= e^{-2\pi y}\int_{N}^{\infty} x^{k+3}e^{-2\pi yx}\,dx = \frac{e^{-2\pi yN}}{2\pi y}\sum_{m=0}^{k+3}\frac{(k+3)!}{m!(2\pi y)^{k+3-m}}N^m$$

Since for any $1 \leq m \leq k+3$

$$\frac{(k+3)\dots(m+1)}{(2\pi y)^{k+3-m}} \leq \left(\frac{k+3}{2\pi y}\right)^{k+3-m} \leq N^{k+3-m}$$

we obtain

$$\sum_{N+1}^{\infty} n^{k+3}e^{-2\pi(n+1)y} \leq \frac{e^{-2\pi yN}}{2\pi y}(k+4)N^{k+3}$$

□

Corollary 4.2.3. Let F_j^k be a cusp form of weight k, $N = \lceil \frac{k+2}{2\pi y} \rceil$ and denote $S_N = \sum_{n=1}^{N} n^{k+3}e^{-2\pi(n+1)y}$. Then

$$\sum_{j=1}^{r}\left(\sum_{n=1}^{\infty} c_n^j n q^n\right)^2 \leq 4r\left(S_N + \frac{e^{-2\pi yN}}{2\pi y}(k+4)N^{k+3}\right)$$

Proof. The proof is obvious by the above lemma. □

Finding a lower bound for $\sum_{j=1}^{r}\left(\sum_{n=1}^{\infty} c_n^j q^n\right)^2$ is a bit more tricky. For this we will use the fact that for any $1 \leq j \leq r$, the function $\left(\sum_{n=1}^{\infty} c_n^j q^n\right)^2$ is a cusp form of weight $2k$ and make use of an upper bound of the sup norm, which will be defined later, on the vector space S_k of cusp forms. To be more clear, let us explain a few things.

Definition 4.2.4. Let $f \in S_k$ be a cusp form of weight k and $z = x + iy$ so that $\operatorname{Im} z = y$. Then

$$\|f\|_\infty = \sup_{z\in\Gamma} y^{k/2}|f(z)|$$

is called the **sup norm** on S_k.

One can easily show that the factor $y^{k/2}$ makes this norm Γ-invariant. The goal from now on will be to find bounds, especially lower bound, for this norm.

Lemma 4.2.5. (see [19]) Let f be an L^2-normalized cusp form of weight k and $\epsilon > 0$. Then

$$(k/2)^{\frac{1}{4}-\epsilon} \ll \|f(z)\|_\infty \ll (k/2)^{\frac{1}{4}+\epsilon} .$$

As a corollary of this lemma, one can give a lower bound for the denominator terms appeared in the formulations in lemma 4.2.1.

Corollary 4.2.6. Let f satisfy the conditions in the lemma above, $\dim S_k = r$ and F_j^k denote the orthonormal basis elements of S_k as before. Then,

(i)
$$(k/2)^{\frac{1}{2}-\epsilon} \ll \sup_{z\in\Gamma} |f(z)|^2 y^k \ll (k/2)^{\frac{1}{2}+\epsilon}$$

(ii)
$$\sup_{z\in\Gamma} \left(\sum_{j=1}^{r} |F_j^k|^2 y^k \right) \gg (k/2)^{\frac{3}{2}-\epsilon}$$

Proof.

(i) This can be easily proved by using the lemma above.

(ii) (see [20, §7.2]) □

By using these results we will be able to give a lower bound for the denominators of the formulas in lemma 4.1.3.

Corollary 4.2.7. Recall the formulations from lemma 4.2.1. Then

(i)
$$\inf_{\delta_1} \frac{1}{\sum\limits_{j=1}^{r} \left(\sum\limits_{n=1}^{\infty} c_n^j q^n \right)^2} \leq \frac{y^k}{(k/2)^{\frac{3}{2}-\epsilon}}$$

$$\inf_{\delta_1} \frac{1}{\left| \sum\limits_{n=1}^{\infty} c_n^i q^n \sum\limits_{n=1}^{\infty} c_n^j q^n \right|} \leq \frac{y^k}{(k/2)^{\frac{1}{2}-\epsilon}}$$

(ii) Same inequalities hold on δ_2 and δ_3 as well.

Proof.

(i) This is due to $\inf \frac{1}{A} = \frac{1}{\sup A}$ where $A \subset (0, \infty)$ and (i) of the corollary above.

(ii) The proof is the same with the first part. $\square$

Finally we can give an upper bound for the $\inf_{\delta_i} \|R_i'\|^2$ for $i = 1, 2, 3$.

Corollary 4.2.8. Let us denote $S_N + \frac{e^{-2\pi y N}}{2\pi y}(k+4)N^{k+3} = B_N$ where $N = \lceil \frac{k+2}{2\pi y} \rceil$ (recall corollary 4.1.5). Then

(i)
$$\inf_{\delta_1} \|R_1'(t)\|^2 \ll \frac{16\pi^2}{y^k} r B_N (k/2)^{\frac{1}{2}-\epsilon}(k/2 + r^2 - r)$$

(ii)
$$\inf_{\delta_2} \|R_1'(t)\|^2 \ll \frac{16\pi^2}{y^k} r B_N (k/2)^{\frac{1}{2}-\epsilon}(k/2 + r^2 - r)$$

(iii)
$$\inf_{\delta_3} \|R_1'(t)\|^2 \ll e^{2i\theta} \frac{16\pi^2}{y^k} r B_N (k/2)^{\frac{1}{2}-\epsilon}(k/2 + r^2 - r)$$
where $\pi/3 \leq \theta \leq \pi/2$

Proof. We first take the $\inf$ of the formulas $*$, $**$ and $***$ on the geodesic segments δ_1, δ_2 and δ_3, respectively. Then we use corollary 4.2.7 and corollary 4.2.3. $\square$

REFERENCES

[1] L. J. P. Kilford, *Modular Forms: A classical and computational introduction*. World Scientific Publishing Company, 2015.

[2] P. Sarnak, *Some applications of modular forms*, vol. 99. Cambridge University Press, 1990.

[3] M. S. Viazovska, “The sphere packing problem in dimension 8,” *Annals of mathematics*, pp. 991–1015, 2017.

[4] D. Hankerson, S. Vanstone, and A. Menezes, “Guide to elliptic curve cryptography, springer-verlag,” *New York*, 2004.

[5] F. K. Rankin and H. P. Swinnerton-Dyer, “On the zeros of eisenstein series,” *Bulletin of the London Mathematical Society*, vol. 2, no. 2, pp. 169–170, 1970.

[6] D. Husemöller, *Elliptic Curves*. Graduate Texts in Mathematics, Springer, 2004.

[7] J. H. Silverman, *The arithmetic of elliptic curves*, vol. 106. Springer, 2009.

[8] T. M. Apostol, *Modular functions and Dirichlet series in number theory*, vol. 41. Springer Science & Business Media, 2012.

[9] J.-P. Serre, *A course in arithmetic*, vol. 7. Springer Science & Business Media, 2012.

[10] R. C. Gunning, *Lectures on modular forms*. No. 48, Princeton University Press, 1962.

[11] F. Diamond and J. M. Shurman, *A first course in modular forms*, vol. 228. Springer, 2005.

[12] J. H. Bruinier, G. Van der Geer, G. Harder, and D. Zagier, *The 1-2-3 of modular forms: lectures at a summer school in Nordfjordeid, Norway*. Springer Science & Business Media, 2008.

[13] S. Lang, *Introduction to modular forms*, vol. 222. Springer Science & Business Media, 2001.

[14] D. Armendáriz, O. Colman, N. Coloma, A. Ghitza, N. C. Ryan, and D. Teran, "Analytic evaluation of hecke eigenvalues for classical modular forms," *arXiv preprint arXiv:1806.01586*, 2018.

[15] P. Sarnak and A. Ghosh, "Real zeros of holomorphic hecke cusp forms," *Journal of the European Mathematical Society*, vol. 14, no. 2, pp. 465–487, 2012.

[16] A. Edelman and E. Kostlan, "How many zeros of a random polynomial are real?," *Bulletin of the American Mathematical Society*, vol. 32, no. 1, pp. 1–37, 1995.

[17] M. Kac, "On the average number of real roots of a random algebraic equation," *Bulletin of the American Mathematical Society*, vol. 49, no. 4, pp. 314–320, 1943.

[18] P. Deligne, "La conjecture de weil. i," *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, vol. 43, pp. 273–307, 1974.

[19] H. Xia, "On $l - \infty$ norms of holomorphic cusp forms," *Journal of Number Theory*, vol. 124, no. 2, pp. 325–327, 2007.

[20] J. S. Friedman, J. Jorgenson, and J. Kramer, "Uniform sup-norm bounds on average for cusp forms of higher weights," *arXiv preprint arXiv:1305.1348*, 2013.

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