Noncoprime action of a cyclic group

GÜLİN ERCAN* and İSMAİL Ş. GÜLOĞLU

Abstract. Let A be a finite nilpotent group acting fixed point freely on the finite (solvable) group G by automorphisms. It is conjectured that the nilpotent length of G is bounded above by $\ell(A)$, the number of primes dividing the order of A counted with multiplicities. In the present paper we consider the case A is cyclic and obtain that the nilpotent length of G is at most $2\ell(A)$ if $|G|$ is odd. More generally we prove that the nilpotent length of G is at most $2\ell(A) + \mathbf{c}(G; A)$ when G is of odd order and A normalizes a Sylow system of G where $\mathbf{c}(G; A)$ denotes the number of trivial A-modules appearing in an A-composition series of G.

1. Introduction

Let H be a finite group. Dade conjectured in his 1969's paper [**3**] that whenever H is solvable and C is a Carter subgroup of H there is a linear function g such that the nilpotent length $h(H)$ is at most $g(\ell(C))$ where $\ell(C)$ denotes the number of (not necessarily distinct) prime divisors of C. If A is a finite nilpotent group acting fixed point freely on the finite group G by automorphisms, that is $C_G(A) = \{g \in G : g^a = g \text{ for all } a \in G\} = 1$, then one can regard A as a Carter subgroup of the semidirect product GA with normal complement G. This allows us to state a special case of the above conjecture as follows.

Let A be a finite nilpotent group acting fixed point freely on the finite group G by automorphisms. Then $h(G)$ is bounded above by a linear function of $\ell(A)$.

Although Dade established an exponential bound in the same paper, no linear bound has been found so far even if A is cyclic. However,

2020 *Mathematics Subject Classification.* 20D45.

Key words and phrases. nilpotent length, automorphism, fixed point free action.

*Corresponding author.

some special cases of this conjecture are known. It should be noted that the best possible bound in this case is $\ell(A)$ due to the fact that for any finite group A there is a finite solvable group G on which A acts coprimely such that $h(G) = \ell(A)$ and $C_G(A) = 1$. All the results in this direction are listed in the survey paper [**12**] of Turull. This conjecture is mostly studied under the assumption that $(|G|, |A|) = 1$ and the best description of the results obtained so far is given in Theorem 2.1 in [**13**] as follows:

Let A be a finite group acting coprimely on the finite solvable group G by automorphisms. Assume that for every subgroup A_0 of A and every A_0-invariant irreducible elementary abelian section S of G there is $v \in S$ with $C_{A_0}(v) = C_{A_0}(S)$. Then $h(G) \leq \ell(A) + \ell(C_G(A))$.

Notice that this yields immediately the best possible bound $\ell(A)$ in case where $C_G(A) = 1$.

As an example to the noncoprime case we can give a result of Turull, namely the following.

Let A be a finite abelian group of squarefree exponent acting fixed point freely on the finite solvable group G by automorphisms. Then $h(G) \leq 5\ell(A)$.

On the same lines the authors obtained in [**5**] as an example of a partial special case that $h(G) \leq \ell(A)$ under the assumption that A is finite abelian of squarefree exponent coprime to 6 acting fixed point freely on the group G of odd order. More recently, Jabara handled the case where A is cyclic in [**7**], and obtained the polynomial bound $7\ell(A)^2$ for $h(G)$.

In 1990 [**1**] Bell and Hartley constructed an elegant example showing that the nilpotentness condition on A cannot be freely dropped in case of a noncoprime action. Namely, they proved the following.

For any finite nonnilpotent group A and a positive integer k, there exists a finite solvable group G on which A acts fixed point freely and $h(G) = k$.

In view of this result the conjecture should be restated as follows.

Let A be a finite nilpotent group acting fixed point freely on the finite solvable group G by automorphisms. Then $h(G) \leq \ell(A)$.

It should be noted that the solvability condition on G allows us to ask about the nilpotent length of the group. However, this condition is guaranteed by results of Rowley [**9**] and Belyaev-Hartley [**2**] in the cases of a coprime fixed point free action of a group and a noncoprime fixed point free action of a nilpotent group, respectively. Hence we remove the solvability condition whenever the action is fixed point free.

It is our aim in the present paper to obtain a result in the noncoprime situation which is similar to Theorem 2.1 in [**13**] mentioned above when A is cyclic. Let A act on the group G, and let $\mathbf{c}(G;A)$ denote the number of trivial A-modules appearing as factors in any A-composition series of G. We prove the following.

Theorem. *Let A be a finite cyclic group acting on the finite group G of odd order. Suppose that A normalizes a Sylow system of G. Then $h(G) \leq 2\ell(A) + \mathbf{c}(G;A)$.*

It is straightforward to show that $\mathbf{c}(G;A) = \ell(C_G(A))$ in case where G is solvable and $(|G|,|A|) = 1$. If A is nilpotent we shall see in Section 3 that $\mathbf{c}(G;A) = 0$ if and only if $C_G(A) = 1$. It follows that as an immediate consequence of the above theorem we have

Corollary. *Let A be a finite cyclic group acting fixed point freely on the finite group G of odd order. Then $h(G) \leq 2\ell(A)$.*

The paper is divided into three sections. Section 2 includes new observations on the existence of homogeneous components and regular modules as well as some known results on the existence of regular orbits. Section 3 is reserved to the proof of the main theorem of the paper.

2. Existence of homogeneous components and regular characters

This section is devoted to some key theorems which form bases for the proofs of the main theorems of this paper. We start with the following result as a preparation of the next theorem.

Theorem 2.1. *Let A be a finite nilpotent group acting on a finite solvable group G. Suppose that G/M is a GA-chief factor of G which is an elementary abelian r-group for a prime r, and that A normalizes a Hall r'-subgroup of GA. Let V be a kGA -module for a field k such that V_G is homogeneous. Then there is a homogeneous component of V_M which is stabilized by A.*

Proof. Assume the contrary. Then V_M is not homogeneous. Let $V_M = W_1 \oplus W_2 \oplus \cdots \oplus W_s$ be the decomposition of V_M into its homogeneous components. As G acts transitively on $\{W_1, W_2, ..., W_s\}$ we see that $GA = N_{GA}(W_1)G$.

As G/M is irreducible A-module and an r-group we see that $A_r \leq C_{GA}(G/M) = C_{N_{GA}(W_1)}(G/M)G = LG$ where $L = C_{N_{GA}(W_1)}(G/M)$. Clearly $M \leq G \cap L$ and $G \cap L$ is normalized by G and $N_{GA}(W_1)$ and hence by $GA = N_{GA}(W_1)G$. Thus $(G \cap L)/M$ is a GA-submodule of the irreducible GA-module G/M and hence $M = G \cap L$. We have $[G, L] \leq M \leq L$ and hence $L \trianglelefteq GA$.

Set $\overline{GA} = GA/L$. We have $\overline{G} \leq O_r(\overline{GA})$. On the other hand, $GA/LG \cong A/(A \cap LG)$ is an r'-group as $A_r \leq A \cap LG$. Thus we have $\overline{G} = O_r(\overline{GA})$ and $\overline{GA} = O_{r,r'}(\overline{GA})$. Furthermore $\overline{G}$ is a minimal normal subgroup of $\overline{GA}$, that is an irreducible $\overline{GA}$-module. In particular $\overline{A_r} \leq \overline{G}$ and hence $\overline{A_r} \leq C_{\overline{G}}(\overline{GA}) = 1$, that is $A_r \leq L \leq N_{GA}(W_1)$.

Let now Q be an A-invariant Hall r'-subgroup of GA. Then $\overline{Q}O_r(\overline{GA}) = \overline{QG} = \overline{GA} = \overline{N_{GA}(W_1)}\overline{G}$. Note that $\overline{N_{GA}(W_1)} \cap \overline{G}$ is trivial. It follows that $\overline{Q}$ and $\overline{N_{GA}(W_1)}$ are conjugate in $\overline{GA}$. Without loss of generality, we may assume that $QL = N_{GA}(W_1)$. Since Q is an A -invariant Hall r'-subgroup of GA we see that $Q = QA_{r'}$ that is $A_{r'} \leq Q \leq N_{GA}(W_1)$ and hence A leaves W_1 invariant. This completes the proof. □

We now present a new result that we shall use as a tool to make up for the lack of the Glauberman's lemma in the case of a noncoprime action.

THEOREM 2.2. *Let A be a finite nilpotent group acting on a finite solvable group G and let V be a kGA-module for a field k such that V_G is homogeneous. If A normalizes a Sylow system of GA, then there is a homogeneous component of V_N which is stabilized by A for any A-invariant normal subgroup N of G.*

PROOF. We consider a series

$$N = M_k < \cdots \leq M_2 < M_1 < M_0 = G$$

where M_i is a maximal normal A-invariant subgroup of M_{i-1} containing N for $i = 2, \dots, k$. Applying Theorem 2.1 to the triple (V, G, M_1) under the action of A we obtain a homogeneous component V_1 of V_{M_1} which is stabilized by A. We apply Theorem 2.1 to the triple (V_1, M_1, M_2) and get an A-invariant homogeneous component V_2 of $(V_1)_{M_2}$. Continuing this process along the remaining terms of the above series we eventually obtain a nontrivial homogeneous N-module U which is A-invariant. It is apparent that A stabilizes the homogeneous component W of V_N such that $U \leq W$. This establishes the claim. □

The following result which is essentialy due to Dade is stated as a proposition in [**4**].

PROPOSITION 2.3. *Let V be a faithful kA-module over a finite field k of characteristic s. Assume that $A = B \times C$ where B is a cyclic s-group and C is an s'-group which has a regular orbit on every C-invariant irreducible section of V. Then A has a regular orbit on V.*

The next result is Theorem 1.1 in [**14**].

PROPOSITION 2.4. *Let A be a nilpotent group and let V be a faithful $\mathbb{F}A$-module over a field of characteristic s where $O_s(A) = 1$. Suppose that A involves no wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2$; and involves no wreath product $\mathbb{Z}_r \wr \mathbb{Z}_r$ for r a Mersenne prime when $s = 2$. Then A has at least one regular orbit on V.*

We are ready now to prove a theorem which concludes the existence of a regular module in a special configuration. Firstly we obtain the following preliminary result.

THEOREM 2.5. *Let PQA be a finite group where P is a p-group and Q is a q-group for distinct primes p and q such that q is not a Fermat prime if $p = 2$. Assume that $P \lhd PQA$, $Q \lhd QA$. Assume further that the following are satisfied:*
(a) Either A is cyclic; or A is a noncyclic nilpotent group which is $\mathbb{Z}_2 \wr \mathbb{Z}_2$-free and $\mathbb{Z}_r \wr \mathbb{Z}_r$-free for all Mersenne primes r.
(b) P is an extraspecial p-group for some prime p where $C_A(P) = 1$ and $Z(P) \leq Z(PQA)$;
(c) Q/Q_0 is of class at most two and of exponent dividing q where $Q_0 = C_Q(P)$; and $A_0 = C_A(Q/Q_0)$ is either 1 or of prime order. Assume that q is coprime to $|A_0|$ and $[P, A_0] = P$ when $A_0 \neq 1$.
(d) $(pq, |A|) = 1$ whenever A is noncyclic.

Let χ be a complex PQA-character such that χ_P is faithful. Then χ_A contains the regular A-character.

PROOF. Let (P, Q, χ) be a counterexample with $|PQ| + \chi(1)$ minimum.

We shall proceed in a series of steps. To simplify the notation we set $G = PQ$.

(1) χ is irreducible.

There exists an irreducible constituent χ_1 of χ which does not contain $Z(P)$ in its kernel, that is $(\chi_1)_P$ is faithful. Then we have $\chi_1 = \chi$ because otherwise χ_1 contains the regular A-character by induction.

(2) χ_P is homogeneous and $Q_0 = 1$.

Notice that $\chi_{Z(P)}$ is homogeneous since $Z(P) \leq Z(GA)$. As is well known the irreducible characters of the extraspecial group P are uniquely determined by their restriction to $Z(P)$ so that $\chi_P = e\theta$ for some faithful irreducible GA-invariant character θ of P and some positive integer e. The coprimeness condition $(|P|, |QA_{p'}|) = 1$ enables us to extend θ in a unique way to an irreducible character $\overline{\theta}$ of $GA_{p'}$ such that $det(\overline{\theta})(x) = 1$ for each $x \in QA_{p'}$ by [[**6**], 8.16]. On the other hand $\theta_1 = \theta \times 1_{Q_0}$ is an irreducible $P \times Q_0$-character with $Q_0 \leq Ker\theta_1$. We can extend θ_1 uniquely to $\overline{\theta}_1 \in Irr(GA_{p'}/Q_0)$ with $det(\overline{\theta}_1)(x) = 1$ for each $x \in QA_{p'}$. Notice that $(\overline{\theta}_1)_P = \theta = \overline{\theta}_P$. So the uniqueness of this extension implies $Q_0 \leq Ker\overline{\theta}$.

Observe that the set $\{\varphi : \varphi \in Irr(GA_{p'})$ such that $\varphi_P = \theta\}$ is A_p-invariant, because $\theta^a = \theta$ for each $a \in A_p$. Since $det(\overline{\theta}^a)(x) = 1$ for each $a \in A_p$, the uniqueness of $\overline{\theta}$ gives $\overline{\theta}^a = \overline{\theta}$. Notice that in case where A is cyclic $\overline{\theta}$ is extendible to an irreducible GA-character, say $\overline{\overline{\theta}}$, by [[**6**], Corollary 11.22]. When A is noncylic, we know by hypothesis (d) that $GA = GA_{p'}$ and we put simply $\overline{\overline{\theta}} = \overline{\theta}$ respectively. Now $\overline{\overline{\theta}}_P = \theta$ and $Q_0 \leq Ker\overline{\theta} = G \cap Ker\overline{\overline{\theta}}$. If $\overline{\overline{\theta}}(1) < \chi(1)$ or $Q_0 \neq 1$, by induction applied to the group GA/Q_0 over $\overline{\overline{\theta}}$ we see that $\overline{\overline{\theta}}_A$ contains the regular A-character. Since χ is a constituent of $\overline{\overline{\theta}}_P|^{GA}$, there exists $\beta \in Irr(GA/P)$ such that $\chi = \overline{\overline{\theta}} \cdot \beta$ by [[**6**], 6.17] and hence $\chi_A = \overline{\overline{\theta}}_A \cdot \beta_A$. We conclude that χ_A contains the regular A-character, while $\overline{\overline{\theta}}_A$ does. Therefore without loss of generality we may assume that $Q_0 = 1$ as claimed.

(3) Theorem follows.

Notice that we can regard $Irr(Q/\Phi(Q))$ as a faithful $\mathbb{F}_q(A/A_0)$-module which is isomorphic to $Q/\Phi(Q)$. Applying Proposition 2.3 and Proposition 2.4 in cases where A/A_0 is cyclic and A/A_0 is noncyclic, respectively, we get a linear character λ_1 of Q such that $C_A(\lambda_1) = A_0$.

Theorem 1.3 in [**13**] applied to the group $P(Q \times A_0)$ over χ shows that one of the following holds:

(i) χ_{QA_0} contains the regular QA_0-character;

(ii) $A_0 = 1$, $p = 2$, and q is a Fermat prime;

(iii) $A_0 \neq 1$, Q is abelian and for some $\zeta \in Irr(QA_0)$, $\chi|_{QA_0} + \zeta$ contains the regular QA_0-character;

(iv) A_0 does not act faithfully on some irreducible A_0-submodule of $P/\Phi(P)$.

By the hypothesis q is not a Fermat prime when $p = 2$. Furthermore $[P, A_0] = P$ if $A_0 \neq 1$. Hence one of (i) or (iii) follows. Note that $A \neq A_0$ (see [**8**] and [**10**]). Let $\lambda \in Irr(A_0)$. Then $C_A(\lambda \otimes \lambda_1) = A_0$ and χ_{QA_0} is A-invariant. Now $\lambda \otimes \lambda_1 \subseteq \chi_{QA_0}$ since at most one irreducible QA_0-character is missing. Set $\nu = \lambda_1$ and $\mu = \lambda \otimes \lambda_1$. Now $\mu|_{A_0} = \lambda, \mu|_Q = \nu,\ C_A(\mu) = A_0$ and $\mu \subseteq \chi_{QA_0}$.

Let $\zeta \in Irr(A)$. Then $\zeta_{A_0} = e\Sigma_{i=1}^n \lambda_i$ where e is some positive integer and the λ_i are distinct irreducible characters of A_0. Set $\mu_i = \nu \otimes \lambda_i$ for each $i = 1, \ldots, n$. Now $\mu_i \in Irr(QA_0)$ and hence contained in χ_{QA_0}. Let $\eta_i \in Irr(QA)$ such that $\mu_i \subseteq \eta_{i_{QA_0}}$ and $\eta_i \subseteq \chi_{QA}$. Then $\eta_i = \mu_i|^{QA}$ as $C_{QA}(\mu_i) = QA_0$. Notice that μ_i are not A-conjugate because the λ_i are all distinct. Thus the η_i are all distinct. Furthermore $\eta_{i_A} = \lambda_i|^A$ and so $[\eta_{i_A}, \zeta] = e$. Hence $\Sigma_{i=1}^n \eta_i \subseteq \chi_{QA}$ and $[\Sigma_{i=1}^n \eta_i, \zeta] = ne = \zeta(1)$. Repeat this argument for all $\zeta \in Irr(A)$ and let χ_1 be the smallest QA-character which contains $\Sigma_{i=1}^n \eta_i$ for all $\zeta \in Irr(A)$. Now $\chi_1 \subseteq \chi_{QA}$ and $[(\chi_1)_A, \zeta] \geq \zeta(1)$. So the regular A-character is contained in $(\chi_1)_A \subseteq \chi_A$. This completes the proof.

$\square$

Another version of Theorem 2.5 in which the condition of being extraspecial on P is weakened can be given as follows. This theorem is stated in a generality which is not necessary in handling the case where A is cyclic with the hope of applying it later to the discussion of the case where A is nilpotent.

THEOREM 2.6. *Let GA be a finite group with $G \trianglelefteq GA$ where G is solvable, A satisfies the condition (a) in Theorem 2.5 and normalizes a Sylow system of G. Let P be a p-subgroup of G, for some prime p, such that $P/Z(P)$ is elementary abelian, $P \trianglelefteq GA$, $\Phi(P) = P'$, $exp(P) = p$ if p is odd, and $P/\Phi(P)$ is completely reducible as a GB-module for any subgroup B of A.*

Let Q be an A-invariant q-subgroup of $C_G(\Phi(P))$ for a prime q which is coprime to $p|A|$and not a Fermat prime if $p = 2$. Assume that $Q/Q_0/Z(Q/Q_0)$ is elementary abelian where $Q_0 = C_Q(P)$, and $QC_G(P/P') \trianglelefteq GA$, and that $[Q, P] = P$ if $P' \neq 1$. Let $A_0 = C_A(Q/Q_0)$. Assume that q is coprime to $|A_0|$ and $[P, A_0] = P$ when $A_0 \neq 1$.

Assume further that the following hold:

(i) $P = [P, B]^G$ for every $B \leq A$ with $\ell(B) \geq 1$,

(ii) if P is nonabelian, $Q = [Q, C]^{N_G(Q)} Q_0$ for every $C \leq A$ with $\ell(C) \geq 2$,

(iii) pq is coprime to $|A|$ when A is noncyclic.

Let χ be a complex GA-character such that $P \not\leq \ker(\chi)$. Then χ_A contains the regular A-character.

PROOF. It should be noted that in case where P is abelian there is no loss in assuming that $Q = 1$. Suppose that the theorem is false and choose a counterexample with $|PQA| + \chi(1)$ minimum. χ contains an irreducible component such that P is not contained in its kernel. So χ is irreducible by the minimality of $|PQA|+\chi(1)$. We may also assume that χ_P is faithful by the minimality of $|PQA|+\chi(1)$.

Suppose that χ_G is not homogeneous. Then there exists a proper subgroup B of A and a GB-character ψ such that $\psi^{GA}=\chi$. We must have $P \not\leq \ker(\psi)$ because otherwise P is contained in the kernel of every GA-conjugate of ψ and hence in $\ker(\chi)$, which is not the case. The minimality of χ implies that ψ_B contains the regular B-character. Then $\chi_A = \psi_B|^A$ and χ_A contains the regular A-character, which is a contradiction. Therefore χ_G is homogeneous.

Suppose that $N \leq Z(P)$ is a normal subgroup of GA such that $[N,Q]=1$. We claim that $N \leq Z(GA)$: Assume the contrary. Then χ_N is not homogeneous. By Theorem 2.2 there exists an A-invariant irreducible constituent θ of χ_N. Then there exists an A-invariant proper subgroup H of G and an irreducible HA-character ψ such that $\psi^{GA} = \chi$, $\theta \subseteq \psi_N$ and $P \not\leq \ker\psi$, and $PQ \leq C_G(N) \leq H$. By the minimality of $|PQA|+\chi(1)$, we get ψ_A contains the regular A-character. This contradiction shows that $N \leq Z(GA)$.

Assume now that P is abelian. Then $Q=1$, and so $P \leq Z(GA)$ by the above paragraph. This contradicts the hypothesis (i) and hence P is not abelian.

Since $P/\Phi(P)$ is GA-completely reducible in any case there exists $E \trianglelefteq GA$ containing $\Phi(P)$ so that

$$P/\Phi(P) = Z(P)/\Phi(P) \oplus E/\Phi(P).$$

Then $P = Z(P)E$ and hence $Z(P)\cap E = Z(E)$. We have $P' = \Phi(P) \leq Z(P)$ by the hypothesis. So $\Phi(P) \leq Z(E)$. Also,

$$E/\Phi(P) \cap Z(P)/\Phi(P) = 1$$

and hence $Z(E) \leq \Phi(P)$. Thus we have $Z(E) = \Phi(P) = P'$. Notice that $\Phi(P) \leq Z(GA)$ by an argument above and hence $\Phi(P) = P'$ is cyclic of prime order. As $E \trianglelefteq P$ we get $\Phi(E) \leq \Phi(P) = Z(E)$. It follows that $Z(E) = E' = \Phi(E) = \Phi(P)$ is cyclic of prime order, that is E is extraspecial with $[Z(E),QA]=1$. Notice that we have $E = [E,B]^G$ for all $B \leq A$ with $\ell(B) \geq 1$. So it holds by induction that $P=E$. Then $[Z(P),Q]=1$ which implies that $Z(P) \leq Z(GA)$. Recall that $|C_A(Q/Q_0)| = 1$ or a prime, by the hypothesis (ii). Applying now Theorem 2.5 to the action of PQA on χ we obtain that χ_A contains the regular A-character, and the claim is established. $\square$

3. PROOF OF THE THEOREM

Definition 3.1. *Let A act on the group solvable group G. We denote the number of trivial A-modules appearing as factors in any A-composition series of G by $\mathbf{c}(G;A)$. More generally, for any normal A-series $1 = N_{k+1} < N_k < N_{k-1} < \cdots < N_1$ of G and A-invariant normal subgroups M_i for $i = 1, \ldots, k$ of G with $N_{i+1} \leq M_i < N_i$ and $P_i = N_i/M_i$, we write $\mathbf{c}(P_k, \ldots, P_1; A)$ for $\sum_{i=1}^{k} \mathbf{c}(P_i; A)$.*

Remark 3.2. *Let A act on G and normalizes a Sylow system of G. Then by a slight modification of Lemma 8.2 of the same paper, one can show the existence of an irreducible A-tower in Turull's sense (see [**11**]), namely the existence of sections $P_i = S_i/T_i$, $i = 1, \ldots, h$, of G where S_i and T_i are subgroups of G such that $T_i \lhd S_i$ and $h = h(G)$ satisfying the following conditions:*

(a) P_i is a nontrivial p_i-group, for some prime p_i,

(b) $\Phi(P_i) \leq Z(P_i), \Phi(\Phi(P_i)) = 1$ and if p_i is odd, then P_i has exponent p_i,

(c) P_i is A-invariant, for $i = 1, \ldots, h$,

(d) $p_i \neq p_{i+1}$, for $i = 1, \ldots, h-1$,

(e) $T_i = Ker(S_i \text{ on } P_{i+1})$, for $i = 1, \ldots, h-1$,

(f) $T_h = 1$ and $S_h \leq F(G)$,

(g) $[\Phi(P_{i+1}), S_i] = 1$, for $i = 1, \ldots, h-1$,

(h) $(\prod_{1 \leq j < i} S_j)A$ acts irreducibly on $\tilde{P}_i$.

*We should also note that if A is a nilpotent group acting fixed point freely on the group G, then A is a Carter subgroup of the semidirect product GA having G as a normal complement, and hence Lemma 8.1 in [**3**] guarantees that A normalizes a Sylow system of G. Furthermore in this case we clearly have $\mathbf{c}(G;A) = 0$ which shows that the Corollary is an immediate consequence of the Theorem.*

Now we proceed to the proof of Theorem. Since A normalizes a Sylow system of G, by the above remark we may assume the existence of an irreducible A-tower $P_1, \ldots, P_h$ with $P_i = S_i/T_i$ satisfying the conditions (a)-(h) for each $i = 1, \ldots, h$. Notice that it is sufficient to establish the following claim in order to complete the proof of the theorem.

Let A be a cyclic group and let $P_1, \ldots, P_h$ be a sequence of A-invariant sections satisfying the conditions $(a), (c), (d), (e)$ of a group G of odd order. Then $h \leq 2\ell + \mathbf{c}(P_h, \ldots, P_1; A)$.

Let $\ell = \ell(A)$ and $h = h(G)$. By the above remark we may assume that $P_1, \ldots, P_h$ is an irreducible A-tower. Set $V = P_h, P = P_{h-1}, Q = S_{h-2}$, $X = PQS_{h-3} \ldots S_1$, and let χ be the XA-character afforded by V.

(1) *We can assume that χ is a complex character by the Fong-Swan theorem.*

(2) $Q = [Q, B]^{S_{h-3} \ldots S_1} Q_0$ for every $B \leq A$ with $\ell(B) \geq 1$, and hence $P = [P, B]^X$ for every $B \leq A$ with $\ell(B) \geq 1$.

Proof. Let $B \leq A$ with $\ell(B) \geq 1$ such that $Q \neq [Q, B]^{S_{h-3} \ldots S_1} Q_0$. Recall that the Frattini factor group of Q/Q_0 is $S_{h-3} \ldots S_1 A$-irreducible. Hence $[Q, B] \leq \Phi(Q) Q_0$, that is $[P_{h-2}/\Phi(P_{h-2}), B] = 1$. It follows that $[P_i, B] = 1$ for each $i < h-2$. Then the sequence

$$P_{h-2}/\Phi(P_{h-2}), P_{h-3}, \ldots, P_1$$

is an A-chain of length $h-2$ centralized by B so that the cyclic group A/B acts on each term of this chain. By induction assumption we have

$$h - 2 \leq 2(\ell - 1) + \mathbf{c}(P_{h-2}/\Phi(P_{h-2}), P_{h-3}, \ldots, P_1; A)$$

which is impossible. Hence $Q = [Q, B]^{S_{h-3} \ldots S_1} Q_0$ for every $B \leq A$ with $\ell(B) \geq 1$.

Next let $B \leq A$ with $\ell(B) \geq 1$ such that $P \neq [P, B]^X$. Set $P_1 = [P, B]^X$. Recall that the Frattini factor group of P is XA-irreducible. Hence $P_1 \leq \Phi(P)$. Now B is trivial on $P/\Phi(P)$. It follows that $[Q/Q_0, B] = 1$, which is not the case. □

(4) *Theorem follows.*

Proof. We are now ready to apply Theorem 2.6 to the action of XA on V as P and Q satisfy the required hypothesis, and obtain the contradiction that χ_A contains the regular A-character, that is $C_V(A) \neq 0$. Since $C_V(A)$ is subnormal subgroup of $S_h \ldots S_1 A$ we see that $\mathbf{c}(P_{h-1} \ldots P_1; A) \leq \mathbf{c}(P_h \ldots P_1; A) - 1$. Thus we have

$$h - 1 \leq 2\ell + \mathbf{c}(P_h \ldots P_1; A) - 1$$

which completes the proof. □

References

[1] S.D. Bell, B. Hartley, *A note on fixed-point-free actions of finite groups*, Quart. J. Math. Oxford Ser. (2), **41** no. 162 (1990) 127–130.

[2] V. V. Belyaev, B. Hartley, *Centralizers of finite nilpotent subgroups in locally finite groups*, Algebra and Logic, **35** (1996) 217–228.

[3] E. C. Dade, *Carter subgroups and Fitting heights of finite solvable groups*, Illinois J. Math., **13** no. 4 (1969) 449-514.

[4] G.Ercan, İ.Ş.Güloğlu, *A Brief Note on the Noncoprime Regular Module Problem*, Ukrainian Mathematical Journal, **72** no.11 (2021) 1837-1841.

[5] G.Ercan, İ.Ş.Güloğlu, *Fixed point free action on groups of odd order,* J. Algebra, **320** no.1 (2008) 426-436.

[6] I.M. Isaacs, *Character theory of finite Groups*, Dover Publications, Inc., New York, 1994.

[7] *The Fitting length of finite soluble groups II: Fixed-point-free automorphisms*, J. Algebra, **487** (2017) 161-172.

[8] P. Hall and G. Higman, *On the p-length of p-soluble groups and reduction theorems for the Burnside problem*, Proc. London Math. Soc. (3) 6 (1956) 1-42.

[9] P.Rowley, *Finite groups admitting a fixed point free automorphism group*, J. Algebra, **174** (1995) 724-727.

[10] E. Shult, *On groups admitting fixed point free abelian operator groups,* Illinois J. Math. **9** (1965) 701- 720.

[11] A. Turull, *Fitting height of groups and of fixed points*, J. Algebra, **86**, Issue 2 (1984) 555-566.

[12] A. Turull, *Character theory and length problems.* (English. English summary) Finite and locally finite groups (Istanbul, 1994), 377–400, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 471, Kluwer Acad. Publ., Dordrecht, 1995.

[13] A. Turull, *Fixed point free action with regular orbits*, J. Reine Angew. Math., **371** (1986) 67-91.

[14] Y. Yang, Regular orbits of nilpotent subgroups of solvable linear groups, J. Algebra, **325** (2011) 56–69.

Department of Mathematics, Middle East Technical University, 06800, Ankara/Turkey

Email address: ercan@metu.edu.tr

Department of Mathematics, Doğuş University, Istanbul, Turkey