NILPOTENT RESIDUAL OF A FINITE GROUP

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Abstract. Let $F$ be a nilpotent group acted on by a group $H$ via automorphisms and let the group $G$ admit the semidirect product $FH$ as a group of automorphisms so that $C_G(F) = 1$. We prove that the order of $\gamma_\infty(G)$, the rank of $\gamma_\infty(G)$ are bounded in terms of the orders of $\gamma_\infty(C_G(H))$ and $H$, the rank of $\gamma_\infty(C_G(H))$ and the order of $H$, respectively in cases where either $FH$ is a Frobenius group; $FH$ is a Frobenius-like group satisfying some certain conditions; or $FH = \langle \alpha, \beta \rangle$ is a dihedral group generated by the involutions $\alpha$ and $\beta$ with $F = \langle \alpha \beta \rangle$ and $H = \langle \alpha \rangle$.

1. Introduction

Throughout all groups are finite. Let a group $A$ act by automorphisms on a group $G$. For any $a \in A$, we denote by $C_G(a)$ the set \( \{ x \in G : x^a = x \} \), and write $C_G(A) = \bigcap_{a \in A} C_G(a)$. In this paper we focus on a certain question related to the strong influence of the structure of such fixed point subgroups on the structure of $G$, and present some new results when the group $A$ is a Frobenius group or a Frobenius-like group or a dihedral group of automorphisms.

In what follows we denote by $A^\#$ the set of all nontrivial elements of $A$, and we say that $A$ acts coprimely on $G$ if $(|A|, |G|) = 1$. Recall that a Frobenius group $A = FH$ with kernel $F$ and complement $H$ can be characterized as a semidirect product of a normal subgroup $F$ by $H$ such that $C_F(h) = 1$ for every $h \in H^\#$. Prompted by Mazurov’s problem 17.72 in the Kourovka Notebook [26], some attention was given to the situation where a Frobenius group $A = FH$ acts by automorphisms on the group $G$. In the case where the kernel $F$ acts fixed-point-freely

2020 Mathematics Subject Classification. 20D45.

Key words and phrases. Frobenius groups, Frobenius-like groups, Dihedral groups, Automorphisms, Nilpotent residual.
on $G$, some results on the structure of $G$ were obtained by Khukhro, Makarenko and Shumyatsky in a series of papers [8], [9], [10], [11], [12], [13], [14]. They observed that various properties of $G$ are in a certain sense close to the corresponding properties of the fixed-point subgroup $C_G(H)$, possibly also depending on $H$. In particular, when $FH$ is metacyclic they proved that if $C_G(H)$ is nilpotent of class $c$, then the nilpotency class of $G$ is bounded in terms of $c$ and $|H|$. In addition, they constructed examples showing that the result on the nilpotency class of $G$ is no longer true in the case of non-metacyclic Frobenius groups. However, recently in [6] it was proved that if $FH$ is supersolvable and $C_G(H)$ is nilpotent of class $c$, then the nilpotency class of $G$ is bounded in terms of $c$ and $|FH|$.

Later on, as a generalization of Frobenius group the concept of a Frobenius-like group was introduced by Ercan and Güloğlu in [16], and their action studied in a series of papers [18], [19], [20], [23], [24], [21]. A finite group $FH$ is said to be Frobenius-like if it has a nontrivial nilpotent normal subgroup $F$ with a nontrivial complement $H$ such that $FH/F'$ is a Frobenius group with Frobenius kernel $F/F'$ and complement $H$ where $F' = [F,F]$. Several results about the properties of a finite group $G$ admitting a Frobenius-like group of automorphisms $FH$ aiming at restrictions on $G$ in terms of $C_G(H)$ and focusing mainly on bounds for the Fitting height and related parameters as a generalization of earlier results obtained for Frobenius groups of automorphisms; and also new theorems for Frobenius-like groups based on new representation-theoretic results. In these papers two special types of Frobenius-like groups have been handled. Namely, Frobenius-like groups $FH$ for which $F'$ is of prime order and is contained in $C_F(H)$; and the Frobenius-like groups $FH$ for which $C_F(H)$ and $H$ are of prime orders, which we call Type I and Type II, respectively throughout the remainder of this paper.

In [25] Shumyatsky showed that the techniques developed in [14] can be used in the study of actions by groups that are not necessarily Frobenius. He considered a dihedral group $D = \langle \alpha, \beta \rangle$ generated by two involutions $\alpha$ and $\beta$ acting on a finite group $G$ in such a manner that $C_G(\alpha\beta) = 1$. In particular, he proved that if $C_G(\alpha)$ and $C_G(\beta)$ are both nilpotent of class $c$, then $G$ is nilpotent and the nilpotency class of $G$ is bounded solely in terms of $c$. In [5], a similar result was obtained for other groups. It should also be noted that in [24] an extension of [25] about the nilpotent length obtained by proving that the nilpotent length of a group $G$ admitting a dihedral group of automorphisms in the same manner is equal to the maximum of the nilpotent lengths of the subgroups $C_G(\alpha)$ and $C_G(\beta)$. 
Throughout we shall use the expression “\((a, b, \ldots)\)-bounded” to abbreviate “bounded from above in terms of \(a, b, \ldots\) only”. Recall that the rank \(r(G)\) of a finite group \(G\) is the minimal number \(r\) such that every subgroup of \(G\) can be generated by at most \(r\) elements. Let \(\gamma_\infty(G)\) denote the nilpotent residual of the group \(G\), that is the intersection of all normal subgroups of \(G\) whose quotients are nilpotent. Recently, in [4], de Melo, Lima and Shumyatsky considered the case where \(A\) is a finite group of prime exponent \(q\) and of order at least \(q^3\) acting on a finite \(q'\)-group \(G\). Assuming that \(|\gamma_\infty(C_G(a))| \leq m\) for any \(a \in A^\#\), they showed that \(\gamma_\infty(G)\) has \((m, q)\)-bounded order. In addition, assuming that the rank of \(\gamma_\infty(C_G(a))\) is at most \(r\) for any \(a \in A^\#\), they proved that the rank of \(\gamma_\infty(G)\) is \((m, q)\)-bounded. Later, in [3], it was proved that the order of \(\gamma_\infty(G)\) can be bounded by a number independent of the order of \(A\).

The purpose of the present article is to study the residual nilpotent of finite groups admitting a Frobenius group, or a Frobenius-like group of Type I and Type II, or a dihedral group as a group of automorphisms. Namely we obtain the following results.

**Theorem A** Let \(FH\) be a Frobenius, or a Frobenius-like group of Type I or Type II, with kernel \(F\) and complement \(H\). Suppose that \(FH\) acts on a finite group \(G\) in such a way that \(C_G(F) = 1\). Then

a) \(|\gamma_\infty(G)|\) is bounded solely in terms of \(|H|\) and \(|\gamma_\infty(C_G(H))|\);

b) the rank of \(\gamma_\infty(G)\) is bounded in terms of \(|H|\) and the rank of \(\gamma_\infty(C_G(H))\).

**Theorem B** Let \(D = \langle \alpha, \beta \rangle\) be a dihedral group generated by two involutions \(\alpha\) and \(\beta\). Suppose that \(D\) acts on a finite group \(G\) in such a manner that \(C_G(\alpha \beta) = 1\). Then

a) \(|\gamma_\infty(G)|\) is bounded solely in terms of \(|\gamma_\infty(C_G(\alpha))|\) and \(|\gamma_\infty(C_G(\beta))|\);

b) the rank of \(\gamma_\infty(G)\) is bounded in terms of the rank of \(\gamma_\infty(C_G(\alpha))\) and \(\gamma_\infty(C_G(\beta))\).

The paper is organized as follows. In Section 2 we list some results to which we appeal frequently. Section 3 is devoted to the proofs of two key propositions which play crucial role in proving Theorem A and Theorem B whose proofs are given in Section 4.

### 2. Preliminaries

If \(A\) is a group of automorphisms of \(G\), we use \([G, A]\) to denote the subgroup generated by elements of the form \(g^{-1}g^a\), with \(g \in G\) and \(a \in A\). Firstly, we recall some well-known facts about coprime action, see for example [7], which will be used without any further references.
Lemma 2.1. Let $Q$ be a group of automorphisms of a finite group $G$ such that $([G], [Q]) = 1$. Then

(a) $G = C_{G}(Q)[G, Q]$.
(b) $Q$ leaves some Sylow $p$-subgroup of $G$ invariant for each prime $p \in \pi(G)$.
(c) $C_{G/N}(Q) = C_{G}(Q)N/N$ for any $Q$-invariant normal subgroup $N$ of $G$.

We list below some facts about the action of Frobenius and Frobenius-like groups. Throughout, a non-Frobenius Frobenius-like group is always considered under the hypothesis below.

Hypothesis* Let $F H$ be a non-Frobenius Frobenius-like group with kernel $F$ and complement $H$. Assume that a Sylow 2-subgroup of $H$ is cyclic and normal, and $F$ has no extraspecial sections of order $p^{2m+1}$ such that $p^{m} + 1 = |H_1|$ for some subgroup $H_1 \leq H$.

It should be noted that Hypothesis* is automatically satisfied if either $|F H|$ is odd or $|H| = 2$.

Theorem 2.2. Suppose that a finite group $G$ admits a Frobenius group or a Frobenius-like group of automorphisms $F H$ with kernel $F$ and complement $H$ such that $C_{G}(F) = 1$. Then $C_{G}(H) \neq 1$ and $r(G)$ is bounded in terms of $r(C_{G}(H))$ and $|H|$.

Proposition 2.3. Let $F H$ be a Frobenius, or a Frobenius-like group of Type I or Type II. Suppose that $F H$ acts on a $q$-group $Q$ for some prime $q$ coprime to the order of $H$ in case $F H$ is not Frobenius. Let $V$ be a $kQF H$-module where $k$ is a field with characteristic not dividing $|QH|$. Suppose further that $F$ acts fixed-point freely on the semidirect product $VQ$. Then we have $C_{V}(H) \neq 0$ and

$$\text{Ker}(C_{Q}(H) \text{ on } C_{V}(H)) = \text{Ker}(C_{Q}(H) \text{ on } V).$$

Proof. See [17] Proposition 2.2 when $F H$ is Frobenius; [18] Proposition C when $F H$ is Frobenius-like of Type I; and [22] Proposition 2.1 when $F H$ is Frobenius-like of Type II. It can be easily checked that [17] Proposition 2.2 is valid when $C_{Q}(F) = 1$ without the coprimeness condition $(|Q|, |F|) = 1$.

The proof of the following theorem can be found in [25] and in [2].

Theorem 2.4. Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by two involutions $\alpha$ and $\beta$. Suppose that $D$ acts on a finite group $G$ in such a manner that $C_{G}(\alpha \beta) = 1$. Then

(a) $G = C_{G}(\alpha)C_{G}(\beta)$;
(b) the rank of $G$ is bounded in terms of the rank of $C_{G}(\alpha)$ and $C_{G}(\beta)$;
Proposition 2.5. Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions $\alpha$ and $\beta$. Suppose that $D$ acts on a $q$-group $Q$ for some prime $q$ and let $V$ be a $kQD$-module for a field $k$ of characteristic different from $q$ such that the group $F = \langle \alpha \beta \rangle$ acts fixed point freely on the semidirect product $VQ$. If $C_Q(\alpha)$ acts nontrivially on $V$ then we have $C_V(\alpha) \neq 0$ and $\text{Ker}(C_Q(\alpha) \text{ on } C_V(\alpha)) = \text{Ker}(C_Q(\alpha) \text{ on } V)$.

Proof. This is Proposition C in [24]. □

The next two results were established in [14, Lemma 1.6].

Lemma 2.6. Suppose that a group $Q$ acts by automorphisms on a group $G$. If $Q = \langle q_1, \ldots, q_n \rangle$, then $[G, Q] = [G, q_1] \cdots [G, q_n]$.

Lemma 2.7. Let $p$ be a prime, $P$ a finite $p$-group and $Q$ a $p'$-group of automorphisms of $P$.

a) If $|[P, q]| \leq m$ for every $q \in Q$, then $|Q|$ and $|[P, Q]|$ are $m$-bounded.

b) If $r([P, q]) \leq m$ for every $q \in Q$, then $r(Q)$ and $r([P, Q])$ are $m$-bounded.

We also need the following fact whose proof can be found in [1].

Lemma 2.8. Let $G$ be a finite group such that $\gamma_\infty(G) \leq F(G)$. Let $P$ be a Sylow $p$-subgroup of $\gamma_\infty(G)$ and $H$ be a Hall $p'$-subgroup of $G$. Then $P = [P, H]$.

3. Key Propositions

We prove below a new proposition which studies the actions of Frobenius and Frobenius-like groups and forms the basis in proving Theorem A.

Proposition 3.1. Assume that $FH$ be a Frobenius group, or a Frobenius-like group of Type I or Type II with kernel $F$ and complement $H$. Suppose that $F$ acts on a $q$-group $Q$ for some prime $q$. Let $V$ be an irreducible $\mathbb{F}_pQFH$-module where $\mathbb{F}_p$ is a field with characteristic $p$ not dividing $|Q|$ such that $F$ acts fixed-point-freely on the semidirect product $VQ$. Additionally, we assume that $q$ is coprime to $|H|$ in case where $FH$ is not Frobenius. Then $r([V, Q])$ is bounded in terms of $r([C_V(H), C_Q(H)])$ and $|H|$.

Proof. Let $r([C_V(H), C_Q(H)]) = s$. We may assume that $V = [V, Q]$ and hence $C_V(Q) = 0$. By Clifford’s Theorem, $V = V_1 \oplus \cdots \oplus V_t$, direct sum of of $Q$-homogeneous components $V_i$, which are transitively
permuted by $FH$. Set $\Omega = \{V_1, \ldots, V_t\}$ and fix an $F$-orbit $\Omega_1$ in $\Omega$. Throughout, $W = \Sigma_{U \in \Omega} U$.

Now, we split the proof into a sequence of steps.

(1) We may assume that $Q$ acts faithfully on $V$. Furthermore $\text{Ker}(C_Q(H) \text{ on } C_V(H)) = \text{Ker}(C_Q(H) \text{ on } V) = 1$.

**Proof.** Suppose that $\text{Ker}(Q \text{ on } V) \neq 1$ and set $\overline{Q} = Q/\text{Ker}(Q \text{ on } V)$. Note that since $C_Q(F) = 1$, $F$ is a Carter subgroup of $QF$ and hence also a Carter subgroup of $\overline{Q}F$ which implies that $C_{\overline{Q}}(F) = 1$. Notice that the equality $C_Q(H) = C_{\overline{Q}}(H)$ holds in case $FH$ is Frobenius (see [14] Theorem 2.3). The same equality holds in case where $FH$ is non-Frobenius due to the coprimeness condition $(q, |H|) = 1$. Then $[C_V(H), C_Q(H)] = [C_V(H), C_{\overline{Q}}(H)]$ and so we may assume that $Q$ acts faithfully on $V$. Notice that by Proposition 2.3 we have

$$\text{Ker}(C_Q(H) \text{ on } C_V(H)) = \text{Ker}(C_Q(H) \text{ on } V) = 1$$

establishing the claim. \hfill \Box

(2) We may assume that $Q = \langle c^F \rangle$ for any nonidentity element $c \in C_{Z(Q)}(H)$ of order $q$. In particular $Q$ is abelian.

**Proof.** We obtain that $C_{Z(Q)}(H) \neq 1$ as $C_Q(F) = 1$ by Proposition 2.3. Let now $1 \neq c \in C_{Z(Q)}(H)$ of order $q$ and consider $\langle c^FH \rangle = \langle c^F \rangle$, the minimal $FH$-invariant subgroup containing $c$. Since $V$ is an irreducible $QFH$-module on which $Q$ acts faithfully we have that $V = [V, \langle c^F \rangle]$. Thus we may assume that $Q = \langle c^F \rangle$ as claimed. \hfill \Box

(3) $V = [V, c] \cdot [V, c^{f_1}] \cdots [V, c^{f_n}]$ where $n$ is a $(s, |H|)$-bounded number. Hence it suffices to bound $r([W, c])$.

**Proof.** Notice that the group $C_Q(H)$ embeds in the automorphism group of $[C_V(H), C_Q(H)]$ by step (1). Then $C_Q(H)$ has $s$-bounded rank by Lemma 2.7. This yields by Theorem 2.2 that $Q$ has $(s, |H|)$-bounded rank. Thus, there exist $f_1, \ldots, f_n$ in $F$ for an $(s, |H|)$-bounded number $n$ such that $Q = \langle c^{f_1}, \ldots, c^{f_n} \rangle$. Now $V = [V, c] \cdot [V, c^{f_1}] \cdots [V, c^{f_n}] = \prod_{i=1}^{n} [V, c]^{f_i}$ by Lemma 2.6. This shows that we need only to bound $r([V, c])$ suitably. In fact it suffices to show that $r([W, c])$ is suitably bounded as $V = \Sigma_{h \in H} W^h$. \hfill \Box

(4) $H_1 = \text{Stab}_H(\Omega_1) \neq 1$. Furthermore the rank of the sum of members of $\Omega_1$ which are not centralized by $c$ and contained in a regular $H_1$-orbit, is suitably bounded.
NILPOTENT RESIDUAL OF A FINITE GROUP

**Proof.** Fix $U \in \Omega_1$ and set $Stab_F(U) = F_1$. Choose a transversal $T$ for $F_1$ in $F$. Let $W = \sum_{t \in T} U^t$ where $T$ is a transversal for $F_1$ in $F$ with $1 \in T$. Then we have $V = \sum_{h \in H} W^h$. Notice that $[V, c] \neq 0$ by (1) which implies that $[W, c] \neq 0$ and hence $[U^t, c] = U^t$ for some $t \in T$. Without loss of generality we may assume that $[U, c] = U$.

Suppose that $Stab_H(\Omega_1) = 1$. Then we also have $Stab_H(U^t) = 1$ for all $t \in T$ and hence the sum $X_t = \sum_{h \in H} U^t$ is direct for all $t \in T$. Now, $U \leq X_1$. It holds that

$$C_{X_1}(H) = \{ \sum_{h \in H} v^h : v \in U\}.$$

Then $|U| = |C_{X_1}(H)| = ||C_{X_1}(H), c|| \leq ||C_V(H), C_Q(H)||$ implies $r(U) \leq s$. On the other hand $V = \bigoplus_{t \in T} X_t$ and

$$[C_V(H), c] = \bigoplus \{ [C_{X_t}(H), c] : t \in T \text{ with } [U^t, c] \neq 0 \} \leq [C_V(H), C_Q(H)].$$

In particular, $\{ t \in T : [U^t, c] \neq 0 \}$ is suitably bounded whence $r([W, c])$ is $(s, |H|)$-bounded. Hence we may assume that $Stab_H(\Omega_1) \neq 1$.

Notice that every element of a regular $H_1$-orbit in $\Omega_1$ lies in a regular $H$-orbit in $\Omega$. Let $U \in \Omega_1$ be contained in a regular $H_1$-orbit of $\Omega_1$. Let $X$ denote the sum of the members of the $H$-orbit of $U$ in $\Omega$, that is $X = \bigoplus_{h \in H} U^h$. Then $C_X(H) = \{ \sum_{h \in H} v^h : v \in U\}$. If $[U, c] \neq 0$ then by repeating the same argument in the above paragraph we show that $r(U) \leq s$ is suitably bounded. On the other hand the number, say $m$, of all $H$-orbits in $\Omega$ containing a member $U$ such that $[U, c] \neq 0$ is suitably bounded because $m \leq r([C_V(H), c]) \leq s$. It follows then that the rank of the sum of members of $\Omega_1$ which are not centralized by $c$ and contained in a regular $H_1$-orbit, is suitably bounded. \hfill $\Box$

(5) We may assume that $FH$ is not Frobenius.

**Proof.** Assume the contrary that $FH$ is Frobenius. Let $H_1 = Stab_H(\Omega_1)$ and pick $U \in \Omega_1$. Set $S = Stab_{FH_1}(U)$ and $F_1 = F \cap S$. Then $|F : F_1| = |\Omega_1| = |FH_1 : S|$ and so $|S : F_1| = |H_1|$. Since $(|F_1|, |H_1|) = 1$, by the Schur-Zassenhaus theorem there exists a complement, say $S_1$ of $F_1$ in $S$ with $|H_1| = |S_1|$. Therefore there exists a conjugate of $U$ which is $H_1$-invariant. There is no loss in assuming that $U$ is $H_1$-invariant. On the other hand if $1 \neq h \in H_1$ and $x \in F$ such that $U^{xh} = U^x$, then $[h, x] \in Stab_F(U) = F_1$ and so $F_1 x = F_1 x^h = (F_1 x)^h$. This implies that $F_1 x \cap C_F(h)$ is nonempty. Now the Frobenius action of $H$ on $F$ forces that $x \in F_1$. This means that for each $x \in F \setminus F_1$ we have $Stab_{FH_1}(U^x) = 1$. Therefore $U$ is the unique member of $\Omega_1$ which is $H_1$-invariant and all the $H_1$-orbits other than $\{U\}$ are regular. By (4), the rank of the sum of all members of $\Omega_1$ other than $U$ is is suitably
bounded. In particular $r(U)$ and hence $r([W, c])$ is suitably bounded in case where $[U^x, c] \neq 0$ for some $x \in F \setminus F_1$. Thus we may assume that $c$ is trivial on $U^x$ for all $x \in F \setminus F_1$. Now we have $[W, c] = [U, c] = U$.

Due to the action by scalars of the abelian group $Q$ on $U$, it holds that $[Q, F_1] \leq C_Q(U)$. We also know that $c^x$ is trivial on $U$ for each $x \in F \setminus F_1$. Since $C_Q(F) = 1$, there are prime divisors of $|F|$ different from $q$. Let $F_q'$ denote the $q'$-Hall subgroup of $F$. Clearly we have $C_Q(F_q') = 1$. Let now $y = \prod_{f \in F_q'} c^f$. Then we have

$$1 = y = \left( \prod_{f \in F_q' \cap F_q' F_1} c^f \right) \left( \prod_{f \in F_q' \setminus F_q'} c^f \right) \in c^{F_q' \cap F_q' F_1} C_Q(U).$$

As a consequence $c \in C_Q(U)$, because $q$ is coprime to $|F_q'|$. This contradiction establishes the claim. \hfill \qed

(6) We may assume that the group $FH$ is Frobenius-like of Type II.

**PROOF.** On the contrary we assume that $FH$ is Frobenius-like of Type I. By (4), we have $H_1 = Stab_H(\Omega_1) \neq 1$. Choose a transversal $T_1$ for $H_1$ in $H$. Now $V = \bigoplus_{h \in T_1} W^h$. Also we can guarantee the existence of a conjugate of $U$ which is $H_1$-invariant by means of the Schur-Zassenhaus Theorem as in (5). There is no loss in assuming that $U$ is $H_1$-invariant.

Set now $Y = \sum_{x \in F} U^x$ and $F_2 = Stab_F(Y)$ and $F_1 = Stab_F(U)$. Clearly, $F_2 = F'F_1$ and $Y$ is $H_1$-invariant. Notice that for all nonidentity $h \in H$, we have $C_F(h) \leq F' \leq F_2$. Assume first that $F = F_2$. This forces that we have $V = Y$. Clearly, $Y \neq U$, that is $F'' \not\leq F_1$, because otherwise $Q = [Q, F] = 1$ due to the scalar action of the abelian group $Q$ on $U$. So $F' \cap F_1 = 1$ which implies that $|F : F_1|$ is a prime. Then $F_1 \leq F$ and $F'' \leq F_1$ which is impossible. Therefore $F \neq F_2$.

If $1 \neq h \in H$ and $t \in F$ such that $Y^th = Y^t$ then $[h, t] \in F_2$. Now, $F_2t = F_2t^h = (F_2t)^h$ and this implies the existence of an element in $F_2t \cap C_F(h)$. Since $C_F(h) \leq F'' \leq F_2$ we get $t \in F_2$. In particular, for each $t \in F \setminus F_2$ we have $Stab_H(Y^t) = 1$.

Let $S$ be a transversal for $F_2$ in $F$. For any $t \in S \setminus F_2$ set $Y_t = Y^t$ and consider $Z_t = \sum_{h \in H} Y_t^h$. Notice that $V = Y \oplus \bigoplus_{t \in S \setminus F_2} Z_t$. As the sum $Z_t$ is direct we have

$$C_{Z_t}(H) = \left\{ \sum_{h \in H} v^h : v \in Y_t \right\}$$

with $|C_{Z_t}(H)| = |Y_t|$. Then $r([Y_t, c]) = r([C_{Z_t}(H), c]) \leq s$ for each $t \in S \setminus F_2$ with $[Y_t, c] \neq 0$. On the other hand,

$$\Sigma\{r([C_{Z_t}(H), c]) : t \in S \text{ with } [Y_t, c] \neq 0\} \leq r([C_{Y}(H), c]) \leq s$$
First assume that $F \cup U \cup V$ have $V = Y \oplus \bigoplus_{t \in S \setminus F_2} Z_t$. Thus we may assume that $c$ is trivial on \( \bigoplus_{t \in S \setminus F_2} Z_t \) and hence $|Y, c| = [Y, c]$.

There are two cases now: We have either $F' \cap F_1 = 1$ or $F' \leq F_1$. First assume that $F' \leq F_1$. Then we get $F_1 = F_2$ because $F_2 = F' F_1$.

Now $U = Y$. Due to the action by scalars of the abelian group $Q$ on $U$, it holds that $[Q, F_1] \leq C_Q(U)$. From this point on we can proceed as in the proof of step (5) and observe that $C_Q(F_{q'}) = 1$. Letting now $y = \prod_{f \in F_{q'}} c^f$, we have

\[
1 = y = ( \prod_{f \in F_1 \cap F'} c^f ) ( \prod_{f \in F_{q'} \setminus F_1} c^f ) \in c^{F_1 \cap F'} C_Q(U),
\]

implying that $c \in C_Q(U)$, because $q$ is coprime to $|F_{q'}|$.

Thus we have $F_1 \cap F' = 1$. First assume that $H_1 = H$. Then $Y$ is $H$-invariant and $F_1 H$ is a Frobenius group. Note that $C_U(F_1) = 1$ as $C_V(F) = 1$, and hence $C_Y(F_1) = 1$ since $F' \leq Z(F)$. We consider now the action of $QF_1 H$ on $Y$ and the fact that $r([C_Y(H), C_Q(H)]) \leq s$.

Then step (5), we obtain that $r(Y) = r([Y, Q])$ is $(s, |H|)$-bounded. Next assume that $H_1 \neq H$. Choose a transversal for $H_1$ in $H$ and set $Y_1 = \Sigma_{h \in T_1} Y^h$. Clearly this sum is direct and hence

\[
C_{Y_1}(H) = \{ \sum_{h \in T_1} v^h : v \in Y \}
\]

with $|[C_{Y_1}(H), c]| = |[Y, c]|$. Then $r([Y, c]) = r([C_{Y_1}(H), c]) \leq s$ establishing claim (6).

(7) The proposition follows.

**Proof.** From now on $FH$ is a Frobenius-like group of Type II, that is, $H$ and $C_F(H)$ are of prime orders. By step (4) we have $H = H_1 \neq Stab_H(\Omega_1)$ since $|H|$ is a prime. Now $V = W$. We may also assume by the Schur-Zassenhaus theorem as in the previous steps that there is an $H$-invariant element, say $U$ in $\Omega$. Let $T$ be a transversal for $F_1 = Stab_F(U)$ in $F$. Then $F = \bigcup_{t \in T} F_1 t$ implies $V = \bigoplus_{t \in T} U^t$.

It should also be noted that we have $|\{ t \in T : [U^t, c] \neq 0 \}|$ is suitably bounded as

\[
[C_V(H), c] = \bigoplus \{ [C_{X_t}(H), c] : t \in T \text{ with } [U^t, c] \neq 0 \} \leq [C_V(H), C_Q(H)]
\]

where $X_t = \bigoplus_{h \in H} U^{th}$.

Let $X$ be the sum of the components of all regular $H$-orbits on $\Omega$, and let $Y$ denote the sum of all $H$-invariant elements of $\Omega$. Then
V = X ⊕ Y. Suppose that \( U^t = U \) for \( t \in T \) and \( 1 \neq h \in H \). Now \( [t, h] \in F_1 \) and so the coset \( F_t \) is fixed by \( H \). Since the orders of \( F \) and \( H \) are relatively prime we may assume that \( t \in C_F(H) \).

Conversely for each \( t \in C_F(H) \), \( U^t \) is \( H \)-invariant. Hence the number of components in \( Y \) is \( |T \cap C_F(H)| = |C_F(H) : C_{F_1}(H)| \) and so we have either \( C_F(H) \leq F_1 \) or not.

If \( C_F(H) \not\leq F_1 \) then \( C_{F_1}(H) = 1 \) whence \( F_1 H \) is Frobenius group acting on \( U \) in such a way that \( C_U(F_1) = 1 \). Then \( r(U) \) is \((s, |H|)-\)bounded by step (5) since \( r([C_U(H), C_Q(H)]) \leq s \) holds. This forces that \( r([V, c]) \) is bounded suitably and hence the claim is established.

Thus we may assume that \( C_F(H) \leq F_1 \). Then \( Y = U \) is the unique \( H \)-invariant \( Q \)-homogeneous component. If \([U^t, c] \neq 0\) for some \( t \in F \setminus F_1 \) we can bound \( r(U) \) and hence \( r([V, c]) \) suitably. Thus we may assume that \( c \) is trivial on \( U^t \) for each \( t \in F \setminus F_1 \). Due to the action of the abelian group \( Q \) on \( U \), it holds that \([Q, F_1] \leq C(Q)\) from this point on we can proceed as in the proof of step (5) and observe that \( C_Q(F_q') = 1 \). Letting now \( y = \prod_{f \in F_q'} c^j \), we have

\[
1 = y = \left( \prod_{f \in F_t \cap F_q'} c^j \right) \left( \prod_{f \in F_q' \setminus F_1} c^j \right) \in c^{|F_t \cap F_q'|} C_Q(U).
\]

implying that \( c \in C_Q(U) \), because \( q \) is coprime to \( |F_q'| \). This final contradiction completes the proof of Proposition 3.1.

The next proposition studies the action of a dihedral group of automorphisms and is essential in proving Theorem B.

**Proposition 3.2.** Let \( D = \langle \alpha, \beta \rangle \) be a dihedral group generated by two involutions \( \alpha \) and \( \beta \). Suppose that \( D \) acts on a \( q \)-group \( Q \) for some prime \( q \). Let \( V \) be an irreducible \( \mathbb{F}_p QD \)-module where \( \mathbb{F}_p \) is a field with characteristic \( p \) not dividing \( |Q| \). Suppose that \( C_V Q(F) = 1 \) where \( F = \langle \alpha \beta \rangle \). If \( \max \{ r([C_V(\alpha), C_Q(\alpha)]), r([C_V(\beta), C_Q(\beta)]) \} \leq s \), then \( r([V, Q]) \) is \( s \)-bounded.

**Proof.** We set \( H = \langle \alpha \rangle \). So \( D = FH \). By Lemma 2.4 and Theorem 2.6 we have \([V, Q] = [V, C_Q(\alpha)][V, C_Q(\beta)]\). Then it is sufficient to bound the rank of \([V, C_Q(H)]\). Following the same steps as in the proof of Proposition 3.1 by replacing Proposition 2.3 by Proposition 2.4, we observe that \( Q \) acts faithfully on \( V \) and \( Q = \langle c^F \rangle \) is abelian with \( c \in C_{Z(Q)}(H) \) of order \( q \). Furthermore \( \text{Ker}(C_Q(H)) \) on \( C_V(H) \) = \( \text{Ker}(C_Q(H)) \) on \( V \) = 1. Note that it suffices to bound \( r([V, c]) \) suitably.

Let \( \Omega \) denote the set of \( Q \)-homogeneous components of the irreducible \( QD \)-module \( V \). Let \( \Omega_1 \) be an \( F \)-orbit of \( \Omega \) and set \( W = \sum_{U \in \Omega} U. \)
Then we have \( V = W + W^\alpha \). Suppose that \( W^\alpha \neq W \). Then for any \( U \in \Omega_1 \) we have \( \text{Stab}_H(U) = 1 \). Let \( T \) be a transversal for \( \text{Stab}_F(U) = F_1 \) in \( F \). It holds that \( V = \sum_{t \in T} X_t \) where \( X_t = U^t + U^{t^\alpha} \). Now \( [V, c] = \sum_{t \in T} [X_t, c] \) and \( C_V(H) = \sum_{t \in T} C_{X_t}(H) \) where \( C_{X_t}(H) = \{ w + w^\alpha : w \in U^t \} \). Since \([V, c] \neq 0\) there exists \( t \in T \) such that \([U^t, c] \neq 0\), that is \([U^t, c] = U^t \). Then \( [C_{X_t}(H), c] = C_{X_t}(H) \). Since \( r([C_V(H), C_Q(H)]) \leq s \) we get \( r(U) = r(C_{X_t}(H)) \leq s \). Furthermore it follows that \( \{ t \in T : [U^t, c] \neq 0 \} \) is \( s \)-bounded and as a consequence \( r([V, c]) \) is suitably bounded. Thus we may assume that \( W^\alpha = W \) which implies that \( \Omega_1 = \Omega \) and \( H \) fixes an element, say \( U \), of \( \Omega \) as desired.

Let \( U^t \in \Omega \) be \( H \)-invariant. Then \([t, \alpha] \in F_1 \). On the other hand \( t^{-1} t^\alpha = t^{-2} \) since \( \alpha \) inverts \( F \). So \( F_1 t \) is an element of \( F/F_1 \) of order at most 2 which implies that the number of \( H \)-invariant elements of \( \Omega \) is at most 2. Let now \( Y \) be the sum of all \( H \)-invariant elements of \( \Omega \). Then \( V = Y \oplus \bigoplus_{i=1}^m X_i \) where \( X_1, \ldots X_m \) are the sums of elements in \( H \)-orbits of length 2. Let \( X_i = U_i \oplus U_i^\alpha \). Notice that if \([U_i, c] \neq 0\) for some \( i \), then we obtain \( r(U) = r(U_i) \leq s \) by a similar argument as above. On the other hand we observe that the number of \( i \) for which \([U_i, c] \neq 0\) is \( s \)-bounded by the hypothesis that \( r([C_V(H), c]) \leq s \). It follows now that \( r([V, c]) \) is suitably bounded in case where \([U_i, c] \neq 0\) for some \( i \).

Thus we may assume that \( c \) centralizes \( \bigoplus_{i=1}^m X_i \) and that \([U, c] = U \). Due to the scalar action by scalars of the abelian group \( Q \) on \( U \), it holds that \([Q, F_1] \leq C_Q(U) \). As \( F_1 \leq FH \), we have \([Q, F_1] \leq C_Q(V) = 1 \). Clearly we have \( C_Q(F_{q'}) = 1 \) where \( F_{q'} \) denotes the Hall \( q' \)-part of \( F \) whose existence is guaranteed by the fact that \( C_Q(F) = 1 \). Let now \( y = \prod_{f \in F_q} c^f \). Then we have \( 1 = y = ( \prod_{f \in F_q \cap F_{q'}} c^f \prod_{f \in F_q \setminus F_1} c^f ) \in c^{[F_q \cap F_{q'}]} C_Q(U) \). As a consequence \( c \in C_Q(U) \), because \( q \) is coprime to \( |F_{q'}| \). This contradiction completes the proof of Proposition 3.2.

\( \square \)

4. Proofs of theorems

Firstly, we shall give a detailed proof for Theorem A part (b). The proof of Theorem A (a) can be easily obtained by just obvious modifications of the proof of part (b).

First, we assume that \( G = PQ \) where \( P \) and \( Q \) are \( FH \)-invariant subgroups such that \( P \) is a normal \( p \)-subgroup for a prime \( p \) and \( Q \) is
a nilpotent $p'$-group with $|[C_p(H), C_Q(H)]| = p^s$. We shall prove that $r(\gamma_\infty(G))$ is $(s, |H|)$-bounded. Clearly $\gamma_\infty(G) = [P, Q]$. Consider an unrefinable $FH$-invariant normal series

$$P = P_1 > P_2 > \cdots > P_k > P_{k+1} = 1.$$  

Note that its factors $P_i/P_{i+1}$ are elementary abelian. Let $V = P_k$. Since $C_V(Q) = 1$, we have that $V = [V, Q]$. We can also assume that $Q$ acts faithfully on $V$. Proposition 3.1 yields that $r(V)$ is $(s, |H|)$-bounded. Set $S_i = P_i/P_{i+1}$. If $[C_{S_i}(H), C_Q(H)] = 1$, then $[S_i, Q] = 1$ by Proposition 2.3. Since $C_P(Q) = 1$ we conclude that each factor $S_i$ contains a nontrivial image of an element of $[C_P(H), C_Q(H)]$. This forces that $k \leq s$. Then we proceed by induction on $k$ to obtain that $r([P, Q])$ is an $(s, |H|)$-bounded number, as desired.

Let $F(G)$ denote the Fitting subgroup of a group $G$. Write $F_0(G) = 1$ and let $F_{i+1}(G)$ be the inverse image of $F(G/F_i(G))$. As is well known, when $G$ is soluble, the least number $h$ such that $F_h(G) = G$ is called the Fitting height $h(G)$ of $G$. Let now $r$ be the rank of $\gamma_\infty(C_G(H))$. Then $C_G(H)$ has $r$-bounded Fitting height (see for example Lemma 1.4 of [15]) and hence $G$ has $(r, |H|)$-bounded Fitting height.

We shall proceed by induction on $h(G)$. Firstly, we consider the case where $h(G) = 2$. Indeed, let $P$ be a Sylow $p$-subgroup of $\gamma_\infty(G)$ and $Q$ an $F'H$-invariant Hall $p'$-subgroup of $G$. Then, by the preceding paragraphs and Lemma 2.3 the rank of $P = [P, Q]$ is $(r, |H|)$-bounded and so the rank of $\gamma_\infty(G)$ is $(r, |H|)$-bounded. Assume next that $h(G) > 2$ and let $N = F_2(G)$ be the second term of the Fitting series of $G$. It is clear that the Fitting height of $G/\gamma_\infty(N)$ is $h - 1$ and $\gamma_\infty(N) \leq \gamma_\infty(G)$. Hence, by induction we have that $\gamma_\infty(G)/\gamma_\infty(N)$ has $(r, |H|)$-bounded rank. As a consequence, it holds that

$$r(\gamma_\infty(G)) \leq r(\gamma_\infty(G)/\gamma_\infty(N)) + r(\gamma_\infty(N))$$

completing the proof of Theorem A(b).

The proof of Theorem B can be directly obtained as in the above argument by replacing Proposition 3.1 by Proposition 3.2 and Proposition 2.3 by Proposition 2.5.

References


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