

NILPOTENT RESIDUAL OF A FINITE GROUP

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ABSTRACT. Let F be a nilpotent group acted on by a group H via automorphisms and let the group G admit the semidirect product FH as a group of automorphisms so that $C_G(F) = 1$. We prove that the order of $\gamma_\infty(G)$, the rank of $\gamma_\infty(G)$ are bounded in terms of the orders of $\gamma_\infty(C_G(H))$ and H , the rank of $\gamma_\infty(C_G(H))$ and the order of H , respectively in cases where either FH is a Frobenius group; FH is a Frobenius-like group satisfying some certain conditions; or $FH = \langle \alpha, \beta \rangle$ is a dihedral group generated by the involutions α and β with $F = \langle \alpha\beta \rangle$ and $H = \langle \alpha \rangle$.

1. Introduction

Throughout all groups are finite. Let a group A act by automorphisms on a group G . For any $a \in A$, we denote by $C_G(a)$ the set $\{x \in G : x^a = x\}$, and write $C_G(A) = \bigcap_{a \in A} C_G(a)$. In this paper we focus on a certain question related to the strong influence of the structure of such fixed point subgroups on the structure of G , and present some new results when the group A is a Frobenius group or a Frobenius-like group or a dihedral group of automorphisms.

In what follows we denote by $A^\#$ the set of all nontrivial elements of A , and we say that A acts coprimely on G if $(|A|, |G|) = 1$. Recall that a Frobenius group $A = FH$ with kernel F and complement H can be characterized as a semidirect product of a normal subgroup F by H such that $C_F(h) = 1$ for every $h \in H^\#$. Prompted by Mazurov's problem 17.72 in the Kourovka Notebook [26], some attention was given to the situation where a Frobenius group $A = FH$ acts by automorphisms on the group G . In the case where the kernel F acts fixed-point-freely

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on G , some results on the structure of G were obtained by Khukhro, Makarenko and Shumyatsky in a series of papers [8], [9], [10], [11], [12], [13], [14]. They observed that various properties of G are in a certain sense close to the corresponding properties of the fixed-point subgroup $C_G(H)$, possibly also depending on H . In particular, when FH is metacyclic they proved that if $C_G(H)$ is nilpotent of class c , then the nilpotency class of G is bounded in terms of c and $|H|$. In addition, they constructed examples showing that the result on the nilpotency class of G is no longer true in the case of non-metacyclic Frobenius groups. However, recently in [6] it was proved that if FH is supersolvable and $C_G(H)$ is nilpotent of class c , then the nilpotency class of G is bounded in terms of c and $|FH|$.

Later on, as a generalization of Frobenius group the concept of a Frobenius-like group was introduced by Ercan and Güloğlu in [16], and their action studied in a series of papers [18], [19],[20],[23],[24],[21]. A finite group FH is said to be Frobenius-like if it has a nontrivial nilpotent normal subgroup F with a nontrivial complement H such that FH/F' is a Frobenius group with Frobenius kernel F/F' and complement H where $F' = [F, F]$. Several results about the properties of a finite group G admitting a Frobenius-like group of automorphisms FH aiming at restrictions on G in terms of $C_G(H)$ and focusing mainly on bounds for the Fitting height and related parameters as a generalization of earlier results obtained for Frobenius groups of automorphisms; and also new theorems for Frobenius-like groups based on new representation-theoretic results. In these papers two special types of Frobenius-like groups have been handled. Namely, Frobenius-like groups FH for which F' is of prime order and is contained in $C_F(H)$; and the Frobenius-like groups FH for which $C_F(H)$ and H are of prime orders, which we call Type I and Type II, respectively throughout the remainder of this paper.

In [25] Shumyatsky showed that the techniques developed in [14] can be used in the study of actions by groups that are not necessarily Frobenius. He considered a dihedral group $D = \langle \alpha, \beta \rangle$ generated by two involutions α and β acting on a finite group G in such a manner that $C_G(\alpha\beta) = 1$. In particular, he proved that if $C_G(\alpha)$ and $C_G(\beta)$ are both nilpotent of class c , then G is nilpotent and the nilpotency class of G is bounded solely in terms of c . In [5], a similar result was obtained for other groups. It should also be noted that in [24] an extension of [25] about the nilpotent length obtained by proving that the nilpotent length of a group G admitting a dihedral group of automorphisms in the same manner is equal to the maximum of the nilpotent lengths of the subgroups $C_G(\alpha)$ and $C_G(\beta)$.

Throughout we shall use the expression “ (a, b, \dots) -bounded” to abbreviate “bounded from above in terms of a, b, \dots only”. Recall that the rank $\mathbf{r}(G)$ of a finite group G is the minimal number r such that every subgroup of G can be generated by at most r elements. Let $\gamma_\infty(G)$ denote the *nilpotent residual* of the group G , that is the intersection of all normal subgroups of G whose quotients are nilpotent. Recently, in [4], de Melo, Lima and Shumyatsky considered the case where A is a finite group of prime exponent q and of order at least q^3 acting on a finite q' -group G . Assuming that $|\gamma_\infty(C_G(a))| \leq m$ for any $a \in A^\#$, they showed that $\gamma_\infty(G)$ has (m, q) -bounded order. In addition, assuming that the rank of $\gamma_\infty(C_G(a))$ is at most r for any $a \in A^\#$, they proved that the rank of $\gamma_\infty(G)$ is (m, q) -bounded. Later, in [3], it was proved that the order of $\gamma_\infty(G)$ can be bounded by a number independent of the order of A .

The purpose of the present article is to study the residual nilpotent of finite groups admitting a Frobenius group, or a Frobenius-like group of Type I and Type II, or a dihedral group as a group of automorphisms. Namely we obtain the following results.

Theorem A Let FH be a Frobenius, or a Frobenius-like group of Type I or Type II, with kernel F and complement H . Suppose that FH acts on a finite group G in such a way that $C_G(F) = 1$. Then

- a) $|\gamma_\infty(G)|$ is bounded solely in terms of $|H|$ and $|\gamma_\infty(C_G(H))|$;
- b) the rank of $\gamma_\infty(G)$ is bounded in terms of $|H|$ and the rank of $\gamma_\infty(C_G(H))$.

Theorem B Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by two involutions α and β . Suppose that D acts on a finite group G in such a manner that $C_G(\alpha\beta) = 1$. Then

- a) $|\gamma_\infty(G)|$ is bounded solely in terms of $|\gamma_\infty(C_G(\alpha))|$ and $|\gamma_\infty(C_G(\beta))|$;
- b) the rank of $\gamma_\infty(G)$ is bounded in terms of the rank of $\gamma_\infty(C_G(\alpha))$ and $\gamma_\infty(C_G(\beta))$.

The paper is organized as follows. In Section 2 we list some results to which we appeal frequently. Section 3 is devoted to the proofs of two key propositions which play crucial role in proving Theorem A and Theorem B whose proofs are given in Section 4.

2. Preliminaries

If A is a group of automorphisms of G , we use $[G, A]$ to denote the subgroup generated by elements of the form $g^{-1}g^a$, with $g \in G$ and $a \in A$. Firstly, we recall some well-known facts about coprime action, see for example [7], which will be used without any further references.

LEMMA 2.1. *Let Q be a group of automorphisms of a finite group G such that $(|G|, |Q|) = 1$. Then*

- (a) $G = C_G(Q)[G, Q]$.
- (b) Q leaves some Sylow p -subgroup of G invariant for each prime $p \in \pi(G)$.
- (c) $C_{G/N}(Q) = C_G(Q)N/N$ for any Q -invariant normal subgroup N of G .

We list below some facts about the action of Frobenius and Frobenius-like groups. Throughout, a non-Frobenius Frobenius-like group is always considered under the hypothesis below.

Hypothesis* Let FH be a non-Frobenius Frobenius-like group with kernel F and complement H . Assume that a Sylow 2-subgroup of H is cyclic and normal, and F has no extraspecial sections of order p^{2m+1} such that $p^m + 1 = |H_1|$ for some subgroup $H_1 \leq H$.

It should be noted that Hypothesis* is automatically satisfied if either $|FH|$ is odd or $|H| = 2$.

THEOREM 2.2. *Suppose that a finite group G admits a Frobenius group or a Frobenius-like group of automorphisms FH with kernel F and complement H such that $C_G(F) = 1$. Then $C_G(H) \neq 1$ and $\mathbf{r}(G)$ is bounded in terms of $\mathbf{r}(C_G(H))$ and $|H|$.*

PROPOSITION 2.3. *Let FH be a Frobenius, or a Frobenius-like group of Type I or Type II. Suppose that FH acts on a q -group Q for some prime q coprime to the order of H in case FH is not Frobenius. Let V be a $kQFH$ -module where k is a field with characteristic not dividing $|QH|$. Suppose further that F acts fixed-point freely on the semidirect product VQ . Then we have $C_V(H) \neq 0$ and*

$$\text{Ker}(C_Q(H) \text{ on } C_V(H)) = \text{Ker}(C_Q(H) \text{ on } V).$$

PROOF. See [17] Proposition 2.2 when FH is Frobenius; [18] Proposition C when FH is Frobenius-like of Type I; and [22] Proposition 2.1 when FH is Frobenius-like of Type II. It can be easily checked that [17] Proposition 2.2 is valid when $C_Q(F) = 1$ without the coprimeness condition $(|Q|, |F|) = 1$. \square

The proof of the following theorem can be found in [25] and in [2].

THEOREM 2.4. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by two involutions α and β . Suppose that D acts on a finite group G in such a manner that $C_G(\alpha\beta) = 1$. Then*

- (a) $G = C_G(\alpha)C_G(\beta)$;
- (b) the rank of G is bounded in terms of the rank of $C_G(\alpha)$ and $C_G(\beta)$;

PROPOSITION 2.5. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by the involutions α and β . Suppose that D acts on a q -group Q for some prime q and let V be a kQD -module for a field k of characteristic different from q such that the group $F = \langle \alpha\beta \rangle$ acts fixed point freely on the semidirect product VQ . If $C_Q(\alpha)$ acts nontrivially on V then we have $C_V(\alpha) \neq 0$ and $\text{Ker}(C_Q(\alpha) \text{ on } C_V(\alpha)) = \text{Ker}(C_Q(\alpha) \text{ on } V)$.*

PROOF. This is Proposition C in [24]. □

The next two results were established in [15, Lemma 1.6] .

LEMMA 2.6. *Suppose that a group Q acts by automorphisms on a group G . If $Q = \langle q_1, \dots, q_n \rangle$, then $[G, Q] = [G, q_1] \cdots [G, q_n]$.*

LEMMA 2.7. *Let p be a prime, P a finite p -group and Q a p' -group of automorphisms of P .*

- a) *If $|[P, q]| \leq m$ for every $q \in Q$, then $|Q|$ and $|[P, Q]|$ are m -bounded.*
- b) *If $r([P, q]) \leq m$ for every $q \in Q$, then $r(Q)$ and $r([P, Q])$ are m -bounded.*

We also need the following fact whose proof can be found in [1].

LEMMA 2.8. *Let G be a finite group such that $\gamma_\infty(G) \leq F(G)$. Let P be a Sylow p -subgroup of $\gamma_\infty(G)$ and H be a Hall p' -subgroup of G . Then $P = [P, H]$.*

3. Key Propositions

We prove below a new proposition which studies the actions of Frobenius and Frobenius-like groups and forms the basis in proving Theorem A.

PROPOSITION 3.1. *Assume that FH be a Frobenius group, or a Frobenius-like group of Type I or Type II with kernel F and complement H . Suppose that FH acts on a q -group Q for some prime q . Let V be an irreducible $\mathbb{F}_p QFH$ -module where \mathbb{F}_p is a field with characteristic p not dividing $|Q|$ such that F acts fixed-point-freely on the semidirect product VQ . Additionally, we assume that q is coprime to $|H|$ in case where FH is not Frobenius. Then $\mathbf{r}([V, Q])$ is bounded in terms of $\mathbf{r}([C_V(H), C_Q(H)])$ and $|H|$.*

PROOF. Let $\mathbf{r}([C_V(H), C_Q(H)]) = s$. We may assume that $V = [V, Q]$ and hence $C_V(Q) = 0$. By Clifford's Theorem, $V = V_1 \oplus \cdots \oplus V_t$, direct sum of of Q -homogeneous components V_i , which are transitively

permuted by FH . Set $\Omega = \{V_1, \dots, V_t\}$ and fix an F -orbit Ω_1 in Ω . Throughout, $W = \Sigma_{U \in \Omega_1} U$.

Now, we split the proof into a sequence of steps.

(1) *We may assume that Q acts faithfully on V . Furthermore $\text{Ker}(C_Q(H)$ on $C_V(H)) = \text{Ker}(C_Q(H)$ on $V) = 1$.*

PROOF. Suppose that $\text{Ker}(Q$ on $V) \neq 1$ and set $\overline{Q} = Q/\text{Ker}(Q$ on $V)$. Note that since $C_Q(F) = 1$, F is a Carter subgroup of QF and hence also a Carter subgroup of $\overline{Q}F$ which implies that $C_{\overline{Q}}(F) = 1$. Notice that the equality $\overline{C_Q(H)} = C_{\overline{Q}}(H)$ holds in case FH is Frobenius (see [14] Theorem 2.3). The same equality holds in case where FH is non-Frobenius due to the coprimeness condition $(q, |H|) = 1$. Then $[C_V(H), C_Q(H)] = [C_V(H), C_{\overline{Q}}(H)]$ and so we may assume that Q acts faithfully on V . Notice that by Proposition 2.3 we have

$$\text{Ker}(C_Q(H)$$
 on $C_V(H)) = \text{Ker}(C_Q(H)$ on $V) = 1$

establishing the claim. \square

(2) *We may assume that $Q = \langle c^F \rangle$ for any nonidentity element $c \in C_{Z(Q)}(H)$ of order q . In particular Q is abelian.*

PROOF. We obtain that $C_{Z(Q)}(H) \neq 1$ as $C_Q(F) = 1$ by Proposition 2.3. Let now $1 \neq c \in C_{Z(Q)}(H)$ of order q and consider $\langle c^{FH} \rangle = \langle c^F \rangle$, the minimal FH -invariant subgroup containing c . Since V is an irreducible QFH -module on which Q acts faithfully we have that $V = [V, \langle c^F \rangle]$. Thus we may assume that $Q = \langle c^F \rangle$ as claimed. \square

(3) *$V = [V, c] \cdot [V, c^{f_1}] \cdots [V, c^{f_n}]$ where n is a $(s, |H|)$ -bounded number. Hence it suffices to bound $\mathbf{r}([W, c])$.*

PROOF. Notice that the group $C_Q(H)$ embeds in the automorphism group of $[C_V(H), C_Q(H)]$ by step (1). Then $C_Q(H)$ has s -bounded rank by Lemma 2.7. This yields by Theorem 2.2 that Q has $(s, |H|)$ -bounded rank. Thus, there exist $f_1 = 1, \dots, f_n$ in F for an $(s, |H|)$ -bounded number n such that $Q = \langle c^{f_1}, \dots, c^{f_n} \rangle$. Now $V = [V, c] \cdot [V, c^{f_2}] \cdots [V, c^{f_n}] = \prod_{i=1}^n [V, c]^{f_i}$ by Lemma 2.6. This shows that we need only to bound $\mathbf{r}([V, c])$ suitably. In fact it suffices to show that $\mathbf{r}([W, c])$ is suitably bounded as $V = \Sigma_{h \in H} W^h$. \square

(4) *$H_1 = \text{Stab}_H(\Omega_1) \neq 1$. Furthermore the rank of the sum of members of Ω_1 which are not centralized by c and contained in a regular H_1 -orbit, is suitably bounded.*

PROOF. Fix $U \in \Omega_1$ and set $Stab_F(U) = F_1$. Choose a transversal T for F_1 in F . Let $W = \sum_{t \in T} U^t$ where T is a transversal for F_1 in F with $1 \in T$. Then we have $V = \sum_{h \in H} W^h$. Notice that $[V, c] \neq 0$ by (1) which implies that $[W, c] \neq 0$ and hence $[U^t, c] = U^t$ for some $t \in T$. Without loss of generality we may assume that $[U, c] = U$.

Suppose that $Stab_H(\Omega_1) = 1$. Then we also have $Stab_H(U^t) = 1$ for all $t \in T$ and hence the sum $X_t = \sum_{h \in H} U^{th}$ is direct for all $t \in T$. Now, $U \leq X_1$. It holds that

$$C_{X_t}(H) = \left\{ \sum_{h \in H} v^h : v \in U^t \right\}.$$

Then $|U| = |C_{X_1}(H)| = |[C_{X_1}(H), c]| \leq |[C_V(H), C_Q(H)]|$ implies $\mathbf{r}(U) \leq s$. On the other hand $V = \bigoplus_{t \in T} X_t$ and

$$[C_V(H), c] = \bigoplus \{ [C_{X_t}(H), c] : t \in T \text{ with } [U^t, c] \neq 0 \} \leq [C_V(H), C_Q(H)].$$

In particular, $\{t \in T : [U^t, c] \neq 0\}$ is suitably bounded whence $\mathbf{r}([W, c])$ is $(s, |H|)$ -bounded. Hence we may assume that $Stab_H(\Omega_1) \neq 1$.

Notice that every element of a regular H_1 -orbit in Ω_1 lies in a regular H -orbit in Ω . Let $U \in \Omega_1$ be contained in a regular H_1 -orbit of Ω_1 . Let X denote the sum of the members of the H -orbit of U in Ω , that is $X = \bigoplus_{h \in H} U^h$. Then $C_X(H) = \{ \sum_{h \in H} v^h : v \in U \}$. If $[U, c] \neq 0$ then by repeating the same argument in the above paragraph we show that $\mathbf{r}(U) \leq s$ is suitably bounded. On the other hand the number, say m , of all H -orbits in Ω containing a member U such that $[U, c] \neq 0$ is suitably bounded because $m \leq \mathbf{r}([C_V(H), c]) \leq s$. It follows then that the rank of the sum of members of Ω_1 which are not centralized by c and contained in a regular H_1 -orbit, is suitably bounded. \square

(5) *We may assume that FH is not Frobenius.*

PROOF. Assume the contrary that FH is Frobenius. Let $H_1 = Stab_H(\Omega_1)$ and pick $U \in \Omega_1$. Set $S = Stab_{FH_1}(U)$ and $F_1 = F \cap S$. Then $|F : F_1| = |\Omega_1| = |FH_1 : S|$ and so $|S : F_1| = |H_1|$. Since $(|F_1|, |H_1|) = 1$, by the Schur-Zassenhaus theorem there exists a complement, say S_1 of F_1 in S with $|H_1| = |S_1|$. Therefore there exists a conjugate of U which is H_1 -invariant. There is no loss in assuming that U is H_1 -invariant. On the other hand if $1 \neq h \in H_1$ and $x \in F$ such that $U^{xh} = U^x$, then $[h, x] \in Stab_F(U) = F_1$ and so $F_1x = F_1x^h = (F_1x)^h$. This implies that $F_1x \cap C_F(h)$ is nonempty. Now the Frobenius action of H on F forces that $x \in F_1$. This means that for each $x \in F \setminus F_1$ we have $Stab_{H_1}(U^x) = 1$. Therefore U is the unique member of Ω_1 which is H_1 -invariant and all the H_1 -orbits other than $\{U\}$ are regular. By (4), the rank of the sum of all members of Ω_1 other than U is suitably

bounded. In particular $\mathbf{r}(U)$ and hence $\mathbf{r}([W, c])$ is suitably bounded in case where $[U^x, c] \neq 0$ for some $x \in F \setminus F_1$. Thus we may assume that c is trivial on U^x for all $x \in F \setminus F_1$. Now we have $[W, c] = [U, c] = U$.

Due to the action by scalars of the abelian group Q on U , it holds that $[Q, F_1] \leq C_Q(U)$. We also know that c^x is trivial on U for each $x \in F \setminus F_1$. Since $C_Q(F) = 1$, there are prime divisors of $|F|$ different from q . Let $F_{q'}$ denote the q' -Hall subgroup of F . Clearly we have $C_Q(F_{q'}) = 1$. Let now $y = \prod_{f \in F_{q'}} c^f$. Then we have

$$1 = y = \left(\prod_{f \in F_1 \cap F_{q'}} c^f \right) \left(\prod_{f \in F_{q'} \setminus F_1} c^f \right) \in c^{|F_1 \cap F_{q'}|} C_Q(U).$$

As a consequence $c \in C_Q(U)$, because q is coprime to $|F_{q'}|$. This contradiction establishes the claim. \square

(6) *We may assume that the group FH is Frobenius-like of Type II.*

PROOF. On the contrary we assume that FH is Frobenius-like of Type I. By (4), we have $H_1 = \text{Stab}_H(\Omega_1) \neq 1$. Choose a transversal T_1 for H_1 in H . Now $V = \bigoplus_{h \in T_1} W^h$. Also we can guarantee the existence of a conjugate of U which is H_1 -invariant by means of the Schur-Zassenhaus Theorem as in (5). There is no loss in assuming that U is H_1 -invariant.

Set now $Y = \sum_{x \in F'} U^x$ and $F_2 = \text{Stab}_F(Y)$ and $F_1 = \text{Stab}_F(U)$. Clearly, $F_2 = F'F_1$ and Y is H_1 -invariant. Notice that for all nonidentity $h \in H$, we have $C_F(h) \leq F' \leq F_2$. Assume first that $F = F_2$. This forces that we have $V = Y$. Clearly, $Y \neq U$, that is $F' \not\leq F_1$, because otherwise $Q = [Q, F] = 1$ due to the scalar action of the abelian group Q on U . So $F' \cap F_1 = 1$ which implies that $|F : F_1|$ is a prime. Then $F_1 \trianglelefteq F$ and $F' \leq F_1$ which is impossible. Therefore $F \neq F_2$.

If $1 \neq h \in H$ and $t \in F$ such that $Y^{th} = Y^t$ then $[h, t] \in F_2$. Now, $F_2 t = F_2 t^h = (F_2 t)^h$ and this implies the existence of an element in $F_2 t \cap C_F(h)$. Since $C_F(h) \leq F' \leq F_2$ we get $t \in F_2$. In particular, for each $t \in F \setminus F_2$ we have $\text{Stab}_H(Y^t) = 1$.

Let S be a transversal for F_2 in F . For any $t \in S \setminus F_2$ set $Y_t = Y^t$ and consider $Z_t = \sum_{h \in H} Y_t^h$. Notice that $V = Y \oplus \bigoplus_{t \in S \setminus F_2} Z_t$. As the sum Z_t is direct we have

$$C_{Z_t}(H) = \left\{ \sum_{h \in H} v^h : v \in Y_t \right\}$$

with $|C_{Z_t}(H)| = |Y_t|$. Then $\mathbf{r}([Y_t, c]) = \mathbf{r}([C_{Z_t}(H), c]) \leq s$ for each $t \in S \setminus F_2$ with $[Y_t, c] \neq 0$. On the other hand,

$$\Sigma \{ \mathbf{r}([C_{Z_t}(H), c]) : t \in S \text{ with } [Y_t, c] \neq 0 \} \leq \mathbf{r}([C_V(H), c]) \leq s$$

whence $|\{t \in S \setminus F_2 : [Y_t, c] \neq 0\}|$ is suitably bounded. So the claim is established if there exists $t \in S \setminus F_2$ such that $[Y_t, c] \neq 0$, since we have $V = Y \oplus \bigoplus_{t \in S \setminus F_2} Z_t$. Thus we may assume that c is trivial on $\bigoplus_{t \in S \setminus F_2} Z_t$ and hence $[V, c] = [Y, c]$.

There are two cases now: We have either $F' \cap F_1 = 1$ or $F' \leq F_1$. First assume that $F' \leq F_1$. Then we get $F_1 = F_2$ because $F_2 = F'F_1$. Now $U = Y$. Due to the action by scalars of the abelian group Q on U , it holds that $[Q, F_1] \leq C_Q(U)$. From this point on we can proceed as in the proof of step (5) and observe that $C_Q(F_{q'}) = 1$. Letting now $y = \prod_{f \in F_{q'}} c^f$, we have

$$1 = y = \left(\prod_{f \in F_1 \cap F_{q'}} c^f \right) \left(\prod_{f \in F_{q'} \setminus F_1} c^f \right) \in c^{|F_1 \cap F_{q'}|} C_Q(U).$$

implying that $c \in C_Q(U)$, because q is coprime to $|F_{q'}|$.

Thus we have $F_1 \cap F' = 1$. First assume that $H_1 = H$. Then Y is H -invariant and F_1H is a Frobenius group. Note that $C_U(F_1) = 1$ as $C_V(F) = 1$, and hence $C_Y(F_1) = 1$ since $F' \leq Z(F)$. We consider now the action of QF_1H on Y and the fact that $\mathbf{r}([C_Y(H), C_Q(H)]) \leq s$. Then step (5), we obtain that $\mathbf{r}(Y) = \mathbf{r}([Y, Q])$ is $(s, |H|)$ -bounded. Next assume that $H_1 \neq H$. Choose a transversal for H_1 in H and set $Y_1 = \sum_{h \in T_1} Y^h$. Clearly this sum is direct and hence

$$C_{Y_1}(H) = \left\{ \sum_{h \in T_1} v^h : v \in Y \right\}$$

with $|[C_{Y_1}(H), c]| = |[Y, c]|$. Then $\mathbf{r}([Y, c]) = \mathbf{r}([C_{Y_1}(H), c]) \leq s$ establishing claim (6). \square

(7) *The proposition follows.*

PROOF. From now on FH is a Frobenius-like group of Type II, that is, H and $C_F(H)$ are of prime orders. By step (4) we have $H = H_1 = \text{Stab}_H(\Omega_1)$ since $|H|$ is a prime. Now $V = W$. We may also assume by the Schur-Zassenhaus theorem as in the previous steps that there is an H -invariant element, say U in Ω . Let T be a transversal for $F_1 = \text{Stab}_F(U)$ in F . Then $F = \bigcup_{t \in T} F_1 t$ implies $V = \bigoplus_{t \in T} U^t$. It should also be noted that we have $|\{t \in T : [U^t, c] \neq 0\}|$ is suitably bounded as

$$[C_V(H), c] = \bigoplus \{[C_{X_t}(H), c] : t \in T \text{ with } [U^t, c] \neq 0\} \leq [C_V(H), C_Q(H)]$$

where $X_t = \bigoplus_{h \in H} U^{th}$.

Let X be the sum of the components of all regular H -orbits on Ω , and let Y denote the sum of all H -invariant elements of Ω . Then

$V = X \oplus Y$. Suppose that $U^{th} = U^t$ for $t \in T$ and $1 \neq h \in H$. Now $[t, h] \in F_1$ and so the coset $F_1 t$ is fixed by H . Since the orders of F and H are relatively prime we may assume that $t \in C_F(H)$. Conversely for each $t \in C_F(H)$, U^t is H -invariant. Hence the number of components in Y is $|T \cap C_F(H)| = |C_F(H) : C_{F_1}(H)|$ and so we have either $C_F(H) \leq F_1$ or not.

If $C_F(H) \not\leq F_1$ then $C_{F_1}(H) = 1$ whence $F_1 H$ is Frobenius group acting on U in such a way that $C_U(F_1) = 1$. Then $\mathbf{r}(U)$ is $(s, |H|)$ -bounded by step (5) since $\mathbf{r}([C_U(H), C_Q(H)]) \leq s$ holds. This forces that $\mathbf{r}([V, c])$ is bounded suitably and hence the claim is established.

Thus we may assume that $C_F(H) \leq F_1$. Then $Y = U$ is the unique H -invariant Q -homogeneous component. If $[U^t, c] \neq 0$ for some $t \in F \setminus F_1$ we can bound $\mathbf{r}(U)$ and hence $\mathbf{r}([V, c])$ suitably. Thus we may assume that c is trivial on U^t for each $t \in F \setminus F_1$. Due to the action of the abelian group Q on U , it holds that $[Q, F_1] \leq C_Q(U)$. From this point on we can proceed as in the proof of step (5) and observe that $C_Q(F_{q'}) = 1$. Letting now $y = \prod_{f \in F_{q'}} c^f$, we have

$$1 = y = \left(\prod_{f \in F_1 \cap F_{q'}} c^f \right) \left(\prod_{f \in F_{q'} \setminus F_1} c^f \right) \in c^{|F_1 \cap F_{q'}|} C_Q(U).$$

implying that $c \in C_Q(U)$, because q is coprime to $|F_{q'}|$. This final contradiction completes the proof of Proposition 3.1. \square

\square

The next proposition studies the action of a dihedral group of automorphisms and is essential in proving Theorem B.

PROPOSITION 3.2. *Let $D = \langle \alpha, \beta \rangle$ be a dihedral group generated by two involutions α and β . Suppose that D acts on a q -group Q for some prime q . Let V be an irreducible $\mathbb{F}_p Q D$ -module where \mathbb{F}_p is a field with characteristic p not dividing $|Q|$. Suppose that $C_{VQ}(F) = 1$ where $F = \langle \alpha \beta \rangle$. If $\max\{\mathbf{r}([C_V(\alpha), C_Q(\alpha)]), \mathbf{r}([C_V(\beta), C_Q(\beta)])\} \leq s$, then $\mathbf{r}([V, Q])$ is s -bounded.*

PROOF. We set $H = \langle \alpha \rangle$. So $D = FH$. By Lemma 2.6 and Theorem 2.4, we have $[V, Q] = [V, C_Q(\alpha)][V, C_Q(\beta)]$. Then it is sufficient to bound the rank of $[V, C_Q(H)]$. Following the same steps as in the proof of Proposition 3.1 by replacing Proposition 2.3 by Proposition 2.4, we observe that Q acts faithfully on V and $Q = \langle c^F \rangle$ is abelian with $c \in C_{Z(Q)}(H)$ of order q . Furthermore $\text{Ker}(C_Q(H) \text{ on } C_V(H)) = \text{Ker}(C_Q(H) \text{ on } V) = 1$. Note that it suffices to bound $\mathbf{r}([V, c])$ suitably.

Let Ω denote the set of Q -homogeneous components of the irreducible QD -module V . Let Ω_1 be an F -orbit of Ω and set $W = \sum_{U \in \Omega_1} U$.

Then we have $V = W + W^\alpha$. Suppose that $W^\alpha \neq W$. Then for any $U \in \Omega_1$ we have $Stab_H(U) = 1$. Let T be a transversal for $Stab_F(U) = F_1$ in F . It holds that $V = \sum_{t \in T} X_t$ where $X_t = U^t + U^{t\alpha}$. Now $[V, c] = \sum_{t \in T} [X_t, c]$ and $C_V(H) = \sum_{t \in T} C_{X_t}(H)$ where $C_{X_t}(H) = \{w + w^\alpha : w \in U^t\}$. Since $[V, c] \neq 0$ there exists $t \in T$ such that $[U^t, c] \neq 0$, that is $[U^t, c] = U^t$. Then $[C_{X_t}(H), c] = C_{X_t}(H)$. Since $\mathbf{r}([C_V(H), C_Q(H)]) \leq s$ we get $\mathbf{r}(U) = \mathbf{r}(C_{X_t}(H)) \leq s$. Furthermore it follows that $|\{t \in T : [U^t, c] \neq 0\}|$ is s -bounded and as a consequence $\mathbf{r}([V, c])$ is suitably bounded. Thus we may assume that $W^\alpha = W$ which implies that $\Omega_1 = \Omega$ and H fixes an element, say U , of Ω as desired.

Let $U^t \in \Omega$ be H -invariant. Then $[t, \alpha] \in F_1$. On the other hand $t^{-1}t^\alpha = t^{-2}$ since α inverts F . So $F_1 t$ is an element of F/F_1 of order at most 2 which implies that the number of H -invariant elements of Ω is at most 2. Let now Y be the sum of all H -invariant elements of Ω . Then $V = Y \oplus \bigoplus_{i=1}^m X_i$ where X_1, \dots, X_m are the sums of elements in H -orbits of length 2. Let $X_i = U_i \oplus U_i^\alpha$. Notice that if $[U_i, c] \neq 0$ for some i , then we obtain $\mathbf{r}(U) = \mathbf{r}(U_i) \leq s$ by a similar argument as above. On the other hand we observe that the number of i for which $[U_i, c] \neq 0$ is s -bounded by the hypothesis that $\mathbf{r}([C_V(H), c]) \leq s$. It follows now that $\mathbf{r}([V, c])$ is suitably bounded in case where $[U_i, c] \neq 0$ for some i .

Thus we may assume that c centralizes $\bigoplus_{i=1}^m X_i$ and that $[U, c] = U$. Due to the scalar action by scalars of the abelian group Q on U , it holds that $[Q, F_1] \leq C_Q(U)$. As $F_1 \trianglelefteq FH$, we have $[Q, F_1] \leq C_Q(V) = 1$. Clearly we have $C_Q(F_{q'}) = 1$ where $F_{q'}$ denotes the Hall q' -part of F whose existence is guaranteed by the fact that $C_Q(F) = 1$. Let now $y = \prod_{f \in F_{q'}} c^f$. Then we have

$$1 = y = \left(\prod_{f \in F_1 \cap F_{q'}} c^f \right) \left(\prod_{f \in F_{q'} \setminus F_1} c^f \right) \in c^{|F_1 \cap F_{q'}|} C_Q(U).$$

As a consequence $c \in C_Q(U)$, because q is coprime to $|F_{q'}|$. This contradiction completes the proof of Proposition 3.2. \square

4. Proofs of theorems

Firstly, we shall give a detailed proof for Theorem A part (b). The proof of Theorem A (a) can be easily obtained by just obvious modifications of the proof of part (b).

First, we assume that $G = PQ$ where P and Q are FH -invariant subgroups such that P is a normal p -subgroup for a prime p and Q is

a nilpotent p' -group with $|[C_P(H), C_Q(H)]| = p^s$. We shall prove that $\mathbf{r}(\gamma_\infty(G))$ is $((s, |H|)$ -bounded. Clearly $\gamma_\infty(G) = [P, Q]$. Consider an unrefinable FH -invariant normal series

$$P = P_1 > P_2 > \cdots > P_k > P_{k+1} = 1.$$

Note that its factors P_i/P_{i+1} are elementary abelian. Let $V = P_k$. Since $C_V(Q) = 1$, we have that $V = [V, Q]$. We can also assume that Q acts faithfully on V . Proposition 3.1 yields that $\mathbf{r}(V)$ is $(s, |H|)$ -bounded. Set $S_i = P_i/P_{i+1}$. If $[C_{S_i}(H), C_Q(H)] = 1$, then $[S_i, Q] = 1$ by Proposition 2.3. Since $C_P(Q) = 1$ we conclude that each factor S_i contains a nontrivial image of an element of $[C_P(H), C_Q(H)]$. This forces that $k \leq s$. Then we proceed by induction on k to obtain that $\mathbf{r}([P, Q])$ is an $(s, |H|)$ -bounded number, as desired.

Let $F(G)$ denote the Fitting subgroup of a group G . Write $F_0(G) = 1$ and let $F_{i+1}(G)$ be the inverse image of $F(G/F_i(G))$. As is well known, when G is soluble, the least number h such that $F_h(G) = G$ is called the Fitting height $h(G)$ of G . Let now r be the rank of $\gamma_\infty(C_G(H))$. Then $C_G(H)$ has r -bounded Fitting height (see for example Lemma 1.4 of [15]) and hence G has $(r, |H|)$ -bounded Fitting height.

We shall proceed by induction on $h(G)$. Firstly, we consider the case where $h(G) = 2$. Indeed, let P be a Sylow p -subgroup of $\gamma_\infty(G)$ and Q an FH -invariant Hall p' -subgroup of G . Then, by the preceding paragraphs and Lemma 2.8, the rank of $P = [P, Q]$ is $(r, |H|)$ -bounded and so the rank of $\gamma_\infty(G)$ is $(r, |H|)$ -bounded. Assume next that $h(G) > 2$ and let $N = F_2(G)$ be the second term of the Fitting series of G . It is clear that the Fitting height of $G/\gamma_\infty(N)$ is $h-1$ and $\gamma_\infty(N) \leq \gamma_\infty(G)$. Hence, by induction we have that $\gamma_\infty(G)/\gamma_\infty(N)$ has $(r, |H|)$ -bounded rank. As a consequence, it holds that

$$\mathbf{r}(\gamma_\infty(G)) \leq \mathbf{r}(\gamma_\infty(G)/\gamma_\infty(N)) + \mathbf{r}(\gamma_\infty(N))$$

completing the proof of Theorem A(b).

The proof of Theorem B can be directly obtained as in the above argument by replacing Proposition 3.1 by Proposition 3.2; and Proposition 2.3 by Proposition 2.5.

References

- [1] C. Acciarri, P. Shumyatsky and A. Thillaisundaram, *Conciseness of coprime commutators in finite groups*, Bull. Aust. Math. **89** (2014), 252-258.
- [2] E. de Melo, *Fitting Height of a Finite Group with a Metabelian Group of Automorphisms*. Communication in Algebra **43** (2015), 4797-4808.
- [3] E. de Melo, *Nilpotent residual and Fitting subgroup of fixed points in finite groups*. J. Group Theory **22** (2019), 1059-1068.

- [4] E. de Melo, A. S. Lima and P. Shumyatsky *Nilpotent residual of fixed points*. Arch. Math **111** (2018) 13-21.
- [5] E. de Melo, J. Caldeira, *On finite groups admitting automorphisms with nilpotent centralizers*. J. Algebra **493** (2018) 185-193.
- [6] E. de Melo, J. Caldeira, *Supersolvable Frobenius groups with nilpotent centralizers*. J. Pure and Applied Algebra **223** (2019) 1210-1216.
- [7] D. Gorenstein, *Finite Groups*, Harper and Row, London, New York, 1991.
- [8] N. Y. Makarenko and P. Shumyatsky, *Frobenius groups as groups of automorphisms*, Proc. Am. Math. Soc. **138** No. 10 (2010) 3425-3436 .
- [9] N. Yu. Makarenko, E. I. Khukhro, and P. Shumyatsky, *Fixed points of Frobenius groups of automorphisms*, Dokl. Akad. Nauk, 437, No. 1 (2011) 20-23.
- [10] E. I. Khukhro, *The nilpotent length of a finite group admitting a Frobenius group of automorphisms with a fixedpoint-free kernel*. Algebra Logika, **49** (2010), 819-833; English transl, Algebra Logic, **49** (2011) 551-560.
- [11] E. I. Khukhro, *Fitting height of a finite group with a Frobenius group of automorphisms*, J. Algebra **366** (2012), 1-11.
- [12] E. I. Khukhro, *Rank and order of a finite group admitting a Frobenius group of automorphisms*. Algebra Logika **52** (2013) 99-108; English transl., Algebra Logic **52** (2013) 72-78.
- [13] E.I. Khukhro EI, N.Yu Makarenko, *Finite groups and Lie rings with a metacyclic Frobenius group of automorphisms*. J. Algebra **386** (2013) 77-104.
- [14] E. I. Khukhro, N. Yu Makarenko and P. Shumyatsky, *Frobenius groups of automorphisms and their fixed points*, Forum Math. **26** (2014), 73-112.
- [15] E. I. Khukhro, P. Shumyatsky, *Finite groups with Engel sinks of bounded rank*, Glasgow Mathematical Journal **60** (2018), 695-701.
- [16] İ.Ş. Güloğlu, G. Ercan, *Action of a Frobenius-like group*, J Algebra **402**, (2014) 533–543.
- [17] G. Ercan, İ.Ş. Güloğlu, E. Ögüt, *Nilpotent Length of a Finite Solvable Group with a Frobenius Group of Automorphisms*, Com. Algebra **42** issue 11, (2014) 4751-4756.
- [18] G. Ercan, İ.Ş. Güloğlu, *Action of a Frobenius-like group with fixed-point-free kernel*. J Group Theory **17**, (2014) 863-873.
- [19] G. Ercan, İ.Ş. Güloğlu, E.I. Khukhro, *Rank and Order of a Finite Group admitting a Frobenius-like Group of Automorphisms*, Algebra and Logic, **53** Issue 3, (2014) 258–265.
- [20] G. Ercan, İ.Ş. Güloğlu, E.I.Khukhro, *Derived length of a Frobenius-like kernel*, J Algebra **412**, (2014) 179-188.
- [21] G. Ercan, İ.Ş. Güloğlu, *Action of a Frobenius-like group with kernel having central derived subgroup*, International Journal of Algebra and Computation **26** No. 6 (2016) 1257–1265.
- [22] G. Ercan, İ.Ş. Güloğlu, *On the influence of fixed point free nilpotent automorphism groups*, Monat. Math. **184** (2017) 531–538.
- [23] G. Ercan, İ.Ş. Güloğlu, E.I. Khukhro, *Frobenius-like groups as groups of automorphisms*. Turk J Math. **38** (2014) 965 – 976.
- [24] G. Ercan, İ.Ş. Güloğlu, (2017) *Finite groups admitting a dihedral group of automorphisms*. Algebra and Discrete Mathematics **23**. Number 2, 223–229.
- [25] P. Shumyatsky, *The dihedral group as group of automorphisms*. J. Algebra **375** (2013), 1-12.

- [26] *Unsolved problems in group theory*. The Kourovka Notebook. 18th edition, Institute of Mathematics, Novosibirsk 2014.

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