ON TWO INSTANCES OF SPECTRAL RIGIDITY

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ABSTRACT

ON TWO INSTANCES OF SPECTRAL RIGIDITY

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It is widely known that the spectrum fails to characterise a Riemannian manifold. There are, however, Riemannian manifolds which admit no non-trivial continuous isospectral deformations. Strongly expected to be a "generic" phenomenon, the absence of non-trivial isospectral deformations is referred to as spectral rigidity. The present work treats two instances therof.

Keywords: Laplace-Beltrami operator, spectrum, isospectrality, isospectral deformation, quadratic forms.

ÖZ

TAYF SEVİYESİNDE KATILIĞA İKİ ÖRNEK

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Riemann uzaylarının tayfları tarafından belirlenemediği iyi bilinmektedir. Buna karşılık bazı Riemann uzayları üzerinde tayfı kıpırdatmayan sürekli şekil değişmeleri de mümkün olmamaktadır. Büyük bir ihtimalle "generic" olduğu tahmin edilen tayf kıpırdatmayan sürekli dönüşümlerin imkansızlığı hadisesi, tayf seviyesinde katılık olarak anılmaktadır. Eldeki çalışmada bu halin iki örneği incelenmektedir.

Anahtar kelimeler: Laplace-Beltrami operatörü, tayf, tayf kıpırdatmayan şekil değiştirme, kuvadratik formlar.

To Caner Şimşir

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CHAPTER 1

INTRODUCTION

Let (M, G) be a Riemannian manifold with Levi-Civita connection ∇. Given a smooth scalar function f on M, the *gradient* of f, denoted by grad f is the vector field on M uniquely determined by the relation

$$\mathsf{G}(\text{ grad } f, Y\ ) = Yf$$

Given a vector field Y on M the *divergence* div Y of Y is a scalar function the value of which at each $p \in M$ is the trace of the linear map

$$(u \longmapsto \nabla_u Y) : T_pM \longmapsto T_pM$$

where ∇ is the Levi - Civita connection on (M, G).

Finally, the *Laplace - Beltrami* operator Δ on (M, G) is defined by,

$$\Delta f = -\text{ div (grad } f) \quad .$$

for each scalar function f.

If $x = (x_i)_{1 \leq i \leq n}$ is any chart on M with

$$\mathsf{G}|_{dom(x)} = \mathsf{G}_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j$$

and

$$\nabla(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

then for any scalar field f

$$grad\ f|_{dom(x)} = \mathsf{G}^{ij}\frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^j}$$

where

$$\mathsf{G}^{ij}\mathsf{G}_{kj} = \delta^i_j$$

On the other hand for a vector field Y with

$$\mathrm{d}iv(Y)|_{dom(x)} = \frac{\partial Y^i}{\partial x^i} + \Gamma^i_{ik}Y^k$$

$$Y|_{dom(x)} = Y^i\frac{\partial}{\partial x^i}$$

thus

$$\begin{aligned}\Delta f|_{dom(x)} &= -\mathrm{div}(\mathrm{grad}\ f)|_{dom(x)} \\ &= -\frac{\partial}{\partial x^i}(\mathsf{G}^{im}\frac{\partial f}{\partial x^m}) - \Gamma^i_{ik}\mathsf{G}^{kn}\frac{\partial f}{\partial x^n}\end{aligned}$$

for any scalar f .

$\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of Δ if there exists $f \in \mathcal{D}(M)$ such that $\Delta f = \lambda f$. A non-vanishing function is called an *eigenfunction* of the Laplace - Beltrami operator Δ if $\Delta f = \lambda f$ for some $\lambda \in \mathbb{C}$. Under these circumstances we refer to f as an eigenfunction of Δ with eigenvalue λ.

Let us illustrate these concepts on the basis of the following two simplest possible examples :

EXAMPLE 1.1 : On the flat circle $(\frac{\mathbb{R}}{L\mathbb{Z}}, \mathrm{d}x \otimes \mathrm{d}x)$ of content $L > 0$, the Laplace - Beltrami operator reduces to

$$\Delta = -\frac{\partial^2}{\partial x^2}$$

It can routinely be checked that the eigenfunctions of Δ are of the form

$$\varphi_n(x) = e^{\frac{2\pi n i}{L}x}$$

with eigenvalues

$$\lambda_n = \frac{4\pi^2 n^2}{L^2}$$

for $n \in \mathbb{Z}$.

EXAMPLE 1.2 : On the real line $(\mathbb{R}, \mathrm{d}x \otimes \mathrm{d}x)$, the Laplace-Beltrami operator is again of the form

$$\Delta = -\frac{\partial^2}{\partial x^2}.$$

In this case *every* real number $\lambda \in \mathbb{R}$ is an eigenvalue of Δ. Eigenfunctions with eigenvalue λ are of the form

$$\varphi(x) = \begin{cases} Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} & \text{for } A, B \in \mathbb{R} \text{ if } \lambda < 0 \\ Ce^{i\sqrt{\lambda}x} + De^{-i\sqrt{\lambda}x} & \text{for } C, D \in \mathbb{R} \text{ if } \lambda > 0 \\ Ex + F & \text{for } E, F \in \mathbb{R} \ \ \lambda = 0 \end{cases}$$

As it can be readily observed on the basis of the above examples, the set of eigenvalues of the Laplace-Beltrami operator on a compact manifold has a markedly different nature from that on a non-compact manifold.

We shall mostly concentrate on compact manifolds on which the set of eigenvalues of the Laplace-Beltrami operator has a well-understood and tidy structure: If (M, G) is a compact Riemannian manifold with the Laplace-Beltrami operator Δ, then the eigenvalues of Δ constitute a countable, discrete and unbounded set of non-negative real numbers. Furthermore the set of eigenfunctions corresponding to an eigenvalue λ span a *finite* dimensional subspace $\mathcal{H}_\lambda$ of $\mathcal{D}(M)$. The dimension of $\mathcal{H}_\lambda$ is the *multiplicity* of λ. $0 \in \mathbb{R}$ always occurs as an eigenvalue. However $\mathcal{H}_0$ consists of constant functions and thus $\dim \mathcal{H}_0 = 1$. In other words, the multiplicity of the eigenvalue 0 is always 1. Thus, in the case of the flat circle of content L each eigenvalue is of the form

$$\lambda_n = \frac{4\pi^2 n^2}{L^2}$$

and

$$\mathcal{H}_{\lambda_n} = \langle e^{\frac{2\pi i n}{L}x}, e^{\frac{-2\pi i n}{L}x} \rangle$$

where $n \in \mathbb{Z}_{\geq 0}$.

We may now offer an elucidation of the expression "spectral geometry" ([BGM]): Spectral geometry done within the framework of Riemannian geometry is essentially the study of Riemannian manifolds on the basis of the information consisting of magnitudes and multiplicities of the eigenvalues of the Laplace-Beltrami operator. It is very important to notice that the "information" in this question encompasses *not only* the magnitudes of the eigenvalues but the multiplicities thereof. Thus, given a compact Riemannian manifold (M, G) we understand the *spectrum* of (M, G) to be the set of eigenvalues of the Laplace-Beltrami operator on (M, G) with each eigenvalue tagged by a number indicating its multiplicity. We shall denote the spectrum of (M, G) by $Sp(M, \mathsf{G})$ or $Sp(M)$ unless confusion is likely. Observe that, in the case of a compact Riemannian manifold (M, G), the spectrum $Sp(M, \mathsf{G})$ may be identified with an increasing sequence

$$0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots$$

tending to $+\infty$. Each eigenvalue occurs in the form of λ_n and is repeated as many times as its multiplicity.

Riemannian manifolds (M, G) and (M', G') are said to be *isometric* if there exists a diffeomorphism $\varphi : M \longmapsto M'$ satisfying

$$\mathsf{G}_p(u, v) = \mathsf{G}'_{\varphi(p)}(T\varphi_p(u), T\varphi_p(v))$$

for $p \in M, \quad u, v \in T_pM$.

(M, G) and (M', G') are said to be *isospectral* if

$$Sp(M, G) = Sp(M', G') \quad .$$

It is clear that isometric Riemannian manifolds are isospectral. In spectral geometry of Riemannian manifolds, the principal problem is to determine the extent to which the isometry class of a Riemannian manifold is determined by its spectrum, ([Kec], [Brooks]).

Today it is known that through numereous examples ([Mil], [Ike], [Eji], [Vis]) that there exist non-isometric Riemannian manifolds which are isospectral.

CHAPTER 2

SPECTRAL GEOMETRY OF FLAT TORI

For $N \geq 1, N \in \mathbb{Z}$, consider $\mathbb{R}^N$ with its additive group and vector space structure over $\mathbb{R}$.

A *lattice* in $\mathbb{R}^N$ is a discrete subgroup of $\mathbb{R}^N$ which contains a basis of $\mathbb{R}^N$.Equivalently,a lattice Λ in $\mathbb{R}^N$ is a subset of $\mathbb{R}^N$ of the form

$$\Lambda = B\mathbb{Z}^N$$

where $B \in \mathbb{R}^{N \times N}$ with $\det B \neq 0$.

Given a lattice, $\Lambda = B\mathbb{Z}^N \subseteq \mathbb{R}^N$ the *dual* of Λ is a lattice $\Lambda^* \subseteq \mathbb{R}^N$ which is defined to be

$$\Lambda^* = (B^{-1})^T \mathbb{Z}^N$$

We note that, the dual Λ^* of Λ can also be characterised by an important property

$$l^* \in \mathbb{R}^N \text{ iff } l^* l^T \in \mathbb{Z}, \quad \forall l \in \Lambda$$

A *flat torus* is a Riemannian manifold with vanishing curvature which is diffeomorphic to a torus.

Each flat torus of dimension N is known to be isometric to $(\frac{\mathbb{R}^N}{\Lambda}, \delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j)$ for some lattice $\Lambda \subseteq \mathbb{R}^N$, [Ch]. We shall denote $\mathbb{T}_\Lambda$ as the flat torus determined by lattice Λ.

Given a flat torus $(\mathbb{T}_\Lambda, \delta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^j)$, it can be checked that the Laplace-Beltrami operator on $\mathbb{T}_\Lambda$ is of the form $\Delta = -\delta_{ij} \frac{\partial^2}{\partial x^i \partial x^j}$ and the eigenfunctions $\varphi_{l^*} = \exp 2\pi l^{*T}x, l^* \in \Lambda^*$. Clearly, the eigenvalue corresponding to φ_{l^*} is $4\pi^2| l^* |^2$. Consequently, the spectrum $Sp(\mathbb{T}_\Lambda)$ is a sequence

$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \ldots$ in which, a quantitiy λ occurs as many times as the number of $l^{*\prime}s$ with $\lambda = | l^* |^2$.

At this step a natural question arises, does there exist any other eigenfunctions ? How do we know that these exhaust all possibilities. The answer "yes!" is affected by showing that $\{\exp 2\pi i l^{*T}x\}_{l^* \in \Lambda^*}$ spans, by a finite linear combinations over $\mathbb{C}$, a dense subset of the algebra of continuous complex-valued functions on $\mathbb{T}_\Lambda$, by using Stone - Weierstraβ Theorem,[BGM].

On a Riemannian manifold there is an object which is closely allied to the spectrum : The *length spectrum* of a Riemannian manifold is the collection of the lengths of its closed geodesics counted with multoplicity of the free homotopy classes in which the geodesic occurs. This is a concept which can quite easily become useless when, for instance, a length occurs in infinitely many free homotopy classes.

In the case of flat tori, the length spectrum is easy to describe : The length spectrum of $\mathbb{T}_\Lambda$ is the sequence

$$0 = \mu_0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \leq \ldots$$

in which quantity μ occurs as many times as the number of $\mu \in \Lambda$ with $\mu =| l |$.

In general the ammount of information carried by the spectrum and the length spectrum may widely differ. However, on flat tori there is a happy coincidence : The information provided by the spectrum is exactly the same as that provided on the length spectrum.

We translate the Λ^*-information into Λ-information via an analytic function F by writing $F(t) = \sum_{n=0}^{\infty} \exp -(\lambda_n t)$ where right hand side is uniformly convergent on each compact subset of $\mathbb{R}_{.>0}$. It can routinely be shown that $F(t) = \Theta_{B^{-1}(B^{-1})^T}(4\pi i t)$ where $\Theta_s(z)$ is the Jacobi-theta function defined by

$$\Theta_s(z) = \sum_{n\in\mathbb{Z}^N} \exp\left[\pi i(n^T S n)z\right]$$

with positive definite, symmetric $S \in \mathbb{R}^{N\times N}$, [SB]. Employing Jacobi-Inversion Formula

$$\Theta_s(z) = \left(\frac{z}{i}\right)^{\frac{-N}{2}} \frac{1}{\sqrt{\det S}} \Theta_{S^{-1}}\left(-\frac{1}{z}\right)$$

with $S = B^{-1}(B^{-1})^T$ and $z = 4\pi i t$, we obtain

$$F(t) = \frac{|\det B|}{(4\pi t)^{\frac{N}{2}}} \sum_{l\in\Lambda} \exp\left(-\frac{l^T t}{4\pi}\right)$$

CHAPTER 3

ISOSPECTRAL FINITENESS OF FLAT TORI

The purpose of this part is to demonstrate that among the N-dimensional flat tori, each isospectrality class contains at most finitely many isometry classes.

To this end we first notice the correlation between this problem and the clasically well investigated problem of the equivalence of quadratic forms.

In the following, given $Q \in \mathbb{R}^{N \times N}$,

$$Q\ [.,.] : \mathbb{R}^N \times \mathbb{R}^N \longmapsto \mathbb{R}$$

will stand for the function defined by

$$Q\ [x, y] = x^T Q y.$$

We will replace $Q\ [x, x]$ with $Q[x]$ for brevity and, again for brevity and by abuse of language, we shall talk about the quadratic form Q instead of $Q[.]$ as a Q is symmetric and positive definite. In particular, for an arbitrary non-singular $A \in \mathbb{R}^{N \times N}$, $A^T A[.]$ is a quadratic form.

Quite generally, we call $f, g : \mathbb{R}^N \longmapsto \mathbb{R}$ *congruent* if there exists a linear isomorphism $\alpha : \mathbb{R}^N \longmapsto \mathbb{R}^N$ with $\alpha(\mathbb{Z}^N) = \mathbb{Z}^N$ such that $g = f \circ \alpha$.

Clearly, quadratic forms Q, Q' are congruent iff there exists a unimodular [1] $K \in \mathbb{R}^{N \times N}$ matrix K such that

$$Q' = K^T Q K$$

.

Each flat torus of dimenion N is isometric to $\mathbb{T}_A = \mathbb{R}^N / \Lambda_A$ for some non-singular $A \in \mathbb{R}^N$, where $\Lambda_A = A\mathbb{Z}^N$, with the ordinary (Euclidian) Riemannian tensor field it inherits from $\mathbb{R}^N$.

PROPOSITION 3.1 : Given non-singular $A, B \in \mathbb{R}^{N \times N}$, the flat tori $\mathbb{T}_A$, $\mathbb{T}_B$ are isometric iff there exist an orthogonal $U \in \mathbb{R}^{N \times N}$ and a unimodular $K \in \mathbb{R}^{N \times N}$ such that

$$B = U\ A\ K$$

Proof : If U, K exist as specified above, the map

$$\varphi : \mathbb{T}_A \longmapsto \mathbb{T}_B$$

defined by

$$\varphi(x + A\mathbb{Z}^N) = Ux + B\mathbb{Z}^N$$

can be checked to be a well defined isometry.

Conversely, if $\varphi : \mathbb{T}_A \longmapsto \mathbb{T}_B$ is an isometry we may assume without loss of generality that

$$\varphi(0 + A\mathbb{Z}^N) = 0 + B\mathbb{Z}^N$$

[1] K is said to be unimodular if it has integer entries and $\det K = \pm 1$.

Let $\tilde{\varphi} : \mathbb{R}^N \longmapsto \mathbb{R}^N$ be the unique lifting of φ with $\tilde{\varphi}(0) = 0$. $\mathbb{R}^N$ is an isometry of $\tilde{\varphi}$ hence equals some orthogonal $U \in \mathbb{R}^{N \times N}$ and

$$U\ A\ \mathbb{Z}^N = B\mathbb{Z}^N$$

This means that UA and B can differ only up to multiplication by a unimodular matrix on the right.

Let us assign to each flat torus $\mathbb{T}_A$ the quadratic form $A^T A$. If $\mathbb{T}_A$ is isometric to $\mathbb{T}_B$, then $B = UAK$ for orthogonal U and unimodular K, consequently

$$B^T B = K^T A^T U^{-1} U A K = K^T A^T A K$$

showing that the quadraric forms $A^T A$ and $B^T B$ are congruent. Therefore, by the above described assignment we have introduced a map from the set of isometry classes of flat tori to the set of congruence classes of quadratic forms.

Observe that this map is a *bijection*: Every non-singular, symmetric, positive definite Q has a (unique) symmetric, positive-definite square root A. Clearly, $A^T A = A^2 = Q$. This shows that the map in question is surjective. On the other hand if $A^T A$ and $B^T B$ are congruent, then

$$B^T\ B = K^T A^T A K$$

for some unimodular K and $U = BK^{-1}A^{-1}$ is orthogonal since

$$U^T\ U = (A^{-1})^T (K^{-1})^T B^T\ B K^{-1} A^{-1} = I$$

We conclude that $B = UAK$ and $\mathbb{T}_A$ is isometric to $\mathbb{T}_B$.

On the other hand, the spectrum of $\mathbb{T}_A$ is deposited in the function

$$\begin{aligned} \sum_{l^* \in \Lambda_A^*} \exp\left[-4\pi^2 l^{*T} l^* t\right] &= \sum_{l^* \in (B^T)^{-1}\mathbb{Z}^N} \exp\left[-4\pi^2 l^{*T} l^* t\right] \\ &= \sum_{n \in \mathbb{Z}^N} \exp\left[-4\pi^2 n^T Q n t\right] \\ &= \Theta_Q(4\pi\ it) \end{aligned}$$

where $Q = (B^TB)^{-1} = B^{-1}(B^T)^{-1}$ and Θ_Q is the Jacobi theta function of the quadratic form Q. Consequently, given non-singular $A, B \in \mathbb{R}^{N\times N}$, in order for the flat tori $\mathbb{T}_A, \mathbb{T}_B$ to be isospectral it is necessary and sufficient that quadratic forms Q, Q' have the same Jacobi theta function. Let's remember the classical interpretation : Two quadratic forms with identical theta fuctions express the same real numbers, (with entries from $\mathbb{Z}^N$) for the same number of times. Quadratic forms with the same theta function will be referred to as *equivalent* quadratic forms.

In the light of the above observations, it is now clear that the problem of classifying isometry classes of flat tori up to isospectrality is equivalent to the problem of classifying congruence classes of quadratic forms up to equivalence.

We shall develope the rest of our treatment in the language of quadratic forms so as to be in a position to draw readily from the well established body of techniques of the so called "Geometry of Numbers"

A set $A \subseteq \mathbb{R}^N$ is set to be *convex* if for any $x, y \in A$ and $\mathbb{R}^N$ with $\alpha + \beta = 1, \quad \alpha x + \beta y \in A$. In other words, a subset of $\mathbb{R}^N$ is called a convex set if it contains the line segment joining any two of its points. $A \subseteq \mathbb{R}^N$ is said to be *symmetric* if for any $x \in A, \quad -x \in A$, too.

Clearly, a convex and symmetric subset of $\mathbb{R}^N$ contains $0 \in \mathbb{R}^N$.

A *Minkowskian domain* in $\mathbb{R}^N$, is a compact, convex, symmetric neighbourhood of $0 \in \mathbb{R}^N$.

A function $f : \mathbb{R}^N \longmapsto \mathbb{R}$ is said to be *convex* if for every $x, y \in \mathbb{R}^N$ and $\mathbb{R}^N$ with $\alpha + \beta = 1, f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$

A *gauge function* is a continuous, convex function $f : \mathbb{R}^N \longmapsto \mathbb{R}$ which obeys

$$f^{-1}(0) = \{0\}$$

$$f(\alpha x) = \mid \alpha \mid \; f(x)$$

for all $x \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$

Remark : A gauge function takes non-negative values only. Moreover, it reduces to zero only at $\vec{0} \in \mathbb{R}^N$.

EXAMPLE 3.1 : Quite typically given a quadratic form Q, $\sqrt{Q[.]}$ is a gauge function.

It is important to notice that Minkowskian domains and gauge functions constitute two languages describing the one and the same thing ! Indeed, given a gauge function f

$$K = K_f = f^{-1}([0,1]) \subseteq \mathbb{R}^N$$

is a Minkowskian domain. Conversely, given A Minkowskian domain K the function

$$f : \mathbb{R}^N \longmapsto \mathbb{R}$$

defined by

$$f(x) = \begin{cases} 0 \quad \text{if } x = 0 \\ \\ \sup \{\xi \mid \quad \xi > 0, \ \xi^{-1}x \notin K\} \end{cases}$$

is a gauge function. It can be routinely checked (Appendix 1) that the above correspondence between Minkowskian domains and gauge functions is well-defined and constitutes a bijection in the sense that

$$f_{K_f} = f$$

and

$$K_{f_K} = K$$

In the following we shall freely pass between gauge functions and Minkowskian domains and employ K, f generically. For instance, let us notice that

$$\lambda K = f^{-1}([0, |\lambda|])$$

for any $\lambda \in \mathbb{R}$.

Note that a continuous function $g : \mathbb{R}^N \longmapsto \mathbb{R}$, which is bounded from below, may fail to attain its infimum on a set $A \subseteq \mathbb{R}^N$. This is so even A is discrete. For instance, for

$$f : \mathbb{R} \longmapsto \mathbb{R}$$

defined by

$$f(x) = \frac{1}{1+x^2}$$

and

$$A = \mathbb{Z} \subseteq \mathbb{R}$$

we have

$$\inf\{f(x) \mid x \in \mathbb{R}\} = 0$$

but for any $x \in \mathbb{Z}$

$$f(x) \neq 0.$$

However, if f is a gauge fuction then f attains its infimum on any discrete set $A \subseteq \mathbb{R}^N$. In this case for any $C > 0$, we note that the set

$$\{\vec{x} \in \mathbb{R}^N : | f(\vec{x}) | \leq C\}$$

is compact. Clearly, it is closed since f is continuous. On the other hand, let

$$R = \inf\{| f(\vec{x}) | : || \vec{x} || = 1\} > 0$$

and notice that for $|| \vec{x} || > C/R$

$$| f(\vec{x}) | = | \, || \vec{x} || \, f(\frac{\vec{x}}{|| \vec{x} ||}) \, | > C$$

Therefore, if

$$| f(\vec{x}) | \leq C$$

then

$$|| \vec{x} || \leq \frac{C}{R}$$

which shows that the set

$$\{\vec{x} \in \mathbb{R}^N : | f(\vec{x} |) \ < C\}$$

is compact. Let

$$\alpha = inf\{f(\vec{x}) \ : \vec{x} \in A\}$$

since

$$M = \{\vec{x} \in \mathbb{R}^N \ : f(\vec{x}) \leq \alpha + 1\}$$

is compact and inf

$$\{f(\vec{x}) \ : \vec{x} \in A\} = \{f(\vec{x}) \ : \vec{x} \in M \cap A\}$$

where $M \cap A$ is finite M being compact and A discrete. Consequently, there exists $\vec{x} \in M \cap A$ such that $\alpha = f(\vec{x})$ =inf$\{f(\vec{x}) : \ \vec{x} \in A\}$.

Given a Minkowskian domain $K \subseteq \mathbb{R}^N$ (equivalently a gauge function f), for each $k = 1, 2, \ldots, N$ we define the *kth minimum* associated with K by

$$\lambda_k = inf\{\lambda \mid \quad \lambda > 0, \dim(\lambda K \cap \mathbb{Z}^N) \geq k\}$$

The numbers $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{R}^N$ are collectively referred to as *successive minima* and clearly satisfy

$$\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N.$$

In the following, our work will heavily depend upon the following celebrated theorem of Minkowski, [Lek].

THEOREM 3.1 : If $K \subseteq \mathbb{R}^N$ a Minkowskian domain with succesive minima $\{\lambda_i\}_{i=1}^N$, then

$$vol(K) \prod_{i=1}^{N} \lambda_k \ \leq 2^N$$

Remark : The above result is known under the name of " the Second Theorem of Minkowski ". Its weaker counterpart is tantamount to the statement

$$vol(K)\lambda_1^N \leq 2^N$$

and is better known in its geometrically very intuitive version to the effect that a Minkowskian domain in $\mathbb{R}^N$ with volume exceeding 2^N contains a non-zero element of $\mathbb{Z}^N$!

A basis $\{v_i\}_{i=1}^N$ of $\mathbb{Z}^N$ is said to be *reduced* with respect to a gauge function f if

$$f(v_1) \leq f(x)$$

for all $x \in \mathbb{Z}^N - \{0\}$ and for every $k = 1, 2, \ldots, N-1$

$$f(v_{k+1}) \leq f(x)$$

for each $x \in \mathbb{Z}^N$ with the property that $\{v_1, v_2, \ldots, v_k, x\}$ is a subset of a basis for $\mathbb{Z}^N$.

Lemma : Every gauge function has a reduced basis in $\mathbb{Z}^N$.

Proof : Choose $\vec{v}_1 \in \mathbb{Z}^N - \{0\}$ such that

$$f(\vec{v}_1) = \inf\{f(x) \mid x \in \mathbb{Z}^N - \{0\}\}$$

Having chosen linearly independent $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{Z}^N$, $k \leq N-1$ which can be completed to a basis for $\mathbb{Z}^N$, choose $\vec{v}_{k+1} \in \mathbb{Z}^N$ to satisfy

$$f(\vec{v}_{k+1}) = \inf\{f(x) \mid \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k, \vec{v}\}\text{is a subset of a basis for}\mathbb{Z}^N\}$$

This inductive process allows us to affect the construction in question.

Corollary : Every gauge function is congruent to some gauge function which admits the standart basis of $\mathbb{Z}^N$ as a reduced basis.

THEOREM 3.2 : Let f be a gauge function on $\mathbb{R}^N$ with successive minima $\{\lambda_k\}_{k=1}^N$, let $\{\vec{v}_i\}_{i=1}^N \subseteq \mathbb{Z}^N$ be a reduced basis for f. If $\mu_k = f(v_k)$ for $k = 1, 2, \ldots, N$, then

$$\lambda_k \leq \mu_k \leq (\frac{3}{2})^{\max 0, k-2}$$

for $k = 1, 2, \ldots, N$.

Proof : Since the set $\{\vec{x} \mid f(\vec{x}) \leq \mu_k\}$ contains $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ which are linearly independent, we conclude that $\lambda_k \leq \mu_k$ for $1 \leq k \leq N$. This completes the left hand side of the inequality.

As $\vec{u}_1$ can be completed to a basis for $\mathbb{Z}^N\}$, we have

$$\lambda_1 = f(\vec{u_1}) \geq f(\vec{v_1}) = \mu_1$$

this shows that $\lambda_1 = \mu_1$. In view of $\lambda_1 \leq \mu_1$ we obtain $\lambda_1 = \mu_1$.

Let $k \geq 2$: There exists $\vec{u}_l \;\; 1 \leq l \leq k-1$ such that $\vec{u}_l \notin Span_{\mathbb{R}}(\vec{v}_1, \ldots, \vec{v}_{k-1})$,

$$\lambda_k = mu_k$$

. We have to consider two cases:

Case 1: $\{\vec{u}_l, \vec{v}_1, \ldots, \vec{v}_{k-1}\}$ can be completed to a basis for $\mathbb{Z}^N$. In this case,

$$\lambda_k \geq \mu_k = f(\vec{u}_1) \geq f(\vec{v}_k) = \mu_k$$

hence, $\lambda_k = \mu_k$.

Case 2: $\{\vec{u_l}, \vec{v_1}, \ldots, \vec{v}_{k-1}\}$ can not be completed to a basis for $\mathbb{Z}^N$. Choosing

$$\vec{v} \in span_{\mathbb{R}}\{\vec{u_l}, \vec{v_1}, \ldots, \vec{v}_{k-1}\}$$

such that

$$\{\vec{v_1}, \ldots, \vec{v}_{k-1}, \vec{v}\}$$

can be completed to a basis for $\mathbb{Z}^N$, we find

$$\vec{u_l} = \alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \ldots + \alpha_{k-1}\vec{v}_{k-1} + m\vec{v}$$

where

$$\alpha_1, \ldots, \alpha_{k-1}, m \in \mathbb{Z}$$

and $\mid m \mid \geq 2$. Consequently,

$$\vec{v} = \alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \ldots + \alpha_{k-1}\vec{v}_{k-1} - \frac{1}{m}\vec{u_l}$$

Since $\vec{v}$ may be replaced with

$$\vec{v} + \beta_1\vec{v}_1 + \ldots + \beta_{k-1}\vec{v}_{k-1}$$

for any $\beta_1, \ldots, \beta_{k-1}$ without loss of generality we may assume that

$$\mid \alpha_i \mid \leq \frac{1}{2} \quad i = 1, 2, \ldots, k-1$$

Since f is a gauge function $f(\vec{v}_i) \leq f(\vec{v})$ by definition of $\vec{v}_i$ for $i = 1, 2, \ldots, k-1$ and

$$\mu_k = f(\vec{v}_k) \leq f(\vec{v}) \leq \mid \alpha_1 \mid \mu_1 \ldots \mid \alpha_{k-1} \mid \mu_k + \frac{1}{m}\lambda_l \leq \frac{1}{2}(\mu_1 + \ldots + \mu_{k-1} + \lambda_k)$$

Collecting the results we find that in all cases

$$\mu_k \leq max(\lambda_k, \frac{1}{2}(\mu_1 + \ldots + \mu_{k-1} + \lambda_k)) \quad k = 2, \ldots, N$$

which yields the desired inequality, by a simple induction.

LEMMA 3.1 : if $Q[.]$ is a quadratic form that admits the standart basis $\{e_i\}_{i=1}^{N}$ as a reduced basis, then

$$\prod_{i=1}^{N} Q(e_i) \leq 4^N \; V_N (\frac{3}{2})^{(N-1)(N-2)} \det(Q)$$

where V_N is the volume of the N-dimensional closed disc of unit radius

Proof : Let K be the Minkowski domain corresponding to $\sqrt{Q[.]}$. Q being symmetric, positive definite, there exist non-singular $A \in \mathbb{R}^{N \times N}$ such that

$$A^T \; Q \; A = I$$

In particular, we have $(\det A^T)^2 = \det Q^{-1}$

$$\begin{aligned} A^{-1}K &= \{A^{-1}x \in \mathbb{R}^N | x^T Q x \leq 1\} \\ A^{-1}K &= \{y \in \mathbb{R}^N \mid y^T A^T Q A y \leq 1\} \\ A^{-1}K &= \{y \in \mathbb{R}^N \mid y^T y \leq 1\} \end{aligned}$$

Consequently,

$$V_N = vol(A^{-1} \; K) = \det(A^{-1}) vol(K)$$

and conclude that

$$vol(K) = \frac{V_N}{\sqrt{\det Q}}$$

Now, let $\{\lambda_i\}_{i=1}^{N}$ be the succesive minima of K. By theorem and the second theorem of Minkowski we obtain

$$\begin{aligned}
\prod_{i=1}^{N} Q[e_i] &\leq \prod_{k=1}^{N}[(\frac{3}{2})^{\max 0,k-2}]^2\lambda_k^2 \\
&= (\frac{3}{2})^{(N-1)(N-2)}\prod_{k=1}^{N}\lambda_k^2 \\
&\leq (\frac{3}{2})^{(N-1)(N-2)}[\frac{2^N}{vol(K)}]^2 \\
&= (\frac{3}{2})^{(N-1)(N-2)}\frac{4^N V \det Q}{V_N}.
\end{aligned}$$

Remark : It is known that $V_N = \pi^{\frac{N}{2}}/\Gamma(\frac{N}{2}+1)$. This, however, needs not concern as here.

THEOREM 3.3 : (M. Kneser) For each $N \in \mathbb{N}$, there exist a positive constant $C_N \in \mathbb{N}$ such that each isospectrality class of flat tori of dimension N is the disjoint union of at most C_N isometry classes of flat tori.

Proof : Proving the above geometric statement amouts to showing that a class of quadratic forms expressing the same real quantities with the same multiplicities, in other words, sharing the same theta function, is the disjoint union of finitely many congruence classes of quadratic forms.

Consider a quadratic form Q. Let $\mathcal{A}$ be a set of quadratic forms obtained by choosing from every equivalence class of quadratic forms congruent to Q, exactly one representative which admits the standart basis $\{e_1, \ldots, e_N\}$ of $\mathbb{Z}^N$ as a reduced basis. We shall further assume that $Q \in \mathcal{A}$. Clearly the cardinality of $\mathcal{A}$ equals to the cardinality of equivalence classes of quadratic forms congruent to Q. Still more explicitly, the cardinality of $\mathcal{A}$ equals the cardinality of isometry classes of flat tori isospectral to the torus $\mathbb{T}_A$ where $A \in \mathbb{R}^{N\times N}$ with $A^T A = Q$.

We prove that this cardinality is finite :

Given $Q' \in \mathcal{A}$, by the Jacobi Inversion formula we have

$$\det Q = \det Q'$$

Since $Q[\mathbb{Z}^N] = Q'[\mathbb{Z}^N]$ we also conclude that

$$\begin{aligned} Q'[e_1] &= \inf\{Q'[x] \mid x \in \mathbb{Z}^N - \{0\}\} \\ &= \inf\{Q[x] \mid x \in \mathbb{Z}^N - \{0\}\} \\ &= Q[e_1] \end{aligned}$$

Since both Q and Q' admit the standart basis as reduced basis.
By the lemma 3.3.1 ,

$$\prod_{k=1}^{N} Q'(e_k) \leq C_N \det(Q)$$

where

$$C_N = (\frac{3}{2})^{(N-1)(N-2)} 2^N \pi^{-\frac{N}{2}} \Gamma(\frac{N}{2}+1)$$

and since $Q[e_1] = Q'[e_1] \leq Q'[e_k]$ for $k = 1, \ldots, N$ we find that

$$Q[e_1]^{N-1} Q'[e_N] \leq C_N \det Q$$

Therefore, for any $k = 1, \ldots, N$

$$Q'[e_k] \leq Q'[e_N] \leq \frac{C_N \det Q}{Q[e_1]^{N-1}}$$

Applying the triangle inequality we obtain

$$\begin{aligned} \sqrt{Q'[e_i + e_j]} &\leq \sqrt{Q'[e_i]} + \sqrt{Q'[e_j]} \\ &\leq 2\sqrt{Q'[e_N]} \\ &\leq \frac{2C_N \det Q}{Q[e_1]^{N-1}} \end{aligned}$$

for all $1 \leq i \leq j \leq N$. Obviously, the same inequality holds if we replace the left handside with $\sqrt{Q'[e_i - e_j]}$. Consequently,

$$Q'[e_i, e_j] = \frac{1}{4}[Q'[e_i + e_j] - Q'[e_i - e_j]]$$

hence

$$Q'[e_i, e_j] \in Q'[\mathbb{Z}^N \times \mathbb{Z}^N] \cap [-L, L]$$

where

$$L = \frac{1}{2}(\frac{C_N \det Q}{Q[e_1]})^2$$

Being the intersection of a discrete set and a compact set, the right hand side is finite. Consequently, $Q'[e_i, e_j]$ can take only finitely many values. This shows that there exists at most finitely many quadratic forms in $\mathcal{A}$.

CHAPTER 4

GENERALITIES ON GEODESIC FLOWS

Let M be a smooth n-dimensional manifold, TM, T^*M be the tangent, cotangent bundles with $\tau : TM \longmapsto M$, $\tau^* : T^*M \longmapsto M$ be the natural tangential, cotangential projections, respectively.

Given a chart $x = (x^i)_{1\leq i\leq n} : dom(x) \subseteq M \longmapsto \mathbb{R}^n$ we may introduce the natural chart $\tilde{x} = (y^i)_{1\leq i\leq 2n}$ on TM defined by

$$dom \quad \tilde{x} = \tau^{-1}(dom \quad x)$$

and

$$y^i = x^i \circ \tau$$

$$y^{i+n}(v) = v^i$$

for $1 \leq i \leq n$, for any $v = \tau^{-1}(dom \quad x)$ with $v = v^i \frac{\partial}{\partial x}|_{\tau(v)}$.

We shall resort to the conventional abuse of notation by replacing $y^i = x^i \circ \tau$ with "x^i" and y^{i+n} with "$\dot{x}^i$" for $1 \leq i \leq n$.

In exactly the same fashion, given a chart $x = (x^i)_{1\leq i\leq 2n}$ on M, we introduce its canonical extension $x^* = (z^i)_{1\leq i\leq n}$ on T^*M defined by

$$dom \quad x^* = \tau^{*-1}(dom \quad x)$$

$$z^i = x^i \circ \tau^* \quad \text{for} \quad 1 \leq i \leq n$$

$$z^{i+n}(\omega) = \omega_i$$

for $1 \leq i \leq n$, for any $\omega \in \tau^{*-1}$ with $\omega = \omega_i \; \mathrm{d}x^i|_{\tau^*(\omega)}$.

Once again we shall resort to the conventional abuse of language and replace $z^i = x^i \circ \tau^*$ with "x^i" and z^{i+n} with $\dot{x}_i$ for $1 \leq i \leq n$

In the presence of a Riemannian tensor field $\mathsf{G} \in \mathcal{X}^{(0,2)}(\mathsf{M})$ on M, the manifolds TM and T^*M become closely related by so called *musical pairing* which consist of maps

$$\sharp : TM \longmapsto T^*M$$

$$\flat : T^*M \longmapsto TM$$

defined by

$$\sharp(u) = \mathsf{G}_{\tau(u)}(u, .) \in T^*_{\tau(u)}M$$

for $u \in M$ and $\flat = \sharp^{-1}$.

Introducing the dual Riemannian tensor field $\mathsf{G}^* \in \mathcal{X}^{(2,0)}(\mathsf{M})$ by

$$\mathsf{G}^*_m(\xi, \eta) = \mathsf{G}_m(\xi^\flat, \eta^\flat)$$

for $\xi, \eta \in T^*M_m$ where we have employed the space saving device of writing functions exponentially. We observe similarly,

$$\mathsf{G}_m(u, v) = \mathsf{G}^*_m(u^\sharp, v^\sharp)$$

for any $u, v \in T_mM$. Given any chart $x = (x^i)_{1 \leq i \leq n}$ on M, if

$$\mathsf{G}|_{dom(x)} = \mathsf{G}_{ij}\ \mathrm{d}x^i \otimes \mathrm{d}x^j$$

then

$$\mathsf{G}^*|_{dom(x)} = \mathsf{G}^{ij}\ \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

where $\mathsf{G}^{ij} : \mathrm{dom}(x) \longmapsto \mathbb{R}$ are smooth functions defined by

$$\mathsf{G}^{ik}\mathsf{G}_{kj} = \delta^i_j$$

for $1 \leq i, j \leq n$.

It can easily be checked that for a vector field A with

$$A\ |_{dom(x)} = A^i \frac{\partial}{\partial x^i}$$

we have

$$A^\sharp \mid_{dom(x)} = \mathsf{G}_{ij} A^j \mathrm{d}x^i$$

and similarly for a covector field Ω with

$$\Omega|_{dom(x)} = \Omega_i \mathrm{d}x^i$$

we have

$$\Omega^\flat \mid_{dom(x)} = \mathsf{G}^{ij} \Omega_j \frac{\partial}{\partial x^i}$$

Finally, the musical pairing can now be described in terms of the canonical charts by

$$\dot{x}_i \circ \sharp = \mathsf{G}_{ij} \dot{x}^j$$

and

$$\dot{x}^j \circ \flat = \mathsf{G}^{ij} \dot{x}_j$$

In the presence of the Riemannian tensor field G the following submanifolds of the tangent and cotangent bundles are of importance. The *sphere bundle* SM of M is defined by

$$SM = \{u \in TM \mid \mathsf{G}_{\tau(u)}(u, u) = 1\}$$

Similarly, *the cosphere bundle* S^*M of M is defined by

$$S^*M = \{\omega \in T^*M \mid \mathsf{G}^*_{\tau^*(\omega)}(\omega, \omega) = 1\}$$

Again, in the presence of a Riemannian tensor field, the tangent bundle TM houses, a very natural flow: The *geodesic flow* associated with (M, G) is the map $\varphi : TM \times \mathbb{R} \longmapsto TM$ which sends $(u, t) \in T_mM \times \mathbb{R}$ into $\dot{\gamma}(t)$ where $\gamma : \mathbb{R} \longmapsto M$ is the geodesic with $\dot{\gamma}(0) = u, \gamma(0) = m$.

Observe that we talk about the flow on the whole of $\mathbb{R}$. This is because the manifold is compact and it is complete in its metric structure.

In the following we shall find it more convenient to use the *cogeodesic flow* $\varphi^* = (\sharp, Id) \circ \varphi \circ (\flat, Id) : T^*M \times \mathbb{R} \longmapsto T^*M$.

It is important to exhibit the form that the geodesic and the cogeodesic flows assume in Hamiltonian formalism.

A *symplectic form* is a closed, non-degenerate two form. A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a symplectic form.

Observe that, the existence of a symplectic form in a manifold M implies that M is even dimensional.

Let (M, ω) be a symplectic manifold and $H : M \longmapsto \mathbb{R}$ function of class C^r $r \geq 1$. The vector field X_H determined by the condition

$$\omega(X_H, Y) = \mathrm{d}H(Y)$$

is called *symplectic gradient* of H.

In the presence of a symplectic structure on a manifold M, a scalar function whereof one considers the symplectic gradient is generically refered to as a *Hamiltonian.*

Observe that T^*M has a natural symplectic structure: The natural symplectic form $\omega \in \Lambda^2(T^*M)$ and the covector field $\alpha \in \Lambda^1(T^*M) = \mathcal{X}^{(0,1)}(T^*M)$ locally defined by

$$\omega \mid_{dom(\bar{x})} = \mathrm{d}\dot{x}_i \wedge \mathrm{d}x^i$$

$$\alpha \mid_{dom(\bar{x})} = \dot{x}^i \; \mathrm{d}x_i$$

These are clearly well-defined and satisfy

$$\omega = \mathrm{d}\alpha$$

ω is clearly a symplectic form with respect to which the symplectic gradient $X_H \in \mathcal{X}(T^*M)$ satisfies

$$X_H \mid_{dom(x)} = \frac{\partial H}{\partial \dot{x}_i} \frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i} \frac{\partial}{\partial \dot{x}_i}$$

Again, it is well-known that along the orbits of the flow generated by X_H, H is a conserved quantity.

THEOREM 4.1 : Let $\epsilon > 0$. Consider a Riemannian manifold (M, G) and a smooth family $H_s : T^*M \longmapsto \mathbb{R}, s \in [-\epsilon, \epsilon]$ of Hamiltonians with the following properties:

(i) Each H_s is homogeneous on the fibers of with constant order independent of $s \in [-\epsilon, \epsilon]$.

(ii) The flow generated by the symplectic gradient of H_s with respect to the canonical symplectic form on T^*M has a closed orbit γ_s parametrised by arclength for each $s \in [-\epsilon, \epsilon]$.

(iii) $H_s \circ \gamma_s \equiv 1$ for each $s \in [-\epsilon, \epsilon]$

(iv) The length of $\tau^*\gamma_s$ is independent of s.

Under these conditions $V = \frac{\partial H_s}{\partial s}\,|_{t=0}$ satisfies

$$\int_\gamma V = 0$$

where $\gamma = \gamma_0$.

Proof : Let L be the common length of γ_s, $s \in [-\epsilon, \epsilon]$. Put

$$\Omega = [-\epsilon, \epsilon] \times [0, L]$$

consider the simplex

$$C : \Omega \longmapsto T^*M$$

defined by

$$C(s,t) = \gamma_s(t)$$

Suppose k is the common order of the Hamiltonian H_s, $s \in [-\epsilon, \epsilon]$ as homogeneous functions of covectors. Using a generic chart $x = (x^i)_{1 \leq i \leq n}$ we find that

$$\begin{aligned} \alpha(\dot{\gamma}_s) &= \dot{x}_i \mathrm{d}x^i (\frac{\partial H}{\partial \dot{x}_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial \dot{x}_i}) \\ &= \dot{x}_i \frac{\partial H}{\partial \dot{x}_i} = kH \equiv k \end{aligned}$$

by Euler's renowed identitiy for homogeneous functions :
Therefore

$$\begin{aligned} \int_C \int \omega &= \int_C \mathrm{d}\alpha \int_{\partial C} \alpha \\ &= \int_0^L \alpha(\dot{\gamma_\epsilon}\mathrm{d}(t)) - \int_0^L \alpha(\dot{\gamma_{-\epsilon}}\mathrm{d}(t)) \\ &= kL - kL = 0 \end{aligned}$$

On the other hand

$$\begin{aligned} \int_C \omega &= \int_\Omega C^*\omega \\ &= \int_{-\epsilon}^{\epsilon}\int_0^L \omega \mid_{C(s,t)} (T_{(s,t)}C(\frac{\partial}{\partial s}), (T_{(s,t)}C(\frac{\partial}{\partial s}))\mathrm{d}s\mathrm{d}t \\ &= \int_{-\epsilon}^{\epsilon}\int_0^L \mathrm{d}H_s \mid_{C(s,t)} (T_{(s,t)}C(\frac{\partial}{\partial s}))\mathrm{d}s\ \mathrm{d}t \\ &= \int_{-\epsilon}^{\epsilon}\int_0^L \mathrm{d}(H_s \circ C) \mid_{(s,t)} (\frac{\partial}{\partial s})\mathrm{d}s\ \mathrm{d}t \\ &= \int_{-\epsilon}^{\epsilon}\int_0^L \frac{\partial(H_s \circ C)}{\partial s} \mid_{(s,t)} \mathrm{d}s\ \mathrm{d}t \end{aligned}$$

Clearly this argument is valid for any $\epsilon', 0 \leq \epsilon' \leq \epsilon$. Finally, since $H_s \circ \gamma_s \equiv 1$ for all $s \in [-\epsilon, \epsilon]$ we can replace $\frac{\partial(H_s \circ C)}{\partial s}$ with $\frac{\partial H_s}{\partial s} \circ C$. Putting

$$f(s) = \int_0^L \frac{\partial H_s}{\partial s} \circ C\mathrm{d}t$$

we not that unless $f(0) = \int_0^L V(\gamma(t))\mathrm{d}t = \int_\gamma = 0$ it is possible to choose $\epsilon \geq \epsilon'$ with

$$0 \neq \int_{-\epsilon'}^{\epsilon'} f(t)\mathrm{d}t = \int_{-\epsilon}^{\epsilon}\int_0^L \frac{\partial(H_s \circ C)}{\partial s}\mathrm{d}s\ \mathrm{d}t$$

On T^*M we shall be considered a specific Hamiltonian, the *canonical Hamiltonian* H of T^*M defined locally by

$$H\mid_{dom(\tilde{x})} = \frac{1}{2}\mathsf{G}^{ij}\ \dot{x}_i\dot{x}_j$$

THEOREM 4.2 : The cogeodesic flow arises as the integral flow of the symplectic gradient of H with respect to ω.

Proof : We shall use local coordinates and write down the differential equation governing the integral flow of the symplectic gradient of H with respect to ω and observe that under the musical pairing they transform into the those equations which govern the geodesic flow:

Suppose

$$\psi : T^*M \times \mathbb{R} \longmapsto \mathbb{T}^*\mathbb{M}$$

is the flow arising from X_H, the symplectic gradient of H with respect to ω. Clearly,

$$X_H|_{dom(\tilde{x})} = \mathsf{G}^{ip}\dot{x}_p\frac{\partial}{\partial x^i} - \frac{1}{2}\frac{\partial \mathsf{G}^{pq}}{\partial x^i}\dot{x}_p\dot{x}_q\frac{\partial}{\partial \dot{x}_i}$$

but $\omega \in dom(\tilde{x}) \subseteq T^*M$, putting

$$A^i(t) = x^i\ (\psi(\omega, t))$$

$$B_i(t) = \dot{x}_i\ (\psi(\omega, t))$$

for brevity, we conclude that

$$\frac{\mathrm{d}A^i}{\mathrm{d}t} = \mathsf{G}^{ip}B_p$$

$$\frac{\mathrm{d}B^i}{\mathrm{d}t} = -\frac{1}{2}\frac{\partial\mathsf{G}^{pq}}{\partial x^i}B_pB_q$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{d}^2A^i}{\mathrm{d}t^2} &= \frac{\partial\mathsf{G}^{ip}}{\partial x^k}\frac{\mathrm{d}A^k}{\mathrm{d}t}B_p + \mathsf{G}^{ip}\frac{\mathrm{d}B_p}{\mathrm{d}t} \\
\frac{\mathrm{d}^2A^i}{\mathrm{d}t^2} &= \frac{\partial\mathsf{G}^{ip}}{\partial x^k}\mathsf{G}^{kq}B_qB_p + \mathsf{G}^{ip}(-\frac{1}{2}\frac{\partial\mathsf{G}^{kl}}{\partial x^p}B_kB_l)
\end{aligned}$$

In view of the fact that $\mathsf{G}^{pq}\mathsf{G}_{qr}\delta^p_r$ we have

$$\frac{\partial\mathsf{G}^{pq}}{\partial x^i}\mathsf{G}_{qr} + \mathsf{G}^{pq}\frac{\partial\mathsf{G}^{qr}}{\partial x^i} = 0$$

and

$$\frac{\partial\mathsf{G}^{pq}}{\partial x^i} = -\mathsf{G}^{ps}\frac{\partial\mathsf{G}_{sm}}{\partial x^i}\mathsf{G}^{mq}$$

Consequently,

$$\begin{aligned}
\frac{\mathrm{d}^2A^i}{\mathrm{d}t^2} &= -\mathsf{G}^{is}\frac{\partial\mathsf{G}_{sm}}{\partial x^k}\mathsf{G}^{mp}\mathsf{G}^{kq}B_qB_p + \frac{1}{2}\mathsf{G}^{ip}\mathsf{G}^{ks}\frac{\partial\mathsf{G}_{sm}}{\partial x^p}\mathsf{G}^{mp}B_kB_l \\
\frac{\mathrm{d}^2A^i}{\mathrm{d}t^2} &= -\mathsf{G}^{is}\frac{\partial\mathsf{G}_{sm}}{\partial x^k}\frac{\mathrm{d}A^k}{\mathrm{d}t}\frac{\mathrm{d}A^m}{\mathrm{d}t} + \frac{1}{2}\mathsf{G}^{ip}\mathsf{G}^{ks}\frac{\partial\mathsf{G}_{sm}}{\partial x^p}\frac{\mathrm{d}A^s}{\mathrm{d}t}\frac{\mathrm{d}A^m}{\mathrm{d}t} \\
\frac{\mathrm{d}^2A^i}{\mathrm{d}t^2} &= -\mathsf{G}^{is}[\frac{\partial\mathsf{G}_{sj}}{\partial x^k} - \frac{1}{2}\frac{\partial\mathsf{G}_{jk}}{\partial x^s}]\frac{\mathrm{d}A^j}{\mathrm{d}t}\frac{\mathrm{d}A^k}{\mathrm{d}t} \\
\frac{\mathrm{d}^2A^i}{\mathrm{d}t^2} &= -\frac{1}{2}\mathsf{G}^{is}[\frac{\partial\mathsf{G}_{sk}}{\partial x^j} + \frac{\partial\mathsf{G}_{js}}{\partial x^k} - \frac{\partial\mathsf{G}_{jk}}{\partial x^s}]\frac{\mathrm{d}A^j}{\mathrm{d}t}\frac{\mathrm{d}A^k}{\mathrm{d}t} \\
\frac{\mathrm{d}^2A^i}{\mathrm{d}t^2} &= -\Gamma^i_{jk}\frac{\mathrm{d}A^j}{\mathrm{d}t}\frac{\mathrm{d}A^k}{\mathrm{d}t}
\end{aligned}$$

which shows that $(\flat, Id)\circ\psi\circ(\sharp, id)$ must be the geodesic flow. Therefore, ψ is the cogeodesic flow.

Geodesic flows, equivalently cogeodesic flows, on Riemannian manifolds of negative sectional curvature are known to be Anosov flows (Anosov 1961) do not concern us directly, except for the following cohomological property:

If $\overrightarrow{X} \in \mathcal{X}(\mathcal{M})$ generates an Anosov flow, for each smooth function $f : M \longmapsto \mathbb{R}$ which has the property

$$\int f(\lambda(t))\mathrm{d}t = 0$$

for each closed orbit of the flow generated by X, there exist a function $g : M \longmapsto \mathbb{R}$ of class C^1 such that $f = Xg$.(Livcic's Theorem), [Liv].

CHAPTER 5

TECHNICAL RESULTS ON THE COSPHERE BUNDLE OF A SURFACE

Let M be a 2-dimensional manifold, which we shall assume to be orientable. The assumption of orientability will cause no loss of generality since most of the following will be of local nature.

From the restriction $\tau^* : S^*M \longmapsto M$ we obtain :

$$T^*\tau^* : T^*M \longmapsto T^*S^*M$$

We define,

$$\omega^1 = T^*\tau^* \mid_{S^*M}$$

which is easily seen to be a cross section of the cotangential projection from T^*S^*M onto S^*M. In other words, ω^1 is a covector field on S^*M, that is $\omega^1 \in \mathcal{X}^*(S^*M)$.

More precisely, for each $\sigma \in S^*M$, the covector $\omega^1 \mid_\sigma$ is the map which sends $\xi \in T_\sigma S^*M$ to $\sigma(T_\sigma\tau^*(\xi))$. In the presence of a chart $x = (x^1, x^2)$ on M it is easily verified that [2]

$$\omega^1 \mid_{dom(\dot{x})} = \dot{x}_1 \mathrm{d}x^1 + \dot{x}_2 \mathrm{d}x^2$$

Since M is orientable there exist a tensor field $\Xi \in \mathcal{X}^{(1,1)}(M)$ which has the effect of rotating all tangent spaces about 90° in "counter-clockwise". To be precise, for each $p \in M$, the map $\Xi_p : T_pM \longmapsto T_pM$ is a linear isomorphism that preserves the inner product G_p and has the property that

$$\mathsf{G}_p(u, \Xi_p(u)) = 0$$

[2] The definition of ω^1 makes sense for a manifold of arbitrary dimension. The expression $\omega^1 = \dot{x}_i \mathrm{d}\, x^i$ retains its validity, too.

and makes $(u, \Xi_p(u))$ into a positively oriented orthogonal frame in T_pM.

Now we define yet another covector field on S^*M : $\omega^2 \in \mathcal{X}^{(1,1)}(S^*M)$ is the covector field, which assigns to each $\sigma \in S^*M$ the covector $\omega^2 \mid_\sigma$ that sends $\xi \in T_\sigma S^*M$ to $\sigma(\Xi_{\tau^*(\sigma)}(T_\sigma \tau^*(\xi)))$. Again, in the presence of a chart $x = (x^1, x^2)$ it can be verified that

$$\omega^2 \mid_{dom(\tilde{x})} = \dot{x}_2 \ \mathrm{d}x^1 - \dot{x}_1 \ \mathrm{d}x^2$$

From this point on we shall employ exclusively positively oriented *isothermal charts* on M, In the presence of a such chart $x = (x^1, x^2)$, there exist a smooth scalar field $\lambda : dom(x) \longmapsto \mathbb{R}$ such that

$$\mathsf{G} \mid_{dom(x)} = \lambda^2 \ \delta_{ij} \ \mathrm{d}x^i \otimes \mathrm{d}x^j$$

similarly,

$$\mathsf{G}^* \mid_{dom(x)} \frac{1}{\lambda^2} \ \delta^{ij} \ \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$$

Consequently, the coordinates $(x^1, x^2, \dot{x}_1, \dot{x}_2)$ of a point on S^*M satisfy the equality

$$(\dot{x}_1)^2 + (\dot{x}_2)^2 = \lambda^2$$

This means that it is possible to choose unique $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ with

$$\dot{x}_1 = \lambda \cos\theta \qquad \dot{x}_2 = \lambda \sin\theta$$

In the light of these observations we assume that S^*M is endowed with smooth atlas consisting of "cylindrical" charts of the form x^1, x^2, θ with

$$dom(x^1, x^2, \theta) = \tau^{*-1}(dom(x^1, x^2)) \cap S^*M$$

and each $\omega \in \tau^{*-1}(dom(x^1, x^2)) \cap S^*M$ being assigned coordinates $x^1 = x^1(\tau^*(m))$, $x^2 = x^2(\tau^*(m))$ and unique $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ with

$$\omega = \lambda(\cos\theta \ \mathrm{d}x^1 + \sin\theta \ \mathrm{d}x^2) \mid_{\tau^*(\omega)}$$

It is important to notice that in the passage from one positively oriented isothermal chart to the other the value of θ changes by a constant independent of the position. Consequently, the local vector field $\partial/\partial\theta$ makes sense globally.

In the following we shall make use of a generic "cylindrical" chart (x, y, θ) and denote the obvious extension of $\frac{\partial}{\partial\theta}$ by $\frac{\partial}{\partial\theta}$, this being only a slight abuse of notation !

Given

$$\omega^1 = \lambda(\cos\theta \mathrm{d}x + \sin\theta)\mathrm{d}y$$

$$\omega^2 = \lambda(\sin\theta \mathrm{d}x - \cos\theta \mathrm{d}y)$$

we obtain

$$\begin{aligned}
\mathrm{d}\omega^1 &= \mathrm{d}\lambda \wedge \frac{\omega^1}{\lambda} + \lambda[-\sin\theta \mathrm{d}\theta \wedge \mathrm{d}x + \cos\theta \mathrm{d}\theta \wedge \mathrm{d}y] \\
\mathrm{d}\omega^1 &= \mathrm{d}(\log\lambda) \wedge \omega^1 + \omega^2 \wedge \mathrm{d}\lambda \\
\mathrm{d}\omega^2 &= \mathrm{d}\lambda \wedge \frac{\omega^2}{\lambda} + \lambda[\cos\theta \mathrm{d}\theta \wedge \mathrm{d}x + \sin\theta \mathrm{d}\theta \wedge \mathrm{d}y] \\
\mathrm{d}\omega^2 &= \mathrm{d}(\log)\lambda \wedge \omega^2 + \mathrm{d}\theta \wedge \omega^1
\end{aligned}$$

Notice that

$$\begin{aligned}
\mathrm{d}(\log\lambda) &= \frac{\partial\log\lambda}{\partial x}\mathrm{d}x + \frac{\partial\log\lambda}{\partial y}\mathrm{d}y \\
&= \frac{\partial\log\lambda}{\partial x}[\frac{\cos\theta}{\lambda}\omega^1 + \frac{\sin\theta}{\lambda}\omega^2] + \frac{\partial\log\lambda}{\partial y}[\frac{\sin\theta}{\lambda}\omega^1 - \frac{\cos\theta}{\lambda}\omega^2]
\end{aligned}$$

from which we conclude that

$$\mathrm{d}(\log\lambda) = A\omega^1 + B\omega^2$$

where

$$\begin{aligned}
A &= \frac{1}{\lambda}[\frac{\partial\log\lambda}{\partial x}\cos\theta + \frac{\partial\log\lambda}{\partial y}\sin\theta] \\
B &= \frac{1}{\lambda}[\frac{\partial\log\lambda}{\partial x}\sin\theta - \frac{\partial\log\lambda}{\partial y}\cos\theta]
\end{aligned}$$

In view of the fact that

$$d\omega^1 = \omega^2 \wedge [d\theta + B\omega^1]$$

$$d\omega^2 = [d\theta - A\omega^2] \wedge \omega^1$$

we write

$$d\omega^1 = \omega^2 \wedge \varphi$$

$$d\omega^2 = \varphi \wedge \omega^1$$

where

$$\varphi = B\omega^1 - A\omega^2 + d\theta$$

Hence

$$\begin{aligned} \varphi &= \frac{1}{\lambda}[\frac{\partial \log \lambda}{\partial x} \sin\theta - \frac{\partial \log \lambda}{\partial y} \cos\theta]\omega^1 \\ &\quad -\frac{1}{\lambda}[\frac{\partial \log \lambda}{\partial x} \cos\theta + \frac{\partial \log \lambda}{\partial y} \sin\theta]\omega^2 + d\theta \end{aligned}$$

or equivalently,

$$\begin{aligned} \varphi &= [-\frac{\partial}{\partial x}(\frac{1}{\lambda}) \sin\theta + \frac{\partial}{\partial y}(\frac{1}{\lambda}) \cos\theta)]\omega^1 \\ &+ [\frac{\partial}{\partial x}(\frac{1}{\lambda} \cos\theta + \frac{\partial}{\partial y}(\frac{1}{\lambda}) \sin\theta]\omega^2 + d\theta \end{aligned}$$

Let $\xi_1, \xi_2, \xi_3 \in \mathcal{X}(\mathcal{S}^*\mathcal{M})$ constitute (in the given order !) the dual of $\omega^1, \omega^2, \varphi$: A simple inspection reveals that

$$\xi_1 = \frac{\cos\theta}{\lambda}\frac{\partial}{\partial x} + \frac{\sin\theta}{\lambda}\frac{\partial}{\partial y} + \{\frac{\partial}{\partial x}(\frac{1}{\lambda})\sin\theta - \frac{\partial}{\partial y}(\frac{1}{\lambda})\cos\theta\}\frac{\partial}{\partial\theta}$$

$$\xi_2 = \frac{\sin\theta}{\lambda}\frac{\partial}{\partial x} - \frac{\cos\theta}{\lambda}\frac{\partial}{\partial y} - \{\frac{\partial}{\partial x}(\frac{1}{\lambda})\cos\theta + \frac{\partial}{\partial y}(\frac{1}{\lambda})\sin\theta\}\frac{\partial}{\partial\theta}$$

$$\xi_3 = \frac{\partial}{\partial\theta}$$

Finally,

$$\begin{aligned}
\mathrm{d}\varphi = & - [\frac{\partial^2}{\partial x^2}(\frac{1}{\lambda})\mathrm{d}x + \frac{\partial^2}{\partial y\partial x}\mathrm{d}y]\sin\theta\wedge\omega^1 \\
& + [\frac{\partial^2}{\partial x\partial y}(\frac{1}{\lambda})\mathrm{d}x + \frac{\partial^2}{\partial y}(\frac{1}{\lambda})\mathrm{d}y]\cos\theta\wedge\omega^1 \\
& - [\frac{\partial}{\partial x}(\frac{1}{\lambda})\cos\theta + \frac{\partial}{\partial y}(\frac{1}{\lambda})\sin\theta]\mathrm{d}\theta\wedge\omega^1 \\
& + [\frac{\partial^2}{\partial x^2}(\frac{1}{\lambda})\mathrm{d}x + \frac{\partial^2}{\partial y\partial x}(\frac{1}{\lambda})\mathrm{d}y]\cos\theta\wedge\omega^2 \\
& + [\frac{\partial^2}{\partial x\partial y}(\frac{1}{\lambda})\mathrm{d}x + \frac{\partial^2}{\partial y}(\frac{1}{\lambda})\mathrm{d}y]\sin\theta\wedge\omega^2 \\
& - [\frac{\partial}{\partial x}(\frac{1}{\lambda})\sin\theta - \frac{\partial}{\partial y}(\frac{1}{\lambda})\cos\theta]\mathrm{d}\theta\wedge\omega^2 \\
& - [\frac{\partial}{\partial x}(\frac{1}{\lambda})\sin\theta - \frac{\partial}{\partial y}(\frac{1}{\lambda})\cos\theta]\mathrm{d}\omega^1 \\
& + [\frac{\partial}{\partial x}(\frac{1}{\lambda})\cos\theta + \frac{\partial}{\partial y}(\frac{1}{\lambda})\sin\theta]\mathrm{d}\omega^2
\end{aligned}$$

$$
\begin{aligned}
d\varphi = &+ (\frac{1}{\lambda})[\frac{\partial^2}{\partial x^2}(\frac{1}{\lambda}) + \frac{\partial^2}{\partial y^2}(\frac{1}{\lambda})] \\
&- [\frac{\partial}{\partial x}(\frac{1}{\lambda}) \sin\theta - \frac{\partial}{\partial y}(\frac{1}{\lambda}) \cos\theta]\ (\omega^2 \wedge \varphi) \\
&+ [\frac{\partial}{\partial x}(\frac{1}{\lambda}) \cos\theta + \frac{\partial}{\partial y}(\frac{1}{\lambda}) \sin\theta](\varphi \wedge \wedge\omega^1)
\end{aligned}
$$

hence,

$$
\begin{aligned}
d\varphi &= [\frac{1}{\lambda}\Delta(\frac{1}{\lambda}) - [\frac{\partial}{\partial x}(\frac{1}{\lambda})]^2 + [\frac{\partial}{\partial y}(\frac{1}{\lambda})]^2]\ \omega^1 \wedge \omega^2 \\
&= [\frac{1}{\lambda}\Delta(\frac{1}{\lambda}) - \frac{1}{\lambda^4}(\frac{\partial\lambda}{\partial x^2})^2 - \frac{1}{\lambda^4}(\frac{\partial\lambda}{\partial y})^2]\omega^1 \wedge \omega^2 \\
&= [\frac{1}{\lambda^4}[(\frac{\partial\lambda}{\partial x})^2(\frac{\partial\lambda}{\partial y})^2] - \frac{1}{\lambda^3}(\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y})]\ \omega^1 \wedge \omega^2 \\
&= \frac{\Delta(\log\lambda)}{\lambda^2}\omega^1 \wedge \omega^2 \\
&= -(K \circ \tau^*)\ \omega^1 \wedge \omega^2
\end{aligned}
$$

In this connection it is important to notice that ξ_1 is exactly the flow which generates the cogeodesic flow :

$$
H = \frac{1}{\lambda^2}(\dot{x}^2 + \dot{y}^2)
$$

we have

$$
\begin{aligned}
X_H &= \frac{\partial H}{\partial \dot{x}_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial \dot{x}_i} \\
X_H &= \frac{\dot{x}}{\lambda^2}\frac{\partial}{\partial x} + \frac{\dot{y}}{\lambda^2}\frac{\partial}{\partial y} + \frac{1}{\lambda^3}\frac{\partial\lambda}{\partial x}(\dot{x}^2 + \dot{y}^2)\frac{\partial}{\partial \dot{x}} + \frac{1}{\lambda^3}\frac{\partial\lambda}{\partial y}(\dot{x}^2 + \dot{y}^2)\frac{\partial}{\partial \dot{y}}
\end{aligned}
$$

Introducing the polar coordinates for the covector part of points in T^*M with

$$
\dot{x} = \lambda\cos\theta
$$

$$\dot{y} = \lambda \sin\theta$$

we have

$$\begin{aligned} \frac{\partial}{\partial\lambda} &= \cos\theta\frac{\partial}{\partial\dot{x}} + \sin\theta\frac{\partial}{\partial\dot{y}} \\ \frac{\partial}{\partial\theta} &= -\lambda\sin\theta\frac{\partial}{\partial\dot{x}} + \lambda\cos\theta\frac{\partial}{\partial\dot{y}} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial\dot{x}} &= \cos\theta\frac{\partial}{\partial\lambda} - \frac{\sin\theta}{\lambda}\frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\dot{y}} &= cos\theta\frac{\partial}{\partial\lambda} + \frac{\cos\theta}{\lambda}\frac{\partial}{\partial\theta} \end{aligned}$$

from which we obtain

$$X_H = \frac{\cos\theta}{\lambda}\frac{\partial}{\partial x} + \frac{\sin\theta}{\lambda}\frac{\partial}{\partial y} + \{\frac{\partial}{\partial x}\frac{1}{\lambda}\sin\theta - \frac{\partial}{\partial y}\frac{1}{\lambda}\cos\theta\}\frac{\partial}{\partial\theta}$$

CHAPTER 6

SPECTRAL RIGIDITY OF 2 DIMENSIONAL RIEMANNIAN MANIFOLDS OF NEGATIVE CURVATURE

The purpose of this section is to prove that a 2-dimensional compact Riemannian manifold with negative sectional curvature cannot undergo non-trivial spectral deformations.

We shall slur over numereous details from analysis and from analysis and try to lay bare the essential ideas. We adopt the notations introduced in the previous sections and consider the vector fields $\xi_1, \xi_2, \frac{\partial}{\partial\theta} \in \mathcal{X}(S^*M)$ which obey

$$\begin{aligned}
[\xi_1, \frac{\partial}{\partial\theta}] &= -\xi_2 \\
[\xi_2, \frac{\partial}{\partial\theta}] &= -\xi_1 \\
[\xi_1, \xi_2] &= (K \circ \tau^*)\frac{\partial}{\partial\theta}
\end{aligned}$$

It will be convenient to pass to the complexified vector fields η_+, η_- defined by

$$\eta_+ = \frac{1}{2}(\xi_1 - i\xi_2)$$

$$\eta_- = \frac{1}{2}(\xi_1 + i\xi_2)$$

It can now be easily checked that

$$\begin{aligned} [\eta_+, \frac{\partial}{\partial\theta}] &= -i\eta_+ \\ [\frac{\partial}{\partial\theta}, \eta_-] &= -i\eta_- \\ [\eta_+, \eta_-] &= \frac{iK\circ\tau^*}{2}\frac{\partial}{\partial\theta} \end{aligned}$$

We shall introduce the volume form

$$\Omega = \omega^1 \wedge \omega^2 \wedge \varphi$$

on S^*M and implicitely work with the Lebesque measure induced by Ω on S^*M and talk about the Hilbert space $L^2(S^*M)$. We observe that the cogeodesic flow preserves Ω, since

$$\begin{aligned} (L_{\xi_1}\Omega)(\xi_1, \xi_2, \frac{\partial}{\partial\theta}) &= \xi_1\Omega(\xi_1, \xi_2, \frac{\partial}{\partial\theta}) + \Omega([\xi_1, \xi_2], \xi_2, \frac{\partial}{\partial\theta}) \\ &+ \Omega(\xi_1, [\xi_1, \xi_2], \frac{\partial}{\partial\theta}) + \Omega(\xi_1, \xi_2, [\xi_1, \frac{\partial}{\partial\theta}]) \\ &= \Omega(\xi_1, (K\circ\tau^*)\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}) = 0 \end{aligned}$$

Therefore $L_\xi\Omega(X, Y, Z) = 0$ for any $X, Y, Z \in \mathcal{X}(S^*M)$. Similarly the flow that rotates the fibers of S^*M, that is, the flow generated by $\frac{\partial}{\partial\theta}$ preserves Ω, too:

$$\begin{aligned} L_{\frac{\partial}{\partial\theta}}\Omega(\xi_1, \xi_2, \frac{\partial}{\partial\theta}) &= \frac{\partial}{\partial\theta}\Omega(\xi_1, \xi_2, \frac{\partial}{\partial\theta}) + \Omega([\frac{\partial}{\partial\theta}, \xi_1], \xi_2, \frac{\partial}{\partial\theta}) \\ &+ \Omega(\xi_1, [\frac{\partial}{\partial\theta}, \xi_2], \frac{\partial}{\partial\theta}) + \Omega(\xi_1, \xi_2, [\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}]) \\ &= \Omega(\xi_1, \xi_2, \frac{\partial}{\partial\theta}) + \Omega(\xi_1, -\xi_2, \frac{\partial}{\partial\theta}) = 0 \end{aligned}$$

This means that both the differential operators $\frac{1}{i}\frac{\partial}{\partial\theta}$ and $\frac{1}{i}\xi_1$ are self-adjoint operators with dense domains on $L^2(S^*M)$.

By an obvious anology with the ordinary Fourier analysis on the circle we see that

$$L^2(S^*M) = \ominus_{n\in\mathbb{Z}}\mathcal{H}_n$$

where each $\mathcal{H}_n$ is a two dimensional space and

$$\frac{1}{i}\frac{\partial}{\partial\theta}\ f = n\ f$$

for each $f \in \mathcal{H}_n, n \in \mathbb{Z}$. In fact $\mathcal{H}_n$ is exactly characterised by this property

Lemma : $\eta_+(\mathcal{H}_n) \subseteq \mathcal{H}_{n+1}$

Proof : Given $f \in \mathcal{H}_n$

$$\begin{aligned} \frac{1}{i}\frac{\partial}{\partial\theta}\eta_+ f &= [\frac{1}{i}\frac{\partial}{\partial\theta}, \eta_+]f + \eta_+\frac{1}{i}\frac{\partial f}{\partial\theta} \\ &= \eta_+ f + n\eta_+ f \\ &= (n+1)\eta_+ f \end{aligned}$$

hence $\eta_+ f \in \mathcal{H}_n$.

Similarly we can show that $\eta_-\mathcal{H}_{n+1} = \mathcal{H}_n$. Let us denote adjoints by $*$:

Lemma : $\eta_+^* = -\eta_-$

Proof : Since $\xi_1 = (\eta_+^* + \eta_-^*)$ and $\frac{1}{i}\xi_1$ is self-adjoint, we have $\xi_1^* = -\xi_1$ Therefore

$$\xi_+^* + \xi_-^* = -\xi_+^* + \xi_-^*$$

As η_+ and η_- move components $\mathcal{H}_n$ of $L^2(S^*M)$ in opposite directions, we conclude that

$$\eta_+^* = -\eta_-$$

$$\eta_-^* = -\eta_+$$

The operator $\eta_+^*\eta_+$ thus be extended by a non-negative self-adjpnt operator on $L^2(S^*M)$ with

$$\eta_+^*\eta_+(\mathcal{H}_n) = \mathcal{H}_n$$

and with a discrete spectrum tending to ∞.

LEMMA 6.1 : If

$$\begin{aligned} a &= \frac{1}{2}\inf\{-K(m) \mid m \in M\} \\ b &= \frac{1}{2}\sup\{-K(m) \mid m \in M\} \end{aligned}$$

then

$$\| \eta_+^* f \|^2 + an \| f \|^2 \leq \| \eta_+^* f \|^2 \leq \| \eta_+^* f \|^2 + bn \| f \|^2$$

for any $f \in \mathcal{H}_n \cap dom(\eta_+) \cap dom(\eta_-)$.

Proof :

$$\begin{aligned} \| \eta_+^* f \|^2 &= < \eta_+^* f, \eta_+^* f > \\ &= < \eta_+^* \eta_+^* f, f > \\ &= < \eta_+ \eta_- f, f > \\ &= < \eta_+ \eta_- f + \frac{K}{2i}\frac{\partial f}{\partial \theta}, f > \\ &= < \eta_-^* \eta_- f, f > - \frac{K}{2} n \| f \|^2 \\ &= < \eta_- f, \eta_- f > - \frac{K}{2} n \| f \|^2 \\ &= \| \eta_- f \|^2 - \frac{K}{2} n \| f \|^2 \end{aligned}$$

which gives the desired inequalities.

LEMMA 6.2 : Consider $f = \sum_{-N \leq n \leq N} f_n \in C^\infty(S^*M)$ with $-N \leq n \leq N$, the integral of f vanishes over each closed orbit of the cogeodesic flow, then there exists $g = \sum_{-(N-1) \leq n \leq (N-1)} g_n$ with $g \in \mathcal{H}_n \cap C^\infty(S^*M)$ such that $f = \xi_1 g$.

Proof : By the theorem of Livčic [Liv], there exists $g = \sum_{n\in\mathbb{Z}} g_n$ with $g_n \in \mathcal{H}_n \cap C^1(S^*M)$ such that

$$\xi_1 f = g$$

This means

$$\eta_+ g_{n-1} + \eta_- g_{n+1} = f_n$$

for each $n \in \mathbb{Z}$. Therefore, for $n \geq N+1$

$$\eta_+ g_{n-1} + \eta_- g_{n+1} = 0$$

Now employing the lemma 6.1 we find that

$$\| \eta_+ g_{n-1} \|^2 = \| \eta_- g_{n+1} \|^2 \leq \| \eta_+ g_{n+1} \|^2 - a(n+1) \| g_{n+1} \|^2$$

for $a > 0$. Consequently,

$$\eta_+ g_{n-1} \| \leq \| \eta_+ g_{n+1} \|$$

As $\lim_{n \longmapsto \infty} \| \eta_+ g_n \| = 0$, we conclude that $\| \eta_+ g_k \| = 0$ for $k \geq N$. Again by Lemma 6.1

$$\| \eta_- g_n \|^2 + an \| g_n \|^2 \leq \| \eta_+ g_n \|^2$$

for $n \geq N$, gives us $g_n = 0$ for $n \geq N$. It remains to show that g_n is smooth, not only C^1. This, however is clear by the recurrence relation

$$\eta_+ g_{n-1} = f_n - \eta_- - g_{n+1}$$

Consider a smooth family $\{\mathsf{G}_{s\in(-\epsilon,\epsilon)}\}$ of Riemannian metrics, let $\{\mathsf{H}_{s\in(-\epsilon,\epsilon)}\}$ be the corresponding family of canonical Hamiltonians on T^*M. Let $V = \frac{\partial H_s}{\partial s} \mid_{s=0} \in C^\infty(T^*M)$

LEMMA 6.3 :

$$V\mid_{S^*M} \in \mathcal{H}_{-2} \ominus \mathcal{H}_0 \oplus \mathcal{H}_2$$

Indeed $V\mid_{S^*M} = V_{-2} + V_0 + V_2$ with $V_k \in \mathcal{H}_k$ for $k \in \{-2, 0, 2\}$ where

$$\overline{V}_2 = V_{-2}, \quad \overline{V}_0 = V_0$$

Proof : We shall control ourselves with observing that H_s is a quadratic form in covector variables with coefficients which are functions of position in M and of the parameter s. The rest is obvious from the following elememtary analogue :
The expression $ax^2 + 2bxy + cy$ with $x^2 + y^2 = 1$ can be written as

$$Az^2 + Bz\overline{z} + C\overline{z}^2$$

where $z = x + iy$

$$\begin{aligned} A &= \frac{a-c-2bi}{4} \\ C &= \frac{a-c+2bi}{4} \\ B &= a-c \end{aligned}$$

Putting $z = \exp i\theta$ we obtain it in the requried form

$$Ae^{2i\theta} + B + Ce^{-2i\theta}$$

where $\overline{A} = C, \overline{B} = B$.

LEMMA 6.4 : If $\varphi : M \times (-\epsilon, \epsilon) \longmapsto M$ is flow generated by the smooth vector field X, then the flow $\tilde{\varphi} : T^*M \times (-\epsilon, \epsilon) \longmapsto T^*M$ be induced by φ, is generated by the symplectic gradient of the Hamiltonian $Z : T^*M \longmapsto \mathbb{R}$ defined by

$$Z(\sigma) = \sigma(X\mid_{\tau^*(\sigma)})$$

with respect to the canonical symplectic form on T^*M.

Proof : We shall do it explicitely by introducing a chart (x, y) on M and considering its canonical extension $(x, y, \dot{x}, \dot{y})$ on T^*M. If

$$X \mid_{dom(x,y)} = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$$

for some $A, B : dom(x, y) \longmapsto \mathbb{R}$ then

$$Z \mid_{(x,y,\dot{x},\dot{y})} = A\dot{x} + B\dot{y}$$

The canonical symplectic form ω satisfies

$$\omega \mid_{dom(x)} = \mathrm{d}\dot{x} \wedge \mathrm{d}x + \mathrm{d}\dot{y} \wedge \mathrm{d}y$$

and the symplectic gradient $\tilde{X}$ of Z with respect to ω is

$$\begin{aligned}
\tilde{X} &= \frac{\partial Z}{\partial \dot{x}}\frac{\partial}{\partial x} + \frac{\partial Z}{\partial \dot{y}\frac{\partial}{\partial y}} - \frac{\partial Z}{\partial x}\frac{\partial}{\partial \dot{x}} - \frac{\partial Z}{\partial y}\frac{\partial}{\partial \dot{y}} \\
&= A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} - (\frac{\partial A}{\partial x}\dot{x} + \frac{\partial B}{\partial x}\dot{y})\frac{\partial}{\partial \dot{x}} - (\frac{\partial A}{\partial y}\dot{x} + \frac{\partial B}{\partial y}\dot{y})\frac{\partial}{\partial \dot{y}}
\end{aligned}$$

we have

$$\tilde{\varphi}(., s) = T^*\varphi(., s)$$

to be quite explicitely for $\sigma \in T^*_m M$, $\sigma_s = \tilde{\varphi}(\sigma, s) \in T^*_{\varphi(m,s)}M$ defined by

$$\sigma_s(u) = \sigma((T_m\varphi(., s))^{-1}(u))$$

for any $u \in T_{\varphi(m,s)}$. Let's assume for simplicity that for a point $m \in dom(x, y)$ with coordinates x, y, the point $\varphi(m, s)$ have coordinates x_s, y_s. Clearly

$$\frac{\partial x_s}{\partial s} \mid_{s=0} = A, \ \frac{\partial y_s}{\partial s} \mid_{s=0} = B$$

Again, suppose that for a point $\sigma \in T^*M$ with coordinates $x, y, \dot{x}, \dot{y}$ let $\sigma_s = \tilde{\varphi}(\sigma, s)$ have coordinates $x_s, y_s, \dot{x}_s, \dot{y}_s$. Obviously.

$$\sigma_s = \dot{x}_s \mathrm{d}x + \dot{y}_s \mathrm{d}y \mid_{\varphi(m,s)}$$

and

$$\begin{aligned}
\dot{x}_s &= \sigma_s(\frac{\partial}{\partial x} \mid_{\varphi(m,s)}) \\
&= \sigma(\frac{\partial x_{-s}}{\partial x}\frac{\partial}{\partial x} + \frac{\partial y_{-s}}{\partial x}\frac{\partial}{\partial y} \mid_m) \\
&= \frac{\partial x_{-s}}{\partial x}\dot{x} + \frac{\partial y_{-s}}{\partial y}\dot{y} \mid_m \\
\dot{y}_s &= \sigma_s(\frac{\partial}{\partial y} \mid_{\varphi(m,s)}) \\
&= \sigma(\frac{\partial x_{-s}}{\partial y}\frac{\partial}{\partial x} + \frac{\partial y_{-s}}{\partial y}\frac{\partial}{\partial y} \mid_m) \\
&= \frac{\partial x_{-s}}{\partial y}\dot{x} + \frac{\partial y_{-s}}{\partial y}\dot{y} \mid_m
\end{aligned}$$

Therefore, the vector field generating $\tilde{\varphi}$ has to be

$$\begin{aligned}
\frac{\partial x_s}{\partial s} \mid_{s=0} \frac{\partial}{\partial x} &+ \frac{\partial y_s}{\partial s} \mid_{s=0} \frac{\partial}{\partial y} \\
&+ \frac{\partial \dot{x}_s}{\partial s} \mid_{s=0} \frac{\partial}{\partial \dot{x}} + \frac{\partial \dot{y}_s}{\partial s} \mid_{s=0} \frac{\partial}{\partial \dot{y}} \\
&= A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} - (\frac{\partial A}{\partial x}\dot{x} + \frac{\partial B}{\partial x}\dot{y})\frac{\partial}{\partial \dot{x}} - (\frac{\partial A}{\partial y}\dot{x} + \frac{\partial B}{\partial y}\dot{y})\frac{\partial}{\partial \dot{y}} \\
&= \tilde{X}
\end{aligned}$$

LEMMA 6.5 :

$$Z \mid_{S^*M} \in \mathcal{H}_{-\infty} \oplus \mathcal{H}_{\infty}$$

Indeed $Z \mid_{S^*M} = Z_{-1} + Z_{+1}$ where $Z_k \in \mathcal{H}_k, k \in \{-1, 1\}$ and $\overline{Z}_1 = Z_{-1}$

Proof : It is sufficient to evoke the analogy already employed in Lemma ..., by noticing that $\dot{Z}$ restricted to a fiber of S^*M behaves as a linear expression of form

$$ax + by$$

where $x, y \in BbbR$ and $x^2 + y^2 = 1$. It can be put into the form

$$Az + B\overline{z}$$

or

$$Ae^{i\theta} + Be^{-i\theta}$$

since $\mid z \mid = 1$ where $z = x + iy,\ A = \frac{1}{2}(a - ib),\ B = \frac{1}{2}(a + ib)$ Note that $\overline{B} = A$.

Remember that, on flat tori the spectrum determines the length spectrum owing to the Jacobi inversion formula for theta functions. It has been asked [K-S] whether it is possble to produce such formulae for more general Riemannian manifolds. This has been done for two dimensional Riemannian manifolds of negative curvature by Y. Colin de Verdiere who has produced the analogues formula

$$\sum_{k\in\mathbb{N}} \exp(-\lambda_k/z) = (\frac{z}{4\pi} Vol(M)) + (\frac{z}{4\pi})^{1/2} \sum_{a\in\mathcal{A}} n_a \exp(-\frac{1}{4} L_a^2 z) + O(1)$$

on z with $Re(z) > 0$ where $0 \leq \lambda_0 < \lambda_1 < \dots$ is the spectrum, $\mathcal{A}$ the set of non-trivial free homotopy classes of loops L_a the length of the unique closed geodesic in $a \in \mathcal{A}$, \⊣ $\in \mathbb{R}_{>\prime}$, $O(1)$ is a function of z which is bounded on $Re(z) \geq A$ for each $A > 0$. This result has been later extended to "generic" Riemannian manifolds of negative curvature, [Ver]. This shows us that an isospectral deformation preserves the length of closed geodesics.

THEOREM 6.1: Let M be a 2-dimensional, smooth compact manifold, $\{\mathsf{G}_t\}_{t\in(-\epsilon,\epsilon)}$ a smooth family of Riemannian metrics of negative curvature. If (M, G_t) and $(M, \mathsf{G}_{t'})$ are isospectral for all $t, t' \in (-\epsilon, \epsilon)$ then they are isometric, too.

Proof : Let $\{\mathsf{H}_s\}_{s\in(-\epsilon,\epsilon)}$ be the corresponding family of canonical Hamiltonians. Let $V = \frac{\partial H_s}{\partial s}|_{s=0}|_{S^*M} \in C^\infty(S^*M)$. By the above mentioned results of Verdiere as s changes closed geodesics vary smoothly without changing their length. Consequently,

$$\int_\gamma V = 0$$

for any closed geodesic γ (of G_0)Let $\xi \in \mathcal{X}(S^*M)$ be the vector field generating the cogeodesic flow, by the theorem of Livčic and Lemma 6.5 there exists

$$Z = Z_{-1} + Z_0 + Z_1 \in C^\infty(S^*M)$$

where

$$Z_k \in \mathcal{H}_k \cap C^\infty(S^*M)$$

for $k \in \{-1, 0, 1\}$ such that

$$V = \xi Z$$

We have alreadily seen that

$$V = V_{-2} + V_0 + V_2$$

with $V_k \in \mathcal{H}_k \cap C^\infty(S^*M)$. We conclude that

$$\eta_+ Z_0 = \eta_- Z_0 = 0$$

since

$$\eta_+ Z_0 \in \mathcal{H}_1, \eta_- Z_0 \in \mathcal{H}_{-1}$$

and

$$\eta_+ Z_0 + \eta_- Z_0 = \xi Z_0$$

On the other hand since $Z_0 \in \mathcal{H}_0, \partial/\partial\theta Z_0 = 0$, we conclude that $\xi_1 Z_0 = \xi_2 Z_0 = \partial/\partial\theta Z_0 = 0$ showing that Z_0 is a constant which we may assume constant, without loss of generality. Now, we repeat this argument for each $s \in (-\epsilon, \epsilon)$ and for

$$V_{-2}^{[s]} + V_0^{[s]} + V_2^{[s]} = V^{[s]} = \frac{\partial H_s}{\partial s} \in C^\infty(S^*M)$$

we obtain

$$Z^{[s]} = Z_{-1}^{[s]} + Z_{+1}^{[s]} \in C^\infty(S^*M)$$

with

$$V^{[s]} = \eta^{[s]} Z^{[s]}$$

in particular

$$V_2^{[s]} = \eta_+^{[s]} Z_1^{[s]}$$

and

$$V_{-2}^{[s]} = \eta_-^{[s]} Z_{-1}^{[s]}$$

by taking conjugates.

Now, since $Z^{[s]}$ is of the form

$$Z^{[s]} = Z_{+1}^{[s]} + Z_{-1}^{[s]}$$

there exist a unique vector field $X^{[i]} \in \mathcal{X}(M)$ such that

$$Z^{[i]}(\sigma) = \sigma^{[i]}(X\mid_{\tau^*(\sigma)})$$

(as in Lemma 6.4) Remember that $Z^{[i]}$ as Hamiltonian gives rise to the flow on T^*M induced from the flow on M generated by $X^{[s]}$. Consider a flow $\varphi : M \times (-\epsilon, \epsilon) \longmapsto M$ which obeys

$$\frac{\partial\varphi(.,s)}{\partial s} = T\varphi(.,s)(X^{[s]} \circ \varphi(0,1)^{-1})$$

and define

$$G'_s = \varphi(.,s)_* G_0$$

Clearly,

$$\frac{\partial}{\partial s} G'_s = X^{[s]} G'_s$$

subject to $G'_0 = G_0$ In view of the fact that

$$\frac{\partial}{\partial s} = G_s = X^{[s]} G_s$$

we conclude that $G_s = G'_s = \varphi(0,s)_* G_0$ showing that (M, G_0) is isometric to (M, G_1).

The purpose of this section is to prove that a 2-dimensional Riemannian manifold cannot undergo non-trivial spectral deformations.

THEOREM 6.2 : It will be convinient to pass to the complexified vector fields η_+, η_- defined by

$$\eta_+ = \frac{1}{2}(\xi_1 - i\xi_2)$$

$$\eta_- = \frac{1}{2}(\xi_1 + \xi_2)$$

where,

$$[\eta_+, \frac{\partial}{\partial\theta}] = -i\eta_+$$

$$[\frac{\partial}{\partial\theta}, \eta_-] = -\eta_-$$

$$[\eta_+, \eta_-] = \frac{iK}{2}\frac{\partial}{\partial\theta}$$

THEOREM 6.3 : $L^2(S^*M) = \oplus_{n\in\mathbb{Z}} H_n$ where $A\varphi = in\varphi$ for each $\varphi \in H_n$. This is merely Fourrier analysis on $S^1 = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$. Each $f \in L^2(S^1)$ can be written as

$$f(x,t) = \sum_{n\in\mathbb{Z}} a_n(x)e^{int}$$

LEMMA 6.6 : If $f \in H_n$, then $\eta_+ f \in H_{n+1}$

LEMMA 6.7 : η_+ and η_- are adjoint operators with respect to the L^2 inner product. That is, $< \eta_+ f, g > = < f, \eta_- g >$ for any $f \in \mathrm{dom}(\eta_+) \subseteq L^2(S^*M)$ and $g \in \mathrm{dom}(\eta_-) \subseteq L$.

LEMMA 6.8 :

$$\Xi = \Xi_0 \in H_2 \oplus H_0 \oplus H_{-2}$$

moreover, $\Xi = f_2 + f_0 + f_{-2}$ with $f_i \in H_i$ and $\overline{f_2} = f_{-2}, f_0$ real. **Essential idea:** Identify for each $p \in MT_p{}^*M$ with $\mathbb{C}$ and $S_p{}^*M$ with $S^1 = \{z \in \mathbb{C} | \mid z \mid = 1\}$

$$\Xi = \begin{cases} ax^2 + 2bxy + cy^2 \quad a, b, c \in \mathbb{R} \\ a\frac{z+\overline{z}}{2}^2 + 2b\frac{z+\overline{z}}{2}\frac{z-\overline{z}}{2i} + c\frac{z-\overline{z}}{2i}^2 \\ (\frac{a}{4} - \frac{c}{4} - \frac{b}{2})z^2 + (\frac{a}{4} - \frac{c}{4} - \frac{b}{2}i)\overline{z}^2 + (a-c)z\overline{z} \end{cases}$$

therefore,

$$\Xi = Le^{2it} + \overline{L}e^{-2it} + M$$

where $M * in\mathbb{R}$

LEMMA 6.9 : $H \in H_1 \oplus H_{-1}$ in fact $H = f_1 + f_{-1} f_i \in H_i$.

THEOREM 6.4 : Let M be a 2-dimensional compact manifold, $\{\mathcal{G}_{t \in J}\}$ a continuous family of Riemannian tensor fields on M. If $\{M, \mathsf{G}_t\}$ and $\{M, \mathsf{G}'_t\}$ are isospectral for all $t, t' \in J$, then they are isometric, too.

CHAPTER 7

CONCLUSION

What is exciting about a manifold that is spectraly rigid? In view of the imposibility of characterising a Riemannian manifold by means of its spectrum, this seem to be the next best thing: A spectrally rigid Riemannian manifold may be understood to be determined by its spectrum, in a local sense. It seems hard to imagine that an isospectrally rigid Riemannian manifold can be arbitrarily approximated by isospectral ones which are isometrically distinct. This brings up the first important question:

1) Is spectrally rigid manifold locally determined by its spectrum ? To be precise : Given a manifold M let us introduce a suitable topology on the set of the Riemannian tensor fields on M. Suppose (M, G) is spectrally rigid. Is it true that there exist a neighbourhood U of G such that for all isospectral $G', G'' \in \mathcal{X}$, the Riemannian manifolds (M, G') and (M, G'') are isometric ?

In my opinion only an affirmative answer to this question can lead relevance to spectral rigidity. Note that, within the class of tori we have this picture, rather trivially.

In this two examples which have been treated in the present work, the passage from isospectrality to isometry depends heavily on the presence of a formula like the Jacobi identity (or Poisson identity) that allows spectrum to dictate the "statistics" of the lengths of closed geodesics. Once present cruel formulae can provide information quite flexibly and have a shape which is unaffected by terms which are " sufficiently tame at infinity". One significant weak point in this line of approach is that the "statistics" of closed geodesics [Ver], that is length spectrum does not make sense generally.

2) Is it possible to modify the concept of length spectrum so as to make it more generally useful ? See [Gor].

3) Is there a generally valid formula of the Jacobi or Poisson which translates spectral information into metric information ?

4) The torus on the one hand and surfaces of negative sectional curvature on the other seem to be quite unrelated instances of spectral rigidity. Is there a common trait ? (Presently, it is nowhere to be seen) Or possibly the kind of spectral rigidity displayed by the torus, depending as it does on finiteness of isometry classes within isospectral classes, a misleading coincidence ? In any case tori seem to belong with the Heisenberg manifolds rather than surfaces of negative sectional curvature !

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APPENDIX 1

MINKOWSKI DOMAINS AND GAUGE FUNCTIONS

Let $f : \mathbb{R}^N \longmapsto \mathbb{R}$ be a gauge function. Consider $K_f = K = f^{-1}([0,1]) \subseteq \mathbb{R}^N$. Clearly K is closed. If $x \in K$ then $f(-x) = f(x) \in [0,1]$ hence $-x \in K$. Therefore, K is symmetric, $f^{-1}([0,1]) \subseteq K$ is an open subset and contains 0. Consequently, K is a neighbourhood of 0. Finally, putting

$$\inf\{f(x) \mid \quad \| x \| = 1\} = R > 0$$

we find that for each $x \in K$

$$\| x \| R \leq \| x \| f(\partial x \| x \|) \leq f(x) \leq 1$$

and hence

$$\| x \| \leq 1/R$$

We conclude that K is bounded and hence compact. Finally for any $x, y \in K$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y) \leq \alpha + \beta = 1$$

from which we conclude that $\alpha x + \beta y \in K$. Thus K is convex.

Conversely, let $K \subseteq \mathbb{R}^N$ be a Minkowski domain. Consider

$$f = f_K : \mathbb{R}^N \longmapsto \mathbb{R}$$

defined by

$$f(x) = \begin{cases} 0 & \text{if} x = 0 \\ \\ \sup\{\xi \mid & \xi > 0, \xi^{-1}x \notin K\} \end{cases}$$

First of all things it is important to notice that f is well-defined: K being compact for every $x \neq 0, \exists \xi > 0$ such that $\xi^{-1}x \notin K$. On the other hand, for any $x \in \mathbb{R}$ and $x \neq 0$

$$\begin{aligned} f(\alpha x) &= \sup\{\xi \mid \xi > 0, \ \xi^{-1}x \notin K\} \\ &= \sup\{\mid \alpha \mid \eta \ \mid \eta > 0, \ \eta^{-1}x \notin K\} \\ &= \mid \alpha \mid \sup\{\eta \mid \ \eta > 0, \ \eta^{-1}x \notin K\} \\ &= \mid \alpha \mid f(x) \end{aligned}$$

Consider, $x, y \in \mathbb{R}^N, \ x, y \neq 0$ and $\alpha, \beta \geq 0, \ \alpha + \beta = 1.$ for every $\epsilon > 0$

$$\begin{aligned} \frac{\alpha x + \beta y}{\alpha f(x) + \beta f(y) + \epsilon} &= \frac{\alpha(f(x) + \epsilon)}{\alpha(f(x) + \epsilon) + \beta(f(y) + \epsilon)} \frac{x}{f(x) + \epsilon} \\ &+ \frac{\beta(f(y) + \epsilon)}{\alpha(f(x) + \epsilon) + \beta(f(y) + \epsilon)} \frac{y}{f(y) + \epsilon} \end{aligned}$$

As $\frac{x}{f(x)+\epsilon}, \frac{y}{f(y)+\epsilon} \in K$ and K is convex we conclude that

$$\frac{\alpha x + \beta y}{\alpha(f(x) + \epsilon) + \beta(f(y) + \epsilon)} \in K$$

Consequently,

$$\alpha(f(x) + \epsilon) + \beta(f(y) + \epsilon) \geq f(\alpha x + \beta y)$$

This being true for arbitrary $\epsilon > 0$, we conclude that

$$f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$$

This inequality is trivially valid if one of x or y vanishes. Therefore, f is a convex function.

For any given $\epsilon > 0$, ϵK is a neighbourhood of $0 \in \mathbb{R}^N$ and for any $x * in \epsilon K$, $x = \epsilon y$ for some $y \in K$ and

$$f(x) = f(\epsilon y) = \epsilon f(y) \leq \epsilon$$

therefore f is continuous at $0 \in \mathbb{R}^N$.

On the other hand, considering

$$x = \frac{1}{2}2(x-y) + \frac{1}{2}2y$$

we can easily deduce from the convexity of f, the inequality

$$| f(x) - f(y) | \leq f(x-y)$$

which combined with the continuity at $0 \in \mathbb{R}^N$ shows that f is continuous.

APPENDIX 2

CURVATURE IN ISOTHERMAL COORDINATES

$$\begin{aligned} H &= \frac{1}{2}\frac{1}{\lambda^2}(\dot{x}^2+\dot{y}^2) \\ \frac{\partial H}{\partial \dot{x}} &= \frac{1}{\lambda^2}\dot{x} \\ \frac{\partial H}{\partial x} &= -\frac{1}{\lambda^3}\frac{\partial \lambda}{\partial x}(\dot{x}^2+\dot{y}^2) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial \dot{x}}\right) &= \left[\frac{1}{\lambda^2}\ddot{x} - \frac{2}{\lambda^3\frac{\partial \lambda}{\partial x}} - \frac{2}{\lambda^3}\frac{\partial \lambda}{\partial y}\dot{x}\dot{y}\right] \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial \dot{x}}\right) &= \frac{1}{\lambda^2}\ddot{x} - \frac{2}{\lambda^3}\left[\frac{\partial \lambda}{\partial x}\dot{x}\dot{x} + \frac{\partial \lambda}{\partial y}\dot{x}\dot{y}\right] \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial H}{\partial \dot{x}}\right) - \frac{\partial H}{\partial x} = 0$$

hence,

$$\ddot{x} - \frac{1}{\lambda}\frac{\partial \lambda}{\partial x}\dot{x}^2 - \frac{2}{\lambda}\frac{\partial \lambda}{\partial y}\dot{x}\dot{y} + \frac{1}{\lambda}\dot{y}^2 = 0$$

thus,

$$\begin{aligned} \Gamma^x_{xx} &= -\frac{1}{\lambda}\frac{\partial \lambda}{\partial x} \\ \Gamma^x_{yx} &= -\frac{1}{\lambda}\frac{\partial \lambda}{\partial y} \\ \Gamma^x_{yy} &= \frac{1}{\lambda}\frac{\partial \lambda}{\partial x} \\ \Gamma^x_{xy} &= -\frac{1}{\lambda}\frac{\partial \lambda}{\partial y} \end{aligned}$$

similarly,

$$\begin{aligned}\Gamma^y_{yy} &= -\frac{1}{\lambda}\frac{\partial\lambda}{\partial y}\\ \Gamma^y_{yx} &= -\frac{1}{\lambda}\frac{\partial\lambda}{\partial x}\\ \Gamma^y_{xx} &= \frac{1}{\lambda}\frac{\partial\lambda}{\partial y}\\ \Gamma^y_{xy} &= -\frac{1}{\lambda}\frac{\partial\lambda}{\partial x}\end{aligned}$$

$$R^1_{212} = \Gamma^x_{yy}\Gamma^x_{xx} + \Gamma^y_{yy}\Gamma^x_{xy} - \Gamma^x_{xy}\Gamma^x_{yx} - \Gamma^y_{xy}\Gamma^x_{yy} + \frac{\partial}{\partial x}\Gamma^x_{yy} - \frac{\partial}{\partial y}\Gamma^x_{xy}$$

therefore,

$$R^1_{212} = -\frac{1}{\lambda^2}(\frac{\partial\lambda}{\partial x})^2 + \frac{1}{\lambda^2}(\frac{\partial\lambda}{\partial y})^2 - \frac{1}{\lambda^2}(\frac{\partial\lambda}{\partial y})^2 + \frac{1}{\lambda^2}(\frac{\partial\lambda}{\partial x})^2 + \frac{\partial}{\partial x}(\frac{1}{\lambda}\frac{\partial\lambda}{\partial x}) + \frac{\partial}{\partial y}(\frac{1}{\lambda}\frac{\partial\lambda}{\partial y})$$

$$R_{1212} = \lambda^2 R^1_{212}$$

hence,

$$K = \frac{R_{1212}}{G_{11}G_{22}} = \frac{\Delta(\log\lambda)}{\lambda^2}$$