ON THE MODULAR CURVE $X(6)$ AND SURFACES ADMITTING GENUS 2 FIBRATIONS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In this thesis, we study the moduli spaces of surfaces admitting nonsmooth genus 2 fibrations with slope $\lambda = 6$, (necessarily) over curves of genus ≥ 1. We determine the dimension and the structure of each connected component of these moduli spaces. Our results fill the gap of earlier work in the literature to complete the picture of the moduli spaces of genus 2 fibrations over curves of genus ≥ 2 except for the case of $\lambda = 4$.

Keywords: Fibrations, moduli spaces

ÖZ

MODÜLER EĞRİ $X(6)$ VE GENUS 2 LİF UZAYI OLAN YÜZEYLER

Karadoğan, Gülay

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Bu tezde, eğimi $\lambda = 6$ olan, düzgün olmayan ve genusu 1'den büyük ya da eşit olan eğriler üzerine genus 2 lif uzayı olan yüzeylerin moduli uzaylarını inceledik. Bu moduli uzayların her bir bağlantılı bileşeninin boyutunu ve yapısını belirledik. Ulaştığımız sonuçlar, $\lambda = 4$ durumu dışında, genusu 1'den büyük olan eğriler üzerine genus 2 lif uzayı olan yüzeylerin moduli uzayları hakkında yapılan çalışmalardaki eksikleri doldurmaktadır.

Anahtar Kelimeler: Lif uzayları, moduli uzaylar

To My Family

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CHAPTER 1

INTRODUCTION

The aim of this thesis is to work out the (slope) $\lambda = 6$ case of the moduli spaces of surfaces of general type admitting genus 2 fibrations. The case of albanese fibrations was studied extensively by Seiler ([5],[6]) and nonalbanese fibrations with $\lambda \neq 6$ and base genus $g \geq 2$ have been subject of ([3],[4]) by Önsiper and Tekinel. General methods developed in the latter work apply for $\lambda = 6$ case, too. However, since $\lambda = 6$ case is related to the modular curve $X(6)$ (as will be explained in Chapter 3) which is of genus 1, we need some modifications in the arguments of ([4]).

In Chapter 2 we recall the definition and basic properties of the modular curve $X(6)$. We also discuss basic existence and moduli questions for nonconstant holomorphic maps $C \longrightarrow X(6)$ from curves C of genus ≥ 1.

In Chapter 3, results of Chapter 2 will be combined with techniques of ([4]) to prove our main results. More precisely, we determine the values of K^2, χ for surfaces admitting genus 2 fibrations with $\lambda = 6$ and for each possible pair (K^2, χ) we determine the connected components of the moduli space of surfaces of general type, parametrizing these fibred surfaces. We also include nontrivial concrete examples.

We work over the complex numbers $\mathbb{C}$ and use the following standard notation:

$$\Gamma(6) = \{A \in SL_2(\mathbb{Z}) : A \simeq I \ mod \ 6\}$$

$$\mathcal{H} = \text{upper half plane in } \mathbb{C}$$

$K(X), \chi(X)$ are the canonical class and the holomorphic Euler characteristic of X.

$c_1(X), c_2(X)$ denote the first and the second chern classes of X, respectively.

$\mathcal{M}_{K^2,\chi}$ is the moduli space of surfaces of general type with invariants K^2, χ.

$j(E)$ is the j-invariant of the elliptic curve E.

All surfaces, considered in this thesis, are minimal surfaces.

CHAPTER 2

THE MODULAR CURVE $X(6)$

We consider the action of the modular group $SL_2(\mathbb{Z})$ on $\mathcal{H}$ via holomorphic automorphisms

$$\begin{array}{rcl} \mathcal{H} \times SL_2(\mathbb{Z}) & \longrightarrow & \mathcal{H} \\ (z, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}) & \longmapsto & \dfrac{az+b}{cz+d} \end{array}$$

$\mathcal{H}/SL_2(\mathbb{Z}) \simeq \mathbb{C}$ is a noncompact Riemann surface which can be compactified to get $\mathbb{P}^1_{\mathbb{C}}$ by adjoining the cusps of $SL_2(\mathbb{Z})$. It is well known that there exists a single cusp which is the orbit of $\mathbb{Q} \cup \{\infty\}$ under the action of $SL_2(\mathbb{Z})$ ([7],p.14). Since $\Gamma(6)$ is a subgroup of $SL_2(\mathbb{Z})$ of finite index ([7], p.22), the modular curve $X(6) = \mathcal{H}^*/\Gamma(6)$, where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$, is a finite covering of $\mathcal{H}^*/SL_2(\mathbb{Z})$. Therefore, it is compact and the cusps arise simply as the orbits obtained from $\mathbb{Q} \cup \{\infty\}$ under the action of $\Gamma(6)$.

In the following lemma, we collect some well-known facts about $X(6)$.

Lemma 2.1. $X(6)$ has 12 cusps and genus $g(X(6))$=1. The j-invariant $j(X(6))$=0.

Proof. We know by ([7], p.22-23) that $\Gamma(N)$ has exactly μ_N/N inequivalent cusps, where

$$\mu_N = \frac{N^3}{2}\prod_{p|N}(1-p^2) \qquad \text{for } N > 2$$

and has genus

$$g(X(N)) = 1 + \frac{\mu_N(N-6)}{12N} \qquad \text{for } N > 1.$$

Taking $N = 6$, we get the first two assertions.

To calculate the j-invariant , we observe that

(i) ∞ is a cusp for $\Gamma(6)$, and

(ii) since $\Gamma(6) \lhd SL_2(\mathbb{Z})$, $SL_2(\mathbb{Z})/\Gamma(6) \simeq SL_2(\mathbb{Z}/6\mathbb{Z})$ acts on $Y(6) = \mathcal{H}/\Gamma(6)$ via automorphisms and hence on $X(6)$. Under this action, the cusp at ∞ is left fixed by the cyclic subgroup $\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rangle$ of $SL_2(\mathbb{Z}/6\mathbb{Z})$. Therefore making $X(6)$ into an elliptic curve by taking the cusp at ∞ as the 0 element for the group law, we see that $X(6)$ has a cyclic subgroup of automorphisms (as an elliptic curve) of order 6. Hence, $j(X(6)) = 0$. $\square$

Next we consider covers of $X(6)$ by Riemann surfaces C. Equivalently we are interested in nonconstant holomorphic maps $C \longrightarrow X(6)$.

Given a surjective holomorphic map $\varphi : C \longrightarrow X(6)$, we know by Riemann-Hurwitz formula that $2g(C) - 2 = R(\varphi)$ where $R(\varphi)$ is the ramification degree of φ. In particular, φ is unramified if and only if $g(C) = 1$. In this case the given cover is necessarily Galois (because $\pi_1(X(6)) = \mathbb{Z} \oplus \mathbb{Z}$ being an abelian group, $\pi_1(C) \lhd \pi_1(X(6))$) and the covering group G is either cyclic or an abelian group with two generators (since $\pi_1(X(6))$ has two generators). In fact (up to translation on $X(6)$), φ is an isogeny of degree n. Recall that an isogeny between two elliptic curves E, E' is a surjective morphism

$$\varphi : E \longrightarrow E'$$

which respects the natural group structures on E, E'. In particular, E' is completely determined by the kernel $Ker(\varphi)$ of φ via the identification

$$E' = E/Ker(\varphi).$$

Using this observation together with Lemma 2.1 we can calculate the number of distinct cyclic unramified covers of $X(6)$ of degree n, for $n > 1$. Thus we obtain a lower bound for the number $N(n)$ of distinct unramified covers of degree n.

Lemma 2.2. Modulo the action of $Aut(X(6))$, $X(6)$ has

$$N(n) \geq \frac{n}{3}\prod_{p|n}(1+\frac{1}{p}) + \frac{2}{3}\nu_3$$

distinct unramified covers of degree n, where

$$\nu_3 = \begin{cases} 0 & \text{if } 9|n \\ \prod_{p|n}(1+(\frac{-3}{p})) & \text{otherwise.} \end{cases}$$

Proof. Using dual isogenies, each cyclic isogeny $X(6) \longrightarrow E$ of degree n will give an isogeny $E \longrightarrow X(6)$ of degree n. Therefore, the number of cyclic isogenies $X(6) \longrightarrow E$ will give a lower bound for $N(n)$. To calculate this lower bound we use the modular curve $X_0(n)$. Recall that $Y_0(n) = \mathcal{H}/\Gamma_0(n)$ classifies pairs (E, G) of elliptic curves E and cyclic subgroups G of order n; therefore, by the construction explained above, $Y_0(n)$ classifies pairs of elliptic curves and cyclic isogenies of degree n. Letting $X_0(n)$ be the compactification of $Y_0(n)$, we have a natural map

$$\varphi : X_0(n) \longrightarrow X(1) = \mathcal{H} \cup \{\infty\}/SL_2(\mathbb{Z})$$

which simply maps the pair (E, G) to $[E] \in X(1)$. Therefore, the number of points over $[E]$ is precisely the number of cyclic isogenies $E \longrightarrow E'$ of degree n. In particular, taking $[X(6)] \in X(1)$ and applying the recipe in ([7],p.23), we obtain the lower bound given in the lemma. To see this, we just observe that since $j(X(6)) = 0$, $X(6)$ corresponds to the image of $\tau = e^{2\pi i/3}$ in $X(1)$ and over this point we have exactly

$$t = \frac{\mu}{3} + \frac{2}{3}\nu_3$$

points where $\mu = n \prod_{p|n} (1 + \frac{1}{p})$ and ν_3 is as given in the statement of the Lemma.

□

Remark: The case of ramified covers $\varphi : C \longrightarrow X(6)$ (which correspond precisely to covers with $g(C) > 1$) is more subtle. Restricting ourselves to Galois covers, we know by a classical construction (cf.[2]) that $X(6)$ has a Galois cover with group G for any finite group G. However, as the ramification degree $R(\varphi)$ depends on the possibilities for subgroups of G, it is impossible to give a general formula for $g(C)$. The case $G = \mathbb{Z}/p\mathbb{Z}$, for some prime p, is obviously an exception for which we can work out $g(C)$ using Riemann-Hurwitz formula, to get

$$2g(C) - 2 = R(\varphi) = k(p-1)$$

where k = number of ramification points. Therefore, $g(C) = \dfrac{k(p-1)+2}{2}$, for such covers. For example, taking $p = 2$ and writing $k = 2\ell$ we see that $X(6)$ has degree 2 covers by curves of genus $g(C) = \ell + 1$ for all $\ell \geq 1$. More generally, the same calculation works for covers of arbitrary degree n which totally ramify over each point of ramification and for such covers we get $g(C) = \dfrac{k(n-1)+2}{2}$. However, when one considers general covers of a given degree, the only method to classify these covers and to calculate the genera of the covering curves is to exploit the relation between covers and monodromy.

CHAPTER 3

GENUS 2 FIBRATIONS WITH SLOPE $\lambda = 6$

In this chapter, we will apply the results obtained in chapter 2 to the study of moduli spaces of surfaces X admitting nonsmooth genus 2 fibrations $X \longrightarrow C$.

We assume that the irregularity $q(X) = g(C) + 1$, so that the fibration is not of albanese type. Furthermore, we take the slope of the fibrations $\lambda = 6$. We recall that the slope of a genus 2 fibrations over a curve C is defined by $K^2 = \lambda\chi + (8 - \lambda)(g(C) - 1)$.

We first note that by Xiao's work ([8]) on genus 2 fibrations, for each elliptic curve E there exists a fibration $S(E,6) \longrightarrow X(6)$ on the modular curve $X(6)$ which is universal in the following sense: any genus 2 fibration $\alpha : X \longrightarrow C$ with $\lambda = 6$ and E as the fixed part of the jacobian fibration corresponding to α, is the minimal desingularization of the pull-back $f^*(S(E,6))$ via a surjective holomorphic map $f : C \longrightarrow X(6)$.

We need to determine the singularities of $f^*(S(E,6))$. Clearly, $f^*(S(E,6))$ has singularities only if f ramifies over some points in the singular locus of

$S(E,6) \to X(6)$. We take such a point $p \in X(6)$ and we let k_i, $i = 1, ..., l$ be the ramification index at $q_i \in f^{-1}(p)$. The surface $f^*(S(E,6)) \to C$ has exactly one singular point on the fiber over q_i , which is of type A_{k_i-1}; this follows from the fact that a singular fiber of $S(E,6) \to X(6)$ is either an elliptic curve with a single node or two smooth elliptic curves intersecting transversally at a single point ([8], Lemme 3.11, Theoreme 3.16). This observation has two important consequences :

1) we can apply *simultaneous desingularization* to any given family of surfaces obtained via a family of holomorphic maps into $X(6)$, and

2) we can calculate the second Chern classes of minimal resolutions to prove ([4], Lemma 1)

Lemma 3.1. Let $f_i : C_i \longrightarrow X(6)$, $i = 1, 2$, be surjective holomorphic maps with $g(C_1) = g(C_2) \geq 1$. Then the induced fibrations $X_i \longrightarrow C_i$ have the same invariants K^2, χ if and only if $\deg(f_1) = \deg(f_2)$.

The proof of ([4], Lemma 1) gives the value of $c_2(X)$ for a fibration X over a curve C of genus g, obtained from a map $C \longrightarrow X(6)$ of degree n:

$$c_2(X) = nc_2(S(E,6)) + 4(g-1).$$

Combining this with Noether's formula

$$12\chi(X) = c_1^2(X) + c_2(X)$$

and the slope formula

$$c_1^2(X) = 6\chi(X) + 2(g-1),$$

we get

$$c_1^2(X) = nc_2(S(E,6)) + 8(g-1) \qquad (1).$$

We note that since X is minimal and $c_1^2(X) > 0$, $c_2(X) > 0$, all of these surfaces are of general type.

Lemma 3.2. Given a deformation $\pi : \mathcal{X} \longrightarrow \mathcal{B}$ over a connected base $\mathcal{B}$ of X which admits a genus 2 fibration $X \longrightarrow C$ with $\lambda = 6$ and $g(C) \geq 2$. Then

(i) Each fiber $\mathcal{X}_b$ of π admits a fibration $\mathcal{X}_b \longrightarrow C_b$ of the same type, with $g(C_b) = g(C)$ and this fibration is unique.

(ii) The degree of the map $C_b \longrightarrow X(6)$ inducing the fibration $\mathcal{X}_b \longrightarrow C_b$ is constant on $\mathcal{B}$.

If $g(C) = 1$, uniqueness in part (i) and (ii) hold, provided that one knows the existence of the fibration $\mathcal{X}_b \longrightarrow C_b$.

Proof. This is a special case of Lemma 2 in ([4]), except for the uniqueness statement for elliptic base C. But for this case, since $\lambda = 6$ we have

$$\begin{aligned} K^2(X) &= 36n \quad \text{for some n} \geq 1 \\ &> 4 \end{aligned}$$

and uniqueness follows from ([8], Proposition 6.4). □

We recall that for the surface $S(E,6)$, we have $K^2(S(E,6)) = 36$, $\chi(S(E,6)) = 6$ and $c_2(S(E,6)) = 36$ ([8], p.53). Therefore, a fibration $X \longrightarrow C$ obtained from an unramified map $C \longrightarrow X(6)$ of degree n, has $K^2(X) = 36n$, $\chi(X) = 6n$. By Lemma 2.2, we have $N(n)$ distinct unramified covers of degree n of $X(6)$. With this notation, we have

Theorem 3.3. The moduli space of surfaces X admitting genus 2 fibrations with $\lambda = 6$ and with $K^2(X) = 36n$, $\chi(X) = 6n$ consists of $N(n)$ disjoint copies of the line $\mathbb{A}^1$, which parameterize precisely the fibrations arising from isogenies $C \longrightarrow X(6)$ of degree n.

Proof. Given $\varphi : C \longrightarrow X(6)$ unramified of degree n, for each elliptic curve E we obtain $X_E \longrightarrow C$ where $X_E = \varphi^*(S(E,6))$. Therefore, by uniqueness of fibrations (Lemma 3.2(i)), for a fixed cover $\varphi_i : C \longrightarrow X(6)$, the modulus of surfaces X admitting fibration over C corresponds to the modulus of elliptic curves via the correspondence $X_E \longrightarrow j(E)$. By the argument in the proof of Theorem 3.5, we know that we can deform $S(E_1,6)$ to $S(E_2,6)$ for any two elliptic curves E_1, E_2. Hence it follows that for each φ_i, we have a single component in $\mathcal{M}_{36n,6n}$ which is $\mathbb{A}^1$.

As $K^2(S(E,6)) = 36$ and $c_2(S(E,6)) = 36$, if we have a curve C of genus $g \geq 2$ and a fibration $X \longrightarrow C$ with slope $\lambda = 6$, then such a fibration will have

$$
\begin{aligned}
c_2 &= 36m + 4(g-1) \\
c_1^2 &= 36m + 8(g-1)
\end{aligned}
$$

for some $m \geq 2$, and hence $c_1^2 \neq c_2$. This shows that in the moduli space $\mathcal{M}_{36n,6n}$ we have only fibrations over elliptic curves. □

Remark: Since fibrations over an elliptic curve need not be preserved under deformation, the components in Theorem 3.3 need not correspond to components of $\mathcal{M}_{K^2,\chi}$, contrary to the case of components in Theorem 3.5 .

We observe that in case of fibered surfaces with given K^2, χ and arising from ramified covers of $X(6)$, the number and the dimension of connected components in $\mathcal{M}_{K^2,\chi}$, parametrizing these fibred surfaces, depend on:

(a) the base genus $g(C)$ of the fibrations,

(b) the degree of the covers $C \longrightarrow X(6)$,

(c) for a fixed base genus g and degree n, on the connected components of $\mathbf{Hol}_n(g, X(6))$ which is the complex space parameterizing holomorphic maps of degree n from curves of genus g onto $X(6)$ (modulo $Aut(X(6))$).

To state our result in this case, we let $\mathcal{M}$ be the moduli space of all genus 2 fibrations with $\lambda = 6$ and with given K^2, χ. We write $\mathcal{M} = \bigsqcup_g \mathcal{M}(g, 6)$ as a disjoint union, where $\mathcal{M}(g,6)$ parameterizes surfaces X in $\mathcal{M}$ fibered over curves of genus $g \geq 2$. We note that, by formula (1), we have only finitely many, if any, $g \geq 2$ for which $\mathcal{M}(g,6) \neq \emptyset$ and once g is fixed, the degree n of the maps $C \longrightarrow X(6)$ is determined. With this notation, we have

Lemma 3.4. Given a fixed elliptic curve E, the map $\varphi : \mathbf{Hol}_n(g, X(6)) \rightarrow \mathcal{M}(g,6)$ defined by sending the class of $f : C \rightarrow X(6)$ to the fibration of type $(E, 6)$ induced by f is holomorphic.

Proof. This is a special case of ([4], Lemma 4). □

Now we can state our main result which describes the components of $\mathcal{M}(g,6)$ and which is a special case of Proposition 5 in [4]; for the sake of completeness, we include the proof given in [4].

Theorem 3.5. Suppose $\mathcal{M}(g,6)$ is nonempty. Then the set of connected components $\{\mathcal{M}(g,i)\}$ of $\mathcal{M}(g,6)$ is in one-to-one correspondence with the set $\{\mathbf{Hol}_{n,i}\}$ of components of $\mathbf{Hol}_n(g, X(6))$, for some $n \geq 1$ and each $\mathcal{M}(g,i)$ is a fiber space over $\varphi(\mathbf{Hol}_{n,i})$ with $\mathbb{A}^1$ as fiber.

Proof. In the construction of the universal fibration $S(E,6) \to X(6)$ given in ([6], p.42), letting $u_1(z) = (z_1, 0)$, $u_6(z) = \frac{1}{d}(z_1, 1)$ vary, we obtain a family $\mathcal{S}(6) \to X(6) \times \mathbb{A}^1$ where for $a \in \mathbb{A}^1$, $\mathcal{S}(6)|_{X(6)\times a} \cong S(E_a, 6)$, E_a being the elliptic curve with j-invariant $j(E_a) = a$.

Now for $f : C \to X(6)$, $(f \times id_{\mathbf{A}_1})^*(\mathcal{S}(6))$ gives a deformation of $f^*(S(E_1, 6))$ to $f^*(S(E,6))$ and hence by simultaneous desingularization we get a deformation of the surface $X(E_1, 6)$ to $X(E,6)$. Therefore, for $[X] \in \mathcal{M}(g,i)$, we may deform the surface X to a surface of type $(E,6)$ corresponding to a point $\varphi(f)$ as described in Lemma 3.4, where we take n to be the degree of the maps inducing the surfaces in $\mathcal{M}(g,i)$ (Lemma 3.2 (ii)). This shows in particular that in each component of $\mathcal{M}(g,6)$ lies image under φ of some component of $\mathbf{Hol}_n$. The proposition will follow once we show that no two components of $\mathbf{Hol}_n$ are mapped to the same component of $\mathcal{M}(g,6)$.

To check this final point, let $[X_1], [X_2] \in \mathcal{M}(g,i)$ correspond to $\varphi(f_1),\ \varphi(f_2)$ for $f_i : X_i \to X(d)$. Then since a deformation of a surface admitting a fibration $\psi : X \to X(d)$ as in this paper, is locally induced from a deformation of the fibering ψ, using any deformation of $[X_1]$ to $[X_2]$ we obtain a deformation of f_1 to some $\overline{f}_2$ which induces a fibration on X_2 necessarily over C_2, by uniqueness of fibrations (Lemma 3.2 (i)). Furthermore, again by uniqueness of fibrations, we have $g \in Aut(C_2),\ h \in Aut(X(6))$ such that $f_2 \circ g = h \circ \overline{f}_2$. Therefore, f_2 and $\overline{f}_2$ and hence $f_1,\ f_2$ lie in the same component of $\mathbf{Hol}_n(g,6)$ ([4], Lemma 3). $\square$

Remark : We note that $\mathcal{M}(g,6) \neq \emptyset$ in $\mathcal{M}_{K^2,\chi}$ only if $K^2 = 36n + 8(g-1)$, $\chi = 6n + (g-1)$ for some $n \geq 2$. However, to see if $\mathcal{M}(g,6) \neq \emptyset$ for a possible value of g, one needs to work out the related monodromy.

Example: We consider surfaces X with

$$\begin{aligned} c_2(X) &= 2c_2(S(6)) + 4 = 76 \qquad \text{and} \\ c_1^2(X) &= 2c_2(S(6)) + 8 = 80. \end{aligned}$$

From the formula,

$$c_2(X) = nc_2(S(6)) + 4(g-1),$$

we see that $g = 2$, $n = 2$ is the only set of possible values for g and $n > 1$. Hence among surfaces X with these values of $c_1^2,\ c_2$, those which admit genus 2 fibrations with $\lambda = 6$ all arise from degree 2 covers of $X(6)$ by genus 2 curves . Such a cover $Y \longrightarrow X(6)$ can be constructed explicitly by taking two distinct points $P_1, P_2 \in X(6)$ and completing the unramified degree 2 cover of $X(6) - \{P_1, P_2\}$

over P_1, P_2. Since $X(6)$ is an elliptic curve , we can permute any pair $\{P_1, P_1'\}$ of points on $X(6)$ via a translation. Therefore, we can fix one of P_1, P_2, say we take $P_1 = 0$

Furthermore, applying inversion, we see that $\{0, P\}$ and $\{0, -P\}$ give rise to the same covering curve. Hence the moduli of all degree 2 covers of $X(6)$ by genus 2 curves corresponds to ($X(6)$- $\{0\ \}$)$/ < -1 >= \mathbb{P}^1 - \{\text{a point}\} \simeq \mathbb{A}^1$. Thus $\mathcal{M}$(2,6) is an $\mathbb{A}^1$ fiber space over $\mathbb{A}^1$.

REFERENCES

[1] M.Namba : Families of Holomorphic Maps on Compact Riemann Surfaces, *LNM767, Springer-Verlag* (**1979**).

[2] M.Namba : On finite galois coverings of projective manifolds, *J. Math. Soc.Japan*, 41 (3) (**1989**), 391-403.

[3] H.Önsiper, On moduli spaces of fiber bundles of curves of genus ≥ 2, *Arch. Math.* 47 (5) (**2000**), 346-348

[4] H.Önsiper- C.Tekinel : On the moduli of surfaces admitting genus 2 fibrations

[5] W.Seiler : Moduli spaces for special surfaces of general type, in *Algebraic Geometry and Its Applications, Springer-Verlag* (**1994**), 161-172

[6] W.Seiler : Moduli of surfaces of general type with a fibration of genus 2 curves, *Math.Ann.*, 301 (**1995**), 771-812

[7] G.Shimura, Introduction to the Arithmetic Theory of Automorphic Functions, *PUP* (**1971**).

[8] G.Xiao, Surfaces Fibrées en Courbes de Genre Deux, LNM 1137, *Springer-Verlag* (**1985**)