DEVELOPING PRE-SERVICE MATHEMATICS TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' THINKING IN GEOMETRIC MEASUREMENT THROUGH PEDAGOGIES OF PRACTICE

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submitted by **BÜŞRA ÇAYLAN ERGENE** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in **Mathematics Education in Mathematics and Science Education, Middle East Technical University** by,

Prof. Dr. N. Emre Altun
Dean, **Graduate School of Natural and Applied Sciences** ________________

Prof. Dr. Mine Işıksal Bostan
Head of the Department, **Mathematics and Science Education** ________________

Prof. Dr. Mine Işıksal Bostan
Supervisor, **Mathematics and Science Education, METU** ________________

Examining Committee Members:

Assoc. Prof. Dr. Sibel Kazak
Mathematics and Science Education, METU ________________

Prof. Dr. Mine Işıksal Bostan
Mathematics and Science Education, METU ________________

Assoc. Prof. Dr. Şerife Sevinç
Mathematics and Science Education, METU ________________

Assoc. Prof. Dr. Fadime Ulusoy
Mathematics and Science Education, Kastamonu Uni. ________________

Assoc. Prof. Dr. Reyhan Tekin Sitrava
Mathematics and Science Education, Kırıkkale Uni. ________________

Date: 20.03.2024

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name Last name : Büşra Çaylan Ergene

Signature :

ABSTRACT

DEVELOPING PRE-SERVICE MATHEMATICS TEACHERS' PROFESSIONAL NOTICING OF STUDENTS' THINKING IN GEOMETRIC MEASUREMENT THROUGH PEDAGOGIES OF PRACTICE

Çaylan Ergene, Büşra
Doctor of Philosophy, Mathematics Education in Mathematics and Science Education
Supervisor : Prof. Dr. Mine Işıksal Bostan

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The purpose of the study was to examine the extent to which pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement changed when they participated in a video-based module situated in the pedagogies of practice framework. In addition, the study aimed to explore how a video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement. The study employed a mixed methods intervention design, and the participants were 32 fourth-year (senior) pre-service teachers enrolled in an elementary mathematics education program at one of the state universities. The data were collected through a noticing questionnaire, reflection papers, individual semi-structured interviews, group discussions, and whole class discussions. Both qualitative and quantitative methods were used to analyze the data. The findings revealed that pre-service teachers' professional noticing skills improved through the pedagogies of practice. Specifically, pre-service teachers' decompositions of practice through analyzing and discussing the students' mathematical thinking in the video clips provided as the representations of practice, as well as their enactment and reflection on their own practices in the task-based interviews they conducted as approximations of practice, contributed to the development of their professional noticing skills. In addition, the statistically significant change in the pre-service teachers' attending to students' solutions, interpreting students' understanding, and deciding how to respond skills in the final questionnaire provided valuable insights into the effectiveness of the video-based module situated in the pedagogies of practice framework.

Keywords: Professional Noticing, Pedagogies of Practice, Geometric Measurement, Pre-service Teacher Education

ÖZ

MATEMATİK ÖĞRETMEN ADAYLARININ ÖĞRENCİLERİN ÖLÇMEYE YÖNELİK DÜŞÜNÜŞLERİNE İLİŞKİN MESLEKİ FARK ETME BECERİLERİNİN UYGULAMA PEDAGOJİLERİ YOLUYLA GELİŞTİRİLMESİ

Büşra Çaylan, Ergene
Doktora, Matematik Eğitimi, Matematik ve Fen Bilimleri Eğitimi
Tez Yöneticisi: Prof. Dr. Mine Işıksal Bostan

Mart 2024, 499 sayfa

Bu çalışmanın amacı, öğretmen adaylarının uygulama pedagojileri destekli video tabanlı bir modüle katıldıktan sonra öğrencilerin ölçme konusundaki matematiksel düşünüşlerine ilişkin mesleki fark etme becerilerinin ne derecede değiştiğini incelemektir. Ayrıca, çalışma, uygulama pedagojileri destekli video tabanlı modülün, öğretmen adaylarının öğrencilerin ölçme konusundaki matematiksel düşünmelerini fark etme becerilerini nasıl desteklediğini ortaya çıkarmayı amaçlamaktadır. Çalışmada karma yöntem müdahale deseni kullanılmıştır ve katılımcılar bir devlet üniversitesinin ilköğretim matematik öğretmenliği programına kayıtlı 32 dördüncü sınıf (son sınıf) öğretmen adayıdır. Fark etme testi, yansıtıcı düşünce raporları, bireysel yarı yapılandırılmış görüşmeler, grup tartışmaları ve sınıf tartışmaları çalışmanın verilerini oluşturmuştur. Veriler hem nicel hem de nitel yöntemler kullanılarak analiz edilmiştir. Bulgular, öğretmen adaylarının mesleki fark etme becerilerinin uygulama pedagojileri aracılığıyla geliştiğini ortaya koymuştur. Özellikle, öğretmen adaylarının uygulamanın temsili olarak sunulan video kliplerde öğrencilerin matematiksel düşünmelerini analiz ederek ve tartışarak uygulamayı ayrıştırmaları ve uygulamanın yaklaşımı olarak gerçekleştirdikleri görev temelli görüşmeler ile bu görüşmelerin analizi öğretmen adaylarının mesleki fark

etme becerilerinin gelişimine katkı sağlamıştır. Ayrıca, son testte öğretmen adaylarının öğrencilerin çözümlerini dikkate alma, öğrencilerin anlayışlarını yorumlama ve nasıl yanıt vereceklerine karar verme becerilerindeki istatistiksel olarak anlamlı değişim, uygulama pedagojileri destekli video tabanlı modülün etkililiğine ilişkin önemli bilgiler sağlamıştır.

Anahtar Kelimeler: Mesleki Fark Etme, Uygulama Pedagojileri, Ölçme, Öğretmen Adayı Eğitimi

To my love Özkan and my baby in the womb

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LIST OF ABBREVIATIONS

ABBREVIATIONS

NCTM: National Council of Teachers of Mathematics

VC: Video Clips

MoNE: Ministry of National Education

PSTs: Pre-service Teachers

BD: Before the Discussions

AD: After the Discussions

CHAPTER 1

INTRODUCTION

Teacher noticing is at the core of teacher expertise and professional competence (Weyers et al., 2023b) because filtering information and making spontaneous decisions for effective teaching and learning are necessary due to the complex and dynamic nature of classroom instruction (Sherin & Star, 2011). Hence, teacher noticing is a prerequisite for effective mathematics instruction, which shapes students' learning progress (Blömeke et al., 2022). In mathematics education literature, even though noticing is conceptualized among researchers differently, it involves attention to and interpretation of students' mathematical thinking in general, the subsequent instructional decisions, and the connection of these interpretations to broader principles of teaching and learning (Jacobs et al., 2010; van Es & Sherin, 2008). For the first time in the literature, van Es and Sherin (2002) characterized noticing as "identifying what is important about a classroom situation, making connections between specific actions and broader principles of teaching and learning, and using what is known to reason about interactions" (Amador et al., 2021, p. 3). Research on teacher noticing has continued (van Es & Sherin, 2006, 2008), and the definition of noticing has been further developed, resulting in Learning to Notice Mathematical Thinking Framework (van Es, 2011). Here, van Es (2011) focused on what and how teachers notice and figured these based on four different levels of expertise by providing descriptors for each. Jacobs et al. (2010) suggested Professional Noticing of Children's Mathematical Thinking Framework, which involves three interconnected skills, i.e., attending to children's mathematical thinking, interpreting that thinking, and deciding how to respond based on that thinking. From this perspective, professional noticing of students' mathematical thinking is not just about identifying what is right or wrong in students' responses; instead, it is related to reasoning about the meaning of students' responses from mathematical and cognitive viewpoints, as well as deciding on proper instructional actions based on students' understanding (Ulusoy & Çakıroğlu, 2021).

A moment that a teacher uses student thinking is a "potentially powerful learning opportunity" (Davis, 1977, p. 360). For this reason, the significance of using student thinking as a basis for teaching is underlined in many research studies (Fennema et al., 1996; Hiebert, 2003) since understanding students' mathematical thinking is essential for teachers to choose and design instructional tasks, modify instruction to meet the needs of students, ask proper questions, gain insight into students' reasoning, identify and eliminate students' difficulties and misconceptions, facilitate classroom discussions and evaluate student progress (Battista, 2017). Moreover, it is necessary for teachers to be able to notice students' thinking first to become involved in instruction that prioritizes student thinking. "Teachers who know about their students' mathematical thinking can support the development of mathematical proficiency" (Franke et al., 2007, p. 229). Considering all this, teachers should be able to identify and interpret student thinking and also integrate student thinking as a foundation for instruction. However, enacting instructional practices that build on students' mathematical thinking is difficult (Sherin, 2002). Teachers, especially novices, often do not recognize opportunities to use student thinking to advance mathematical understanding (Stockero & Van Zoest, 2013). The difficulty of using students' mathematical thinking can be due to the complexity of recognizing and interpreting that thinking (Leatham et al., 2015). It is the same for pre-service teachers who expectedly have difficulty in noticing students' mathematical thinking (Jacobs et al., 2010). Recognizing and interpreting the details in students' mathematical thinking in a classroom is not an easy learning task for pre-service teachers (Sherin & van Es, 2009) because they do not have enough experience and expertise that practicing teachers would bring to the classroom (Huang & Li, 2009). Moreover, research on deciding how to respond skills of pre-service teachers indicated that pre-service teachers were either inclined to re-teach a concept (Cooper, 2009) or to show how to make correct calculations (Son, 2013) without providing responses that promoted students' learning (Stockero et al., 2017b). In addition, professional noticing of student thinking cannot develop by itself, and pre-service teachers are not expected to gain this skill at the beginning of the teaching profession. Furthermore, even though getting teaching experience throughout the years promotes the improvement of attending and interpreting skills, it is insufficient for the improvement of the responding skill because this skill requires not only attending to students' strategies and interpreting their understanding but also knowing mathematical development of students to ascertain the proper next step (Jacobs et al., 2010).Therefore, it is important to develop the professional noticing skills of pre-service teachers before they start teaching.

Focusing on a particular content domain is crucial for gaining insight into pre-service teachers' noticing of students' mathematical thinking (Walkoe, 2015). One of the content domains in which students' difficulties and misconceptions were reported in the research studies is the geometric measurement, i.e., length, area, and volume measurement (e.g., Barrett et al., 2017; Curry et al., 2006; Martin & Strutchens, 2000; Tan-Sisman & Aksu, 2016). For example, in length measurement, students have difficulties and misconceptions about the use of units such as iterating units by leaving gaps or overlaps, use of different units or the same units in different sizes, and use of a ruler such as measuring with a ruler by starting with 1, citing the number on the ruler corresponding to the endpoint of the object or counting numbers next to marks on a ruler (Bragg & Outhred, 2004; Clements & Sarama, 2009; Clements & Stephan, 2004; Lehrer, 2003). There are also students' misconceptions related to the perimeter, such as conceptualization of perimeter as adding the sides or distance around, adding the lengths of two sides, believing that the perimeter does not change in the case of the rearrangement of a shape, confusion about perimeter and area, use of units of area or volume for the perimeter, counting dots while finding the perimeter of a shape presented on dot paper or counting the surrounding squares when a shape is presented on a grid (Charles et al., 2004; Proulx, 2021a, 2021b; Tan-Sisman & Aksu, 2016; Vighi & Marchini, 2011). Moreover, in area measurement, students' misconceptions are about the use of units, such as confusing linear and square units, inability to understand the inverse relationship between the unit size and the number of units, misunderstanding array structure, using the area formula of the rectangle for the area of other shapes, difficulties in calculating the area of composite shapes such as calculating the perimeter of the composite shapes instead of the area or adding the base and height to calculate their area, confusing perimeter and area concepts or using perimeter and area formulas interchangeably, establishing an incorrect relationship between perimeter and area such as figures with the same perimeter must have the same area (Carpenter & Lewis, 1976; Chappell & Thompson, 1999; Patahuddin et al., 2018; Reynolds & Wheatley, 1996; Simon & Blume, 1994; Tsamir & Mandel, 2000; Wiest, 2005; Zacharos, 2006). Similarly, students also have misconceptions about surface area measurement related to inappropriate use of the surface area formula, difficulties in calculating the area of a net of a three-dimensional object, confusion between surface area and volume, establishing an incorrect relationship between surface area and volume (Lehmann, 2022; Lim et al., 2019; Pitta-Pantazi & Christou, 2010; Sáiz & Figueras, 2009; Seah & Horne, 2020; Tan Sisman & Aksu, 2016). In volume measurement, students' misconceptions are about focusing on a single dimension to reason about the volume of a three-dimensional object, difficulties in determining how the dimensions affect volume, positioning cubes by leaving gaps and overlaps while packing cuboid, enumerating cubes in a three-dimensional array incorrectly such as counting only visible cubes by ignoring the invisible cubes, or counting the visible faces of cubes shown, and difficulty in interpreting two-dimensional representations of three-dimensional objects (Battista & Clements, 1996; Ben-Haim et al., 1985; Curry & Outhred, 2005; French, 2004; Piaget, 1968; Piaget et al., 1960). Consequently, it is necessary for teachers to understand students' mathematical thinking and misconceptions about geometric measurement and also difficulties in it (Lehrer, 2003) because teachers who recognize the misconceptions and know how to remedy them can improve students' learning (Jaworski, 2004). Teachers who have knowledge of students' conceptions and misconceptions are more prepared to make well-informed choices within the classroom. In order to be able to attend to students' strategies, interpret their understanding, and respond on the basis of their understanding, teachers should understand how students reason while working on the given task and what their difficulties and misconceptions are. If teachers learn to pay attention to, interpret, and respond to students before they begin teaching, their future teaching will potentially result in enhanced student learning. Thus, it is important to develop pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement.

Having students solve tasks on geometric measurement using concrete models may give pre-service teachers more information about their mathematical thinking and understanding. Intentionally designing tasks that involve concrete models can help to provide pre-service teachers with details about students' way of thinking. As Piaget (1975) pointed out, students are more proficient in understanding and doing actions than in explaining verbally. Therefore, allowing students to represent their mathematical ideas with concrete models gives teachers information about students' understanding that cannot be effectively assessed through paper-and-pencil tasks (Ontario Ministry of Education, 2008). Similarly, a study by Martin (2007) reported that students' actions and responses with concrete models provided more information about their thinking than their answers in paper-and-pencil tasks, and hence, she underlined the value of concrete models for students' understanding of perimeter, area, surface area, and volume measurement and the value of explaining that understanding. Thus, providing pre-service teachers with opportunities to analyze students' mathematical thinking through geometric measurement tasks involving concrete models can help them recognize significant aspects of geometric measurement, reason about it regarding students' strategies, and decide what course of action to take based on students' understanding.

Although noticing students' mathematical thinking is challenging for pre-service teachers (Jacobs et al., 2010), it "is a learnable practice" (Jacobs & Spangler, 2017, p. 772). Moreover, it is asserted that pre-service teachers' professional noticing skills should be improved before starting to teach (van Es & Sherin, 2002). Taking part in a carefully designed content-specific professional development program that includes active learning opportunities for teachers over extended periods may likely change teacher practice positively and enhance student learning (Darling-Hammond et al., 2017; Desimone, 2009). Accordingly, there are several research studies in which interventions and deliberate scaffolds were used to develop pre-service teachers' professional noticing skills in the literature (e.g., Ivars et al., 2020; Sánchez-Matamoros et al., 2015; Schack et al., 2013). For example, Ivars et al. (2020) demonstrated how pre-service teachers' professional noticing developed in an environment in which students' learning trajectory was used as a scaffold and how using students' learning trajectory helped pre-service teachers attend to students' mathematical understanding, interpret it, and provide a response for proper instruction to support students' learning. Furthermore, Sánchez-Matamoros et al. (2015) reported that when pre-service teachers participated in a seven-session teacher training module based on the development of the understanding of the derivative concept, pre-service teachers' ability to notice the signs of the students' understanding developed. Thus, it can be asserted that guided reflection and scaffolding activities are promising regarding the development of pre-service teachers' professional noticing skills.

Professional noticing of students' mathematical thinking is one of the practices that is crucial for effective mathematics instruction. Grossman et al. (2009) asserted that "for pre-service teachers to learn to engage in the complex practice, they may need opportunities first to distinguish and then, to practice, the different components that go into professional work prior to integrating them fully" (p. 2068). Correspondingly, activities based on pedagogies of practice can enable pre-service teachers to learn and improve teaching practices and begin to recognize significant aspects of teaching practices (Arbaugh et al., 2021). Through pedagogies of practice, pre-service teachers can interact with representations of practice, use particular decomposition of practice, and engage in approximations of practice (Grossman et al., 2009). More specifically, representations of practice portray one or more aspects of practice in specific ways, such as videos or written cases. In decompositions of practice, represented practice is disintegrated into components. Approximation of practices refers to having experiences close to real practices that replicate the complexity of practice (Grossman et al., 2009). Each pedagogy (representation, decomposition, and approximation) can make additional aspects of practice for pre-service teachers to understand and explore. Much of the previous research on noticing predominantly included representation of practice (e.g., Gonzalez et al., 2016; Kosko et al., 2021; Walkoe & Levin, 2018), such as classroom videos, animations/comics, and students' written works. Representations of practice prepare novice teachers for future practice and support them in developing the way they understand the practice by presenting the specific cases of that practice (Grossman et al., 2009). Research revealed that representations of practice assist the professional expertise of pre-service teachers by allowing them to attend, interpret, and decide how to respond to events that take place in the classroom (Amador et al., 2016; van Es & Sherin, 2002). In particular, deliberately choosing and using classroom artifacts as representations of practice can support teachers in recognizing students' thinking (Goldsmith & Seago, 2011). In this regard, using video clips of task-based student interviews can immerse pre-service teachers in students' mathematical thinking. Especially, the authenticity of the representation, due to the involvement of concrete models while students are solving tasks, can give more insight into students' mathematical thinking by making student thinking more explicit. As ideas are evaluated both individually and collaboratively by pre-service teachers, this process can make students' thinking public. However, examining representations of practice will be insufficient to prepare pre-service teachers for complex practices like professional noticing of students' mathematical thinking (Ghousseini & Herbst, 2016). "Use of representations of practice can be extended through decompositions and approximations of practice" (Austin & Kosko, 2022, p. 61). Building on this, decompositions of practice can enable pre-service teachers to reflect on noticing students' mathematical thinking from the teacher's point of view. Moreover, considering the importance of allowing teachers to engage in individual reflection as well as collaborative discussions on videos (Zhang et al., 2011), discussions around video clips about attending, interpreting, and deciding how to respond components, in addition to individual analysis become significant in a decomposition of practice.

In this way, pre-service teachers can become aware of the students' strategies, misconceptions, and difficulties, comprehend students' different ways of thinking and understanding, and recognize particular instructional actions. Thus, both representations and decompositions of practice can provide pre-service teachers with opportunities to learn to notice students' mathematical thinking by complementarily working. However, mathematics education courses in teacher education programs generally try to enhance pre-service teachers' knowledge and foster their beliefs about teaching by leaving the development of practical skills mostly to field experiences (Grossman & McDonald, 2008). In this sense, approximations of practice that follow can enable pre-service teachers to put the concepts and ideas emphasized in the representation and decomposition into practice and to reflect on their own practices. Through approximations of practice, pre-service teachers are "engaged in doing the interactive work of teaching" (Howell & Mikeska, 2021, p. 11). Therefore, approximations of practice are effective approaches for pre-service teachers to learn how to respond to students and be better prepared to do so (Baldinger & Campbell, 2021). Approximations of practice as pedagogical approaches focus on the design and implementation of activities in which pre-service teachers deal with various aspects of teaching (Zeichner, 2012). Yet, a few research studies on pre-service teachers' professional noticing used approximations of practice such as animations (e.g., Amador et al., 2016) or rehearsals (McDuffie et al., 2014), in which virtual characters or peers are considered as students by pre-service teachers. In these cases, pre-service teachers do not have a chance to experience ways of understanding and responding to students based on their social and cultural backgrounds (Sapkota & Max, 2023). Therefore, approximations like task-based student interviews that maintain complexity and make the practice more authentic by including pre-service teachers in tasks similar to those done in school settings come into prominence. In this regard, through approximations of practice, pre-service teachers have a chance to elicit students' mathematical thinking and use evidence of students' way of thinking (Estapa et al., 2018). These kinds of approximations can also involve pre-service teachers' mathematical knowledge in order to utilize it in the light of specific content and specific student understanding (Ball & Cohen, 1999). Additionally, in responsive teaching, “teachers’ instructional decisions about what to pursue and how to pursue are continuously adjusted during instruction in response to children’s content-specific thinking” (Jacobs & Empson, 2016, p. 185). Having pre-service teachers engage with students’ thinking and respond to students’ thinking by eliciting through interviews around tasks they designed beforehand suggests that approximations of practice contribute to nurturing pre-service teachers’ responsive teaching. Accordingly, pedagogies of practice make learning complex practice possible through action and reflection and can provide pre-service teachers with opportunities to practice different aspects, including reflective and interactive aspects of teaching (Grossman et al., 2009). Moreover, pedagogies of practice also allow pre-service teachers to explore the teacher’s work and develop the teacher’s perspective (Ghousseini & Herbst, 2016). Thus, pedagogies of practice can provide a useful framework for the implementation of professional noticing in a teacher education program for the development of pre-service teachers (Fisher et al., 2018). Deliberate implementation of pedagogies of practice can enable pre-service teachers to learn to notice students’ mathematical thinking and improve their professional noticing skills. To conclude, it is believed that a module based on video clips of students’ mathematical thinking in geometric measurement tasks involving the use of concrete models, which is situated in the pedagogies of practice framework, may provide a scaffold for the development of professional noticing skills of pre-service teachers.

1.1 Purpose of the Study and Research Questions

The purpose of the study is to examine the extent to which pre-service teachers’ professional noticing of students’ mathematical thinking in geometric measurement changes when they participate in a video-based module situated in the pedagogies of practice framework. In addition, it aims to explore how a video-based module situated in the pedagogies of practice framework supports pre-service teachers’ professional noticing of students’ mathematical thinking in geometric measurement. For these purposes, the following research questions are formulated:

1. To what extent do pre-service teachers’ professional noticing of students’ mathematical thinking in perimeter-area and volume-surface area measurement change as they participate in a video-based module situated in the pedagogies of practice framework?

1.1. Is the change in pre-service teachers’ professional noticing of students’ mathematical thinking in perimeter-area and volume-surface area measurement from pre-test to post-test statistically significant?

2. How does a video-based module situated in the pedagogies of practice framework support pre-service teachers’ professional noticing of students’ mathematical thinking in perimeter-area and volume-surface area measurement?

1.2 Significance of the Study

It is necessary for teachers to be able to first notice students’ thinking in order to become involved in instruction that prioritizes student thinking. Teachers who notice students’ mathematical thinking can provide them with opportunities for learning by designing suitable tasks (Mason, 2011). Therefore, teachers should pay attention to and diagnose students’ mathematical thinking from students’ explanations and justifications, and they should make inferences about students’ thinking and then take pedagogical actions (Luna et al., 2009). In order to be able to attend to students’ strategies, interpret their understanding, and respond on the basis of their understanding, teachers should understand how students reason while working on the given task and what difficulties and misconceptions they experience.

In recent years, teacher educators have emphasized the importance of pre-service teachers’ ability to attend, interpret, and respond to students’ thinking. Therefore, pre-service teachers’ noticing of student thinking was examined in various mathematical domains, including pattern generalization (Callejo & Zapatera, 2017; Mouhayar, 2019; Özel, 2019); derivative (Sánchez-Matamoros et al., 2015), proportional reasoning (Fernández et al., 2012; Son, 2013), equal sign and equality (van den Kieboom et al., 2017), fractions (Estapa et al., 2018; Ivars et al., 2020; Lee, 2021; Tekin-Sitrava et al., 2022; Tyminski et al., 2021), geometry (Baldinger, 2020; Guner & Akyuz, 2020; Ulusoy & Çakıroğlu, 2021), arithmetic (Dick, 2013; Fisher et al., 2018; Jacobs et al., 2010; Kalinec-Craig et al., 2021; Kosko et al., 2020; Schack et al., 2013; Ulusoy, 2020; Warshauer et al., 2021); statistics (Shin, 2020), solving equations (Lesseig et al., 2016; Monson et al., 2020), algebraic thinking (Walkoe, 2013) and measurement (Caylan Ergene & Isiksal Bostan, 2022; Girit Yildiz et al., 2023; Moreno et al., 2021). Two research studies on measurement specifically focused on length measurement (Caylan Ergene & Isiksal Bostan, 2022; Moreno et al., 2021). The other research study conducted by Girit Yildiz et al. (2023) was limited to two sessions, although it included all geometric measurement types (length, area, and volume measurement). For this reason, the present study aims to enhance the understanding of the mathematics education community regarding how pre-service teachers notice students' mathematical thinking in geometric measurement, i.e., length, area, and volume measurement, for a longer period of time.

Students can communicate mathematical ideas with concrete models more clearly than paper-and-pencil tasks (Ontario Ministry of Education, 2008). In addition, students become active in their learning process by engaging them in mathematical tasks through concrete experiences (Karol, 1991). Thus, concrete models enable teachers to understand what students think (Skemp, 1989). Therefore, presenting students with concrete models while they are solving tasks in task-based interviews can provide more information about students' mathematical thinking to pre-service teachers rather than presenting students' written work about geometric measurement. In this way, pre-service teachers can better recognize the mathematically significant details in students' thinking, which in turn may support pre-service teachers' professional noticing skills. Moreover, the literature review has shown that previous studies of teacher noticing have not dealt with how pre-service teachers notice students' mathematical thinking, especially while students are engaging in mathematical tasks using concrete models. That is, little is known about how pre-service teachers attend, interpret, and respond to students' mathematical thinking on tasks in which students use concrete models. Thus, it is noteworthy to shed light on how pre-service teachers notice students' mathematical thinking while engaging in geometric measurement tasks using concrete models. Accordingly, the findings of this study are expected to give valuable information for teacher educators.

Research showed that pre-service teachers experience difficulty in identifying students' misconceptions (Turnuklu & Yesildere, 2007). Furthermore, even if they could identify the reasons for students' misconceptions, they could not decide on proper instructional actions in order to help students overcome these misconceptions. Yet, as future teachers, pre-service teachers should know students' difficulties in and misconceptions about geometric measurement and be aware of how to eliminate them in order to prevent students' misconceptions and difficulties before they actually start teaching. Professional noticing is not an expertise that novice or experienced teachers normally have; instead, teachers should acquire the ability to notice by getting support (Estapa et al., 2018). Based on the empirical findings of previous research, it can be claimed that without practice, it is not possible for pre-service teachers to develop their ability to attend to, interpret, and respond to student thinking. Researchers made different attempts to improve pre-service teachers' professional noticing skills through deliberate scaffolds and interventions (Ivars et al., 2020; Sánchez-Matamoros et al., 2015; Schack et al., 2013). Considering the importance of learning to teach, developing the ability to notice professionally should be the primary focus of teacher education programs (Star & Strickland, 2008). Teacher education programs mainly include practices that concentrate on pre-active aspects of teaching (e.g., lesson planning) rather than reflective aspects, which contain in-the-moment decisions depending on professional noticing (Grossman et al., 2009). In this regard, pedagogies of practice can create a reflective setting for pre-service teachers and scaffold them for future teaching practices in real classrooms (Schack et al., 2013). Integration of pedagogies of practice with professional noticing as an approach different from the interventions in the previous research studies can be one of the ways to support pre-service teachers' noticing of students' mathematical thinking by providing a basis for improvement in professional skills of pre-service teachers as they enter upon their career (Amador et al., 2017). In this way, interacting with students can help pre-service teachers enhance their understanding of how students think about mathematics (Fisher et al., 2018). Moreover, in most of the studies, representations of practice are used without incorporating decompositions and approximations of practice. However, the importance of all three opportunities for pre-service teachers' learning is underlined (Grossman et al., 2009). Thus, all components of the pedagogies of practice framework may provide a scaffold for the development of professional noticing skills of pre-service teachers.

Researchers have often preferred using video cases as a representation of practice to examine and sharpen pre-service teachers' professional noticing skills, which generally included classroom videos (Sherin et al., 2011; van Es & Sherin, 2008). However, teachers may pay attention to other aspects (e.g., classroom management and pedagogy) in the classroom context rather than students' mathematical thinking when classroom videos are used (Star & Strickland, 2008). Since these videos consist of multiple dimensions (e.g., students, teacher, mathematical thinking, pedagogy, climate, management), noticing important events may be challenging (Superfine et al., 2015). Besides, because of the complex nature of classroom environments, students' mathematical thinking may not be discernible in classroom videos (Mitchell & Marin, 2015). Therefore, videos focusing on student thinking can potentially develop pre-service teachers' professional noticing skills because they help zoom in on students' particular thinking about mathematical concepts. In addition, through video clips of different students, pre-service teachers can realize how mathematical concepts are understood differently and compare different students' thinking (Jacobs et al., 2010). Moreover, the use of video clips containing specific important misconceptions about geometric measurement can enable pre-service teachers to explore these misconceptions and the possible causes of these misconceptions (Hill & Collopy, 2003), which they will probably encounter while teaching in the future, and to be aware of how to eliminate them when they observe before entering the teaching profession. Moreover, existing videos can generate some technical problems with videotaping, and they can be out of context (Girit Yildiz et al., 2023). In this way, deliberately produced video clips can prevent some problems that may occur when using existing videos, including poor content due to video recording, camera effects, and audio problems (Girit Yildiz et al., 2023). Furthermore, the fact that the video clips do not include any distractions unrelated to mathematics and teaching may help preservice teachers focus on students' mathematical thinking and enable researchers to explore their professional noticing skills in-depth. In addition, when pre-service and in-service teachers do not take part in the video clips and the video clips only involve the researcher and middle school students, pre-service teachers can freely discuss students' mathematical thinking in the video clips without having to criticize their peers in the video clips (Girit Yildiz et al., 2023). Thus, instead of taking video clips directly from the literature, intentionally producing video clips that focus on different students' thinking through task-based interviews with middle school students and using them as representations of practice rather than videos of whole lessons can reveal students' mathematical thinking with their actions clearly, which in turn can support pre-service teachers' professional noticing skills and contribute to the literature.

In addition to viewing video clips of different students' mathematical thinking as a representation of practice, analyzing these video clips both individually and collaboratively as a decomposition of practice is important for pre-service teachers to improve their professional noticing skills and learn to notice students' mathematical thinking by focusing on components of professional noticing. Through these practices, preservice teachers can have a chance to discuss the mathematical details in students' mathematical thinking in the video clips, make sense of students' understanding, and suggest instructional actions to eliminate students' misconceptions and extend students' understanding through discussions.

Accordingly, a discussion environment may help pre-service teachers listen and learn their peers' ideas, respond to them, make various comments on students' mathematical understanding, and make different instructional suggestions based on students' understanding in a discussion setting (Ulusoy & Çakıroğlu, 2021). In this regard, discussing their ideas with their peers in both a group and whole-class setting can be valuable because pre-service teachers can share and discuss their ideas more comfortably in small groups. Then, they can bring the ideas that emerge in the group discussion to the whole class discussion. The whole-class discussion may provide pre-service teachers with an opportunity to have an idea of the mathematical details that are not discussed in the group and to learn different instructional actions that are not offered in the group discussion. Furthermore, by focusing on a comment, pre-service teachers can do a detailed analysis together and develop new ideas about students' mathematical thinking together during the discussions (Walkoe, 2015). From this point of view, the present study can provide insight into teacher education programs in terms of the impact of analyzing and discussing students' mathematical thinking in video clips on pre-service teachers' professional noticing skills.

In addition to the video clips, another way to improve pre-service teachers' professional noticing skills is clinical interviews with students conducted by pre-service teachers. Teachers need to elicit individual student thinking, which is not always the case in the classroom due to the existence of other students (Heng & Sudarshan, 2013). Therefore, clinical interviews are significant in providing individual interaction with students. Through clinical interviews, students' mathematical thinking can be elicited, and consequently, teachers can better understand the way students think and their strategies to solve problems (Heng & Sudarshan, 2013). In relation to this statement, in this study, while conducting clinical interviews, pre-service teachers were not only able to observe how students solve mathematical problems, but they could also learn students' strategies from their answers to interview questions. Research showed that teachers who conducted clinical interviews with students adopted student thinking-centered instruction, making discussing different ways of solving problems in class possible (Buschman, 2001; Jacobs et al., 2006). Therefore, conducting one-to-one interviews with students while they are engaging in mathematical tasks can be a way to unveil each student's mathematical thinking and conceptions (Copeland, 1984). In this way, teachers can determine students' strategies and attempts to solve the given task (Heng & Sudarshan, 2013). Furthermore, one-to-one interviews with students can promote teachers' motivation to understand student thinking (Buschman, 2001; Heng & Sudarshan, 2013; Jacobs et al., 2006). Hence, clinical interviews can be an effective tool for uncovering students' mathematical thinking and promoting teachers' skills of eliciting and responding to student thinking (Heng & Sudarshan, 2013; Jacobs et al., 2006; McDonough et al., 2002). Accordingly, giving pre-service teachers opportunities to conduct one-to-one task-based interviews with students as an approximation of practice can help them elicit students' mathematical thinking, interpret students' understanding, and decide how to respond based on that understanding.

1.3 Definition of Important Terms

Professional noticing of children's mathematical thinking is expertise consisting of three interrelated skills: attending to children's strategies, interpreting children's understanding, and deciding how to respond on the basis of children's understandings (Jacobs et al., 2010).

Attending to children's strategies is identifying mathematically important details in children's strategies (Jacobs et al., 2010). In the present study, pre-service teachers' attending skills are measured through their responses to the attending prompt in the noticing questionnaire and reflection papers, as well as their explanations in group discussions, whole-class discussions and semi-structured interviews.

Interpreting children's mathematical understanding is making sense of children's mathematical understanding using the details in the children's strategies (Jacobs et al., 2010). In this study, pre-service teachers' interpreting skills are measured through their responses to the interpreting prompt in the noticing questionnaire and reflection papers, as well as their explanations in group discussions, whole-class discussions and semi-structured interviews.

Deciding how to respond on the basis of children's understanding is reasoning that is used while deciding how to respond to children based on their understanding (Jacobs et al., 2010). In the light of this definition, in this study, pre-service teachers' deciding how to respond skills are measured through their responses to the deciding prompt in the noticing questionnaire and reflection papers, as well as their explanations in group discussions, whole-class discussions and semi-structured interviews. To improve pre-service teachers' professional noticing skills, video-based module of this study is situated in the pedagogies of practice framework which was proposed by Grossman et al. (2009).

Pedagogies of practice framework was developed to support pre-service teachers' learning the practice of teaching. The framework includes three key concepts, which are representations, decompositions, and approximations of practice for understanding the pedagogies of practice in professional education.

The first concept, *representations of practice* include the ways that practice is represented, which makes the practice visible to novices (Grossman et al., 2009). In this study, video clips of task-based interviews conducted with middle school students by the researcher were used as representations of practice.

Secondly, *decompositions of practice* refer to breaking down practice into meaningful components for teaching and learning purposes (Grossman et al., 2009). In the present study, pre-service teachers' individual analysis of the video clips in the sessions and discussions around these video clips about professional noticing components, i.e., attending, interpreting, and deciding how to respond, served as a decomposition of practice.

Finally, *approximations of practice* mean opportunities that are provided for novices to participate in practices that are more or less close to a profession's practices

(Grossman et al., 2009). In the current study, as approximations of practice, pre-service teachers conducted task-based interviews in which middle school students worked on the tasks the pre-service teachers designed beforehand.

Geometric measurement is the "assignment of a numerical value to an attribute of an object" (NCTM, 2000, p. 44), which includes the length, area, and volume measurement. In the present study, the content domain of the video clips and tasks designed by pre-service teachers was based on geometric measurement. This study explores pre-service teachers' professional noticing of students' thinking about perimeter measurement, area measurement, surface area measurement, and volume measurement, all of which take place at the middle school level according to the Turkish mathematics curriculum (MoNE, 2018).

CHAPTER 2

LITERATURE REVIEW

The purpose of the study is to examine the extent to which pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement changes when they participate in a video-based module situated in the pedagogies of practice framework. In addition, it aims to explore how a video-based module situated in the pedagogies of practice framework supports pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement. Based on these purposes, teacher noticing, geometric measurement, and pedagogies of practice are the main issues addressed in this chapter.

2.1 Teacher Noticing

In the general sense, noticing is "a term used in everyday language to indicate the act of observing or recognizing something" (Jacobs et al., 2018, p. 1). Particular forms of noticing that a profession portrays require recognizing aspects of a practice that are valued by a specific group (van Es & Sherin, 2008). In a systematic literature review of research on teacher noticing, König et al. (2022) specified four main perspectives that influence research on teacher noticing. The first one is a cognitive-psychological perspective of teacher noticing. From this perspective, noticing is a set of cognitive processes in which teachers engage, and it is based on attending to specific important events and making sense of these events (Sherin & Star, 2011). This perspective emerged in teacher education and professional development programs with the analysis of videos of teachers' own lessons (van Es & Sherin, 2002). Here, the researchers focused on the significant events teachers notice, their interpretations, and how they establish a relationship between particular situations and teaching and learning principles (van Es & Sherin, 2002). By considering student thinking as the central point, Jacobs et al. (2010) described three processes of noticing. This approach focuses on teachers' noticing students' mathematical thinking in which teachers attend to students' strategies, interpret their understanding, and make in-the-moment decisions about how to proceed in the lesson. Similarly, Kaiser et al. (2015) proposed perception, interpretation, and decision-making as components of noticing. Stockero et al. (2017a) categorized noticing as (a) noticing among instances and (b) noticing within an instance. In the studies regarding noticing among instances, teachers choose the ones they consider significant in a classroom video. Then, they explain why they consider those specific instances important (Star & Strickland, 2008; van Es & Sherin, 2008). On the other hand, in the studies that focus on noticing within an instance, teachers are asked to analyze a particular instance of students' mathematical thinking (e.g., Jacobs et al., 2010). Researchers can utilize these conceptualizations according to their purposes in order to understand different aspects of teachers' noticing.

From the cognitive-psychological perspective, teacher noticing as one of the key components of teachers' professional competence (Scheiner, 2016) involves whether a teacher attends or does not attend to what happens in class and to significant aspects of classroom environments. Teachers cannot make sense of events that take place during the class if they do not notice these classroom events. That is, teachers can make sense of what they actually notice (Star & Strickland, 2008). However, teachers can attend to some events, but not all of them (Erickson, 2011). For instance, when teachers pay attention to classroom management, they may not pay attention to students' thinking at the same time. Therefore, teachers should be able to decide what is important at any moment among the numerous events that happen simultaneously in classrooms. According to van Es and Sherin (2008), several factors may explain why teachers attend to particular things among many others, i.e., what teachers value and notice during their teachings are affected by these factors. These include educational background, teaching experiences, knowledge, views about the nature of mathematics, and perceptions regarding the activity in which they are involved. For example, if a teacher gives priority to classroom management, s/he might notice discipline problems in the classroom at first (Guler et al., 2020). Thus, recognizing the limitation of human perception, teachers should acknowledge that they need to pay attention to particular aspects of instructional practices while ignoring others (König et al., 2022).

The second perspective is a socio-cultural perspective of teacher noticing, which looks at noticing from the social aspect, and this kind of noticing was addressed as a professional vision. In this perspective, recognizing significant events is a socially situated activity rather than a psychological process (Goodwin, 1994). Recently, by taking equity-oriented approaches, research studies stressed the necessity of teaching practices that evaluate underprivileged students' abilities (e.g., Louie et al., 2021). Furthermore, van Es et al. (2022) highlighted equity and offered an equity-focused aspect of noticing, incorporating teachers' noticing of the historical and cultural development of students' learning.

The third perspective is a discipline-specific perspective of teacher noticing, proposed by Mason (2002). It is a general perspective, but since the examples given by Mason (2002) are mathematical in nature, the impact of it is primarily addressed by mathematics education scholars. This perspective attempts to increase teachers' awareness through four interdependent actions: systematic reflection, recognizing, preparing, and noticing and validating (Mason, 2002). In this way, a retrospective moment of noticing is brought into the moment so that selection can be made for responding rather than responding by the force of habit (Mason, 2002). Different from the previous perspectives, this practice-oriented approach predicates noticing on teachers' preparation for noticing, the act of noticing, and the practice of reflection on noticing.

The fourth perspective is an expertise-related perspective of teacher noticing with a focal point on the differences between novice and expert teachers (Berliner, 2001). Berliner (1988), the pioneer in expertise research, differentiated the development of novice and expert teachers' skills at different stages, which implies that expert and novice teachers' performances might differ at the noticing components. For instance, there were differences between novice and expert teachers in terms of observing, recognizing, and monitoring events (Sabers et al., 1991). In addition, novices experienced difficulties in interpreting the classroom events and explaining what was going on in the classroom (König et al., 2022).

In summary, focusing on the teacher as an individual and conceptualizing noticing mainly as an individual mind is common in the cognitive-psychological, discipline-specific, and expertise-related perspectives. On the other hand, from the socio-cultural perspective, noticing is accepted as society's function rather than the individual teacher. However, this does not show the separation or opposite of the individual mind from society (König et al., 2022). In the present study, the cognitive-psychological perspective of teacher noticing was adopted.

2.1.1 Frameworks for Teacher Noticing

In this part, the learning to notice framework (van Es & Sherin, 2002; van Es, 2011; van Es & Sherin, 2021) and professional noticing of children's mathematical thinking framework (Jacobs et al., 2010) are presented, respectively.

2.1.1.1 Learning to Notice Framework

The learning to notice framework involves "identifying what is important or noteworthy about a classroom situation, making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent, and using what one knows about the context to reason about classroom interactions" (van Es & Sherin, 2002, p. 573). The first aspect of the framework is about specifying significant teaching situations during the class. The second aspect is about considering classroom situations as not independent but related to learning and teaching principles. The third aspect is related to utilizing knowledge of content and knowledge of students in accordance with context while reasoning about situations as they occur. By using this framework, the same researchers conducted several research studies and analyzed the data according to five dimensions, consisting of actor, topic, stance, specificity, and evidence (van Es & Sherin, 2008, 2010). The actor is the people about whom teachers comment, such as students, teachers, or others. The topic is what teachers make comments on, such as the pedagogy the teacher uses, students' mathematical thinking, the climate of the classroom environment, classroom management, or other topics. The stance is how teachers examine events, that is, whether they describe, interpret, or evaluate events. The specificity is the degree of comments teachers make about events, which can be general or specific. The evidence is whether or not teachers' comments come from videos; in other words, whether they are video-based or non-video-based.

After 2002, van Es (2011) proposed a framework for learning to notice student thinking. This framework consists of two dimensions and four levels. What teachers notice and how teachers notice are the dimensions, and the levels that indicate teachers' noticing abilities in the framework are Level 1 (Baseline), Level 2 (Mixed), Level 3 (Focused), and Level 4 (Extended) (van Es, 2011). The dimension of what teachers notice is about attending to the significant events in the classroom, while the dimension of how teachers notice is about making sense of and interpreting these events by providing evidence. Attending and interpreting noteworthy classroom events are important because teachers have to make quick decisions during lessons, notice students' mathematical thinking, and utilize this to improve the lesson as the lesson progresses (Sherin & van Es, 2005). In level 1, teachers attend to the classroom environment as a whole, express general opinions regarding the situation by providing insufficient evidence, and make descriptive and evaluative comments. In level 2, teachers mainly focus on pedagogy, but they start to attend to students' mathematical thinking. They still hold general impressions but also point out significant events. They make evaluative comments like in level 1, but they add some interpretive ones to their comments. They start to provide particular situations as evidence at this level. In level 3, teachers shift their focus from other issues in the classroom to students' mathematical thinking, which differentiates level 3 from the first two levels. They make interpretive comments, underline particular significant events, and provide detailed explanations regarding students' mathematical thinking. In level 4, different from the other levels, teachers point out the relationship between student thinking and teacher pedagogy. They also relate events to learning and teaching principles and, alternatively, suggest pedagogical solutions according to their interpretations. The difference between this framework and the previous one is that improvement in teachers' noticing skills of student thinking can be shown as teachers' progression from lower to higher levels.

More recently, van Es and Sherin (2021) revised the framework by elaborating the construct of teacher noticing. The researchers expanded the attending and interpreting components in the previous framework and proposed a new component called shaping. In this revised framework, the attending component involves two parts, which are identifying significant aspects of classroom events and neglecting selected features of these events. That is, attending requires taking a closer look at some features while ignoring other aspects of the classroom environment. The interpreting component consists of two parts: making sense of an event with the help of knowledge and experience and taking an inquiring stance. In other words, in addition to making sense of the event, interpreting is also about seeing the event as worth trying to unravel. The third component, shaping, refers to creating interactions to obtain additional information, which allows teachers to further attend to and interpret student thinking. For instance, shaping might require asking a question to students or looking at students' writing to obtain more information. To sum up, the learning to notice framework (van Es, 2011; van Es & Sherin, 2002; van Es & Sherin, 2021) focuses on the diversity of what teachers notice and how teachers notice. On the other hand, Jacobs et al. (2010) developed the professional noticing of children's mathematical thinking framework that focuses specifically on children's mathematical thinking by giving less attention to the diversity of what teachers notice and how teachers notice.

2.1.1.2 Professional Noticing of Children's Mathematical Thinking Framework

Many events occur simultaneously in classrooms, which are complex environments. While it is helpful to notice a variety of instances, it can be difficult to pay attention to every single instance in the class. Hence, Jacobs et al. (2010) "attended less to the variety of what teachers notice and more to how, and the extent to which, teachers notice children's mathematical thinking" (p. 171). With this focus, in the cross-sectional study about whole-number operations, the researchers collected data from 131 pre-service teachers and experienced in-service K-3 teachers with different experiences, using a classroom video clip and a set of written student work included strategies children used while solving problems. Jacobs et al. (2010) called this particular form of teacher noticing professional noticing of children's mathematical thinking. Here, the focus is on the extended level (level 4) of learning to notice the student thinking framework proposed by van Es (2011) rather than other levels.

The professional noticing of children's mathematical thinking framework comprises three skills: attending to children's strategies, interpreting children's understanding, and deciding how to respond based on children's understanding (Jacobs et al., 2010). The key difference between this framework and the learning to notice framework is that the professional noticing of children's mathematical thinking framework includes the skill of deciding how to respond on the basis of students' understanding. The first skill in the professional noticing of children's mathematical thinking framework is attending to students' strategies, which means recognizing mathematically noteworthy details in the students' strategies (Jacobs et al., 2010). This skill is essential since identifying the strategies gives the teacher information about students' understanding by providing gateways. This understanding can be used as a basis for teaching, which demonstrates the importance of noticing details in students' strategies (Jacobs et al., 2010). After identifying all mathematically significant details in children's strategies, the researchers categorized the attending to children's strategies in two categories: evidence and lack of evidence. Teachers who were able to provide most of the details for at least two of three strategies were accepted as evidence of attending to children's strategies (Jacobs et al., 2010).

Interpreting students' understanding as the second skill refers to the ability to make inferences regarding students' understanding using the details in the students' strategies, i.e., making sense of students' mathematical thinking (Jacobs et al., 2010). Since teachers cannot directly see students' thinking, they create mental models regarding students' mathematics based on their expressions and actions (Stefe & Thompson, 2000). Here, the point is not to produce one perfect interpretation, but teachers are expected to ground their interpretations on evidence from students' work rather than making judgments without supporting any evidence. Jacobs et al. (2010) classified responses regarding interpretations of children's understanding under three categories: robust evidence, limited evidence, and lack of evidence based on the extent of evidence provided.

The third skill, deciding how to respond on the basis of students' understanding, is determining how to take instructional actions based on students' understanding (Jacobs et al., 2010). Here, the emphasis is on the intended response, not on the actual implementation of the response. In other words, as a next step, teachers are expected to suggest hypothetical instructional actions that could result from attending to and interpreting students' thinking (e.g., Krupa et al., 2017; Santagata, 2011). This skill catches the incorporated relationship between the information gained from teachers' observations and interpretation of student thinking and their immediate response plans (Jacobs et al., 2010). As teachers identify the details of student thinking and provide robust interpretation, they can give more appropriate and detailed instructional responses (Ulusoy, 2020). Deciding how to respond entails reasoning about a possible response, which serves as a means to ascertain the next instructional step possibly to extend student thinking (Jacobs et al., 2010; Smith & Sherin, 2019). There is no single correct response, but teachers with this skill should be able to use what they have learned about students' understanding of a certain situation. Furthermore, there should be consistency between teachers' responses and the research on students' mathematical development (Jacobs et al., 2010). Moreover, deciding on the basis of students' understanding necessitates knowledge about the easy and difficult aspects of the concept, common errors related to the concept, and the appropriate techniques and representations used while introducing the concept. While making an in-the-moment decision, which occurs when a student provides a mathematical explanation, the teacher analyses the student's mathematical thinking and associates specific situations with what s/he already knows about the student's mathematical development (Franke et al., 2007; Lampert, 2001). "This type of in-the-moment decision making is in contrast to the long-term decision making (or planning) that teachers do after school when they are not interacting with children" (Jacobs et al., 2010, p. 173). Jacobs et al. (2010) created three categories for the responses on deciding how to respond on the basis of children's understanding based on the extent of the evidence: robust evidence, limited evidence, and lack of evidence.

The three skills, i.e., attending, interpreting, and deciding, are intertwined, which means that their development is interdependent (Jacobs et al., 2010). Thus, professional noticing is more than attending to a situation that attracts the teacher's attention. It directs the teacher's attention to the student's thinking, and then, the teacher interprets the student's understanding and devises a response. That is, teachers can decide how to respond as long as they interpret the student's understanding, and interpretations can be based on the student's understanding as long as the teacher can attend to the student's strategies (Jacobs et al., 2010).

When the frameworks for teacher noticing are considered from a general perspective, it is observed that frameworks, except that proposed by Jacobs et al. (2010), were developed to examine teachers' noticing of various aspects in a classroom environment. The framework developed by Jacobs et al. (2010) focuses completely on noticing students' mathematical thinking, and it was constructed to examine teachers' professional noticing of students' mathematical thinking. Since the aim of the present study is to explore how pre-service teachers notice students' mathematical thinking in geometric measurement and develop pre-service teachers' noticing of students' mathematical thinking in this context, the rest of the thesis continues by adopting the professional noticing of children's mathematical thinking framework (Jacobs et al., 2010).

2.1.2 Importance of Noticing Students' Mathematical Thinking

Research indicates that many teachers experience difficulties in eliciting student thinking and making students' thinking explicit. Consequently, they may not realize that students can have their own nonstandard mathematical strategies and ideas that may differ from teachers' way of mathematical thinking (Empson & Jacobs, 2008). In addition, effective listening to students and responding to numerous factors particular to students' thinking is not easy work (Empson & Jacobs, 2008). As a result, teachers superficially ask students whether they know the facts they memorized or whether students can perform what has been taught by rarely probing students' thinking (Black et al., 2004). NCTM (2000) indicated that "effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning" (p.10). In this regard, teachers can attend to what students think and what is significant about that thinking, how students engage with tasks, what makes them interested and motivated, and how they use what they already know to build new knowledge and understanding. Thus, understanding the mathematical thinking of students is crucial for selecting and designing instructional tasks, asking proper questions, facilitating classroom discussions, modifying instruction to meet the needs of students, understanding students' reasoning, assessing students' learning progress, and recognizing and remedying students' learning difficulties (Battista, 2017). For instance, teachers first identify and interpret students' understanding of the material in order to make adaptations in their instructions to address students' needs. In his study, Choppin (2011) observed that teachers who attended to students' thinking could adapt tasks to retain the complexity of the task concerned with students' opportunities to understand mathematical concepts. Moreover, Schoenfeld (2011a) stressed that highly accomplished teachers adapt lessons based on their discoveries about their students.

Teachers are required to listen to students' mathematical thinking, establish a connection with their own and other students' thinking, and decide how to respond (Kooloos et al., 2022). Thus, noticing students' thinking is essential for teachers to be able to make in-the-moment decisions (Richards et al., 2020). Philipp et al. (2014) emphasized the difference between professional noticing and knowledge and beliefs by indicating noticing as "an interactive, practice-based process rather than a category of cognitive resource" (p. 466). However, although they are different, mathematical knowledge and noticing are theoretically and empirically connected (Thomas et al., 2017). Paying attention to significant events in students' work and making inferences about students' thinking require knowledge about mathematics and students' mathematics (Dreher & Kuntze, 2015; Stockero et al., 2017b; Yang et al., 2019). When teachers fail to understand the rationale behind students' misconceptions, mistakes, or alternative strategies, they may tend to ignore them instead of trying to elicit further students' thinking (Kilic & Dogan, 2022). Furthermore, Ball et al. (2008) also emphasized that teachers should have content knowledge not only to teach but also to understand and interpret students' mathematics. In addition to attending and interpreting, decision-making is also related to teacher conceptions, including knowledge (Kooloos et al., 2022). Teachers interpret student thinking and take action to develop students' understanding based on their experience and knowledge (Kilic & Dogan, 2022). Furthermore, making proper instructional moves as a response to students' thinking necessitates knowledge about effective scaffolding practices (Kilic, 2018; van Zoest et al., 2017). That is, teachers should know different ways to help students progress in their mathematical thinking and which is best for a specific circumstance and a student.Thus, the relationship between mathematical knowledge for teaching and professional noticing skills should be nurtured for effective learning environments (van Zoest et al., 2017).

Prioritizing teachers' attention to students' thinking is essential for the improvement of classroom instruction (Gonzalez & Skultety, 2018). In a similar vein, Mason (2011) maintains that developing teachers' noticing of student thinking is important for recognizing classroom events and responding to them in the future. As teachers recognize events, they can foresee similar situations, which allows them to establish routines for responding (Wallin, 2015). In this way, by being equipped with various strategies to deal with classroom events and contemplate activity-related or possible student questions through planning lessons, teachers can be well-prepared for situations that may occur in the classrooms (Smith & Stein, 2011). Therefore, teachers' noticing skills of student thinking should be improved before they actually start teaching (van Es & Sherin, 2002). According to Fennema et al. (1996), supporting teachers to understand students' mathematical thinking processes is the only way to enhance mathematics instruction and student learning. Providing teachers with plenty of opportunities to uncover students' mathematical thinking by discussing with their colleagues can help them learn more about students' way of thinking and, as a result, improve the quality of instruction.

The ability to provide evidence for student thinking is a significant step for pre-service teachers (Sleep & Boerst, 2012) as it bases their decision-making on students' thinking and reasoning (Darling-Hammond & Bransford, 2005). Similarly, Levin et al. (2009) pointed out that when the issue of noticing student thinking is not prioritized before pre-service teachers start to create routines, they might create routines that do not necessarily concentrate on student thinking. Based on this, it is possible to claim that pre-service teachers can develop their noticing skills if opportunities are provided regarding noticing in teacher education programs. Improving pre-service teachers' noticing skills can support them to deliver better instruction at the beginning of their profession. Improved noticing skills enable pre-service teachers to obtain more from their observation of mentor teachers during their teaching practice (Star & Strickland, 2008), reflect reasonably upon their own teaching (Llinares & Valls, 2010), and make better instructional decisions during their practice (Sherin & van Es, 2005). Research revealed that pre-service teachers' noticing of students' mathematical thinking is explored in specific content domains. One of the content domains that students experience difficulties in understanding is geometric measurement (length, area, and volume measurement), and hence, this study explores pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement. While preparing the video clips and student solutions in the noticing questionnaire, we took into account the misconceptions that students experience in geometric measurement.

2.2 Geometric Measurement

Geometry provides contexts for activities related to measurement, and measurement provides a means to quantify different attributes of geometric shapes. Geometric measurement refers to concepts of measurement, including length, area, and volume, which are one, two, and three-dimensional measures, respectively, and it is different from other types of measurement, such as weight and time. Measurement is important for understanding the construction of shapes, determining objects' sizes, utilizing the coordinate system to ascertain locations, and identifying transformations (Battista, 2007). Measurement is the relationship between the unit of measure and the quantity to be measured, in which the unit and the quantity have attributes of the same type (Lee & Lee, 2021). In the process of measurement, the unit to be used is chosen, and the number of units required between the two points is detected. Measuring includes two ideas: (a) there is an inverse relationship between the unit size and the required number of units, and (b) it is essential to use the same unit throughout the measuring process (Clements & Stephan, 2004). Accordingly, focusing on students' conceptual understanding of units can be considered fruitful in forecasting and expressing changes in their conceptual understanding regarding measurement (Outhred et al., 2003). The measurement of length, area, and volume has similar measurement principles, and developmental progressions for geometric measurement have common attributes (Curry & Outhred, 2005).

However, research on students' understanding of measurement revealed that when students cannot make sense of what they are doing while measuring, they might experience difficulties and have misconceptions about geometric measurement. For example, students' difficulties in early grades consist of overlapping units or leaving gaps between units, using units having different sizes, counting the same units twice, or skipping units. In older grades, students have difficulty in making a transition from physical filling to visualization and using more abstract techniques (Bragg & Outhred, 2000; Outhred & Mitchelmore, 2000). Students' difficulties and misconceptions in understanding concepts of measurement can be related to teachers' way of teaching, which involves introducing concepts of measurement through formulas rather than utilizing students' conceptions of measurement (Thompson et al., 1994). That is, students' difficulty and poor understanding can be related to focusing on the procedure instead of the underlying concepts in teaching measurement. Moreover, students' misunderstanding of measurement can be due to stressing how to measure rather than what to measure (Grant & Kline, 2003). Limited curriculum materials and limited knowledge of teachers can be the reasons why a procedural understanding of measurement is focused on more than conceptual understanding (Runnalls & Hong, 2020; Smith et al., 2006). However, traditional measurement instruction is not sufficient to support students in constructing the concepts that constitute the basis for understanding measurement (Clements & Stephan, 2004). As a result, most of the students utilize instruments for measurement by rote and use formulas while solving problems in order to obtain results without understanding the meaning and the reasoning underlying the formula and conceptual ideas that rationalize the procedures (Clements & Battista, 1992; Irwin et al., 2004; Sherman & Randolph, 2004). Geometric measurement in one dimension refers to length measurement, and in the following part, the importance of length measurement, foundational concepts for length measurement, and students' misconceptions and difficulties in length measurement are provided.

2.2.1 Length Measurement

The concept of length is critical in both daily life and formal geometry. People frequently use lengths in everyday life to describe the size of objects and the distance that is covered. The fact that measurement includes the main concepts puts length measurement in an important place in geometric measurement. Hence, students' lack of understanding of length measurement may prevent them from learning the basic concepts of measurement (Martin, 2007) and jeopardize their' understanding of geometry in high school (Battista, 2007). If students do not understand length measurement conceptually, they will also have difficulty gaining an understanding of the measurement of perimeters and realizing that the perimeter is a one-dimensional measure as well as length (Coskun et al., 2021). Consequently, it can be said that students' deep understanding of length measurement is important. In Turkey, according to the Turkish mathematics curriculum (MoNE, 2018), students encounter length measurement for the first time at the first-grade level, and there are objectives related to length measurement at each grade level (1st, 2nd, 3rd, and 4th grade levels) until the end of the primary school and in the first year of the middle school (5th grade level).

"Understanding of the attribute, conservation, transitivity, equal partitioning, iteration of a standard unit, accumulation of distance, origin, and relation to number" (Sarama & Clements, 2009, p. 164) concepts constitute the basis for children's understanding of length measurement. Attribute is "understanding that lengths span fixed distances" (Lee & Cross Francis, 2016, p. 220). Conservation is the understanding that even though an object is moved, its length is the same. Transitivity is the understanding that if object X and object Y have equal lengths and object Y and object Z also have equal lengths, then the lengths of object X and object Z are also equal to each other. Equal partitioning is realizing that an object can be mentally partitioned into units of the same sizes (Sarama & Clements, 2009). In order to gain insight into students' understanding of partitioning, students can be asked to explain the meaning of the marks on the ruler. Unit iteration is repeatedly locating the length of a small unit along the length of an object and realizing that a small unit is part of the length of the object. Accumulation of distance means comprehending that the number indicates the covered space when all iterations of a unit are counted. The other concept, origin, refers to realizing that any point on the ruler can be the origin while measuring. The last concept is the relation to numbers, which means that the relationship between the number and measurement refers to recognizing that counting discrete units is a cornerstone for measuring. That is, counting helps students develop concepts of measurement (Clements & Stephan, 2004). Although there is a debate about the sequence of these concepts and at which age the development of these concepts occurs, researchers agree that these concepts constitute the foundation for length measurement and should be taken into account in length measurement instructions. Even though length is an important concept, research revealed that elementary and middle school students experience difficulties in understanding and learning length measurement (Barrett et al., 2006; Battista, 2006), as presented below.

2.2.1.1 Students' misconceptions and difficulties in length measurement

Students' difficulties in understanding concepts of length measurement can be related to teachers' way of teaching, which involves introducing concepts of length measurement through formulas rather than utilizing students' conceptions of length measurement (Thompson et al., 1994). Traditional length measurement instruction is not sufficient to support students in constructing the concepts that constitute the basis for understanding length measurement (Clements & Stephan, 2004). Students' misunderstanding of measurement can be due to stressing how to measure rather than what to measure (Grant & Kline, 2003). That is, students' difficulty and poor understanding can be related to focusing on procedures instead of the underlying concepts in teaching measurement. Moreover, activities related to measurement in typical textbooks include questions that ask for the number of units; for example, "How many paper clips does the pencil measure?" These activities encourage students to give an answer expressed in numbers (Kamii & Clark, 1997). In addition, limited curriculum materials and knowledge of teachers can be the reasons why the procedural understanding of measurement is focused on more than conceptual understanding (Runnalls & Hong, 2020; Smith et al., 2006). Furthermore, since measuring requires physical work, assessing students' conceptual understanding is ambiguous to teachers. Hence, teachers tend to teach students to measure using a ruler. However, length measurement is "more than just learning how to use a ruler" (Smith et al., 2006, p. 30). The physical activity of the measuring procedure and using the marks on the ruler can conceal conceptual understanding (Stephan & Clements, 2003). As a result, most of the students utilize instruments for measurement by rote and use formulas while solving problems related to perimeter calculations in order to obtain results without understanding the meaning and the reasoning underlying the formula and conceptual ideas that rationalize the procedures (Clements & Battista, 1992; Irwin et al., 2004; Sherman & Randolph, 2004).

In length measurement, one of the misconceptions and difficulties is related to the usage of units. Students iterate a unit by leaving gaps between units or overlapping units while measuring (Lehrer, 2003). Many students regard iterating as merely putting units end to end but not as covering the length without gaps (Clements & Sarama, 2009; Clements & Stephan, 2004). The fact that children are not disturbed by the gaps or overlaps between the units while measuring can be an indicator of not considering the units as parts of the whole length (Kamii, 2006). Furthermore, many students use different units, such as both pencils and paper clips, or they use the same units in different sizes, such as big and small paper clips, at the same time while measuring because they think that the overall length is covered in any case (Lehrer, 2003). In a study conducted by Curry et al. (2006), it was found that although students did not accept using units in different sizes while measuring area and volume, they did not see a problem with utilizing units in different sizes while measuring length. It can be said that using identical units in order to measure area stemmed from students' reasoning that putting different-sized tiles together to cover an area was not possible. Therefore, the researchers pointed out that students lacked the necessary understanding of using identical units, and they did not have a clear understanding of what they measured.

In addition to the usage of units, students can have difficulties and misconceptions about using a ruler. Lehrer (2003) stated that most students measure with a ruler by starting with 1, not zero, and only a small number of students can comprehend that if they count intervals while measuring the length of an object, any number on the ruler can function as a starting point. Moreover, when they have a starting point different from zero, they cite the number on the ruler corresponding to the endpoint of the object as the length of the object. That is, they count numbers next to marks on a ruler rather than focusing on spaces between marks (Bragg & Outhred, 2004). Additionally, it was found that even the correct usage of the ruler might not show students' deep understanding of linear measurement in complex tasks. Although students may know how to use the ruler in simple measuring tasks, they might not understand the relationship between the process of measuring and linear units (Hiebert, 1984). Furthermore, Bragg and Outhred (2000) asserted that when students do not comprehend the construction of rulers, they do not have the essential knowledge to associate length measurement with number lines. The present study particularly focused on perimeter measurement, which is also a one-dimensional measure of geometric measurement.

2.2.1.2 Perimeter measurement and students' misconceptions and difficulties in perimeter measurement

In Turkey, students encounter perimeter measurement for the first time at the third-grade level. In addition, objectives related to perimeter measurement at the fourth and fifth-grade levels are included in the Turkish mathematics curriculum (MoNE, 2018). The perimeter is a "one-dimensional property of a two-dimensional figure" (Proulx, 2021b, p. 25). Therefore, the perimeter should be conceived as "a unidimensional measure (1D) of a length belonging to and describing a two-dimensional (2D) figure" rather than adding the sides or distance around it (Proulx, 2021a, p. 31). However, almost all middle school mathematics curricula describe perimeter as "the distance around a figure" (Charles et al., 2004, p. 441). Here, the meaning of the figure is not explicit. In instructions on the perimeter concept, polygons are demonstrated to students, and they are asked to measure the distance around the figure through a string. Students wrap the string around, measure it, and realize that this measure gives the perimeter of the figure. However, for the perimeter of a punctured square or perforated shape, the definition of the distance around does not work. Thus, defining the perimeter as the length or measure of the boundary of a connected region or two-dimensional figure would be more appropriate (Danielson, 2005). In addition, most of the textbooks include how to calculate the perimeter with various examples. Without making a connection with the meaning, the perimeter can be seen as just a formula and exercises on addition or multiplication (Vighi & Marchini, 2011). Yet, the perimeter is not a simple concept that just requires measuring and computing. Difficulties in understanding the perimeter stem from the presence of the formula to specify the perimeter. Giving procedures such as adding the lengths affects students' conceptualization of perimeter (Proulx, 2021a). This reduction to calculations brings into prominence the arithmetic aspects of the problem by disregarding the geometric aspect. Formula-based approaches without conceptual understanding cause students to have misconceptions, such as adding the lengths of two sides, i.e., the length and width of the polygon, or multiplying by two after adding the length of all sides (Tan-Sisman & Aksu, 2016). Furthermore, the researchers found that some students believed that the perimeter does not change in the case of the rearrangement of a shape. The researchers also observed students' confusion about perimeter and area and students' use of units of area or volume for the perimeter. Some of the students made judgments regarding the appearance of the shapes by visual comparison. Moreover, some counted dots while finding the perimeter of the shapes presented on dot paper. Similarly, when a shape is presented on a grid, the perimeter is sometimes considered the surrounding squares by students (Vighi & Marchini, 2011). Students also have trouble deciding when to use units and square units due to a lack of knowledge about what numbers represent. They mask their lack of knowledge by memorizing formulas and putting numbers, knowing that they must somehow use the numbers given on the sides in the figures in a formula to get the correct result (Moyer, 2001b). In addition to perimeter measurement, area measurement, which is a two-dimensional measure of geometric measurement, is in the scope of the present study. The following part continues with the importance of area measurement, foundational concepts for area measurement, and students' misconceptions and difficulties in area measurement.

2.2.2 Area Measurement

The area concept is one of the significant concepts in mathematics because it enhances students' understanding of spatial measurement and provides a basis for the improvement of students' understanding of other concepts, such as multiplication and fractions (Outhred & Mitchelmore, 2000; Sarama & Clements, 2009). In Turkey, according to the Turkish mathematics curriculum (MoNE, 2018), students encounter area measurement for the first time at the third-grade level, and there are various objectives related to area measurement at each grade level (3^{rd}, 4^{th}, 5^{th}, 6^{th} and 7^{th} grade levels) until the end of the seventh-grade level.

"Finding the area of a region can be thought of as tiling (or equal partitioning) a region with a two-dimensional unit of measure" (Sarama & Clements, 2009, p. 294). Hence, in area measurement, students are expected to cover the surface using square tiles and determine the number of unit squares required to cover that surface (Barrett & Clements, 2003; Clements & Sarama, 2009; Kamii & Kysh, 2006). When a rectangle is covered with unit squares, which results in an array/grid, students can count composite units in a row and column and realize that they can multiply rows and columns to find all units.

Foundational concepts in area measurement are transitivity, the relation between number and measurement, understanding the attribute of area, equal partitioning, units of area and unit iteration, structuring an array, conservation, and linear measurement (Sarama & Clements, 2009). Transitivity and relation to number work in length measurement and area measurement in a similar way. Understanding the attribute of the area means quantifying a two-dimensional surface that is bounded. Equal partitioning is splitting a two-dimensional surface into parts that have equal areas. Unit iteration is properly tiling a region with two-dimensional units. Structuring an array is described below:

> "The region must be covered by a number of congruent units without overlap or leaving gaps, and that a covering of units can be represented by an array in which rows (and columns) are aligned parallel to the sides of the rectangle, with equal numbers of units in each" (Outhred & Mitchemore, 2004, p. 465).

The main idea of being competent at area measurement is enumerating arrays of squares meaningfully (Battista, 2004). A lack of understanding of array structure may cause students not to use the area formula meaningfully and may result in confusing area and perimeter concepts, such as counting units around the shape while finding area (Sarama & Clements, 2009). Conservation is the fact that the area does not change when the parts of the shape are rearranged. The following part is concerned with the students' misconceptions and difficulties in area measurement.

2.2.2.1 Students' misconceptions and difficulties in area measurement

One of the misconceptions and difficulties in understanding area measurement is related to the use of units, similar to the length measurement. Research indicated that students might fail to recognize the relationship between the attribute of the unit and the attribute of the object being measured (Nunes et al., 1993). As a result, they may confuse linear and square units (Chappell & Thompson, 1999; Reynolds & Wheatley, 1996; Simon & Blume, 1994). Furthermore, students have difficulty understanding that in order to conserve area, there should be an inverse relationship between the unit size and the number of units (Carpenter & Lewis, 1976). Moreover, it was seen that even though students could count units along arrays, they had difficulty realizing that the number of units represents the area of the rectangular shape (Carpenter et al., 1980). In addition to the use of units, difficulties in understanding area measurement also originate in the use of the area formula.

Using the area formula "requires an understanding that the procedure yields the number of square units that fill the space inside each shape" (Lehmann, 2023, p.535). The use of the area formula, particularly at the very beginning of the subject, causes difficulties in understanding area measurement (Zacharos, 2006). Regarding the use of the area formula, three significant limitations were revealed (Baturo & Nason, 1996): strengthening the perception of the area as concerning the boundary of a shape, avoiding creating the unit of area, and ignoring the array notion. Research showed that students experience difficulties in understanding area as space inside a figure (Maher & Beattys, 1986), and as a result, when it is asked what area is, they mention measuring the area and its formula. Also, while finding the area of a rectangle, students use the formula by rote without explaining the rationale behind the formula (Huang & Witz, 2013). It is argued that an inadequate understanding of area measurement is due to the rote application of the area formula for rectangles (Simon & Blume, 1994). Moreover, it was seen that students could use the formula for finding the area of a rectangle while finding areas of other shapes (Zacharos, 2006). In addition, procedures that focus on area calculation cause difficulties in calculating the area of composite shapes (Patahuddin et al., 2018). Erroneous strategies were observed in students' solutions to tasks involving the measurement of composite tasks. To illustrate, students applied the rectangular area formula to calculate the area of composite shapes inappropriately (Hirstein, 1981; Zacharos, 2006), calculated the perimeter of the given composite shapes instead of area (Hirstein, 1981), or added the base and height in the composite shapes to calculate their area (Zacharos, 2006). Similar difficulties were also reported in the research studies conducted with pre-service mathematics teachers (Baturo & Nason,1996; Reinke, 1997). These erroneous strategies stemmed from teaching that gives priority to area formulas and a lack of conceptual understanding regarding area (Patahuddin et al., 2018). Thus, it is necessary to make connections between the formula and corresponding elements in the figure for understanding area measurement formulas. When the explicit connections are not provided, and the emphasis is on computations such as putting numbers on the formula area=base×height (Zacharos, 2006), students can experience difficulties and have misconceptions (e.g., while finding the area of an obtuse triangle, use of a side rather than height) (Fuys et al., 1988).

Furthermore, students usually confuse perimeter and area when these concepts are taught just by using a set of procedures or formulas (Moyer, 2001), and they use perimeter and area formulas interchangeably (Cavanagh, 2008; Orhan, 2013; Tan-Sisman & Aksu, 2009). They think that figures with the same perimeter must have the same area (Tsamir & Mandel, 2000), and if the area of a figure increases or decreases, the perimeter of the figure also increases or decreases and vice versa (Tirosh & Stavy, 1999). It is suggested that this misconception is based on the intuitive rules (More A-More B, Same A-Same B) (Stavy et al., 2002). Pre-service teachers who are expected to teach these concepts also have similar misconceptions (Livy et al., 2012; Wanner, 2019). Similarly, many students believe that no matter what the shape is, a fixed perimeter covers the same area (Wiest, 2005). Like area measurement, the surface area can be challenging for many students when it is not understood conceptually. Therefore, the following part lays out surface area measurement, which is in the scope of the present study.

2.2.2.2 Surface area measurement and students' misconceptions and difficulties in surface area measurement

In the Turkish mathematics curriculum (MoNE, 2018), objectives related to surface area measurement take place at the fifth-grade and eighth-grade levels. Students learn the surface area of a rectangular prism at the fifth-grade level, while they learn the surface area of the right circular cylinder at the eighth-grade level. The surface area of three-dimensional objects can be found through two strategies: one uses a formula (2lw + 2lh + 2hw), and the other calculates the area of the net of the objects. These strategies hold two opposite perspectives. Some of the researchers consider surface area extension of two-dimensional measurement, i.e., application of area measurement (Battista, 2012; Kim et al., 2017). This perspective underestimates the significance of three-dimensional reasoning in surface area, and hence, calculating the area of a net of a three-dimensional object might yield difficulties for students (Lehmann, 2022). Focus on the formula to calculate surface area can be accountable for students' struggles since that kind of teaching hinders students from reasoning about three-dimensional objects' geometric measurements (Seah & Horne, 2018). In this way, students may not develop a conceptual understanding of the surface area and understand what they are measuring while calculating surface area. On the other hand, the researchers who adopt the second perspective consider the surface area part of three-dimensional thinking (Pittalis & Christou, 2010). Using a formula without reasoning about the properties of prisms may not help students while they are learning to calculate surface area, and it is likely to result in memorization of the formula or inappropriate use of the formula (Pitta-Pantazi & Christou, 2010; Seah & Horne, 2020). Moreover, Tan Sisman and Aksu (2016) found that sixth grade students believed that a prism had more than one surface area. In addition, students' confusion between surface area and volume (Lim et al., 2019; Tan Sisman & Aksu, 2016) and incorrect beliefs regarding the relationship between surface area and volume (Sáiz & Figueras, 2009) were reported in the research studies. It is suggested that students should be given opportunities to examine the properties of three-dimensional objects by using concrete models or dynamic geometry prior to the development of their own strategies in surface area calculation (Dogruer & Akyuz, 2020; Moore, 2018). Students' confusion of surface area with volume reveals the importance of conceptual understanding of volume measurement in addition to surface area measurement. The part below reviews the literature related to volume measurement, which is a three-dimensional measure of geometric measurement.

2.2.3 Volume Measurement

Volume measurement is an important topic from elementary to high school grades because it provides a context for enhancing students' knowledge of arithmetic, geometric reasoning, and spatial structuring (Battista, 2003; Lehrer, 2003). In the Turkish mathematics curriculum, objectives related to volume measurement take place at the sixth-grade and eighth-grade levels. At the sixth-grade level, students learn the volume of a rectangular prism, specifically the relationship between the number of unit cubes completely filling a rectangular prism and the volume of the rectangular prism, constructing different rectangular prisms with the same volume with unit cubes and the volume formula for a rectangular prism. In addition, at this grade level, students are expected to associate units of liquid measurement with units of volume measurement. At the eighth-grade level, students learn the volume of a right circular cylinder. Piaget and Inhelder (1956) asserted that students could have three different meanings of the concept of volume: An interior volume is the number of unit cubes that make up an object or the amount of substance held within the boundaries. Occupied volume is the amount of occupied space of an object. Complementary volume is the amount of displaced water if an object is immersed in water. Comprehending all these types and their coordination can play a significant role in understanding the volume concept because all of them are related to "the measurement of the amount that quantifies an attribute (volume) of a three-dimensional figure" (Zembat, 2007, p. 208).

In school mathematics, there are two approaches to volume measurement: Volume as filling and volume as packing. Volume as filling is filling the three-dimensional object by iterating a fluid unit that occupies the container (Clements & Sarama, 2009; Curry & Outhred, 2005). It was challenging for students to compare the amount of liquids required to fill containers that differ in size and shape (Smith & Barrett, 2017). Regarding students' conceptions of volume as filling, research showed that students think volume decreases when the liquid is poured into a wider cup (Piaget, 1968; Piaget et al., 1960). This prevailing usage of a single dimension to make a judgment about the volume of the three-dimensional object was called the centration hypothesis. That is, students are inclined to focus on height rather than base area and experience difficulties in determining how the dimensions affect volume. Hence, it seems that competence about filling volume and competence about length develops concurrently because both require one-dimensional thinking (Sarama & Clements, 2009).

Volume as packing is multiplying the number of units that cover a three-dimensional object's base and the number of layers (Curry et al., 2006). Here, students are expected "to decide on the attribute to be measured, select a unit that has that attribute" and finally "compare units, by filling, […] with the attribute of the object being measured" (Van de Walle, 2004, p.317). Volume measurement necessitates more complex reasoning than measuring one and two dimensions because of the inclusion of a third dimension (Lehrer, 2003). It is also suggested that area understanding is a necessary condition for packing volume understanding. Research revealed that students correctly reason about problems related to volume concepts differently (Battista & Clements, 1996). Volume concept is conceptualized by students as composed of individual unit cubes or as composed of rows or columns of unit cubes. In the latter, cubes are conceived as space-filling rather than layers. Here, students can use column/row iteration. In column/row iteration, skip counting is utilized to obtain the total number after finding the number of cubes in a row/column. There are also some students who conceptualize volume as composed of layers. Here, students can enumerate cubes by multiplying or adding. In layer multiplying, after determining the number of cubes in a vertical/horizontal layer, this number is multiplied by the number of layers. In layer adding, addition or skip counting is used to find the total number of unit cubes after getting the number of cubes in a vertical/horizontal layer (Battista & Clements, 1996). All conceptualizations include understanding three dimensions in volume, even if the problems are given in written form or as two-dimensional representations (Rupnow et al., 2022). Students generally start by counting the individual units and continue with counting layers and multiplying the base area by height or multiplying the number of layers by the volume of the base. This shift in student reasoning from using the individual unit to the use of units of units, like in the area, and units of units of units, like in volume, is a significant development (Smith & Barrett, 2017). Vasilyeva et al. (2013) asked students the number of unit cubes in the given tasks involved gridded and without gridded representations of the arrays. The researchers concluded that students used more appropriate strategies when the grids were drawn on the objects. This showed that students' strategies depended on the representations in the given task. Furthermore, Finesilver (2017) found that at the beginning, students counted the faces of the cubes in the presented tasks without considering the cubes that were inside and back. Then, they started to prefer layering strategies in the case that arrays were presented in different colors. Several researchers highlighted the effects of virtual manipulatives (Panorkou & Pratt, 2016; Panorkou, 2019). In these studies, students explored that pulling the prism's base, which is two-dimensional, through a virtual manipulative created a volume that is three-dimensional. This prevented common student difficulties such as neglecting invisible cubes and double counting the cubes at corners. In a similar vein, a dynamic virtual manipulative was influential in eliminating students' challenges and creating new understanding regarding volume (Rupnow et al., 2022). However, students' lack of conceptual understanding causes them to have misconceptions and difficulties in volume measurement, which are described in the following part.

2.2.3.1 Students' misconceptions and difficulties in volume measurement

Students with an advanced level of understanding can relate length, width, and height dimensions in a rectangular prism to the number of cubes in vertical/horizontal layers (van Dine et al., 2017). A lack of connection between the volume formula and layer structure may push students to use the formula without knowing why it works (Sarama & Clements, 2009). Students' inability to visualize the three-dimensional structure and associate the number of cubes with the volume formula may cause students to utilize the formula by rote (Vasilyeva et al., 2013). Battista and Clements

(1996, 1998) asserted that learning the volume formula without conceptual understanding was more likely to result in difficulties and misconceptions about volume measurement. Even though students could make the calculations they aimed at, most could not interpret the results they reached. Hence, rote memorization and application of formulas hinder students from conceptualizing volume and students' spatial structuring in learning volume (Battista, 2002). Spatial structuring is "establishing units, establishing relationships between units (…), and recognizing that a subset of the units, if repeated properly, can generate the whole set" (Battista & Clements, 1996, p. 282). Spatial structuring is necessary to construct a mental model and enumerate the array. The rote memorization of formulas also causes confusion about volume and surface area, i.e., focusing on the surface area by ignoring the hidden cubes and difficulty interpreting two-dimensional representations of three-dimensional objects (French, 2004). In addition, even though students could use a formula for volume, they could not make a connection between the formula and the spatial structuring of the cube building (Battista & Clements, 1998). Curry and Outhred (2005) asserted that packing volume is more difficult than the concepts of length and area for children and found that while packing a big cuboid by enumerating small cubes, students experienced difficulty in the position of cubes and left gaps and overlaps. The packing volume approach requires structuring a three-dimensional array of cubes (Battista & Clements, 1998). Therefore, meaningfully enumerating cubes is fundamental for becoming competent at volume measurement (Battista, 2004). In order to enumerate cubes in a three-dimensional array, students must have spatial reasoning skills (Fujita et al., 2020). Spatial reasoning, which includes both spatial structuring and spatial visualization, is related to mentally building and manipulating objects, decomposing them into parts, and establishing relationships between the parts (Battista et al., 2018). Low spatial reasoning skills may yield to counting only visible cubes and ignoring the invisible cubes (Battista & Clements, 1996). In a study conducted by Ben-Chaim et al. (1988), it was reported that when students were asked to find the volume of a rectangular prism, they either counted the number of visible cubes or the visible faces of cubes shown in the diagram. This situation can be an indicator of students' poor spatial visualization skills. Spatial visualization entails the mental creation and manipulation of images of objects to enable reasoning about objects (Fujita et al., 2020). Similarly, Ben-Haim et al. (1985) examined how students visualized arrays of cubes. The researchers documented that students counted the visible faces by disregarding the three-dimensional nature of objects. That is, students related cubes to the faces, which resulted in double counting the cubes at the corners.

In order to eliminate students' misconceptions and errors and to enhance their understanding of geometric measurement, pre-service teachers should be able to notice students' thinking, and they should also know the reasons underlying these misconceptions and difficulties and how to overcome them before they begin their teaching profession. Therefore, the investigation of pre-service teachers' professional noticing of students' mathematical thinking is significant. The part below reviews the literature related to studies about teachers' noticing in the content domain of geometry and measurement.

2.3 Studies about Teacher Noticing in the Content Domain of Geometry and Measurement

A search of the literature revealed few studies that explored pre-service or in-service teachers' professional noticing of students' mathematical thinking in geometry and measurement. One study with in-service teachers by Haj-Yahya (2022) examined how teachers' noticing abilities changed when they were exposed to both theoretical and empirical information about pedagogical aspects of geometric thinking. Forty-one in-service teachers first solved two tasks on geometry. Secondly, they were presented with empirical research on geometric concepts and asked to solve the tasks in the articles. Thirdly, they watched a lesson video and responded to three questions corresponding to each professional noticing component. In order to collect the data, a questionnaire with open-ended questions was implemented to the teachers. Findings showed that engaging teachers in theoretical and empirical articles affected their professional noticing skills in a positive manner. Moreover, after the intervention, teachers focused more on the specific difficulties in the tasks, made specific interpretations regarding these difficulties, and provided specific responses to overcome these difficulties.

In another study conducted with in-service teachers, Dışbudak Kuru et al. (2022) specifically examined middle school mathematics teachers' professional noticing skills in volume measurement. For this reason, the researchers presented a problem on the volume of the rectangular prism involving a student's correct solution strategy to 35 teachers, and they were asked to respond to attending, interpreting, and deciding prompts in a written task. Findings revealed that teachers were better at attending to students' strategies than interpreting students' understanding and deciding how to respond based on the student's understanding. Therefore, the researchers concluded that teachers' interpreting and deciding skills need to be developed.

Different from the previous ones, the following research studies involved pre-service teachers as participants. A qualitative study by Baldinger (2020) examined pre-service secondary teachers' reasoning about students' written solutions. Eight pre-service teachers solved two high school mathematical tasks; one was about algebra, and the other was about geometry, and then, they analyzed students' solutions to the same tasks regarding students' understanding in task-based interviews. Findings indicated the use of three reasoning strategies by the pre-service teachers: mathematical reasoning, pedagogical reasoning, and reasoning through self-comparison. The researcher particularly emphasized self-comparison in which the pre-service teachers compared their own solutions with the students' solutions and highlighted the use of this strategy to notice students' mathematical thinking in written work.

Guner and Akyuz (2020) designed a professional development model and examined how a pre-service teacher noticed students' mathematical thinking in the context of a lesson study in geometry. In the study, a pre-service teacher engaged in four lesson study cycles consisting of planning, teaching, and reflecting phases. Findings indicated that the pre-service teacher's noticing was at Level 1 and Level 2 in the first and second cycles, but she started to notice at Level 2 and Level 3 in the third and fourth cycles. Collaboration between the group members, the pre-service teacher's observation of and reflection on the lessons, feedback of the classroom teacher, and focus on the particular related topics led the pre-service teacher to notice at higher levels. Thus, the researchers concluded that participating in a lesson study was effective in improving the pre-service teacher's noticing of students' mathematical thinking.

In another study, Ulusoy and Çakıroğlu (2021) investigated pre-service teachers' noticing of students' understanding through micro-case video design in the context of geometry. Micro-case videos were produced by the researchers through interviews with seventh-grade students and included students' responses to questions about definitions, properties, identification, and drawing of quadrilaterals, as well as hierarchical relations between the quadrilaterals. Eight pre-service teachers individually wrote reflection papers regarding their attending of students' understanding of trapezoids in the micro-case video, interpretations of students' understanding, and instructional suggestions based on students' understanding. Then, the pre-service teachers participated in group discussions. The same procedure was followed for the second micro-case video. After the discussions were completed, they individually wrote post-discussion reflection papers in which they offered instructional suggestions. Findings showed that the pre-service teachers had difficulties in responding in the individual analysis since their suggestions were general, i.e., they were not based on students' understanding in the videos. In addition, even though they could identify mathematical elements in the students' solutions, they could not use these while interpreting students' understanding. However, after the group discussions, there was an improvement in pre-service teachers' professional noticing of students' understanding.

By focusing on pre-service kindergarten teachers, Moreno et al. (2021) explored the development of pre-service kindergarten teachers' professional noticing of students' mathematical thinking in length measurement. They also examined how the pre-service teachers used a learning trajectory to learn to notice students' mathematical thinking. Forty-seven pre-service teachers took part in a teaching experiment consisting of five sessions, with each session being 100 min. The teaching experiment was based on learning and teaching length measurement between the ages of three and six years old. During the sessions, pre-service teachers analyzed the videos and narratives, worked in small groups, and discussed as a whole class. Information regarding the learning trajectory, including length magnitude and measurement (Sarama & Clements, 2009), was provided to them. Findings revealed that the pre-service teachers utilized the learning trajectory in three ways while noticing teaching situations. This resulted in five changes in the development of pre-service teachers' noticing. For instance, the pre-service teachers who could not identify any mathematical elements started to identify some elements and interpret some students' mathematical thinking using these at the end of the teaching experiment (change 1). Or, the pre-service teachers who could identify some mathematical elements, interpret some students' mathematical thinking using these elements, and suggest tasks for these students started to identify all elements, interpret all students' mathematical thinking, and offer tasks for all students to progress in the learning. Thus, the researchers recommended using the learning trajectory as a tool in teacher education programs to develop pre-service teachers' noticing skills.

Caylan Ergene and Isiksal Bostan (2022) particularly focused on the levels-of-sophistication framework and investigated the role of that framework in the improvement of pre-service teachers' professional noticing skills in length measurement, mainly perimeter. The researchers designed tasks on perimeter measurement, and each task involved two different student solutions reflecting different levels of reasoning. At the beginning of the study, to identify the initial professional noticing skills, three pre-service teachers analyzed students' solutions to the tasks and responded to the noticing prompts regarding attending, interpreting, and deciding how to respond in the pre-interviews. Then, they participated in an intervention consisting of four sessions based on the levels-of-sophistication framework (Battista, 2006). Information about the foundational concepts of length measurement and the conceptual framework was given during the sessions. The pre-service teachers also analyzed students’ solutions involving non-measurement and measurement reasoning individually, and then they discussed students’ strategies and understanding. After the intervention was completed, to ascertain the final professional noticing skills, post-interviews were conducted by following a procedure similar to the pre-interview. Findings showed that the pre-service teachers were in a better position for attending, interpreting, and deciding how to respond skills in the post-interviews. This reveals the potential of intervention based on the levels-of-sophistication framework to improve pre-service teachers’ professional noticing skills.

A recent study by Girit Yildiz et al. (2023) involved area and volume measurement in addition to length measurement and explored preservice teachers’ noticing of students’ misconceptions in measurement in a video-case-based professional development environment. By acting as teachers and students, the researchers produced ten video cases, including students’ misconceptions. Thirty pre-service teachers, after viewing the video cases, discussed misconceptions, the causes of the misconceptions, and how to eliminate these misconceptions as a class. Following this, they also wrote their suggestions for remedying the misconceptions individually. The findings revealed that the professional development environment helped the pre-service teachers identify, interpret, and make suggestions for the misconceptions. They could describe the misconceptions and provide suggestions based on the conceptual understanding of students. However, they experienced difficulties in interpreting students’ understanding and providing robust evidence for interpreting.

In summary, it has been shown from the review of studies about teachers’ noticing in the content domain of geometry and measurement that a limited number of research studies examined particularly pre-service teachers’ professional noticing of students’ mathematical thinking in geometric measurement. Additionally, research shows that it is very important for students to be involved with concrete models on a one-to-one basis. However, the literature review reveals that the studies on teacher noticing in the content domain of geometry and measurement did not include concrete models. These models are pivotal in providing pre-service teachers with detailed insight into students' ways of thinking. Therefore, this study incorporates concrete models to explore pre-service teachers' professional noticing of students' mathematical thinking while students were engaged in tasks using concrete models in geometric measurement. Thus, the next part delves into the literature related to concrete models.

2.4 Concrete Models

The use of concrete models is one of the strategies used to represent mathematical ideas, and its benefit to the conceptual understanding of students in mathematics is acknowledged (Moyer, 2001a). Concrete models were defined by researchers differently in the literature. According to Moyer (2001a), concrete models are objects designed to represent abstract mathematical concepts concretely. Uttal et al. (1997) highlighted that concrete models are particularly designed for students learning mathematics. Hynes (1986) asserted that concrete models "incorporate mathematical concepts, appeal to several senses, and can be touched and moved around by students" (p. 11). Similarly, Yeatts (1991) defined concrete models as objects that can be rearranged and moved by students. Perry and Howard (1997) described concrete models as objects that are used along with hands-on activities by students for visual exploration. On the other hand, according to Van de Walle et al. (2010), concrete models can be in the form of object, picture, or drawing that is utilized to represent abstract mathematical concepts. Marshall and Swan (2008) asserted that concrete models could be structured, such as based-ten blocks, algebra tiles, pattern blocks, and geoboards, or unstructured, such as popsicle sticks, buttons, and paper folding. In the present study, concrete models involve both structured and unstructured physical objects.

Research showed that using concrete models in class rather than only abstract symbols leads to more success, especially when combined with proper classroom strategies (Carbonneau et al., 2013; Sarama & Clements, 2016). In addition to achievement, there are many benefits of using concrete models for students while learning mathematics. Concrete models help students establish a connection between the concrete environment and abstract mathematics (Yeatts, 1991) and relationships between mathematical concepts (Balka, 1993; Nevin, 1993). Visualizing mathematical concepts, concrete models yield an understanding of the mathematical concepts (Crawford & Brown, 2003). Moreover, by providing students with experience in visualizing and organizing information, using concrete models supports the development of students' analytical and spatial thinking skills (Balka, 1993; Heddens, 1997). In this way, concrete models enhance students' conceptual understanding of mathematical concepts by allowing them to have meaningful experiences (Silver et al., 2009). In addition, students become active in their learning process by engaging them in mathematical tasks through concrete experiences (Karol, 1991). Concrete models also enable teachers to understand what students think (Skemp, 1989). Observing students while they are interacting with concrete models is an effective way to determine whether learning is happening.

Concrete models can have a critical role, but they must be used cautiously to build a strong understanding (Thompson & Thompson, 1990). Research indicated that using concrete models does not engender better learning all the time (Stein & Bovalino, 2001). That is, students' use of concrete models does not guarantee student learning because "students may hold, move, and arrange physical objects without thinking about the concepts" (Sarama & Clements, 2016, p. 74). Therefore, time should be given to students to examine and become familiar with concrete models in order to prevent them from focusing on the model itself. In addition, even though concrete models provide support and have a significant role in learning, they do not directly convey mathematical ideas to students (Sarama & Clements, 2016). In order to promote learning by relating the concrete models to abstract symbols, teachers are required to consider the type of concrete models and when and how to use them

(Simon, 2022). Hence, students should be encouraged toward a specific goal and to focus on the mathematical concept (Marshall & Swan, 2008), and they should reflect on their actions with models to create meaning (Sarama & Clements, 2016). Sowell (1989) emphasized the long-term use of the models and the supervision of teachers who are knowledgeable about the use of the models for the increase in students' mathematics achievement. Concrete models are useful if appropriately utilized by teachers who know how to use them to encourage students to construct their own mathematical knowledge (McNeil & Jarvin, 2007). The incorporation of concrete models into lessons does not simply improve student learning but how teachers use them to represent or illustrate mathematical concepts (Quigley, 2021). Therefore, it is necessary for teachers to have pedagogical content knowledge to increase the possibility of students' understanding of the new concept (Carpenter et al., 1996).

Thus, the use of concrete models makes students more active while engaging in tasks. In this way, students can provide more information about their thinking by using concrete models. This situation can facilitate pre-service teachers' noticing of students' mathematical thinking and also enable pre-service teachers to explain this thinking better. In this sense, deliberately selecting and utilizing classroom artifacts that showcase instances of students' mathematical thinking can greatly aid pre-service teachers in recognizing and understanding students' mathematical thinking. Through individual and collaborative evaluation of students' thinking, pre-service teachers can gain valuable insights and reflect on students' mathematical thinking. It is also crucial for pre-service teachers to actively put into practice the concepts and ideas they have learned through this process and to reflect on their own practice. By integrating these strategies, pre-service teachers can effectively develop their ability to notice students' mathematical thinking. In this regard, the pedagogies of practice framework developed by Grossmann et al. (2009) can provide a context for a systematic and deliberate way of teaching pre-service teachers professional noticing by focusing their attention on students' thinking. Providing pre-service teachers with opportunities to notice students' mathematical thinking by analyzing video clips of students' task-based interviews and engaging in this practice guided by this framework can develop their professional noticing skills. Accordingly, the next part describes the pedagogies of practice used in the present study as a pedagogical framework.

2.5 Pedagogies of Practice

Practice-based approaches see teaching both as a resource for and an essential element of learning to teach (McDonald et al., 2014). Grossman et al. (2009) offered a framework for the teaching of practice in the university context. The framework consists of three key concepts, which are representations, decompositions, and approximations for understanding the pedagogies of practice. Grossman et al. (2009) asserted that these concepts are helpful while teaching complex practices to novices, and there should be more opportunities for pre-service teachers in these areas to enable them to provide effective teaching in the future.

Representation is making practice that is being learned visible to novices (Grossman et al., 2009). Videos of instruction that illustrate the teaching of particular concepts, videos of interviews with students that demonstrate students' mathematical strategies, examples of lesson plans, unit plans or examples of curriculum materials, classroom observations, or written cases of students' work are examples of representations of practice in mathematics education (Tyminski et al., 2014). That is, representations show the practice that is being learned and can be in various formats, such as videos, vignettes, or other recordings of that practice. Representations "provide novices with opportunities to develop ways of seeing and understanding professional practice" (Grossman et al., 2009, p. 2065). Thus, through representations of practice, pre-service teachers can see cases of the practice they are preparing, and teacher educators can support pre-service teachers in comprehending elements of practice (Grossman et al., 2009). Grossman et al. (2009) stressed that since representations and approximations may rarely catch up with the entire practice, there is a need for the decomposition of practice to be incorporated by instructors.

Decomposition is breaking down practice into meaningful components for teaching and learning purposes (Grossman et al., 2009). Grossman et al. (2009) argued that providing opportunities to focus and deal with individual parts of the practice is fruitful because it enables novices to distinguish and understand divided components before incorporating them into complex practice. Decomposition implies that "part of the work of professional education lies in identifying components that are integral to practice and that can be improved through targeted instruction" (Grossman et al., 2009, p. 2069). While pre-service teachers are learning a complex and new practice, the decomposition of practice can promote their learning experiences. In the studies conducted with pre-service teachers, researchers used decompositions of practice with different frameworks in order to "characterize aspects of the practice such as orchestrating conversations (Smith & Stein, 2011), noticing student learning (van Es, 2011), or selecting tasks (Stein et al., 2000)" (Sztajn et al., 2020, p. 2). These studies suggested that using decompositions for designing and analyzing teacher education and combining decompositions with frameworks can promote the learning of such practices (Sztajn et al., 2020).

Class discussions of videos and annotation technologies that allow interactive "markup and comment" on videos and animations can be given as examples for the decomposition of practice. Particularly, using videos to decompose a practice in teacher education can facilitate pre-service teachers' adaptation to the complexities of students' thinking, competencies of students in learning mathematics, and multiple mathematical knowledge bases of students (Star & Strickland, 2008; Stockero et al., 2015). In this way, participating in classroom discussions based on videos can help pre-service teachers attend to student thinking and plan their future lessons on eliciting and responding to student thinking (Calandra & Rich, 2015). Although representations and decomposition practice are partially effective, they are not enough to make pre-service teachers ready to perform the teaching profession (Lampert et al., 2010, 2013) and hence, combining representations and decompositions with approximations of practice results in learning to do the work of teaching in addition to learning about the work of teachers (Grossman et al., 2009).

Approximation of practice is simulating the practice that is “related, but not identical, to the work of practicing professionals” (Grossman, 2011, p. 2840). Microteaching, role-plays, student teaching, rehearsal, and unit and lesson planning can be approximations of practice. Representations show practice, while approximations engage novices in these practices. The difference between representation and approximation depends on the role of novices as observers or actors. In addition, approximations that do not retain actual practices’ complexity are considered less authentic (Grossman et al., 2009). Research showed that simulations and role plays were mainly used as approximations of practice. In these studies, pre-service teachers generally implemented lessons in a virtual environment or classroom by regarding virtual characters or their peers as students. However, pre-service teachers cannot experience interactions with students through these approximations (Sapkota & Max, 2021). Virtual environments are accepted as less authentic since they do not maintain the complexity of teaching (Janssen et al., 2015). In addition, pre-service teachers in animated classrooms have few opportunities to make in-the-moment decisions and respond to real students in real environments (de Araujo et al., 2015). On the other hand, approximations that engage pre-service teachers in tasks similar to those implemented in school settings are considered more authentic. Therefore, in the present study, practicing task-based interviews with students as an approximation may provide pre-service teachers with opportunities to predict and respond to students’ thinking. This leads to more authentic practice by increasing complexity and student interactions.

Pedagogies of practice offer opportunities for pre-service teachers to learn knowledge of content and students, particular techniques to manage teaching, and the complexities of the work of teaching (Ghousseini & Herbst, 2016). Integrating pedagogies of practice into courses in teacher education programs enables pre-service teachers to find out and develop teaching practices and start noticing significant aspects of these practices (Arbaugh et al., 2021). Research indicated that pedagogies of practice were used to improve various teaching practices, including curriculum enactment (e.g., Earnest & Amador, 2019); leading classroom discussions (e.g., Baldinger et al., 2016; Ghousseini & Herbst, 2016; Smith & Stein, 2011; Tyminski et al., 2014); analyzing and reflecting on teaching (Lampert, 2013) and professional noticing (e.g., Estapa et al., 2018; Schack et al., 2013). For example, Earnest and Amador (2019) found that planning and enacting a lesson through simulation was helpful for pre-service teachers to learn how to utilize the curriculum to design instruction. Tyminski et al. (2014) noted that pedagogies of practice helped pre-service teachers develop teaching practices to orchestrate mathematical discussions. In a similar vein, Ghousseini and Herbst (2016) found that pedagogies of practice supported pre-service teachers in finding out how to lead mathematical discussions. Arbaugh et al. (2021) reported that engaging in pedagogies of practice broadened pre-service teachers' visions of mathematics instruction. Research indicated the positive impacts of pedagogies on practice not only on pre-service teachers but also on novice teachers. To illustrate, Lampert et al. (2013) found that during rehearsals, novice teachers learned specific features of teaching, including eliciting and responding to students' performance. Baldinger et al. (2016) reported that pedagogies of practice were effective on novice teachers' skills regarding managing discussions toward a mathematical point as well as eliciting and using student thinking.

Pedagogies of practice not only bring about change in pre-service teachers' professional vision and help them to learn teaching practices but also enable them to master professional noticing. In the literature, a few studies examined pre-service teachers' professional noticing in the context of pedagogies of practice. For instance, Schack et al. (2013) examined the development of pre-service elementary teachers' professional noticing of children's early numeracy through a module they developed. In the study, researchers presented video excerpts of diagnostic interviews of children as representations of practice, discussions around video clips about attending, interpreting, and deciding about the next instructional steps acted as a decomposition of practice, and conducting at least one diagnostic interview with a child acted as an approximation of practice. The researchers used nested levels of decomposition offered by Boerst et al. (2011) in a module for pre-service primary teachers to decompose the practice of professional noticing of children's mathematical thinking. They emphasized that decomposing professional noticing into three components, i.e., attending, interpreting, and deciding how to respond, and focusing on each in a progressively nested manner in the module improved pre-service teachers' professional noticing skills. In another study, Estapa et al. (2018) investigated the professional noticing of pre-service teachers within and across pedagogies of practice. Researchers used video medium as a representation, written medium (selecting one moment from the video and explaining this selection in writing) as a decomposition, and animated medium (creation of animation) as an approximation of practice. The findings of the study indicated that the pedagogies of practice was effective in developing the professional noticing skills of pre-service teachers because they learned how to attend, interpret, and respond to student thinking.

Video clips that emphasize student thinking have the potential to enhance pre-service teachers' professional noticing skills by providing focused insights into students' particular ways of thinking about mathematical concepts. Moreover, the absence of distractions unrelated to mathematics and teaching within these video clips may facilitate pre-service teachers' concentration on students' mathematical thinking, allowing researchers to delve deeply into their professional noticing abilities. Deliberately producing video clips that spotlight various students' thinking through task-based interviews with middle school students and utilizing them as representations of practice rather than videos of entire lessons, can effectively unveil students' mathematical thinking processes through their actions. This approach, in turn, can significantly bolster pre-service teachers' professional noticing skills. Therefore, in the present study, video clips produced by the researcher through task-based interviews with middle school students were used as representations of practice. For this reason, in the part that follows, a review of the literature related to the use of videos as a representation of practice is presented.

2.5.1 Use of videos as representations of practice

In the research studies conducted to examine pre-service teachers' professional noticing skills, the researchers used different artifacts, including students' written work (Callejo & Zapatera, 2017; Ivars et al., 2020; Monson et al., 2020; Sánchez-Matamoros et al., 2015; Shin, 2020), lesson video clips (Kosko et al., 2020; Stockero et al., 2017b; Ulusoy, 2020; Warshauer et al., 2019), video excerpts of clinical interviews (Fisher et al., 2018; Schack et al., 2013; Ulusoy & Çakıroğlu, 2020), or both student work and video clips (Fernández et al., 2012; Jacobs et al., 2010). Considering the insufficient pedagogical experience of pre-service teachers, support is necessary for pre-service teachers to notice students' thinking (Jacobs et al., 2010; Pang, 2011). With the increasing accessibility and commonness of video technology, videos have been used in professional development to improve pre-service and in-service teachers' noticing skills (Barnhart & van Es, 2015; McDuffie et al., 2014; Roller, 2016; Santagata, 2011; Star & Strickland, 2008; Star et al., 2011; Stockero et al., 2017b; van Es, 2011; van Es & Sherin, 2002; van Es & Sherin, 2008; Walkoe, 2015; Watkins & Portsmore, 2021). Videos were used in two formats in the research studies, including whole lesson videos or video clips. Among the lesson videos, videos were presented to portray teachers' classroom practices or teachers' own videos. On the other hand, video clips involved only students who solved mathematical problems (Santagata et al., 2021). Furthermore, videos can focus on various mathematical contents. To illustrate, in the study conducted by Bruckmaier et al. (2016), videos on inequalities were viewed and analyzed by secondary mathematics teachers. In the video club, Walkoe (2015) provided videos on algebra to examine teachers' noticing of students' algebraic thinking. Thus, in professional development programs, video excerpts can be selected to deal with specific characteristics of teaching and learning that are desired to be examined. Researchers ascertained significant changes over time, both in the topics that teachers discussed and in the way they were handled in the video clubs. For instance, there was a shift from attention to teachers' actions to students' mathematical thinking. Moreover, teachers started to analyze students' thinking in detail and linked the discussions with student thinking (Borko et al., 2008).

Videos can be used as a tool to nurture teachers' learning since the real-time aspect of classroom teaching is maintained (Brophy, 2004). In this sense, video cases can be more favorable to enhancing the knowledge and skills of teachers in terms of observation of teaching and understanding the student's mathematical thinking process compared to written cases (Fukkink et al., 2011). Reflecting on videos enables teachers to analyze instruction in a classroom context, depicting the richness of instruction (Santagata et al., 2021). Moreover, they allow teachers to examine instructional interactions several times and for various purposes closely (Larison et al., 2022) and to focus on student ideas and teaching actions (Sherin & Han, 2004). Pausing and re-watching particular video segments permit teachers to understand better students' solution strategies and mathematical thinking (Calandra et al., 2009). This feature makes it possible to think about students' solutions by providing time without requiring immediate responses (Sherin & Han, 2004). In addition, videos enable teachers to see students physically working on the problem (Gonzalez & Skultety, 2018) and to reflect, analyze, and think about alternative approaches in a common context (Borko et al., 2008). Through video-based professional development programs, teachers can develop different types of knowledge that can be applied during instruction (Kersting et al., 2010). Research revealed that teachers who analyzed student thinking in the video were better at responding to students' thinking during instruction (Cohen, 2004).

van Es et al. (2017) reported that as pre-service teachers viewed and analyzed more videos, they paid more attention to students' mathematical thinking by justifying their interpretations. Particularly, specific interventions combined with video activities bring pre-service teachers an opportunity to utilize their knowledge in learning environments and improve their professional noticing skills. Teacher educators can produce video excerpts addressing specific aspects of learning and teaching. Pausing, replaying, or manipulating videos to focus conversations on these aspects assists collaborative learning. Through analysis of videos, pre-service teachers can focus on student thinking and interact with student learning (Friel & Carboni, 2000). Roller (2016) emphasized that videos provide pre-service teachers with the opportunity to have professional conversations regarding teaching with university instructors. Incorporating video discussion environments in teacher education programs enables pre-service teachers to encounter the realities of teaching and develop noticing skills (Johnson & Cotterman, 2015). Moreover, discussing videos with their peers promotes collaborative learning and analytical skills in various methods and techniques (Borko et al., 2008). Pre-service teachers listen and respond to their peers' suggestions, comments, and opinions, and they can think about opposite ideas during discussions (Sherin & Han, 2004). Thus, they can gain new ideas regarding responding based on students' thinking (Walkoe, 2015).

However, watching videos does not always provide teachers with new insight regarding practice (Brophy, 2004). That is, videos alone cannot be sufficient for teachers' learning to notice. Therefore, videos should be watched with the aim in mind to be useful in teacher education, and videos should be guided and scaffolded to achieve specific learning goals (Erickson, 2007). While used in professional development programs, video clips should be deliberately chosen to meet particular goals (e.g., developing teachers' knowledge and improving their skills to notice students' incorrect solutions). They should be integrated into activities designed to support teachers' progress (Seidel et al., 2005). In brief, teacher learning can be promoted by intentionally determining and choosing prompts, video clips, and tools (Kang & van Es, 2019).

Teachers' responses to video analysis provide researchers with information regarding how teachers think (Star et al., 2011). In order to support learning to notice, structured frameworks, including prompts to guide teachers' video analysis or open-ended prompts, were used in the research studies. The frameworks were either based on research about students' learning of particular mathematical concepts (Fisher et al., 2018) or helped teachers engage in details of interactions (Walkoe & Levin, 2018). In some research studies (van Es & Sherin, 2008), open-ended prompts (e.g., What do you notice?) were also preferred. van Es et al. (2014) suggested facilitation moves consisting of orienting the group to the video analysis task, sustaining an inquiry stance, maintaining a focus on the video and the mathematics, and supporting group collaboration. It is asserted that using these moves can help teachers analyze and discuss what they notice in videos. Prompts that direct attention while watching videos can be helpful to pre-service teachers in reducing overload (Schworm & Renkl, 2007). Estapa and Amador (2023) stressed the relationship between the video, the prompts that are asked, and the learning outcome. Asking a specific prompt after watching the video directs attention to a particular aspect of the video; hence, noticing is related to that particular aspect. In other words, asking for specific prompts will likely result in more specific noticing. Using specific prompts does not always guarantee the development of noticing, but they give researchers an idea about how to assess noticing. In this way, researchers can deduce noticing levels or development in noticing.

Research revealed that teacher noticing was measured through materials from instructional practice, often with video clips along with written prompts (e.g., Jacobs et al., 2010; Kaiser et al., 2015). By viewing video clips that represent instructional practice, teachers are involved in cognitive processes that are likely to be encountered in their own instruction (Weyers et al., 2023a). Designing videos and prompts that point to students' mathematical thinking on specific mathematical domains can promote teachers' noticing skills. In addition, videos that include only students encourage teachers to comment on the details of students' mathematical thinking (Amador et al., 2022). Thus, in this study, by deliberately producing video clips on geometric measurement through individual task-based interviews with students and using them as representations of practice, it was attempted to improve pre-service teachers' professional noticing of students' mathematical thinking. Representations of practice show the practice, while approximations of practice engage novices in that practice. Approximations such as task-based student interviews, which retain complexity and authenticity by involving pre-service teachers in activities akin to those encountered in real school settings, gain prominence. In this respect, the inclusion of these practices provides pre-service teachers with opportunities to elicit students' mathematical thinking and to utilize evidence of their thinking processes. By engaging pre-service teachers in interactions that prompt them to respond to students' thinking through pre-designed tasks, these approximations of practice play a crucial role in fostering particularly their deciding how to respond skills. Hence, task-based student interviews as approximations of practice are provided in the following part.

2.5.2 Task-based student interviews as an approximation of practice

Several research studies examined the effect of one-to-one student interviews on in-service teachers' noticing of student thinking (Buschman, 2001; Heng & Sudarshan, 2013; Jacobs et al., 2006; Kose, 2021) or used to improve pre-service teachers' noticing skills (Krupa et al., 2017; Lesseig et al., 2016). Buschman (2001) examined the impact of conducting student interviews on teachers' understanding of students' mathematical thinking. A group of elementary in-service teachers took part in workshops about how to conduct clinical interviews, and then they conducted one-to-one interviews with the students. These helped teachers modify the instruction based on the students' needs and enable students to explain and discuss the strategies they used while solving the problems. It was concluded that the interviews assisted teachers in adopting a more student-centered teaching approach. Similarly, Jacobs et al. (2006) investigated how the use of video recordings of student interviews conducted by teachers facilitates discussions. Each of the eighteen elementary in-service teachers conducted interviews with students who were working on a problem and then selected and brought one of the videos to discuss with other teachers. During the discussions, teachers developed ideas regarding the students' understanding and the next instructional step. In this way, they left focusing on the (in)correctness of solutions and could realize different student strategies. Moreover, Heng and Sudarshan (2013) attempted to gain clinical interviewing skills to understand students' mathematical thinking. The researchers and teachers worked together to design task-based interviews. Teachers conducted task-based interviews in class with both individual and small groups of students, during which students solved mathematical problems. After completing this, pre- and post-project interviews were conducted with the teachers to reveal their pedagogical content knowledge, beliefs, and views about teaching mathematics and the use of clinical interviews. Researchers found that understanding student thinking through clinical interviews enabled teachers to plan for better teaching. More recently, Kose (2021) designed a professional development program to improve teachers' noticing of student thinking. With this aim, the researcher analyzed three middle school teachers' responses to noticing prompts and how they discussed the chosen videos of interviews during the sessions. The findings revealed that teachers were able to attend to student thinking and interpret student understanding more extensively over the course of time. However, teachers could not show the desired improvement in responding to students' mathematical thinking component of professional noticing.

Rather than the in-service teachers, Krupa et al. (2017) explored secondary pre-service teachers' professional noticing of student mathematical thinking through a model focused on student interviews. Thirty-two pre-service teachers viewed videos on student thinking by providing answers to the noticing prompts. Afterward, they conducted interviews with a secondary school student using the interview protocol they had prepared beforehand. They analyzed students' solutions based on the rubrics provided and reflected on the interviews. Following this, post-assessment, which required viewing a video and giving answers to the questions corresponding to the noticing prompts, was implemented. Findings showed that there was an improvement in the pre-service teachers' attending to students' thinking and interpreting students' understanding, but the change was not observed in their deciding how to respond skills.

In view of all that has been mentioned so far, one may suppose that task-based student interviews have been used to enable teachers to assess student understanding in-depth. Conducting task-based student interviews provides pre-service teachers with a powerful tool for gaining insight into student thinking. By engaging students in these interviews, pre-service teachers can uncover misconceptions, gaps in understanding, and effective strategies used by students to solve the tasks. Through these interviews, teachers can refine their questioning techniques, improve their ability to elicit student thinking, and develop strategies for responding effectively to student thinking. Therefore, in this study, pre-service teachers conducted individual task-based interviews with middle school students as approximations of practice. In this way, they had a chance to elicit students' mathematical thinking, interpret students' understanding, and decide how to respond based on students' understanding.

2.6 Summary of literature review

Teacher noticing, which is one of the key components of teachers' professional competence (Scheiner, 2016), is crucial for effective mathematics instruction (Sherin et al., 2011). The literature on teacher noticing highlighted several frameworks (Jacobs et al., 2010; van Es, 2011; van Es & Sherin, 2002; van Es & Sherin, 2021). The learning to notice framework developed by Van Es and Sherin (2002) involves "identifying what is important or noteworthy about a classroom situation, making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent, and using what one knows about the context to reason about classroom interactions" (p. 573). Later, van Es (2011) proposed a framework for learning to notice student thinking. This framework consists of two dimensions and four levels. This framework differed from the previous one in terms of revealing the improvement in teachers' noticing skills of student thinking as teachers progress from lower to higher levels. More recently, van Es and Sherin (2021) revised the framework by elaborating the construct of teacher noticing. The researchers expanded the attending and interpreting components in the previous framework and proposed a new component called shaping. In brief, the learning to notice framework (van Es, 2011; van Es & Sherin, 2002; van Es & Sherin, 2021) focused on the diversity of what teachers notice and how teachers notice. On the other hand, the professional noticing of children's mathematical thinking framework developed by Jacobs et al. (2010) focuses specifically on children's mathematical thinking by giving less attention to the diversity of what teachers notice and how teachers notice. The professional noticing of children's mathematical thinking framework comprises three skills: attending to children's strategies, interpreting children's understanding, and deciding how to respond based on children's understanding (Jacobs et al., 2010). Since the aim of the present study is to explore the extent to which pre-service teachers notice students' mathematical thinking in geometric measurement and to develop pre-service teachers' noticing of students' mathematical thinking in this context, the professional noticing of children's mathematical thinking framework is adopted.

Unfortunately, pre-service teachers experience difficulties in noticing students' mathematical thinking (Jacobs et al., 2010). However, pre-service teachers can develop their noticing skills if opportunities are provided regarding noticing in teacher education programs (Ivars et al., 2020; Sánchez-Matamoros et al., 2015; Schack et al., 2013). Thus, improving pre-service teachers' professional noticing skills can support them to deliver better instruction at the beginning of their profession. As one of the content domains, measurement is important for understanding the construction of shapes, determining objects' sizes, utilizing the coordinate system to ascertain locations, and identifying transformations (Battista, 2007). However, students experience misconceptions and difficulties in geometric measurement, which involves length, area, and volume measurement (e.g., Barrett et al., 2017; Curry et al., 2006; Martin & Strutchens, 2000; Tan-Sisman & Aksu, 2016). Accordingly, considering both the importance of geometric measurement and students' misconceptions and difficulties in geometric measurement, exploring and developing pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement is important. In addition, students become active in their learning process by engaging them in mathematical tasks through concrete experiences (Karol, 1991), and concrete models also enable teachers to understand what students think (Skemp, 1989). Thus, students' use of concrete models while engaging in tasks can provide more information about students' mathematical thinking, which may facilitate pre-service teachers' noticing of students' mathematical thinking and also enable pre-service teachers to explain that thinking better. Therefore, this study incorporates concrete models to explore pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement.

Grossman et al. (2009) asserted that "for pre-service teachers to learn to engage in the complex practice, they may need opportunities first to distinguish and then, to practice, the different components that go into professional work prior to integrating them fully" (p. 2068). Accordingly, the pedagogies of practice framework developed by Grossmann et al. (2009) can provide a context for a systematic and deliberate way of teaching pre-service teachers professional noticing by focusing their attention on student thinking. The framework consists of three key concepts, which are representations, decompositions, and approximations for understanding the pedagogies of practice. Representation is making practice that is being learned visible to novices (Grossman et al., 2009). Video clips focusing on student thinking can potentially develop teachers' professional noticing skills because they help zoom in on students' particular thinking about mathematical concepts. In addition, through video clips of different students, pre-service teachers can realize how mathematical concepts are understood differently and compare different students' thinking (Jacobs et al., 2010). Decomposition is breaking down practice into meaningful components for teaching and learning purposes (Grossman et al., 2009). Analysis of video clips and discussions around attending to students' mathematical thinking, interpreting students' understanding, and deciding how to respond, which are the components of professional noticing of students' mathematical thinking, can help pre-service teachers learn the professional noticing practice and improve their professional noticing skills. Approximation of practice is simulating the practice that is "related, but not identical, to the work of practicing professionals" (Grossman, 2011, p. 2840). Clinical interviews, as approximations of practice, can be effective tools for uncovering students' mathematical thinking and promoting teachers' skills of eliciting and responding to student thinking (Heng & Sudarshan, 2013; Jacobs et al., 2006; McDonough et al., 2002). Particularly, student interviews around mathematical tasks preserve the complexity of teaching and lead to more authentic practice. In this way, pre-service teachers can experience interactions with students engaging in approximations of practice. Hence, the present study aims to develop pre-service teachers' professional noticing skills through the video-based module situated in the pedagogies of practice framework. This module enables pre-service teachers to learn how to notice students' mathematical thinking by viewing video clips that illustrate students' mathematical thinking and breaking down the noticing practice and video clips both individually and collaboratively. Additionally, the module offers additional opportunities for practice by incorporating interactive student-teacher engagements that go beyond what is depicted solely in the video clips.

CHAPTER 3

METHODOLOGY

The purpose of this study was to examine the extent to which pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement changed when they participated in a video-based module situated in the pedagogies of practice framework. In addition, the study aimed to explore how a video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement. In this regard, in this chapter, the context and participants, data collection tools and procedure, data analysis, the trustworthiness of the study, ethical considerations, and limitations of the study are presented.

3.1 Research Design

In the present study, answers to two research questions were sought. Table 3.1 summarizes the research questions, and corresponding research methods and data collection tools to find answers to these research questions. Details are given below.

Table 3. 1 Research questions, corresponding research methods, and data collection tools

Research Questions	Research Method	Data Collection Tools
1. To what extent do pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement change as they participate in a video-based module situated in the pedagogies of practice framework?	Qualitative	Noticing questionnaire
1.1. Is the change in pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement from pre-test to post-test statistically significant?	Quantitative	Noticing questionnaire
2. How does a video-based module situated in the pedagogies of practice framework support pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement?	Qualitative	Reflection paper 1 Reflection paper 2 Reflection paper 3 Semi-structured interviews Group discussions Whole-class discussions

In order to find answers to the research questions, a mixed methods experimental (intervention) design was used (Creswell & Creswell, 2018). This design enabled the researcher to collect and analyze both quantitative and qualitative data and integrate the information within an intervention. The first aim of the present study was to examine the extent to which pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement changed when they participated in a video-based module situated in the pedagogies of practice framework. With this aim, an Initial Noticing Questionnaire was given to 32 pre-service teachers at the beginning of the study as a pre-test to identify their existing professional noticing skills in geometric measurement. Then, these pre-service teachers participated in an intervention, which is a video-based module situated in the pedagogies of practice framework. At the end of the study, a questionnaire that was identical to the initial one, named the Final Noticing Questionnaire, was given to the same pre-service teachers as a post-test to determine their final professional noticing skills in geometric measurement. Frequency analysis and statistical analysis was then carried out to find out whether the improvement in the final noticing questionnaire observed was statistically significant. Thus, the data obtained from the questionnaires constituted the quantitative data of the study. In addition to the quantitative data, in the present study, the mixed methods intervention design added qualitative data into the intervention so that the qualitative data was the secondary source of data embedded in the experimental pretest-posttest data collection (Sandelowski, 1996). The design implemented the qualitative strand during the experiment, which is named convergent core design (Creswell & Creswell, 2018). The second aim of the present study was to explore how a video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement. That is, the qualitative data were embedded during the experiment to understand how the intervention supported pre-service teachers' professional noticing of students' mathematical thinking. With this aim, the qualitative data were collected at multiple points in time during the intervention. Six pre-service teachers who were in the same discussion group were focused, and the reflection papers individually written by these pre-service teachers in each session as they decomposed the represented practice and in the approximation of practice stage were examined to reveal the improvement in pre-service teachers' professional noticing skills in depth and to track changes on an individual level more consistently. Thus, the data obtained from the reflection papers, semi-structured interviews, and group and whole-class discussions constituted the qualitative data of the study. In this way, by employing various data collection tools, I tried to portray a whole picture of pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement.

3.2 Context of the Study

The context of the study was an elementary mathematics teacher education program, which is a four-year undergraduate program in one of the state universities in Turkey. Pre-service teachers who graduate from this program become mathematics teachers in middle schools (grades 5-8). The program includes mathematics courses (e.g., Linear Algebra, Calculus, and Analytic Geometry), educational courses (e.g., Introduction to Education, Educational Psychology, Educational Sociology), methods of teaching mathematics courses, and elective courses (e.g., general culture). In the first two years, pre-service teachers take most of the mathematics and educational courses. Starting from the third year, they take methods of teaching mathematics courses, including Teaching Geometry and Measurement, Teaching Numbers, Teaching Algebra, and Teaching Probability and Statistics. In the last year of the program, they take Teaching Practice (I and II) courses, and within the context of these courses, they gain teaching experience by doing an internship in schools affiliated with the Ministry of National Education.

3.3 Participants of the Study

Participants of the study were 32 fourth-year (senior) pre-service teachers (22 females and 10 males) enrolled in an elementary mathematics teacher education program at a state university in Sakarya, Turkey. Fourth-year pre-service teachers were preferred since they had already completed methods of teaching mathematics courses before the study. Pretests were given to 32 participants at the beginning of the study. The highest CGPA was 3.59, and the lowest CGPA was 2.88 among the participants. Firstly, they were ordered according to their CGPAs from high to low.

Secondly, they were divided into three clusters. Before the intervention, six discussion groups involving 5 or 6 participants were formed according to their cumulative grade point averages (CGPA) by selecting from the clusters (high, moderate, and low) and according to their genders with the idea that differences in the participants' backgrounds will enrich the discussions. A group of participants among these six groups was focused on for in-depth exploration since the participants in this group were the most willing and active ones during the discussions. These participants were named A, B, E, G, K and S. There were two males (A and E) and four females in this group (B, G, K and S); two had high GPAs (A and B), two had moderate GPAs (G and E), and two had low GPAs (S and K).

3.4 Data Collection Procedure

Figure 3.1 presents an overview the design of the study. As can be seen from the figure, at the beginning of the study, an initial noticing questionnaire was given to 32 participants as a pre-test in order to identify their existing professional noticing skills. After that, all participants were asked to read assigned articles about clinical interviews (Didiş Kabar & Tataroğlu Tasdan, 2019; Hunting, 1997; Özaltun Çelik & Bukova Güzel, 2020) before the class. In the class, the participants and the researcher discussed what a clinical interview is, the purpose of a clinical interview, how to conduct it, and the type of questions that are asked in clinical interviews. Following this, the participants took part in seven intervention sessions on geometric measurement. Before each session, the geometric measurement task in the video clips, i.e., the task of the week, was presented to the participants as an out-of-class assignment, and they were asked to solve the task before the class to become familiar with the task. In each session, before viewing the video clips, the researcher provided some background information about the students in the video clips (e.g., gender and grade level). As *representations of practice,* the participants viewed video clips in which different students solved the same tasks on projection without interruption in order. It is believed that video clips as a representation of practice may provide novices with opportunities to attend and discuss significant features of students' mathematical thinking that can be difficult to notice in a real classroom environment (Schack et al., 2013; Fisher et al., 2018). While viewing the video clips, the participants were encouraged to take notes on what they attended.

As *decompositions of practice*, after viewing the video clips, the participants individually and collaboratively analyzed the video clips, respectively. Firstly, the participants sat on the desks that were separated during the individual analysis of video clips, and they were told that they could use their notes while writing if they wanted to but that they could not talk with each other. In this way, by answering the prompts related to attending, interpreting, and deciding how to respond, they wrote a reflection paper 1. Secondly, group discussions regarding participants' attention features in the video clips, interpretations of students' mathematical thinking, and decisions about the next diagnostic or instructional steps on the basis of students' understanding were held. For the group discussions, the participants in the same group came together to talk to their peers and articulate their ideas. Thirdly, a whole class discussion led by the researcher regarding participants' attention features in the video clips, interpretations of students' mathematical thinking, and decisions about the next diagnostic or instructional steps on the basis of students' understanding were held. Finally, at the end of each session, i.e., after the discussions were completed, the participants were asked to write a reflection paper 2 individually by typing changes in and additions to their reflections after participating in discussions, if any. The same procedure was followed for each session, and the participants viewed, analyzed, and discussed the video clips for 2 hours each week. All intervention sessions were completed in seven consecutive weeks. Sessions were conducted in a classroom that has a seating capacity of 80 people at the Faculty of Education. At the end of the seven sessions, interviews were conducted with the selected six participants who were in the same discussion group regarding the reflection papers as a formative assessment to uncover the change in their responses to the noticing prompts.

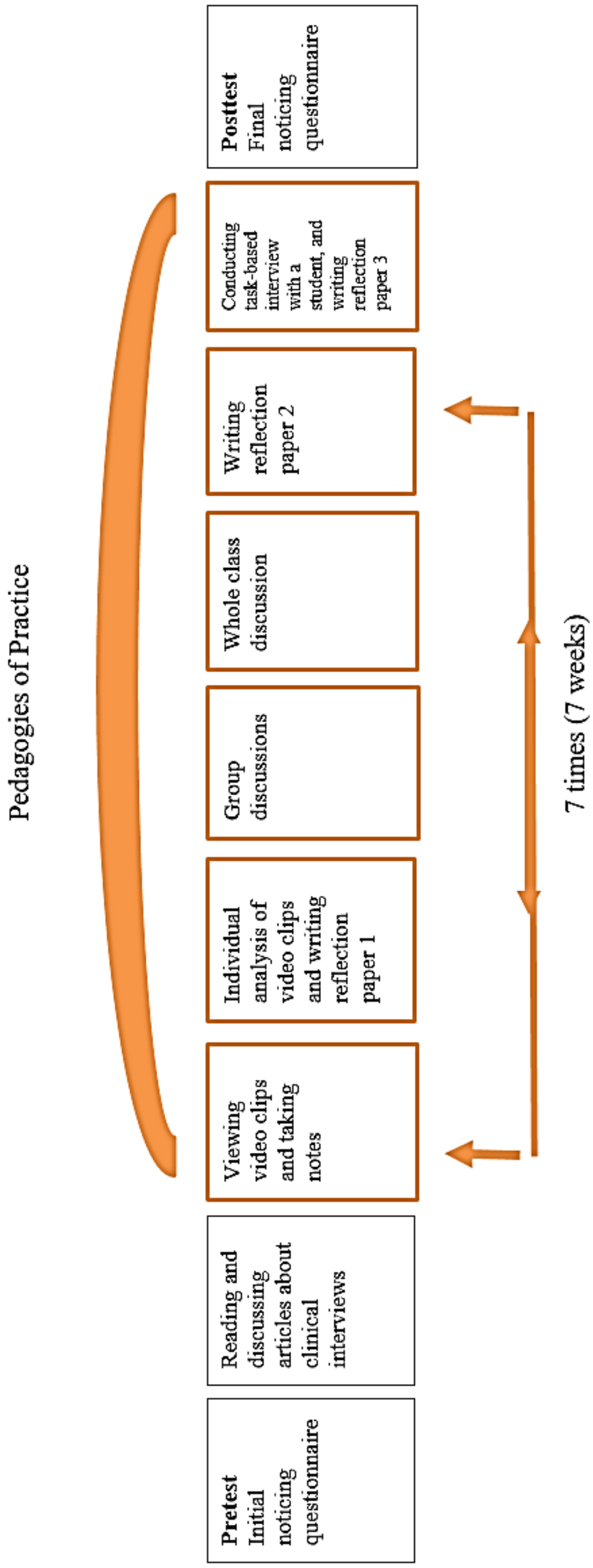


Figure 3. 1 Overview of the design of the study

At the beginning of the study, the participants were informed that they were expected to design a geometric measurement task. Throughout the semester, they worked on it and prepared a task. They got some feedback from the researcher and revised the tasks based on the comments and suggestions. All tasks included the use of concrete models. Task-based interviews surpass standard paper and pencil tests by providing evidence of students' thinking processes regarding mathematical concepts, procedures and reasoning (Confrey & Lipton, 1985). In the present study, attention was drawn to the task-based interviews conducted between the researcher and the middle school students while the pre-service teachers were watching the video clips. In addition, the pre-service teachers continuously observed the students during the teaching practice and after the *representation* and *decomposition of practice* were completed, they conducted task-based interviews with a middle school student that they chose as *approximations of practice*. During the interviews, they presented the tasks and wanted the students to solve the tasks by asking the questions they had prepared beforehand in the interview protocol. In this way, they had a chance to observe and interpret, i.e., assess the students' mathematical learning (Hunting, 1997). They recorded a video of the interviews. After the task-based interviews were completed, they viewed the video records of the interviews. They were expected to individually analyze the students' mathematical thinking in the video and answer the questions in reflection paper 3. After the approximation of practice was completed, a final noticing questionnaire, identical to the initial questionnaire, was given to the participants as a post-test in order to identify their final professional noticing skills.

3.5 Data Collection Tools

In the present study, geometric measurement tasks, video clips produced through task-based interviews with middle school students around the geometric measurement tasks, reflection paper 1, which represents pre-service teachers' individual analysis of video clips, group and whole-class discussions, reflection paper 2 about the changes and additions made by pre-service teachers after the discussions, reflection paper 3 which portrays pre-service teachers' professional noticing of students' mathematical thinking they elicited through task-based interviews, semi-structured interviews conducted with pre-service teachers regarding the reflection papers, group discussions, and whole-class discussions, and initial and final noticing questionnaire given to pre-service teachers at the beginning and at the end of the study were the data collection tools. Figure 3.2 presents the data collection procedure of the study. Details regarding the data collection tools are provided below.

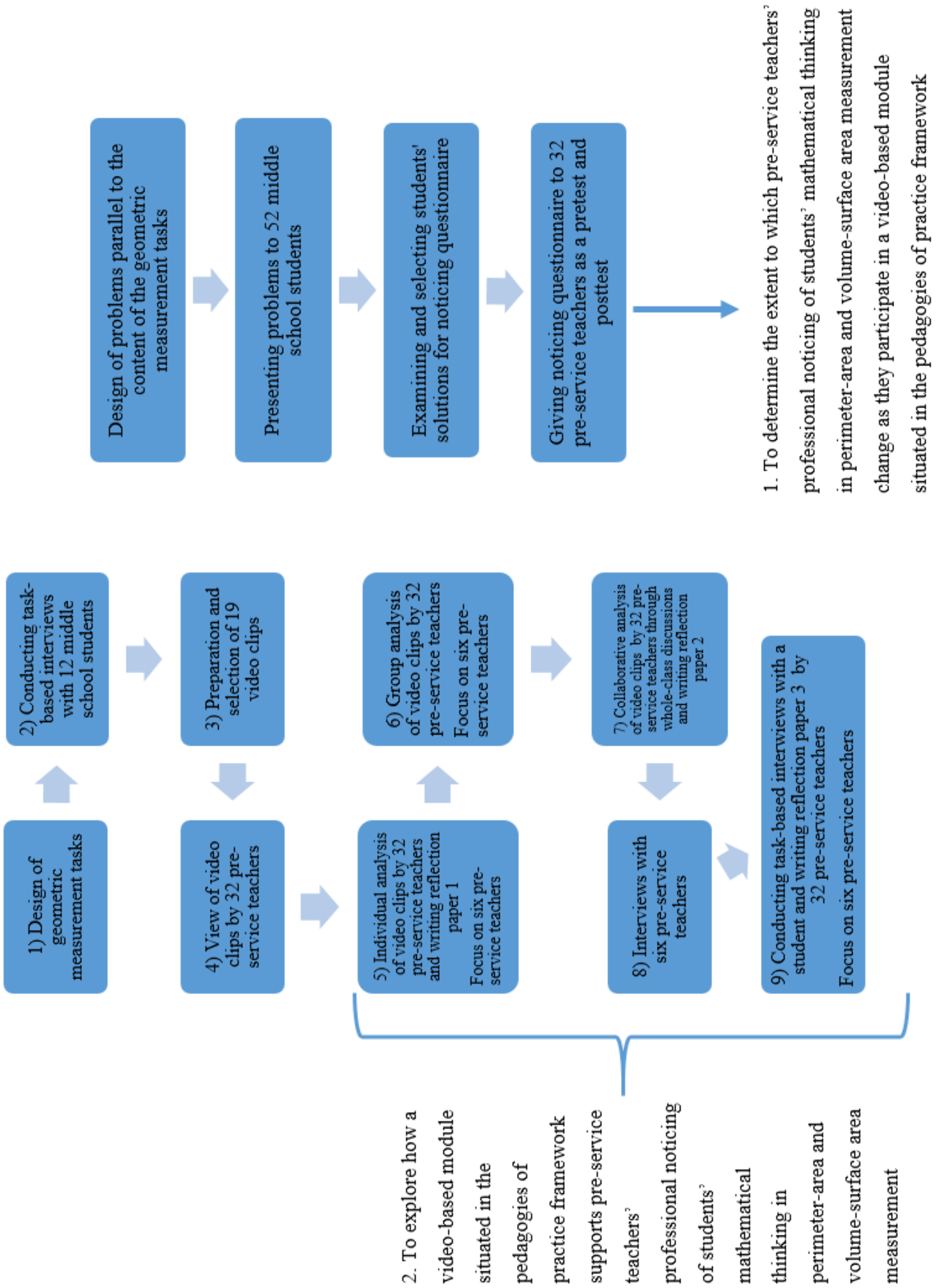


Figure 3. 2 Data collection procedure

3.5.1 Geometric Measurement Tasks

As presented in step 1 in Figure 3.2, for the video-based module, five tasks related to the perimeter, area, volume and surface area measurement were designed. Geometric measurement tasks were developed to engage students in action and dialogue, which in turn opens a window into students' mathematical thinking (Hunting, 1997). In addition, tasks deliberately involved concrete models since tasks with concrete models "provide greater opportunity to observe students' actions along with their verbal explanations and comments" (Hunting, 1997, p. 152). Moreover, the tasks or the ideas of the tasks used in this study were based on research, as the tasks used in the research have generally been subjected to rigorous analysis. Table 3.2 shows the content of the geometric measurement tasks developed for the study.

Table 3. 2 Content of the geometric measurement tasks

	Based on
Task 1	Change in perimeter
Task 2	A fixed area-changing perimeter situation/area conservation
Task 3	A fixed perimeter-changing area situation
Task 4	Part 1: Comparing the volume of two prisms
	Part 2: Enumeration of cubes to measure the volume
Task 5	Part 1: A fixed volume-changing surface area situation
	Part 2: Change in surface area

Task 1 was about length, particularly perimeter measurement; Task 2 and Task 3 were about perimeter and area measurement; Task 4 was about volume measurement, and Task 5 was about both volume and surface area measurement. Task 4 and Task 5 consisted of two parts. The tasks were listed in this way due to the confirmation of the general teaching sequence of the length-area-volume topics (Curry & Outhred, 2005). Moreover, by adopting the perspective that considers the surface area part of three-dimensional thinking (Pittalis & Christou, 2010), the task about surface area followed the task on volume in this study.

Task 1 was adapted from Lo et al. (2019) and modified by the researcher. This task is based on the intuitive rules offered by Tirosh and Stavy (1999), which is about the direct relationship between perimeter and area misconception. This task requires students to respond regarding the change in the perimeter of a rectangular paper when a piece of it is cut. They are also asked to show by cutting the paper and justify their reasoning. Afterward, if a different result is possible, they are expected to demonstrate by cutting the paper and justifying their reasoning. Figure 3.3 shows the questions for Task 1.

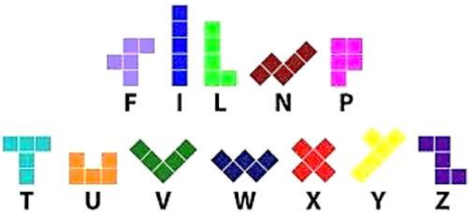
The student is shown a rectangular sheet of paper (A4 size) and asked the following questions:

- If I cut a piece of this paper, what happens to its perimeter/how does its perimeter change?
- (According to the student's answer) Can you cut this paper so that the perimeter decreases/increases/remains the same?
- (After the cutting is completed) Why do you think the perimeter is decreasing/increasing/not changing?
- Is it possible to cut a piece of paper and get a different result from what you just said?
- (If the student says no) Why do you think so?
- (If the student says yes) Can you show me how (on another paper)?
- (After the cutting is completed) Why do you think the perimeter is decreasing/increasing/not changing?

Figure 3. 3 The questions for Task 1

Task 2 is about area conservation, i.e., fixed area but changing perimeter, which was adapted from the problem that exists in learner.org and modified by the researcher. This task requires students to find the perimeter and area of the twelve pentomino pieces given and then to construct shapes having the smallest and largest perimeters using two pentomino pieces of their choosing. The questions for Task 2 are presented in Figure 3.4.

The student is given 12 pentomino pieces and asked to analyze them. Then, the following questions are asked to the student:



- What can you say about the areas of these pieces? Why do you think so?
- What can you say about the perimeter of these pieces? Why do you think so?
- (If the student realizes that the perimeter of 11 pieces is 12 units while the perimeter of the P piece is 10 units) What could be the reason for this?
- Can you choose two pieces and put them together to construct the shapes with the largest perimeter and smallest perimeter possible?
- Why did you choose these pieces? /Is there a special reason for choosing these pieces?
- What can you say about the areas of the shapes you constructed?
- What can you say about the perimeters of the shapes you constructed?
- What did you pay attention to while constructing the shapes with the largest/smallest perimeter?
- Why do you think the shape you construct has the largest/smallest possible perimeter?
- (If the student realizes that shapes have the same area) Why do the perimeters of the shapes you construct change while their areas are the same?

Figure 3. 4 The questions for Task 2

Different from Task 2, Task 3 is about fixed perimeter but changing area, and it was adapted from Sanfeliz (2019). This task requires students to find the perimeter and area of a figure comprising concrete unit squares and reach a perimeter of 16 units by adding squares. In this way, they are expected to find the shape with the largest area. Then, by forming a rectangle whose short side is three units and the long side is four units in length, students are expected to reach a perimeter of 16 units by removing the squares. However, while doing this, they are required to not eliminate an entire row or column and to connect squares through the sides. Figure 3.5 presents the questions for Task 3.

The figure on the right is formed with unit squares, and the student is asked the following questions:

- What is the perimeter of the shape?
- What is the area of the shape?
- Can you add a square(s) to the shape until you reach a perimeter of 16 units? What is the area of the new shape?
- Can you find other shapes with a perimeter of 16 units?
- Which shape with a perimeter of 16 units has the maximum area?

Then, the student is asked to form a rectangle with squares with a short side of 3 units and a long side of 4 units. The following questions are directed to the student:

- What is the perimeter of the rectangle?
- What is the area of the rectangle?

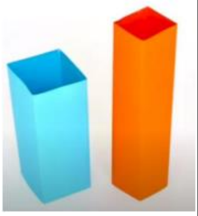
After the student determines the perimeter of the rectangle as 14 units and the area as 12 square units, he/she is asked to form shapes with a perimeter of 16 units by removing squares from the rectangle. At this stage, the student needs to pay attention to the following: 1) The unit squares must be connected to each other at the edges. 2) It is forbidden to take a whole row or column. The student is asked the area of the rectangle when each 16 units of perimeter is reached.

- Can you find the shapes with a perimeter of 16 units by removing a square(s) from the rectangle? What is the area of the new shape?
- How many squares can you remove at most to get a shape with a perimeter of 16 units?

Figure 3. 5 The questions for Task 3

Task 4 is based on the Same A-Same B intuitive rule proposed by Tirosh and Stavy (1999), which is about the direct relationship between area and volume misconception. The first part of this task requires students to compare the volume of two prisms, one short and wide and the other tall and narrow, obtained by folding two identical rectangular (A4 size) sheets of paper. They are also expected to reason about the other two prisms obtained in the same way but using a larger paper (A3 size) this time and put the four prisms in order by volume. The second part of this task requires students to find the volume of the prisms using unit cubes. The questions for Task 4 are presented in Figure 3.6.

Two A4 papers in blue and orange colours is shown to the student. The blue A4 paper is folded with its short sides over each other, and a blue square prism is obtained as shown in the figure. Then, the orange A4 paper is folded with the long sides over each other, and an orange square prism is obtained. The student is shown this pair of prisms and asked the following questions:



- What can you say about the volumes of two different prisms made of the same size paper? Which holds more/fewer objects (e.g., rice)? Or do they get an equal amount? Why do you think so?
- Does it matter how I fold the paper to form the prism? Does the volume of the prism depend on how I fold the paper?

Then, A3 paper is shown to the student, and the following questions are asked:

- What would happen if I used A3 paper instead of A4 paper? How would I get prisms compared to those I formed with A4 paper? Why do you think so?

As in the previous step, a square prism is obtained by folding the A3 paper with the short sides overlapping each other. Then, another A3 paper is folded with the long sides over each other, and another square prism is obtained. The student is asked to put the four prisms in order from largest to smallest volume and explain why he/she makes such an order. For this, the student is asked the following questions:

- Can you compare the volumes of these four prisms? Can you put the prisms in order from largest to smallest volume? Why did you make such an order?

Then, the student is given unit cubes and asked to find the total number of unit cubes that the prisms can take. For this, the student is asked the following questions:

- How can you find how many cubes are needed to fill the prisms made of A4 paper?
- What can you say about the volumes of these prisms?

Figure 3. 6 The questions for Task 4

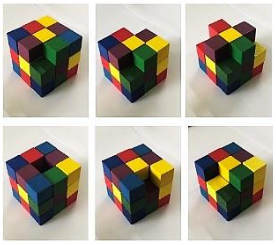
The first part of Task 5 is about different prisms having the same volume but different surface areas, which was adapted from Haylock (2010). In this part, students are expected to build different prisms with 12 unit cubes by identifying the volume and surface area of these prisms. They are also asked to build the prism with the least surface area using 24 unit cubes. The second part of the task is about change in surface area, which was adapted from Burger et al. (2014). In this part, students are required to explain the change in the surface area of a cube consisting of 27 unit cubes as one, two, or four unit cubes are removed from different parts of the top layer of the cube. Figure 3.7 shows the questions for Task 5.

The student is given unit cubes and asked to build as many different prisms as he/she can using only 12 cubes. After the student has built the prisms, the following questions are asked:

- How many different prisms can you build with 12 unit cubes?
- What can you say about the volumes of the prisms you built?
- What can you say about the surface areas of the prisms you built?
- How does arranging unit cubes differently change the surface area?
- How do you build a prism using 24 cubes to get the smallest surface area?

Then, the student is given 27 unit cubes and asked to build a cube.

- What can you say about the volume of the cube you built?
- What can you say about the surface area of the cube you built?



Then, unit cube(s) is removed respectively, as in the figure, and the student is asked to explain how the surface area changes.

- Can you explain how the surface area changes? Why do you think so?
- (If the student realizes that the surface area increases or does not change in the given cases) Is there a case in which the surface area decreases by removing the cubes?
- (If the student says no) Can you explain why you think so?
- (If the student says yes) Can you show in which case the surface area decreases?

Figure 3. 7 The questions for Task 5

3.5.2 Task-based interviews with students around geometric measurement tasks

As presented in step 2 in Figure 3.2, in order to produce video clips involving students' mathematical thinking in geometric measurement for the intervention sessions, I conducted task-based interviews with 12 middle school students (5th-8th grade level/11-14 years old) in the summer of 2021. They voluntarily took part in the study and attended different public schools. According to the Turkish mathematics curriculum (MoNE, 2018), objectives related to perimeter and area measurement take part at the third-grade level for the first time, and there are several objectives until the end of the seventh grade. Particularly, in the seventh grade, students examine the perimeter of different rectangles with the same area and the area of different rectangles with the same perimeter. Surface area is introduced at the fifth-grade level for the first time, and students learn how to calculate the surface area of a rectangular prism. Moreover, students encounter volume at the sixth-grade level for the first time and learn that the number of unit cubes that completely fill a rectangular prism gives the volume of that prism. In this way, they are also expected to construct the volume formula for the rectangular prism in this grade level. Accordingly, by considering which grade level the objectives in the curriculum are and the content of the tasks, I preferred to conduct task-based interviews with middle school students who had already learned the basic concepts of perimeter, area, volume, and surface area measurement before the study. For instance, since students learn the relationship between the perimeter and area at the seventh-grade level, I conducted task-based interviews with students who had completed at least seventh grade for Task 2 and Task 3.

Task-based interviews were conducted in a suitable and quiet room. Tasks and concrete models were presented to the students, and they were asked to solve tasks by using concrete models. During the interviews, the researcher asked the questions she prepared beforehand. However, based on the students' responses, she also asked follow-up questions in order to elicit students' mathematical thinking. All interviews were video recorded using a single camera to capture students’ responses and actions.

3.5.3 Preparation and selection of video clips

As presented in step 3 in Figure 3.2, among the task-based interviews, some of them were edited and selected in order to prepare proper video clips for the intervention sessions. By watching all the video recordings, I identified the mathematically significant details in each student’s thinking. While selecting video clips, I considered whether students’ ideas were understandable and if students had misconceptions, whether they were clear enough to be identified by pre-service teachers. I selected the video clips in such a way that each video clip involved different students' mathematical thinking on the same task. In this way, for each task, there were two or three video clips, i.e., two or three different student solutions. In total, 19 video clips were produced. The students in the video clips were enumerated from S1 to S9. Some students in some video clips were the same.

Moreover, by considering the suggestion that long video-clips are ineffective in terms of productive discussions around video clips (Sherin et al., 2009), I prepared the duration of the video clips to be no more than 7 minutes. Usually, each video clip contained the whole interview with a student about a task. In some cases, I reduced the duration of the video clip by excluding the times when the students did not do or say anything, i.e., when the thinking process was too long. Details of the video clips are presented in Table 3.3.

Table 3. 3 Details of the video clips

Week	Session	Task	Concept	Video Clip	Length (min.)	Student	Gender	Grade Level (had finished)
W1	1	T1	Perimeter	VC1	1.27	S1	Female	7
				VC2	2.20	S2	Male	6
				VC3	2.32	S3	Male	6
W2	2	T2	Perimeter and area	VC4	4.09	S1	Female	7
				VC5	2.57	S4	Female	7
				VC6	3.05	S5	Male	8
W3	3	T3	Perimeter and area	VC7	6.33	S6	Female	7
				VC8	5.09	S7	Male	7
W4	4	T4-P1	Volume	VC9	2.34	S1	Female	7
				VC10	2.25	S2	Male	6
				VC11	1.40	S8	Female	6
W5	5	T4-P2	Volume	VC12	0.46	S1	Female	7
				VC13	1.57	S2	Male	6
				VC14	2.08	S8	Female	6
W6	6	T5-P1	Volume and surface area	VC15	2.42	S5	Male	8
				VC16	3.53	S3	Male	6
W7	7	T5-P2	Surface area	VC17	1.20	S7	Male	7
				VC18	2.09	S5	Male	8
				VC19	3.33	S9	Male	8

As presented in step 4 in Figure 3.2, during the intervention sessions, participants viewed these video clips on projection as a representation of practice without interruption. In each session, participants viewed video clips of different students solving one task (a task of the week). In the first session, the participants viewed three video clips (VC1, VC2, and VC3) about Task 1, which was about perimeter measurement. The length of the video clips varied between 1.27 and 2.32. While the student in VC1 (S1) was female, the students in VC2 (S2) and in VC3 (S3) were male. S1 had just finished the seventh grade, and S2 and S3 had just finished the sixth grade at the time of the study. In the second session, the participants viewed three video clips (VC4, VC5, and VC6) about Task 2, which was about perimeter and area measurement. The length of the video clips varied between 2.57 and 4.09.

The student in VC5 (S4) was female, whereas the student in VC6 (S5) was male. S4 had just finished the seventh grade, and S5 had just finished the eighth grade at the time of the study. S1 in VC4 was the student in VC1. In the third session, the participants viewed two video clips (VC7 and VC8) about Task 3, which was about perimeter and area measurement. The lengths of the video clips were 6.33 and 5.09, respectively. The student in VC7 (S6) was female, and the student in VC8 (S7) was male. Both students had just finished the seventh grade at the time of the study. In the fourth session, the participants viewed three video clips (VC9, VC10, and VC11) about the first part of Task 4, which was about volume measurement. The length of the video clips varied between 1.40 and 2.34. The student in VC11 (S8) was female and had just finished sixth grade at the time of the study. S1 in VC9 was the student in VC1 and VC4. S2 in VC10 was the student in VC2. In the fifth session, the participants viewed three video clips (VC12, VC13, and VC14) about the second part of Task 4. The length of the video clips varied between 0.46 between 2.08. The students in these video clips were the students in the video clips from the previous session, as the task in this session was a continuation of Task 4. In the sixth session, the participants viewed two video clips (VC15 and VC16) about the first part of Task 5, which was about volume and surface area measurement. The lengths of the video clips were 2.42 and 3.53, respectively. S5 in VC15 was the student in VC6, and S3 in VC16 was the student in VC3. In the seventh session, the participants viewed three video clips (VC17, VC18, and VC19) about the second part of Task 5, which was about the surface area measurement. The length of the video clips varied between 1.20 and 3.33. S7 in VC17 was the student in VC8, and S5 in VC18 was the student in VC6 and VC15. The student in VC19 (S9) was male and had just finished eighth grade at the time of the study.

3.5.4 Students' solutions in the video clips

In the study, there were seven intervention sessions, and in each session, participants viewed two or three video clips as presented in step 4 in Figure 3.2. The following part presents the students' solutions in the video clips in each session.

3.5.4.1 Students' solutions in the video clips in the first session

In the first session, VC1, VC2, and VC3 on the perimeter concept were prepared for the pre-service teachers. In VC1, S1 split the paper in half and considered that the perimeter was halved since the paper was split. Figure 3.8 shows S1's way of cutting the paper. In addition, she thought that when a piece of paper is cut, the perimeter always decreases. Here, the student had a misconception stemming from the intuitive rules. That is, she believed that less area results in less perimeter, and there is a direct relationship between area and perimeter, which was a significant mathematical detail in this video clip. This video clip was involved in the session to make the pre-service teachers recognize this common student misconception.

Figure 3. 8 The solution of S1 in VC1

In VC2, S2 cut a square piece from the inside of the shape and responded that the perimeter did not change because there was no change on the outside of the shape. This was the most unexpected and surprising event among the video clips in the first session. Considering that the pre-service teachers may also have limited knowledge of this, viewing and analyzing this video clip for them was useful. He got a perforated shape but did not think the new boundaries formed inside the shape were included in the perimeter. The student believed that the perimeter was related to the outside of the shape, which caused the student to give an incorrect answer. In the second part, he responded that the perimeter could be reduced, cut off the paper along the short side, and justified his reasoning by obtaining a smaller paper. Yet, to him, increasing the perimeter by cutting a piece of paper was not possible. Here, he had a misconception hailed from the intuitive rules, similar to S1. S2's ways of cutting the paper are presented in Figure 3.9.

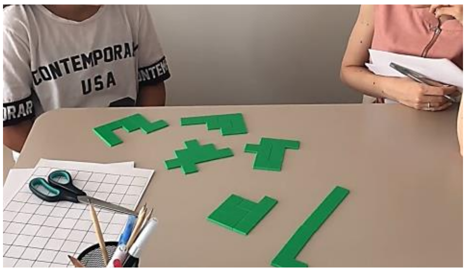
Figure 3. 9 The solution of S2 in VC2

In VC3, S3 could demonstrate all possible cases correctly by justifying his responses. Firstly, he cut off a triangle at a corner and responded that the perimeter decreased. Since he has not learned the triangle inequality yet, he justified this response by cutting two strips from the right sides of the triangle, adding them end to end, and comparing this segment with the hypothenuse of the triangle. Secondly, he maintained that keeping the perimeter constant was also possible and cut a square piece from the corner. He justified his response by emphasizing the congruence of the opposite sides of the square. Lastly, he cut off a square from the side and asserted that increasing the perimeter was possible in this way due to the increase in the number of sides that contribute to the perimeter. S3's ways of cutting the paper are set out in Figure 3.10. This video clip was useful for the pre-service teachers to help them realize different possible cases and different ways of cutting and broaden their perspectives.

Figure 3. 10 The solution of S3 in VC3

3.5.4.2 Students' solutions in the video clips in the second session

In the second session, there were three video clips, VC4, VC5, and VC6, on the perimeter and area concepts. S1 in VC4 thought the area of all pentomino pieces was equal due to the same number of unit squares that comprise the pieces. For the perimeter of the pieces, she asserted that all the sides of the squares should be added. While finding the perimeter of two pieces, she counted all sides, including the inside of the pieces, and got 16 units. This was the most noteworthy mathematical detail in this video clip. Then, she thought that the perimeter of all pieces was the same. For the definition of a perimeter, she stated that if the length of one side is six units, it is multiplied by how many sides it has; that is, six is multiplied by 4. In a similar way, for the area, she thought that the length of one side is multiplied by the length of the other, i.e., six is multiplied by six. Here, she explained how to find the perimeter and area of the shape using the formula rather than what these concepts mean. Furthermore, giving examples over the square shows her conceptualizing regarding why she added all the sides to find the perimeter of the shapes. Afterward, she considered that obtaining the shapes with the largest and smallest perimeters using two pieces was not possible because shapes would have the same perimeter due to the same number of squares. Then, she put two different pieces randomly and obtained six shapes. For one shape, she calculated the perimeter as 30 units, whereas for another shape, she found 29 units. She was puzzled when she got different results for the perimeter of the two shapes and tried to calculate the perimeter of the pieces individually and then added the results, but she found 31 units this time. Consequently, she deduced that a change in shapes may change the perimeter of the shapes. Figure 3.11 presents the solution of S1 in VC4. This video clip was important for pre-service teachers to become aware of instability in students' decisions.



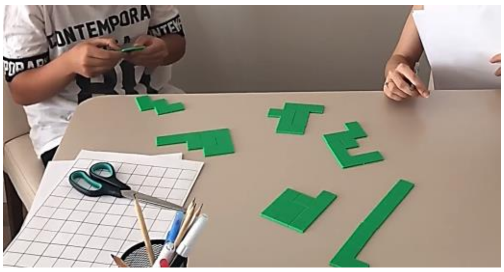


Figure 3. 11 The solution of S1 in VC4

The transcription of VC5 is provided in Table 3.4 as an example. This video clip consisted of many significant unexpected mathematical details for the pre-service teachers. S4, in this video clip, could not correctly find the perimeter and area of the pentomino pieces because of her overreliance on the perimeter and area formula for rectangular shapes. She overgeneralized the area formula for the rectangular shapes to the irregular shapes and multiplied the length of two sides she determined to find the area of the pieces. As a result, she did not recognize that the area was about covering, and all pieces had the same area because they included the same number of unit squares. While finding the perimeter of the shapes, she added the length of the two sides she specified. Consequently, the student did not know that the perimeter was related to the boundaries of the shape. Moreover, she believed that to calculate the perimeter of a shape, the shape had to have short and long sides; otherwise, its perimeter could not be found. This video clip was useful to illustrate how the student lacked a conceptual understanding of perimeter and area concepts.

Table 3. 4 The dialogue between the researcher and S4 in VC5

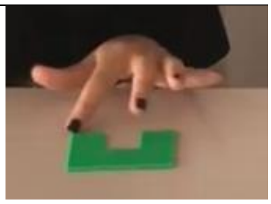
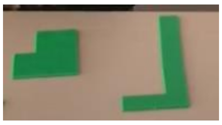
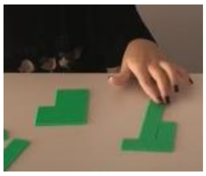
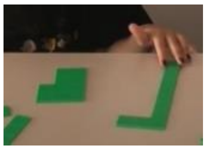
Time interval	Content	Dialogue
00:00-1:13	Finding the perimeter and area of the pentomino pieces	 R: What can you say about the areas of these 12 pentomino pieces? S4: The area is 6 (pointing to U piece) R: How did you decide? S4: By multiplying. I multiplied the short side by the long side. There are two units here; this is 3.  R: What about this? (pointing to P piece) S4: This is also 6. Here is 3, and here is 2. R: What can you say about their perimeters? S4: We can find by adding. R: How can you find the area of this? (pointing to Y piece) S4: There’s no short side. It had to be here to be counted. We can’t find the perimeter.  R: How should the shape be to calculate the perimeter? S4: It could be like this (pointing to U piece) R: You say it has both the short side and the long side. S4: Yes.  R: Do you mean there is no short side because of a gap here? (pointing to Y piece) S4: Yes. R: Do you think the perimeter of this can be found? (pointing to Z piece) S4: Yes. Five. The short side is two units, and the long side is three units. I added them.

Table 3.4 (cont'd)

1:13- 2:57	Constructing shapes having the smallest and largest perimeters using two pentomino pieces	R: Can you construct the shape having the smallest possible perimeter by selecting two of these pieces?  S4: These pieces can be (putting P and U pieces together) R: What is the perimeter of the shape? S4:7 R: How did you find it? S4: The long side is four, and the short side is 3. I found it by adding.  R: Can you construct the shape having the largest possible perimeter by selecting two of these pieces? S4: By placing these (V and I) in this way, we can get a larger perimeter. R: How many units is the perimeter of this shape? S4: 11. R: How did you find it? S4: Adding the long side and the short side. This is three, this is eight. I get 11.  R: What can you say about the areas of the shapes you constructed? How many units is the area of this shape? (P and U pieces) S4: 12. I multiplied four by three. R: What about the area of this? (V and I pieces)  S4: 24. I multiplied eight by three. R: Is there a reason for choosing these pieces? S4: No. R: Did you choose them randomly? S4: Yes. R: What did you pay attention to while placing the pieces?  S4: I make sure that the perimeter is the largest. If I place it like this, I will get a shorter perimeter. It was bigger when I placed it like this.

In VC6, S5 declared that all pieces have the five unit squares; therefore, they have the same area. For the perimeter of the pieces, he responded that their perimeters might differ because of differences in the spreads of the pieces. He illustrated his response by giving P and I pieces as examples by realizing that the P piece's perimeter was smaller because squares were more compact in the P piece, but they were more spread in the I piece. He correctly formed the shapes with the smallest and largest perimeters and correctly computed the perimeters of these shapes. For the former, he put P and V pieces together and found 14 units; for the latter, he put U and I together and found 22 units. He maintained that he consciously selected the pieces to form the shapes because the P and V pieces fitted together perfectly, and the squares were less spread. On the other hand, in a shape with the largest perimeter, he stated that the squares were spread more. He also correctly identified the area of the shapes as 10 square units. He was aware that the same area of the shapes was related to the number of squares that make the shape, while differences in their perimeters were associated with the spread of the shapes. Figure 3.12 shows the solution of S5 in VC6. This video clip was important for the pre-service teachers since it clearly shows that shapes with the same area may have different perimeters, which was what Task 2 was based on.

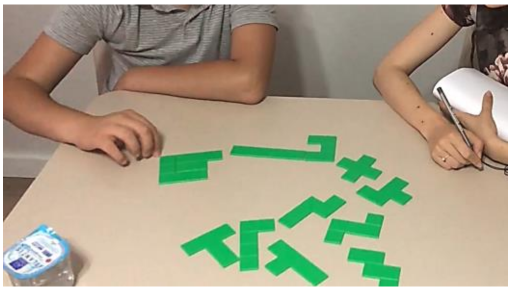


Figure 3. 12 The solution of S5 in VC6

3.5.4.3 Students' solutions in the video clips in the third session

Session 3 included two video clips, VC7 and VC8, on perimeter and area concepts. S6 in VC7 correctly identified the perimeter and area of the given figure. In addition, she could reach the perimeter of 16 units by adding two squares and also demonstrated two shapes with perimeters of 16 units and an area of eight and nine square units. In order to reach the maximum area, she was aware that she needed to add squares to the corners because, in this way, the perimeter did not change. After adding to the corners, she added two rows to the bottom of the shape and obtained a square with an area of 16 square units. She was sure that this was the shape with the maximum area with a perimeter of 16 units because she realized that if she added one more square to any place, the perimeter would change. Then, after forming a rectangle with a perimeter of 14 units and an area of 12 square units, by removing one square from the side, she reached the perimeter of 16 units. She also could demonstrate other possible shapes in the case of removing one square. By removing one square from the side and one square from the corner, she could show all possible shapes with a perimeter of 16 units and an area of 10 square units. She removed three (one square from the side and two squares from the corners) and four squares (one square from the side and three squares from the corners) by following the same way. She concluded that she could keep the perimeter constant by removing a maximum of five squares. The solution of S6 in VC7 is presented in Figure 3.13.

Figure 3. 13 The solution of S6 in VC7

In VC8, S7 correctly identified the perimeter as 12 units by counting the sides one by one. However, while finding the shape with a perimeter of 16 units, he added a square to the top side of the shape and thought that three sides were added, and hence, the perimeter was 15 units. Then, he moved the same square to the bottom right corner and specified the perimeter as 14 units due to adding two sides. By adding a square to the left upper corner, he thought he obtained a shape with a perimeter of 16 units. However, he actually got a rectangle with a perimeter of 12 units. This might be an unexpected situation for the pre-service teachers. The student ignored the disappeared sides when the squares were added and considered how many sides were added. For this reason, he could not realize that the maximum area could be obtained when the shape was a square. Instead, he found the shape with a maximum area of 15 square units in which one square was missing from the corner. Since he thought adding a square to the corner increased the perimeter by two units, he did not add one more square. Figure 3.14 shows the solution of S7 in VC8.

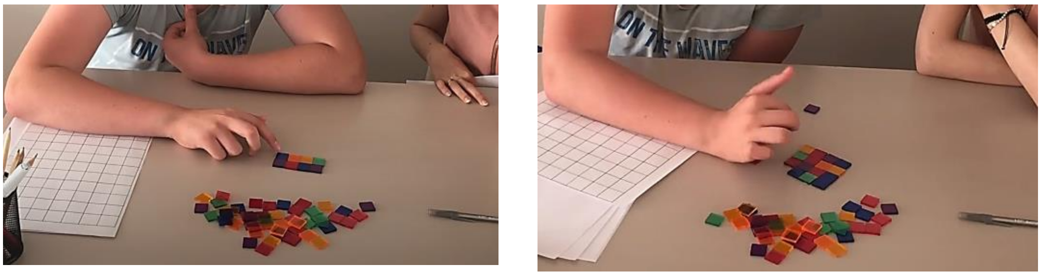


Figure 3. 14 The solution of S7 in VC8

3.5.4.4 Students' solutions in the video clips in the fourth session

In the fourth session, three video clips, VC9, VC10, and VC11, were on the volume concept. S1 in VC9 claimed that both prisms get an equal amount of objects because they were obtained using the same paper. Therefore, she added that the volume of the prisms did not depend on how the paper was folded. The misconception of the student originated in the Same A-Same B intuitive rule. Here, the area of the papers was conserved, but the volume of the prisms obtained using the same papers was not conserved. The volume of the prisms changed based on how the paper was folded. This video clip was involved in the session to make the pre-service teachers recognize this misconception and its cause, which constituted the noteworthy mathematical detail in this video clip. When asked to reason about the prisms obtained using the larger paper, she asserted that these prisms would take more objects than the previous ones because of the larger size of the paper. While putting the prisms in order from largest to smallest volume, she argued that the volume of the prisms made of A3 paper was the same and the volume of the prisms made of A4 paper was the same. However, she declared that the volume of those made of A3 paper was greater than that of those made of A4 paper. The solution of S1 in VC9 is set out in Figure 3.15.

Figure 3. 15 The solution of S1 in VC9

In VC10, S2, firstly, he thought that the short and wide prism held more objects, i.e., it had a larger volume because it was wider than the other prism. In the beginning, S2 had correct reasoning, but then, he changed his mind and fell into the intuitive rule trap like S1. Surprisingly, he maintained that the volume did not change depending on how the paper was folded because both prisms were made of the same size paper. Moreover, he added that both held an equal amount of objects, and only the way of folding differed. For the prisms made of A3 paper, he correctly reasoned that they would be bigger and have a larger volume than the previous ones. His reasoning for the volume of the bigger prisms was consistent with his reasoning for the volume of the smaller prism. That is, he thought that the prisms made of A3 paper held an equal amount of objects. He put the prisms in order in the same way as S1. The solution of S2 in VC10 is shown in Figure 3.16. This video clip was valuable for the pre-service teachers to make them realize the instability in students' decisions and how intuitive rules were effective in students' decisions.

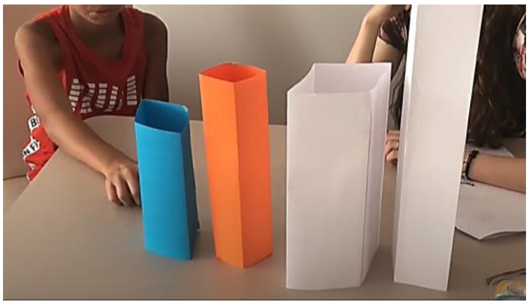


Figure 3. 16 The solution of S2 in VC10

In VC11, S8's reasoning was different from the previous students' reasoning because she believed that the tall and narrow prism had a larger volume because it was higher. Moreover, only S8 responded that the volume depended on how the paper was folded. For the prisms made of A3 paper, she thought that the volume would be larger because of the larger size of the paper. She compared the short and wide prism made of A3 paper with the same kind of prism made of A4 paper. She asserted that the volume of the one made of A3 was larger because A3 paper was larger in size, and the prism was higher than the other. In a similar vein, she compared the tall and narrow prism made of A3 paper with the same type of prism made of A4 paper. She claimed that the height of the prism made of A4 paper was shorter and the size of A4 paper was smaller. She added that the size of the paper changed the volume even though both were folded on the long side. The ordering she made was consistent with her reasoning because she put the prisms in order from largest to smallest volume as follows: the tall and narrow prism made of A3 paper, the short and wide prism made of A3 paper, the tall and narrow prism made of A4 paper and the short and wide prism made of A4 paper. While putting the prisms in order, she considered both the size of the paper and the height of the prisms. Figure 3.17 presents the solution of S8 in VC11. This video clip included a different misconception from the previous video clips as a noteworthy event. The student only focused on the height of the prisms, ignoring the other dimensions while making judgments regarding the volume of the prisms. Therefore, it was valuable to make pre-service teachers recognize this misconception.

Figure 3. 17 The solution of S8 in VC11

3.5.4.5 Students' solutions in the video clips in the fifth session

The fifth session involved VC12, VC13, and VC14, which were the continuation of the video clips in the fourth session. In these video clips, students measured the volume of the prisms made of A4-size paper using the unit cubes. In VC12, S1 tried to fill the prisms by randomly throwing the unit cubes into the prisms. Figure 3.18 provides the solution of S1 in VC12. She incorrectly found that the short and wide prism held 52 unit cubes and the tall and narrow prism held 29 unit cubes and believed that these gave the volume of the prisms. She did not attempt to construct layers and iterate these layers. Therefore, her approach was a volume as filling rather than a packing approach. This situation resulted in leaving gaps and overlaps between the cubes and did not give the actual volume of the prisms. S1's way of measuring volume may be an important, unexpected mathematical detail for the pre-service teachers.

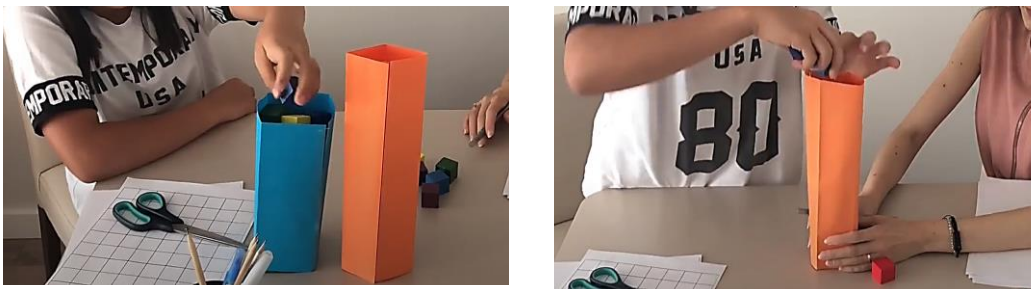


Figure 3. 18 The solution of S1 in VC12

In VC13, S2 first tried to determine how many cubes could be stacked along the height of each prism. Secondly, he identified the number of cubes fitting in one face of the prism. Thirdly, he multiplied that number by the number of faces. In this way, for the short and wide prism, after finding the height to be 12 units, he thought that one column consisting of 12 unit cubes would fit in the face of the prism. By multiplying 12 by four, he found that the prism held 48 unit cubes. Although the result was correct, he accidentally reached this result since four cubes could be placed at the base of the prism, which was equal to the number of faces. He found the height as nine units incorrectly rather than eight units for the tall and narrow prism and multiplied nine by 12. He thought that three columns consisting of 9 unit cubes would fit in one face. Since there were four faces, he claimed that 12 of these columns would fill the prisms. In this way, by multiplying 12 by 9, he asserted that 108 unit cubes would fill the prism. The solution of S2 in VC13 is presented in Figure 3.19. He found more than the actual result because he did not consider the common cubes on the adjacent faces and double counted the cubes at the corners. The student lacked spatial visualization because his conceptualization was based on faces. Using this strategy might result in finding the lateral area of the prism rather than the volume. S2's strategy was a noteworthy mathematical detail for the pre-service teachers regarding revealing the spatial structuring among students.

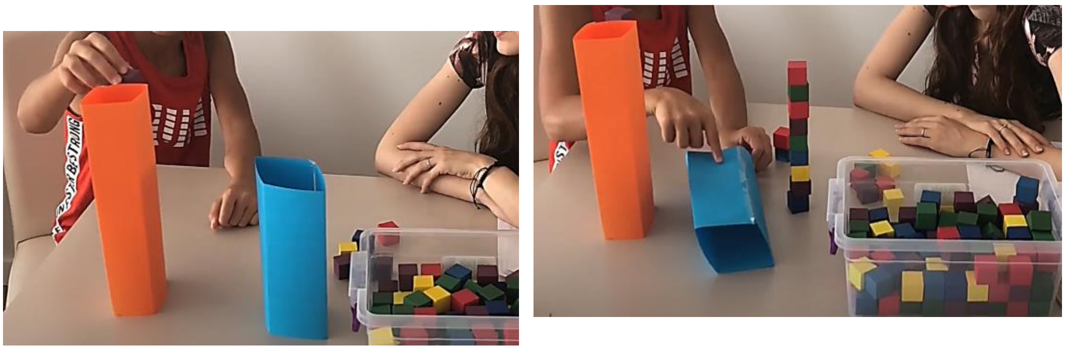


Figure 3. 19 The solution of S2 in VC13

In VC14, S8 built two prisms using unit cubes. For the short and wide prism, she first constructed the first layer, i.e., the base, and decided it consisted of nine unit cubes. Then, she stacked this layer along the height of 8 units. After building the prism, she first found the number of cubes in a vertical layer, and then she multiplied this number by the number of layers. That is, she multiplied 24 by three. Similarly, for the tall and narrow prism, firstly, she constructed the first layer consisting of four unit cubes. Then, she built the prism by stacking the layer along the height, i.e., 12 units. In order to find the total number of unit cubes, she counted the number of cubes in a vertical layer as 24 unit cubes. By multiplying 24 by two, she found the number of cubes that filled the prism. The solution of S8 in VC14 is set out in Figure 3.20. The layer multiplying strategy that the student used was one of the correct strategies that could be used while enumerating cubes, which shows the student's conceptualization of cubes as sets that were arranged in layers. Therefore, the video clip was valuable for the pre-service teachers to make them realize this strategy.

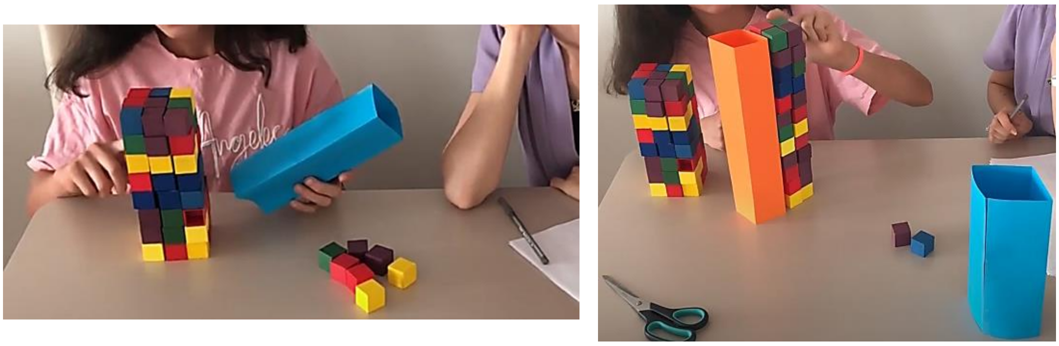


Figure 3. 20 The solution of S8 in VC14

3.5.4.6 Students' solutions in the video clips in the sixth session

In the sixth session, video clips VC15 and VC16 were on the concepts of volume and surface area. VC15 involved many significant mathematical details for the pre-service teachers. In this video clip, S5 first built a prism with dimensions of three units in length, two units in width, and two units in height. The second prism he built had two units in length, two in width, and three in height. Then, he realized they were not different prisms because rotating the first prism gave the second prism. Different from these, he built a prism with six units in length, two units in width, and one in height. Consequently, he could build two different prisms. The student was aware that it was the same prism even if its position changed, and a different prism could be obtained by changing the dimensions. For the volume of these prisms, he multiplied three numbers corresponding to the dimensions of the prisms, i.e., length, width, and height, and found their volume to be 12 cubic units. Here, the student did not associate the volume with the number of cubes and used the volume formula even though he did not explicitly refer to it. For the surface area of the prisms, he added the area of two bases and four lateral faces and got the correct result. He knew the surface area might change by arranging the cubes differently. He thought a smaller surface area could be obtained by making the prism more compact. Moreover, he built the prism with the smallest surface area with three units in length, two units in width, and four units in height using 24 unit cubes. Here, his reasoning was consistent with the previous one because he justified his action by being compact of the prism. Figure 3.21 shows the solution of S5 in VC15.

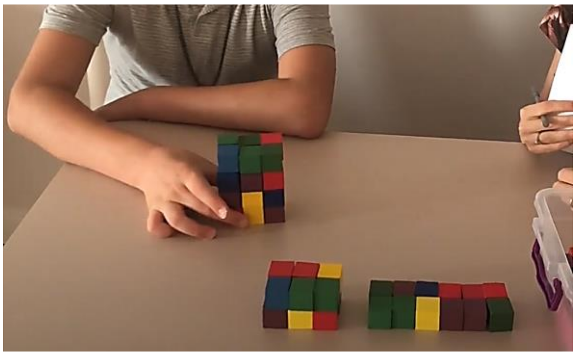


Figure 3. 21 The solution of S5 in VC15

In VC16, at the beginning, S3 built two prisms; one was the rotated version of the other, like S5. However, after realizing they were identical prisms, he could build all four possible prisms using 12 unit cubes. For the volume of the prisms, he correctly reasoned that all prisms had the same volume because of including the same number of unit cubes. For the surface area of the prisms, he made judgments regarding the area of a face that touched the floor. He thought the surface area changed each time, as the face touching the floor changed when a prism was rotated differently. To him, the surface area changed depending on the position of the prism. Surprisingly, he claimed that when the prism (one unit in length, one unit in width, and 12 units in height) was rotated, its surface area would change, but it would not be a prism anymore. In this case, he believed the cubes would be lined up side by side, which would not constitute a prism. While building a prism with the smallest surface area using 24 unit cubes, he stacked the cubes one by one along the height of 24 units. He claimed that the prism had the smallest surface area because one square touched the floor. This was expected since it was consistent with his reasoning about the surface area of the prisms that he built using 12 unit cubes. Figure 3.22 presents the solution of S3 in VC16.

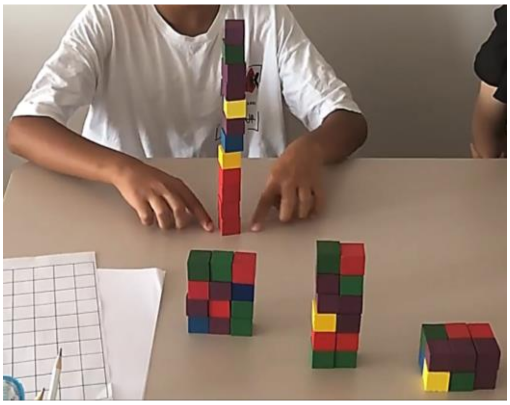


Figure 3. 22 The solution of S3 in VC16

3.5.4.7 Students' solutions in the video clips in the seventh session

The seventh session involved three video clips, VC17, VC18, and VC19, on the surface area. In VC17, S7 specified the surface area of the cube as 9 unit cubes. The student only considered the area of one face, the top face of the large cube, rather than all faces. He asserted that when one cube was removed, the surface decreased by one, and when two cubes were removed, the surface area decreased by two. For the change in the surface area, he thought that the more cubes were removed, the smaller the surface area would be. He claimed the surface area would always decrease when the cubes were removed. However, he added that the surface area would not change only in one case, which was removing the interior cube in the large cube. He justified his response by stating that appearance did not change. The solution of S7 in VC17 is shown in Figure 3.23.

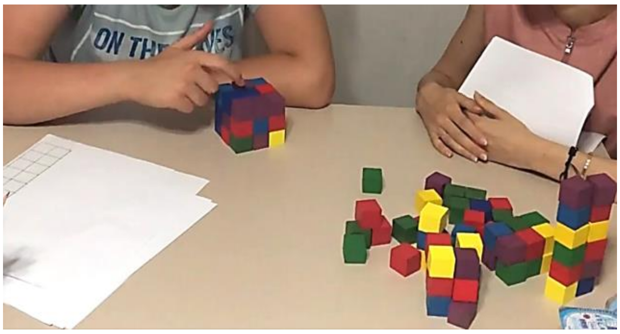


Figure 3. 23 The solution of S7 in VC17

In VC18, S5 correctly computed the surface area of the large cube as 54 unit cubes. However, for the change in the surface area, he thought the surface area would decrease by the number of visible faces of the removed cubes. Here, the student did not discern the new faces that appeared when the cubes were removed. He maintained that the surface area would always decrease when the cubes were removed, and it was not possible to increase the surface area or keep it constant. The solution of S5 in VC18 is presented in Figure 3.24.

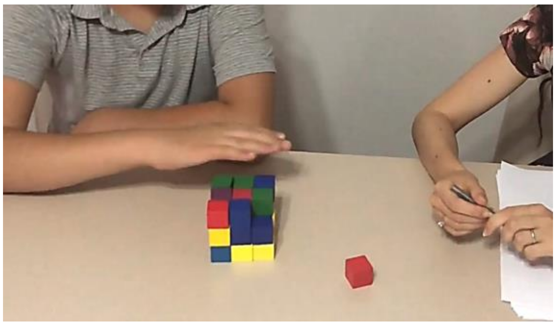


Figure 3. 24 The solution of S5 in VC18

In VC19, S9 first computed the area of one face as nine unit cubes and multiplied this number by six since the cube had six faces, and he correctly found the surface area as 54 unit cubes. In each case, he considered how many faces contributed to the surface area before removal and how many new faces appeared after removal. In this way, he correctly determined the surface area either increased or did not change for each case. When asked whether reducing surface area was possible by removing the cubes, he realized the need for removing the cubes with the largest number of visible faces. Therefore, he focused on the corners and removed three cubes from the top right part of the large cube. He was aware that initially, the number of faces contributing to the surface area was eight, and this number decreased to six when the cubes were removed. The solution of S9 in VC19 is set out in Figure 3.25.

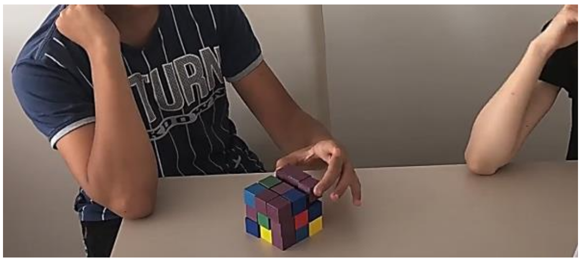


Figure 3. 25 The solution of S9 in VC19

3.5.5 Reflection papers

In the present study, reflection papers were one of the data collection tools used to explore how a video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement. During each intervention session, the participants were expected to write three reflection papers individually, one after watching the video clips, one after the discussions and one after conducting task-based interviews with middle school students. All of 32 participants wrote these reflection papers, but in the present study, reflection papers of six pre-service teachers who were in the same discussion group were examined to reveal the improvement in pre-service teachers' professional noticing skills in depth and to track changes on an individual level more consistently. After viewing the video clips in the sessions, the participants individually analyzed the video clips and wrote reflection paper 1 by using the notes they took while viewing the video clips. In this reflection paper, the participants were asked to respond to three prompts adapted from Jacobs et al. (2010) and modified by the researcher as in the following:

i. What did you notice about the student's mathematical thinking in the video? Explain the student's thinking process and how he/she solved the task by supporting it with examples from the video.

ii. Please explain in detail what you learned about the student's understanding based on his/her response.

iii. Pretend that you are the teacher of this student. What(s) would you do to develop the student's mathematical thinking/understanding and eliminate his/her misconceptions (if any)? Please explain in detail.

The purpose of the first prompt is to explore the extent to which pre-service teachers attend to students' mathematical thinking, the second prompt tries to explore how pre-service teachers interpret students' understanding, and the third prompt aims to examine pre-service teachers' instructional actions to eliminate students' misconceptions and extend their mathematical thinking. Writing a reflection paper 1 took approximately 40 minutes. Reflection papers were collected after the participants completed them to initiate group discussions. Moreover, at the end of each session, participants were asked to write a second reflection paper (reflection paper 2) by typing changes in and additions to their reflections after participating in discussions, if any, to reveal the influence of peers' ideas on pre-service teachers' professional noticing of students' mathematical thinking**.** In reflection paper 2, the participants were asked to respond to the following question:

i. How did the discussion environment influence your thoughts? If there have been any changes in your thoughts or if you have anything to add after the group and whole-class discussion, explain it by giving examples.

They individually wrote reflection paper 2 in the classroom and delivered them as hard copies to the researcher while leaving the classroom. Writing a reflection paper 2 took approximately 15 minutes. Furthermore, after the participants conducted task-based interviews with middle school students by using the tasks they designed, they were asked to analyze the interviewed students' mathematical thinking and write a reflection paper 3 to determine their professional noticing skills in approximation of practice. In reflection paper 3, the participants were expected to respond to the following questions:

i. What did you notice about the student's mathematical thinking in the task-based interview you conducted? Explain the student's thinking process and how he/she solved the task you designed by supporting it with examples.

ii. Please explain in detail what you learned about the student's understanding based on his/her response.

iii. Pretend that you are the teacher of this student. What(s) would you do to develop the student's mathematical thinking/understanding and eliminate his/her misconceptions (if any)? Please explain.

iv. Were there any surprising/unexpected aspects of (that you could not predict) the student's solution? Please explain.

v. If you were to do the interview again, what would you change (e.g., in the task you designed, in the questions you asked, etc.)? Please explain.

In reflection paper 3, as in reflection paper 1, the purpose of the first prompt is to explore the extent to which pre-service teachers attend to students' mathematical thinking, the second prompt tries to explore how pre-service teachers interpret students' understanding and the third prompt aims to examine pre-service teachers' instructional actions to eliminate students' misconceptions and extend their mathematical thinking. The other two prompts were included in the reflection paper in order for pre-service teachers to evaluate the interviews they conducted. Writing a reflection paper 3 was given to the participants as an out of class assignment. The Turkish version of the questions in reflection paper 1, reflection paper 2 and reflection paper 3 is provided in Appendix A, Appendix B and Appendix C, respectively.

3.5.6 Semi-structured interviews

Individual semi-structured interviews were conducted with the six participants who were in the same discussion group after the seven sessions were completed regarding the reflection papers as a formative assessment to uncover the change in their responses to noticing prompts and understand the development in the pre-service teachers' professional noticing skills. There was no time limitation, but the interviews were completed in approximately 20 minutes. During the interviews, reflection papers 1, which were written by the participants in seven intervention sessions, were presented to them and they were asked to examine what they wrote each week. After the participants examined, the following questions were directed to the participants:

i. Now, when you look at what you have written in each week in more detail, do you see an improvement in yourself? If yes, please explain.

ii. Do you think that from now on, if you encounter different student solutions about geometric measurement, you will be better able to attend to the students' thinking, interpret it and respond to the students? If yes, please explain.

iii. Were the discussions useful? If yes, what were the benefits of the discussions?

The Turkish version of the sample interview questions is provided in Appendix D.

3.5.7 Group discussions and whole-class discussions

After the participants completed the individual analysis of video clips and wrote reflection paper 1, group discussions were initiated. There were six discussion groups. Participants shared their ideas with their peers in groups of 5 or 6. They discussed the students' mathematical thinking, understanding of the concepts in the video clips, misconceptions, and suggestions for the next instructional steps to eliminate students' misconceptions or to extend students' understanding.

Discussions in each group were audio-recorded. During the discussions, the researcher walked around the groups to observe the participants unobtrusively without interrupting them. Group discussions took approximately 15 minutes, depending on the content of the video clips. After the group discussions were completed, the whole-class discussion was initiated. Whole class discussions were led by the researcher regarding the participants' attention features in the video clips, interpretations of students' mathematical thinking, and decisions about the next diagnostic or instructional steps based on students' understanding. During the whole-class discussions, the researcher did not prompt the participants to a particular response. However, she asked some questions to facilitate the discussions, such as "*What did you notice about the student's mathematical thinking in the video clip?*" or "*Is there anything else you noticed*?". All whole-class discussions were video-recorded and took approximately 30 minutes. Group discussions and whole-class discussions were implemented in similar ways in each session.

3.5.8 Noticing Questionnaire

Noticing questionnaire was given to 32 pre-service teachers as a pretest at the beginning of the study and as a posttest at the end of the study. The questionnaire, which was implemented as a pretest, was named the initial noticing questionnaire and its aim was to determine the pre-service teachers' existing professional noticing skills in geometric measurement. The questionnaire, which was implemented as a posttest was named the final noticing questionnaire since the aim of it was to determine their final professional noticing skills measurement. Detailed information on how the noticing questionnaire was developed and implemented is given below.

3.5.8.1 Development of Noticing Questionnaire

In order to develop a noticing questionnaire, firstly, five problems (nine problems with sub-problems), which were parallel to the content of the geometric measurement tasks were developed. Problem 1 examines students' knowledge about perimeter, which was adapted from Kellogg (2010) and modified by the researcher (Figure 3.26). In this problem, one of the given figures is cut without changing its width and length, and another figure is obtained. Students are asked to compare the perimeter of these two figures. In order to respond to the problem correctly, students have to realize that cutting the piece off from the figure in this way preserves the perimeter. The problem provides students with an opportunity to realize how and where the piece is cut affects the perimeter of the shape.

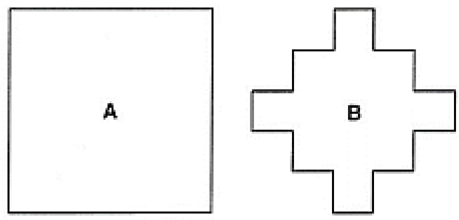
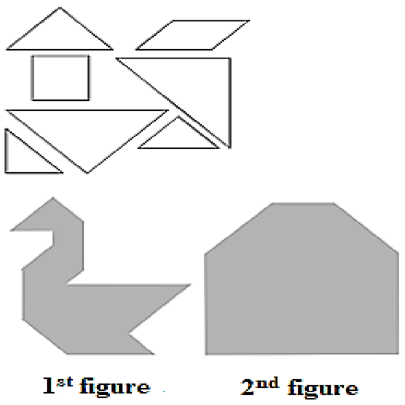


Figure A is cut in such a way that its width and length do not change, and Figure B is obtained.
What can you say when you compare the perimeters of the figures? Justify your response.

Figure 3. 26 Problem 1

Problem 2 examines students' knowledge about perimeter and area measurement, which was adapted from Steele (2013) and modified by the researcher (Figure 3.27). This problem asks students to compare the perimeter and area of the given two figures obtained by arranging the tangram pieces differently. In responding to the problem, students have to recognize that different arrangements of the tiles do not change the area; that is, the area is conserved. However, different perimeters can be found through different arrangements of the tiles. The problem provides students with an opportunity to realize the non-constant relationship between area and perimeter.



Using all seven tangram pieces, the following two figures are obtained.

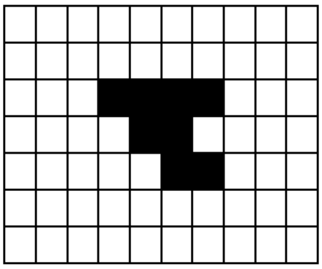
a) What can you say when you compare the perimeters of the figures? Justify your response.

b) What can you say when you compare the area of the figures? Justify your response.

Figure 3. 27 Problem 2

Problem 3 assesses students' knowledge about perimeter and area measurement, which was adapted from Lappan et al. (2014) and modified by the researcher. In this problem, students are asked to determine the least and maximum number of squares that can be added without changing the perimeter of the given figure, to show the added square(s) by painting on the figure, and to find the area of the new figures formed when the squares are added (Figure 3.28). While doing this, students should pay attention to the following: The squares should come into contact from the edges/be connected to each other from the edges. In responding to the problem, students have to recognize that a figure with a fixed perimeter can have multiple areas, understand how adding squares affects perimeter and area, and where to add squares to keep the perimeter constant. Similar to the second problem, this problem provides students with an opportunity to realize the non-constant relationship between area and perimeter.

Add square(s) to the figure given below in a way that the perimeter of the figure does not change (The squares should contact from the edges/ be connected to each other from the edges)



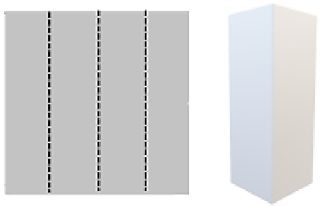
a) What would be the least number of squares you can add? Show where you added the square(s) by painting and find the area of the new figure.
b) What would be the maximum number of squares you can add? Show where you added the square(s) by painting and find the area of the new figure.

Figure 3. 28 Problem 3

Problem 4 assesses students' knowledge about volume measurement (Figure 3.29). In this problem, students are asked to compare the volume of a prism and a cube, which are made using the same paper. To respond correctly, student have to realize that although the lateral area of the objects is the same, the volume changes depending on how the papers are folded. The problem provides students with an opportunity to realize the impact of changes in the dimensions on volume.

Making a Prism

A square-shaped paper is divided into four equal parts by folding along the lines. Then, an open square prism is obtained by combining the edges.

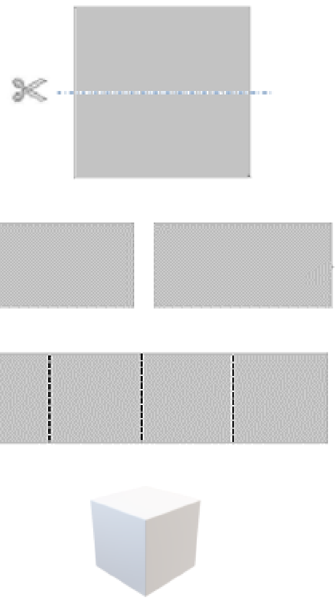


Making a Cube

Another square-shaped paper of the same size is divided into two equal parts by cutting horizontally from the middle.

The cut parts are combined in a way that short edges are coincident, and a rectangle is created.

This rectangular paper is divided into four equal parts by folding along the lines. Then, an open cube is obtained by combining the edges.



What can you say when you compare the volume of the prism and cube? Justify your response.

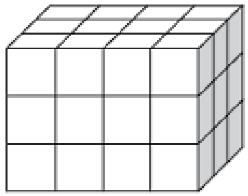
Figure 3. 29 Problem 4

Problem 5 explores students' knowledge about volume and surface area measurement (Figure 3.30). In this problem, students are asked to determine the height of the new prisms formed by using all the cubes resulting from the breaking of the given prism into unit cubes. They are also asked to identify the prism with the greatest volume and the prism with the greatest surface area. Students have to discern that prisms can have the same volume but a different surface area to respond to the problem correctly. The problem allows them to realize the non-constant relationship between volume and surface area. The Turkish version of the problems is provided in Appendix E.

a) The prism (a) is broken into unit cubes, and they are used again to form the prisms (b) and (c), whose first layers are already given in Figure 1 and Figure 2, respectively. What will the height of the resulting prism (b) and (c) be? Please explain.

b) Which of the three prisms (a-b-c) has the most significant volume? Justify your response.

c) Which of the three prisms (a-b-c) has the greatest surface area? Justify your response.


(a)

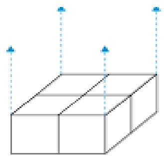


Figure 1

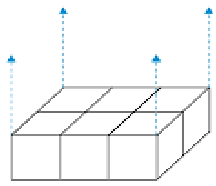


Figure 2

Figure 3. 30 Problem 5

In order to explore how pre-service teachers attend to students’ solutions, interpret students’ understanding, and respond based on students’ understanding, there was a need for different student solutions. Therefore, to obtain different solutions that involve different characteristics of understanding, the problems were presented to middle school students. Students had to complete the seventh grade to be able to answer all the problems. Therefore, it was decided to present the problems to the eighth grade students who were accessible to the researcher. These students were asked whether they would like to participate, and the problems were given to those who were volunteers. Fifty-two students in a public middle school in Sakarya solved the problems. After the students’ solutions were obtained, they were examined to develop the noticing questionnaire. To explore pre-service teachers’ professional noticing skills through the noticing questionnaire, different student solutions were selected in a way that both correct and incorrect student solutions were included in the questionnaire. Two or three student solutions that reflect different characteristics of understanding were provided under each problem in the questionnaire to enable pre-service teachers to recognize the differences between the student solutions and the ways they respond to certain aspects of understanding (Sánchez-Matamoros et al., 2019). The students whose solutions were included in the questionnaire were named Ada, Tuna, Sare, Can, Hazal, Yelda, Utku, Mert, Bade, Eylem, and Kaan (pseudonyms) respectively. English translations of the solutions are given here. The original solutions in Turkish are provided in Appendix F. Students' solutions to Problem 1 are given in Figure 3.31.

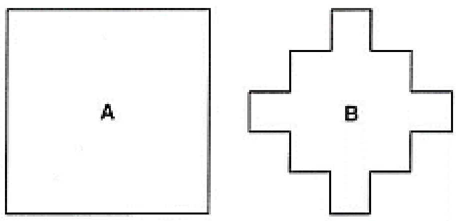
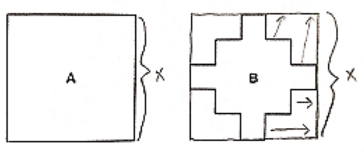


Figure A is cut in such a way that its width and length do not change, and Figure B is obtained.
What can you say when you compare the perimeters of the figures? Justify your response.

Students Ada, Tuna and Sare responded as follows:

Ada: The perimeter of B is larger because a curved path is longer than a straight path.



Tuna: The perimeter of B is smaller because B is the cut version of A.

Sare: A and B have equal perimeter. In B the perimeter is inside.

Figure 3. 31 Students' solutions to Problem 1

In this problem, among three students' solutions, while the solutions of Ada and Tuna were incorrect, Sare's solution was correct. Ada thought that shape B had a larger perimeter because the number of sides increased after cutting. She believed that more sides led to a larger perimeter, i.e., there was a direct relationship between the number of sides and perimeter. Here, the area of the shape decreased due to the cutting, but its perimeter did not change. Tuna, however, accepted that there was a direct relationship between perimeter and area, considering that the perimeter of the shape decreased as its area decreased. Sare transformed shape B into shape A by moving the newly formed sides on the cut parts with arrows, thus matching the sides of the shapes A and B and showing that the perimeter of the shape did not change. For Problem 2, students' solutions are provided in Figure 3.32.

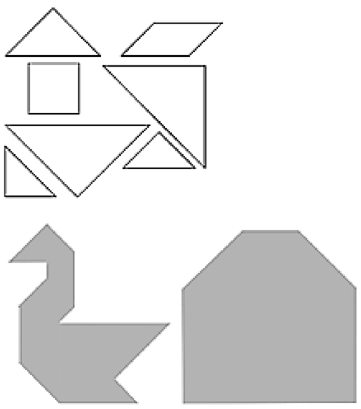
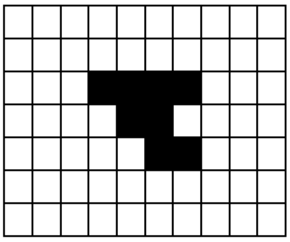
 **1st figure** **2nd figure**	Using all seven tangram pieces, the following two figures are obtained. a) What can you say when you compare the perimeters of the figures? Justify your response. **Can:** In figure 1, most of the sides of each piece form the perimeter of the figure. In figure 2, the perimeter of figure 1 is larger because more sides of each piece are inside the figure. **Hazal:** The perimeters of figure 1 and figure 2 are equal. Only the locations of the tangram pieces are different. b) What can you say when you compare the area of the figures? Justify your response. **Can:** Since two figures consist of the same parts, their areas are equal. **Hazal:** I think the area of figure 2 is larger because figure 2 is wider.

Figure 3. 32 Students' solutions to Problem 2

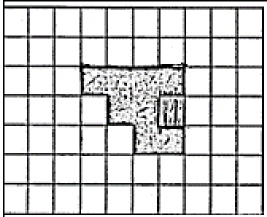
In the second problem, Can's solutions to both sub-problems were correct, whereas Hazal's solutions were incorrect. In sub-problem *a*, although the same tangram pieces were used, how the pieces were placed affected the perimeter. Can was aware that in the first figure, the pieces were placed in such a way that most of the sides were outside the shape and enlarged its perimeter. Since the pieces used were the same, Hazal thought that the perimeters of the shapes were the same. Here, the area was conserved, but Hazal thought that the perimeter was conserved. The perimeters of the figures were affected by the location of the pieces and how they were placed. Therefore, Hazal confused the concepts of area and perimeter.

In sub-problem *b*, since two figures were made up of the same pieces, Can accepted their areas as equal. Since both figures consisted of the same pieces (no change in pieces and the number of pieces), the area was conserved. It was not affected by how the pieces were placed (provided there was no gap and no overlap). That is, the sum of the area of the individual pieces was equal to the area of the figure formed by the individual pieces. Hazal focused on the appearances of the figures and thought that the second figure had a larger area because it looked wider/bigger. She was not aware of the area conservation. The area formula in the form of length x width may have caused the student to think this way. The student may have given such an answer, thinking that the width directly increased the area of the figure due to the formula. Students' solutions to Problem 3 are presented in Figure 3.33.

Add square(s) to the figure given below in a way that the perimeter of the figure does not change (The squares should contact from the edges/ be connected to each other from the edges)



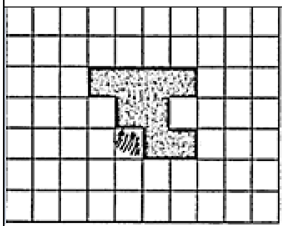
a) What would be the least number of squares you can add? Show where you added the square(s) by painting and find the area of the new figure.



Yelda:

1 square can be painted.

Its area becomes 9 square units and increases.



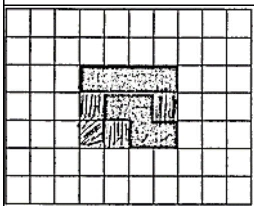
Utku:

1 square can be used.

3 times 4 = 12

12-3= 9 square units

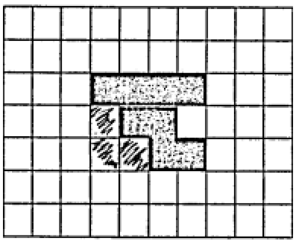
b) What would be the maximum number of squares you can add? Show where you added the square(s) by painting and find the area of the new figure.



Yelda:

We can add 4 squares.

Its area becomes 12 square units and increases.



Utku:

3 squares can be used.

3 times 4 = 12

12-1= 11 square units

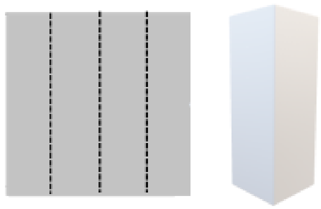
Figure 3. 33 Students' solutions to Problem 3

In the third problem, the solutions of Yelda to both sub-problems were incorrect. On the other hand, the solution of Utku to sub-problem *a* was correct, but his solution to sub-problem *b* was partially correct. In sub-problem *a*, the answer given by Yelda that at least one square can be painted was correct, but the place where she painted the square was incorrect because the added square changed the perimeter of the figure. Before adding the square, the effect of that part on the perimeter was three units, whereas after adding the square, the effect on the perimeter was one unit. Therefore, after adding the square, there was a decrease of two units in the perimeter; that is, the perimeter of the figure decreased from 16 units to 14 units. The answer given by Utku was that at least one square can be added, and the place where he painted the square was correct. The added square did not change the perimeter of the figure. Before and after adding the square, the effect of that part on the perimeter was two units.

In sub-problem *b*, in addition to the square she painted in sub-problem *a*, Yelda added three more squares to the lower left corner of the figure and found the maximum number of squares to be added as four. Although the three squares that were added later did not change the perimeter of the figure, the student's solution was incorrect because the first square that was added changed the perimeter. Moreover, the maximum number of squares that could be added was eight, not four. Utku added three squares to the lower left corner and found the maximum number of squares to be three. The student's answer was partially correct. Although the added three squares did not change the perimeter, this was not the maximum number of squares that could be added without changing the perimeter. Students' solutions to Problem 4 are given in Figure 3.34.

Making a Prism

A square-shaped paper is divided into four equal parts by folding along the lines. Then, an open square prism is obtained by combining the edges.

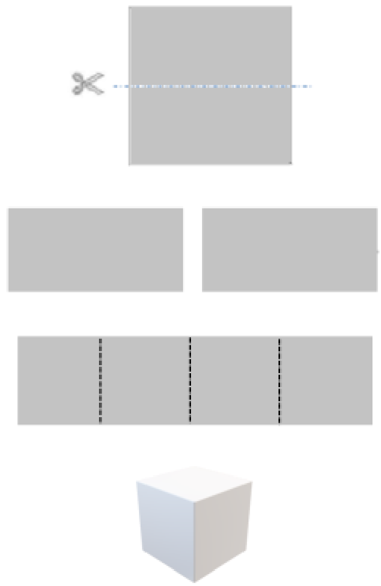


Making a Cube

Another square-shaped paper of the same size is divided into two equal parts by cutting horizontally from the middle.

The cut parts are combined in a way that short edges are coincident, and a rectangle is created.

This rectangular paper is divided into four equal parts by folding along the lines. Then, an open cube is obtained by combining the edges.



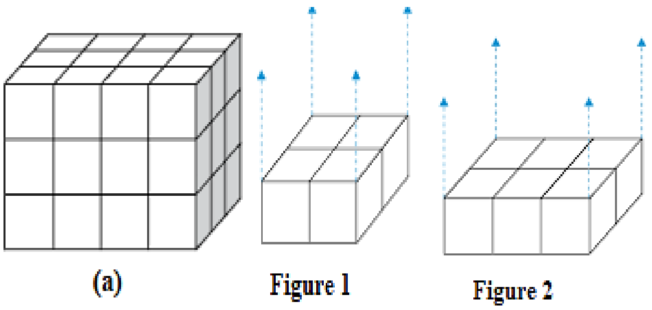
What can you say when you compare the volume of the prism and cube? Justify your response.

Mert: Their areas are equal, so their volumes must also be equal.

Bade: When I think of a bowl of rice, if it covers the whole cube, the cube is half the height of the prism, so the prism will be half full (if we pour the same rice into the prism). That is why the volume of the prism is larger.

Figure 3. 34 Students' solutions to Problem 4

In the fourth problem, both students' solutions were incorrect. A prism and a cube were formed by folding papers of the same size in different ways. Since both prisms were made using the same papers, the area of the papers was conserved when the prisms were formed, but the volume of the prisms was not conserved. The volume of the cube was twice the volume of the prism. Mert generalized the conservation of area to the conservation of volume. He had the same area-same volume misconception. In other words, he thought the volume of the prisms formed using papers with the same area should be the same. Bade considered the volume of the prism to be larger. She thought that the volume of the prism was two times the volume of the cube, saying that if we pour the rice that fills the whole cube into the prism, it reaches half. However, the volume of the cube was twice the volume of the prism. Here, the student focused only on the heights of the prism and the cube and made a comparison accordingly. She ignored the other dimensions (width and length) of the prism and cube. For Problem 5, students' solutions are provided in Figure 3.35.



The prism (a) is broken into unit cubes, and they are used again to form the prisms (b) and (c), whose first layers are already given in Figure 1 and Figure 2, respectively. What will the height of the resulting prism (b) and (c) be? Please explain.

Eylem:

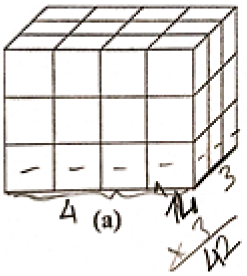
9 units in height

36 | 4
-36 | 9
00

6 units in height

36 | 6
36 | 6
00

Kaan:



10 units in height

42 | 4
-4 | 10
02

7 units in height

42 | 6
-42 | 7
00

Which of the three prisms (a-b-c) has the greatest volume? Justify your response.

Eylem: Their volumes are equal because the same number of cubes are used in prisms.

Kaan: Prism (a) because it has the widest base.

Which of the three prisms (a-b-c) has the greatest surface area? Justify your response.

Eylem: Prism (a) because it has the maximum number of cubes on its top face.

A=4x3=12
B=2x2=4
C=3x2=6

Kaan: Prism (b) has the largest surface area because it is the higher than the other prisms.

Figure 3. 35 Students' solutions to Problem 5

In the fifth problem, Eylem's solution to sub-problem *a* was correct, while Kaan's solutions was incorrect. In sub-problem *b*, Eylem's solution was correct, but Kaan's solution was incorrect. Eylem's solution to sub-problem *c* was incorrect, while Kaan's solution was partially correct. In sub-problem *a,* Eylem correctly found the number of unit cubes forming prism (a) to be 36 unit cubes. In prism (b), there were four cubes in a layer. Therefore, she divided 36 by 4 to find how many of these layers were iterated because the number of iterations gave the height of the prism (b). She correctly found this as nine units. Likewise, in a prism (c), there were six cubes in a layer. Hence, she divided 36 by 6 to find how many layers were iterated. The number of iterations gave the height of the prism (c). She correctly found this as six units. On the other hand, Kaan found the number of cubes forming the prism (a) incorrectly as 42 cubes. He first counted the squares on the visible faces while finding the number of cubes in the given layer. He thought that there would be four unit cubes in the front and three unit cubes on the right; in the same way, he thought that there would be seven unit cubes on the back and left, and found the number of cubes in one layer as 14. Since he counted the squares on the visible faces in the layer, he counted the cubes in the corners twice but did not count the two invisible cubes in the middle. This shows that he thought two-dimensionally rather than three and did not have an understanding of spatial structuring. After finding 14 unit cubes in a layer, he multiplied 14 by three since there were three layers and found the total number of cubes to be 42. Although the division operations he performed to find the heights of prisms (b) and (c) were correct, the solution was not correct because he incorrectly found the total number of cubes in prism (a).

In sub-problem *b*, Eylem thought that since all prisms consisted of an equal number of unit cubes, their volumes were equal. She recognized that the number of cubes that made up an object gave the volume of that object. Kaan thought the volume of prism (a) was larger because of its larger width. Here, the student focused only on one dimension, ignoring other dimensions. In sub-problem *c,* Eylem found the area of a face by calculating only the number of squares on the top face, but for the surface area, she would have to find the area of all the faces and add them up. She considered the surface area as the area of only one face. Kaan's solution was partially correct.

The surface area of prism (b) was larger than the surface area of the other prisms. In prism (b), fewer faces were contacted, and a higher number of faces faced outward, which enlarged the surface area. This was because the prism (b) was higher. However, this is not always the case. When a prism whose height is greater than its length and width is tilted, its height decreases, but its surface area does not change. Therefore, it would be more accurate to express it with the number of faces that affects the surface area rather than the height.

After the students' solutions were provided under each problem, expert opinions were taken for the content and format of the instrument, and necessary revisions were made to the instrument. Then, a pilot study was conducted with two pre-service teachers who were not participants in the main study, and based on the pilot study, the questionnaire was revised, and it was ready for administration to the participants. In the questionnaire, the participants were asked to respond to three prompts adapted from Jacobs et al. (2010) and modified by the researcher for each student solution in each problem as in the following:

i. Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?

ii. Please explain in detail what you learned about the student's understanding based on his/her response.

iii. Pretend that you are the teacher of this student. If the student's solution is correct, what problem(s) might you pose next to help the student progress in his/her mathematical thinking/understanding? Why? If the student has a misconception, what do you do to eliminate his/her misconception? Please explain in detail.

The participants were required to answer these open-ended prompts in writing. The purpose of the first prompt is to explore the extent to which pre-service teachers attended to students' strategies, the second prompt tries to explore how pre-service teachers interpret students' understanding, and the third prompt aims to examine pre-service teachers' instructional actions to eliminate students' misconceptions and extend their mathematical thinking. The Turkish version of the noticing prompts in the questionnaire is provided in Appendix G.

3.6 Data Analysis

Noticing questionnaires, reflection papers, semi-structured interviews, group discussions and whole-class discussions constituted the data of the present study. In order to analyze the data obtained from the questionnaires, professional noticing of children's mathematical thinking framework developed by Jacobs et al. (2010) was modified by considering the data collected, and a more detailed categorization was made for each component based on the data collected. Furthermore, an open coding method was used to ascertain the characteristics of the levels of each component. Participants' responses to each of the three prompts in the questionnaires were grouped and compared using a constant comparative method (Strauss & Corbin, 1994) according to whether and how they (i) attend to students' strategies, (ii) interpret students' understanding, and (iii) respond on the basis of students' understanding.

The first component of the professional noticing framework, i.e., attending to students' solutions, originally included two levels, which are evidence and lack of evidence (Jacobs et al., 2010). In the present study, in order to represent all responses, the categorization developed by Tekin Sitrava et al. (2022) was taken into account, and two more levels, limited evidence, and substantial evidence, were added between the lack of evidence and robust evidence. As a result, participants' attending skills were examined at four levels: lack of evidence, limited evidence, substantial evidence, and robust evidence. Table 3.5 shows the characteristics of the levels in attending to each student's solution.

Table 3. 5 The characteristics of the levels in attending to each student's solution in the questionnaire

Attending to students' solutions	
No attention	
Lack of evidence	○ identify the student's solution incorrectly ○ provide inappropriate evidence ○ rephrase the student's solution ○ identify the student's solution only as correct/incorrect without providing any details
Limited evidence	○ describe incorrect solution without mathematical properties ○ provide a general description of a correct solution ○ provide a description of the solution superficial about the concept or without making a connection with the concept ○ identify the student's incorrect solution but an inability to recognize misconception in the student's solution ○ provide a description of some, but not all, of the concepts in the solution ○ provide a description of either of the concepts in the solution without making a connection between them ○ identify the student's correct solution but an inability to associate the student's justification with the concept
Substantial evidence	○ recognize the student's ideas and misconceptions to some extent in an incorrect solution ○ describe the student's correct solution shortly with mathematical properties ○ provide a short description of the student's solution using all concepts ○ provide a short description of the student's solution by making a connection between the concepts
Robust evidence	○ recognize the student's ideas and misconceptions in an incorrect solution ○ describe the student's solution in detail with mathematical properties ○ provide a detailed description of all concepts in the student's solution

The second component of the professional noticing framework, i.e., interpreting students' understanding, originally included three levels, which are lack of evidence, limited evidence, and robust evidence (Jacobs et al., 2010). In the current study, in order to represent all responses, the categorization developed by Tekin Sitrava et al.

(2022) was taken into account, and one more level, substantial evidence, was added between limited evidence and robust evidence. Therefore, participants' interpreting skills were examined at four levels: lack of evidence, limited evidence, substantial evidence, and robust evidence. The characteristics of the levels in interpreting each student's understanding are presented in Table 3.6.

Table 3. 6 The characteristics of the levels in interpreting each student's understanding in the questionnaire

Interpreting students' understanding	
No interpretation	
Lack of evidence	○ provide an incorrect interpretation of the student's understanding ○ provide an irrelevant interpretation ○ state that I didn't understand what the student did/thought about ○ make comments about only the student understood or not
Limited evidence	○ point out the student's mistake ○ make comments about the student's understanding without any mathematical properties ○ blame the student (e.g., for memorization or lack of knowledge) ○ make comments about the student's understanding by mentioning either of the concepts without mentioning all of them
Substantial evidence	○ provide a valid justification for the student's understanding of the concept without much detail ○ make comments about the student's understanding by mentioning both concepts but without mentioning the relationship between them ○ make comments about the student's understanding by mentioning both concepts without much detail
Robust evidence	○ provide a detailed explanation about the possible reasoning behind the student's mathematics

The third component of the professional noticing framework, i.e., deciding how to respond, originally consisted of three levels, which are lack of evidence, limited evidence, and robust evidence (Jacobs et al., 2010). In the present study, in order to represent all responses, two more levels, named medium evidence and substantial evidence, were added between the limited evidence and robust evidence. The substantial evidence was adapted from Tekin Sitrava et al. (2022), and medium evidence was added by the researcher. In this way, participants' skills of deciding how to respond were examined at five levels: lack of evidence, limited evidence, medium evidence, substantial evidence, and robust evidence. The characteristics of the levels in deciding how to respond to each student are provided in Table 3.7.

Table 3. 7 The characteristics of the levels in deciding how to respond to each student in the questionnaire

Deciding how to respond based on the students' understanding component	
No response	
Lack of evidence	○ provide an inappropriate suggestion for the concept(s) in the problem ○ provide an inappropriate suggestion as a result of incorrect attention
Limited evidence	○ provide a teacher-centered suggestion ○ provide a general suggestion ○ ask the student for clarification of thinking ○ ask a factual question(s) ○ ask a question(s) that is not clear on how it extends the student's thinking
Medium evidence	○ provide an orientation with questions/activities to answer ○ provide a procedural understanding-focused suggestion ○ provide an activity that helps the student realize an incorrect answer but is not sufficient to eliminate the student's misconception ○ ask probing question(s)
Substantial evidence	○ provide a specific suggestion that helps the student overcome his/her misconception without much detail ○ provide a specific suggestion that extends the student's understanding without much detail
Robust evidence	○ provide a detailed suggestion that extends the student's understanding ○ provide a detailed suggestion that makes the student understand

After determining the participants' attending, interpreting, and deciding how to respond levels based on the responses they provided to noticing prompts for each student solution in the noticing questionnaire, the responses were examined once more as a whole to ascertain a single level for the noticing components and to come up with a level in each problem. To do this, participants' responses to each noticing prompt for two or three students' solutions were considered together. Thus, it was possible to compare the participants' levels for attending, interpreting, and deciding how to respond in each problem in the initial and final noticing questionnaire. Levels of participants' professional noticing skills in the questionnaire are set out in Table 3.8.

Table 3. 8 Levels of participants' professional noticing skills in the questionnaire

	Component	Description
Attending	No attention	
	Lack	Identify none of the mathematical properties in the students' solutions
	Limited	Identify mathematical properties in one of two students' solutions to some extent Identify mathematical properties in one or two of three students' solutions to some extent
	Substantial	Identify mathematical properties in both students' solutions to some extent Identify mathematical properties in one of the students' solutions by explaining in detail Identify mathematical properties in three students' solutions by explaining at most one of them in detail Identify mathematical properties in two of the three students' solutions by explaining in detail
	Robust	Identify mathematical properties in both students' solutions by explaining at least one of them in detail Identify mathematical properties in three students' solutions by explaining at least two of them in detail
Interpreting	No interpretation	
	Lack	Inability to provide a valid justification for any students' understanding
	Limited	Provide a valid justification for one of two students' understanding without providing much detail Provide a valid justification for one or two of three students' understanding to some extent
	Substantial	Provide a valid justification for both students' understanding without providing much detail Provide a valid justification for one of the student's understanding by explaining in detail Provide a valid justification for three students' understanding by explaining at most one of them in detail Provide a valid justification for two of the three students' understanding by explaining in detail
	Robust	Provide a valid justification for both students' understanding by explaining at least one of them in detail Provide a valid justification for three students' understanding by explaining at least two of them in detail

Table 3.8 (cont'd)

Deciding how to respond	No response	
	Lack	Provide inappropriate suggestions for students
	Limited	Provide suggestions that are not sufficient to eliminate the students' misconceptions or extend the students' understanding
	Medium	Provide specific suggestions for one of two students to eliminate the student's misconception or extend the student's understanding without providing much detail Provide specific suggestions for one or two of three students to eliminate the student's misconception or extend the student's understanding without providing much detail
	Substantial	Provide specific suggestions for both students to help the students overcome their misconceptions or extend the students' understanding without providing much detail Provide specific suggestions for one of two students to eliminate the students' misconceptions or extend their understanding by explaining in detail Provide specific suggestions for three students to help the students overcome their misconceptions or extend the students' understanding, but explain at most one of them in detail Provide specific suggestions for two of the three students to help students overcome their misconceptions or extend their understanding by explaining in detail
	Robust	Provide suggestions for both students to make the students understand or extend their understanding by explaining at least one of them in detail Provide suggestions for three students to make the students understand or extend their understanding by explaining at least two of them in detail

In order to find an answer to the first research question and to ascertain the extent to which pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement changes when they participate in a video-based module situated in the pedagogies of practice framework, the participants' levels for each noticing component in each problem were determined based on the table above. Then, the frequency analysis was conducted by calculating the frequency and percentage of responses for each level of three noticing components in each problem in the initial and final noticing questionnaire, and these were presented in the bar charts in the findings section. In addition, statistical analysis was performed using statistical analysis software to find out whether the improvement in the final noticing questionnaire observed through the frequencies and percentages was statistically significant. Since the participants were measured on two occasions, i.e., the initial noticing questionnaire and final noticing questionnaire, and the professional noticing responses represented ordinal data, a Wilcoxon Signed Ranks Test was used. The aim of this was to determine whether there was a statistically significant difference between the three components of professional noticing framework for each problem in the initial and final noticing questionnaire and thus to answer the sub-research question: "Is the change in pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement from pre-test to post-test statistically significant?"

Reflection paper 1, written individually by six participants in each session, was analyzed to explore in depth the changes in the participants' professional noticing skills. In this way, it was possible to track changes on an individual level more consistently. Furthermore, to uncover the impact of the discussions on the participants' professional noticing skills, reflection paper 2 was analyzed. Moreover, to reveal the participants' professional noticing skills in the context of the task-based interviews, reflection paper 3 was analyzed. The participants' attending to each student's solution, interpreting each student's understanding, and deciding how to respond to each student in the reflection papers were determined through the modified version of the professional noticing framework (Jacobs et al., 2010), as presented in Table 3.9.

Table 3. 9 Levels of participants' professional noticing skills for each student's solution in the video clips

	Component	Description
Attending	Lack	Identify the student's solution incorrectly Identify the student's solution only as correct/incorrect without providing any details
	Limited	Describe the student's solution without mathematical properties (general descriptions)
	Substantial	Identify mathematically significant details in the student's solution to some extent
	Robust	Identify all mathematically significant details in the student's solution
Interpreting	No interpretation	
	Lack	Provide an incorrect interpretation of the student's understanding Make comments about only the student understood or not
	Limited	Make comments about the student's understanding without any mathematical properties (in broad terms) Provide limited justification for the student's understanding (e.g., blaming the student for lack of knowledge)
	Substantial	Provide justification about the possible reasoning behind the student's solution without much detail
	Robust	Provide a detailed explanation about the possible reasoning behind the student's solution
Deciding how to respond	No response	
	Lack	Provide an inappropriate suggestion as a result of incorrect attend
	Limited	Provide a teacher-centered suggestion Provide a general suggestion
	Medium	Provide a procedural understanding-focused suggestion Provide an orientation with questions/activities to answer
	Substantial	Provide a specific suggestion that helps the student overcome misconceptions or extend the student's understanding without much detail
	Robust	Provide a detailed suggestion that makes the student understand or extends the student's understanding

Additionally, after determining the participants' attending, interpreting, and deciding how to respond levels based on the responses they provided to noticing prompts for each student's solution in the video clips in reflection paper 1, the responses were examined once more to ascertain a single level for the noticing components and to come up with a level in each session. To do this, participants' responses to each noticing prompt for two or three students' solutions to the same task in the video clips were considered together. Thus, it was possible to compare the participants' levels for attending, interpreting, and deciding how to respond throughout the sessions. Levels of participants' professional noticing skills in reflection paper 1 are shown in Table 3.10.

Table 3. 10 Levels of participants' professional noticing skills in reflection paper 1

	Component	Description
Attending	Lack	Identify none of the mathematically significant details in students' solutions
	Limited	Identify mathematically significant details in one of two students' solutions to some extent Identify mathematically significant details in one or two of three students' solutions to some extent
	Substantial	Identify mathematically significant details in all students' solutions to some extent Identify all mathematically significant details in one of three students' solutions and those in two students' solutions to some extent
	Robust	Identify all mathematically significant details in two of three students' solutions and those in one student's solution to some extent Identify all mathematically significant details in one of two student's solutions and those in the other student's solution to some extent Identify all mathematically significant details in both students' solutions
Interpreting	Lack	Inability to provide a valid justification for any student's understanding
	Limited	Provide a valid justification for one of two students' understanding to some extent Provide a valid justification for one or two of three students' understanding to some extent
	Substantial	Provide a valid justification for all students' understanding to some extent Provide a valid justification for one of two students' understanding Provide a valid justification for three students' understanding with two students' understanding to some extent
	Robust	Provide a valid justification for all students' understanding with one student's understanding to some extent Provide a valid justification for both students' understanding

Table 3.10 (cont'd)

Deciding how to respond	No response	
	Lack	Provide inappropriate suggestions for students
	Limited	Provide suggestions that are not sufficient to eliminate the student's misconception or extend the student's understanding
	Medium	Provide a specific suggestion for one of two students to eliminate the student's misconception or extend the student's understanding without providing much detail Provide a specific suggestion for one of three students to eliminate the student's misconception or extend the student's understanding without providing much detail Provide a specific suggestion for one of three students to eliminate the student's misconception or extend the student's understanding by explaining in detail
	Substantial	Provide specific suggestions for both students to help the student overcome their misconceptions or extend the student's understanding without providing much detail Provide specific suggestions for one of two students to eliminate the student's misconception or extend the student's understanding by explaining in detail Provide specific suggestions for three students to help the student overcome their misconceptions or extend the student's understanding but explain at most one of them in detail Provide specific suggestions for two of three students to eliminate the student's misconception or extend the student's understanding by explaining at least one of them in detail
	Robust	Provide specific suggestions for both students to make the student understand or extend the student's understanding by explaining at least one of them in detail Provide specific suggestions for three students to make the student understand or extend the student's understanding by explaining at least two of them in detail

3.7 Trustworthiness of the Study

Trustworthiness is a way for researchers to convince themselves and readers that the research findings are remarkable (Lincoln & Guba, 1985). In order to ensure the trustworthiness of a qualitative research study, a researcher should meet the trustworthiness criteria, which are credibility, transferability, dependability, and confirmability (Lincoln & Guba, 1985). These concepts correspond to internal validity, external validity, reliability, and objectivity in quantitative research.

3.7.1 Credibility

Internal validity aims to make sure that a test measures what it is supposed to measure. Credibility, in preference to internal validity in qualitative research, is one of the main criteria that researchers should meet (Lincoln & Guba, 1985). For credibility, Lincoln and Guba (1985) suggested some activities, including prolonged engagement, persistent observation, and triangulation, which lead to credible findings and interpretations as well as peer debriefing, negative case analysis, referential adequacy, and member checking. In the present study, the first technique used to increase credibility was *prolonged engagement*. It is emphasized that the researcher cannot overcome distortions if the participants do not accept the researcher as a member of the group (Lincoln & Guba, 1985). When this study was carried out, the participants were fourth-grade students, and the participants and the researcher had known each other since the beginning of their undergraduate studies, i.e., for three years. This provided the researcher with an opportunity to invest sufficient time with the participants to get to know them and build trust.

The present study employed triangulation as a second technique for establishing credibility. Data source triangulation was implemented to enhance credibility by utilizing multiple data sources, including questionnaires, reflection papers, interviews, and group and whole-class discussions. Furthermore, method triangulation was employed due to the study's use of mixed methods, combining both qualitative and quantitative approaches.

The third technique used to establish credibility was *member checking*. Member checking is testing data, categories, or interpretations with participants from whom the data were collected (Lincoln & Guba, 1985). It allows participants to add information by enabling them to remember the things they did not mention before. In addition, member checking enables participants to correct errors and challenge the things that are perceived as incorrect interpretations. In this study, I discussed the six participants' responses during the semi-structured interviews for clarification of their written responses in the reflection papers. For example, I asked, "You wrote…Do you mean…?" In this way, I tested the correctness of the interpretations I made from their responses.

3.7.2 Dependability

Reliability is related to the extent to which the same findings are obtained. That is, replication of inquiry yields the same results. However, when human beings are involved, the repetition of observations and measurements can be misleading since human behavior is dynamic. Therefore, this is not the case in qualitative research because replication of qualitative research may not give the same results (Merriam, 1995). In qualitative research, consistency between the findings of the study and the data collected matter (Merriam, 1995). In other words, dependability is ensured in qualitative research study if "the findings are consistent with the data presented" (Merriam & Tisdell, 2016, p. 252). To strengthen the dependability of a qualitative study, the researcher is required to describe in detail the data collection process and the emergence of categories (Merriam, 1988) so that other researchers can benefit from the report as a guide while replicating the study (Goetz & LeCompte, 1984). In addition, in-depth exploration enables the reader to evaluate whether the researcher has followed appropriate research methods (Shenton, 2004). In the present study, to establish dependability, the data collection process and the emergence of categories were presented clearly and in detail.

Inter-coder agreement is “a process whereby one or more additional coders analyze the qualitative database, provide codes for the data base, and compare the results of coders for the amount of agreement on the codes.” (Creswell, 2016, p.576). To ensure inter-coder agreement in this study, a random sample of data was analyzed by another researcher who works on teacher noticing. After the researcher and the co-coder analyzed the data individually, the researcher’s coding and the co-coder’s coding were compared. The supervisor monitored the results while prompting code clarifications and new codes and provided a critical perspective. The inter-rater reliability was calculated as 91.4%. using the formula suggested by Miles and Huberman (1994). Inconsistencies were discussed to reach a consensus.

3.7.3 Transferability

Transferability in qualitative research refers to external validity in quantitative research. External validity means “the extent to which the findings of a study can be applied to other situations” (Merriam, 1995, p. 57), i.e., generalizability. The aim of qualitative research is to understand a phenomenon in depth rather than generalize the findings. In addition, in qualitative research, it is not possible to reveal that findings can be applied to other populations or situations because of the small number of participants involved (Shenton, 2004). In order to ensure the transferability of a qualitative study, a thick description can be provided. Describing a phenomenon for other researchers who think about making a transfer helps the reader ascertain whether the transfer of the findings is possible (Merriam, 1995). In this way, they can understand the phenomenon and compare the phenomenon presented in the report with the instances that appeared in their situations (Shenton, 2004). The findings of a qualitative study can be understood within its context. Factors viewed as insignificant by the researcher and hence not covered in the research report may be crucial for readers. Researchers underlined the importance of providing information regarding the number of participants, data collection methods, the number and length of the data collection sessions, and the time period during which the data was collected (Cole & Gardner, 1979; Pitts, 1994). Thus, in the present study, I provided a thick description of the study for other researchers by explaining the context of the study, how to select participants, data collection tools, and the procedure in the methodology part in detail.

3.7.4 Confirmability

Confirmability is a concern in qualitative research, which corresponds to objectivity in quantitative research. Guba and Lincoln (1989) asserted that confirmability can be ensured when credibility, dependability, and transferability are established. Confirmability is about explicitly showing the findings and interpretation of the researcher deduced from the research data as well as showing how to draw conclusions and interpretations (Tobin & Begley, 2004). Researchers should provide the reasons for their choices regarding theory and methodology to help readers figure out the decisions that were made (Koch, 1994). Here, the findings of the study should be the result of participants' experiences and opinions, not the researcher's preferences. Triangulation has an important role in increasing confirmability to eliminate the impact of researcher bias. Miles and Huberman (1994) stressed that the main criterion for confirmability is the degree to which the researcher accepts his or her own predispositions. The beliefs that underlie the decisions taken should be stated in the research report, and the reasons for choosing one approach and the limitations of the techniques used should be acknowledged (Shenton, 2004). In the present study, confirmability was ensured through triangulation and describing how the data were collected and analyzed in a transparent manner.

3.8 Ethical Considerations

In order to do ethical research, researchers are required to carry out certain procedures (Fraenkel et al., 2011). Therefore, ethical approval was obtained from the METU Applied Ethics Research Center, the human research ethics committee, to carry out the research. Ethical approval is presented in Appendix H. In addition, informed consent was obtained from the parents of middle school students to conduct task-based interviews (Appendix I). Even if the parents accepted participation, the students were not coerced into participating in the study. Rather, the interviews were conducted with those who were willing and voluntarily agreed to take part. Moreover, to ensure confidentiality, the real names of the students in the video clips and pre-service teachers were not used in the report of the study. Instead, numbers with letters were assigned to the students as S1, S2…S9, and to the pre-service teachers as P1, P2…P32. In addition, pseudonyms were used for the middle school students whose solutions were included in the noticing questionnaire. At the beginning of the study, the participants were informed regarding all aspects of the research. That is, they were provided detailed information about the video recording of sessions, features of video clips, individual analysis of video clips, group discussions, whole-class discussions, reflection papers, questionnaires, task designs, and task-based interviews with a middle school student. They were also informed about the right to withdraw from participating in the study at any time. Furthermore, the pre-service teachers' responses to the noticing prompts in the questionnaires and reflection papers and the recordings of the interviews that followed, i.e., the data of the study, were not accessible to anyone other than the researcher both during the data collection process and after the data were collected. The preservice teachers were assured that the data collected would be held in confidence. Due to the nature of the study, neither the students nor the pre-service teachers were at any kind of risk, and hence, the study was exempt from the possibility of harm to the participants.

3.9 Researcher's Role and Bias

The researcher had an active role in data collection process, particularly, design of geometric measurement tasks, producing and selecting video clips, development of noticing questionnaire and reflection papers, interviewing and observing the participants, and data analysis process as well as interacting and meeting directly with the participants in the present study. Moreover, qualitative research requires researchers to be reflexive before and during the research process. Being reflexively does not mean that ignoring or avoiding researchers' their own biases but researchers should reflect and articulate their own subjectivities, including perspectives and biases (Sutton & Austin, 2015). In this way, readers can better understand the filters through which data are collected and analyzed, and findings are presented. In this respect, bias and subjectivity are not negative, but are unavoidable; consequently, it is important that these are expressed in advance in a clear and coherent manner for readers (Sutton & Austin, 2015). Although avoiding bias is probably impossible, qualitative researchers should try to find ways to minimize researcher bias and increase the validity of their research studies (Fraenkel et al., 2011). In the present study, I was both the implementer, i.e., the teacher, and the data collector, i.e., the researcher. Thus, researcher bias can be a risk factor in the present study. It is suggested that independent recording can reduce this threat. Hence, in order to reduce the researcher's bias, both audiotaping and videotaping were used to check the researcher's observations against these. In addition, bias can occur when the researcher asks leading questions. To prevent this, in this study, the researcher tried to facilitate the flow of discussions and increase productivity by asking some questions, but she avoided prompting the participants to a particular response. Furthermore, the researcher checked the interpretations she made on participants' responses with the participants to reduce the researcher bias in the current study. Besides, by making the purpose of the research clear for the participants, ensuring confidentiality, making the participants feel comfortable during the data collection process, the researcher tried to reduce researcher bias in the present study. In addition, researchers should support the conclusions they reached with direct quotations of the participants to reveal that the themes emerged from what participants present rather than the researcher's mind (Sutton & Austin, 2015). Hence, in the present study, while presenting the findings, quotations from questionnaires, reflection papers, discussions, and interviews were provided to support the conclusions.

3.10 Limitations and Delimitations of the Study

Fourth-year (senior) pre-service teachers in an elementary mathematics teacher education program in a state university took part in the present study. Since fourth-year pre-service teachers had already completed the methods of teaching mathematics courses before the study, they were intentionally included in the study. This situation was one of the delimitations of the study, and the findings of the study were limited to the responses of these participants. Additionally, in this study, one of the data collection tools was the reflection papers through which the data were collected from all participants. However, the presented findings were limited to the reflection papers written by six preservice teachers who were in the same group due to handling a huge amount of data. In this way, the aim was to reveal how the video-based module situated in pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking.

The number of video clips and the time period of the sessions might create delimitation of the study. This study was limited to the decomposition of 19 video clips, which were chosen by the researcher among the video recordings of task-based interviews, and these video clips were viewed by the participants in seven intervention sessions. While selecting the video clips for the intervention sessions, variety in the student's mathematical thinking in terms of the important concepts in geometric measurement was considered, and both correct and incorrect student reasoning were included. Moreover, while choosing the video clips with incorrect student reasoning, the researcher paid attention to including common and important misconceptions identified based on the literature. Considering the time period of the sessions, two or three video clips were involved in each session. Moreover, the same procedure was also followed while developing the noticing questionnaire, which included both correct and incorrect student solutions and diversity in the incorrect solutions regarding crucial issues in geometric measurement.

CHAPTER 4

FINDINGS

The purpose of this study was to examine the extent to which pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement changed when they participated in a video-based module situated in the pedagogies of practice framework. In addition, the study aimed to explore how a video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in geometric measurement. More specifically, pre-service teachers' professional noticing of students' mathematical thinking was investigated in perimeter-area and surface area-volume measurement. Therefore, the findings are presented under two headings: perimeter-area measurement and volume-surface area measurement contexts. The parts of the chapter are organized according to the order of the research questions of this study. Table 4.1 presents the research questions, corresponding data collection tools, and data analysis.

Table 4. 1 Research questions, corresponding data collection tools, and data analysis

Research Questions	Themes	Skills	Data Collection Tools	Data Analysis	Codes
1. To what extent do pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement change as they participate in a video-based module situated in the pedagogies of practice framework?	Perimeter-area relationship Volume-surface area relationship	Attending Interpreting Deciding how to respond	Noticing questionnaire (perimeter and area: first, second, and third problems volume and surface area: fourth and fifth problems)	Content analysis (Qualitative)	Modified version of the Professional Noticing Framework + open coding (lack, limited, medium, substantial, robust)
1.1. Is the change in pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement from pre-test to post-test statistically significant?	Perimeter-area relationship Volume-surface area relationship	Attending Interpreting Deciding how to respond	Noticing questionnaire (perimeter and area: first, second, and third problems volume and surface area: fourth and fifth problems)	Wilcoxon Signed Ranks Test (Quantitative)	

Table 4.1 (cont'd)

2. How does a video-based module situated in the pedagogies of practice framework support pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement?	Representation-Decomposition of practice Approximation of practice	Attending Interpreting Deciding how to respond	Reflection paper 1 Reflection paper 2 Reflection paper 3 Semi-structured interviews Group discussions Whole-class discussions	Content Analysis (Qualitative)	Modified version of the Professional Noticing Framework + open coding (lack, limited, medium, substantial, robust)

The study has two phases. In the first phase, to find an answer to the first research question, “To what extent do pre-service teachers’ professional noticing of students’ mathematical thinking in perimeter-area and volume-surface area measurement change as they participate in a video-based module situated in the pedagogies of practice framework?”, content analysis was done, and pre-service teachers’ professional noticing skills in initial and final noticing questionnaires are presented by comparing them and supported by examples of pre-service teachers’ responses. Moreover, to find an answer to the sub-research question “Is the change in pre-service teachers’ professional noticing of students’ mathematical thinking in perimeter-area and volume-surface area measurement from pre-test to post-test statistically significant?”, Wilcoxon Signed Rank test was conducted, and whether the difference between the pre-service teachers’ professional noticing skills in the initial and final noticing questionnaire was significant is revealed. In the second phase, to find an answer to the third research question, “How does a video-based module situated in the pedagogies of practice framework support pre-service teachers’ professional noticing of students’ mathematical thinking in perimeter-area and volume-surface area measurement?”, content analysis was done, and how pre-service teachers’ professional noticing skills in the representation-decomposition of practice and approximation of practice stages improved are revealed by providing evidence from semi-structured interviews, group discussions, and whole-class discussions.

4.1 Changes in pre-service teachers’ professional noticing

The implementation of the initial and final noting questionnaire as a pre-test and post-test provided an opportunity to examine the changes and improvement in the pre-service teachers’ attending, interpreting, and responding skills. In this section, it was examined whether there was an improvement from the pretest to the posttest for 32 pre-service teachers. The findings in the context of perimeter-area measurement and volume-surface area measurement regarding the three components of professional noticing, i.e., attending, interpreting, and deciding how to respond, are presented with the frequencies and percentages of the pre-service teachers' responses for each category supported by examples of pre-service teachers' responses and statistical analysis.

4.1.1 Changes in pre-service teachers' professional noticing in the context of perimeter-area measurement

In this part, in the context of perimeter-area measurement, the findings related to the problems of perimeter-area relationship in the initial and final noticing questionnaire under the headings of attending, interpreting, and deciding how to respond are given. The problems in the context of perimeter-area measurement were designed as area changes when the perimeter is fixed and perimeter changes when the area is fixed. At the end of the part, the difference between the initial and final noticing questionnaire for attending, interpreting, and deciding how to respond skills in the context of perimeter-area measurement is given in total.

4.1.1.1 Changes in pre-service teachers' attending to students' solutions in the context of perimeter-area measurement

Pre-service teachers' attending to students' solutions in the context of perimeter-area measurement is measured by their responses to attending prompts related to students' solutions in problems 1, 2, and 3 of the noticing questionnaire. The pre-service teachers' responses to the attending to students' solutions prompt in the noticing questionnaire were examined at four levels: robust level of evidence, substantial level of evidence, limited level of evidence, and lack of evidence of attending to students' solutions. Furthermore, some of the pre-service teachers did not respond to the attending prompt. This was categorized as no attention. All three problems about perimeter-area measurement in the noticing questionnaire were gathered under the theme of perimeter-area relationship. Therefore, the findings regarding the pre-service teachers' attending skills are presented under the title of pre-service teachers' attending to students' solutions about the perimeter-area relationship in the following part.

4.1.1.1.1 Pre-service teachers' attending to students' solutions about the perimeter-area relationship

In the noticing questionnaire, the first three problems (five problems with sub-problems) were about the perimeter-area relationship. The first problem and the third problem were about a fixed perimeter-changing area situation, while the second problem was about a fixed area-changing perimeter situation. The findings for the first problem and the third problem, which are related to a fixed perimeter-changing area situation, and the findings for the second problem, which is related to a fixed area-changing perimeter situation, are presented below, respectively.

4.1.1.1.1.1 Pre-service teachers' attending to students' solutions about a fixed perimeter-changing area situation

The first problem in the noticing questionnaire was named change in perimeter, and the pre-service teachers were expected to attend to three different students' solutions to this problem. Figure 4.1 shows the frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the change in perimeter problem in the initial and final noticing questionnaire. In this problem, the no attention category did not emerge. That is, all pre-service teachers provided responses for the attending component both in the initial and final questionnaire. In the initial questionnaire, almost one-third of the pre-service teachers (n=10) provided a lack of evidence, and almost two-thirds of them (n=21) provided a limited level of evidence. In addition, there were not any pre-service teachers who provided a substantial level of evidence, and only 1 of 32 pre-service teachers (%3) could provide a robust level of evidence. On the other hand, in the final questionnaire, there was a decrease in the percentage of pre-service teachers who provided a lack of evidence or limited evidence of attending. Moreover, more than half of the pre-service teachers (n=17) provided a substantial level of evidence, whereas no pre-service teachers had a substantial level of response in the initial questionnaire. In this way, after the intervention, 22 of 32 pre-service teachers' responses (69%) demonstrated at the top two levels, substantial or robust levels of evidence of attending to students' solutions. This shows that there was an improvement in the pre-service teachers' responses in the final questionnaire for attending to students' solutions in the change in the perimeter problem. As an example of the improvement in attending, pre-service teacher 11's responses to the attending prompt in the initial and final questionnaire are given below.

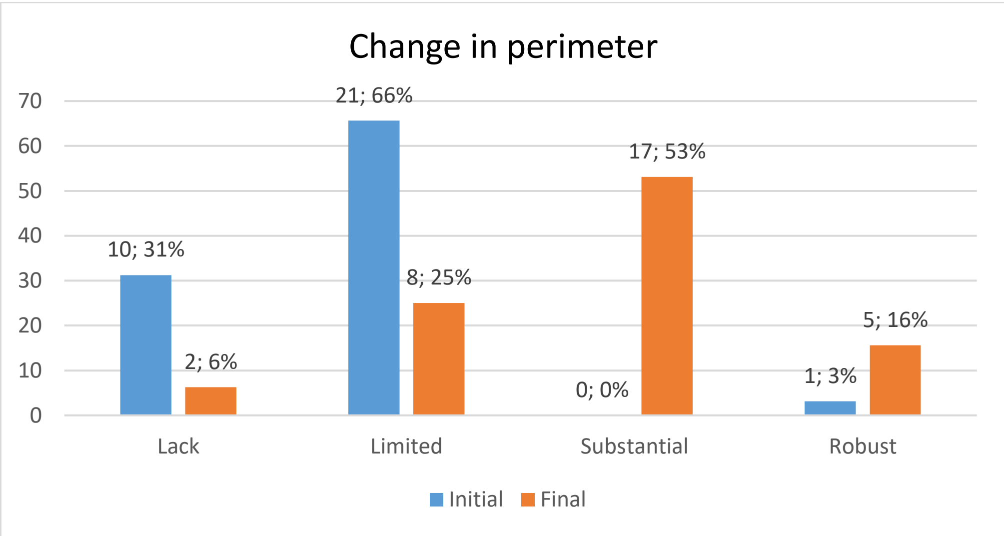


Figure 4. 1 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the change in perimeter problem in the initial and final noticing questionnaire

Figure 4.2 shows the response of pre-service teacher 11 to the change in perimeter problem in the initial noticing questionnaire.

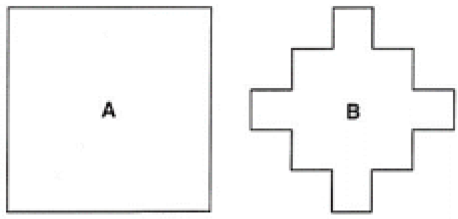


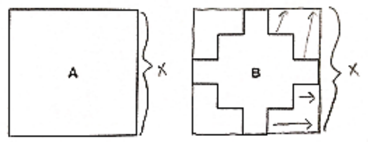
Figure A is cut in such a way that its width and length do not change, and Figure B is obtained.
What can you say when you compare the perimeters of the figures? Justify your response.

Students Ada, Tuna, and Sare responded as follows:

Ada: The perimeter of B is larger because a curved path is longer than a straight path.

Tuna: The perimeter of B is smaller because B is the cut version of A.

Sare: A and B have an equal perimeter. In B, the perimeter is inside.



(i) Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?

Ada made a visual comparison directly without making any operation on the figure. The solution is incorrect.

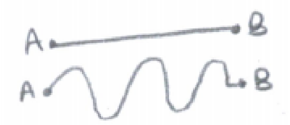
Tuna tried to establish a part-whole relationship. He said that by cutting a whole, it becomes smaller. However, this approach is wrong because the problem asks us to compare the perimeter of the figures. Therefore, his solution is incorrect.

Sare's solution is correct. By completing the cut shape, Sare showed that the perimeter of the shapes is equal.

Figure 4. 2 The response of pre-service teacher 11 to the change in perimeter problem in the initial noticing questionnaire

Here, pre-service teacher 11 attended to Ada's solution by providing a *limited* level of evidence because although the pre-service identified the student's incorrect solution, she could not recognize the misconception in the student's solution. Moreover, she described the incorrect solution without mentioning sides, length, or perimeter concepts. Pre-service teacher 11's response regarding Tuna's solution also demonstrated a *limited* level of evidence since the pre-service teacher again failed to recognize the student's misconception, i.e., the perimeter of the shape decreased as its area decreased, although she could identify the incorrect solution. Furthermore, she did not describe the solution using mathematical properties. She attended to Sare's solution by providing a *limited* level of evidence because she provided a general description of the correct solution. Overall, the response to the attending prompt given by the pre-service teacher regarding each student's solution together indicates that she provided a *lack* of evidence of attending in the change in the perimeter problem because she identified none of the mathematical properties in the students' solutions.

Response of pre-service teacher 11 in the final noticing questionnaire:

> 
>
>
> Ada confused the given situation with the situation when calculating the length of the shortest path. Under normal conditions, the curved one is longer in the two paths with the same starting and ending points. But this is not the case in this problem. Ada failed to notice the corresponding lengths in the two figures. Hence, the solution is incorrect.
>
> Tuna thinks that when a shape is cut, its perimeter decreases. This is true for the area. Although cutting parts from the whole always reduces the area, this is not always the case for the perimeter, as in this problem, which resulted in an incorrect answer. The perimeter may decrease, increase, or remain unchanged depending on how the paper is cut.
>
> Sare completed Figure B, and by moving the sides of the B to the right, left, up, and down, she obtained Figure A. She realized that there was no increase or decrease in the number of sides. She came to the conclusion that the perimeters of Figure A and Figure B are the same and gave the correct answer.

Unlike the initial questionnaire, pre-service teacher 11 provided higher levels of evidence of attending to each student's solution in the final questionnaire. While attending to Ada's solution, the pre-service teacher provided a *substantial* level of evidence because she recognized the student's idea and misconception, but she did not connect what she explained with the lengths of sides and perimeter in the problem. For the solutions of Tuna and Sare, the pre-service teacher provided a *robust* level of evidence for attending. She was able to recognize the misconception of Tuna regarding the direct relationship between perimeter and area and to describe the correct solution of Sare in detail with the student's transformation of shape B into shape A strategy by relating with the sides of two shapes. Overall, when the response to the attending prompt given by the pre-service teacher regarding each student's solution is considered together, the pre-service teacher provided a *robust* level of evidence for attending in this problem because she identified the mathematical properties in three students' solutions and explained two of them (the solutions of Tuna and Sare) in detail. While the pre-service teacher provided a *lack* of evidence of attending in the initial questionnaire, the response she provided showed a *robust* level of evidence in the final questionnaire.

The third problem in the noticing questionnaire was about a fixed perimeter-changing area situation. This problem had two sub-problems. The first sub-problem was named the minimum area for a fixed perimeter, and the pre-service teachers were expected to attend to two different students' solutions to this problem. Figure 4.3 below illustrates the frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the minimum area for a fixed perimeter problem in the initial and final noticing questionnaire. In the initial questionnaire, half of the responses (n=16) showed a lack of evidence of attending. Only a minority of the pre-service teachers (n=3) provided a robust level of evidence. Surprisingly, the percentage of limited and substantial evidence provided by the pre-service teachers in the initial and final questionnaires was equal. On the other hand, there was a decrease in the percentage of responses showing a lack of evidence and an increase in that demonstrated robust evidence of attending in the final questionnaire. This improvement can be illustrated by the responses of pre-service teacher 23 to the attending prompt in the initial and final questionnaire.

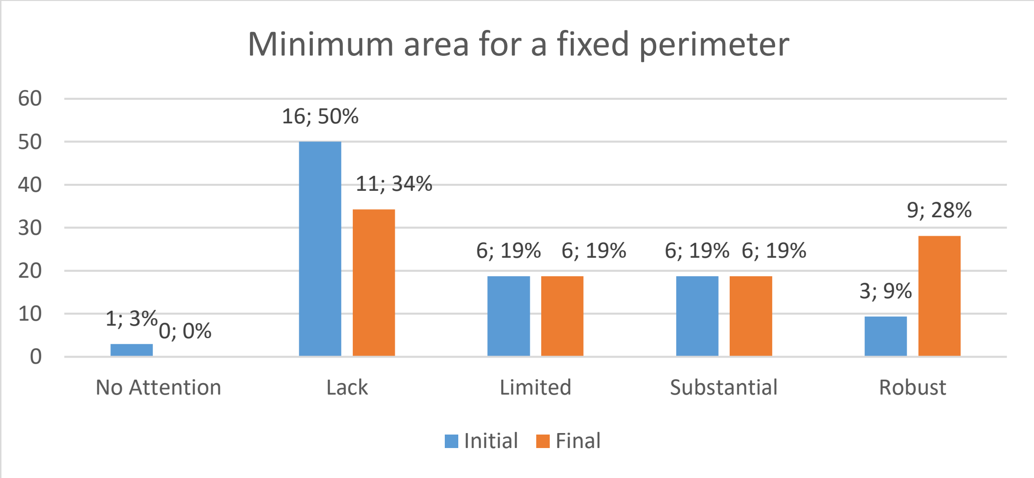
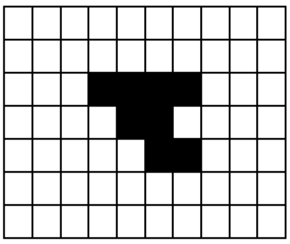


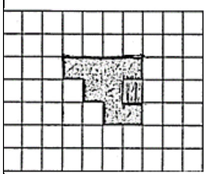
Figure 4. 3 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the minimum area for a fixed perimeter problem in the initial and final noticing questionnaire

The response of pre-service teacher 23 to the minimum area for a fixed perimeter problem in the initial noticing questionnaire is provided in Figure 4.4.

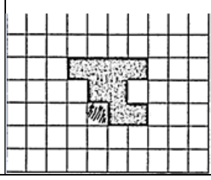
Add square(s) to the figure given below in a way that the perimeter of the figure does not change (The squares should contact from the edges/ be connected to each other from the edges)



a) What would be the least number of squares you can add? Show where you added the square(s) by painting and find the area of the new figure.



Yelda:
1 square can be painted.
Its area becomes 9 square units and increases.



Utku:
1 square can be used.
3 times 4 = 12
12-3= 9 square units

i. Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?

Yelda said that the least number of squares she can add is 1. Her solution is correct. Utku also answered the problem correctly.

Figure 4. 4 The response of pre-service teacher 23 to the minimum area for a fixed perimeter problem in the initial noticing questionnaire

Pre-service teacher 23 provided a *lack* of evidence of attending to each student's solution because she incorrectly identified the solution of Yelda and identified the solution of Utku as correct without providing any details. Consequently, the response of pre-service teacher 23 to the attending prompt in the initial questionnaire showed a *lack* of evidence due to her inability to identify the mathematical properties in the students' solutions.

Response of pre-service teacher 23 in the final noticing questionnaire:

> While solving the problem, Utku paid attention to the sides of the figure and made the number of disappeared sides and the number of added sides the same. His solution is correct because the number of sides included in the perimeter was two units in the first case and the second case. In this way, Utku did not change the perimeter of the shape with the square he added to the shape.
>
> Yelda, on the other hand, preferred to add 1 square to the empty space on the right side by thinking that the perimeter would not change. Her solution was incorrect because, in the first case, three sides were included in the perimeter, but after adding the square, only one side was included in the perimeter. This resulted in a decrease in the perimeter of the shape from 16 units to 14 units.

The pre-service teacher described the solution of Utku in detail by relating it to the sides and perimeter concept. She also recognized the reason underlying the incorrect solution of Yelda, that is, the added square changed the perimeter of the figure. Hence, the response of pre-service teacher 23 showed a *robust* level of evidence for attending to each student's solution. Accordingly, since the pre-service teacher identified the mathematical properties in both students' solutions and explained them in detail, her response in the final questionnaire demonstrated a *robust* level of evidence for attending.

The second sub-problem was named maximum area for a fixed perimeter, and the pre-service teachers were expected to attend to two different students' solutions to this problem. Figure 4.5 shows the frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the maximum area for a fixed perimeter problem in the initial and final noticing questionnaire. In the initial questionnaire, 6 of 32 pre-service teachers (19%) did not attempt to attend to students' solutions. More than half of the pre-service teachers (n=18) provided a lack of or limited evidence of attending. The percentage of responses showing substantial and robust evidence was equal. In contrast, in the final questionnaire, there was a decrease in the percentage of the responses that demonstrated a lack and limited level of evidence, whereas there was an increase in the percentage of the responses that demonstrated a substantial and robust level of evidence. It is worth noting that the robust evidence of attending with the highest percentage was provided for this problem in the context of perimeter and area measurement in the final questionnaire. One of the pre-service teachers' (pre-service teacher 28) responses to the attending prompt in the initial and final noticing questionnaire was reported to portray the improvement.

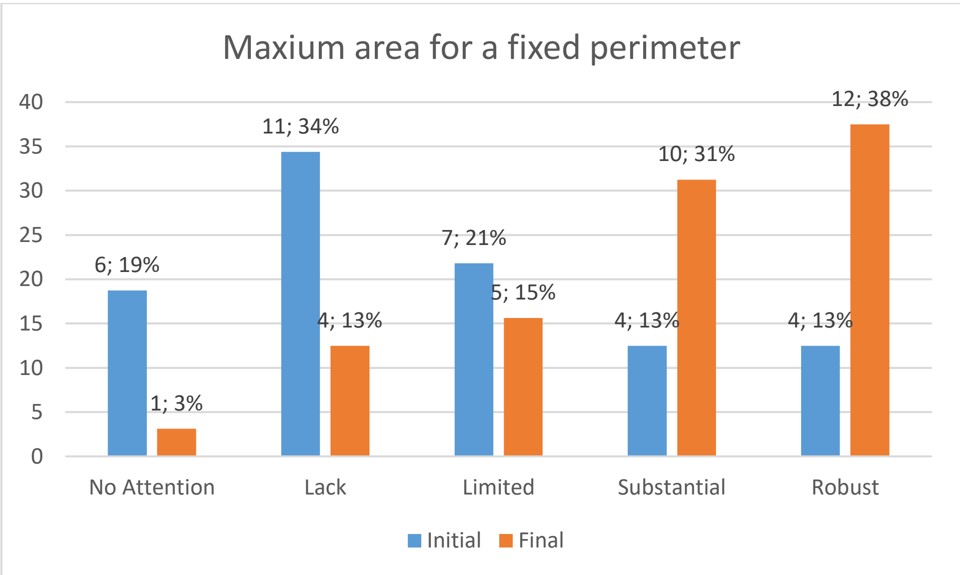


Figure 4. 5 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the maximum area for a fixed perimeter problem in the initial and final noticing questionnaire

Figure 4.6 shows the response of pre-service teacher 28 to the maximum area for a fixed perimeter problem in the initial noticing questionnaire.

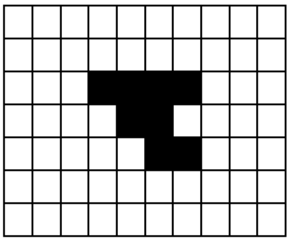
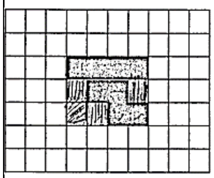
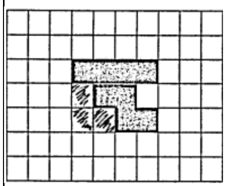
<table>
<tr><td colspan="3">Add square(s) to the figure given below in a way that the perimeter of the figure does not change (The squares should contact from the edges/ be connected to each other from the edges)</td></tr>
<tr><td rowspan="3"></td><td colspan="2">b) What would be the maximum number of squares you can add? Show where you added the square(s) by painting and find the area of the new figure.</td></tr>
<tr><td></td><td>Yelda:
We can add 4 squares.
Its area becomes 12 square units and increases.</td></tr>
<tr><td></td><td>Utku:
3 squares can be used.
3 times 4 = 12
12-1= 11 square units</td></tr>
<tr><td colspan="3">(i) Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student’s solution is correct? Why?
Yelda tried to complete the shape into a rectangle. Her solution is wrong.
Utku did not change the perimeter with the unit squares he added. However, he did not realize that more squares could be added.</td></tr>
</table>

Figure 4. 6 The response of pre-service teacher 28 to the maximum area for a fixed perimeter problem in the initial noticing questionnaire

Pre-service teacher 28 provided *limited* evidence of attending to the solution of Yelda since he did not make a connection with the perimeter concept. Moreover, the pre-service teacher provided a short description of the solution of Tuna regarding both the perimeter and maximum area of the shape, but he did not provide much detail regarding why the added squares keep the perimeter constant and how many more squares can be added. Therefore, he provided *substantial* evidence of attending to the solution of Utku since he described the student's solution shortly. Therefore, the response of pre-service teacher 28 in the initial questionnaire showed a *limited* level of attending because he identified mathematical properties in one of two students' solutions to some extent.

Response of pre-service teacher 28 in the final noticing questionnaire:

> Yelda answered the problem incorrectly. She made the same mistake she made in the previous problem. She reduced the number of sides contributing to the perimeter. In this case, the perimeter is reduced. Moreover, the maximum area is obtained when a square is created. Yet, the student formed a rectangle by adding four squares.
>
> Utku added three squares to the bottom left corner. The places where he added the squares are correct because the perimeter does not change, but three is not the maximum number of squares that can be added. He could have formed a 4x4 square because the perimeter does not change either in this case. Therefore, he would have to paint another five unit squares in addition to what he painted.

The response of pre-service teacher 28 demonstrated a *robust* level of evidence for attending to each student's solution. He described the solution of Utku in detail by mentioning both the perimeter concept and the maximum number of squares to be added. He also recognized why the solution of Yelda was incorrect by emphasizing the perimeter of the figure and the maximum area. Consequently, the pre-service teacher identified the mathematical properties in both students' solutions and explained them in detail, which shows a *robust* level of evidence for attending.

4.1.1.1.1.2 Pre-service teachers' attending to students' solutions about a fixed area-changing perimeter situation

The second problem in the noticing questionnaire was about a fixed area-changing perimeter situation. This problem had two sub-problems. The first sub-problem was named perimeter comparison, and the pre-service teachers were expected to attend to two different students' solutions to this problem. The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the perimeter comparison problem in the initial and final noticing questionnaire are presented in Figure 4.7. In the initial questionnaire, 9 of 32 pre-service teachers (28%) did not attempt to attend to students' solutions. Furthermore, more than half of them (n=18) provided either a lack of evidence or limited evidence. A minority of the pre-service teachers' responses (n=5) showed a substantial or robust level of attending. In contrast, all pre-service teachers attended to the students' solutions in the final questionnaire. Moreover, there was a decrease in the percentage of pre-service teachers who provided a lack of evidence (n=2) and limited evidence of attending (n=6). Of interest here is the increase in the percentage of the responses that showed substantial and robust levels. That is, three-quarters of the pre-service teachers (n=24) provided high levels of evidence, i.e., substantial or robust levels of evidence. This finding suggests an improvement in the responses that the pre-service teachers provided to the attending prompt in the perimeter comparison problem in the final questionnaire. Pre-service teacher 20's responses to the attending prompt in the initial and final noticing questionnaire illustrate this improvement.

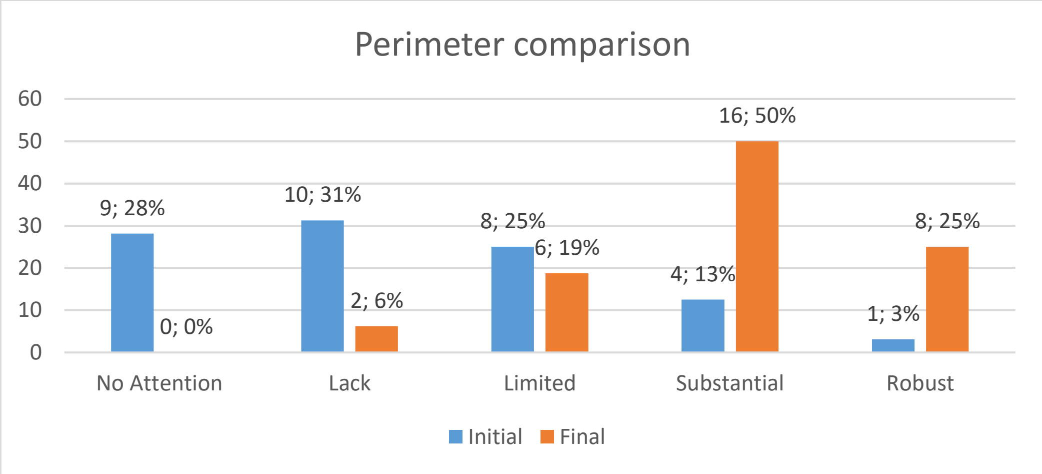
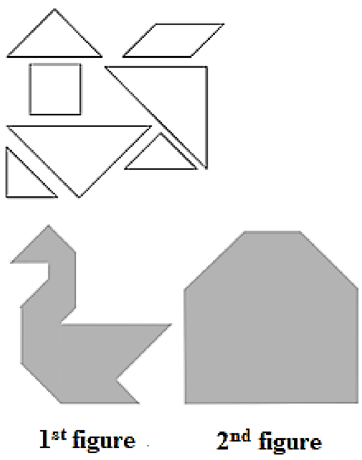


Figure 4. 7 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the perimeter comparison problem in the initial and final noticing questionnaire

The response of pre-service teacher 20 to the perimeter comparison problem in the initial noticing questionnaire is presented in Figure 4.8.

<table>
<tr>
<td>
</td>
<td>Using all seven tangram pieces, the following two figures are obtained.

a) What can you say when you compare the perimeters of the figures? Justify your response.
Can: In Figure 1, most of the sides of each piece form the perimeter of the figure. In Figure 2, the perimeter of Figure 1 is larger because more sides of each piece are inside the figure.
Hazal: The perimeters of Figure 1 and Figure 2 are equal. Only the locations of the tangram pieces are different.</td>
</tr>
<tr>
<td colspan="2">(i) Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?
Can thought that the perimeter of the first figure was larger because more shapes were used in the first figure. His solution is incorrect. Hazal's approach to the problem is correct.</td>
</tr>
</table>

Figure 4. 8 The response of pre-service teacher 20 to perimeter comparison problem in the initial noticing questionnaire

Pre-service teacher 20 provided a *lack* of evidence of attending to each student's solution because she incorrectly identified the students' solutions. Therefore, the response of pre-service teacher 20 to the attending prompt in the initial questionnaire showed a *lack* of evidence of attending since she was not able to identify any mathematical properties in the students' solutions.

Response of pre-service teacher 20 in the final noticing questionnaire:

> Can answered the problem correctly. The tangram pieces are combined in different ways, which changes the perimeter. As Can thinks in the first figure, more sides of pieces contribute to the perimeter. This results in a greater perimeter. On the other hand, the second figure exposes fewer sides of pieces outside.
>
> Hazal gave the wrong answer. She thinks that the perimeter of the shape will not change because the same pieces are used, but this is valid for the area. She did not think that the different positions of the tangram pieces would change the perimeter of the shape.

Pre-service teacher 20 described the solution of Can in detail by relating it to the perimeter concept and recognized the misconception of confusion of the perimeter and area concepts underlying the incorrect solution of Hazal. Therefore, the response of pre-service teacher 20 demonstrated a *robust* level of evidence for attending to each student's solution. Accordingly, the pre-service teacher provided a *robust* level of evidence for attending in the final questionnaire because she identified the mathematical properties in both students' solutions by explaining them in detail.

The second sub-problem was named area comparison, and the pre-service teachers were expected to attend to two different students' solutions to this problem. Figure 4.9 presents the frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in area comparison in the initial and final noticing questionnaire. In the initial questionnaire, four of 32 pre-service teachers (13%) did not attempt to attend to students' solutions. Moreover, half of the responses (n=16) showed a lack of evidence of attending. A low percentage of the pre-service teachers (n=3) provided a substantial level of evidence. Interestingly, none of the pre-service teachers provided a robust level of evidence for attending in the initial questionnaire, but four of them (13%) did so in the final questionnaire. There was also an increase in the percentage of responses that demonstrated a substantial level of evidence in the final questionnaire. There was a slight improvement in the attending skills of the pre-service teachers in this problem compared to the previous problems. In response to the attending prompt in the area comparison problem, the responses of pre-service teacher 13 in the initial and final questionnaire were provided as an example.

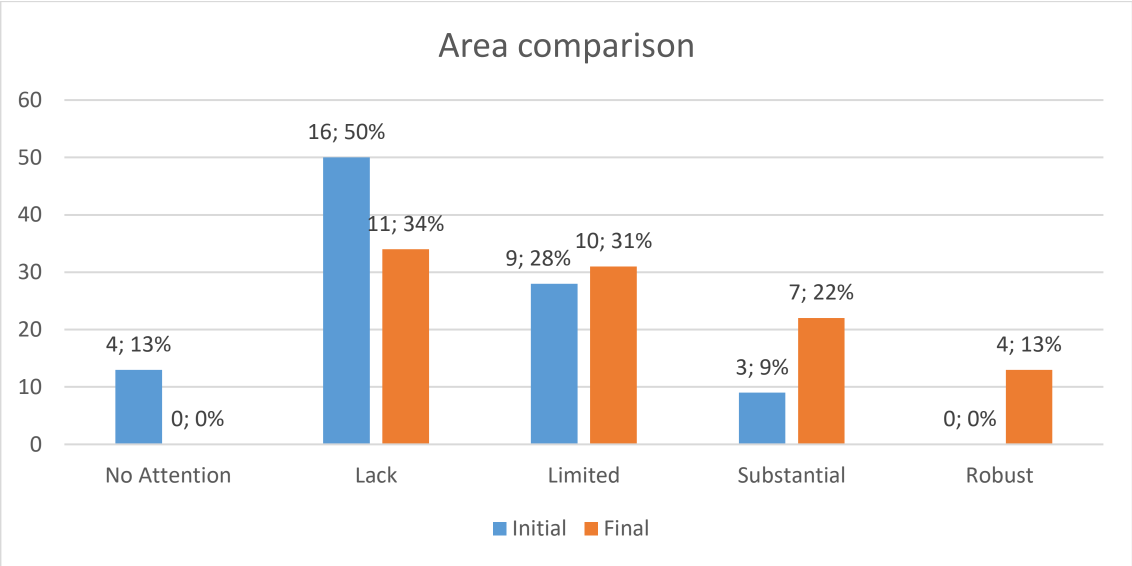
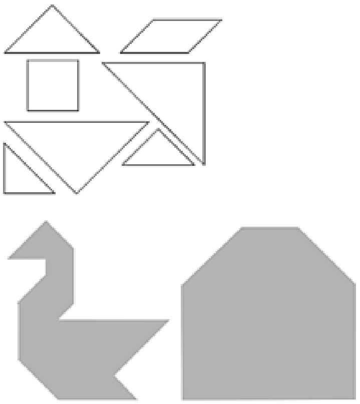


Figure 4. 9 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the area comparison problem in the initial and final noticing questionnaire

Figure 4.10 presents the response of pre-service teacher 13 to the area comparison problem in the initial noticing questionnaire.

<table>
<tr>
<td>
1st figure 2nd figure</td>
<td>Using all seven tangram pieces, the following two figures are obtained.

b) What can you say when you compare the area of the figures? Justify your response.
Can: Since the two figures consist of the same parts, their areas are equal.

Hazal: I think the area of Figure 2 is larger because Figure 2 is wider.</td>
</tr>
<tr>
<td colspan="2">(i) Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?
Can approached the problem by considering a part-whole relationship. His solution is correct.
Hazal looked at the problem in terms of width and length and dealt with it as a whole. Her solution is incorrect.</td>
</tr>
</table>

Figure 4. 10 The response of pre-service teacher 13 to area comparison problem in the initial noticing questionnaire

Pre-service teacher 13 provided a *limited* level of evidence for each student's solution because she provided a general description of the solutions without making a connection with the area concept. Therefore, the response of the pre-service teacher in the initial questionnaire demonstrated a *lack* of evidence of attending since it did not involve any mathematical properties.

Response of pre-service teacher 13 in the final noticing questionnaire:

> Can thinks that since the pieces that make up the shapes do not change, the area of the two figures will be the same. The solution is correct. Because due to area conservation, the sum of the areas of the individual pieces of the shape gives the area of the whole shape.

Hazal focuses on the width of the shape because she thinks of the area as the multiplication of width and length. She believes that the greater the width, the greater the area. The solution is incorrect since the sum of the areas of the parts gives the area of the shape due to area conservation.

Pre-service teacher 13 described the solution of Can in detail by relating it to the area concept and recognized the misconception of Hazal that width directly increased the area of the figure without thinking about area conservation, which underlines her incorrect solution by providing *robust* evidence for each student's solution. Therefore, the response of pre-service teacher 13 showed a *robust* level of evidence for attending in the final questionnaire because she identified the mathematical properties in both students' solutions by explaining them in detail.

Figure 4.11 shows the frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the context of perimeter and area measurement in the initial and final noticing questionnaire.

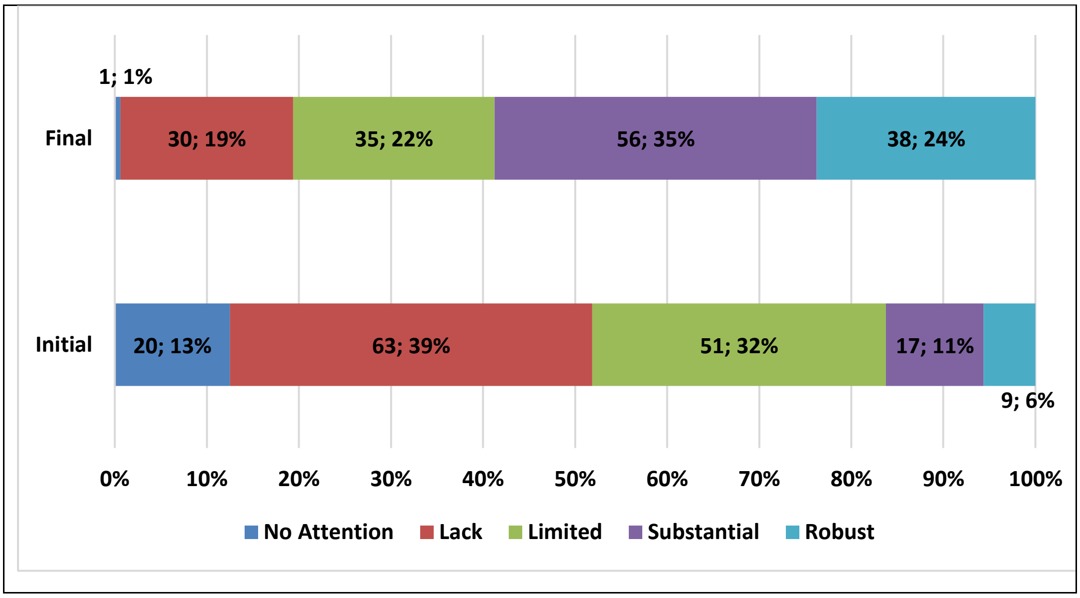


Figure 4. 11 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the context of perimeter and area measurement in the initial and final noticing questionnaire

When we look at the pre-service teachers' responses to the attending prompt in the context of perimeter and area measurement as a whole, it is seen that in the initial questionnaire, the category with the highest percentage was lack of evidence, followed by limited evidence with the second highest percentage, as indicated in Figure 4.11. In addition, the lowest percentage of responses belonged to the category of robust evidence, followed by the category of substantial evidence with the second lowest percentage. On the other hand, in the final questionnaire, the category of substantial evidence represented the largest portion. Robust evidence was the second highest level of evidence provided by the pre-service teachers. These findings reveal the improvement in the pre-service teachers' responses to the prompt of attending to students' solutions in the context of perimeter and area measurement, that is, the improvement in their attending skills in the final questionnaire.

4.1.1.2 Changes in pre-service teachers' interpreting students' understanding in the context of perimeter-area measurement

Pre-service teachers' interpreting students' understanding in the context of perimeter-area measurement is measured by their responses to interpreting prompts related to students' solutions in problems 1, 2, and 3 of the noticing questionnaire. The pre-service teachers' responses to the interpreting students' understanding prompt in the noticing questionnaire were examined at four levels: robust level of evidence, substantial level of evidence, limited level of evidence, and lack of evidence of interpreting students' understanding. In addition, some of the pre-service teachers did not provide an answer to the interpreting prompt. This was categorized as no interpretation. All three problems about perimeter-area measurement in the noticing questionnaire were gathered under the theme of perimeter-area relationship. Therefore, the findings regarding the pre-service teachers' interpreting skills are presented under the title of pre-service teachers' interpreting students' understanding of the perimeter-area relationship in the following part.

4.1.1.2.1 Pre-service teachers' interpreting students' understanding about the perimeter-area relationship

In the noticing questionnaire, the first three problems (five problems with sub-problems) were about the perimeter-area relationship. The first problem and the third problem were about a fixed perimeter-changing area situation, while the second problem was about a fixed area-changing perimeter situation. The findings for the first problem and the third problem, which are related to a fixed perimeter-changing area situation, and the findings for the second problem, which is related to a fixed area-changing perimeter situation, are presented below, respectively.

4.1.1.2.1.1 Pre-service teachers' interpreting students' understanding about a fixed perimeter-changing area situation

The first problem in the noticing questionnaire was named change in perimeter, and the pre-service teachers were expected to interpret three different students' understanding (see Figure 4.2). Figure 4.12 shows the frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding of a change in perimeter problem in the initial and final noticing questionnaire. Although all pre-service teachers responded to the attending prompt in this problem in both the initial and final questionnaire, four of 32 pre-service teachers (13%) did not attempt to interpret students' understanding in the initial questionnaire. A quarter of the pre-service teachers (n=8) provided a lack of evidence, and a quarter of them (n=8) provided substantial evidence. None of the pre-service teachers provided a robust level of evidence in the initial questionnaire for interpreting. The percentage of pre-service teachers who provided limited evidence of interpreting was the highest among the levels. On the other hand, in the final questionnaire, only one of 32 responses (3%) showed a lack of evidence. Furthermore, there was an increase in the percentage of responses showing a substantial level of evidence, and almost half of the pre-service teachers (n=15) provided such evidence. Some of the pre-service teachers (n=5) provided a robust level of evidence in the final questionnaire, even though none of the responses showed a robust level of evidence in the initial one. As an example of the improvement in interpreting, responses of pre-service teacher 9 to the interpreting prompt in the initial and final questionnaire can be given.

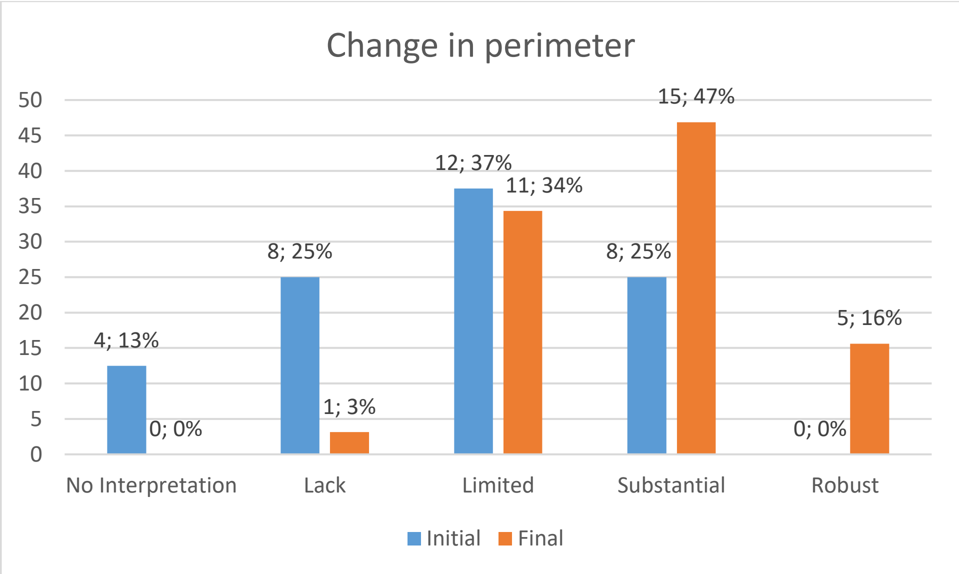


Figure 4. 12 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the change in perimeter problem in the initial and final noticing questionnaire

Response of pre-service teacher 9 in the initial noticing questionnaire:

> Ada solved the problem by relating it to daily life. However, when numbers and operations are involved, Ada may experience difficulty. Moreover, since Ada will answer all the problems in this way with straightforward logic, she will tend to memorize rather than comprehend the concept and forget the concept after a while.
>
> Tuna has learned the subject by rote. He confuses the concepts because he does not understand them conceptually. Therefore, solving problems like this will often be by chance.
>
> Sare understood perimeter measurement.

Pre-service teacher 9 interpreted understanding of Ada by providing a *limited* level of evidence because her comment did not involve any mathematical properties regarding perimeter measurement. Furthermore, the explanation for the understanding of Tuna also showed a *limited* level of evidence since the pre-service teachers blamed the student for memorization. For Sare, pre-service teacher 9 provided a *lack* of evidence because she made comments about whether only the student understood or not. Overall, considering the response of the pre-service teacher to the interpreting prompt regarding each student's understanding, it is seen that she provided a *lack* of evidence of interpreting due to her inability to provide a valid justification for any student's understanding.

Response of pre-service teacher 9 in the final noticing questionnaire:

> Ada thinks that the perimeter of shape B will be longer because the shape is curved. She has associated this with daily life and keeps it in his head as a memorized rule. In fact, she knows the length, but he does not associate the perimeter with the sides. He does not realize that when the shape is cut, the new sides come back into place.
>
> Tuna thinks that the perimeter of B is smaller because B is a cut version of A. When a piece is cut, the area always decreases, but this is not always true for the perimeter. Tuna thinks that as the area decreases, the perimeter also decreases. That is, the area and perimeter will increase or decrease in the same way. He believes that there is a direct relationship between area and perimeter.
>
> Sare compared the sides of two shapes by moving the sides in shape B with arrows. Hence, she understood that the perimeter is the length of the boundaries of a shape.

Unlike the initial questionnaire, pre-service teacher 9 provided higher levels of evidence while interpreting each student's understanding in the final questionnaire. The explanation regarding understanding of Ada showed a *substantial* level of evidence because she provided a valid justification for the student's understanding of perimeter without much detail. For the understanding of Tuna, the pre-service teacher provided a detailed explanation about possible reasoning behind student mathematics by interpreting that the student wrongly explores the relationship between the perimeter and area. Therefore, her comment demonstrated a *robust* level of evidence. Pre-service teacher 9's comment on the understanding of Sare showed a *substantial* level of evidence because she provided a valid justification for the student's understanding of the perimeter concept without much detail. Overall, the response of pre-service teacher 9 to the interpreting prompt in the change in perimeter problem in the final noticing questionnaire demonstrated a *substantial* level of evidence since she provided a valid justification for three students' understanding by explaining one of them in detail.

The third problem in the noticing questionnaire was about a fixed perimeter-changing area situation. This problem had two sub-problems. The first sub-problem was named the minimum area for a fixed perimeter, and the pre-service teachers were expected to interpret two different students' understanding (see Figure 4.4). The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the minimum area for a fixed perimeter problem in the initial and final noticing questionnaire are presented in Figure 4.13. In the initial questionnaire, almost one-third of the pre-service teachers (n=11) did not attempt to interpret students' understanding. The percentage of those who provided substantial and robust levels of evidence was low. An equal percentage of lack of evidence was provided by the pre-service teachers in the initial and final questionnaires. In the final questionnaire, all pre-service teachers provided an interpretation of students' understanding. Additionally, the percentage of responses showing substantial and robust levels of evidence increased. To illustrate the improvement, the responses of pre-service teacher 27 to the initial and final noticing questionnaire are given below.

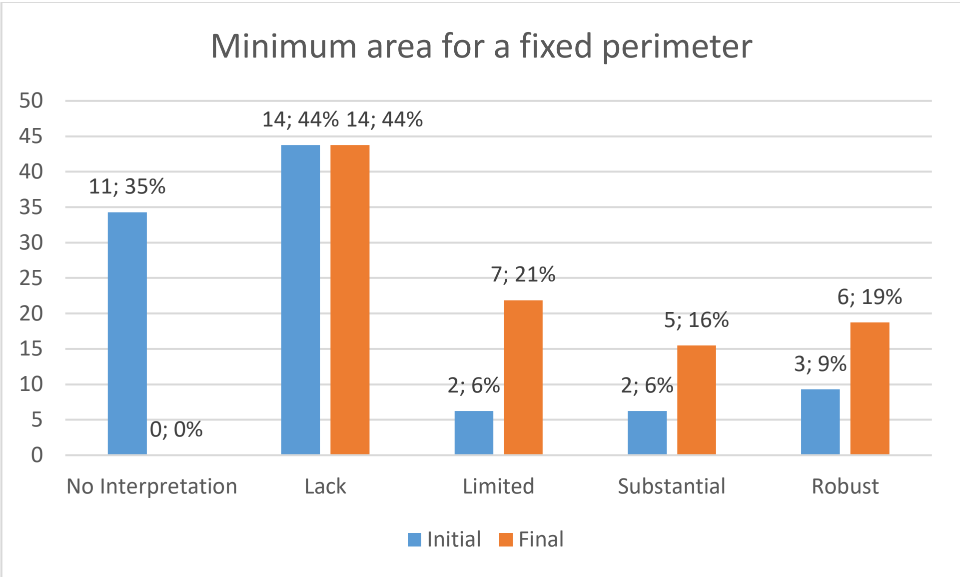


Figure 4. 13 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the minimum area for a fixed perimeter problem in the initial and final noticing questionnaire

Response of pre-service teacher 27 in the initial noticing questionnaire:

> Utku understood the concept, but Yelda did not.

Here, pre-service teacher 27 made comments about whether only the students understood or not. Hence, her response showed a *lack* of evidence for interpreting due to her inability to provide a valid justification for any student's understanding in the initial questionnaire.

Response of pre-service teacher 27 in the final noticing questionnaire:

> Yelda did not understand the perimeter concept because when she added the square, the perimeter of the resulting shape changed. The student could not see that she had reduced the number of sides. She did not realize that the number of disappeared and appeared sides must be equal when the square is added so that the perimeter does not change.
>
> Utku knows what the perimeter is because he sees that the overlapping sides of the added unit square and the outer sides complement each other.

Pre-service teacher 27 made a detailed explanation about the possible reasoning behind Yelda's mathematics. Hence, she provided a *robust* level of evidence for interpreting the understanding of Yelda. Moreover, she provided a justification for Utku's understanding of the relationship between sides and perimeter without much detail. Therefore, she provided a *substantial* level of evidence for interpreting the understanding of Utku. Overall, the response of pre-service teacher 27 to the interpreting prompt in the final questionnaire demonstrated a *robust* level of interpreting since she provided a valid justification for both students' understanding by explaining Yelda's understanding in detail.

The third problem in the noticing questionnaire was about a fixed perimeter-changing area situation. This problem had two sub-problems. The second sub-problem was named maximum area for a fixed perimeter, and the pre-service teachers were expected to interpret two different students' understanding (see Figure 4.6). Figure 4.14 below illustrates the frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the maximum area for a fixed perimeter problem in the initial and final noticing questionnaire.

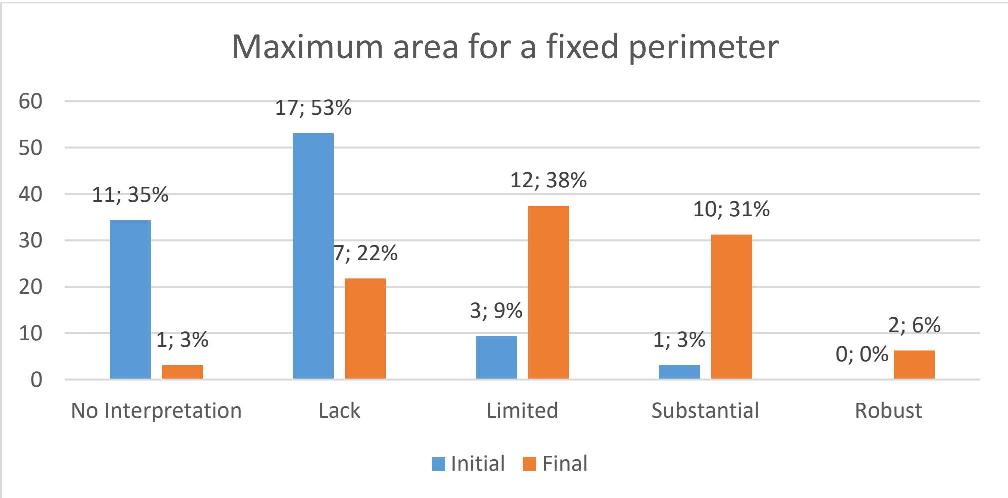


Figure 4. 14 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the maximum area for a fixed perimeter problem in the initial and final noticing questionnaire

In the initial questionnaire, almost one-third of the pre-service teachers (n=11) did not provide any interpretation of students' understanding, like in minimum area for a fixed perimeter problem. More than half of the pre-service teachers (n=17) provided a lack of evidence. Furthermore, only one of 32 responses (3%) showed a substantial level of evidence, and none of the pre-service teachers could provide a robust level of interpreting, like in the perimeter comparison problem. On the other hand, in the final questionnaire, there was a decrease in the percentage of the responses that showed a lack of evidence, whereas there was an increase in the percentage of the responses demonstrating substantial and robust levels of evidence. Almost one-third of the pre-service teachers (n=10) provided a substantial level of evidence in the final questionnaire. As an example, the responses of pre-service teacher 31 to the interpreting prompt in the initial and final noticing questionnaire are presented below.

Response of pre-service teacher 31 in the initial noticing questionnaire:

> Like in the minimum area problem, Yelda again did not take into account that the perimeter would not change. I think Yelda did it wrong because she did not read the problem carefully.

Pre-service teacher 31 did not provide an interpretation of Utku's understanding. While interpreting the understanding of Yelda, pre-service teacher 31 provided a *limited* level of evidence because her comments about student understanding did not involve any mathematical properties. The pre-service teacher attributed the incorrect answer of Yelda to the student's carelessness. Accordingly, she could not provide a valid justification for any student's understanding, and hence, pre-service teacher 31's response in the initial questionnaire showed a *lack* of evidence.

Response of pre-service teacher 31 in the final noticing questionnaire:

> She just filled the empty squares and tried to complete the shape and stopped after reaching the rectangular shape. She did not realize the reduced number of sides. She does not know how to make an addition so that the perimeter does not change.

> Utku partially understood. He knows what perimeter is. He did not change the perimeter with the unit squares he added, but there is a missing point where he says I can add a maximum of 3 squares. He could not think that the perimeter would not change by adding more squares.

The response of pre-service teacher 31 demonstrated a *substantial* level of evidence for interpreting each student's understanding. The pre-service teacher provided valid justifications for both students' understanding without much detail. The comment on Utku involved the student's understanding of the perimeter and number of added squares, but the pre-service teacher did not mention why the student thought it was not possible to add more squares. In addition, while interpreting the understanding of Yelda, the pre-service teacher did not provide a detailed explanation regarding the student's understanding of the relationship between sides and perimeter. Overall, the response of pre-service teacher 31 to the interpreting prompt in the final questionnaire showed a *substantial* level of interpreting since she provided a valid justification for both students' understanding without much detail.

4.1.1.2.1.2 Pre-service teachers' interpreting students' understanding about a fixed area-changing perimeter situation

The second problem in the noticing questionnaire was about a fixed area-changing perimeter situation. This problem had two sub-problems. The first sub-problem was named perimeter comparison, and the pre-service teachers were expected to interpret two different students' understanding (see Figure 4.8). Figure 4.15 presents the frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding of the perimeter comparison problem in the initial and final noticing questionnaire. In the initial questionnaire, half of the pre-service teachers (n=16) could not interpret the students' understanding. Moreover, almost one-third of the pre-service teachers (n=10) provided a lack of evidence of interpreting. Only one of the pre-service teachers' responses (3%) showed a substantial level of evidence. Surprisingly, none of the pre-service teachers could provide a robust level of interpreting. In contrast to the initial questionnaire, however, six of 32 pre-service teachers (19%) provided a robust level of interpreting, and 11 of them (34%) provided a substantial level of interpreting in the final questionnaire. Pre-service teacher 16's responses to the interpreting prompt in the initial and final noticing questionnaire illustrate this improvement.

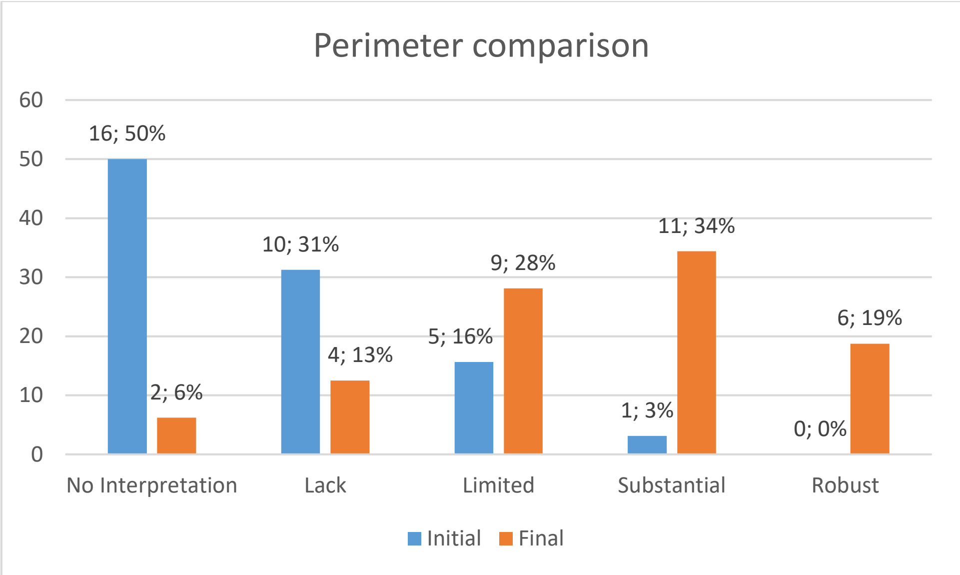


Figure 4. 15 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the perimeter comparison problem in the initial and final noticing questionnaire

Response of pre-service teacher 16 in the initial noticing questionnaire:

> Can understood the perimeter concept because he made correct predictions by generalizing about the length.
>
> Hazal did not understand length measurement. She does not know the definition of the perimeter.

The pre-service teacher made a comment about the understanding of Can without any mathematical properties. In addition, she blamed Hazal for a lack of knowledge about the perimeter concept. Consequently, pre-service teacher 16 provided *limited* evidence while interpreting the understanding of each student. Thus, the response of the pre-service teacher showed a *lack* of evidence in the initial questionnaire since she could not provide a valid justification for any student's understanding.

Response of pre-service teacher 16 in the final noticing questionnaire:

> Can understood the concept of perimeter because he thinks that the perimeter of Figure 1 is larger because more sides constitute the boundaries of the shape. He knows that the perimeter is the boundary of the shape and that the perimeter will change depending on the size and number of sides included.
>
> Hazal said that since the pieces are the same, the perimeter is the same. Hazal confused the perimeter with the concept of area. Therefore, she did not understand the concept of perimeter. It was not clear to Hazal that the perimeter is the total length of the boundaries.

The explanation of pre-service teacher 16 regarding the understanding of Can reveals that she provided a detailed explanation of possible reasoning behind student mathematics, which shows a *robust* level of evidence for interpreting. Furthermore, she provided a valid justification for Hazal's understanding of the concept without much detail. Accordingly, since the pre-service teacher provided a valid justification for both students' understanding by explaining one of them in detail, her response to the interpreting prompt in the final questionnaire demonstrated a *robust* level of evidence.

The second sub-problem was named area comparison, and the pre-service teachers were expected to interpret two different students' understanding (see Figure 4.10). Figure 4.16 presents the frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the area comparison problem in the initial and final noticing questionnaire.

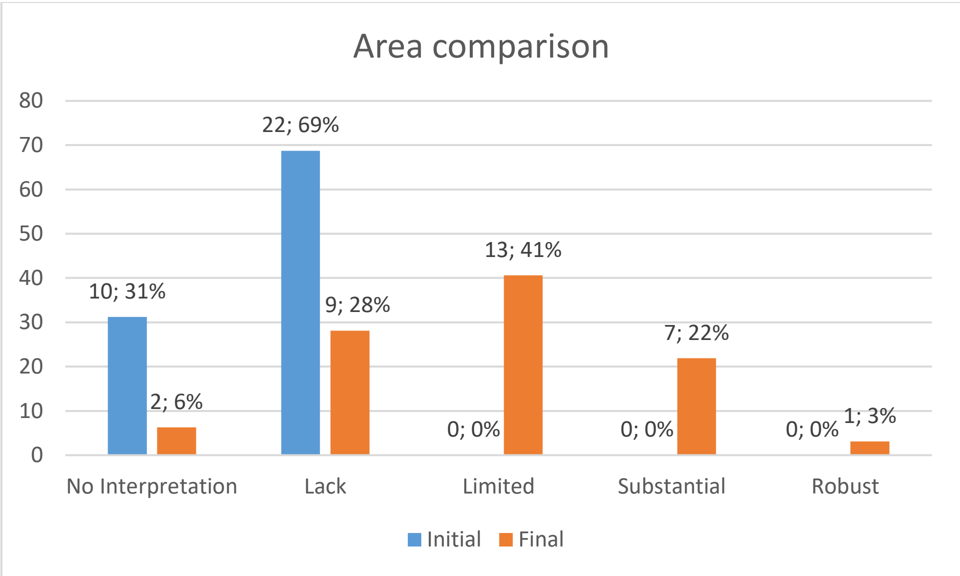


Figure 4. 16 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the area comparison problem in the initial and final noticing questionnaire

The most surprising aspect of the data in Figure 4.16 is that in the initial questionnaire, the pre-service teachers either did not provide an interpretation of students' understanding or provided a lack of evidence of interpreting. Almost one-third of the pre-service teachers (n=10) did not attempt to interpret the students' understanding. Moreover, almost two-thirds of the pre-service teachers (n=22) provided a lack of evidence of interpreting. By contrast, there was an improvement in the responses to the final questionnaire. One-quarter of the pre-service teachers (n=8) provided high levels of evidence (substantial evidence or robust evidence). The responses of pre-service teacher 22 to the initial and final noticing questionnaires are given below to illustrate the improvement.

Response of pre-service teacher 22 in the initial noticing questionnaire:

> Based on the solution of Can, I can say that he understood the area concept because the answer he gave shows that his ability to think from part to whole has developed.

Pre-service teacher 22 only provided an interpretation of the understanding of Can in the initial questionnaire. Her response showed a *limited* level of evidence for interpreting because her comment on student understanding did not involve any mathematical properties. That is, she did not mention how the thinking from part to whole, which she stated was related to the area concept. Her response to the interpreting prompt in this problem showed a *lack* of evidence since she could not provide a valid justification for any student's understanding.

Response of pre-service teacher 22 in the final noticing questionnaire:

> Can understood that the area is the number of unit squares needed to cover a surface because he thinks that the whole formed by the same pieces will have the same area. The student knows that area is conserved because the shapes are made up of the same pieces, even if they are arranged in different positions.
>
> Hazal does not know the meaning of the area and cannot think of the shape in parts.

Pre-service teacher 22 provided a *robust* level of evidence while interpreting the understanding of Can because she provided a detailed explanation about possible reasoning behind student mathematics. Moreover, while interpreting the understanding of Hazal, the pre-service teacher blamed the student for a lack of knowledge, which resulted in a *limited* level of evidence. Therefore, in the final questionnaire, considering the interpretation of each student's understanding together reveals that pre-service teacher 22 provided a *substantial* level of evidence because she provided a valid justification for one of the students' understanding by explaining in detail.

The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the context of perimeter and area measurement in the initial and final noticing questionnaire are presented in Figure 4.17.

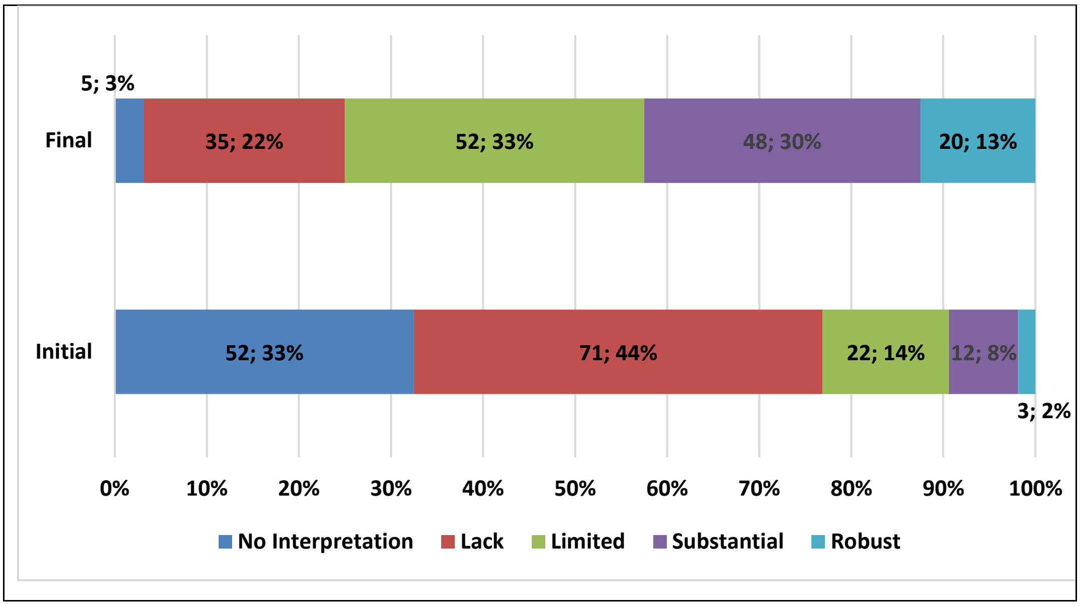


Figure 4. 17 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the context of perimeter and area measurement in the initial and final noticing questionnaire

It is apparent from the figure above that in the initial questionnaire, 44% of the responses, with the highest percentage, to the interpreting prompt in the context of perimeter and area measurement showed a lack of evidence, and a third of the responses did not include any interpretation. In contrast, it is evident that there was a significant decrease in the percentage of responses, showing a lack of evidence in the final questionnaire. Almost one-third of the responses demonstrated a substantial level of evidence. These findings reveal an improvement in the pre-service teachers' interpreting skills in the context of perimeter and area measurement.

4.1.1.3 Changes in pre-service teachers' deciding how to respond on the basis of students' understanding in the context of perimeter-area measurement

Pre-service teachers' deciding how to respond on the basis of students' understanding in the context of perimeter-area measurement is measured by their responses to deciding prompts related to students' solutions in problems 1, 2, and 3 of the noticing questionnaire. The suggestions provided by the pre-service teachers to the deciding how to respond prompt in the noticing questionnaire were examined at five levels: robust level of evidence, substantial level of evidence, medium level of evidence, limited level of evidence, and lack of evidence of deciding how to respond based on the students' understanding. In addition, some of the pre-service teachers did not provide a suggestion, which was categorized as a no response. All three problems about perimeter-area measurement in the noticing questionnaire were gathered under the theme of perimeter-area relationship. Therefore, the findings regarding the pre-service teachers' deciding how to respond skills are presented under the title of pre-service teachers' deciding how to respond on the basis of students' understanding about the perimeter-area relationship in the following part.

4.1.1.3.1 Pre-service teachers' deciding how to respond on the basis of students' understanding about the perimeter-area relationship

In the noticing questionnaire, the first three problems (five problems with sub-problems) were about the perimeter-area relationship. The first problem and the third problem were about a fixed perimeter-changing area situation, while the second problem was about a fixed area-changing perimeter situation. The findings for the first problem and the third problem, which are related to a fixed perimeter-changing area situation, and the findings for the second problem, which is related to a fixed area-changing perimeter situation, are presented below, respectively.

4.1.1.3.1.1 Pre-service teachers' deciding how to respond on the basis of students' understanding about a fixed perimeter-changing area situation

The first problem in the noticing was named change in perimeter, and the pre-service teachers were expected to decide how to respond to three different students based on their understanding (see Figure 4.2). Figure 4.18 presents the frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the change in perimeter problem in the initial and final noticing questionnaire.

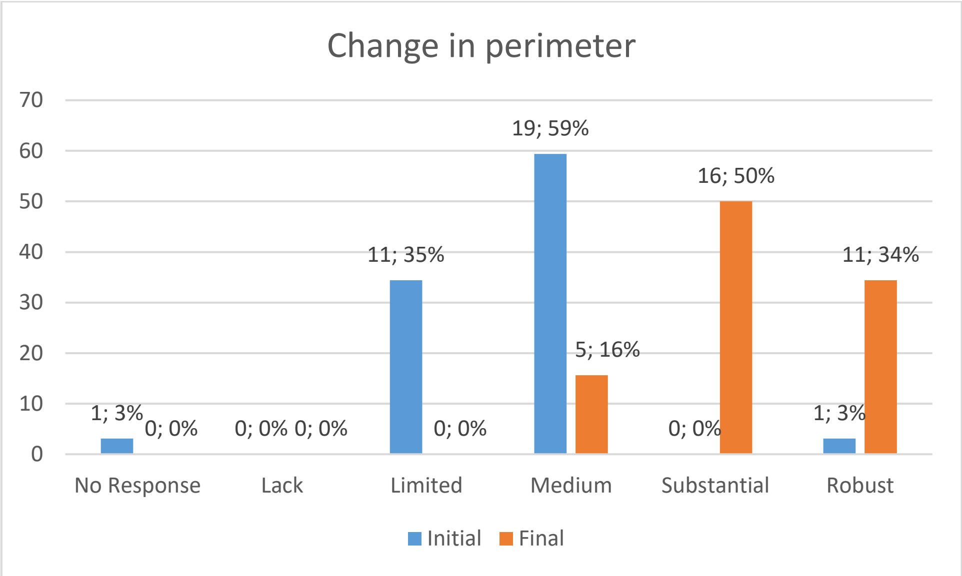


Figure 4. 18 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the change in perimeter problem in the initial and final noticing questionnaire

In the initial questionnaire, almost one-third of the pre-service teachers (n=11) provided limited evidence. Moreover, more than half of the pre-service teachers (n=19) responded to students by providing a medium level of evidence. Only one of them (3%) responded by providing a robust level of evidence. Interestingly, none of the pre-service teachers provided a lack of evidence and substantial evidence in the initial questionnaire. In the final questionnaire, pre-service teachers provided a medium, substantial, and robust level of evidence. There was a decrease in the percentage of pre-service teachers who provided a medium level of evidence. Half of the suggestions (n=16) showed a substantial level of evidence, and almost one-third of the suggestions (n=11) showed a robust level of evidence. That is, in the final questionnaire, 27 of 32 pre-service teachers' suggestions (84%) demonstrated at the top two levels, substantial or robust levels of evidence. This reveals the improvement in the quality of the suggestions provided by the pre-service teachers to the deciding how to respond prompt in the final questionnaire. To illustrate the improvement, how pre-service teacher 10 responded to students in the initial and final noticing questionnaire is given as follows.

Response of pre-service teacher 10 in the initial noticing questionnaire:

> Ada did not understand the concept of perimeter. I would give more explanatory examples from daily life.
>
> Tuna did not understand the concepts of perimeter and length. The student can be asked to measure the perimeter of the shapes with the ruler.
>
> More complex problems can be given to Sare. Since the student already has a conceptual understanding, she will be able to solve problems on this subject.

Pre-service teacher 10's suggestion as a response to Ada showed a *limited* level of evidence because she provided a teacher-centered suggestion. Her response to Tuna demonstrated a *medium* level of evidence because measuring the perimeter of the shape with a ruler may help the student realize his incorrect answer but is insufficient to eliminate the student's misconception, which is believing that there is a direct relationship between the perimeter and area. The pre-service teacher's suggestion for Sare, i.e., giving complex problems, showed a *limited* level of evidence since it was a general suggestion. Overall, the suggestions offered by pre-service teacher 10 were not sufficient to eliminate the misconceptions of Ada and Tuna and extend the understanding of Sare. Accordingly, pre-service teacher 10 provided a *limited* level of evidence for deciding how to respond prompt in the initial questionnaire.

Response of pre-service teacher 10 in the final noticing questionnaire:

> By focusing on the curves, Ada sees more sides in shape B, and hence, she thinks that the perimeter of shape B is larger. To eliminate her misconception, I would ask Ada to create shapes A and B on the geoboard using wire. Then, I would ask the student to compare the perimeters again by opening the wires. In this way, the student can realize that the perimeter has not changed. Afterward, I would ask her to create the same shapes with toothpicks and

> transform shape B into shape A by moving the toothpicks. Since she uses the same number of toothpicks, she can understand that the perimeter does not change, and only the sides are moved in the figure.
>
> Tuna thinks that when the area decreases, the perimeter decreases, too. In order to eliminate the student's misconception, a few examples where the perimeter remains the same, but the area changes are given. By using string and unit squares, the student is made to realize that although the area changes, the perimeter can remain the same.
>
> Sare could see that the perimeter of the shapes is the same by moving the sides. Therefore, in order to extend the student's understanding, I would ask her, "Imagine cutting shape A to form a different shape C. Do you think it is possible for the perimeter of shape C to be larger than that of shape A? Or is it possible that the perimeter of the shape C is greater than that of the shape A. In this way, I would enable the student to explore the situations in which the perimeter increases and decreases by cutting the paper.

In the final questionnaire, pre-service teacher 10 provided a detailed suggestion for Ada to make the student understand and for Sare to extend the student's understanding. Therefore, her suggestions for these two students showed a *robust* level of evidence. While responding to Tuna, the pre-service teacher came up with a specific suggestion that helped the student overcome his misconception, but she did not provide much detail regarding the process, that is, how the student realized the change of area while the perimeter remained the same. Hence, the suggestion made for Tuna demonstrated a *substantial* level of evidence. Overall, pre-service teacher 10's suggestions for three students were based on making Ada and Tuna understand and extending Sare's understanding, and she explained two of the suggestions in detail. Thus, pre-service teacher 10 provided a *robust* level of evidence for deciding how to respond prompt in the final questionnaire.

The third problem in the noticing questionnaire was about a fixed perimeter-changing area situation. This problem had two sub-problems. The first sub-problem was named the minimum area for a fixed perimeter, and the pre-service teachers were expected to decide how to respond to two different students based on their understanding (see Figure 4.4). The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the minimum area for a fixed perimeter problem in the initial and final noticing questionnaires are set out in Figure 4.19. In the initial questionnaire, five of 32 pre-service teachers (16%) did not provide any suggestions; by contrast, in the final questionnaire, all pre-service teachers provided suggestions. More than half of the suggestions (n=17) showed a limited level of evidence in the initial questionnaire. On the other hand, in the final questionnaire, there was a dramatic decrease in the percentage of the suggestions, showing a limited level of evidence. Moreover, in the initial questionnaire, a low percentage of the pre-service teachers (n=3) provided high levels of evidence (substantial or robust level of evidence), and only one of them (3%) showed a robust level of evidence. In contrast, the data regarding the final questionnaire in Figure 4.21 represents an increase in the percentage of the suggestions, demonstrating a robust level of evidence. Furthermore, in the final questionnaire, nearly half of the pre-service teachers' suggestions (n=15) were at the level of substantial evidence, while this rate was only 6% in the initial questionnaire (n=2). As an example, the suggestions of pre-service teacher 3 to the deciding how to respond prompt in the initial and final noticing questionnaire are presented below.

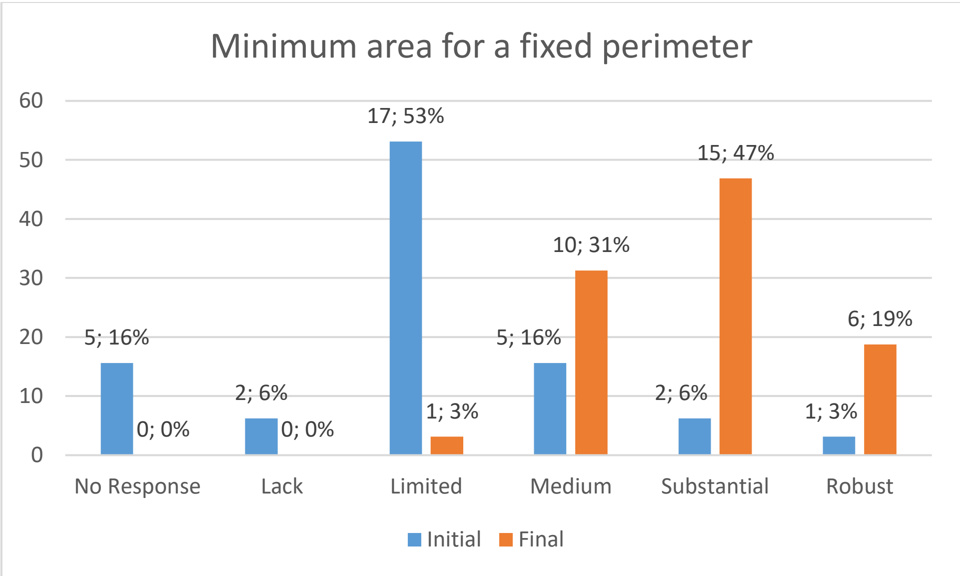


Figure 4. 19 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the minimum area for a fixed perimeter problem in the initial and final noticing questionnaire

Response of pre-service teacher 3 in the initial noticing questionnaire:

> Yelda can be made to calculate the perimeter by proceeding from simple problems to complex ones.
>
> Utku can be asked for more complex problems.

Pre-service teacher 3 offered general suggestions for both Yelda and Utku in the initial questionnaire. Hence, her response to each student showed a *limited* level of evidence. Thus, since the suggestions of pre-service teacher 3 were not sufficient to eliminate Yelda's misconception and extend Utku's understanding, she provided a *limited* level of evidence to deciding how to respond prompt in the initial questionnaire.

Response of pre-service teacher 3 in the final noticing questionnaire:

> I would first ask Yelda the perimeter of the shape before adding the square. Then, I would ask her about the perimeter after adding it so that she would realize that the perimeters were not equal. Afterward, I would ask her where

> the square is added so that the perimeter does not change and ask her to calculate it by trial and error.
>
> Utku was able to add one square without changing the perimeter. I would ask the student to decrease and increase the perimeter of the shape by displacing the unit squares so that the area of the shape would not change. In this case, the student has to think about both the disappearing and appearing sides where he removes the square and the disappearing and appearing sides where he adds the square. Thus, I can extend the student's understanding of the perimeter.

An excerpt from the final questionnaire indicates that pre-service teacher 3 provided orientations for Yelda with questions to answer. Hence, the pre-service teacher responded to the student by providing a *medium* level of evidence. For Utku, she provided a detailed suggestion that extended the student's understanding. Therefore, the suggestion for Utku demonstrated a *robust* level of evidence. Overall, pre-service teacher 3 provided a specific suggestion for Uku, i.e., one of two students, by explaining in detail. Thus, her response to the deciding how to respond prompt in the final questionnaire showed a *substantial* level of evidence.

The second sub-problem was named the maximum area for a fixed perimeter, and the pre-service teachers were expected to decide how to respond to two different students based on their understanding (see Figure 4.6). Figure 4.20 presents the frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the maximum area for a fixed perimeter problem in the initial and final noticing questionnaire. In the initial questionnaire, one-quarter of the pre-service teachers (n=8) did not provide any suggestions. Six percent of the pre-service teachers (n=2) provided a lack of evidence in the initial questionnaire, whereas none of the suggestions demonstrated a lack of evidence in the final questionnaire. Furthermore, about two-thirds of the suggestions (n=20) showed a limited level of evidence. Although there was a decrease in the percentage of the suggestions showing a limited level of evidence in the final questionnaire, the highest percentage was still obtained at the limited level. The percentage of the suggestions that demonstrated a medium and substantial level of evidence was equal, and a really small proportion in the initial questionnaire. In contrast, one-quarter of the suggestions (n=8) showed a substantial level of evidence in the final questionnaire. Interestingly, robust evidence of responding with the lowest percentage was provided for this problem in the final questionnaire. There was a slight improvement in the responding skills of the pre-service teachers in this problem compared to the previous problems. The suggestions of pre-service teacher 14 as a response to the deciding how to respond prompt in the maximum area for a fixed perimeter problem in the initial and final questionnaire are provided as an example below.

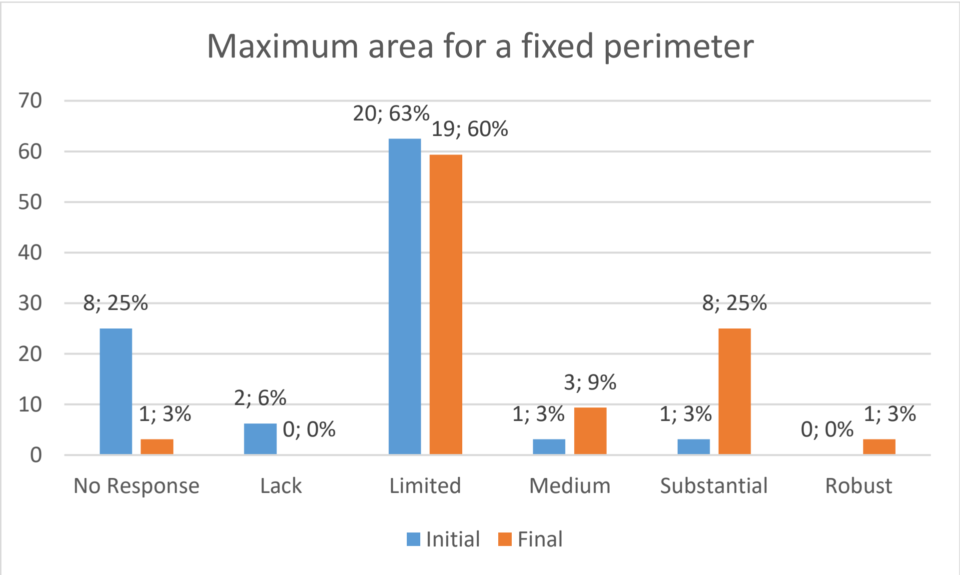


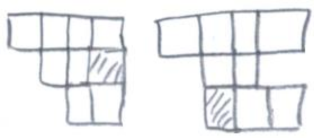
Figure 4. 20 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the maximum area for a fixed perimeter problem in the initial and final noticing questionnaire

Response of pre-service teacher 14 in the initial noticing questionnaire:

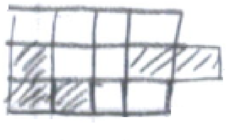
> I would explain the perimeter concept to Yelda again and show the calculation of the perimeter through sample questions.
>
> I would ask Utku how many squares should be added at least and at most so that the perimeter of the shape is 20 units.

Here, pre-service teacher 14 provided a *limited* level of evidence while responding to Yelda because her suggestion was teacher centred. Moreover, the pre-service teacher identified the solution of Utku incorrectly because she considered Utku's solution to be correct. Based on this, she provided a suggestion to extend the student's understanding. However, even though the three squares that Utku added did not change the perimeter of the shape, it was not the maximum number of squares that could be added. Therefore, the suggestion offered by the pre-service teacher for Utku was an inappropriate suggestion as a result of incorrect attention, which led to a *lack* of evidence for responding. Overall, the pre-service teacher provided suggestions that were not sufficient to eliminate the students' misconceptions. Thus, her response to the deciding how to respond prompt showed a *limited* level of evidence in the initial questionnaire.

Response of pre-service teacher 14 in the final noticing questionnaire:

> In order to eliminate the misconception of Yelda about the perimeter, I would concretely give the shapes consisting of the unit squares. I would ask her to calculate the perimeter without adding a unit square; then, I would ask her to calculate the perimeter when she adds a unit square. Afterward, I try to make the student realize
>
> 
>
> how the perimeter changes when we change the location of the square. For example, I would ask the student what the perimeter is when we place the unit square in a different place, as in the next figure, and ask the student to compare the perimeter of the two shapes. After the student realizes that the perimeter increased, I would ask the student, "Why do you think the perimeter has increased?". I would also ask for the number of newly appeared and disappeared sides. In this way, the student will see how the location of the unit squares affects the perimeter. When making additions, she can produce solutions accordingly and reach the shape with a maximum area by herself.

I would ask Utku, “Can we add other unit squares? Can a maximum of three squares be added?



Then, I would ask, “If we add two more unit squares, what can you say about the perimeter?”

With questions like these, I try to lead the student to the shape with maximum area.

In the final questionnaire, pre-service teacher 14 provided a detailed suggestion that made Yelda understand how the location of the unit squares affects the perimeter of the shape and how the sides are related to the perimeter. Consequently, the suggestion of the pre-service teacher provided for Yelda showed a *robust* level of evidence. In addition, pre-service teacher 14’s suggestion for Utku was based on orienting the student with questions to the answer, i.e., the shape with the maximum area. Hence, the suggestion for Utku demonstrated a *medium* level of evidence. Overall, pre-service teacher 14 provided a detailed specific suggestion for Yelda, one of two students, to eliminate the student’s misconception. Accordingly, her response to the deciding how to respond prompt in the final questionnaire showed a *substantial* level of evidence.

4.1.1.3.1.2 Pre-service teachers’ deciding how to respond on the basis of students’ understanding about a fixed area-changing perimeter situation

The second problem in the noticing questionnaire was about a fixed area-changing perimeter situation. This problem had two sub-problems. The first sub-problem was named perimeter comparison, and the pre-service teachers were expected to decide how to respond to two different students based on their understanding (see Figure 4.8). The frequency and percentage of the pre-service teachers’ suggestions for each category of deciding how to respond in the perimeter comparison problem in the initial and final noticing questionnaire is presented in Figure 4.21. In the initial questionnaire, almost one-third of the pre-service teachers (n=11) did not provide any suggestions, and almost one-third of the suggestions (n=11) showed a limited level of evidence. A low percentage of the pre-service teachers (n=3) provided high levels of evidence (substantial or robust level of evidence), whereas almost three-quarters of them (n=23) did so in the final questionnaire. In addition, more than half the suggestions (n=17) showed a robust level of evidence in the final questionnaire. It is worth noting that robust evidence of responding with the highest percentage was provided for this problem in the context of perimeter and area measurement in the final questionnaire. The improvement in the quality of the suggestions provided by pre-service teachers to the deciding how to respond prompt can be illustrated by the responses of pre-service teacher 15 in the initial and final questionnaire.

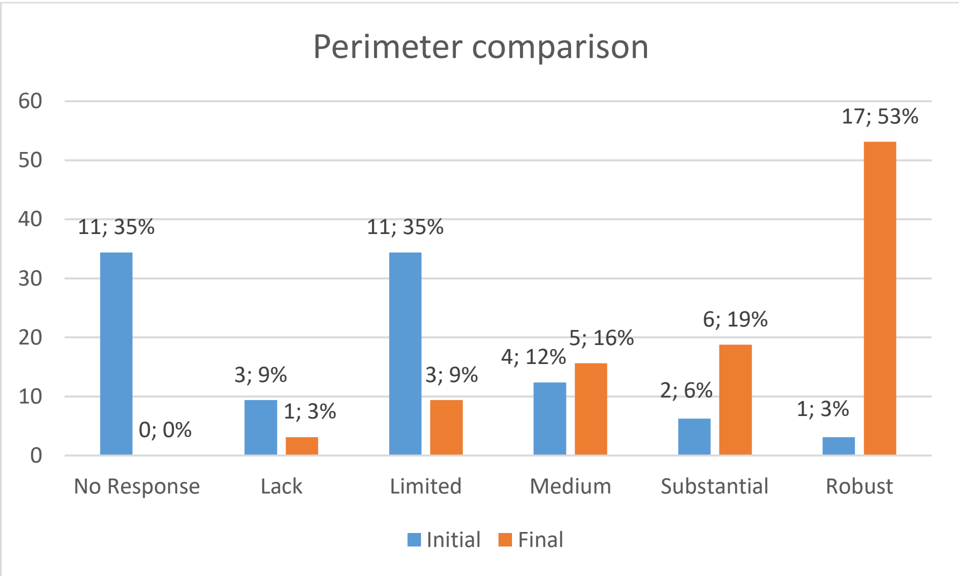


Figure 4. 21 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the perimeter comparison problem in the initial and final noticing questionnaire

Response of pre-service teacher 15 in the initial noticing questionnaire:

> Can thought that the longer the given shapes were, the larger the perimeter would be. In order for the student to understand the subject, I would talk about the properties of the tangram and how it is formed. I would facilitate the student's understanding of the subject with concrete materials. I would show that the shapes obtained are actually formed by combining the same parts.

> Hazal's solution is correct. In the next step, I would ask the student to concretely create tangram pieces from cardboard and compare the perimeter of different shapes created by changing the places of these pieces.

In this excerpt, pre-service teacher 15's suggestion for each student showed a *lack* of evidence for responding because she provided inappropriate suggestions as a result of incorrect attention to the students' solutions. Therefore, the pre-service teacher provided a *lack* of evidence for deciding how to respond in the initial questionnaire.

Response of pre-service teacher 15 in the final noticing questionnaire:

> In order to extend the understanding of Can, the student can be asked to create new shapes and compare their perimeters. In the problem, already constructed shapes were given and asked to compare their perimeters. In the next step, I would wonder whether the student could construct the shapes himself according to the given criteria for the perimeter. Therefore, I would ask him the following questions and answer them using the tangram pieces provided:
>
> Can you create two shapes with the same perimeter?
> Can you create two shapes with different perimeters using the same pieces?
> Can you create shapes with perimeters smaller than the perimeter of the second figure?
> Why do you think the perimeters of the shapes change?
>
> Hazal can be asked to calculate the perimeter of the shapes using a string and then compare the perimeter by creating different shapes with the same pieces. In this way, the student can realize that the differences in the perimeter depend on the positioning of the pieces, even though the same pieces are used.

In the final questionnaire, pre-service teacher 15 responded to Can by providing a *robust* level of evidence because the suggestion offered was detailed and aimed at extending the student's understanding. For Hazal, she made a specific suggestion to eliminate the student's misconception without providing much detail regarding how the student makes sense of the contribution of the length of sides to the perimeter. For this reason, this suggestion showed a *substantial* level of evidence. Thus, pre-service teacher 15 provided specific suggestions for both students, one to extend the understanding of Can and one to make Hazal understand and explained one of them in detail. The response of pre-service teacher 15 to the deciding how to respond prompt in the final questionnaire demonstrated a *robust* level of evidence.

The second problem in the noticing questionnaire was about a fixed area-changing perimeter situation. This problem had two sub-problems. The second sub-problem was named area comparison, and the pre-service teachers were expected to decide how to respond to two different students based on their understanding (see Figure 4.10). Figure 4.22 below illustrates the frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the area comparison problem in the initial and final noticing questionnaire. In the initial questionnaire, six of 32 pre-service teachers (19%) did not provide any suggestions. More than one-third of the suggestions (n=12) demonstrated a limited level of evidence, and one-quarter of the suggestions (n=8) showed a medium level of evidence. A small percentage of the pre-service teachers (n=5) provided high levels of evidence (substantial or robust level of evidence). Only one of them (3%) showed a robust level of evidence in the initial questionnaire; by contrast, half of the pre-service teachers (n=16) responded to the students by providing a robust level of evidence in the final questionnaire. In this way, 22 of the suggestions (69%) showed high levels of evidence (substantial or robust level of evidence). Moreover, the percentage of the suggestions demonstrating limited evidence decreased considerably in the final questionnaire. As an example, the suggestions of pre-service teacher 29 to the deciding how to respond prompt in the initial and final noticing questionnaire are presented below.

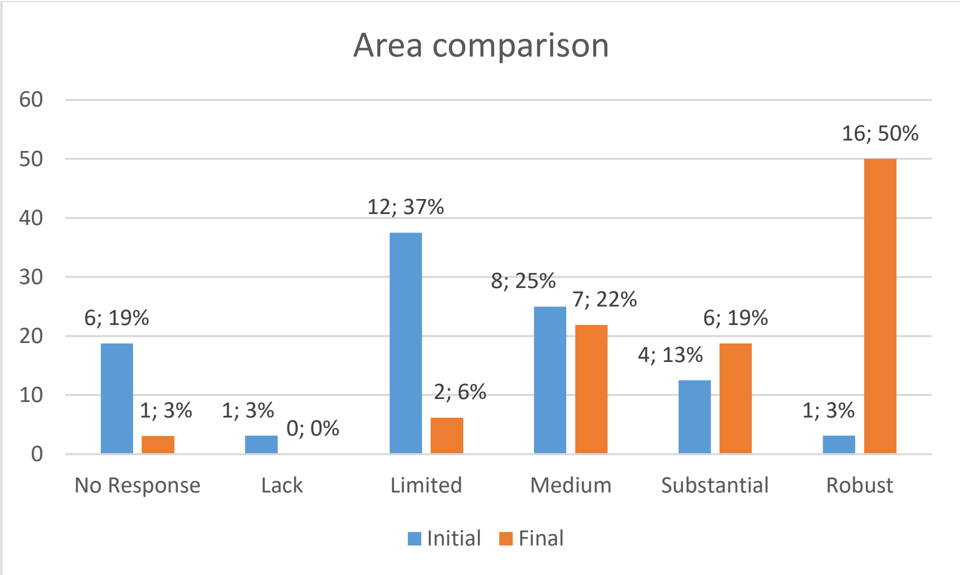


Figure 4. 22 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the area comparison problem in the initial and final noticing questionnaire

Response of pre-service teacher 29 in the initial noticing questionnaire:

> I would ask Can to create different shapes with equal areas.
>
> For Hazal, an activity that requires finding the areas of the whole and the parts of the shape can be done.

The excerpt from the initial questionnaire reveals that pre-service teacher 29 responded to Can by providing a *limited* level of evidence. Here, it is not clear how the question asked will extend the student's understanding of area concepts because Can already recognized that the different shapes provided in the problem had the same area. Furthermore, pre-service teacher 29 provided a specific suggestion to eliminate the misconception of Hazal, but she did not provide much detail about how the mentioned activity would help the student. Therefore, the suggestion for Hazal showed a *substantial* level of evidence. Overall, pre-service teacher 29 provided a *medium* level of evidence for deciding how to respond in the initial questionnaire since the pre-service teacher could provide a specific suggestion for one of the students without providing much detail.

Response of pre-service teacher 29 in the final noticing questionnaire:

> Can knows that the areas of the shapes are equal because they are made of the same pieces. To extend the understanding of the student, I want him to make an estimate and compare the area of the shapes by removing different pieces from the shapes. There is a relationship between the area of the tangram pieces. For example, the middle triangle, square, and parallelogram each consist of two small triangles, so the area of each is twice the area of the small triangle. I wonder, when I remove a square from one shape and a middle triangle from the other shape, the student realizes that the shapes still have the same area.
>
> Hazal looks at the shapes in terms of their width and thinks that the area of the second shape is larger. I should make the student realize the conservation of the area. By giving regular shapes on the gridded paper, I would ask Hazal about the areas of shapes and want her to calculate their areas by counting the squares. Then, I would form composite shapes such as envelopes, fields, and so on by combining these shapes. I would ask the student to calculate the areas of these shapes and make a comparison between the area of individual shapes and the area of composite shapes. Thus, the student can realize the conservation of the area.

In the final questionnaire, pre-service teacher 29 was able to provide a detailed suggestion for Can to extend the student's understanding of the area concept in contrast to the initial questionnaire. Hence, the suggestion for Can showed a robust level of evidence. Moreover, pre-service teacher 29 elaborated on her suggestion for Hazal in the initial questionnaire and provided a detailed suggestion that helped the student overcome her misconception and make the student understand. Therefore, the suggestion for Can demonstrated a robust level of evidence. Overall, pre-service teacher 29 provided a suggestion to extend the understanding of Can and a suggestion for making Hazal understand by explaining both of them in detail. Thus, pre-service teacher 29 provided a robust level of evidence for deciding how to respond prompt in the final questionnaire.

The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the context of perimeter and area measurement in the initial and final noticing questionnaire are summarized in Figure 4.23.

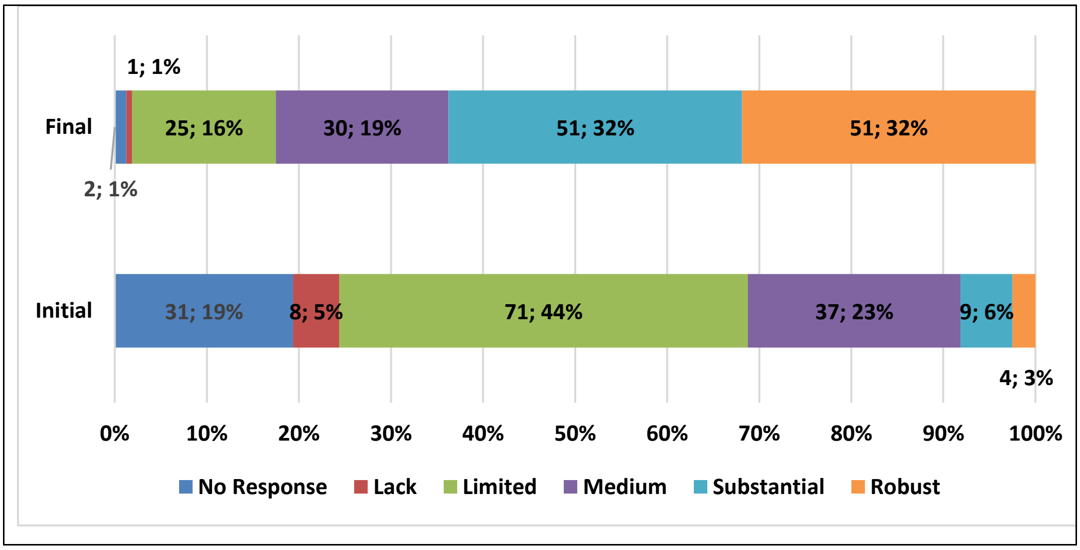


Figure 4. 23 The frequencies and percentages of the pre-service teachers' suggestions for each category of deciding how to respond in the context of perimeter and area measurement in the initial and final noticing questionnaire

From the chart in Figure 4.23, it can be seen that by far, the highest percentage of the suggestions showed a limited level of evidence in the context of perimeter and area, which was followed by a medium level of evidence with the second highest percentage in the initial questionnaire. In addition, almost one-fifth of the responses did not include any suggestions. On the other hand, what is striking about the data in the figure in the final questionnaire is the equal and highest percentage of suggestions that demonstrated substantial and robust levels of evidence. In this way, about two-thirds of the suggestions showed high levels of evidence in the final questionnaire. This was followed by the category of medium evidence with the third highest percentage. Moreover, what stands out in this figure is the significant decrease in the percentage of suggestions in the categories of limited evidence and no response in the final questionnaire.

4.1.1.4 Statistical analysis in the context of perimeter-area measurement

The findings presented above show that there is an improvement in the pre-service teachers' professional noticing skills in the context of perimeter-area measurement in the final questionnaire. This part presents the findings on whether this change is statistically significant, that is, an answer to the research question, "Is the change in pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement from pre-test to post-test statistically significant?" A noticing questionnaire including three problems (five problems with the sub-problems) about perimeter-area relationship was implemented on 32 pre-service teachers as a pre-test to identify their initial professional noticing skills and as a post-test after the intervention to determine their final professional noticing skills in the context of perimeter and area measurement. The present study involved one group, with each participant in the group being measured twice. Moreover, the criteria used to evaluate the professional noticing responses represented ordinal data (rank or ordered categories). Hence, to determine whether there was a statistically significant difference between the pre-test and post-test, A Wilcoxon Signed Ranks Test as a non-parametric test was preferred as an alternative to the repeated-measures t-test. A Wilcoxon Signed Ranks Test was conducted for each problem to determine whether statistically significant gains were found for each component of professional noticing: attending, interpreting, and responding in the context of perimeter and area measurement. The results of the test regarding the attending component are presented in Table 4.2.

Table 4. 2 Results of Wilcoxon signed ranks test regarding attending comparing pre-test and post-test in the context of perimeter and area measurement

	Problems	z	p	r
	Perimeter-area relationship			
Attending	Change in perimeter	-4.249	.000	.53
	Perimeter comparison	-4.304	.000	.54
	Area comparison	-2.855	.004	.36
	Minimum area for a fixed perimeter	-2.223	.026	.28
	Maximum area for a fixed perimeter	-3.657	.000	.46

As can be seen from the table above, for the attending component, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers in the change in perimeter problem following participation in the intervention, $z = -4.249$, $p < .001$, with a large effect size ($r = .53$). In the perimeter comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers following participation in the intervention, $z = -4.304$, $p < .001$, with a large effect size ($r = .54$). In the area comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers following participation in the intervention, $z = -2.855$, $p = .004$, with a medium effect size ($r = .36$). In the minimum area for a fixed perimeter problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers following participation in the intervention, $z = -2.223$, $p = .026$, with a small effect size ($r = .28$). In the maximum area for a fixed perimeter problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers following participation in the intervention, $z = -3.657$, $p < .001$, with a medium effect size ($r = .46$). Table 4.3 compares the pre-service teachers' initial and final levels in each problem for the attending component in the context of perimeter and area measurement.

Table 4. 3 Comparison of pre-service teachers' initial and final levels in each problem for the attending component in the context of perimeter and area measurement

			N	Mean Rank	Sum of Ranks
Attention	Change in perimeter Final-Initial	Negative ranks	2[a]	9.50	19.00
		Positive ranks	25[b]	14.36	359.00
		Ties	5[c]		
		Total	32		
	Perimeter comparison Final-Initial	Negative ranks	1[a]	26.00	26.00
		Positive ranks	29[b]	15.14	439.00
		Ties	2[c]		
		Total	32		
	Area comparison Final-Initial	Negative ranks	2[a]	6.75	13.50
		Positive ranks	14[b]	8.75	122.50
		Ties	16[c]		
		Total	32		
	Minimum area for a fixed perimeter Final-Initial	Negative ranks	5[a]	11.90	59.50
		Positive ranks	17[b]	11.38	193.50
		Ties	10[c]		
		Total	32		
	Maximum area for a fixed perimeter Final-Initial	Negative ranks	4[a]	8.38	33.50
		Positive ranks	22[b]	14.43	317.50
		Ties	6[c]		
		Total	32		

As shown in Table 4.3, for the attending component, in the change in perimeter problem, 25 pre-service teachers' levels increased after the intervention. Five pre-service teachers' levels of attending did not change, and there was a decrease in two pre-service teachers' levels. In the perimeter comparison problem, 29 pre-service teachers increased their levels of attending in the post-test. There was a decrease in one pre-service teacher's level and no change in two pre-service teachers' levels. In the area comparison problem, 14 pre-service teachers provided higher levels of evidence in the post-test than in the pre-test. Sixteen pre-service teachers remained at the same level, and there was a decrease in two pre-service teachers' levels in the post-test. In the minimum area for a fixed perimeter problem, 17 pre-service teachers' levels increased after the intervention, whereas five pre-service teachers' levels decreased. Ten pre-service teachers' levels did not change. In the maximum area for a fixed perimeter problem, an increase was observed in 22 pre-service teachers' levels in the post-test. Four pre-service teachers provided lower levels of evidence, and six pre-service teachers provided the same levels of evidence in the post-test. The results of the Wilcoxon Signed Ranks Test regarding the interpreting component are presented in Table 4.4.

Table 4. 4 Results of Wilcoxon signed ranks test regarding interpreting comparing pre-test and post-test in the context of perimeter and area measurement

	Problems	z	p	r
	Perimeter-area relationship			
Interpreting	Change in perimeter	-4.047	.000	.51
	Perimeter comparison	-4.535	.000	.57
	Area comparison	-4.468	.000	.56
	Minimum area for a fixed perimeter	-2.984	.003	.38
	Maximum area for a fixed perimeter	-4.279	.000	.54

As can be seen from the table above, for the interpreting component, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers in the change in perimeter problem following participation in the intervention, $z = -4.047$, $p < .001$, with a large effect size ($r = .51$). In the perimeter comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers following participation in the intervention, $z = -4.535$, $p < .001$, with a large effect size ($r = .57$). In the area comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers following participation in the intervention, $z = -4.468$, $p < .001$, with a large effect size ($r = .56$). In the minimum area for a fixed perimeter problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers following participation in the intervention, $z = -2.984$, $p = .003$, with a medium effect size ($r = .38$). In the maximum area for a fixed perimeter problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers following participation in the intervention, $z = -4.279$, $p < .001$, with a large effect size ($r = .54$). Table 4.5 provides the comparison of the pre-service teachers' initial and final levels in each problem for the interpreting component in the context of perimeter and area measurement.

Table 4. 5 Comparison of pre-service teachers' initial and final levels in each problem for the interpreting component in the context of perimeter and area measurement

			N	Mean Rank	Sum of Ranks
Interpretation	Change in perimeter Final-Initial	Negative ranks	2[a]	9.50	19.00
		Positive ranks	23[b]	13.30	306.00
		Ties	7[c]		
		Total	32		
	Perimeter comparison Final-Initial	Negative ranks	1[a]	16.00	16.00
		Positive ranks	29[b]	15.48	449.00
		Ties	2[c]		
		Total	32		
	Area comparison Final-Initial	Negative ranks	1[a]	9.00	9.00
		Positive ranks	26[b]	14.19	369.00
		Ties	5[c]		
		Total	32		
	Minimum area for a fixed perimeter Final-Initial	Negative ranks	3[a]	12.00	36.00
		Positive ranks	19[b]	11.42	217.00
		Ties	10[c]		
		Total	32		
	Maximum area for a fixed perimeter Final-Initial	Negative ranks	1[a]	6.00	6.00
		Positive ranks	24[b]	13.29	319.00
		Ties	7[c]		
		Total	32		

For the interpreting component, in the change in perimeter problem, 23 pre-service teachers increased their levels in the post-test. Two pre-service teachers provided lower levels of evidence in the post-test compared to the pre-test. Seven pre-service teachers' levels remained the same. In the perimeter comparison problem, 29 pre-service teachers provided higher levels of evidence for interpreting in the post-test. There was a decrease in only two pre-service teachers' levels in the post-test, and two pre-service teachers' levels remained the same. In the area comparison problem, an increase was observed in 26 pre-service teachers' levels in the post-test. There was a decrease in one pre-service teacher's level and no change in five pre-service teachers' levels. In the minimum area for a fixed perimeter problem, 19 pre-service teachers provided higher levels of evidence, while three pre-service teachers provided lower levels of evidence. Ten pre-service teachers remained at the same level in the post-test. In the maximum area for a fixed perimeter problem, 24 pre-service teachers increased their levels in the post-test. Only one pre-service teacher's level decreased, and seven pre-service teachers stayed at the same level in the post-test. The results of the Wilcoxon Signed Ranks Test regarding the responding component are presented in Table 4.6.

Table 4. 6 Results of Wilcoxon signed ranks test regarding responding comparing pre-test and post-test in the context of perimeter and area measurement

	Problems	z	p	r
	Perimeter-area relationship			
Responding	Change in perimeter	-4.742	.000	.59
	Perimeter comparison	-4.493	.000	.56
	Area comparison	-4.177	.000	.52
	Minimum area for a fixed perimeter	-4.533	.000	.57
	Maximum area for a fixed perimeter	-3.404	.001	.43

It can be seen from the data in Table 4.6 that for the responding component, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers in the change in perimeter problem following participation in the intervention, $z = -4.742$, $p < .001$, with a large effect size ($r = .59$). In the perimeter comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers following participation in the intervention, $z = -4.493$, $p < .001$, with a large effect size ($r = .56$). In the area comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers following participation in the intervention, $z = -4.177$, $p < .001$, with a large effect size ($r = .52$). In the minimum area for a fixed perimeter problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers following participation in the intervention, $z = -4.533$, $p < .001$, with a large effect size ($r = .57$). In the maximum area for a fixed perimeter problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers following participation in the intervention, $z = -3.404$, $p = .001$, with a medium effect size ($r = .43$). Table 4.7 presents the comparison of the pre-service teachers' initial and final levels in each problem for the responding component in the context of perimeter and area measurement.

Table 4. 7 Comparison of pre-service teachers' initial and final levels in each problem for the responding component in the context of perimeter and area measurement

			N	Mean Rank	Sum of Ranks
Response	Change in perimeter Final-Initial	Negative ranks	1[a]	7.00	7.00
		Positive ranks	29[b]	15.79	458.00
		Ties	2[c]		
		Total	32		
	Perimeter comparison Final-Initial	Negative ranks	1[a]	11.00	11.00
		Positive ranks	28[b]	15.14	424.00
		Ties	3[c]		
		Total	32		
	Area comparison Final-Initial	Negative ranks	4[a]	5.38	21.50
		Positive ranks	24[b]	16.02	384.50
		Ties	4[c]		
		Total	32		
	Minimum area for a fixed perimeter Final-Initial	Negative ranks	1[a]	6.00	6.00
		Positive ranks	27[b]	14.81	400.00
		Ties	4[c]		
		Total	32		
	Maximum area for a fixed perimeter Final-Initial	Negative ranks	2[a]	6.50	13.00
		Positive ranks	17[b]	10.41	177.00
		Ties	13[c]		
		Total	32		

a. Initial > Final
b. Initial < Final
c. Initial = Final

For the responding component, in the change in perimeter problem, 29 pre-service teachers provided higher levels of evidence in the post-test. While there was no change in two pre-service teachers' levels, there was a decrease in the level of one pre-service teacher in the post-test. In the perimeter comparison problem, 28 pre-service teachers' levels of responding increased. One pre-service teacher provided lower levels of evidence in the post-test, and three pre-service teachers' levels did not change. In the area comparison problem, there was an increase in the levels of 24 pre-service teachers in the post-test. While the levels of four of the remaining eight pre-service teachers in the post-test decreased, there was no change in the level of four pre-service teachers. In the minimum area for a fixed perimeter problem, an increase was observed in 27 pre-service teachers' levels in the post-test. Four pre-service teachers remained at the same level, and there was a decrease in one pre-service teacher's level in the post-test. In the maximum area for a fixed perimeter problem, 17 pre-service teachers provided higher levels of evidence in the post-test, whereas 13 pre-service teachers' levels did not change. On the other hand, the levels of two pre-service teachers decreased.

Overall, statistically significant increases were obtained in each problem for all three components in the context of perimeter and area measurement. As a result, the statistical analysis revealed that improvements represent a statistically significant change in the post-test in the context of perimeter and area measurement. Accordingly, the findings indicated that the pre-service teachers significantly improved on attending, interpreting, and responding components when they participated in the video-based module situated in the pedagogies of practice framework. The greatest improvement was observed in the responding component, followed by the interpreting component. The least improvement among the components was seen in the attending component in the context of perimeter and area measurement. The highest increase in levels of attending and interpreting was noted in the perimeter comparison problem, whereas the highest increase in the levels of responding was discerned in a change in the perimeter problem.

4.1.2 Changes in pre-service teachers' professional noticing in the context of volume-surface area measurement

In this part, in the context of volume-surface area measurement, the findings related to the problems of the volume-surface area relationship in the initial and final noticing questionnaire under the headings of attending, interpreting, and deciding how to respond are given. The problems in the context of volume-surface area measurement were designed as volume changes when the lateral area is fixed, and surface area changes when the volume is fixed. At the end of the part, the difference between the initial and final noticing questionnaire for attending, interpreting, and deciding how to respond skills in the context of volume-surface area measurement is given in total.

4.1.2.1 Changes in pre-service teachers' attending to students' solutions in the context of volume-surface area measurement

Pre-service teachers' attending to students' solutions in the context of volume-surface area measurement is measured by their responses to attending prompts related to students' solutions in problems 4 and 5 of the noticing questionnaire. The pre-service teachers' responses to the attending to students' solutions prompt in the noticing questionnaire were explored at four levels: robust level of evidence, substantial level of evidence, limited level of evidence, and lack of evidence of attending to students' solutions. Moreover, there were some pre-service teachers who did not provide a response to the attending prompt. This was categorized as no attention. Two problems about volume-surface area measurement in the noticing questionnaire were gathered under the theme of the volume-surface area relationship. Therefore, the findings regarding the pre-service teachers' attending skills are presented under the title of pre-service teachers' attending to students' solutions about the volume-surface area relationship in the following part.

4.1.2.1.1 Pre-service teachers' attending to students' solutions about the volume-surface area relationship

In the noticing questionnaire, the last two problems (four problems with sub-problems) were about the volume-surface area relationship. The fourth problem was about a fixed lateral area-changing volume situation, while the fifth problem was about a fixed volume-changing surface area situation. The findings for the fourth problem, which is related to a fixed lateral area-changing volume situation, and the findings for the fifth problem, which is related to a fixed volume-changing surface area situation, are presented below, respectively.

4.1.2.1.1.1 Pre-service teachers' attending to students' solutions about a fixed lateral area-changing volume situation

The fourth problem in the noticing questionnaire was named volume comparison of paper prisms, and the pre-service teachers were expected to attend to two different students' solutions to this problem. The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the volume comparison of paper prisms problem in the initial and final noticing questionnaire is set out in Figure 4.24. In the initial questionnaire, three of 32 pre-service teachers (9%) did not attempt to attend to students' solutions, whereas all pre-service teachers attended to the students' solutions in the final questionnaire. Moreover, while 14 of 32 responses (44%) in the initial questionnaire showed a lack of evidence, in the final questionnaire, there was a decrease in the percentage of those showing a lack of evidence. Nearly one-third of the responses (n=10) in the initial questionnaire demonstrated a limited level of evidence, which was fallen in the final questionnaire. A low percentage of the responses (n=5) showed a substantial or robust level of attending in the initial questionnaire. On the other hand, the majority of the pre-service teachers could provide high levels of evidence in the final questionnaire. That is, almost one-third of the responses (n=10) demonstrated a substantial level of evidence, and almost one-third of those (n=10) showed a robust level of evidence. This improvement in the pre-service teachers' attending skills in volume comparison of paper prisms problem can be exemplified in the responses of pre-service teacher 19 in the initial and final noticing questionnaire.

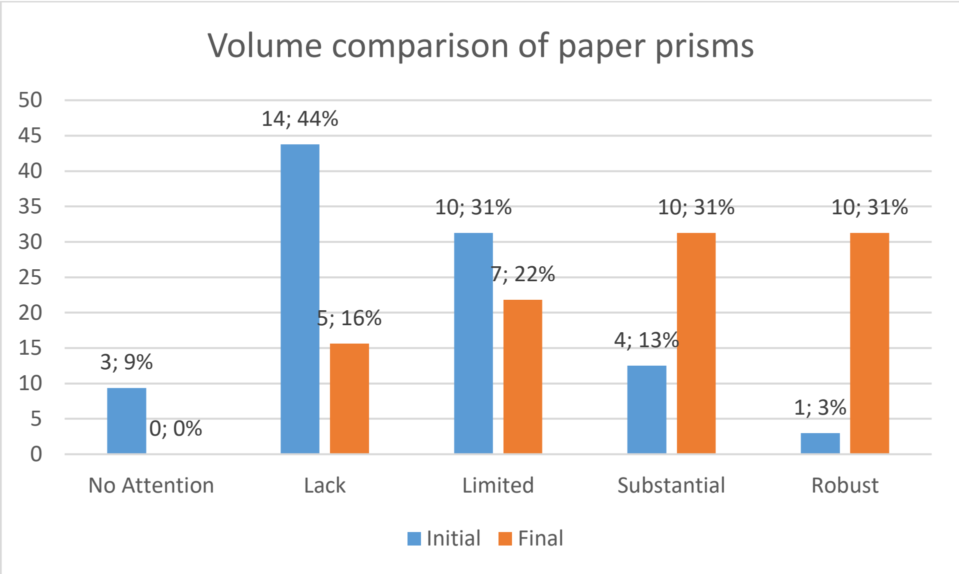
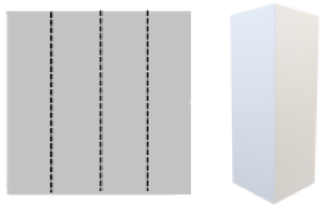


Figure 4. 24 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the volume comparison of paper prisms problem in the initial and final noticing questionnaire

The response of pre-service teacher 19 to the volume comparison of paper prisms problem in the initial noticing questionnaire is shown in Figure 4.25.

Making a Prism

A square-shaped paper is divided into four equal parts by folding along the lines. Then, an open square prism is obtained by combining the edges.

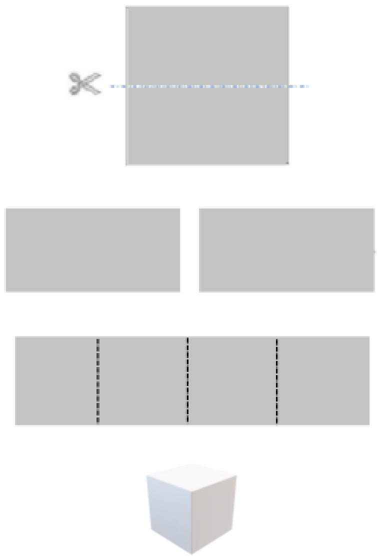


Making a Cube

Another square-shaped paper of the same size is divided into two equal parts by cutting horizontally from the middle.

The cut parts are combined in a way that short edges are coincident, and a rectangle is created.

This rectangular paper is divided into four equal parts by folding along the lines. Then, an open cube is obtained by combining the edges.



What can you say when you compare the volume of the prism and cube? Justify your response.

Mert: Their areas are equal, so their volumes must also be equal.

Bade: When I think of a bowl of rice, if it covers the whole cube, the cube is half the height of the prism, so the prism will be half full (if we pour the same rice into the prism). That is why the volume of the prism is larger.

i. Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?

The solution of Mert is incorrect. Since the base areas of the objects are equal, the student said that their volumes are equal. He neglected the height.

Bade's way of thinking is good, but his solution is wrong because the prism and the cube have no volume due to their open bases.

Figure 4. 25 The response of pre-service teacher 19 to volume comparison of paper prisms problem in the initial noticing questionnaire

Pre-service teacher 19 attended to each student's solution by providing a *lack* of evidence since she provided inappropriate evidence for both students' solutions. Accordingly, the pre-service teacher could not identify any of the mathematical properties in the students' solutions, and hence, her response to the attending prompt in the initial questionnaire showed a *lack* of evidence.

Response of pre-service teacher 19 in the final noticing questionnaire:

> Mert compared the volume of the objects based on the areas of the papers that make up the objects. The student thinks that since the size of the paper used is the same, the areas are the same, and the prism and the cube formed with this paper will have the same volume. However, although the papers used are in the same area, the length, width, and height of the objects are important for the volume, and this varies depending on how the papers are folded. Therefore, the solution is incorrect because the volume of the prism and the cube are not equal.
>
> Bade said that half of the prism would be filled since the height of the prism is half of that of the cube. Bade only included the height when calculating the volume and ignored the base area. Therefore, he thought that the volume of the prism was larger because the height of the prism was greater than that of the cube. His solution is wrong because the volume of the cube is larger.

In the final questionnaire, pre-service teacher 19 was able to recognize each student's ideas and misconceptions, and therefore, she provided a *robust* level of evidence for attending to each student's solution. Thus, the response of pre-service teacher 19 to the attending prompt in the final questionnaire demonstrated a *robust* level of evidence since she identified mathematical properties in both students' solutions by explaining them in detail.

4.1.2.1.1.2 Pre-service teachers' attending to students' solutions about a fixed volume-changing surface area situation

The fifth problem in the noticing questionnaire was about a fixed volume-changing surface area situation. This problem had three sub-problems. The first sub-problem was named height of prisms, and the pre-service teachers were expected to attend to two different students' solutions to this problem. Figure 4.26 presents the frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the height of prisms problem in the initial and final noticing questionnaire. In this problem, the no attention category did not emerge, and all pre-service teachers provided responses to the attending prompt both in the initial and final questionnaire. In the initial questionnaire, an equal percentage of lack of evidence and limited evidence (44%) was provided by the pre-service teachers in the initial questionnaire. A small percentage of the responses (n=4) showed a substantial level of evidence, and there were no pre-service teachers who provided a robust level of evidence in the initial questionnaire. In the final questionnaire, there was a decrease in the percentage of the responses, demonstrating a lack and limited level of evidence. Furthermore, none of the pre-service teachers provided a robust level of evidence for attending in the initial questionnaire, but 12 of them (38%) did so in the final questionnaire. It is worth noting that robust evidence of responding with the highest percentage was provided for this problem in the context of volume and surface area measurement in the final questionnaire. This improvement is evident in the responses of pre-service teacher 4 to the attending prompt in the initial and final noticing questionnaire as follows:

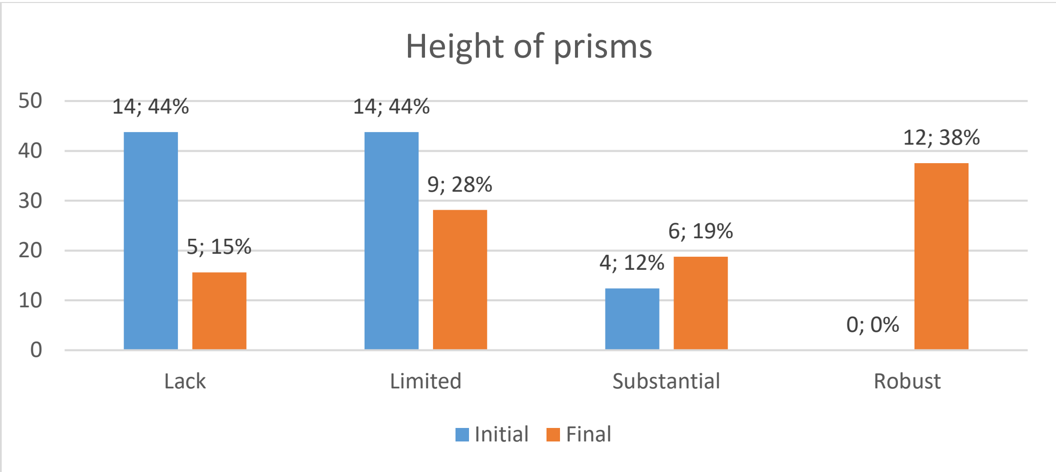
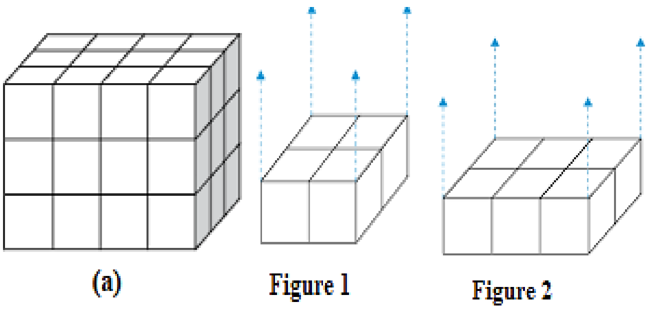


Figure 4. 26 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the height of prisms problem in the initial and final noticing questionnaire

Figure 4.27 presents the response of pre-service teacher 4 to the height of prisms problem in the initial noticing questionnaire.



he prism (a) is broken into unit cubes, and they are used again to form the prisms (b) and (c), whose first layers are already given in Figure 1 and Figure 2, respectively. What will the height of the resulting prism (b) and (c) be? Please explain.

Eylem:

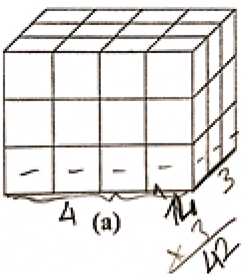
9 units in height

36 | 4
-36 | 9
00

6 units in height

36 | 6
36 | 6
00

Kaan:



10 units in height

42 | 4
-4 | 10
02

7 units in height

42 | 6
-42 | 7
00

(i) Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?

The solution of Eylem is correct, but Kaan incorrectly calculated the number of cubes in the prism. He counted 14 instead of 12, so his answer is wrong. In fact, if he counted 12, his answer would also be correct. I could not see any misconception in Kaan.

Figure 4. 27 The response of pre-service teacher 4 to height of prisms problem in the initial noticing questionnaire

Pre-service teacher 4 attended to the solution of Eylem by providing a *lack* of evidence because the pre-service teacher identified the student's solution as correct without providing any details. In addition, although pre-service teacher 4 identified the incorrect solution of Kaan, she was not able to recognize the misconception in the student's solution because she thought it was just due to a calculation error. Therefore, the pre-service teacher provided a *limited* level of evidence for attending to the solution of Kaan. Together, these responses indicate that pre-service teacher 4 provided a *lack* of evidence of attending in the initial questionnaire since the pre-service teacher did not identify any of the mathematical properties in the students' solutions.

Response of pre-service teacher 4 in the final noticing questionnaire:

> Eylem correctly found the number of unit cubes in the prism (a) to be 36. Then, to find the height of prism (b) and (c), she divided the total number of cubes by the number of cubes in a layer. That is, he divided 36 by 4 and 6. In this way, she found the number of iterations by dividing the total number of cubes by the number of cubes in the layer. The solution is correct because the number of iterations of the layer gives the height.
>
> First of all, Kaan counted the visible faces of the cubes in the first layer, and he found the number of cubes in a layer as 14, thinking that there were four cubes in the front, three cubes on the right, and also four cubes on the back and three cubes on the left. Then, to find the number of cubes in a prism, he multiplied 14 by three and found 42 because there were three layers. Finally, he divided 42 by four and six, respectively, the number of cubes in a layer. The student's solution is wrong. Since he initially incorrectly calculated the number of cubes in prism (a) due to counting faces, he later found the heights of prism (b) and prism (c) incorrectly.

In the final questionnaire, pre-service teacher 4 recognized the faced-based strategy of Kaan in an incorrect solution and described the correct solution of Eylem in detail by relating to a total number of cubes and the number of layers. Hence, she provided a *robust* level of evidence for attending to each student's solution. Taken together, the response of pre-service teacher 4 to the attending prompt in the final questionnaire showed a *robust* level of evidence since she identified mathematical properties in both students' solutions by explaining in detail.

The second sub-problem was named volume comparison with unit cubes, and the pre-service teachers were expected to attend to two different students' solutions to this problem. Figure 4.28 below illustrates the frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the volume comparison with unit cubes problem in the initial and final noticing questionnaire. In the initial questionnaire, seven of the pre-service teachers (22%) did not attempt to attend to students' solutions, whereas all pre-service teachers attended to the students' solutions in the final questionnaire. Moreover, just over one-third of the pre-service teachers (n=12) provided a lack of evidence in the initial questionnaire, but this percentage was reduced by half in the final questionnaire. In the initial questionnaire, a low percentage (n=3) of the responses demonstrated high levels of evidence (substantial or robust level of evidence). On the other hand, there was an increase in the percentage of the responses showing high levels of evidence. In other words, in the final questionnaire, nine of 32 pre-service teachers (28%) provided a substantial level of evidence, and five of those (16%) provided a robust level of evidence. The evidence of improvement can be clearly seen in the responses provided by pre-service teacher 24 to the attending prompt in the initial and final noticing questionnaire as follows:

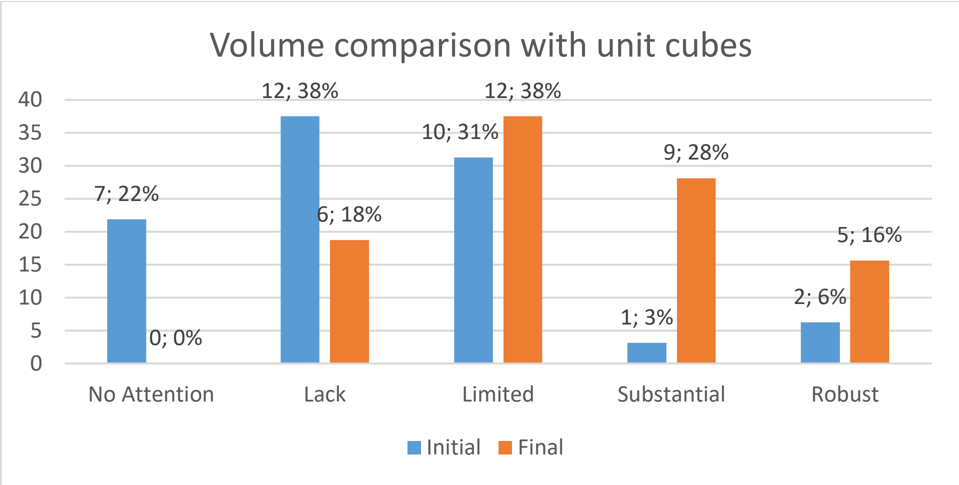
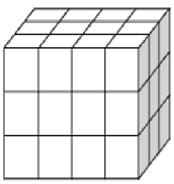


Figure 4. 28 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the volume comparison with unit cubes problem in the initial and final noticing questionnaire

The response of pre-service teacher 24 to volume comparison with unit cubes problem in the initial noticing questionnaire is provided in Figure 4.29.



(a)

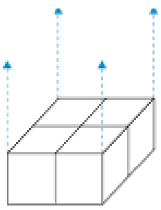


Figure 1

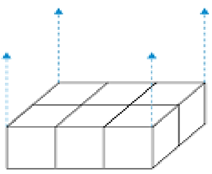


Figure 2

The prism (a) is broken into unit cubes, and they are used again to form the prisms (b) and (c), whose first layers are already given in Figure 1 and Figure 2, respectively. What will the height of the resulting prism (b) and (c) be? Please explain.

b) Which of the three prisms (a-b-c) has the greatest volume? Justify your response.

Eylem: Their volumes are equal because the same number of cubes are used.

Kaan: Prism (a) because it has the widest base.

i. Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?

The solution of Eylem is correct. She said that their volumes were equal because equal numbers of cubes were used. The solution of Kaan is incorrect because he made a comparison only by looking at the base.

Figure 4. 29 The response of pre-service teacher 24 to volume comparison with unit cubes problem in the initial noticing questionnaire

The excerpt reveals that pre-service teacher 24 attended to the solution of Eylem by providing a *lack* of evidence because she rephrased the student's statement. Furthermore, while attending to the solution of Kaan, the pre-service teacher described the solution without making a connection with the volume concept. Hence, her response showed a *limited* level of evidence. Overall, pre-service teacher 24 provided a *lack* of evidence for attending in this problem in the initial questionnaire since she was not able to identify any of the mathematical properties in the students' solutions.

Response of pre-service teacher 24 in the final noticing questionnaire:

> Kaan's solution is incorrect because he compared the volume of the prisms according to the area of the face touching the floor. He said that the volume of the prism (a) was larger because, in that prism, the base was wider, i.e., more squares touched the floor than in other prisms.
>
> The solution of Eylem is correct because the number of unit cubes used gives the volume of the object. Since the prisms were formed using the same number of unit cubes, the student concluded that their volumes were equal.

In the final questionnaire, pre-service teacher 24 provided a *substantial* level of evidence while attending to each student's solution because she recognized Kaan's misconceptions to some extent by mentioning the relationship between the number of unit cubes and volume and described Eylem's correct solution shortly by relating the size of the bases to the volume of prisms. Taken together, the response of pre-service teacher 24 to the attending prompt in the final questionnaire showed a *substantial* level of evidence because the pre-service teacher identified mathematical properties in both students' solutions to some extent.

The third sub-problem was named surface area comparison, and the pre-service teachers were expected to attend to two different students' solutions to this problem. The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the surface area comparison problem in the initial and final noticing questionnaire are set out in Figure 4.30. More than half of the pre-service teachers (n=19) did not provide any response to the attending prompt in the initial questionnaire, while all pre-service teachers attended to the students' solutions in the final questionnaire. Only two of 32 pre-service teachers (6%) provided a substantial level of evidence in the initial questionnaire, but almost one-third of them (n=10) did so in the final questionnaire. Additionally, none of the responses demonstrated a robust level of evidence in the initial questionnaire, but five of the pre-service teachers (16%) were able to provide a robust level of evidence in the final questionnaire. As an example, responses of pre-service teacher 26 to the attending prompt in the initial and final questionnaire are provided below.

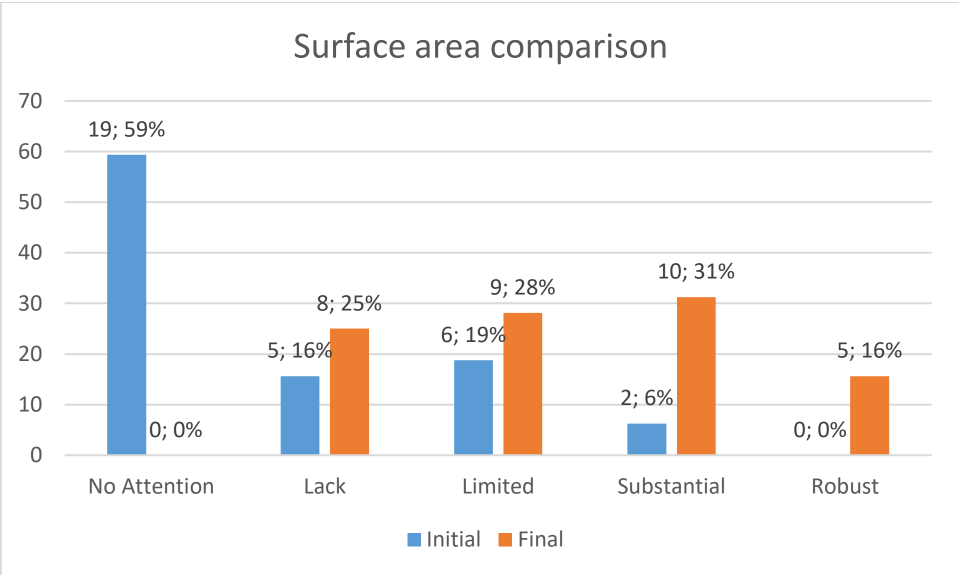
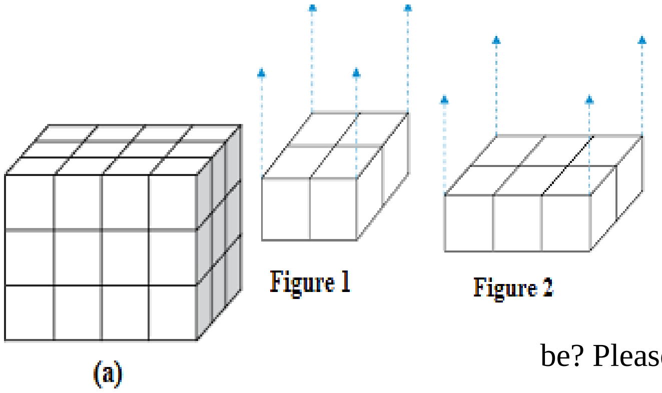


Figure 4. 30 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the surface area comparison problem in the initial and final noticing questionnaire

Figure 4.31 shows the response of pre-service teacher 26 to the surface area comparison problem in the initial noticing questionnaire.



The prism (a) is broken into unit cubes, and they are used again to form the prisms (b) and (c), whose first layers are already given in Figure 1 and Figure 2, respectively. What will the height of the resulting prism (b) and (c) be? Please explain.

a) Which of the three prisms (a-b-c) has the greatest surface area? Justify your response.

Eylem: Prism (a) because it has the maximum number of cubes on its top face.

A=4x3=12
B=2x2=4
C=3x2=6

Kaan: Prism (b) has the largest surface area because it is higher than the other prisms.

i. Please describe what each student did in response to this problem/how (s)he solved the problem. Do you think the student's solution is correct? Why?

The solution of Eylem is correct. By finding the area occupied by prism a, b, and c one by one, she found the prism with the largest surface area as prism a.

Kaan answered the problem incorrectly. He answered the problem by associating the surface area only with the height of the prism.

Figure 4. 31 The response of pre-service teacher 26 to surface area comparison problem in the initial noticing questionnaire

Pre-service teacher 26 attended to each student's solution by providing a *lack* of evidence because she identified each student's solution incorrectly. Thus, the response of pre-service teacher 26 to the attending prompt in the initial questionnaire demonstrated a *lack* of evidence since the pre-service teacher was not able to identify any of the mathematical properties in the students' solutions.

Response of pre-service teacher 26 in the final noticing questionnaire:

> The solution of Eylem is incorrect. She thought of the surface area of the prism as the area of a face. To find the surface area, the student multiplied the length of the short side by the length of the long side using the area formula of the rectangle. In this way, she arrived at the result over the area of a single face, which is the area of the top face. Yet, she would have to calculate the area of all faces to find the surface area of the prisms.
>
> Kaan answered the problem partially correctly. He said that prism (b) has the largest surface area because it is the highest one. Yes, prism (b) has the largest surface area among the given prisms. However, the student did not justify his answer by associating the surface area with the faces of the prism. He probably thought that prism (b) has the largest surface area because fewer faces are in contact in prism (b), and more faces of the unit cubes are visible, which contributes to the surface area.

In the final questionnaire, pre-service teacher 26 was able to recognize Eylem's incorrect solution that is calculating the area of one face, i.e., top face, for the surface area of the prisms and Kaan's partial correct solution that is relating the high of prisms with surface area of prisms. Hence, she provided a *robust* level of evidence for attending to each student's solution. Thus, since the pre-service teacher identified mathematical properties in both students' solutions by explaining them in detail, her response to the attending prompt in the final questionnaire showed a *robust* level of evidence.

Figure 4.32 presents the frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the context of volume and surface area measurement in the initial and final noticing questionnaire.

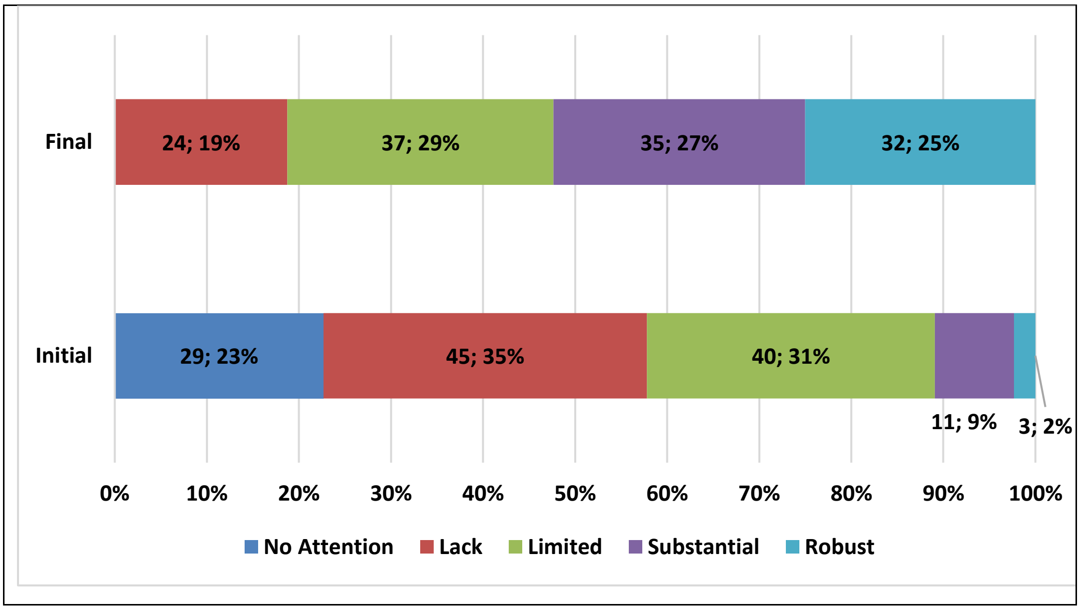


Figure 4. 32 The frequency and percentage of the pre-service teachers' responses for each category of attending to students' solutions in the context of volume and surface area measurement in the initial and final noticing questionnaire

The pie chart above illustrates the proportion of different categories of attending to students' solutions in the context of volume and surface area measurement. In the initial questionnaire, more than one-third of the responses showed a lack of evidence, with the highest percentage. This was followed by the category of limited evidence and no attention, respectively. Furthermore, 23% of the responses included no attention in the initial questionnaire, whereas in the final questionnaire, all participants attended to each student's solution in the context of volume and surface area measurement. In the final questionnaire, more than half of the responses showed high levels of evidence, with one-quarter of the responses in the category of robust evidence. These findings display the improvement in the pre-service teachers' attending skills in the context of volume and surface area measurement in the final questionnaire.

4.1.2.2 Changes in pre-service teachers' interpreting students' understanding in the context of volume-surface area measurement

Pre-service teachers' interpreting students' understanding in the context of volume-surface area measurement is measured by their responses to interpreting prompts related to students' solutions in problems 4 and 5 of the noticing questionnaire. The pre-service teachers' responses to the interpreting students' understanding prompt in the noticing questionnaire were examined at four levels: a robust level of evidence, a substantial level of evidence, a limited level of evidence, and a lack of evidence of interpreting students' understanding. In addition, some of the pre-service teachers did not provide an answer to the interpreting prompt. This was categorized as no interpretation. Two problems about volume-surface area measurement in the noticing questionnaire were gathered under the theme of the volume-surface area relationship. Therefore, the findings regarding the pre-service teachers' interpreting skills are presented under the title of pre-service teachers' interpreting students' solutions about the volume-surface area relationship in the following part.

4.1.2.2.1 Pre-service teachers' interpreting students' understanding about the volume-surface area relationship

In the noticing questionnaire, the last two problems (four problems with sub-problems) were about the volume-surface area relationship. The fourth problem was about a fixed lateral area-changing volume situation, while the fifth problem was about a fixed volume-changing surface area situation. The findings for the fourth problem, which is related to a fixed lateral area-changing volume situation, and the findings for the fifth problem, which is related to a fixed volume-changing surface area situation, are presented below, respectively.

4.1.2.2.1.1 Pre-service teachers' interpreting students' understanding about a fixed lateral area-changing volume situation

The fourth problem in the noticing questionnaire was about a fixed lateral area-changing volume situation. The problem was named volume comparison of paper prisms, and the pre-service teachers were expected to interpret two different students' understanding (see Figure 4.25). The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the volume comparison of paper prisms problem in the initial and final noticing questionnaire are presented in Figure 4.33. In the initial questionnaire, almost one-third of the pre-service teachers (n=11) did not attempt to interpret students' understanding in the initial questionnaire, whereas, in the final questionnaire, all pre-service teachers provided an interpretation of students' understanding. Furthermore, about one-third of the pre-service teachers (n=11) provided a lack of evidence of interpreting, and a quarter of the responses (n=8) showed a limited level of evidence. A small percentage of the responses (n=2) showed a substantial level of evidence in the initial questionnaire, while the percentage showing substantial evidence increased slightly in the final questionnaire. In addition, none of the pre-service teachers provided a robust level of evidence in the initial questionnaire, but nearly a quarter of them (n=7) did so in the final questionnaire. It is worth noting that robust evidence of interpreting with the highest percentage (22%) was provided for this problem in the context of volume and surface area measurement in the final questionnaire. The improvement in the quality of the pre-service teachers' responses to the interpreting prompt is evident in the responses of pre-service teacher 12 in the initial and final questionnaire, which is given below.

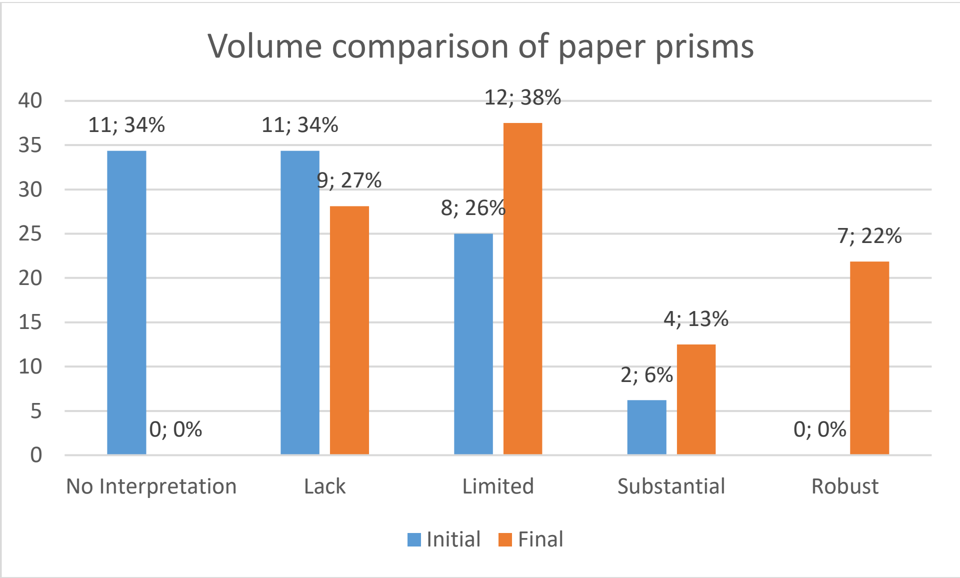


Figure 4. 33 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the volume comparison of paper prisms problem in the initial and final noticing questionnaire

Response of pre-service teacher 12 in the initial noticing questionnaire:

> Mert lacks an understanding of the subject. He tries to apply his memorized knowledge to every problem. He has fallen into the misconception that their volumes will be equal by generalizing from some problems he may have seen before.
>
> Bade lacks knowledge about volume calculation. She assumed that the volume depends only on the height.

The excerpt reveals that pre-service teacher 12 made a comment about the understanding of Mert without any mathematical properties. Therefore, her response demonstrated a *limited* level of evidence for interpreting the understanding of Mert. Moreover, the pre-service teacher's comment on Bade's understanding of volume involved only height. That is, she made a comment on the student's understanding by mentioning the concept of height without mentioning all of them, i.e., width and length or base area. Hence, the response of pre-service teacher 12 showed a *limited* level of evidence for interpreting the understanding of Bade. Overall, pre-service teacher 12 provided a *lack* of evidence to the interpreting prompt in the initial questionnaire since the pre-service teacher failed to provide a valid justification for any student's understanding.

Response of pre-service teacher 12 in the final noticing questionnaire:

> Since the prisms are made of the same papers, Mert may have thought that the volumes are equal by generalizing the conservation of area to the volume.
>
> Bade compared the volumes based on height. She gave such an answer by relating the volume to the height but not to the base area.

In the final questionnaire, pre-service teacher 12 provided a valid justification for understanding of Mert without much detail. Hence, her response showed a *substantial* level of evidence for interpreting Mert's understanding. Furthermore, in contrast to the initial questionnaire, pre-service teacher 12 made a comment about understanding of Bade by mentioning both concepts, i.e., height and base area, but without much detail. Therefore, the pre-service provided a *substantial* level of evidence for interpreting the understanding of Bade. Taken together, pre-service teacher 12 provided a valid justification for both students' understanding without providing much detail. Thus, her response to the interpreting prompt in the final questionnaire showed a *substantial* level of evidence.

4.1.2.2.1.2 Pre-service teachers' interpreting students' understanding about a fixed volume-changing surface area situation

The fifth problem in the noticing questionnaire was about a fixed volume-changing surface area situation. This problem had three sub-problems. The first sub-problem was named height of prisms, and the pre-service teachers were expected to interpret two different students' understanding (see Figure 4.27). The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the height of prisms problem in the initial and final noticing questionnaire is set out in Figure 4.34. In the initial questionnaire, a quarter of the pre-service teachers (n=8) did not attempt to interpret students' understanding. The majority of the responses showed a lack of evidence; in contrast, the percentage of those demonstrating such evidence declined in the final questionnaire. Furthermore, a small percentage of the responses (n=2) showed a substantial level of evidence, and there was no response showing a robust level of evidence in the initial questionnaire. On the other hand, about half of the pre-service teachers (n=15) provided a substantial level of evidence, and three of them (9%) provided a robust level of evidence in the final questionnaire. In this way, more than half of the responses (n=18) demonstrated high levels of evidence. The responses of pre-service teacher 17 to the interpreting prompt in the initial and final questionnaire are provided to illustrate this improvement as follows:

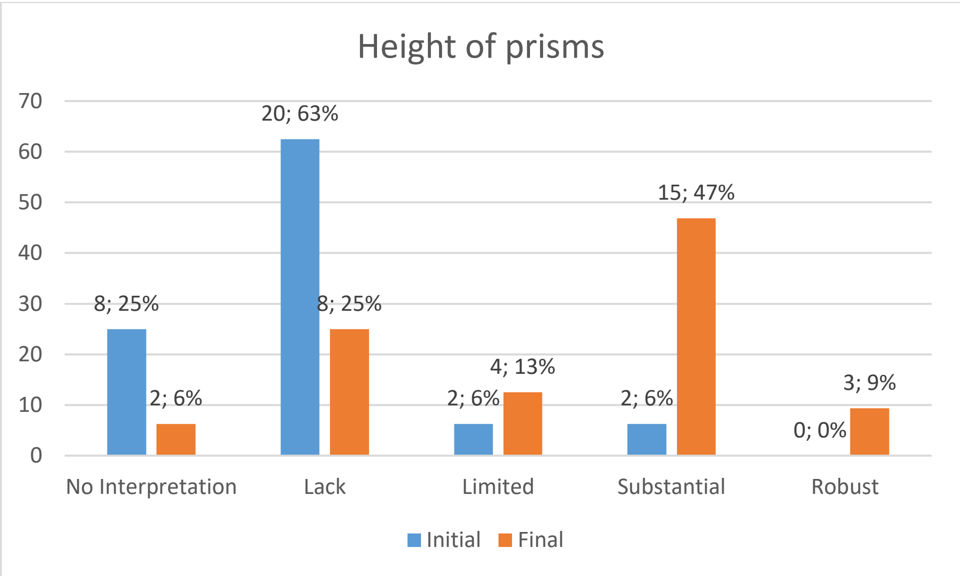


Figure 4. 34 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the height of prisms problem in the initial and final noticing questionnaire

Response of pre-service teacher 17 in the initial noticing questionnaire:

> According to the solution and answer given by Eylem, it is seen that she understood the subject.

I could not understand what approach Kaan was trying to take.

Pre-service teacher 17 made a comment that only Eylem understood, and hence, her response showed a *lack* of evidence of interpreting. In addition, the pre-service stated that she did not understand what Kaan did. Therefore, her response demonstrated a *lack* of evidence of interpreting Kaan's understanding. Taken together, pre-service teacher 17 provided a *lack* of evidence for the interpreting prompt in the initial questionnaire because the pre-service teacher was not able to provide a valid justification for any student's understanding.

Response of pre-service teacher 17 in the final noticing questionnaire:

> Eylem not only correctly found how many unit cubes the given prism consisted of but also created new prisms with the number of unit cubes she found and found the height correctly, which shows that the student understood the subject. Here, the student found the number of layers consisting of 4 cubes in prism (b) and six cubes in prism (c) and described this as height. By dividing the total number of unit cubes by the number of cubes in a layer, he actually found how many times these layers were iterated. Thus, the student was able to establish the relationship between height and layers. He knows that height is the number of iterations of the layer.
>
> Kaan did not understand the subject because while finding the number of the cubes in prism a, i.e., the volume of the prism, he counted the visible faces of cubes in the first layer instead of the cubes themselves. This resulted in counting the same cubes in the corners two times. He has poor spatial reasoning skills.

In the final questionnaire, pre-service teacher 17 provided a detailed, valid justification for understanding of both Eylem and Kaan. Therefore, her response demonstrated a *robust* level of evidence for interpreting the understanding of each student. Overall, pre-service teacher 17 provided a valid justification for both students' understanding by explaining them in detail. Thus, her response to the interpreting prompt in the final questionnaire showed a *robust* level of evidence.

The second sub-problem was named volume comparison with unit cubes, and the pre-service teachers were expected to interpret two different students' understanding (see Figure 4.29). The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the volume comparison with unit cubes problem in the initial and final noticing questionnaire are presented in Figure 4.35. In the initial questionnaire, an equal percentage of no interpretation and lack of evidence (44%) was provided by the pre-service teachers in the initial questionnaire. In the final questionnaire, all pre-service teachers provided an interpretation of students' understanding. Yet, half of the responses (n=16) showed a lack of evidence, and nearly one-third of the responses (n=10) showed limited evidence. Furthermore, only one of 32 responses (3%) showed a substantial level of evidence, and none of the pre-service teachers provided a robust level of evidence in the initial questionnaire. Interestingly, robust evidence of interpreting with the lowest percentage was provided for this problem in the final questionnaire. On the other hand, almost one-fifth of the responses (n=6) demonstrated high levels of evidence (substantial or robust evidence). The responses of pre-service teacher 25 to the interpreting prompt in the initial and final questionnaire illustrate the improvement clearly as follows:

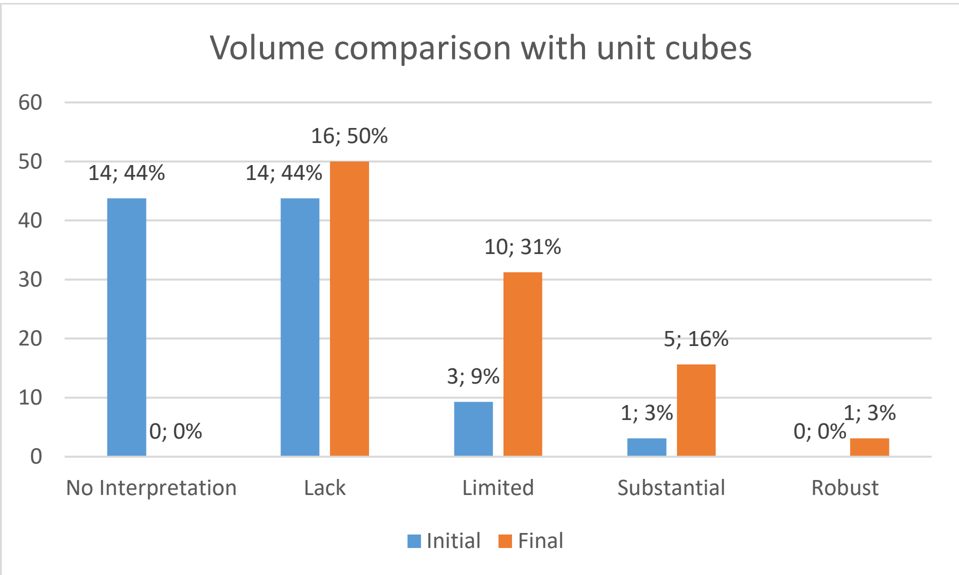


Figure 4. 35 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the volume comparison with unit cubes problem in the initial and final noticing questionnaire

Response of pre-service teacher 25 in the initial noticing questionnaire:

> I think Eylem understood the concept of volume, but Kaan could not comprehend the volume concept.

The excerpt reveals that pre-service teacher 25 made comments about whether only the student understood the concept or not. Therefore, his response demonstrated a *lack* of evidence of interpreting each student's understanding. Overall, the pre-service teacher failed to provide a valid justification for any student's understanding. Consequently, the response provided by pre-service teacher 25 to the interpreting prompt in the final questionnaire showed a *lack* of evidence.

Response of pre-service teacher 25 in the final noticing questionnaire:

> Eylem knows that the number of cubes that make up the prism is the volume of the prism. She knows that all three prisms made using the same cubes will have the same volume. Thus, she understood the volume concept.
>
> Kaan does not know the concept of volume or knows it incompletely. He just established a relationship between the base area and the volume, but he did not use the height. It may be due to the fact that the student learned the volume based on the formula of base area x height.

The excerpt from the final questionnaire indicates that pre-service teacher 25 provided a valid justification for each student's understanding of the volume concept without much detail. Hence, his response showed a *substantial* level of evidence for interpreting each student's understanding. The pre-service teacher did not mention the conservation of volume or no use of formula while interpreting the understanding of Eylem. Moreover, he did not mention Kaan's understanding of the relationship between the unit cubes and the volume and did not provide details on how the volume formula is related to the student's understanding. These made the responses less detailed and led to the categorization of each response as a *substantial* level. Taken together, pre-service teacher 25 provided a valid justification for both students' understanding without providing much detail. Thus, his response to the interpreting prompt in the final questionnaire showed a *substantial* level of evidence.

The third sub-problem was named surface area comparison, and the pre-service teachers were expected to interpret two different students' understanding (see Figure 4.31). The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the surface area comparison problem in the initial and final noticing questionnaire are set out in Figure 4.36. In the initial questionnaire, most of the pre-service teachers (n=25) could not provide any interpretation of students' understanding, and almost one-fifth of them (n=6) provided a lack of evidence. In addition, none of the pre-service teachers provided a substantial and robust level of evidence. In the final questionnaire, all pre-service teachers provided an interpretation of students' understanding. Yet, 14 of 32 responses (44%) showed limited evidence. By contrast, almost one-third of the responses (n=11) showed high levels of evidence (substantial or robust levels). That is, one-quarter of the pre-service teachers (n=8) provided a substantial level of evidence, and three of the pre-service teachers (9%) provided a robust level of evidence in the final questionnaire. The improvement in the quality of the pre-service teachers' responses to the interpreting prompt is exemplified in the responses of pre-service teacher 8 in the initial and final questionnaire as follows:

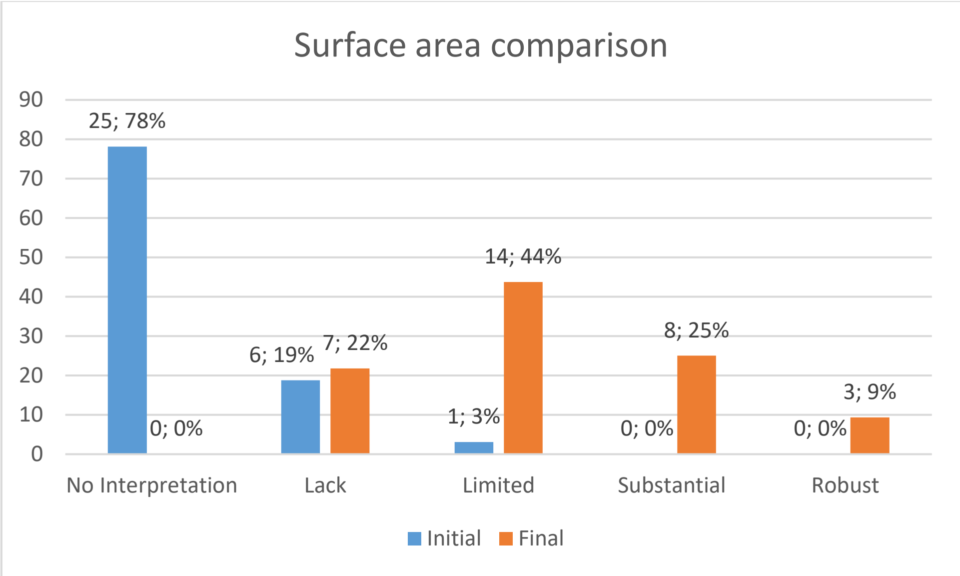


Figure 4. 36 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the surface area comparison problem in the initial and final noticing questionnaire

In the initial questionnaire, pre-service teacher 8 could not provide an interpretation of both students' understanding. On the other hand, in the final questionnaire, she interpreted the students' understanding by providing a *substantial* level of evidence, which is explained below.

Response of pre-service teacher 8 in the final noticing questionnaire:

> I think Eylem did not understand because she confused the surface area and base area.
>
> I think Kaan partially understood the concept of surface area. Although he gave the correct answer, he related the surface area to the number of layers or the height rather than the number of visible faces of unit cubes. Therefore, it would be more accurate to justify his response by saying that the surface area would be larger by considering that the visible faces of the unit cubes would increase as the height increases because, in this case, more faces face outward.

The response of pre-service teacher 8 indicates that the pre-service teacher blamed Eylem for confusing two concepts. Hence, pre-service teacher 8 provided a *limited* level of evidence for interpreting the understanding of Eylem. Moreover, the pre-service teacher provided a detailed explanation about the possible reasoning behind the solution of Kaan. Therefore, her response showed a *robust* level of evidence for interpreting the understanding of Kaan. Taken together, pre-service teacher 8 provided a valid justification for one of the student's (Kaan) understanding by explaining in detail. Thus, the response of pre-service teacher 8 to the interpreting prompt in the final questionnaire demonstrated a *substantial* level of evidence.

Figure 4.37 provides the frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the context of volume and surface area measurement in the initial and final noticing questionnaire.

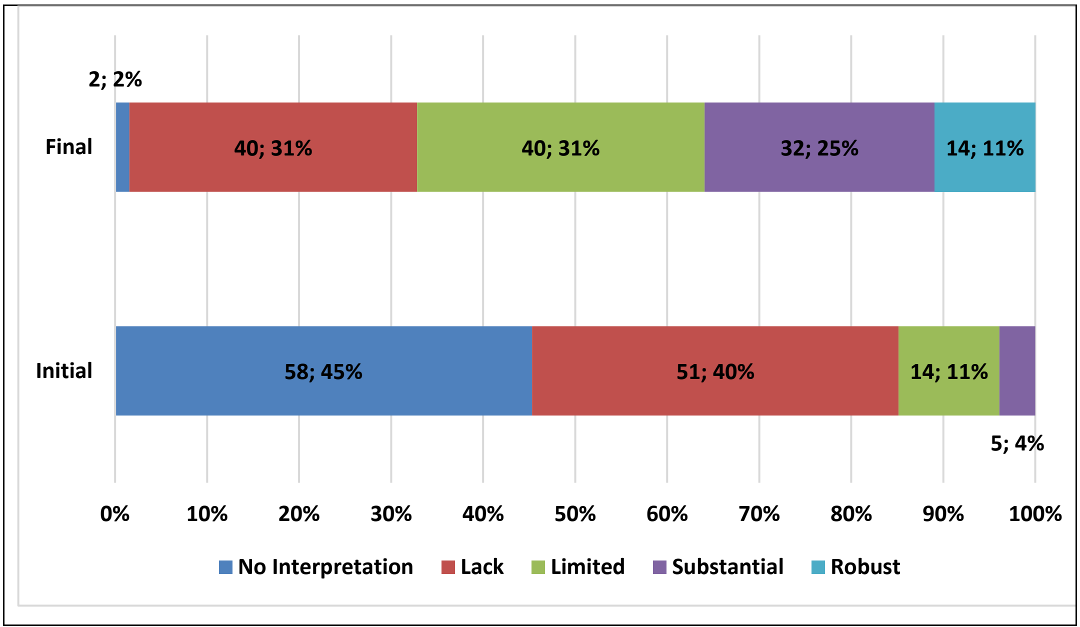


Figure 4. 37 The frequency and percentage of the pre-service teachers' responses for each category of interpreting students' understanding in the context of volume and surface area measurement in the initial and final noticing questionnaire

A closer inspection of the figure above shows that 45% of the responses included no interpretation, with the highest percentage, which was followed by the responses demonstrating a lack of evidence in the context of volume and surface area measurement in the initial questionnaire. Moreover, none of the responses showed a robust level of evidence in the initial questionnaire, while 11% did in the final questionnaire. In addition, a quarter of the responses in the final questionnaire demonstrated a substantial level of evidence, compared to 4 percent in the initial questionnaire. Thus, there was an improvement in the pre-service teachers' interpreting skills in the context of volume and surface area measurement.

4.1.2.3 Changes in pre-service teachers' deciding how to respond on the basis of students' understanding in the context of volume-surface area measurement

Pre-service teachers' deciding how to respond on the basis of students' understanding in the context of volume-surface area measurement is measured by their responses to deciding prompts related to students' solutions in problems 4 and 5 of the noticing questionnaire. In the current study, the suggestions of the pre-service teachers to deciding how to respond prompt in the noticing questionnaire were examined at five levels: robust level of evidence, substantial level of evidence, medium level of evidence, limited level of evidence, and lack of evidence of deciding how to respond based on the students' understanding. There were also some pre-service teachers who did not provide a suggestion, which was categorized as a no response. Two problems about volume-surface area measurement in the noticing questionnaire were gathered under the theme of the volume-surface area relationship. Therefore, the findings regarding the pre-service teachers' deciding how to respond skills are presented under the title of pre-service teachers' deciding how to respond on the basis of students' understanding about the volume-surface area relationship in the following part.

4.1.2.3.1 Pre-service teachers' deciding how to respond on the basis of students' understanding about the volume-surface area relationship

In the noticing questionnaire, the last two problems (four problems with sub-problems) were about the volume-surface area relationship. The fourth problem was about a fixed lateral area-changing volume situation, while the fifth problem was about a fixed volume-changing surface area situation. The findings for the fourth problem, which is related to a fixed lateral area-changing volume situation, and the findings for the fifth problem, which is related to a fixed volume-changing surface area situation, are presented below, respectively.

4.1.2.3.1.1 Pre-service teachers' deciding how to respond on the basis of students' understanding about a fixed lateral area-changing volume situation

The fourth problem in the noticing questionnaire was about a fixed lateral area-changing volume situation. The problem was named volume comparison of paper prisms, and the pre-service teachers were expected to decide how to respond to two different students based on their understanding (see Figure 4.25). Figure 4.38 presents the frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the volume comparison of the paper prisms problem in the initial and final noticing questionnaire.

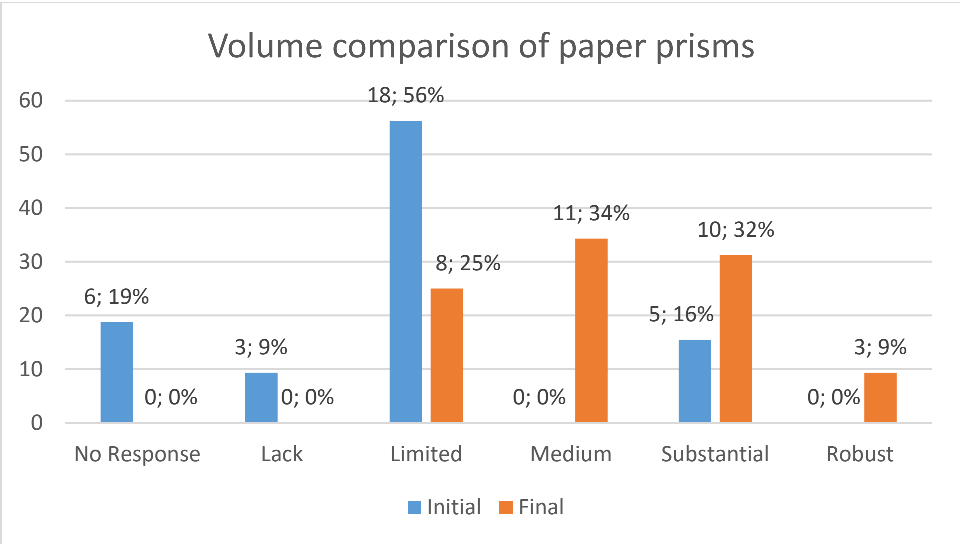


Figure 4. 38 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the volume comparison of paper prisms problem in the initial and final noticing questionnaire

In the initial questionnaire, almost one-fifth of the pre-service teachers (n=6) did not provide any suggestions; in contrast, all pre-service teachers responded to the students in the final questionnaire. Furthermore, more than half of the pre-service teachers (n=18) provided a limited level of evidence in the initial questionnaire, whereas there was a decrease in the percentage of those who provided such evidence in the final questionnaire. Five of the suggestions (16%) demonstrated a substantial level of evidence in the initial questionnaire. On the other hand, the percentage of suggestions showing substantial evidence increased and reached 31% in the final questionnaire. Interestingly, none of the pre-service teachers provided a medium and robust level of evidence in the initial questionnaire. By contrast, nearly one-third of the suggestions (n=11) showed a medium level of evidence, and three of the pre-service teachers (9%) could provide a robust level of evidence in the final questionnaire. As an example, the suggestions of pre-service teacher 7 to the deciding how to respond prompt in the initial and final noticing questionnaire are presented below.

Response of pre-service teacher 7 in the initial noticing questionnaire:

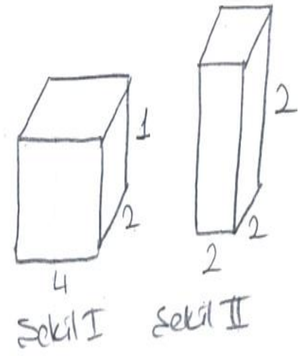
I would create a prism and a cube in this way and show Mert the difference in volume by filling them with something.

I ensure that Bade corrects the information she has learned incorrectly by going through the volume concept again.

The excerpt from the initial questionnaire reveals that pre-service teacher 7 provided teacher-centered suggestions for each student. Hence, the suggestion of the pre-service teacher for each student showed a *limited* level of evidence. Overall, the suggestions provided by pre-service teacher 7 were not sufficient to eliminate the students' misconceptions. Thus, his response to the deciding how to respond prompt in the initial questionnaire showed a *limited* level of evidence.

Response of pre-service teacher 7 in the final noticing questionnaire:

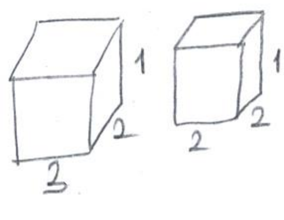
If I were his teacher, I would create a prism and a cube from the same paper in this way for Mert to discover. I would ask him to fill the prism with rice. Then, I would ask him to pour the rice from the prism into the cube. When the cube is filled with rice, he will realize that the volumes of the prism and cube are not equal to each other and that the volume of the cube is larger.



Bade neglects the base area and makes a conclusion based on the heights of the prisms. In order to make the student realize the importance of the base area, I would give the prisms below and ask her to compare their volumes by filling them with rice.

According to Bade's answer, when we fill the prism in Figure 1 with rice and pour it into Figure 2, it should reach half of it, but it does not.

Since the heights are the same here, according to Bade's answer, the prisms should be filled with the same amount of rice, but they are not. I would draw attention to the importance of the base area with these two different examples.



In the final questionnaire, the suggestion of pre-service teacher 7 for Mert was based on orienting the student with activities to the answer. Therefore, the suggestion for Mert demonstrated a *medium* level of evidence. Additionally, pre-service teacher 7 provided a specific suggestion that helped Bade overcome her misconception without much detail. That is, he did not provide any detail regarding how the student would realize the effect of the base area on volume. Consequently, the suggestion for Bade showed a *substantial* level of evidence. Taken together, pre-service teacher 7 provided specific suggestions for one of two students (Bade) to eliminate the student's misconception without providing much detail. Thus, the response of the pre-service teacher to the deciding how to respond prompt in the final questionnaire showed a *medium* level of evidence.

4.1.2.3.1.2 Pre-service teachers' deciding how to respond on the basis of students' understanding about a fixed volume-changing surface area situation

The fifth problem in the noticing questionnaire was about a fixed volume-changing surface area situation. This problem had three sub-problems. The first sub-problem was named height of prisms, and pre-service teachers were expected to decide how to respond to two different students based on their understanding (see Figure 4.27). The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the height of prisms problem in the initial and final noticing questionnaire is set out in Figure 4.39. In the initial questionnaire, about one-fifth of the pre-service teachers (n=6) did not provide any suggestions. Surprisingly, none of the pre-service teachers provided a lack of evidence of responding in the initial and final questionnaires. Furthermore, the majority of the suggestions (n=21) showed a limited level of evidence in the initial questionnaire, whereas the percentage of those showing such evidence decreased in the final questionnaire. The percentage of the suggestions demonstrating medium and substantial levels of evidence was small and equal to each other in the initial questionnaire. Moreover, only one of the pre-service teachers (3%) provided a robust level of evidence. On the other hand, in the final questionnaire, more than half of the pre-service teachers' suggestions (n=17) demonstrated at the top two levels, substantial or robust levels of evidence of responding to students' solutions. The percentage of the pre-service teachers who provided a substantial level of evidence for responding was the highest among the levels in the final questionnaire. In other words, more than one-third of the suggestions (n=12) showed a substantial level of evidence. To illustrate the improvement, how pre-service teacher 18 responded to students in the initial and final noticing questionnaire is given as follows.

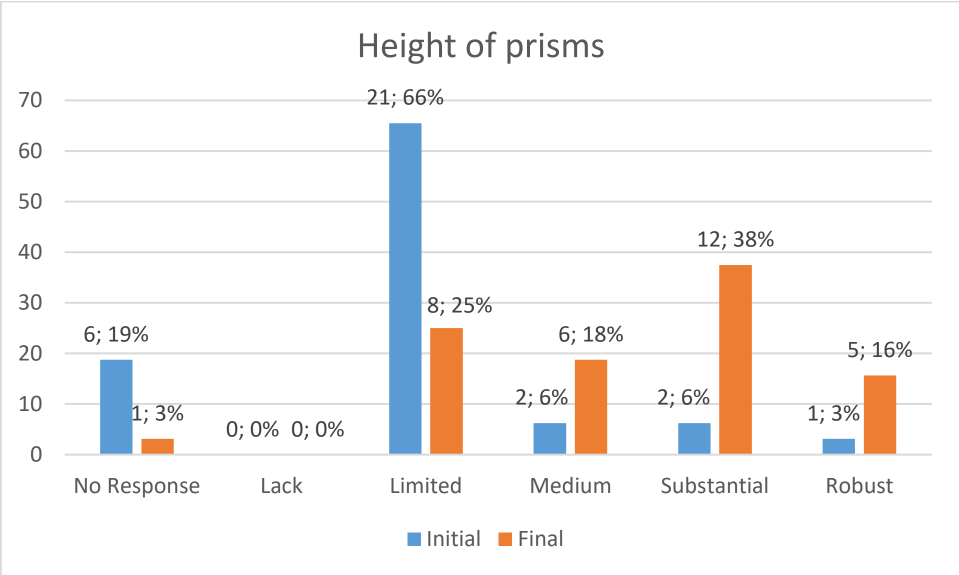


Figure 4. 39 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the height of prisms problem in the initial and final noticing questionnaire

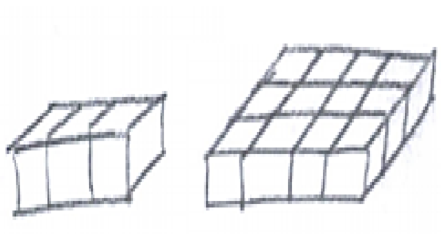
Response of pre-service teacher 18 in the initial noticing questionnaire:

> If I were the teacher of Eylem, I would make her reach the volume formula by doing exercises that would lead to the volume formula.
>
> The concept of volume should be re-explained to Kaan with models. It should be explained in a concrete way in order to eliminate the student's misconception.

Pre-service teacher 18's suggestion as a response to Eylem demonstrated a *medium* level of evidence since the suggestion, which was aimed at directing the student to the formula, was procedural understanding focused. Moreover, the pre-service teacher suggested direct instruction for Kaan, which was a teacher-centered suggestion. Therefore, her response to Kaan showed a *limited* level of evidence. Overall, the suggestions provided by pre-service teacher 18 were not sufficient to eliminate the misconception of Kaan and extend the understanding of Eylem. Thus, pre-service teacher 18 provided a *limited* level of evidence for deciding how to respond prompt in the initial questionnaire.

Response of pre-service teacher 18 in the final noticing questionnaire:

> I would ask Eylem to create prisms of different volumes with the same base and prisms of different volumes with the same height. Then, I would ask the student, "How do the base and height affect the volume?"
>
> If I were Kaan's teacher, I would give prisms with a height of one unit and ask him to find the number of unit cubes in these objects. If he could not find it, I would ask him to find it by breaking the prisms into unit cubes.



In the final questionnaire, pre-service teacher 18 offered a specific suggestion for each student, but she did not provide much detail about the suggestions. That is, the pre-service teacher did not provide a rationale for the suggestions and did not mention how the suggestions extended the understanding of Eylem and helped Kaan overcome his misconception. Therefore, the suggestion made for each student showed a *substantial* level of evidence. Taken together, pre-service teacher 18 provided specific suggestions for both students without providing much detail. Accordingly, her response to the deciding how to respond prompt in the final questionnaire demonstrated a *substantial* level of evidence.

The second sub-problem was named volume comparison with unit cubes, and the pre-service teachers were expected to decide how to respond to two different students based on their understanding (see Figure 4.29). The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the volume comparison with unit cubes problem in the initial and final noticing questionnaire are presented in Figure 4.40. In the initial questionnaire, more than one-third of the pre-service teachers (n=12) could not provide any suggestions. Lack of evidence was only provided in the initial questionnaire with a small percentage (n=1). Interestingly, an equal percentage of limited level of evidence was provided by the pre-service teachers in the initial and final questionnaire. About one-fifth of the suggestions (n=7) showed a medium level of evidence in the initial questionnaire, while a decrease in the percentage of those showing such evidence in the final questionnaire was observed. In the final questionnaire, nine of the suggestions (28%) were at the level of substantial evidence, while this frequency was only one in the initial questionnaire (3%). Additionally, none of the pre-service teachers provided a robust level of evidence for responding in the initial questionnaire, but five of them (16%) did so in the final questionnaire. The suggestions of pre-service teacher 23 as a response to the deciding how to respond prompt in the initial and final questionnaire are provided as an example below.

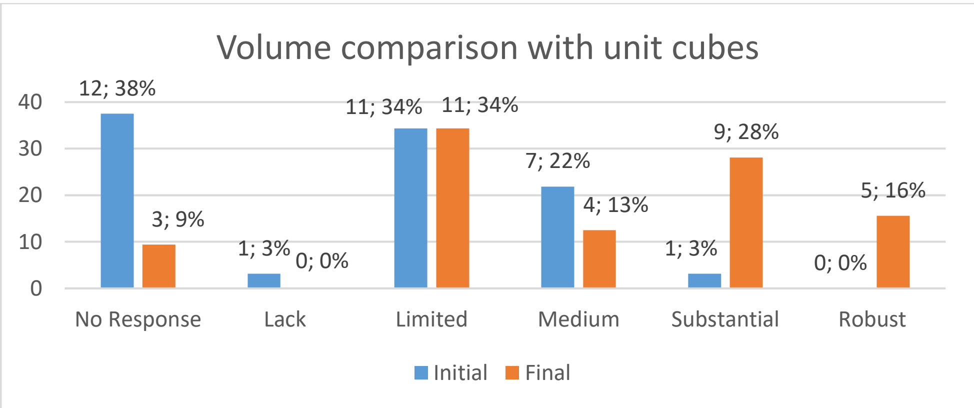


Figure 4. 40 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in volume comparison with unit cubes problem in the initial and final noticing questionnaire

Response of pre-service teacher 23 in the initial noticing questionnaire:

> I would ask a more difficult question in the same style to extend the understanding of Eylem.
>
> In order to eliminate the misconception of Kaan, I would give him prisms with the same bases and different heights and ask him to order them by volume.

The excerpt from the initial questionnaire reveals that pre-service teacher 23's suggestion for Eylem was a general suggestion, which shows a *limited* level of evidence. Moreover, pre-service teacher 23 provided a specific suggestion to eliminate the misconception of Kaan without much detail. In other words, although the suggestion was specific enough, it did not include a rationale and detail about how this suggestion would eliminate the student's misconception. Hence, the suggestion for Kaan showed a *substantial* level of evidence. Overall, the pre-service teacher provided a specific suggestion for one of two students to eliminate the student's misconception without providing much detail. Thus, the response of pre-service teacher 23 to the deciding how to respond prompt in the initial questionnaire showed a *medium* level of evidence.

Response of pre-service teacher 30 in the final noticing questionnaire:

> In the next step, I would like Eylem to compare the volume of two prisms with the same dimensions but consisting of different unit cubes in size. In this case, the volumes will be the same, but the number of unit cubes used will be different.
>
> Kaan focuses on the base and ignores the height. I would give two prisms with the same base and different heights and ask Kaan to compare their volume. Most probably, he will say that they are equal. Then, the two prisms are filled with rice from two containers, including the same amount of rice. After the student realizes which one takes more, I would ask him why that gets more rice. The relationship with height can be made to be noticed. In the same way, when the same volume prisms with different base areas are filled

> with rice, he may stop comparing according to the base area. Then, by placing unit cubes into these prisms, he may realize that they take the same number of unit cubes and, therefore, the volume is the same.

In the final questionnaire, pre-service teacher 23 could provide a specific suggestion to extend the understanding of Eylem in contrast to the initial questionnaire. Yet, the pre-service teacher did not mention the rationale of the suggestion, which resulted in categorizing the suggestion as a *substantial* level of evidence. In addition, pre-service teacher 23 elaborated on her suggestion for Kaan in the initial questionnaire and provided a detailed suggestion that helped the student overcome his misconception and make the student understand. Consequently, the suggestion for Kaan demonstrated a *robus*t level of evidence. Together, these suggestions indicate that pre-service teacher 23 provided a *robust* level of evidence for responding in the final questionnaire since the pre-service teacher's suggestions were based on extending the understanding of Eylem and making Kaan understand by explaining the second suggestion in detail.

The third sub-problem was named surface area comparison, and the pre-service teachers were expected to decide how to respond to two different students based on their understanding (see Figure 4.31). Figure 4.41 presents the frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the surface area comparison problem in the initial and final noticing questionnaire. In the initial questionnaire, more than two-thirds of pre-service teachers (n=22) did not provide any suggestions. Moreover, nearly one-third of the suggestions (n=10) showed a limited level of evidence. Like the height of the prisms problem, none of the pre-service teachers provided a lack of evidence of responding in the initial and final questionnaires. The most surprising aspect of the data in Figure 4.41 is that in the initial questionnaire, the pre-service teachers either did not provide a suggestion or provided a limited level of evidence for responding. On the other hand, the vast majority of the pre-service teachers (n=26) provided high levels of evidence, i.e., substantial or robust levels of evidence in the final questionnaire. It is worth noting that robust evidence of responding with the highest percentage (n=13) was provided for this problem in the context of volume and surface area measurement in the final questionnaire. The responses of pre-service teacher 14 to the initial and final noticing questionnaires are given below to illustrate the improvement.

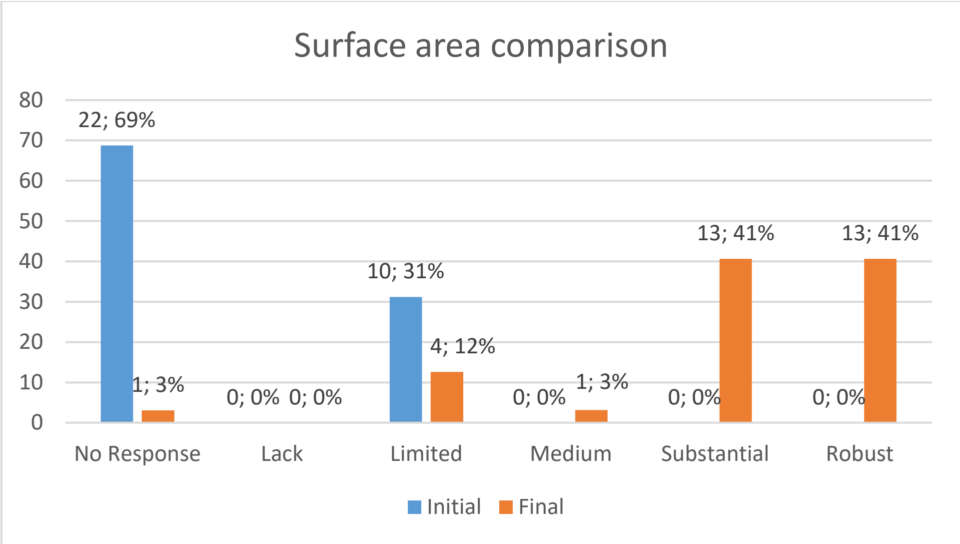


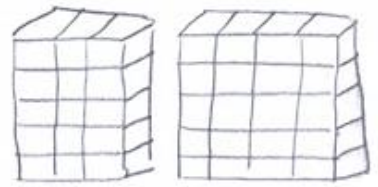
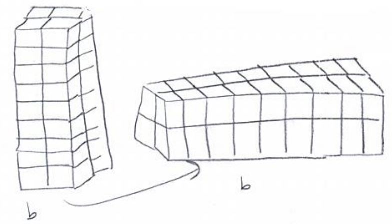
Figure 4. 41 The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the surface area comparison problem in the initial and final noticing questionnaire

Response of pre-service teacher 14 in the initial noticing questionnaire:

> The concept of the surface should be taught to Eylem by supporting concrete models.
>
> Kaan gave the correct answer to the surface area problem, but only because there are more layers. I would give examples of prisms of equal height with different surface areas.

In this excerpt, pre-service teacher 14's suggestion for Eylem was based on direct instruction. Moreover, her suggestion for Kaan was related to the teacher's demonstration of prisms with the same surface area but different heights. As a result, pre-service teacher 14's suggestions for each student showed *limited* evidence of responding because she provided teacher-centered suggestions. Overall, the pre-service teacher provided a *limited* level of evidence for deciding how to respond prompt in the initial questionnaire.

Response of pre-service teacher 14 in the final noticing questionnaire:

> Eylem thinks that the surface area is the area of the top face. She does not include the other faces. For this, I throw the object into the container full of paint and take it out. I ask the action which parts of the prism are painted. In this way, I would try to make the student realize that surface area is the sum of the areas of all the faces outside the prism, and other faces should be taken into account in the surface area.
>
> I would give Kaan two prisms with the same height but different bases and ask if their surface areas are the same.
>
> 
>
> Alternatively, I would ask if the surface area changes by turning prism (b), which has more layers.
>
> 
>

In the final questionnaire, pre-service teacher 14 responded to Eylem by providing a *robust* level of evidence because the suggestion offered was detailed and aimed at eliminating the student's misconception. For Kaan, the pre-service teacher made a specific suggestion to extend the understanding of the student without providing a rationale for the suggestion. For this reason, this suggestion showed a *substantial* level of evidence. Taken together, pre-service teacher 14 provided specific suggestions for both students, one to extend Kaan's understanding and one to make Eylem understand by explaining one of them in detail. Accordingly, the response of pre-service teacher 14 to the deciding how to respond prompt in the final questionnaire showed a *robust* level of evidence.

The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the context of volume and surface area measurement in the initial and final noticing questionnaire are summarized in Figure 4.42.

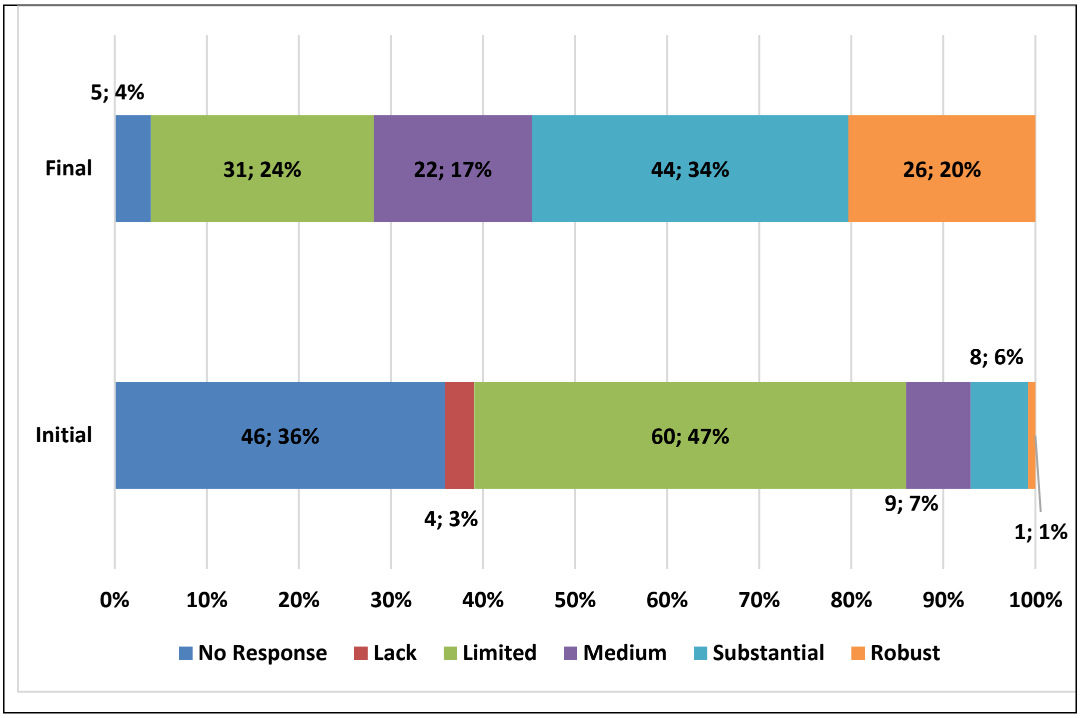


Figure 4. 42. The frequency and percentage of the pre-service teachers' suggestions for each category of deciding how to respond in the context of volume and surface area measurement in the initial and final noticing questionnaire

From the data in Figure 4.42, it is apparent that more than one-third of the responses in the initial questionnaire did not include any suggestions. Moreover, the highest percentage of the suggestions, i.e., nearly half of the responses, showed a limited level of evidence in the context of volume and surface area measurement. On the other hand, in the initial questionnaire, six percent of the responses were in the category of substantial evidence, whereas in the final questionnaire, this rose to one-third. Additionally, only one percent of the responses showed a robust level of evidence, while responses demonstrating such evidence stood at 20% in the final questionnaire. In this way, more than half of the suggestions showed a high level of evidence in the final questionnaire. Furthermore, there is a clear trend of decreasing in the percentages of responses belonging to the categories of no response, lack of evidence, and limited evidence in the final questionnaire. Accordingly, the increase in the percentage of suggestions showing high levels of evidence and the decrease in the percentage of those showing low levels of evidence in the final questionnaire reveals an improvement in the pre-service teachers' responding skills in the context of volume and surface area measurement.

4.1.2.4 Statistical analysis in the context of volume-surface area measurement

The findings presented above reveal that there is an improvement in the pre-service teachers' professional noticing skills in the context of volume-surface area measurement in the final questionnaire. The findings on whether this change is statistically significant are presented in this part, that is, an answer to the research question, "Is the change in pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area and volume-surface area measurement from pre-test to post-test statistically significant?" A noticing questionnaire including two problems (four problems with the sub-problems) about volume-surface area relationship was implemented on 32 pre-service teachers as a pre-test to identify their initial professional noticing skills and as a post-test after the intervention to determine their final professional noticing skills in the context of volume and surface area measurement. One group of participants was involved in the present study, and each participant in the group was measured twice. Furthermore, the criteria used to evaluate the professional noticing responses represented ordinal data (rank or ordered categories). Therefore, a Wilcoxon Signed Ranks Test, which is a non-parametric alternative to the repeated-measures t-test, was conducted for each problem to ascertain whether there was a statistically significant difference between the three components of professional noticing in the pre-test and post-test in the context of volume and surface area measurement. The results obtained from the statistical analysis regarding the attending component are set out in Table 4.8.

Table 4. 8 Results of Wilcoxon signed ranks test regarding attending comparing pre-test and post-test in the context of volume and surface area measurement

	Problems	z	p	r
	Volume-surface area relationship			
Attending	Volume comparison of paper prisms	-3.492	.000	.44
	Height of prisms	-3.833	.000	.48
	Volume comparison with unit cubes	-3.438	.001	.43
	Surface area comparison	-4.352	.000	.54

The results, as shown in Table 4.8, indicated that for the attending component, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers in the volume comparison of paper prisms problem following participation in the intervention, $z = -3.492$, $p < .001$, with a medium effect size ($r = .44$). In the height of prisms problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers following participation in the intervention, $z = -3.833$, $p < .001$, with a medium effect size ($r = .48$). In the volume comparison with unit cubes problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers following participation in the intervention, $z = -3.438$, $p = .001$, with a medium effect size ($r = .43$). In the surface area comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of attending of the pre-service teachers following participation in the intervention, $z = -4.352$, $p < .001$, with a large effect size ($r = .54$). Table 4.9 provides the comparison of pre-service teachers' initial and final levels in each problem for the attending component in the context of volume and surface area measurement.

Table 4. 9 Comparison of pre-service teachers' initial and final levels in each problem for the attending component in the context of volume and surface area measurement

			N	Mean Rank	Sum of Ranks
Attention	Volume comparison of paper prisms Final-Initial	Negative ranks	4[a]	14.75	59.00
		Positive ranks	25[b]	15.04	376.00
		Ties	3[c]		
		Total	32		
	Height of prisms Final-Initial	Negative ranks	3[a]	8.00	24.00
		Positive ranks	22[b]	13.68	301.00
		Ties	7[c]		
		Total	32		
	Volume comparison with unit cubes Final-Initial	Negative ranks	4[a]	7.88	31.50
		Positive ranks	20[b]	13.43	268.50
		Ties	8[c]		
		Total	32		
	Surface area comparison Final-Initial	Negative ranks	1[a]	14.50	14.50
		Positive ranks	27[b]	14.50	391.50
		Ties	4[c]		
		Total	32		

It is apparent from the above table that for the attending component, in the volume comparison of the paper prisms problem, 25 pre-service teachers' levels of attending increased in the post-test. Three pre-service teachers' levels did not change, and a decrease was observed in four pre-service teachers' levels. In the height of the prisms problem, 22 pre-service teachers' levels increased after the intervention. There was no change in seven pre-service teachers' levels, and three pre-service teachers' levels of attending decreased. In the volume comparison with the unit cubes problem, 20 pre-service teachers provided higher levels of evidence in the post-test than in the pre-test. There was a decrease in four pre-service teacher's levels and no change in eight pre-service teachers' levels. In the surface area comparison problem, 27 pre-service teachers' levels increased after the intervention, whereas one pre-service teacher's level decreased. Four pre-service teachers' levels did not change in the post-test. Table 4.10 presents the results of the Wilcoxon Signed Ranks Test regarding the interpreting component.

Table 4. 10 Results of Wilcoxon signed ranks test regarding interpreting comparing pre-test and post-test in the context of volume and surface area measurement

	Problems	z	p	r
	Volume-surface area relationship			
Interpreting	Volume comparison of paper prisms	-3.761	.000	.47
	Height of prisms	-3.807	.000	.48
	Volume comparison with unit cubes	-3.841	.000	.48
	Surface area comparison	-4.670	.000	.58

For the interpreting component, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers in the volume comparison of paper prisms problem following participation in the intervention, $z = –3.761$, $p < .001$, with a medium effect size ($r =.47$). In the height of prisms problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers following participation in the intervention, $z = –3.807$, $p < .001$, with a medium effect size ($r =.48$). In the volume comparison with unit cubes problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers following participation in the intervention, $z = –3.841$, $p < .001$, with a medium effect size ($r =.48$). In the surface area comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of interpreting of the pre-service teachers following participation in the intervention, $z = –4.670$, $p < .001$, with a large effect size ($r =.58$). Table 4.11 provides the comparison of the pre-service teachers' initial and final levels in each problem for the interpreting component in the context of volume and surface area measurement.

Table 4. 11 Comparison of pre-service teachers' initial and final levels in each problem for the interpreting component in the context of volume and surface area measurement

			N	Mean Rank	Sum of Ranks
Interpretation	Volume comparison of paper prisms Final-Initial	Negative ranks	3[a]	7.00	21.00
		Positive ranks	21[b]	13.29	279.00
		Ties	8[c]		
		Total	32		
	Height of prisms Final-Initial	Negative ranks	3[a]	11.00	33.00
		Positive ranks	24[b]	14.38	345.00
		Ties	5[c]		
		Total	32		
	Volume comparison with unit cubes Final-Initial	Negative ranks	2[a]	14.50	29.00
		Positive ranks	24[b]	13.42	322.00
		Ties	6[c]		
		Total	32		
	Surface area comparison Final-Initial	Negative ranks	0[a]	.00	.00
		Positive ranks	28[b]	14.50	406.00
		Ties	4[c]		
		Total	32		

For the interpreting component, in the volume comparison of the paper prisms problem, an increase was observed in 21 pre-service teachers' levels in the post-test. There was a decrease in three pre-service teachers' levels in the post-test, and eight pre-service teachers' levels remained the same. In the height of the prisms problem, 24 pre-service teachers provided higher levels of evidence, while three pre-service teachers provided lower levels of evidence. Five pre-service teachers remained at the same level in the post-test. In the volume comparison with the unit cubes problem, 24 pre-service teachers provided higher levels of evidence for interpreting in the post-test. There was a decrease in two pre-service teacher's levels and no change in six pre-service teachers' levels. In the surface area comparison problem, 28 pre-service teachers increased their levels. Surprisingly, none of the pre-service teachers' levels decreased. Four pre-service teachers provided the same levels of evidence in the post-test. The results of the Wilcoxon Signed Ranks Test regarding the responding component are shown in Table 4.12.

Table 4. 12 Results of Wilcoxon signed ranks test regarding responding comparing pre-test and post-test in the context of volume and surface area measurement

	Problems	z	p	r
	Volume-surface area relationship			
Responding	Volume comparison of paper prisms	-3.817	.000	.48
	Height of prisms	-3.332	.001	.42
	Volume comparison with unit cubes	-3.450	.001	.43
	Surface area comparison	-4.812	.000	.60

For the responding component, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers in the volume comparison of paper prisms problem following participation in the intervention, $z = –3.817$, $p < .001$, with a medium effect size ($r = .48$). In the height of prisms problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers following participation in the intervention, $z = –3.332$, $p = .001$, with a medium effect size ($r = .42$). In the volume comparison with unit cubes problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers following participation in the intervention, $z = –3.450$, $p = .001$, with a medium effect size ($r = .43$). In the surface area comparison problem, a Wilcoxon Signed Rank Test revealed a statistically significant increase in the levels of responding of the pre-service teachers following participation in the intervention, $z = –4.812$, $p < .001$, with a large effect size ($r = .60$). Table 4.13 presents the comparison of the pre-service teachers' initial and final levels in each problem for the responding component in the context of volume and surface area measurement.

Table 4. 13 Comparison of pre-service teachers' initial and final levels in each problem for the responding component in the context of volume and surface area measurement

			N	Mean Rank	Sum of Ranks
Response	Volume comparison of paper prisms Final-Initial	Negative ranks	2[a]	9.25	18.50
		Positive ranks	22[b]	12.80	281.50
		Ties	8[c]		
		Total	32		
	Height of prisms Final-Initial	Negative ranks	3[a]	11.50	34.50
		Positive ranks	21[b]	12.64	265.50
		Ties	8[c]		
		Total	32		
	Volume comparison with unit cubes Final-Initial	Negative ranks	4[a]	10.38	41.50
		Positive ranks	22[b]	14.07	309.50
		Ties	6[c]		
		Total	32		
	Surface area comparison Final-Initial	Negative ranks	1[a]	4.50	4.50
		Positive ranks	30[b]	16.38	491.50
		Ties	1[c]		
		Total	32		

a. Initial > Final
b. Initial < Final
c. Initial = Final

For the responding component, in the volume comparison of the paper prisms problem, 22 pre-service teachers' levels increased after the intervention, whereas two pre-service teachers' levels decreased. Eight pre-service teachers' levels did not change. In the height of the prisms problem, an increase was observed in 21 pre-service teachers' levels in the post-test. While there was no change in eight pre-service teachers' levels, there was a decrease in the level of three pre-service teachers. In the volume comparison with the unit cubes problem, there was an increase in the levels of 22 pre-service teachers in the post-test. While the levels of four of the remaining ten pre-service teachers in the post-test decreased, there was no change in the level of six pre-service teachers. In the surface area comparison problem, 30 pre-service teachers provided higher levels of evidence in the post-test. One pre-service teacher remained at the same level, and there was a decrease in one pre-service teacher's level.

Consequently, there was a statistically significant increase in each problem for all three components in the context of volume and surface area measurement. The statistical analysis showed that improvements represent a statistically significant change in the post-test in the context of volume and surface area measurement. Thus, the pre-service teachers significantly improved in attending, interpreting and responding components when they participated in the video-based module situated in the pedagogies of practice framework. The strongest improvement was observed in the interpreting component, followed by the responding component and the attending component, respectively, in the context of volume and surface area measurement. Interestingly, the highest increase in the levels of the components of attending, interpreting, and responding was noted in the same problem, i.e., the surface area comparison problem.

4.2 The influence of pedagogies of practice on the development of pre-service teachers' professional noticing skills

For the first research question, the relationship between 32 pre-service teachers' responses to initial noticing questionnaire (pre-test) and final noticing questionnaire (post-test) was examined to determine the extent to which pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement changed when they participated in a video-based module situated in the pedagogies of practice framework (Figure 4.43). The findings showed that there was an improvement in pre-service teachers' professional noticing skills in final noticing questionnaire and the change from pre-test to post-test was statistically significant in both perimeter-area and volume-surface area measurement. The aim of the second research question was to examine how a video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement. Since the intervention is based on the pedagogies of practice, the evidence from the video-based module situated in the pedagogies of practice framework are presented to reveal how the module support pre-service teachers' professional noticing skills.

Pedagogies of practice include representation of practice, decomposition of practice and approximation of practice. As the focus is on the process, which is pedagogies of practice, pre-service teachers' professional noticing skills are examined separately in the decomposition of represented practice and in the approximation of practice. Therefore, findings are presented under the headings of representation-decomposition of practice and approximation of practice for six pre-service teachers who were in the same discussion group to present how they attend to students' mathematical thinking, interpret students' understanding, and decide how to respond based on students' understanding. In the first part, the findings for perimeter-area measurement are given under the headings of representation-decomposition of practice and approximation of practice as attending, interpreting and deciding how to respond. In the second part, the findings for volume-surface area measurement are given as attending, interpreting and deciding how to respond under the headings of representation-decomposition of practice and approximation of practice. Reflection papers, semi-structured interviews, group discussions, and whole-class discussions are provided as evidence that indicates improvement in pre-service teachers' professional noticing skills.

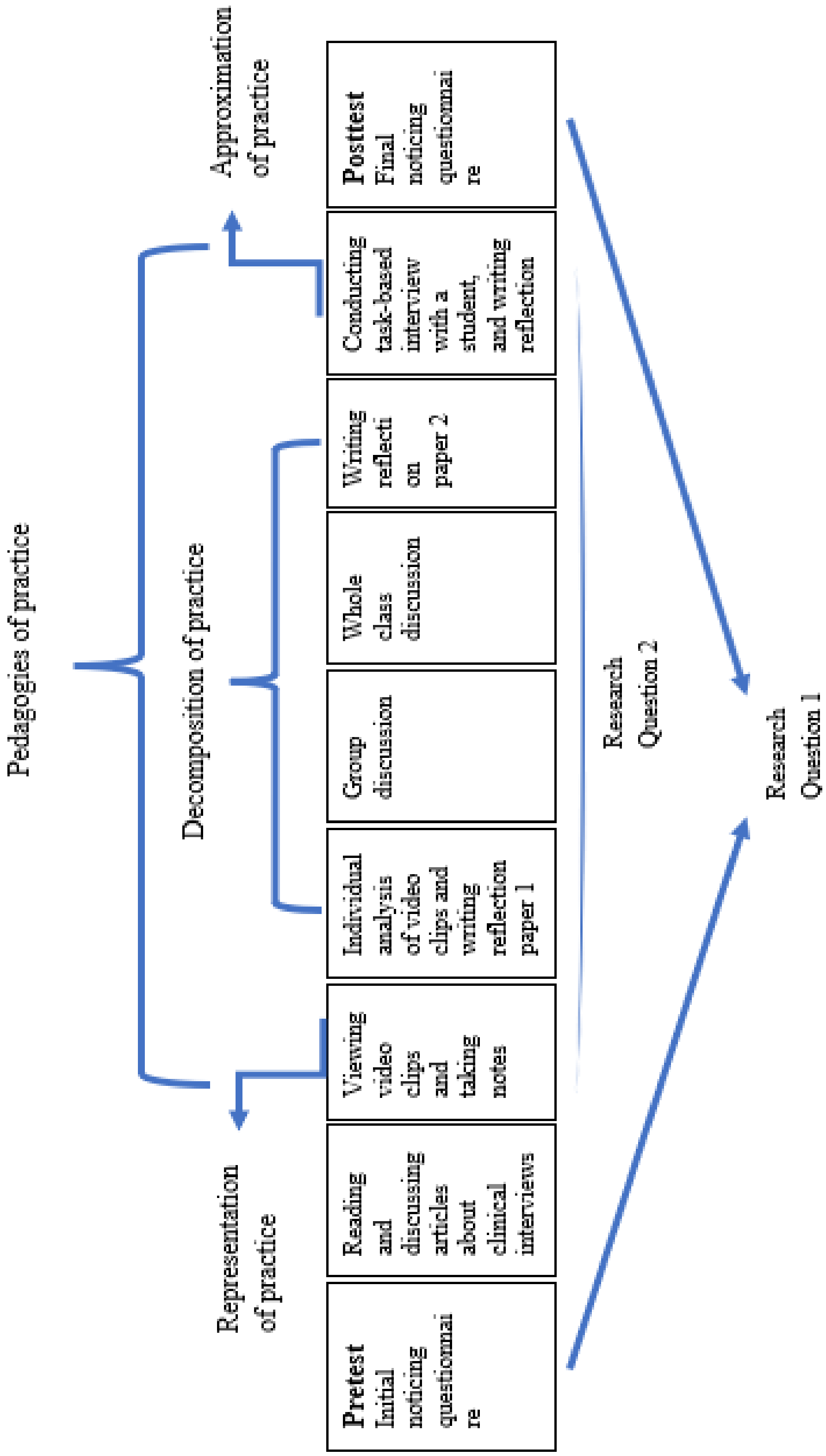


Figure 4. 43 Research questions and corresponding procedures

4.2.1 The influence of pedagogies of practice on the development of pre-service teachers' professional noticing skills in the context of perimeter-area measurement

In this part, pre-service teachers' attending, interpreting and deciding how to respond skills at the representation-decomposition of practice and approximation of practice stages regarding perimeter-area measurement are presented in order to show how a video-based module situated in the pedagogies of practice framework support pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement. This part provides the findings for six pre-service teachers' responses to the attending, interpreting and deciding how to respond prompts in the reflection papers in the context of perimeter-area measurement under the headings of representation-decomposition of practice and approximation of practice, supported by semi-structured interviews and discussions.

4.2.1.1 The influence of representation-decomposition of practice on the development of pre-service teachers' professional noticing skills in the context of perimeter-area measurement

This part provides pre-service teachers' attending, interpreting and deciding how to respond skills at the representation-decomposition of practice stage of pedagogies of practice in perimeter-area measurement to demonstrate how a video-based module situated in the pedagogies of practice framework support pre-service teachers' professional noticing of students' mathematical thinking. The video clips in the first three sessions were about perimeter and area measurement. The pre-service teachers viewed eight video clips during the three sessions as representations of practice (Table 4.14).

Table 4. 14 Video clips and their contents in the context of perimeter-area measurement

Perimeter and area measurement			
Sessions	Video Clips	Students in the video clips	Content
Session 1	VC1	S1	Change in perimeter
	VC2	S2	
	VC3	S3	
Session 2	VC4	S1	A fixed area-changing perimeter situation (area conservation)
	VC5	S4	
	VC6	S5	
Session 3	VC7	S6	A fixed perimeter-changing area situation
	VC8	S7	

As decompositions of practice, after viewing, they individually analyzed the video clips by writing reflection paper 1, discussed students' mathematical thinking in the video clips regarding attending, interpreting, and deciding how to respond components, and wrote reflection paper 2 after the discussions. At the end of the intervention sessions, semi-structured interviews were conducted regarding the reflection papers as a formative assessment to uncover the change in their responses to the noticing prompts and understand the development in the pre-service teachers' professional noticing skills.

4.2.1.1.1 Attending to the video clips in the representation-decomposition of practice in the context of perimeter-area measurement

This part presents the extent to which pre-service teachers attend to the mathematical details in the video clips in the representation-decomposition of practice stage in the context of perimeter-area measurement. Table 4.15 shows the pre-service teachers' attending skills before the discussions (reflection paper 1) and after the discussions (reflection paper 2) in the first three sessions, which are related to perimeter and area measurement.

Table 4. 15 Pre-service teachers' attending skills in individual video analysis in the context of perimeter and area measurement

	1st session		2nd session		3rd session	
	BD	AD	BD	AD	BD	AD
A	Lack	Substantial	Limited	Limited	Limited	Robust
B	Limited	Substantial	Limited	Substantial	Substantial	Substantial
E	Substantial	Substantial	Robust	Robust	Robust	Robust
G	Lack	Limited	Substantial	Substantial	Robust	Robust
K	Lack	Substantial	Limited	Substantial	Limited	Substantial
S	Robust	Robust	Substantial	Substantial	Robust	Robust

BD: Before the Discussion AD: After the Discussion

In the first session, which focuses on change in perimeter, in VC1, S1 split the paper in half and considered that the perimeter was halved since the paper was split. The student also believed that when a piece of paper is cut, the perimeter always decreases. In VC2, S2 cut a square piece from the inside of the shape and stated that the perimeter did not change because there was no change on the outside of the shape. The student also thought the perimeter could be reduced, cut off the paper along the short side, and justified his reasoning by obtaining a smaller paper. For the student, increasing the perimeter by cutting a piece of paper was not possible. In VC3, S3 could demonstrate all possible cases correctly by justifying his responses. The student cut off a triangle at a corner and responded that the perimeter decreased. S3 cut a square piece from the corner and maintained that keeping the perimeter constant was also possible. Lastly, the student cut off a square from the side and asserted that increasing the perimeter was possible.

As can be seen from the table above, pre-service teacher S provided the highest level of evidence for attending, i.e., robust evidence, in the group when the pre-service teachers individually analyzed the video clips, which was followed by pre-service teacher E, who provided substantial evidence. Pre-service teacher S identified all mathematically significant details in two students' solutions (S1 and S2), whereas she identified those in one student's solution (S3) to some extent. Pre-service teacher E identified all mathematically significant details in S3's solution, while he identified those in other students' solutions (S1 and S2) to some extent. Other pre-service teachers provided low levels of evidence for attending, i.e., lack or limited evidence. Pre-service teacher A, pre-service teacher G, and pre-service teacher K identified none of the mathematically significant details in the three students' solutions. Pre-service teacher B identified mathematically significant details in one of the three students' solutions, i.e., S3's solution, to some extent. On the other hand, four pre-service teachers increased their levels of attending to students' mathematical thinking through the discussions because their explanations in reflection paper 2 showed higher levels of evidence than those in reflection paper 1. Pre-service teacher S and pre-service teacher E, who already provided a high level of evidence, i.e., robust and substantial evidence, respectively, in the individual analysis, did not add any explanations regarding what they attended to in reflection paper 2. Pre-service teacher A and pre-service teacher K, who identified none of the mathematically significant details in three students' solutions, and pre-service teacher B, who identified mathematically significant details in one of three students' solutions to some extent in individual analysis, could notice most of the mathematical details in the students' solutions through the discussions. They provided substantial evidence in reflection paper 2 after the discussions. Pre-service teacher G, who did not recognize any mathematically significant details in the individual analysis, improved her attending response in reflection paper 2, but less than the others. To illustrate the improvement in the attending responses of the pre-service teachers, pre-service teacher A's written explanations in reflection paper 1 and reflection paper 2, supported by the discussions, are given below:

> S1 did not consider different possibilities. She thought that the shape could only be cut in the middle. Even in this case, she incorrectly justified the reason for the decrease. (Pre-service teacher A, Session 1, VC1, Reflection paper 1)

> Even if S2's answer is correct, his reasoning is wrong. (Pre-service teacher A, Session 1, VC2, Reflection paper 1)

> S3 justified all his answers correctly. (Pre-service teacher A, Session 1, VC3, Reflection paper 1)

The explanations of pre-service teacher A in reflection paper 1 indicate that he provided a general description of S1's solution. That is, he described the solution without mentioning the mathematical details. He focused on the student's way of cutting and failed to recognize the student's misconception. Moreover, he incorrectly identified the solution of S2. The pre-service teacher accepted the incorrect response of the student as correct. Furthermore, pre-service teacher A only identified S3's solution as correct without any explanations. The following dialogue from the group discussion around VC1 shows how pre-service teacher S and pre-service teacher E, with the higher level of attending response in the group, helped other group members, including pre-service teacher A, to notice the mathematically significant details in S1's solution:

> G: The first thing that attracted my attention was that the student cut the paper right down the middle.

> A: When asked if there could be another cut, she again splits one piece in the middle; this time, she obtains four pieces.

> E: She says splitting the paper into four parts reduces the perimeter again. I think the student associates the concepts of area and perimeter.

> S: I think so, too.

> B: Hmm. You are right. So, she thinks the perimeter will always decrease due to the decrease in the area as the paper is cut.

> G: Yes, the student starts from the area.

> A: She says that the perimeter decreases; in fact, when the paper is cut, the perimeter indeed decreases. When asked why, she said that the paper was

split in half. She justifies her reasoning incorrectly. Again, as you said, she seems to justify it with the area.

S: Yes. The perimeter decreases, but it is not halved.

(Group discussion, Session 1, VC1)

The expressions of pre-service teacher E and pre-service teacher S helped pre-service teacher A shift his focus from the student's way of cutting the paper to the student's misconception about perimeter-area relationship. This mathematically significant detail in S1's solution was also mentioned in the whole-class discussions by the pre-service teachers in other groups. In addition, during the group discussion around VC2, the pre-service teachers realized S2's incorrect response through the comments of pre-service teacher S and pre-service teacher E as follows:

B: S2 knows that the perimeter is related to the outside of the shape because he knows that the piece cut from the inside will not change the perimeter. He has no misconception about this.

S: But why? Isn't the perimeter the total length of the boundaries surrounding the shape? In the calculation of the perimeter, we must also consider the interior.

E: Yes, I think so. The student didn't consider those newly formed four sides.

S: Yes, he should consider that too.

A: Is that included in the perimeter?

G: I think this is included in this new shape.

S: Perimeter is the sum of the boundaries that make up the shape.

K: I've learned now.

A: Me too.

B: I thought that the student's answer was correct. I got it now.

A: I thought that area would decrease, but the perimeter would not change. Yet, you are right.

(Group discussion, Session 1, VC2)

Before the discussions, four pre-service teachers thought the perimeter was related to the outside of the shape. Hence, they thought that cutting a piece from the inside of the shape would not change the perimeter. The excerpt from the discussion shows that these pre-service teachers could notice the student's incorrect answer that they did not notice in the individual analysis. This mathematically significant detail in S2's solution was also mentioned in the whole-class discussions by the pre-service teachers in other groups. Furthermore, during the group discussion around VC3, the pre-service teachers discussed the mathematically significant details in S3's solution as provided below:

S: The student shows that the perimeter does not change by cutting a square from the corner.

E: Here, he realizes that the sides affecting the perimeter do not change.

S: Then, by cutting a square from the middle of one side, he also shows that the perimeter increases.

E: Yes, he says there was one side before cutting and two more sides after cutting.

A: He justifies all his answers correctly, as you said.

K: After cutting a triangle from the corner, he brings the two sides together and measures by comparing the total length of this segment with the length of the newly formed side, which is hypothenuse. We know that the perimeter decreases directly from the triangle inequality. Yet, the student has not yet learnt this, so he shows it that way.

(Group discussion, Session 1, VC3)

Consequently, the pre-service teachers had an opportunity to notice the mathematically significant details in the students' solutions or the students' incorrect answers, which were not noticed in the individual analysis, through the discussions. As one of these pre-service teachers, pre-service teacher A provided higher levels of evidence in his written explanations in reflection paper 2 compared to reflection paper 1 by improving his comments for each video clip as follows:

> I did not recognize that the S1 had established a relationship between the perimeter and the area. My friends said the perimeter decreases because the area decreases during the discussions. I thought that the perimeter decreased because the size of the paper decreased. I did not associate it with the area before the discussions. (Pre-service teacher A, Session 1, VC1, Reflection paper 2)
>
> I thought that when we cut a piece from the inside of a shape, the perimeter would not change, so I considered that the student's answer was correct. Yet, during the discussions, I learned that the perimeter is related to the boundaries so that the perimeter will increase. (Pre-service teacher A, Session 1, VC2, Reflection paper 2)
>
> I did not realize how S3 determined the perimeter for the triangle he cut from the corner of the rectangle before the discussions. The student reached this result by bringing the two sides side by side and comparing them with the length of the hypotenuse. (Pre-service teacher A, Session 1, VC3, Reflection paper 2)

Both group and whole class discussions provided pre-service teacher A with opportunities to realize these students' ideas and misconceptions. The attending response of pre-service teacher A in reflection paper 2 indicates that by referring to the usefulness of the discussions, pre-service teacher A focused on the mathematically significant details in the students' solutions. In this way, although at the beginning of the first session, pre-service teacher A could not identify any of the mathematically significant details in the three students' solutions, at the end of the session, by identifying the mathematically significant details in the three students' solutions to some extent, pre-service teacher A could improve his level of attending from lack to substantial.

In the second session, which focuses on a fixed area-changing perimeter situation (area conservation), S1 in VC4 thought the area of all pentomino pieces was equal. While finding the perimeter of two pieces, the student counted all sides, including the inside of the pieces. S1 considered that obtaining the shapes with the largest and smallest perimeters using two pieces was not possible because shapes would have the same perimeter due to the same number of squares. However, after calculating perimeters by putting two different pieces randomly, she deduced that a change in shapes may change the perimeter of the shapes. S4 in VC5 multiplied the length of two sides she determined to find the area of the pieces. While finding the perimeter of the shapes, the student added the length of the two sides she specified. In VC6, S5 declared that all pieces have the same area. For the perimeter, the student responded that their perimeters might differ because of differences in the spreads of the pieces. S5 correctly formed the shapes with the smallest and largest perimeters and correctly computed the perimeters of these shapes.

As shown in Table 4.15, pre-service teacher E provided the highest level of evidence for attending in the group when the pre-service teachers individually analyzed the video clips, which was followed by pre-service teacher G and pre-service teacher S. E identified all mathematically significant details in two students' solutions (S4 and S5) whereas he identified those in one student's solution (S1) to some extent. Pre-service teacher G and pre-service teacher S identified all mathematically significant details in one student's solution, while they identified those in two students' solutions to some extent. The other three pre-service teachers provided limited evidence. That is, they identified mathematically significant details in one or two of the three students' solutions to some extent. In this session, none of the pre-service teachers' attending responses in reflection paper 1 demonstrated a lack of evidence. In general, the pre-service teachers could provide higher levels of evidence in this session in their individual analysis compared to the first session. Two pre-service teachers (pre-service teacher B and pre-service teacher K) increased their level of attending to students' mathematical thinking through the discussions because their written responses regarding attending in reflection paper 2 demonstrated higher levels of evidence than in reflection paper 1. Pre-service teacher G and pre-service teacher S, who provided substantial evidence in the individual analysis, did not add any explanations regarding what they attended to in reflection paper 2. Pre-service teacher E identified other mathematical details in S1's solution through the discussions in addition to all mathematical details he identified in the solutions of S4 and S5 individual analysis. Since his attending response was already categorized as robust evidence, he stayed at the same level after the discussions. Only pre-service teacher A could not increase his level of attending in reflection paper 2. Even though he identified the mathematical details in S4's solution to some extent, this was not enough for him to move to the upper level. As an example of the improvement in the attending response, explanations of pre-service teacher B in reflection paper 1 and reflection paper 2 supported by the discussions are provided as follows:

> The student stated that the perimeters of the shapes with equal areas would also be equal, but then she correctly calculated the perimeters by combining the pieces in different ways and realized that they could have different perimeters. (Pre-service teacher B, Session 2, VC4, Reflection paper 1)

The comment above reveals that pre-service teacher B incorrectly identified S1's solution in the individual analysis of VC4. The pre-service teacher could not recognize the student's incorrect strategy while finding the perimeter of the pentomino pieces. However, she could notice the student's incorrect answer that was not noticed in the individual analysis through the discussion given below:

> B: The student counted the unit squares while finding the area of the pieces.
>
> G: She said that since the numbers of the squares are the same, the area of the pieces is also the same.

K: She even calculated the area by multiplying the length of two sides. She said that 1 time 1 equals 1, so the area of one square is 1. Yet when the perimeter got involved, she got confused.

S: When she first looked at them individually, she said they all have the same area since the number of squares is the same. Then, when she puts the pieces together, she says they all have the same perimeter because they have the same number of squares.

B: But then she corrects it and says that new shapes with different perimeters can be formed by combining them in different ways. She notices it later.

A: When calculating the perimeter, she also counts the sides inside the shapes.

G: Yes. She also says that she counted some of the sides inside and some of them she didn't. When two squares meet, two sides overlap, and she counts them as one. Moreover, when asked about the definition of the perimeter, she gave an example directly from the square. She says, "I find the length of a side and multiply it by four." And for the area, she says, "I find the length of one side and multiply it by the length of the other." She explains directly over the square as if the side lengths had to be equal.

(Group discussion, Session 2, VC4)

The group discussion enabled pre-service teacher E and pre-service teacher K to notice the other mathematical details they missed in the individual analysis and pre-service teacher B to notice the student's incorrect strategy while finding the perimeter of the shapes. This mathematically significant detail in S1's solution, i.e., counting the sides inside of the shapes, was also mentioned in the whole-class discussions by the pre-service teachers in other groups. Thus, pre-service teacher B, who incorrectly identified S1's solution in individual analysis, noted the mathematically significant detail about the student's calculating the perimeter of the pieces after the discussions as follows:

> I thought the student had calculated the perimeter correctly, but during the discussions, I realized that she had also counted the sides inside of the pieces. (Pre-service teacher B, Session 2, VC4, Reflection paper 2)

In this regard, although pre-service teacher B identified the mathematically significant details in two of the three students' solutions (S4 and S5) to some extent at the beginning of the second session, she identified the mathematically significant details in the three students' solutions to some extent at the end of the session by increasing her level of attending from limited to substantial.

In the third session, which focuses on a fixed perimeter-changing area situation, S6 in VC7 correctly identified the perimeter and area of the given figure. The student could reach the perimeter of 16 units by adding two squares and also demonstrated two shapes with perimeters of 16 units and an area of eight and nine square units. As a shape with the maximum area, S6 formed a square with an area of 16 square units. After forming a rectangle with a perimeter of 14 units and an area of 12 square units, by removing one, two, three, four and five squares, the student could show all possible shapes with a perimeter of 16 units. S7 in VC8 correctly identified the perimeter and area of the given figure. However, the student formed a rectangle with a perimeter of 12 units and thought he obtained a shape with a perimeter of 16 units. Instead of a 4x4 square, S7 formed a shape with an area of 15 square units with one square missing from its corner as a shape with maximum area.

As shown in Table 4.15, pre-service teacher E, pre-service teacher G and pre-service teacher S provided the highest level of evidence, i.e., robust, for attending in the group when the pre-service teachers individually analyzed the video clips. This was followed by pre-service teacher B, who provided substantial evidence. Pre-service teacher E, pre-service teacher G and pre-service teacher S identified mathematically significant details in both students' solutions in the third session while identifying all details in one solution. Pre-service teacher B identified the mathematically significant details in both solutions to some extent. Other pre-service teachers, pre-service teacher A and pre-service teacher K, provided limited evidence. They identified mathematically significant details in one of the two students' solutions to some extent. These pre-service teachers' levels of attending to students' mathematical thinking increased through the discussions because they provided robust and substantial evidence, respectively, in reflection paper 2. Pre-service teacher B, pre-service teacher E, pre-service teacher G and pre-service teacher S, who already provided a high level of evidence, i.e., robust and substantial evidence in the individual analysis, did not add any explanations regarding what they attended to in reflection paper 2. Pre-service teacher K's written explanations in reflection paper 1 and reflection paper 2, supported by the discussions, are presented below to represent the improvement in the attending response of the pre-service teachers.

> S7 acted randomly in his trials for the changing situations of the perimeter and area and could not explain the reasons behind it. (Pre-service teacher K, Session 3, VC8, Reflection paper 1)

The excerpt from reflection paper 1 reveals that the pre-service teacher provided a general description of S7's solution without mentioning the mathematical details. Fortunately, pre-service teacher K had a chance to notice the mathematically significant details regarding S7's finding the perimeter of the shapes through the discussions around VC8 as follows:

> B: I noticed most clearly in the student's solution that he cannot realize that the square removed from the rectangle's corners would not change the perimeter.
>
> S: He doesn't think about the contribution of a square in a corner to the perimeter. There are two sides. When I add 1 square here, there are still two sides. He can't make this inference. He could not say that the perimeter does not change.
>
> E: Furthermore, he correctly finds the perimeter of the figure as 12 units. Then, he adds one square above it. Since it has three sides, he adds three directly to its perimeter and says 15.
>
> K: Hmm. You are all right. He doesn't see the sides that disappear when the squares are added.

S: He does not subtract or add consciously. If he were, he would find the maximum area as 16 square units instead of 15. He stopped there because he thought that adding one more square would increase the perimeter by two units.

(Group discussion, Session 3, VC8)

Pre-service teacher K focused on the mathematically significant detail in the student's solution through the group discussion. This mathematically significant detail in S7's solution, i.e., inability to recognize the sides that disappear when the squares are added, was also mentioned in the whole-class discussions by the pre-service teachers in other groups. Accordingly, she wrote the following comment in reflection paper 2:

> I didn't understand why he said 16 units for the perimeter of this shape. Then, I noticed that he thought two units in each square contributed to the perimeter, but he didn't think the disappeared sides when the squares were added. I realized this during the discussions. (Pre-service teacher K, Session 3, VC8, Reflection paper 2)

The explanation of pre-service teacher K reveals that she could also identify the mathematically significant details in the solution of S7 to some extent at the end of the third session. Thus, her level of attending increased from limited to substantial since she was able to identify the mathematically significant details in both students' solutions to some extent.

In summary, the findings indicate that regarding the attending to the mathematical details in the video clips in the context of perimeter-area measurement, improvement was observed in five pre-service teachers' attending skills (pre-service teacher A, pre-service teacher B, pre-service teacher E, pre-service teacher G, and pre-service teacher K). Pre-service teacher K, who provided lack of evidence in the individual analysis at the beginning of the first session, provided limited evidence in the individual analysis in the third session. Moreover, pre-service teacher A and pre-service teacher G, who, like pre-service teacher K, provided lack of evidence in the individual analysis at the beginning of the first session, were able to provide robust evidence in the individual analysis in the third session. Pre-service teacher B, who was slightly better than these pre-service teachers in the individual analysis at the beginning of the first session, provided substantial evidence in her individual analysis in the third session. Pre-service teacher E, who already provided substantial evidence, which is one of the high levels of evidence, in the individual analysis at the beginning of the first session, improved his attending skill even more and provided robust evidence in the individual analysis in the third session. Pre-service teacher S, who had the highest attending skill in the group initially, continued this and provided robust evidence again in the individual analysis in the third session.

4.2.1.1.2 Interpreting students' understanding in the video clips in the representation-decomposition of practice in the context of perimeter-area measurement

This part presents how pre-service teachers interpret students' understanding in the video clips in the representation-decomposition of practice stage in the context of perimeter-area measurement. The pre-service teachers' interpreting skills before the discussions (reflection paper 1) and after the discussions (reflection paper 2) in the first three sessions, which are related to perimeter and area measurement, are shown in Table 4.16.

Table 4. 16 Pre-service teachers' interpreting skills in individual video analysis in the context of perimeter and area measurement

	1st session		2nd session		3rd session	
	BD	AD	BD	AD	BD	AD
A	Limited	Limited	Limited	Substantial	Limited	Limited
B	Limited	Substantial	Lack	Substantial	Substantial	Substantial
E	Limited	Limited	Limited	Limited	Substantial	Substantial
G	Lack	Lack	Lack	Limited	Lack	Lack
K	Lack	Limited	Lack	Limited	Limited	Limited
S	Substantial	Substantial	Substantial	Substantial	Substantial	Substantial

BD: Before the Discussion AD: After the Discussion

In the first session, which focuses on change in perimeter, it can be seen from the data in Table 4.16 that pre-service teacher S provided the highest level of evidence for interpreting students' understanding in the group in the individual analysis of the video clips as in attending to the mathematical details in the students' solutions. This pre-service teacher provided a valid justification for one student's understanding (S2) while she provided a valid justification for two students' understanding (S1 and S3) to some extent. Other pre-service teachers provided low levels of evidence for interpreting, i.e., lack or limited evidence. Pre-service teacher G and pre-service teacher K failed to provide a valid justification for any student's understanding. Pre-service teacher A, pre-service teacher B and pre-service teacher E could provide a valid justification for one of the three students' understanding to some extent. Two pre-service teachers (pre-service teacher B and pre-service teacher K) could increase their levels of interpreting students' understanding through the discussions since their explanations demonstrated higher levels of evidence in reflection paper 2 compared to reflection paper 1. Pre-service teacher S, who already provided substantial evidence in the individual analysis, did not add any explanations regarding students' understanding in reflection paper 2. Other pre-service teachers could not increase their level of interpreting in reflection paper 2. Among them, pre-service teacher E provided a valid justification for S1's understanding in reflection paper 2, but this was not enough for him to move to the upper level. Written explanations of pre-service teacher K, who made progress at the end of the discussions, in reflection paper 1 and reflection paper 2 are presented below to illustrate the improvement in the interpreting:

> S1 cannot evaluate the subject for different examples and have difficulty thinking differently. (Pre-service teacher K, Session 1, VC1, Reflection paper 1)
>
> S2 understands that the perimeter is related to the length of the sides, but like S1, he cannot think of different situations. (Pre-service teacher K, Session 1, VC2, Reflection paper 1)

The explanations of pre-service teacher K in reflection paper 1 reveal that she made comments about S1's understanding without any mathematical properties (in broad terms). Moreover, she provided an incorrect interpretation of the understanding of S2. The whole-class discussion flow around VC1 shows how the focus of the discussion shifted from the student's way of cutting to the reasoning behind the student's answer as follows:

> P30: The perimeter is not halved here. It is decreasing, but not half. The length of the long sides is halved, but the student could not notice that there was no change in the length of the short sides.
>
> P31: The student always cuts properly. She thinks she could divide the shapes into two and four. She couldn't think differently. She gave short answers.
>
> P7: The student lacks multidimensional thinking. She said the perimeter would only decrease. She couldn't think of cases where the perimeter would increase or not change. While she was cutting, she tried to cut an even piece.
>
> K: The student sees the cutting of a piece as a reduction. She thinks it is constantly decreasing. It is correct regarding the area, but it may not always be correct for the perimeter.

> P15: The area will decrease when we split the paper in half. I think the student has a misconception about the relationship between the area and the perimeter. The student lacks conceptual understanding.
>
> P21: The student overgeneralized the area feature to the perimeter. She has a misconception that when you cut the paper in half, both the area and the perimeter are halved.
>
> (Whole-class discussion, Session 1, VC1)

The expressions of pre-service teacher K, P15 and P21 divert the discussion to the reasons for the student's answer, and they focused on the student's understanding of the perimeter and its relationship with the area. Furthermore, the whole-class discussion around VC2 indicates that the pre-service teachers offered possible reasons for S2's incorrect response as follows:

> G: He cut a piece from the inside of the paper but did not consider it even though the shape's new form was different after cutting. He still considered the outside of the paper.
>
> P18: The student thought of the perimeter only in relation to the outside of the shape; that is, when the inside piece was cut, he did not add the interior sides to the perimeter. Inner sides were formed, but he did not take these into account. I think he has a misconception about the perimeter.
>
> P30: Well, the teacher may have misled the student by calculating the perimeters of uniform shapes in lessons. The student may not have encountered such a case before.
>
> P31: We have seen the perimeter of a rectangle as the sum of its two short sides and two long sides in school. Therefore, in this case, there was no decrease in either the long or short sides since he cut the square from the inside. Initially, I thought like the student and said the perimeter would not change.

> P15: There may be a deficiency due to the definition. We define the perimeter as the sum of two short sides and two long sides, so the student may have the misconception due to this.
>
> (Whole-class discussion, Session 1, VC2)

The comments of P30, P31, and P15 indicated the limitation of the student's previous learning experiences and the presentation of the definition of the perimeter as the possible reasons for the student's incorrect answer. This dialogue contributed to the pre-service teachers' noticing in terms of interpreting the student's understanding. As a result, one of the pre-service teachers, pre-service teacher K, could provide higher levels of evidence in reflection paper 2 by improving her comments for two video clips as follows:

> In the whole-class discussion, I realized that S1 established a direct proportion between the area and the perimeter. She thought that the decrease in the area would also affect the perimeter by decreasing it. (Pre-service teacher K, Session 1, VC1, Reflection paper 2)
>
> S2 thinks that if there is no change in the sides, there is no change in the perimeter. He believes that the perimeter is only related to the outside of the shape. I became aware during the discussions that this can be because the student learned the perimeter of a rectangle as the sum of the lengths of two short sides and two long sides. (Pre-service teacher K, Session 1, VC2, Reflection paper 2)

The comment of pre-service teacher K in reflection paper 2 shows that by providing a valid justification for two of the students' understanding to some extent, she increased her level of interpreting at the end of the first session.

In the second session, which focuses on a fixed area-changing perimeter situation (area conservation), as shown in Table 4.16, pre-service teacher S provided the highest level of evidence for interpreting students' understanding in the group in the individual analysis of the video clips as in the first session. Other pre-service teachers provided low levels of evidence for interpreting, i.e., lack or limited evidence. Pre-service teacher B, pre-service teacher G and pre-service teacher K failed to provide a valid justification for any student's understanding. Pre-service teacher A and pre-service teacher E provided a valid justification for one of the three students' understanding to some extent. Four pre-service teachers (pre-service teacher A, pre-service teacher B, pre-service teacher G and pre-service teacher K) could increase their levels of interpreting students' understanding through the discussions because their explanations showed higher levels of evidence in reflection paper 2 than in reflection paper 1. Pre-service teacher S, who provided substantial evidence, and pre-service teacher E, who provided limited evidence in their individual analysis, provided a valid justification for S1's understanding to some extent in reflection paper 2. However, this was not enough for them to move to the upper level. Written explanations of pre-service teacher A in reflection paper 1 and reflection paper 2, supported by the discussions, are given below to show the improvement in the quality of her interpreting response.

> She understood the area because her mathematical ideas about the area were consistent and correct. However, she did not understand the perimeter. Although her mathematical ideas about the perimeter were consistent, they were incorrect. (Pre-service teacher A, Session 2, VC4, Reflection paper 1)

The comment of pre-service teacher A in reflection paper 1 reveals that she provided limited justification for the student's understanding because he blamed the student for lack of knowledge of the perimeter concept. The following dialogue from the whole-class discussion shows how the pre-service teachers provided different interpretations regarding the student's understanding of the perimeter and area.

> P21: Students may learn the perimeter as the sum of short and long sides at school. Here, she sees that the shape consists of squares. She may think she should add all the sides of this square. We find it by adding the short sides and the long sides. This is how the perimeter is presented at school. She thinks that she should consider all the sides of the square. Since it focuses on the number of sides, it also counts the ones inside.

…

> G: Her answer to the definition of perimeter and area attracted my attention the most. It is as if she defines the perimeter and area directly over the square. As if all sides must be equal when we give a shape. She said I find the length of one side, multiply it by four, and it's the perimeter. She multiplies the length of a side by the number of sides. For the area, she said I find the length of one side and multiply it by the length of the other.
>
> P21: It is related to how it is taught. It may be because examples of equal side lengths were always given this way in teaching. Examples of regular shapes might have been given as if all lengths are always equal to each other. This is due to the uniform examples given by the teacher. If the teacher only gave an example over a square, the student associated it with that square. She directly multiplies the length of a side by how many sides it has.
>
> P10: Therefore, she has no conceptual understanding of the perimeter. Her definition is based on a formula. She goes through the formula by memorizing it. She does not know the definition of the perimeter. She gives an answer about calculating the perimeter. How to calculate it.
>
> (Whole-class discussion, Session 2, VC4)

The explanations of pre-service teachers focused on why the student counted the inner sides while calculating the perimeters of the shapes and they mentioned the student's previous learning experience on perimeter related to the formula-based approach during the whole-class discussions. In this way, pre-service teacher A elaborated his interpretation by mentioning these in reflection paper 2 as follows:

> The student counted the inner sides while calculating the perimeter of the shapes. In the class discussion, I became aware that one of the reasons for this misconception could be the formula-based presentation of the perimeter concept, in which the length of a side is multiplied by the number of sides rather than the definition of the perimeter. (Pre-service teacher A, Session 2, VC4, Reflection paper 2)

The excerpt above demonstrates that pre-service teacher A could explain the possible reasoning behind the student's understanding of the perimeter as the formula-based presentation of the perimeter concept, i.e., the length of a side is multiplied by the number of sides, in reflection paper 2. Thus, pre-service teacher A could provide substantial evidence at the end of the second session. The following expression of pre-service teacher A during the interviews also indicates how the implementations in the sessions supported him in interpreting students' understanding:

> I have realized that without conceptual understanding, students can easily fall into misconceptions only with their procedural knowledge and rote memorization. They can generalize what they know by rote, and these generalizations lead them to misconceptions. They can establish linear relationships between different subjects. They associate subjects with each other erroneously, such as "This was valid in that subject, so it is similarly valid here". This is because meaningful learning has not taken place, and they do not have well-established ideas about concepts. (Pre-service teacher A, Interview)

It seems that the implementations have had a positive influence on the pre-service teacher's awareness and understanding of the role of conceptual understanding in mathematics education. The pre-service teacher appears to be more cognizant of potential challenges and is likely to be more intentional in fostering meaningful learning experiences for his future students. The acknowledgment that misconceptions can arise due to a lack of meaningful learning suggests that the pre-service teacher is recognizing the potential impact of their teaching practices on student learning outcomes. This awareness may stem from insights gained during the implementations.

In summary, the findings show that three pre-service teachers improved their interpreting students' understanding in the context of perimeter-area measurement. Pre-service teacher K, who provided lack of evidence in the individual analysis at the beginning of the first session, provided limited evidence in her individual analysis in the third session. Pre-service teacher B and pre-service teacher E, who provided limited evidence in their individual analysis at the beginning of the first session, improved their interpreting skills and provided substantial evidence in their individual analysis in the third session. Unfortunately, pre-service teacher A and pre-service teacher G could not show any improvement in their interpreting skills in the context of perimeter-area measurement. Pre-service teacher S, who had the highest interpreting skill in the group initially by providing substantial evidence at the beginning of the first session, continued to do so in the other sessions.

4.2.1.1.3 Deciding how to respond based on students' understanding in the video clips in the representation-decomposition of practice in the context of perimeter-area measurement

This part presents pre-service teachers' deciding how to respond based on students' understanding in the video clips in the representation-decomposition of practice stage in the context of perimeter-area measurement. The pre-service teachers' deciding how to respond skills before the discussions (reflection paper 1) and after the discussions (reflection paper 2) in the first three sessions, which are related to perimeter and area measurement, are shown in Table 4.17.

Table 4. 17 Pre-service teachers' deciding how to respond skills in individual video analysis in the context of perimeter and area measurement

	1st session		2nd session		3rd session	
	BD	AD	BD	AD	BD	AD
A	Limited	Medium	Limited	Robust	Substantial	Substantial
B	Limited	Substantial	Medium	Robust	Medium	Substantial
E	No response	Medium	Medium	Robust	Limited	Substantial
G	Limited	Medium	Medium	Robust	Limited	Substantial
K	Limited	Substantial	Medium	Substantial	Medium	Robust
S	Medium	Substantial	Medium	Robust	Medium	Substantial

BD: Before the discussions AD: After the discussions

In the first session, which focuses on change in perimeter, as shown in Table 4.17, pre-service teacher S provided the highest level of evidence for deciding how to respond in the individual analysis of the video clips in the group, as in attending to the mathematical details in the students' solutions and interpreting students' understanding. The pre-service teacher provided a specific suggestion for only one student (S1) to eliminate the student's misconception. Four pre-service teachers (pre-service teacher A, pre-service teacher B, pre-service teacher G and pre-service teacher K) provided limited evidence for deciding. Their suggestions were insufficient to eliminate the students' misconceptions or extend their understanding. One pre-service teacher, pre-service teacher E, could not provide suggestions for any students. All pre-service teachers increased their levels of deciding how to respond to students through the discussions because their suggestions in reflection paper 2 showed higher levels of evidence than in reflection paper 1. For example, even though pre-service teacher S had the highest level of deciding response in the individual analysis in the group, she could not provide any suggestion for S3 to extend the student's understanding. Fortunately, she could increase her level of deciding through the whole-class discussion, which is given below:

> P29: He will cut a triangle on the short side inwards—an equilateral triangle. One side of the triangle will be the short side. He thought it would decrease when he cut a triangle. How would the perimeter change if he cut an equilateral triangle this time?
>
> P25: The student determines the sides of the piece he cuts directly. He brings it here. He says the perimeter has decreased because the remaining pieces are at the bottom. If we create an equilateral triangle, he will understand this easily. We can give shapes that he cannot measure and answer immediately by just looking at the shape.
>
> P13: Yes, I agree with you. In the next step, I want him to cut the edge like a circle. I would ask how he would measure it. This time, I would have it measured with a string. In this way, I would like him to expand his thinking.

P4: I would like him to cut zigzag like a ladder. Then, the number of sides will increase a lot. Will he say that the perimeter will not change again?

P12: We can ask how the perimeter changes if we cut a piece from the inside of the paper as the first student did.

(Whole-class discussion, Session 1, VC3)

The dialogue shows that the pre-service teachers mainly focused on cutting the paper in a different way. Since the student had already compared the perimeters of the shapes before and after cutting linearly, P25, P13, and P4 suggested different cutting ways. These suggestions influenced all the other pre-service teachers regarding alternative instructional actions. As a result, one of the pre-service teachers, pre-service teacher S, provided a suggestion to extend the student's understanding after the discussions in reflection paper 2 as follows:

> The student always cut straight, and he could compare the perimeters. Yet, he can cut in different ways. I had not thought about this. During the discussions, I realized that the student could cut in different ways, such as zigzag, and in this case, I would wonder how to reason about the change in the perimeter in the next step. (Pre-service teacher S, Session 1, VC3, Reflection paper 2)

The explanation of pre-service teacher S in reflection paper 2 reveals that the whole-class discussion helped the pre-service teacher broaden her viewpoint. In this way, she increased her level of deciding at the end of the first session. In addition, the following expression of pre-service teacher S reveals that the discussions fostered her in terms of extending students' understanding as follows:

> At first, if there was a misconception, I focused on what I could do to eliminate it. I felt like I didn't need to do anything if the student's solution was correct. This was due to my difficulty in making suggestions for extending. It was easier to eliminate the misconception than to extend understanding. It was much harder to go over something the student knew. In the beginning, I didn't think about extending at all. Then, I started paying

> attention to it. Our discussions were effective because we exchanged ideas with each other. (Pre-service teacher S, Interview)

This excerpt reveals the pre-service teacher's journey from initially focusing on misconception elimination to recognizing the importance of extending understanding. The pre-service teacher acknowledged that extending understanding, especially when students already knew a concept, was more challenging. This difficulty may be related to finding ways to deepen understanding or provide enriching experiences for students who have already grasped the concept. The passage indicates a shift in the pre-service teacher's perspective over time. She became more conscious of the need to extend understanding and began to pay attention to this aspect of teaching. This suggests a growing awareness and an evolving mindset during the implementations by emphasizing the value of discussions, suggesting that the pre-service teacher found benefit in sharing thoughts and insights with others.

In the second session, which focuses on a fixed area-changing perimeter situation (area conservation), as shown in Table 4.17, all pre-service teachers, except pre-service teacher A, provided a medium level of evidence for deciding how to respond in the individual analysis of the video clips. Pre-service teacher A provided a limited level of evidence because the suggestions offered by him were not sufficient to eliminate the student's misconception or extend the student's understanding. Other pre-service teachers provided specific suggestions for one of the three students. All pre-service teachers could increase their levels of deciding how to respond to students through the discussions because five pre-service teachers' suggestions demonstrated robust evidence, and one pre-service teacher (pre-service teacher K)'s suggestion showed substantial evidence in reflection paper 2. To illustrate the improvement in the quality of the deciding responses after the discussions, the suggestions of pre-service teacher E in reflection paper 1 and reflection paper 2 are presented below.

I would explain the concept of perimeter again and tell them that they need to count the outer edges of the shape to calculate the perimeter. (Pre-service teacher E, Session 2, VC4, Reflection paper 1)

The student only knows how to calculate the area and perimeter in regular shapes such as squares and rectangles and thinks that the area and perimeter of all shapes will be found that way. Firstly, I would change this wrong perception and ask her questions about how to calculate the area and perimeter in different shapes. I make her count, find out that the perimeter is the sum of all sides, and make her realize that she is wrong. (Pre-service teacher E, Session 2, VC5, Reflection paper 1)

In reflection paper 1, pre-service teacher E provided direct instruction, i.e., explaining the concept and telling the student what to do, for S1 in VC4. Moreover, pre-service teacher E provided a specific suggestion for S4 in VC5, but he did not give information about what kind of shapes he asked the student to calculate the area and perimeter. Furthermore, the pre-service teacher could not provide any suggestions to extend the understanding of S5 in VC6. During the whole-class discussion below, the pre-service teachers provided different suggestions to make S1 in VC4 understand the perimeter and how to measure the perimeter of the shapes.

K: I would take such a shape (one of the pentominoes); I can both draw it on the board and measure its perimeter with a string. I would first have the student calculate the units on the board. Then, I give her a string; she will realize that she cannot count from the inside; he will only take the outside, and she will probably find the correct perimeter.

…

K: I would create a discussion environment as to why there is a difference between the results and make her see where she is wrong.

P28: The student said that as the shapes change, the perimeter changes. We can create two different shapes with equal perimeters using these pentominoes. She can see that the perimeters are equal when she wraps the string around the shapes.

…

(Whole-class discussion, Session 2, VC4)

At the beginning of the discussion, in their instructional actions, pre-service teachers generally focused on the definition of the perimeter and examples from daily life and considered that materials such as geoboard could be useful to develop the student's understanding of the perimeter. Then, K and P28 suggested asking the student to measure the perimeter of the shapes with a string. K argued that if she would ask the student to measure the perimeter of a pentomino piece with a string, the student would not count the sides inside. P28 then improved this suggestion and stated that the student would also recognize different shapes with the same perimeter created with pentominoes by measuring them with a string. Moreover, during the whole-class discussion around VC5, the pre-service teachers suggested different teaching moves to help S4 overcome her misconceptions about the perimeter and area as follows:

P23: When calculating the area of the U shape, she multiplied by three times 2. I would give this shape and give a filled shape of this shape. I would ask her to calculate the perimeter and the area of both shapes. She would probably find them the same. Then, I would say, "You found them both the same, but look, there is 1 square missing. So, why are they the same?"

P13: My suggestion is similar to the suggestion of P23. While calculating the area of shapes, she applies the same procedure to all shapes by multiplying the lengths of two sides without paying attention to how the shape is. To eliminate this misconception, I would give a 3x3 square and a different shape with three by three sides consisting of unit squares. To find the area of the two, she would do the same operation three times three and find 9. Then, I would ask her to count the unit squares because the area is the number of unit squares that cover the shape. Since the number of unit squares is different, she would realize that their area is different.

> S: I would ask her to calculate the area of the U shape by breaking it into pieces. When she breaks it into pieces, there will be two rectangles and 1 square. She can easily calculate their area by multiplying the lengths of the sides. Then, she will find the sum of the area of the pieces 5 square units. Yet, in her previous calculation, she calculated the area of U shape six by multiplying three by 2. In this way, she would realize that she is mistaken, and I can eliminate her misconception.
>
> …
>
> (Whole-class discussion, Session 2, VC5)

The suggestions provided in the whole-class discussion helped the pre-service teachers become aware of the alternative instructional actions based on S4's understanding of the perimeter and area. Particularly, P23 and P13 suggested asking the student to compare the perimeter and area of a square/rectangle shape with the shape of squares missing so that the side lengths do not change. In addition, pre-service teachers tried to extend the understanding of S5 in VC6 by focusing on the relationship between the perimeter and the sides and providing the counterexamples as follows:

> P13: He made a generalization based on the spread of the shapes. The more spread the shape is, the bigger the perimeter is. I would give a spread and relatively compact shape with the same perimeter consisting of three pieces. I would ask which one has a bigger perimeter. At first glance, he would say the spread one. Then, I would ask him to find the perimeter of the shapes. In this way, he would realize that the perimeters are the same.
>
> P23: I would give an example of one that is more spread but has a smaller perimeter. It will look more spread, but its perimeter will be smaller. The other one will be more compact, but its perimeter will be larger. I would ask him to calculate their perimeters.
>
> P21: He didn't express the perimeter with the sides as a boundary. We talked last week about the inner and the outer perimeter. When we create a shape, it

> remains empty from the inside; I thought of what he would say. In this case, the perimeter of the shape may be greater, although the shape is less spread.
>
> …
>
> (Whole-class discussion, Session 2, VC6)

By concentrating on what is appropriate to develop the student's understanding, pre-service teachers tried to change the perception of the student that the more spread the shape is, the bigger the perimeter is. All the discussions around VC4, VC5 and VC6 above was useful for pre-service teacher E because he increased the quality of his comments regarding deciding how to respond after the discussions as follows:

> I think one of the actions suggested in both group and whole-class discussions will be useful to eliminate the student's misconceptions about the perimeter. To make the student realize that the perimeter is the sum of the boundaries surrounding the shape, we can teach the concept of the perimeter by wrapping a string around the outside of the shape and making her realize that the string cannot surround the inner sides. Thus, she realizes that the length of the string will give the perimeter. When we wrap the string around both shapes and bring the strings side by side, she sees that the perimeter of both shapes is the same. (Pre-service teacher E, Session 2, VC4, Reflection paper 2)

> Since the student is looking for a whole shape, I realized during the whole-class discussion that it is appropriate to give examples where she can compare the shape with its filled whole form. I would give the student the shape and the filled shape of it. If she gives the same answer, I ask why she did not consider the indented parts. (Pre-service teacher E, Session 2, VC5, Reflection paper 2)

> The student's answer is correct, but this may not always lead him to the correct answer. For this, we can give counterexamples to the student. For example, shapes with different spreads of equal perimeters and shapes with a larger perimeter despite being more compact. We can also use three pieces, as suggested during the whole-class discussion. In this way, the student can

> more easily realize that the perimeter is related to the sides rather than whether the shape is compact or spread. (Pre-service teacher E, Session 2, VC6, Reflection paper 2)

In reflection paper 2, pre-service teacher E focused on eliminating the misconceptions of S1 and S4 and extending the understanding of S5 regarding the perimeter concept. He provided specific suggestions for three students to make two of them understand and extend one of the student's understanding by explaining in detail. Thus, he increased her level of deciding at the end of the second session.

In the third session, which focuses on a fixed perimeter-changing area situation, as shown in Table 4.17, pre-service teacher A provided the highest level of evidence, i.e., substantial, for deciding how to respond in the individual analysis of the video clips in the group. Three pre-service teachers (pre-service teacher B, pre-service teacher K and pre-service teacher S) provided medium evidence. They provided specific suggestions for one of the two students. Two pre-service teachers (pre-service teacher E and pre-service teacher G) provided limited evidence because their suggestions were insufficient to eliminate the students' misconceptions or extend their understanding. Five pre-service teachers increased their levels of deciding through the discussions since they provided higher levels of evidence in reflection paper 2 compared to reflection paper 1. At the end of the third session in reflection paper 2, five pre-service teachers' suggestions demonstrated substantial evidence, and one pre-service teacher (pre-service teacher K)'s suggestion showed robust evidence. To illustrate the improvement in the quality of the deciding responses after the discussions, the suggestions provided by pre-service teacher G in reflection paper 1 and reflection paper 2 are given below.

> The student's way of thinking is correct; she can make correct inferences. Therefore, I think it is necessary to force her in this regard. More complex shapes should be created with the student. I would wonder what kind of a way she would follow when finding the perimeter of more complex shapes. (Pre-service teacher G, Session 3, VC7, Reflection paper 1)

> I would create a shape including common sides and explain that they should not be included in the perimeter. (Pre-service teacher G, Session 3, VC8, Reflection paper 1)

In reflection paper 1, pre-service teacher G provided a general suggestion for S6 in VC7 because she did not explain what the complex shapes are. Moreover, the pre-service teacher provided direct instruction for S7 in VC8. Yet, these suggestions were not sufficient to eliminate the misconception of S7 and extend the understanding of S6. Fortunately, the discussions around these video clips helped the pre-service teachers become aware of the alternative suggestions based on the student's understanding in the video clips as follows:

> B: The student does the calculations correctly in regular shapes such as squares; she knows the area and perimeter correctly, but one wonders if she can accurately calculate irregular shapes.
>
> S: Assume that she calculated the area of this shape as seven square units. She realized that the perimeter did not change. She could find all of them in the same way, even if she created different shapes.
>
> K: I would focus on shapes in such a way that the sides would not coincide with each other. I would give two shapes; one has more sides. I would ask these questions for different shapes, not in the form of unit squares.
>
> E: For example, side with indentations and things like that.
>
> Others: Exactly.
>
> A: I would like to hear from her about our conclusion on this activity. The sentence that the area of shapes with the same perimeter can be different.
>
> S: Very logical. It didn't come to my mind to ask which conclusion we reached.
>
> G: Me neither.
>
> (Group discussion, Session 3, VC7)

During the group discussion around VC7, pre-service teachers focused on extending the understanding of S6 by suggesting asking the student to calculate the perimeter and area of irregular shapes, different shapes that not consisting of unit squares. Different from these suggestions, pre-service teacher A suggested asking the student the conclusion she reached at the end of the task. Furthermore, during the whole-class discussions, pre-service teachers offered various suggestions to eliminate the misconception of S7 about the perimeter concept as follows:

> P28: We wrap something like a string around this shape. Then, we open the rope. We add one square to the corner and wrap the same rope around the shape again. It can also be done by removing one square from the corner. When the student sees that the length of the ropes is the same, he will realize that the perimeter is the same.

> P31: Instead of the string, he can see that the perimeter remains the same by counting. For example, he found the perimeter of 12 units. He added two squares and found the perimeter of 14 units. Then, he added two squares and found the perimeter of 16 units. He will see that the perimeter is still 12 units if we ask him to count.

> P21: Yes, I think the same. In the beginning, I would ask the student to count the perimeter of the shape. Then, after adding the squares to the corners, I would ask him if he could find the perimeter by counting one by one. In this way, he will realize that the perimeter was 12 units at the beginning, and when he added a square to the corner, the perimeter became 12 units. The student can infer that I have kept the perimeter of this shape the same. When I ask, "Why did the perimeter remain the same?" I encourage him to think. The student can overcome his misconception by matching the inner and outer sides. A few more such cases can be provided. Ultimately, he found the maximum area to be 15 square units. However, he will have learned that the perimeter does not change when he adds squares to the corners. I ask him to think again. Accordingly, by adding a square to the corner, the student can find the maximum area as 16 square units.

…

K: I would give a 3x3 shape and ask the student to calculate the perimeter of the shape. I take squares from the corners one by one, one square from the corners, two squares from the corners, and four squares from the corners, and ask the student to calculate the perimeter again. The student will find that all of these have the same perimeter. Thus, I enable the student to arrive at a generalization that when the squares from the corners are removed, the perimeter does not change.

P32: On the contrary, we can give the shape where the squares are removed from the corners. We ask the student to add the squares one by one and calculate the perimeter in each case. In this way, he can realize that the perimeter does not change.

…

P13: I would start with simple examples first. For example, I give a two by 2 square consisting of four unit squares. I would ask him to remove one square and count the sides one by one. In the same way, I would ask him to add squares and count the same way. I ask him what changes when we add and remove squares. Then, I make it more complex and do the same for the 3x4 and 4x4 shapes. So, I would like the student to arrive at a generalization. In each case, I ask how the number of sides has changed and ask the student to record it in the table.

(Whole-class discussion, Session 3, VC8)

In the whole-class discussion around VC8, the pre-service teachers tried to make the student aware of when the perimeter increased, decreased and did not change by removing or adding squares step by step and asking what happened to the perimeter in each case. Thus, pre-service teacher G had an opportunity to hear different instructional actions, which in turn reflected in her suggestions after the discussions as follows:

> I liked the suggestion of pre-service teacher A in the group discussion. To extend the student's understanding, "What conclusion are we reaching with this task? Can we make a generalization from here?" can be asked. In this way, it can be ensured that the student realizes which conclusion the task leads her to. (Pre-service teacher G, Session 3, VC7, Reflection paper 2)

> I realized in the discussions that it can be progressed through question-answer for the student. Questions such as why the perimeter has changed can be asked at that moment by adding and subtracting squares in shapes from simple to complex. In this way, the student can realize his own mistakes. (Pre-service teacher G, Session 3, VC8, Reflection paper 2)

In reflection paper 2, while G offered a suggestion for S7 in VC8 based on orienting the student with questions to answer, she provided a specific suggestion for S6 in VC7 to extend the student's understanding by explaining in detail. Hence, at the end of the third session, her level of deciding increased from limited to substantial. Moreover, pre-service teacher G also mentioned how the sessions helped her improve deciding how to respond skill as follows:

> The things I wrote in the first weeks were very general. Towards the end, especially in the discussions, more specific things started to form in my mind from what my friends said. I would not have developed so much without the discussion part. Because I was using what I heard there in the following weeks. I was relating it. I think, "We did it like this last week. It could be like this". Towards the end, I started to give more solution-oriented suggestions than in the first weeks. What are misconceptions and the questions that reveal these misconceptions, how do we generate them, and what kind of questions can we ask students to think without directing them to the correct answer... I feel closer to teaching now that I have learnt the answers to all these. (Pre-service teacher G, Interview)

This excerpt illustrates the pre-service teacher's journey from general ideas to more specific, solution-oriented thinking, with a notable emphasis on the role of discussions. It reflects a positive evolution in the pre-service teacher's professional development over the course of the implementations. The pre-service teacher highlights learning about misconceptions, the questions that reveal them, and strategies for generating questions that encourage student thinking without directing them to the correct answer. This indicates a deepened understanding of pedagogical strategies and the importance of probing questions in teaching.

In summary, what emerges from the findings reported here is that four pre-service teachers improved their deciding how to respond skills in the context of perimeter-area measurement. Pre-service teacher E could not provide any suggestions based on the students' understanding in his individual analysis at the beginning of the first session, but he could provide medium evidence and limited evidence for deciding how to respond in the second and third sessions when he individually analyzed the video clips. Pre-service teacher B and pre-service teacher K, who provided limited evidence in their individual analysis at the beginning of the first session, provided medium evidence for deciding how to respond in the third session. Pre-service teacher A who provided limited evidence in the individual analysis at the beginning of the first session, was able to provide substantial evidence in his individual analysis in the third session. Pre-service teacher S, who had the highest deciding skill in the group initially, continued this and provided medium evidence again in the individual analysis in the third session.

4.2.1.2 The influence of approximation of practice on the development of pre-service teachers' professional noticing skills in the context of perimeter-area measurement

In this part, pre-service teachers' attending, interpreting and deciding how to respond skills at the approximation of practice stage of pedagogies of practice are presented to show how a video-based module situated in the pedagogies of practice framework supports pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement. As an approximation of practice, pre-service teachers conducted individual task-based interviews by using the tasks they had previously designed with middle school students of their choosing and video-recorded the interviews. Following the interview, they reflected on student's responses and actions by watching the videos and individually analyzed the students' mathematical thinking they elicited by using the professional noticing framework. Finally, in reflection paper 3, they responded to three noticing prompts about attending to students' mathematical thinking, interpreting students' understanding and deciding how to respond, as well as to the questions about surprising/unexpected aspects of the student's solution and what they would change if they were to do the interview again. Three pre-service teachers, pre-service teacher B, pre-service teacher K and pre-service teacher S, designed tasks on perimeter and area measurement. Therefore, in this part, findings for these pre-service teachers are presented. Table 4.18 shows the contents of the tasks the three pre-service teachers designed.

Table 4. 18 Content of the tasks the pre-service teachers designed in the context of perimeter and area measurement

PSTs	Content of the task	Concept
B	○ Constructing a rectangle and a square with four tangram pieces (one large triangle, one medium triangle and two small triangles) and asking to compare their area and perimeter ○ Removing a small triangle from the square and asking how the perimeter and area of the shape change ○ Removing a small triangle from the square, adding it next to the shape, and asking how the perimeter and area of the shape change	Perimeter and area
K	○ Giving large rectangular cardboard and small congruent rectangular pieces ○ Asking to find the perimeter of the cardboard with the small pieces ○ Asking to find the area of cardboard with the small pieces	Perimeter and area
S	○ Giving three tangram pieces (square, parallelogram and medium triangle) and asking to compare the area of these pieces ○ Asking to construct the previous shapes by using two small triangles and to compare the area of the shapes once again ○ Asking to construct the large triangle by using a square, a parallelogram, a medium triangle and two small triangles ○ Asking to find the relationship between the area of the shapes ○ Constructing different shapes with pieces and asking to compare the area of the shapes and to put the shapes in order by area	Area

4.2.1.2.1 Attending to students' mathematical thinking in the approximation of practice in the context of perimeter-area measurement

In order to investigate the extent to which pre-service teachers attend to students' mathematical thinking in the context of perimeter-area measurement, the mathematical details in the solutions of the interviewed students were first determined by the researcher. The mathematical details in the interviewed students' solutions are provided in Table 4.19.

Table 4. 19 Mathematical details in the interviewed students' solutions in the context of perimeter and area measurement

Students interviewed by the pre-service teachers	Students' misconceptions/difficulties in perimeter and area measurement
The student interviewed by pre-service teacher B	Believing the same pieces-same area-same perimeter Relating the area of a shape with a spread of pieces rather than the area of pieces forming the shape (no area conservation) Ignore newly formed sides when the piece is removed while comparing the perimeter of shapes
The student interviewed by pre-service teacher K	Confusing area and perimeter (calculating area instead of perimeter) Ignoring the use of the same units while finding the perimeter Finding different results for the perimeter and area of the shape Multiplying the perimeter by 2 to find the area of the shape
The student interviewed by pre-service teacher S	Relating area of shape with appearance/size Failure to realize that area is related to covering (the area of a shape is the sum of the area of individual sub-pieces)

As can be seen from the table above, all three students who were interviewed had misconceptions similar to those in the video clips that the pre-service teachers viewed as representations of practice and analyzed as decompositions of practice in the sessions. While analyzing the task-based interviews they conducted, each pre-service teacher identified all mathematically significant details in the student's solution by providing specific evidence for the mathematical details in the solutions. Hence, all three pre-service teachers' responses to the attending prompt showed a robust level of evidence in reflection paper 3. Figure 4.44 shows the response of pre-service teacher B to the attending prompt in reflection paper 3.

…When I asked the student, “Which of the given shapes has a larger area?” The student correctly said, “The area of the shapes is equal because they are made up of the same pieces.” For the area, the student said, “Since they are made of the same pieces, their perimeters are equal.”

When I removed one of the tangram pieces forming the square shape and asked how the area changed, the student first declared that the area decreased by pointing to the empty space… because one side was missing now (pointing to the missing side). She did not consider the newly formed inward sides of the shape when the piece was removed. Then, when I added the removed piece next to the shape and asked how the perimeter changed compared to the previous one, she stated that the perimeter increased because sides not previously included in the perimeter were now included by pointing to the hypotenuse and the side of the triangle…For the change in area, she said, “The shape covered a larger area; it increased.” She also changed her mind about the area and said, “Then, the area does not change when the piece is removed” by focusing on the spread of the shape rather than the number of pieces.

Figure 4. 44 Pre-service teacher B’s attending to interviewed student’s mathematical thinking

As can be seen from the figure above, pre-service teacher B provided the mathematical details in the student’s solution and mentioned the student’s misconceptions about both perimeter and area, which are the same area-same perimeter, relating the area of a shape with a spread of pieces, and ignoring newly formed sides in the perimeter. The pre-service teachers also attended to the student’s work based on surprise by responding to the question in reflection paper 3, *“Were there any surprising/unexpected situations (that you could not predict) in the student’s solution? Please explain.”* As a response to this question, pre-service teacher B stated that she was surprised at the student’s correct response regarding the equality of the area of the square and rectangle at the beginning because she expected the student to say that the area of the rectangle is greater than the area of the square based on the student's answer at the end of the interview since rectangle seems more spread. Furthermore, the pre-service teachers also responded to the question in reflection paper 3, *"If you were to do the interview again, what would you change (e.g., in the task you designed, in the questions you asked, etc.)? Please explain."* In the responses, all three pre-service emphasized the importance of questioning. Pre-service teacher B wrote in reflection paper 3:

> In the first question, the student said that the perimeter of a square and a rectangle are the same because they are made of the same pieces. In response to this answer, by replacing the position of a piece in one of the shapes, I would ask, "Is the perimeter the same again? Can you create shapes with the same area and different perimeters?" In addition, the shapes consisted of exactly the same pieces. Yet, the same areas can be obtained with different numbers of pieces. For example, the area of two small triangles is equal to the area of one medium triangle. I wonder what she would say about the area and perimeter of the shapes with equal areas if they consisted of different numbers of pieces. I would ask this if I conducted the interview again. (Pre-service teacher B, Reflection paper 3)

Figure 4.45 presents the response of pre-service teacher K to the attending prompt in reflection paper 3.

When I asked the student to calculate the perimeter of the large cardboard, he arranged the pieces horizontally to cover the entire surface of the large rectangle…He found the perimeter of this shape 24 units by multiplying the sides 6 and 4. He didn't pay attention to the relationship between the side lengths. Rather, he calculated all of them to be the same unit.	
When the student was asked if he could find the perimeter in a different way, he arranged the pieces vertically. When I asked the student to find the perimeter of the figure again, he counted the small rectangles one by one incorrectly and answered 23, thinking that the number of small pieces was the perimeter. About the possibility of the perimeter of the same shape being different, he stated that he didn't know the answer. For the area of the shape, he multiplied 23, the result he found for the perimeter, by two and got 46. I think the student used the formula "add two sides and multiply by 2", which can be used to find the perimeter of the rectangle, for the area.	
When I asked the student if he could find the area of the shape in another way, he arranged the small rectangles horizontally. This time, he multiplied the result of 24 for the perimeter by two and got 48. The student calculated the area based on the result of the perimeter he found.	

Figure 4. 45 Pre-service teacher K's attending to interviewed student's mathematical thinking

As shown in Figure 4.45, pre-service teacher K provided the mathematical details in the student's solution, i.e., confusing area and perimeter, ignoring the use of the same units while finding the perimeter, finding different results for the perimeter and area of the same shape, and multiplying the perimeter by 2 to find the area of the shape. The pre-service teacher also explained how the student found the perimeter and area of the large rectangular cardboard using the small rectangular pieces. While attending to the student's work based on surprise in reflection paper 3, pre-service teacher K stated that she was surprised at the student's confusing perimeter and area concepts. In addition, the student found different results for both the perimeter and area of the same shape when he used the same unit in different ways, which pre-service teacher K did not expect. She was also surprised when the student multiplied the perimeter of the shape that he found by two in order to find the area. Regarding the question of what they would change if they had the interview again, pre-service teacher K wrote:

> If I could do the interview again, I could enhance my understanding regarding student's mathematical thinking more deeply by asking detailed questions about some of the student's actions during the activity. (Pre-service teacher K, Reflection paper 3)

She asserted that the designed task was based on units. However, the implementation of the task proceeded in a different way due to the student's misconceptions and lack of knowledge about the perimeter and area. Therefore, she would ask the student to express the units. The student found different results by changing the arrangement of the small rectangles while measuring the perimeter. Yet, he asserted that he used the number of rectangles for the perimeter. Hence, pre-service teacher K would ask how he got different results, although he used the same number of rectangles while measuring both ways. Similarly, she would ask him to compare two different answers he found for the area of the rectangle. When pre-service teacher K asked the student to calculate the area, he could not answer first and then suggested multiplying by two. Therefore, by asking the student why he multiplied by two, pre-service teacher K stated that she would be sure if the number two was strictly related to the perimeter formula or if he was thinking of something else. Pre-service teacher K declared that the student could be asked which shape other than the rectangle could be used to measure the area of the large rectangle and how he could measure it with this shape. The following are the questions that K pre-service teacher would ask the student in reflection paper 3:

- Can you express the units?
- You used the same number of rectangles in the figure but found different results for the perimeter. What could be the reason for this?
- Why do you multiply by 2 to find the area?
- The area of the same figure was different when you placed the small rectangle in a different way. Why did this happen?

- Do you think that another shape could be used to calculate the perimeter and area of the large rectangle? If yes, which shape would you use? Can you explain the reasons?

The response of pre-service teacher S to the attending prompt in reflection paper 3 is presented in Figure 4.46.

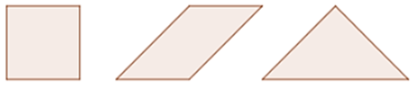
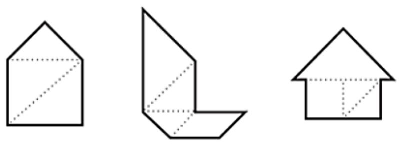
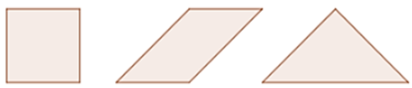
When I asked the student about the area of the square, parallelogram and medium triangle, he looked for a numerical value. He said, "There is no numerical data at the moment, so I will make a comparison according to their sizes. Firstly, parallelogram, then triangle, then square." He thought that the area of the parallelogram would be the largest, considering the spread of the shapes over the surface.	
When I gave the student two small triangles and asked him to create previous shapes and record how many triangles he used, he was able to do it correctly. Then, he formed the large triangle with two triangles and one parallelogram. When I asked him how many triangles in total formed the large triangle, he could correctly answer four triangles by counting.	
While comparing the area of the shapes I gave, the student stated that he would look at the number of triangles and the size of the triangles. He said, "…this one (1) is the biggest. Then this one (2), then this one (3)." …He stated that he looked for large and small triangles and sorted them accordingly. However, the student sorted the shapes correctly by chance because he only considered the number and size of the triangles. He did not calculate the area of each shape in terms of the number of small triangles.	
In the end, when I asked him again about the area of the square, parallelogram and medium triangle…he now sorted them triangle, then the parallelogram, then the square. Although he recorded, he did not consider the number of triangles while forming the shape.	

Figure 4. 46 Pre-service teacher S's attending to interviewed student's mathematical thinking

The figure above shows how pre-service teacher S provided all the mathematical details regarding area measurement in the student's solution. S's response to the question related to the student's work based on surprise in reflection paper 3 indicated that she was surprised at the student's failure to try to build a relationship between the area of a small triangle and the area of a square, parallelogram and a medium triangle. In addition, pre-service teacher S stated that if she could do the interview again, she would ask the student to compare the area of the shapes again after constructing the square, parallelogram and medium triangle. In reflection paper 3, she wrote:

> This was a question I wanted to ask during the interview, but the student could not compare the area of the shapes correctly. I realized this after watching the video. I would like to draw more attention to the fact that he used two small triangles while constructing the mentioned shapes. Even though he recorded it, he didn't realize it. (Pre-service teacher S, Reflection paper 3)

She also added that the student could not establish a connection between the area of the figures. Therefore, she stated that she would ask, *"What kind of relationship is there between the area of the shapes?"* In this way, she believed that she would draw the student's attention more to the area relation between the shapes.

In summary, all three pre-service teachers (pre-service teacher B, pre-service teacher K and pre-service teacher S) who designed tasks on perimeter-area measurement, identified all the mathematically significant details in the student's solution by providing specific evidence for the mathematical details in the solutions when analyzing the task-based interviews they conducted. Therefore, all three of the pre-service teachers' responses to the attending prompt in reflection paper 3 showed a robust level of evidence.

4.2.1.2.2 Interpreting students' understanding in the approximation of practice in the context of perimeter-area measurement

In this part, how pre-service teachers interpret the understanding of interviewed students in the approximation of practice stage of pedagogies of practice in the context of perimeter-area measurement is presented. All three pre-service teachers could interpret the students' mathematical understanding by considering the students' strategies, difficulties and misconceptions in perimeter and area measurement in detail in reflection paper 3. In this way, they provided a robust level of evidence for interpreting students' understanding in the approximation of practice. Figure 4.47 illustrates the response of pre-service teacher B to the interpreting prompt in reflection paper 3.

> In the beginning, the student gave correct answers to the questions about the area. She said that since the shapes are made of the same tangram pieces, their areas will be equal. However, in the last question, she thought that when we removed the piece and added it next to the shape, the area would increase because the shape was more spread… Since the square is a more organized and compact shape compared to the shape formed later, she thought that it occupied less area. When I removed the piece, the appearance of the shape did not change much; it was still like a square, so the student thought that the area did not change, but when I added it next to the shape, the shape became more spread. That's why she said the area increased. Initially, she made inferences about the area according to the number of pieces, but later, she gave wrong answers based on the spread of the shape on the surface. She could not think about the principle of area conservation and needs improvement regarding the area of irregular shapes. In addition, she associates the perimeter with the area. Since the shapes are composed of the same pieces, she believes that the perimeters are equal since the areas are equal. She thinks that there is a linear relationship between area and perimeter. She did not think that the perimeter could change according to how the sides of the pieces that make up the shape are inside or outside.

Figure 4. 47 Pre-service teacher B's interpretation of the interviewed student's mathematical understanding

The response of pre-service teacher B to the interpreting prompt in reflection paper 3 contained statements providing robust evidence of the student's understanding. The pre-service teacher gave what she thought the student did not know and was not able to do, i.e., area conservation and perimeter depends on the sides and changes according to the position of pieces and the possible reasoning behind the student's solution, i.e., believing that there is a direct relationship between perimeter and area and relating the area with the spread of the shapes. Moreover, there was evidence from the student's solution to show how she came to that understanding. That is, she used details from the student's solution to support evidence-based interpretations. The response of pre-service teacher K to the interpreting prompt in reflection paper 3 is presented in Figure 4.48.

> When I asked about the perimeter of the cardboard, the student made an area calculation instead of a perimeter calculation. When I asked about the area of the cardboard, he applied the rule of multiplication by two when calculating the perimeter of regular shapes…This shows that the student does not know the differences between the concepts of area and perimeter. When the student was asked to calculate the perimeter, he found an answer 24 by multiplying 6 and 4 for the first case. However, when the side lengths of the small rectangles are examined, it is seen that the side lengths he multiplied are not equal to each other. From this point, it is seen that the student calculated with the help of small rectangles and, in this way, did not take into account the unit lengths and did not comprehend the importance of using unit squares when calculating. When the student was asked to calculate the perimeter, he answered by multiplying the side lengths for the first figure, but when asked to calculate it in another way, he counted the rectangles this time. In teaching, a generalization is shown that the perimeter of regular shapes such as rectangles is found by adding the lengths of the two sides and multiplying by two. I think that the student did this for the area, which can be done for the perimeter…

Figure 4. 48 Pre-service teacher K's interpretation of the interviewed student's mathematical understanding

The comment of pre-service teacher K regarding interpreting the student's understanding reveals that the pre-service teacher gave details of exactly what the student said and did. Such details by pointing to the student's understanding resulted in the response being categorized as robust level. She explained the possible reasoning behind the student's solution regarding the area of the shape as incorrect generalization of perimeter formula to the area. The pre-service teacher was aware that the student had a lack of conceptional understanding of perimeter and area concepts and used units in perimeter and area measurement. Thus, she drew inferences about the student's mathematical understanding, and she also pointed out the student's specific difficulty measuring both perimeter and area of the cardboard. The response of pre-service teacher S to the interpreting prompt in reflection paper 3 is shown in Figure 4.49.

When I asked about the area of the shapes, the student couldn't compare them directly because there was no numerical value. Then, she made a comparison according to their dimensions. She thought that the area of the parallelogram was the largest, considering its spread over the surface and that the square was the smallest because it was more compact. She constructed the desired shapes using the pieces and noted the result. Yet, she did not realize that the notes she had recorded gave the area of shapes. The reason she didn't notice may be that she didn't see the area as covering... If she was aware of this, she would be able to say the area of the shapes in terms of triangles…When she constructed the shapes with the small triangles, she could not associate the area of the shapes with the number of small triangles. The student may have learned the area based on formulas without conceptual understanding. For this reason, she regarded the area as a numerical operation, not as covering.

Figure 4. 49 Pre-service teacher S's interpretation of the interviewed student's mathematical understanding

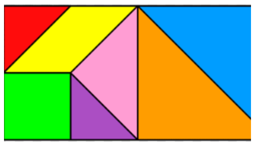
It can be seen from the comment of pre-service teacher S in Figure 4.49 that it included specific evidence in the form of descriptions of the actions of the student to support the claims being made. The pre-service teacher used details from the student's solution to infer explanations about the student's incorrect response, suggesting that she might not have a concept of covering or might have learned the area based on formulas. For this, she provided evidence from the student's solution that the student considered the spread of shapes over the surface while making judgments about the area and the inability to make a connection between the area of the pieces and the area of the shapes. This extent of evidence was categorized as robust, which was direct evidence of the student's thinking.

In summary, all three pre-service teachers (pre-service teacher B, pre-service teacher K and pre-service teacher S) who designed tasks on perimeter-area measurement were able to interpret the students' mathematical understanding by considering the students' strategies, difficulties and misconceptions in perimeter and area measurement in detail in reflection paper 3. Thus, they provided a robust level of evidence for interpreting students' understanding in the approximation of practice.

4.2.1.2.3 Deciding how to respond based on students' understanding in the approximation of practice in the context of perimeter-area measurement

In this part, pre-service teachers' deciding how to respond based on the understanding of interviewed students in the approximation of practice stage of pedagogies of practice in the context of perimeter-area measurement is presented. All three pre-service teachers offered specific instructional suggestions based on students' misconceptions and difficulties they identified to eliminate them and make students understand by providing rationale and detail in reflection paper 3. Therefore, their responses to deciding how to respond prompt were categorized as a robust level of evidence. Figure 4.50 presents the suggestion provided by pre-service teacher B in reflection paper 3.

The student had three misconceptions. The first one was the same area-same perimeter misconception. The student thought that the perimeters of the given square and rectangle shapes were the same by associating them with the area. In order to eliminate this misconception, I would give her string, and I would ask the student to measure the perimeter of both of the given shapes with the help of a string. When she sees that the perimeter of the rectangle is greater than that of the square, I would ask her, “Why do you think the perimeters are not the same?” Using the same pieces, I would also construct another shape in which the perimeter would be greater than the perimeter of the rectangle. I would ask the student to measure it with the string as well and compare the lengths of the string. She would see that even if the area was the same, the perimeter would not and infer that the shapes with the same area can have different perimeters.



The second misconception was associating area with the spread of pieces due to not knowing the principle of area conservation. For this, I would draw the shapes on grid paper and ask the student to count squares, or I would ask the student to cover the shapes with unit squares. For example, I would ask her to use all of the given tangram pieces to form a regular square (Figure 1) and cover it with unit squares. Then, I would ask her to rearrange this square to form an irregular shape (Figure 2) and cover it with unit squares. Then, I would ask her to rearrange this shape to form another irregular shape (Figure 3) and do the same thing for this shape. She will see that an equal number of unit squares are used in all cases. After doing this, I would ask, “Why do you think the area of all three shapes is the same?” In this way, she can realize that the area of the shapes is equal because the area is conserved, and the area is not related to how much the shapes are spread. Rather, the area is related to the number of unit squares required to cover the shapes.

Figure 1

Figure 2

Figure 3

The third misconception was that she did not include the newly formed sides in the perimeter. To eliminate it, I would want her to paint the boundaries that surround the shape. She would keep going until she got to where she started. It means that the newly formed sides are also included in the perimeter. Then she might say I should consider the interior sides, too. Hence, she would realize that she should consider all sides and therefore deduce that the perimeter increases when the piece was removed.

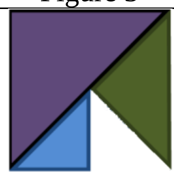


Figure 4. 50 Pre-service teacher B’s deciding how to respond based on the interviewed student’s understanding

As shown in Figure 4.50, in reflection paper 3, pre-service teacher B proposed three suggestions based on her interpretation of the student's understanding to eliminate the student's misconceptions she identified. To help the student overcome the same area-same perimeter misconception, pre-service teacher B suggested that the student measure the perimeter of square and rectangle shapes and another shape with the largest perimeter using a string and compare the lengths of the string. In this way, pre-service teacher B thought the student would realize the shapes necessarily do not have the same perimeter even though their areas are the same and they consisted of the same pieces. Moreover, to eliminate the student's misconception about associating area with the spread of pieces and to make the student understand the area conservation, pre-service teacher B suggested that the student cover the three shapes consisting of the same pieces with the unit squares. Thus, pre-service teacher B believed the student would realize that an equal number of unit squares were used in all three cases, and hence, the area is conserved no matter how much the spread of the shape is. Furthermore, to make the student realize the newly formed sides when the piece was removed from the shape, pre-service teacher B suggested that the student paint the boundaries that surround the shape. Accordingly, she considered that the student would recognize the interior sides and include them in the perimeter. The suggestion of pre-service teacher K in reflection paper 3 is presented in Figure 4.51.

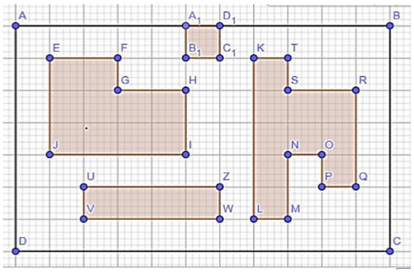
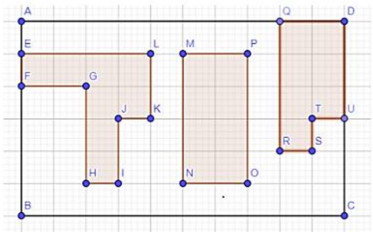
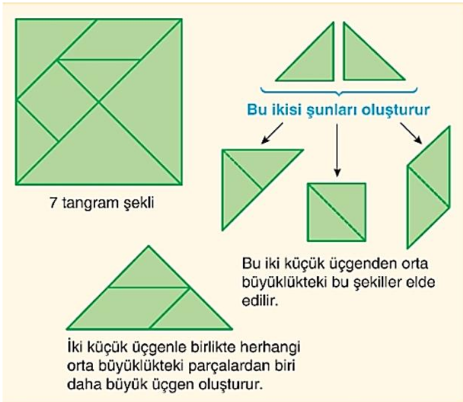
…Since the student does not know the definition of the perimeter, I would ask questions to make the student feel the importance and meaning of the perimeter. For example, I will plant trees at equal intervals on the sides of a small field, but I don't know how many trees are needed. What do you think I can do about this? Or I would create a conversation about the borders of a country. Why do you think borders are important to us? What do people do to measure this? I would ask such questions. Then, I would ask the student to compare the perimeters of the shapes on the geometry board by creating a square, rectangle and combination of these shapes or by drawing these shapes on grid paper and giving unit squares. Here, I don't aim for the student to give a numerical value but to understand the concept of perimeter. Then, I would ask the student to find the perimeter of these shapes with the help of unit squares. I would also ask the difference by asking the student to do the same thing with rectangles once again. Here, I would draw attention to the fact that the units used in calculating the perimeter should have equal side lengths. Thus, the student would have realized the importance of using unit squares. I would then ask the student to measure the sides of the shape with a ruler and add them up so that the student grasps the idea of the standard unit of measurement. Thus, the use of equal units of measurement is emphasized once again.	
For area, I would first ensure that the concept is understood by showing its importance in daily life because the student doesn't know what area means. For this, I would give examples of covering or painting a surface. For example, when we want to cover a table or a book, which one needs more material? Then, I would ask the student to calculate the area by placing squares on the given shapes. Here, I would make him pay attention to the side lengths of the squares in terms of perimeter and the number of squares in terms of area...Then, I would ask the student to find the areas of squares, rectangles and other shapes as in the perimeter calculation.	

Figure 4. 51 Pre-service teacher K's deciding how to respond based on the interviewed student's understanding

As can be seen from the figure above, pre-service teacher K offered some alternative teaching approaches based on her interpretation of the student's understanding. She first aimed to make the student understand what perimeter is. For this, she suggested asking several questions about the perimeter to help the student grasp the meaning of the perimeter. Then, she stated that she would want the student to compare the perimeters of the square, rectangle and combination of these on the geometry board or on grid paper by using unit squares. She would also want the student to do the same thing by using rectangles this time. In this way, she aimed to help the student realize the importance of using units with equal side lengths. Lastly, by asking the student to measure the perimeter with a ruler, she would aim to give the student an idea of the standard unit of measurement. Furthermore, for the area, she first suggested that the student comprehend the area concept by giving examples related to covering or painting a surface. Then, she would ask the student to calculate the area by placing squares on the given shapes. Here, she wanted the student to measure the area of the shapes by using unit squares and also to measure the perimeter of the shapes. In this way, she wanted the student to understand that area is related to the number of squares and perimeter is related to the side lengths of the squares. Figure 4.52 shows the suggestion of pre-service teacher S in reflection paper 3.

The student was unable to establish a relationship between the areas of the shapes and the area of the small triangle and could not express the areas of the shapes in terms of the area of the small triangle. To eliminate this misconception, questions can be redirected to the student by giving numerical values based on the student’s thinking. For example, the area of a small triangle can be given as 10 square centimetres. In this way, the student can realize that the area of the square, parallelogram and medium triangle is 20 square centimetres since the numerical value is given this time (the student can easily calculate the area of the shapes with the notes she recorded while creating these shapes). This is followed by the question, “How many small triangles did you use to create this shape?” and the student is expected to answer two…Then, the large triangle is formed, and since she knows that it is formed from four small triangles, she can calculate its area as the area of four small triangles, i.e., 40 square centimetres...It is then asked how these calculations relate to the area of the shapes when a numerical value is not given... What would the area of this shape be if I had not given it numerical values?... In this way, the student is made to realize that the area is a covering and how many objects cover the surface. In this way, the student can say the area of the shape as the number of objects covered, even if no numerical value is given...



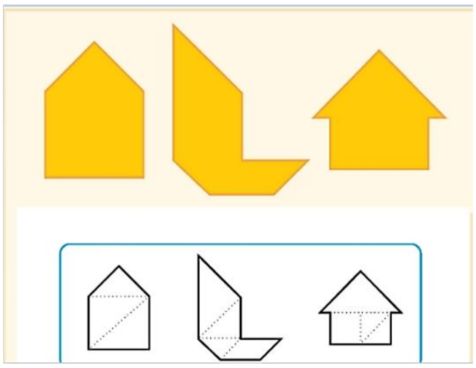


Figure 4. 52 Pre-service teacher S’s deciding how to respond based on the interviewed student’s understanding

From the figure above we can see that the aim of the suggestion of pre-service teacher S was to make the student realize the relationship between the area of the tangram pieces. To do this, she suggested providing the area of the small triangle as numerical. She believed that based on the area of the small triangle, the student could determine the area of a square, parallelogram and medium first and then the area of a large triangle and finally the area of the given shapes, which are the combination of different tangram pieces. In this way, the pre-service teacher thought that the student could realize that the area of the shape is the sum of the area of the pieces that constituted that shape.

In summary, all three pre-service teachers (pre-service teacher B, pre-service teacher K and pre-service teacher S) who designed tasks on perimeter-area measurement, provided specific instructional suggestions based on the students' misconceptions and difficulties they identified in order to eliminate them and help the students understand, by providing rationale and details in reflection paper 3. Therefore, their responses to the deciding how to respond prompt in reflection paper 3 showed a robust level of evidence.

4.2.2 The influence of pedagogies of practice on the development of pre-service teachers' professional noticing skills in the context of volume-surface area measurement

The aim of the second research question was to examine how a video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement. In the above part, pre-service teachers' attending, interpreting and deciding how to respond skills at the representation-decomposition of practice and approximation of practice stages regarding perimeter-area measurement were presented in order to show how a video-based module situated in the pedagogies of practice framework support pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement. In this part, pre-service teachers' attending, interpreting and deciding how to respond skills at the representation-decomposition of practice and approximation of practice stages regarding volume-surface area measurement are presented in order to show how a video-based module situated in the pedagogies of practice framework support pre-service teachers' professional noticing of students' mathematical thinking in volume-surface area measurement. This part provides the findings for six pre-service teachers' responses to the attending, interpreting and deciding how to respond prompts in the reflection papers in the context of volume-surface area measurement under the headings of representation-decomposition of practice and approximation of practice, supported by semi-structured interviews and discussions.

4.2.2.1 The influence of representation-decomposition of practice on the development of pre-service teachers' professional noticing skills in the context of volume-surface area measurement

This part provides pre-service teachers' attending, interpreting and deciding how to respond skills at the representation-decomposition of practice stage of pedagogies of practice in volume-surface area measurement to demonstrate how a video-based module situated in the pedagogies of practice framework support pre-service teachers' professional noticing of students' mathematical thinking. The video clips in the fourth, fifth, sixth and seventh sessions were related to volume and surface area measurement. The pre-service teachers viewed eleven video clips as representations of practice during the four sessions (Table 4.20).

Table 4. 20 Video clips and their contents in the context of volume-surface area measurement

Volume and surface area measurement			
Sessions	Video Clips	Students in the video clips	Content
Session 4	VC9 VC10 VC11	S1 S2 S8	Comparing the volume of two prisms
Session 5	VC12 VC13 VC14	S1 S2 S8	Enumeration of cubes to measure the volume
Session 6	VC15 VC16	S5 S3	A fixed volume-changing surface area situation
Session 7	VC17 VC18 VC19	S7 S5 S9	Change in surface area

As decompositions of practice, they individually analyzed the video clips after viewing them and wrote reflection paper 1. Then, they discussed the students' mathematical thinking in the video clips in terms of the components of attending, interpreting, and deciding how to respond and wrote reflection paper 2 after the discussions. At the end of the seven sessions, semi-structured interviews were conducted regarding the reflection papers as a formative assessment to uncover the change in their responses to the noticing prompts and understand the development in the pre-service teachers' professional noticing skills.

4.2.2.1.1 Attending to the video clips in the representation-decomposition of practice in the context of volume-surface area measurement

This part presents the extent to which pre-service teachers attend to the mathematical details in the video clips in the representation-decomposition of practice stage in the context of volume-surface area measurement. Table 4.21 shows the pre-service teachers' attending skills before the discussions (reflection paper 1) and after the discussions (reflection paper 2) in the four sessions on volume and surface area measurement.

Table 4. 21 Pre-service teachers' attending skills in individual video analysis in the context of volume and surface area measurement

	4th session		5th session		6th session		7th session	
	BD	AD	BD	AD	BD	AD	BD	AD
A	Substantial	Substantial	Substantial	Substantial	Robust	Robust	Substantial	Substantial
B	Substantial	Robust	Substantial	Robust	Substantial	Robust	Substantial	Substantial
E	Substantial	Substantial	Substantial	Substantial	Robust	Robust	Robust	Robust
G	Substantial	Substantial	Limited	Substantial	Substantial	Substantial	Substantial	Substantial
K	Lack	Robust	Substantial	Substantial	Substantial	Substantial	Substantial	Robust
S	Robust	Robust	Limited	Substantial	Robust	Robust	Substantial	Substantial

BD: Before the discussions AD: After the discussions

In the fourth session, which focuses on comparing the volume of two prisms, S1 in VC9 claimed that both prisms get an equal amount of objects because they were obtained using the same paper. For the prisms obtained using the larger paper, the student asserted that these prisms would take more objects than the previous ones because of the larger size of the paper. S2 in VC10, firstly, thought that the short and wide prism held more objects, i.e., it had a larger volume because it was wider than the other prism. Yet, then, the student changed his mind and answered in the same way as S1. S8 in VC11 believed that the tall and narrow prism had a larger volume because it was higher. For the prisms made of A3 paper, the student thought that the volume would be larger because of the larger size of the paper. While putting the prisms in order from largest to smallest volume, S8 considered both size of the papers and height of the prisms.

As shown in Table 4.21, pre-service teacher S provided the highest level of evidence for attending, i.e., robust evidence, in the group when the pre-service teachers individually analyzed the video clips. This pre-service teacher identified all mathematically significant details in all three students' solutions, which was followed by four pre-service teachers (pre-service teacher A, pre-service teacher B, pre-service teacher E and pre-service teacher G) who provided substantial evidence. On the other hand, pre-service teacher K provided a lack of evidence; she identified none of the mathematically significant details in the three students' solutions. Two pre-service teachers, pre-service teacher B and pre-service teacher K, increased their levels of attending to students' mathematical thinking through the discussions by providing higher levels of evidence in reflection paper 2 than in reflection paper 1. Pre-service teacher E, pre-service teacher G and pre-service teacher S, who already provided a high level of evidence, i.e., substantial and robust evidence in the individual analysis, did not add any explanations regarding what they attended to in reflection paper 2. After the discussions, pre-service teacher A identified all mathematically significant details in S2's solution in VC10, but this was not enough for him to move to the upper level. To illustrate the improvement in the quality of the attending responses, pre-service teacher K's written explanations in reflection paper 1 and reflection paper 2, supported by the discussions, are provided below:

> The student approached the task qualitatively, not quantitatively, because she did not give reasons in her answers to the questions. (Pre-service teacher K, Session 4, VC9, Reflection paper 1)
>
> The student answered by just looking at it, which one would get more without explaining why. He changed his answers frequently because he gave his answers without a reason. (Pre-service teacher K, Session 4, VC10, Reflection paper 1)
>
> The student knows that there may be differences in the subject. She says that the volume depends on how the prism is folded, but she is wrong when it comes to practice. (Pre-service teacher K, Session 4, VC11, Reflection paper 1)

In reflection paper 1, pre-service teacher K provided a general description for each student's solution because she described the solutions without relating the volume concept. Hence, she could not identify mathematically significant details in the students' solutions. Whole-class discussion around VC9 provided an opportunity for the pre-service teacher to notice the mathematically significant details in S1 as follows:

> P30: She said their volumes are equal since the papers are the same. I thought he associated this with the area. It seemed to me that she thought like an area. P3: Yes, she associates it with the area. The areas of the papers are the same, but the volumes of the prisms are not the same. Since we use the same paper, the area is conserved, but the volume is not.
>
> Researcher: What about prisms made of A3 paper?
>
> P28: She said their volume is larger because the paper is bigger. Again, since the area of A3 paper is larger than A4 paper, she thinks their volumes are also larger.

P3: Since she perceives the area of A3 as larger, she thinks that the volume of the prism made of A3 is larger than A4, but since she thinks the same paper leads to the same volume, she considers A3s as equal among themselves.

(Whole-class discussion, Session 4, VC9)

The whole-class discussion enabled the pre-service teachers to become aware of the S1's associating volume with the area of the papers. In a similar vein, even though S2 had correct reasoning at the beginning, at the end of the interview, he started to think like the previous student and associated the volume of the prisms with the papers. Whole-class discussion around VC10 helped the pre-service teachers realize how S2's thinking changed during the interview as follows:

P30: He said orange would overflow more quickly when filled with rice because it is thin and long.

P13: He believes that the highest one will overflow faster because the base is narrow. His first thought is correct.

P21: Yes, first, he says that the volume of blue will be bigger because blue is wider. The volume of the other will be smaller because it is thinner. When asked if it depends on how the paper is folded, he said it doesn't. Then, he gets confused and says they are the same. He says, "I forgot they are made of the same paper." He falls into a similar thought again.

E: He first says that if the base is wider, it takes more rice. He says the wider one takes more rice because the base is wider. Then, he says, like S1, that it does not change according to how we fold the paper. If it is the same paper, it takes the same amount of rice.

P15: Here again, he may have associated it with the area. Then he says their volumes are the same because they are the same paper as S1.

(Whole-class discussion, Session 4, VC10)

During the whole-class discussion, the pre-service teachers realized that although, at the beginning, S2 provided the correct answer, then he responded in the same way S1 did. Moreover, the pre-service teachers had a chance to become aware of the mathematically significant details in S8's solution that they missed in their individual analysis through the whole-class discussion around VC11 as follows:

> B: She was the only one who stated that the volume would change according to the way of folding. The other two students said it did not depend. According to her, to obtain a prism with a larger volume, it is necessary to fold it from the long side to be higher.
>
> P13: She makes a direct relationship with height. She says that whichever one is higher, its volume is bigger. She thinks that height is the determining criterion.
>
> P10: She only associated the volume with height. She does not associate it with the base area.
>
> …
>
> P16: She said that the size of the paper changes the volume. She may have done something based on this. A3 is larger, so she thinks the volume of the prisms obtained from it will be larger.
>
> P23: At first, she says that the volume of the prism made of A3 is bigger than A4 because A3 is bigger. Then, she sorts prisms made of A3 according to their heights and A4s in the same way.
>
> P21: Yes, at first, she said that the volume prisms made of A3 are bigger because of the size. When comparing, she compares the long ones among themselves and the short ones among themselves. When comparing the long orange and white, the A4 takes less than the A3 because of its smaller size because it is a bigger piece of paper. Then, she also considered the height. Here, she does not sort them according to only height. It would be wrong to say that she only looks at the height.

P32: When comparing the prisms made of A3 and A4, she considered the areas of the paper. She considered the height while comparing the prisms made of the same paper. She compares the long ones within themselves and the short ones within themselves. She looks at the height when the dimensions are the same. When the sizes are different, she looks at both the prism's height and the size of the paper.

(Whole-class discussion, Session 4, VC11)

During the whole-class discussion around VC11, the pre-service teachers had the opportunity to realize S8' association between volume and height of prisms for prisms made of the same paper. As a result, discussions around the video clips in the fourth session contributed to pre-service teacher K because she provided what she noticed during the discussions after the discussions in reflection paper 2 as follows:

> The student approached the task intuitively and gave an incorrect answer. When comparing the prisms of the same paper, I became aware in both group and whole-class discussions that the student associated the volume with the area because of the size; hence, the papers' area was the same. When they were made of the same paper, their areas did not change while their volumes changed. I realized that the student's misconception was that when the area is conserved, the volume is also conserved. (Pre-service teacher K, Session 4, V9, Reflection paper 2)

> The student answered correctly initially because he said the blue prism would take more rice and have a larger volume. He stated that the reason was that it was wider. Then, he changed his mind, and as the first student, he thought that the volume would not change according to the folding of the paper and said that the volumes were equal. The discussion environment made me realize what I had not realized. I had never thought that the students' answers could be related to the area. (Pre-service teacher K, Session 4, VC10, Reflection paper 2)

> In the whole-class discussion, I recognized that the student gave an incorrect answer because she said that the orange prism has a larger volume because it is higher. Unlike other students, she stated that the prisms' volumes vary depending on how the paper is folded. For the volume of prisms made of A3 paper, she said that since the size of that paper is larger, the volumes of prisms made with this paper are larger than other prisms. (Pre-service teacher K, Session 4, VC11, Reflection paper 2)

Both group and whole class discussions enabled pre-service teacher K to recognize students' misconceptions. Pre-service teacher K's attending response in reflection paper 2 refers to the usefulness of the discussions, indicating that at the end of the fourth session, she became aware of all mathematically significant details in the students' solutions. In this way, she increased her level of attending from lack to robust.

In the fifth session, which focuses on enumeration of cubes to measure the volume, S1 in VC12 tried to fill the prisms by randomly throwing the unit cubes into the prisms without attempting to construct layers and iterate these layers. S2 in VC13 first tried to determine how many cubes could be stacked along the height of each prism. Then, the student identified the number of cubes fitting in one face of the prism and multiplied that number by the number of faces. S8 in VC14 built two prisms using unit cubes. The student first constructed the first layer, i.e., the base, and then, she stacked this layer along the height. While finding the total number of unit cubes, she first found the number of cubes in a vertical layer, and then she multiplied this number by the number of layers.

As shown in Table 4.21, four pre-service teachers (pre-service teacher A, pre-service teacher B, pre-service teacher E and pre-service teacher K) provided substantial evidence when the pre-service teachers individually analyzed the video clips. On the other hand, pre-service teacher G and pre-service teacher S provided limited evidence. Fortunately, these pre-service teachers provided higher levels of evidence in reflection paper 2 than in reflection paper 1. In this way, they increased their levels of attending to students' mathematical thinking from limited to substantial through the discussions. Pre-service teacher A could identify all mathematically significant details in S2's solution in VC13 after the discussions. However, this was not enough for him to move to the upper level. In addition, pre-service teacher E and pre-service teacher K, who already provided substantial evidence in the individual analysis, did not add any explanations regarding what they attended to in reflection paper 2. Among the two pre-service teachers who provided lower levels of evidence than others in the group, the comments of pre-service teacher S in reflection paper 1 and reflection paper 2, supported by the discussions below, illustrate the increase in the quality of the attending responses.

> The student correctly calculated the volume of the orange prism but the volume of the blue one incorrectly. He arranged the cubes to fill the prism, not randomly. He has no misconception in this regard. (Pre-service teacher S, Session 5, VC13, Reflection paper 1)
>
> She arranged the cubes by considering the shape's form and did not make a mistake in the calculation. (Pre-service teacher S, Session 5, VC14, Reflection paper 1)

In reflection paper 1, pre-service teacher S identified S2's solution in VC13 incorrectly. Moreover, the pre-service teacher provided a general description of S8's solution in VC14 without mentioning the student's strategy while measuring the volume of the prisms. The whole-class discussion around VC13 provided below helped pre-service teacher S to become aware of the student's strategy:

> P7: There are four faces. He says that 12 cubes will fit in total. In fact, there are nine unit cubes in the base, but he calculates the base as 12 unit cubes because he thinks differently.
>
> P8: He finds the result bigger. According to what he did, he counted the corners twice.
>
> P16: He thinks that two of the faces of the same unit cube are different unit cubes. He counts one cube twice. The two faces of the unit cube intersect

with the two faces of the prism when we place it. He thinks of them all separately.

P18: But he doesn't count the column in the middle inside.

P30: He counted the corners twice and obtained 32 from 8x4. He didn't count a column in the middle. He overcounted +24 from +32-8 if the height was right.

(Whole-class discussion, Session 5, VC13)

During the whole-class discussion, pre-service teacher S had a chance to recognize the student's incorrect strategy based on the faces of the prisms. Moreover, the whole-class discussion around VC14 enabled pre-service teacher S to become aware of the mathematically significant details in the student's solution as follows:

P10: She counts the ones in front of the blue three by three and finds 24. Then, she says there are also behind it and counts the ones behind it.

P32: There are 24 cubes and 24 cubes are iterated three times. So, she says three times 24.

P12: She calculates one block, then the other blocks. She says there are three blocks. So, three times 24.

P28: She actually divides it into layers. First, she finds how many unit cubes are in the layer. 24 is the number of unit cubes in a layer. Then, she finds how many times the layer is iterated. There are three layers. Then 24 times 3.

…

P9: For orange, she took 24 as a layer. Then she made 24 plus 24, that is, 24 times 2. Since there are two layers, she multiplied it.

P28: Here, she takes it as a vertical layer. As a horizontal layer, she first forms the base, including nine cubes, and there will be eight layers.

(Whole-class discussion, Session 5, VC14)

During the whole-class discussion around VC14, pre-service teacher S had the opportunity to realize the student's correct strategy based on layers to calculate the volume of the prisms. In this way, through the discussions, pre-service teacher S realized what she could not notice individually and corrected what she noticed wrongly in the individual analysis. Accordingly, pre-service teacher S reflected on the mathematically significant details in reflection paper 2 that she became aware of during the discussions after the discussions as follows:

> Initially, I thought the student had made a calculation error and found the wrong result. Yet, during the whole-class discussion, I realized that the student counted the columns in the corners two times and did not count the cubes in the middle. Thus, I understood why he found different and bigger results. (Pre-service teacher S, Session 5, VC13, Reflection paper 2)

> In the whole-class discussion, I noticed that the student made the volume calculation over the layers. She divides the cubes into layers. Then, she finds how many of these layers are and multiplies the number of layers by the number of cubes in the layer. (Pre-service teacher S, Session 5, VC14, Reflection paper 2)

At the end of the fifth session, pre-service teacher S focused on the mathematically significant details in the students' solutions by referring to the usefulness of the discussions. Thus, by identifying the mathematically significant details in the three students' solutions to some extent, pre-service teacher S could increase her level of attending from limited to substantial.

In the sixth session, which focuses on a fixed volume-changing surface area situation, S5 in VC15 could build two different prisms using 12 unit cubes. For the volume of these prisms, the student multiplied three numbers corresponding to the dimensions of the prisms, i.e., length, width, and height. For the surface area of the prisms, S5 added the area of two bases and four lateral faces and got the correct result. The student built the prism with the smallest surface area with three units in length, two units in width, and four units in height using 24 unit cubes. S3 in VC16 built all four possible prisms using 12 unit cubes. For the volume of the prisms, the student correctly reasoned that all prisms had the same volume because of including the same number of unit cubes. For the surface area of the prisms, S3 made judgments regarding the area of a face that touched the floor. While building a prism with the smallest surface area using 24 unit cubes, the student stacked the cubes one by one along the height of 24 units.

As shown in Table 4.21, all pre-service teachers provided a high level of evidence for attending, i.e., substantial and robust evidence, in the group when they individually analyzed the video clips. Three pre-service teachers' (pre-service teacher A, pre-service teacher E and pre-service teacher S) explanations showed a robust level of evidence, whereas the explanations of the other three (pre-service teacher B, pre-service teacher G and pre-service teacher K) demonstrated a substantial level of evidence. Only pre-service teacher B increased her level of attending to students' mathematical thinking through the discussions by providing robust evidence in reflection paper 2. Other pre-service teachers did not add any explanations regarding what they attended to in reflection paper 2. Pre-service teacher B's written explanations in reflection paper 1 and reflection paper 2, supported by the discussions, are presented below to show the improvement in the quality of her attending responses:

> He correctly formed different prisms with the given unit cubes. He said that the volume of all of them is equal because they consist of the same number of unit cubes. He considers the surface area as the base area and even the face that touches the floor (Pre-service teacher B, Session 6, VC16, Reflection paper 1)

In reflection paper 1, pre-service teacher B identified mathematically significant details in S3's solution in VC16 to some extent. Discussions provided her the opportunity to notice other details in addition to what she already noticed, as follows:

> P7: S3 created more prisms than the other student. He could build all four prisms. 3x2x2, 1x1x12, 2x1x6 and 3x1x4.

…

P21: He arranged 12 unit cubes on top of each other. He said it was a prism when it was upright, but when the same prism was laid on its side and placed on the ground on different sides, he said it was not a prism. Because he said, we're lining them up side by side.

…

P28: As a surface area, he thinks of the number of squares on the face where it touches the ground. He said 3x2x2 is the largest because there are six squares on the floor. He said 1x1x12 is the smallest, whereas actually, it is the opposite.

K: Among the other prisms, one had three squares, one had two and one had 1 square touching the floor. He sorts them accordingly.

…

P13: To build a prism with the smallest surface area, he stacked 24 cubes on top of each other and obtained 1x1x24. One cube at the base. However, this is the prism with the largest surface area.

P30: In this case, only one unit square touches the ground. Since it occupies one unit square, he formed that prism, believing it had the smallest surface area.

(Whole-class discussion, Session 6, VC16)

During the whole-class discussion, the pre-service teachers mentioned the prisms the student built using 12 cubes, how he found the surface area of the prisms, and how and why he built the prism with the smallest surface area using 24 cubes. These helped pre-service teacher B to become aware of the mathematically significant details she missed in the individual analysis because she could provide these in reflection paper 2 after the discussions as follows:

The student said that when the prism is turned on its side, that is, when its position is changed, it is still the same prism and that we could build at most four prisms with 12 cubes. Yet, during the discussions, I realized that when

> the prism formed by stacking 12 cubes on top of each other is turned on its side, he believes it will no longer be a prism. He thought that the cubes stood separately when stacked side by side. Additionally, he considers the surface area as the area of the face touching the floor. He believes the surface area would change when the prism turned on its side. With 24 cubes, he stacked the cubes on top of each other to have the smallest surface area since in this case one unit square touches the floor. (Pre-service teacher B, Session 6, VC16, Reflection paper 2)

The comment of pre-service teacher B in reflection paper 2 shows that at the end of the sixth session, pre-service teacher B was aware of all mathematically significant details in S3's solution. Thus, pre-service teacher B increased her level of attending from substantial to robust.

In the last session, i.e., the seventh session, which focuses on change in surface area, S7 in VC17 specified the surface area of the cube as 9 unit cubes. The student only considered the area of one face, the top face of the large cube, rather than all faces. For the change in the surface area, S7 thought when one cube was removed, the surface decreased by one, and when two cubes were removed, the surface area decreased by two. The student claimed the surface area would always decrease when the cubes were removed. S5 in VC18 correctly computed the surface area of the large cube as 54 unit cubes. For the change in the surface area, the student thought the surface area would decrease by the number of visible faces of the removed cubes. S5 maintained that the surface area would always decrease when the cubes were removed. S9 in VC19 correctly found the surface area as 54 unit cubes. In each case, the student considered how many faces contributed to the surface area before removal and how many new faces appeared after removal. In this way, S9 correctly determined the surface area either increased or did not change for each case. To reduce the surface area, the student focused on the corners and removed three cubes from the top right part of the large cube.

As shown in Table 4.21, all pre-service teachers provided high levels of evidence while individually attending to students' mathematical thinking. Pre-service teacher

E provided a robust level of evidence, and others provided a substantial level of evidence. Only pre-service teacher K increased her level of attending to students' mathematical thinking through the discussions since she provided robust evidence in reflection paper 2. Other pre-service teachers did not add any explanations regarding what they attended to in reflection paper 2. Pre-service teacher K's written explanations in reflection paper 1 and reflection paper 2, supported by the discussions, are as follows:

> While solving the task, the student correctly stated that the volume was 27 cubic units since the large cube consisted of 27 unit cubes. He did not refer to the formula. Regarding the surface area, the student had a misconception because he took a single face as the surface area. He also thinks the surface area always decreases when the cubes are removed. (Pre-service teacher K, Session 7, VC17, Reflection paper 1)

In individual analysis, pre-service teacher K identified mathematically significant details in S7's solution in VC17 to some extent. Both group and whole-class discussions around VC17 enabled the pre-service teacher to become aware of the details she could not realize. The following excerpt from the group discussion shows how the group members discussed the details in the student's solution:

> E: He calculated the volume correctly. He found the surface area to be nine cubic units. He only calculated one face.
>
> S: He says unit cube. I think he associates the surface area with the volume. He should say unit square.
>
> E: Because he says that when you remove one cube, you also decrease its surface area by one.
>
> Others: Yes (Agreeing by nodding heads)
>
> B: He says 1 unit cube is lost. There is actually a misconception there. He says that if we remove only the cube from the center inside, the surface area does not change, but in all other cases, it changes.

G: He even says it decreases.

B: Yes, he says it decreases. There is a contradiction here, actually.

A: Because he thinks in terms of volume.

K: But when he removes the cubes, the volume will decrease again, which he realizes.

A: He said the surface area doesn't change when we remove the cube from the center.

G: In this case, he believes the appearance does not change.

(Group discussion, Session 7, VC17)

During the group discussion, pre-service teacher K had a chance to fully realize the student's misconception regarding the surface area, which in turn resulted in the improvement in the quality of her attending response after the discussions in reflection paper 2 as follows:

> I had not realized the units expressed by the student (cubic units for surface area). The discussions helped me to understand the misconception about the volume and surface area relationship that the student had. Moreover, I had not realized that the student said the surface area stays the same if a unit cube is taken from the inside before the discussions. From here, I understood that the student focused on the appearance of the object. (Pre-service teacher K, Session 7, VC17, Reflection paper 2)

At the end of the seventh session, pre-service teacher K was able to attend to all mathematically significant details in the student's solution. Thus, she increased her level of attending from substantial to robust.

Overall, the findings indicate that improvement was observe in two pre-service teachers' attending skills in the context of volume-surface area measurement. Pre-service teacher K, who provided lack of evidence in the individual analysis at the beginning of the fourth session, provided substantial evidence for attending in the individual analysis in the seventh session. Furthermore, pre-service teacher E, who already provided substantial evidence, in his individual analysis at the beginning of the fourth session, improved his attending skill even more and provided robust evidence in the individual analysis in the seventh session. The other pre-service teachers who provided substantial or robust evidence in their individual analysis at the beginning of the fourth session continued to do so until the end of the seventh session, generally providing the same level of evidence.

The following graphs (Figure 4.53) show the pre-service teachers' individual attending skills throughout the sessions.

Figure 4. 53 Pre-service teachers' levels of attending to the mathematical details in the video clips

From the graphs we can see that in the early sessions, the attending skills of pre-service teacher A, pre-service teacher K and pre-service teacher G were low. Pre-service teacher A starting from the fourth session and pre-service teacher K starting from the fifth session could provide high levels of evidence for attending in their individual analysis, and they were able to sustain it. Pre-service teacher G was able to provide high levels of evidence starting from the second session. Even though she provided limited evidence in the fifth session, she maintained the high levels in the other sessions. Pre-service teacher G provided a lack of evidence only in the first session, and she did not provide such kind of evidence in the other sessions. Other pre-service teachers were in a better position in the early sessions. Pre-service teacher B was able to provide high levels of evidence starting from the third session, and she sustained it since she provided substantial evidence in the next sessions. Pre-service teacher E and pre-service teacher S were the pre-service teachers who provided the highest levels of evidence for attending because they could provide high levels of evidence from the first session. Moreover, there was a decrease in pre-service teacher K, pre-service teacher G and pre-service teacher E's levels of attending in the fourth session when the contents of the video clips changed from perimeter and area to the volume measurement. However, they were able to provide high levels of evidence in the late sessions.

4.2.2.1.2 Interpreting students' understanding in the video clips in the representation-decomposition of practice in the context of volume-surface area measurement

This part presents how pre-service teachers interpret students' understanding in the video clips in the representation-decomposition of practice stage in the context of volume-surface area measurement. Table 4.22 provides an overview of the pre-service teachers' interpreting skills before the discussions (reflection paper 1) and after the discussions (reflection paper 2) in the four sessions related to volume and surface area measurement.

Table 4. 22 Pre-service teachers' interpreting skills in individual video analysis in the context of volume and surface area measurement

	4th session		5th session		6th session		7th session	
	BD	AD	BD	AD	BD	AD	BD	AD
A	Limited	Limited	Lack	Substantial	Robust	Robust	Substantial	Substantial
B	Substantial	Robust	Limited	Substantial	Substantial	Robust	Limited	Limited
E	Substantial	Robust	Substantial	Substantial	Substantial	Robust	Substantial	Substantial
G	Substantial	Substantial	Limited	Substantial	Limited	Robust	Limited	Substantial
K	Lack	Limited	Lack	Limited	Substantial	Substantial	Limited	Substantial
S	Limited	Limited	Limited	Robust	Robust	Robust	Substantial	Substantial

BD: Before the Discussion AD: After the Discussion

In the fourth session, which focuses on comparing the volume of two prisms, as can be seen from the data in Table 4.22, pre-service teacher B, pre-service teacher E and pre-service teacher G provided substantial evidence in the individual analysis, that is, higher levels of evidence than the others in the group. Other pre-service teachers provided low levels of evidence for interpreting, i.e., lack or limited evidence. Pre-service teacher K failed to provide a valid justification for any student's understanding. Pre-service teacher A and pre-service teacher S provided a valid justification for two of the three students' understanding to some extent. After the discussions, pre-service teacher A, pre-service teacher G and pre-service teacher S did not add any explanations regarding students' understanding in reflection paper 2. Hence, they could not increase their attending levels at the end of the sessions. The explanations of pre-service teacher B, pre-service teacher E and pre-service teacher K showed higher levels of evidence in reflection paper 2 than in reflection paper 1. Therefore, they increased their levels of interpreting students' understanding through the discussions. Explanations of pre-service teacher E in reflection paper 1 and reflection paper 2 are given below to present the improvement in the quality of the responses after the discussions.

> The student does not know the volume conceptually. The fact that she thinks it will have the same volume when made of the same paper may be due to the area-volume misconception. (Pre-service teacher E, Session 4, VC9, Reflection paper 1)

> S2 fails to realize that volume is related to both height and base area at the same time. Like S1, his saying that the same papers have the same volume indicates that there may be an area-volume misconception. (Pre-service teacher E, Session 4, VC10, Reflection paper 1)

In the individual analysis in reflection paper 1, pre-service teacher E provided a valid justification for both students' understanding to some extent. His comment involved the area concept as a reason for the students' misconceptions, but he did not provide details. The following excerpts from the whole-class discussions around VC9 helped pre-service teacher E elaborate his interpretations of students' understanding.

> P30: She said their volumes are equal since the papers are the same. I thought she associated this with the area. It seemed to me that she thought like an area.
>
> P3: Yes, she associates it with the area. The areas of the papers are the same, but the volumes of the prisms are not the same. Since we use the same paper, the area is conserved, but the volume is not.
>
> …
>
> P30: She has an incorrect understanding because she generalizes that they have the same volume if they have the same area.
>
> …
>
> (Whole-class discussion, Session 4, VC9)

The pre-service teachers discussed the reasoning behind the students' solutions during the whole-class discussion. In this way, pre-service teacher E elaborated his interpretation by providing details about the students' understanding regarding the area-volume relationship in reflection paper 2 as follows:

> I realized that the students answered this way based on the idea that if the area is the same, the volume is the same. The area is conserved, but they are unaware that the volume is not conserved. I understood in the whole-class discussion why the students stated that the volume would not change regardless of how the paper is folded because they associate it with the concept of area. (Pre-service teacher E, Session 4, VC9-10, Reflection paper 2)

As seen from the comment of the pre-service teacher in reflection paper 2, by elaborating on his interpretations of students' understanding of area-volume relationship, pre-service teacher E increased her level of interpreting from substantial to robust at the end of the fourth session.

In the fifth session, which focuses on enumeration of cubes to measure the volume, as shown in Table 4.22, pre-service teacher E was the one who provided the highest level of evidence, i.e., substantial evidence, in the individual analysis. Other pre-service teachers provided either lack or limited evidence. Pre-service teacher A and pre-service teacher K failed to provide a valid justification for any student's understanding, and hence, their interpreting response showed a lack of evidence. Moreover, pre-service teacher B, pre-service teacher G and pre-service teacher S provided a valid justification for one or two of the three students' understanding to some extent, which resulted in a limited level of evidence. Pre-service teacher E, who already provided substantial evidence in reflection paper 1, did not add any explanations regarding students' understanding in reflection paper 2. The other five pre-service teachers increased their levels of interpreting through the discussions. As an example, pre-service teacher G's responses to the interpreting prompt in reflection paper 1, the whole-class discussion around VC14 and the addition she made in reflection paper 2 are given as follows:

> The student knows the components that make up the volume, but instead of using the formula, she preferred to arrange cubes until the whole prism was filled. It would have been sufficient to determine only the base and height. (Pre-service teacher G, Session 5, VC14, Reflection paper 1)

In individual analysis, pre-service teacher G focused on students' building of the whole prism with the unit cubes rather than the student's conceptualization of the set of cubes. In this sense, the whole-class discussion presented below enabled her to realize the reasoning behind the student's solution.

> P4: For blue, she first forms the base three by three and finds 9. She places it inside and checks it. Then she adds it on top of it along the height. The height is 8. She got it right.
>
> …
>
> P12: She calculates one block, then the other blocks. She says there are three blocks. So, three times 24.
>
> P28: She actually divides it into layers. First, she finds how many unit cubes are in the layer. 24 is the number of unit cubes in a layer. Then, she finds how many times the layer is iterated. There are three layers. Then 24 times 3.
>
> P25: In orange, she first finds the base. She puts four on the base. She says it takes four and iterates it again along the height of 12.
>
> P9: For orange, she took 24 as a layer. Then she made 24 plus 24, that is, 24 times 2. Since there are two layers, she multiplied it.
>
> …
>
> S: She arranged the cubes systematically and found the layers. She multiplied the number of unit cubes in the layer by the number of layers. She has no misconceptions regarding this. She proceeded with a correct strategy.

> P13: The student has a sufficient conceptual level. She understands that the volume is calculated as the iteration of the layers. She knows how to find the volume with unit cubes.
>
> (Whole-class discussion, Session 5, VC14)

The whole-class discussion focused on a layering strategy that the student used, which reflects on pre-service teacher G's comments in reflection paper 2 after the discussions as follows:

> During the discussions, I realized that the student found the volume by dividing the cubes into layers, and her understanding was based on a layering approach. (Pre-service teacher G, Session 5, VC14, Reflection paper 2)

As a result, at the end of the fifth session, pre-service teacher G increased her level of interpreting from limited to substantial.

In the sixth session, which focuses on a fixed volume-changing surface area situation, as shown in Table 4.22, five pre-service teachers, except pre-service teacher G, provided high levels of evidence, i.e., substantial or robust evidence in the individual analysis. Pre-service teacher G provided limited evidence by providing a valid justification for one of the two students' understanding to some extent. After the discussions, pre-service teacher K and pre-service teacher S did not add any explanations regarding students' understanding in reflection paper 2. Pre-service teacher A added explanations, but since his interpreting response was already categorized as robust evidence, he stayed at the same level after the discussions. The other three pre-service teachers (pre-service teacher B, pre-service teacher E and pre-service teacher G) increased their interpreting levels through the discussions and provided higher levels of evidence in reflection paper 2. To illustrate the improvement in the quality of interpreting responses, explanations of pre-service teacher G in reflection paper 1 and reflection paper 2, supported by the discussions, are given below.

> The student knows how to calculate the surface area, but his comprehension of this subject is insufficient, especially for subjects requiring thinking and

creating shapes in the mind. In the first question, while building different prisms, he built the first prisms that came to mind because he thought in a limited way. (Pre-service teacher G, Session 6, VC15, Reflection paper 1)

In individual analysis, pre-service teacher G made comments in broad terms, and her comments about student's understanding did not involve students' conceptualization of volume and surface area. The discussion environment provided pre-service teacher G with opportunities to focus on possible reasons and mathematical details as follows:

> P30: While finding the volume of the prisms, instead of counting the number of unit cubes, he calculated using the formula.
>
> P14: Yes, he knows how to use the formula for volume, but he doesn't know that the number of unit cubes gives the volume. There are 12 unit cubes. He does not say that the volume is 12 cubic units and the volume of two prisms is equal. He does not say I used 12 unit cubes when calculating the volume, so the volume is 12 cubic units. Rather, he multiplies the length, width and height of each prism.
>
> …
>
> B: He knows that prisms do not change according to the position. For the prism to change, its dimensions must change because he realized that the first two prisms he built were the same.
>
> G: The only thing that the student failed was building different prisms. He created two prisms with the dimensions of 3x2x2 and 6x2x1. Yet, four prisms can be built with 12 cubes.
>
> …
>
> G: The student justified the difference in the surface area of the two prisms by the fact one was more compact, and the other was more spread over.
>
> P13: He was aware that the surface area could increase or decrease. He said the surface area will be smaller if the prism is more compact.

…

P17: He says the prism should be more compact to get less surface area. He was trying to get closer to the cube. He chooses numbers closer to each other to be closer to the cube. But he expresses it as compact.

G: He thinks prisms need to be more spread over to have a larger surface area. In fact, the number of faces touching each other in common should be as small as possible to obtain a larger surface area.

P4: To make the surface area smaller, the faces must hide each other. That is, the faces must be in common. That's why he tried to make the prism more compact.

P21: When the prism is more compact, we hide the faces facing outwards inside. For example, there was that longest one. Four faces of the cubes appear when we stack 12 cubes on top of each other. When we make the prism more compact, those faces face inwards, and we intersect them. So they are not visible on the outside. For this reason, he built the prism with the dimensions of three units, four units and two units using 24 units cubes.

…

(Whole-class discussion, Session 6, VC15)

This dialogue contributed to pre-service teacher G's professional noticing in terms of interpreting the student's understanding. She mentioned these contributions in reflection paper 2 after the discussions as follows:

He knows that the volume of the prism can be found with width x length x height and that the surface area is the sum of the areas of the faces that make up the shape by adding the area of the bottom, top and lateral faces. Yet, he has difficulty in building prisms with 12 unit cubes. In this regard, I recognized during the discussions that he cannot think about what values he can find to obtain a prism of 12 unit cubes with width x length x height. He knew the surface area would decrease depending on the number of unit

> squares on the faces, so he tried to create more compact shapes. While finding the volumes, he does the operation separately for both prisms and cannot directly say that they have the same volume due to the same number of unit cubes, which shows that he cannot associate the volume with the number of unit cubes. (Pre-service teacher G, Session 6, VC15, Reflection paper 2)

The comment of pre-service teacher G after the discussions revealed that she noticed the underlying reasons for the student's difficulty in building different prisms and for the student's aim to build more compact prisms to obtain a smaller surface area. Pre-service teacher G also mentioned how the sessions helped her in terms of interpreting students' understanding during the interview as follows:

> I realized that the reason for similar misconceptions that have been going on for years can also be attributed to the same type of lectures of teachers based on rote memorization. In my opinion, before starting to teach any subject, there is a need to find out which mistakes students make in that subject, and the teaching process should be shaped accordingly. During the sessions, I have learnt that students can make mistakes on abstract concepts as a result of explanations made without using concrete materials. (Pre-service teacher G, Interview)

This excerpt highlights the pre-service teacher's insights into the root causes of misconceptions, emphasizing the role of teaching methods in their perpetuation. The pre-service teacher recognizes a potential link between longstanding misconceptions and the teaching methods employed by teachers. Specifically, the emphasis on rote memorization is stressed as a contributing factor. This suggests an awareness of the impact of teaching approaches on students' conceptual understanding.

In the seventh session, which focuses on change in surface area, as presented in Table 4.22, pre-service teacher A, pre-service teacher E and pre-service teacher S provided substantial evidence in the individual analysis, that is, higher levels of evidence than the others in the group. Other pre-service teachers, pre-service teacher B, pre-service teacher G and pre-service teacher K, provided limited evidence. They provided a valid justification for two of the three students' understanding to some extent. Only pre-service teacher G and pre-service teacher K added explanations regarding students' understanding in reflection paper 2, which resulted in an increase in their interpreting levels after the discussions. To illustrate, the explanations of pre-service teacher K in reflection paper 1 and reflection paper 2 with the discussions are provided below:

> The student takes a holistic approach to the surface area of the object without considering the new faces that appear when the cube is removed. Since he defends this idea in every situation, he fails to realize that the surface area may increase or not change. (Pre-service teacher K, Session 7, VC18, Reflection paper 1)

In individual analysis, pre-service teacher K provided a valid justification for the student's understanding of surface area to some extent. During the discussions, the pre-service teachers put forward different viewpoints about the student's understanding of the surface area as follows:

> G: He was not adding the new faces. He was only subtracting. For example, when he takes one cube from the corner, he realizes three faces appear but does not add the new faces.
>
> P7: That's why he says the surface area decreases every time the cubes are removed. He says it always decreases as he takes the faces that appear.
>
> P21: When he takes one from the corner, he realizes that there are three visible faces, but he thinks that when it goes away, all three faces will go away. He thinks that, in any case, the surface area decreases, and it cannot increase or remain unchanged.
>
> P13: He thinks about the faces in the first form. He cannot calculate the faces formed inside when we remove a cube. He is not aware of the new faces. He says 54, then when he removes it from the corner, he can see that three faces have disappeared but not the appeared faces. He does not see the new faces coming from the bottom; he sees the missing ones.

> G: He's taking the space that the cube occupies there. He thinks as if it's completely removed when it's removed. He doesn't look at the faces of those small cubes.
>
> S: It is prism filled inside because we work with unit cubes. The student only thinks about the outer surface when we remove the cubes. He does not take into account the faces formed. He only considers the removed faces; he does not see those faces formed again when we remove the cubes.
>
> A: It may also be due to prototype examples. In the lesson, surface area may have been constantly taught over regular shapes. He may have seen only the calculation in prototype shapes before.
>
> (Whole-class discussion, Session 7, VC18)

During the whole-class discussion, the pre-service teachers mentioned the possible reasons for the student's incorrect answer regarding the surface area that surface area always decreases when the cubes are removed. As a result, pre-service teacher K mentioned these in reflection paper 2 after the discussions.

> In group and whole-class discussions, I became aware that the student considers the surface area of regular shapes. This may be because the lesson may have always been on prototype shapes. Moreover, I noticed that the student focused on the space after the cube was removed, not on the faces for surface area (Pre-service teacher K, Session 7, VC18, Reflection paper 2)

Pre-service teacher K elaborated on her interpretations of S5's understanding in VC18 about surface area. Thus, at the end of the seventh session, pre-service teacher K increased her level of interpreting from limited to substantial. She also expressed her improvement in interpreting students' understanding and how the implementations in the sessions were useful in this regard as follows:

> When I think about myself, I know that there is a lot of difference between the beginning and now. When the student makes a mistake, it is not only wrong, but it can go to different places as to why he/she did wrong. I realized

> this more. Group discussion was also useful, but especially whole-class discussion was more effective. There were many different thoughts there. From now on, I think I will be able to recognize and interpret better if I come across different student solutions related to geometric measurement. (Pre-service teacher K, Interview)

This excerpt from the interview reveals the pre-service teacher's self-reflection on professional growth and the effectiveness of discussions in enhancing her ability to understand and interpret student mistakes. There is an acknowledgment that mistakes can have various underlying reasons. The pre-service teacher expresses confidence in her ability to recognize and interpret different student solutions, particularly those related to geometric measurement. This indicates a positive anticipation of future interactions with student work and a belief that her ability to understand and interpret different student solutions has improved.

In a similar vein, the explanation of pre-service teacher B also indicates how the discussion environment enabled her to become aware of the possible reasons behind the students' solutions:

> When I individually analyzed the video clips, I generally didn't provide justification for what I wrote; I did not explain how it happened and why. I knew that the student had a misconception, but I could not understand the reason behind it. I noticed underlying reasons for these misconceptions during the discussions. (Pre-service teacher B, Interview)

The pre-service teacher reflects on a gap in her initial analysis – the absence of justification for her observations. She notes a shift in her understanding during discussions, recognizing the underlying reasons for student misconceptions. This implies that the collaborative nature of discussions provided insights that might not have been apparent during individual analysis.

In summary, the findings indicate that three pre-service teachers improved their interpreting students' understanding in the context of volume-surface area measurement. Pre-service teacher K, who provided lack of evidence in the individual analysis at the beginning of the fourth session, provided limited evidence in her individual analysis in the seventh session. Two pre-service teachers, pre-service teacher A and pre-service teacher S, who provided limited evidence in their individual analysis at the beginning of the fourth session, improved their interpreting skills and provided substantial evidence in their individual analysis in the seventh session. Pre-service teacher B, who provided substantial evidence in her individual analysis at the beginning of the fourth session, provided either limited evidence or substantial evidence in the other sessions. Pre-service teacher E who provided substantial evidence at the beginning of the fourth session, continued to do so in the other sessions.

The following graphs in Figure 4.54 show the pre-service teachers' individual interpreting skills throughout the sessions.

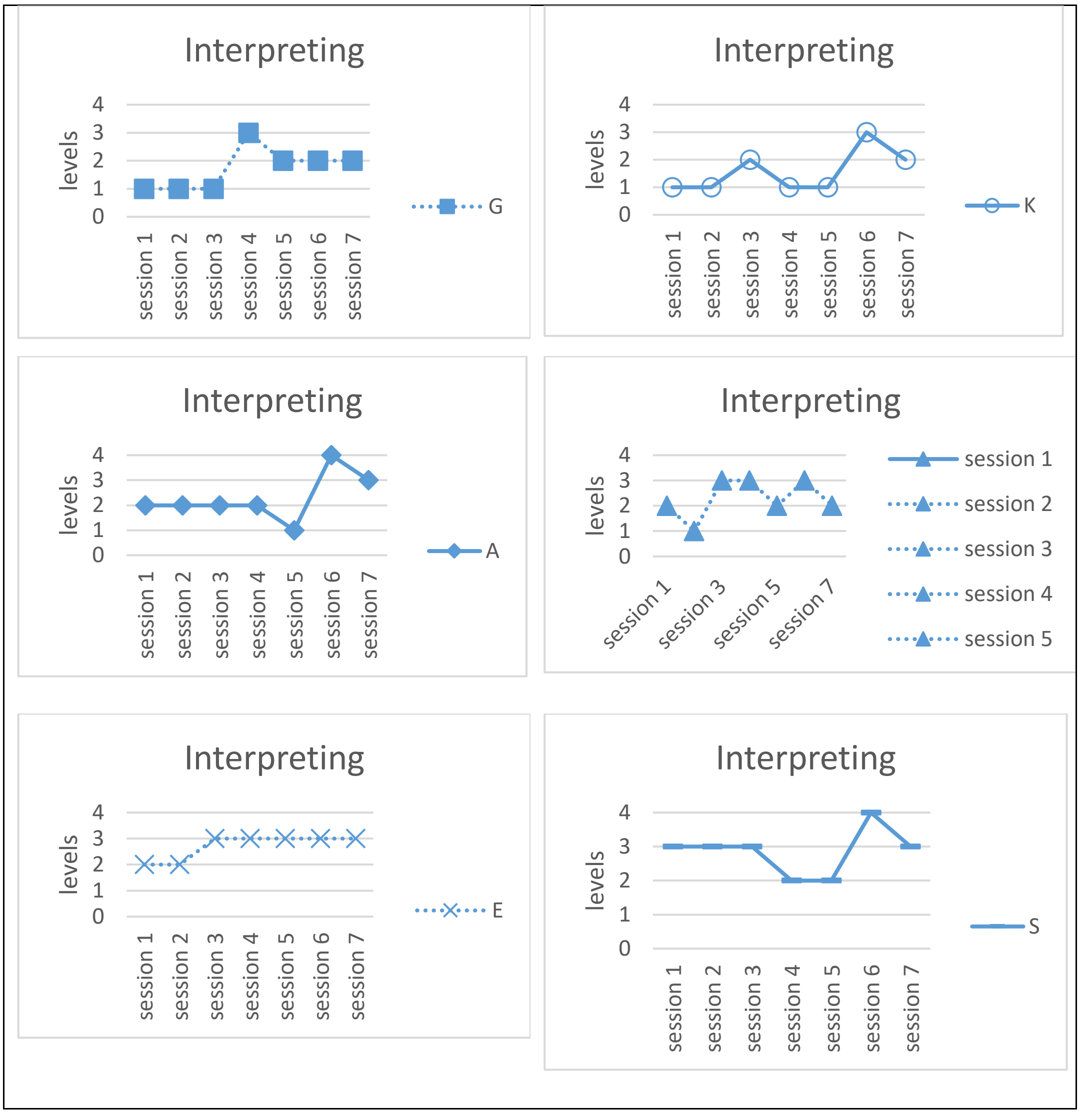


Figure 4. 54 Pre-service teachers' levels of interpreting students' understanding in the video clips

The interpreting skills of the pre-service teachers were lower than the attending skills in general. The interpreting skills of pre-service teacher A, pre-service teacher B, pre-service teacher G, pre-service teacher K and pre-service teacher S fluctuated according to the sessions, but the interpreting skill of pre-service teacher E was stable starting from the third session. Nonetheless, there was an improvement in the pre-service teachers' interpreting skills. Pre-service teacher G provided a lack of evidence in the early sessions, while she provided limited evidence in the late sessions. Pre-service teacher K provided a lack of evidence in the early sessions, whereas she provided limited and substantial evidence in the late sessions. Pre-service teacher B provided a lack of evidence and limited evidence in the early sessions, while she provided limited and substantial evidence in the late sessions. Pre-service teacher A provided limited evidence in the early sessions, but he could provide substantial and robust evidence in the late sessions. Pre-service teacher E provided limited evidence in the early sessions, whereas he provided substantial evidence from the third session. Pre-service teacher S provided substantial evidence in the early sessions. Although there was a decrease in the levels of evidence that she provided in the fourth and fifth sessions when the content of the session changed to volume measurement from perimeter and area measurement, she increased her level of interpreting again in the late sessions by providing substantial and robust evidence.

4.2.2.1.3 Deciding how to respond based on students' understanding in the video clips in the representation-decomposition of practice in the context of volume-surface area measurement

This part presents pre-service teachers' deciding how to respond based on students' understanding in the video clips in the representation-decomposition of practice stage in the context of volume-surface area measurement. The pre-service teachers' deciding how to respond skills before the discussions (reflection paper 1) and after the discussions (reflection paper 2) in the four sessions on volume and surface area measurement are set out in Table 4.23.

Table 4. 23 Pre-service teachers' deciding how to respond skills in individual video analysis in the context of volume and surface area measurement

	4th session		5th session		6th session		7th session	
	BD	AD	BD	AD	BD	AD	BD	AD
A	Limited	Medium	Medium	Medium	Substantial	Robust	Substantial	Robust
B	Substantial	Substantial	Robust	Robust	Substantial	Robust	Substantial	Robust
E	Limited	Medium	Medium	Robust	Substantial	Robust	Substantial	Substantial
G	Substantial	Substantial	Medium	Robust	Substantial	Robust	Robust	Robust
K	Medium	Substantial	Limited	Substantial	Robust	Robust	Substantial	Substantial
S	Limited	Medium	Substantial	Robust	Robust	Robust	Substantial	Robust

BD: Before the Discussion AD: After the Discussion

In the fourth session, which focuses on comparing the volume of two prisms, as shown in Table 4.23, pre-service teacher B and pre-service teacher G provided the highest level of evidence, i.e., substantial evidence, for deciding how to respond in the individual analysis of the video clips in the group. This was followed by pre-service teacher K, who provided medium evidence. The other three pre-service teachers, pre-service teacher A, pre-service teacher E and pre-service teacher S, provided limited evidence for deciding; their suggestions were insufficient to eliminate the students' misconceptions or extend their understanding. All pre-service teachers made additions in reflection paper 2. Four pre-service teachers increased their levels of deciding how to respond to students through the discussions by providing higher levels of evidence in reflection paper 2. Among the other two pre-service teachers, pre-service teacher B provided a detailed suggestion to make S8 in VC11 understand, and pre-service teacher G provided a detailed suggestion to make S1 in VC9 understand. Yet, these were not enough for them to move to the upper level, so they stayed at the same level. To illustrate the improvement in the quality of the deciding responses, the suggestions of pre-service teacher A in reflection paper 1 and reflection paper 2 are provided below.

> After filling the prisms with equal amounts of rice and letting the student see that their volumes are different, I would explain the subject of volume. (Pre-service teacher A, Session 4, VC10, Reflection paper 1)

In the individual analysis, the suggestion of pre-service teacher A was based on direct instruction, i.e., demonstrating the difference between volumes and explaining the subject. Discussion environment helped the pre-service teacher hear alternative instructional actions based on the student's understanding as follows:

> K: We can ask about rice filling. We take rice in two identical containers. We pour them both together. We ask questions such as why do you think this happened, what do you think it depends on, in which case their volumes would be equal?
>
> P13: The student was interpreting the volume according to the width at the beginning so we could show objects with the same width but different heights (blue and tall white prisms). I wonder if he interprets it only according to the width; does he consider the height? He thinks the tall one is narrower, but the tall one will have greater volume when the base areas are the same. However, he may think it has a smaller volume since it is tall. At that time, we can show that it has a larger volume. We put rice in the short one and ask the student to pour it into the other. The space remains in the taller one, but it is not filled. So, he will realize that the taller one has a larger volume when the width is the same because it is higher.
>
> P21: For the same base-different height and the same height-different base, we can move on to the virtual material after the concrete material and proceed through GeoGebra. For example, we create a prism there and add a slider. We change the base by keeping the height; it also calculates the volume there. We change the base from the slider; the height remains the same; we change the height, and the base remains the same. Thus, he can realize that the change

> in width affects the volume more than the change in height and that the wider prism of the same paper has a larger volume.
>
> (Whole-class discussion, Session 4, VC10)

In the whole-class discussion, the pre-service teachers focused on instructional actions to make the student realize that the volume of the prisms made of the same paper is not equal and that the width contributes to the volume more than the height. The instructional actions offered during the discussion are reflected in the suggestion of pre-service teacher A after the discussions as follows:

> Firstly, I give the student rice and ask him to fill the orange prism, and then I ask him to fill the blue prism with the same amount of rice. In this way, he can see that they do not take the same amount of rice, so their volumes are not the same. Then I give prisms with the same width but different heights and ask him to find the volumes. Then, I give prisms with different widths and the same height and ask the student to find their volumes. I ask, "What did you notice from this? What kind of relationship can there be between volume and height and volume and width? With these questions, I try to make the student realize that width affects volume more than height. Thus, I think I can eliminate the student's misconception with this suggestion offered in the whole-class discussion. (Pre-service teacher A, Session 4, VC10, Reflection paper 2)

The suggestion of pre-service teacher A in reflection paper 2 reveals that he could provide a detailed suggestion to make the student understand with the help of a whole-class discussion. In this way, he increased his level of deciding at the end of the fourth session.

In the fifth session, which focuses on enumeration of cubes to measure the volume, as shown in Table 4.23, pre-service teacher B provided robust evidence, the highest level of evidence for deciding how to respond in the group in the individual analysis of the video clips. Pre-service teacher S followed this by providing substantial evidence. Three pre-service teachers (pre-service teacher A, pre-service teacher E and pre-service teacher G) provided medium evidence, and only the deciding response of pre-service teacher K showed a limited level of evidence. Four pre-service teachers made additions in reflection paper 2. Pre-service teacher B, who already provided robust evidence in reflection paper 1, did not add any explanations regarding deciding how to respond to the students in reflection paper 2. All four pre-service teachers who made additions increased their levels of deciding how to respond to students through the discussions in reflection paper 2. Among them, the suggestions of pre-service teacher E, pre-service teacher G and pre-service teacher S showed robust evidence and those of pre-service teacher K showed substantial evidence. As an example of the improvement in the quality of the suggestions, pre-service teacher S's deciding responses in reflection paper 1 and reflection 2 are given below.

> I would explain to the student about volume and show how to calculate volume with unit cubes. (Pre-service teacher S, Session 5, VC13, Reflection paper 1)

Before the discussions, pre-service teacher S suggested direct instruction about explaining the subject and volume calculation. Fortunately, discussions enabled the pre-service teacher to recognize different alternative instructional moves suggested by others as follows:

> P28: He said three columns fit in each face of the prisms. We ask him to arrange the unit cubes and create that shape the way he thinks. Then, we ask him to pass the paper prism over the object he had built. In this way, we let him see whether it passes and how much more it is.
>
> P32: Yes, I thought similarly. I would ask him to create his object and put the blue prism on it. It would not fit into the blue prism. I would make him realize that he had counted the cubes twice by asking where the excess could come from. Then, I would ask him to do it again.
>
> P12: He found 108 cubes. When we tell him what the volume is and ask him to put them back in, he may realize that the cubes are too many.

> …
>
> P28: We can give directions to the student in the same way as S8 did. We ask him to form the base first. For example, he will put four cubes across the width. He will put the paper prism on top of it. He will look at it; it is too wide. He will remove it, add it again and complete it to 9. He will say okay, and that is it. We can ask the student to form the base first and complete the shape with the height in the same way. After building it, if we divide it into columns and make him count it, he may realize that the cube in the corners is the same.
>
> …
>
> (Whole-class discussion, Session 5, VC13)

The instructional actions offered focused on the comparison of the students' building and the actual prism influenced pre-service teacher S, which can be seen in the suggestion provided by her after the discussions in reflection paper 2 as follows:

> I would like the student to create what he thinks for all the faces, as suggested in the whole-class discussion. In this case, there will be a gap inside the object. I would ask questions such as whether it gives us the object's volume. I want him to understand that we need to fill the prism with unit cubes so there is no space inside. (Pre-service teacher S, Session 5, VC13, Reflection paper 2)

Thus, at the end of the fifth session, pre-service teacher S increased her level of deciding from substantial to robust by providing specific suggestions for three students and explaining them in detail.

In the sixth session, which focuses on a fixed volume-changing surface area situation, as shown in Table 4.23, two pre-service teachers (pre-service teacher K and pre-service teacher S) provided robust evidence, while the other four pre-service teachers provided substantial evidence. Four pre-service teachers made additions in reflection paper 2. Pre-service teacher K, who already provided robust evidence in reflection paper 1, did not add any explanations regarding deciding how to respond to the students in reflection paper 2. Only pre-service teacher S, who provided robust evidence in the individual analysis, offered another suggestion after the discussions as follows:

> I think the subject of volume can be associated with factors and multiples, as suggested in group and whole-class discussions. The multipliers of the volume should be the dimensions of the prism. The student may have a deficiency in associating multipliers with the volume. Different prisms can be given to the student, and he is asked to find width, length and height and to record. The student is made to realize that their product equals the volume. Then, he is asked if he thinks more prisms can be built. The student can be asked how many prisms can be built again. (Pre-service teacher S, Session 6, VC15, Reflection paper 2)

In this way, pre-service teacher S had a chance to become aware of the alternative instructional actions as a response to S5. All four pre-service teachers who provided substantial evidence in reflection paper 1 increased their levels of deciding how to respond to students through the discussions by providing robust evidence in reflection paper 2. Among these pre-service teachers, in the individual analysis, the suggestion of pre-service teacher E was based on extending the student's understanding of surface area as follows:

> The student can correctly find the surface area of prisms formed with unit cubes. In the next step, I would like the student to compare the surface areas of different geometric objects. (Pre-service teacher E, Session 6, VC15, Reflection paper 1)

In this explanation, pre-service teacher E provided a rationale for the suggestion, but he did not provide detail regarding the different geometric objects, which led to the categorization of the deciding response as a substantial level of evidence. After the discussions, the suggestion of pre-service teacher E was similar to the one of S, which shows the influence of the discussions around VC15 given below:

P11: The other student stacked 12 cubes on top of each other. However, he thought it was not a prism since the cubes were put side by side. We can show this to this student and ask, “Can such a prism be built?”. In this way, we can help him gain a different perspective.
P27: By stacking 12 cubes on top of each other, we can ask him if this is a prism. Because the student had not thought of this prism.

P31: The student can be given a prism consisting of 12 unit cubes different from the prisms built by the student and asked whether this can also be a prism. Then, he can be asked to build the rest himself.

P4: We want him to obtain different prisms so we can give him a feature. For example, we can ask, “Can you build a prism with the maximum height?” We can observe whether the student put all cubes on top of each other this time.

K: I would make him relate the factors and multiples while building prisms. He should pay attention to the fact that the dimensions of the prism should be factors of 12. For example, why couldn’t we make a prism with a base area of 5 square units? We look at the factors of 12 while building a prism. Therefore, we can make the student find the factors of 12 and then ask if other prisms can be built using these factors.

…

(Whole-class discussion, Session 6, VC15)

The first five suggestions given above were related to the student’s building of prisms. Then, the discussion was diverted to the student’s formula-based approach while finding the volume of the prisms and the suggestions were based on making the student realize the relationship between the volume and the number of unit cubes. Afterwards, the pre-service teachers offered suggestions to extend the student’s understanding of the surface area. Thus, during the whole-class discussion, different kinds of alternative instructional actions were suggested, and the pre-service teachers had an opportunity to recognize alternative decisions based on the student’s understanding. Particularly, the suggestion of pre-service teacher K, which was also offered in the group discussion, influenced the post-discussion decision of E as follows:

> As suggested during the discussions, activities can be included for the student to build different prisms based on the subject of factors and multiples by drawing attention to the factors of 12 while building the other prisms. It will be helpful to ask questions such as: What did you pay attention to while forming the bases? Why does the result come out the same when you multiply the dimensions? Why could we not build a prism with a dimension of 5 units? (Pre-service teacher E, Session 6, VC15, Reflection paper 2)

The instructional move provided by pre-service teacher E in reflection paper 2 shows that he could provide a detailed suggestion to make the student understand of the relationship between the number of unit cubes and the dimensions of prisms with the help of group and whole-class discussions. In this way, pre-service teacher E increased his level of deciding from substantial to robust at the end of the sixth session.

In the seventh session, which focuses on change in surface area, as shown in Table 4.23, one pre-service teacher, pre-service teacher G, provided robust evidence. The other five pre-service teachers provided substantial evidence. Three pre-service teachers (pre-service teacher A, pre-service teacher B and pre-service teacher S) made additions in reflection paper 2, and they all increased their levels of deciding how to respond to students through the discussions by providing robust evidence in reflection paper 2. To illustrate the improvement in the quality of the deciding responses, the suggestions provided by pre-service teacher A before and after the discussions are provided below:

> I would make the student think about the faces created by the removed cubes and make him realize that they are now the surface of the new object. (Pre-service teacher A, Session 7, VC18, Reflection paper 1)

In the individual analysis, pre-service teacher A did not explain how he would help the student recognize the new faces included in the surface area. The discussions around VC18 guide the pre-service teacher regarding how he will do this as follows:

> E: If we ask him to glue the whole object with sticky unit squares. That way, we can make him realize. He will cover all the visible faces.
>
> P8: Yes, you are right. We ask, how many unit squares did you use for the prism to cover the whole surface? He will use 54. Then, we will remove one unit cube from the corner, and if we ask him to cover the whole surface by using unit squares again, he will see that 54 unit squares are used again.
>
> P15: He was removing one unit cube, so there were 51 left, as he said. Before we remove it, we ask him to paint everywhere the same color. Then, when the cubes are removed, he may notice the unpainted part, maybe because it is a different color.
>
> P27: I would dip the cube in a paint can, and all faces are painted. I remove the cubes; he will see the unpainted faces. In this way, he will notice the exposed faces.
>
> …
>
> (Whole-class discussion, Session 7, VC18)

Particularly, the suggestions of pre-service teacher E and P27 influenced pre-service teacher A. The suggestion of pre-service teacher E was also discussed in the group discussion. Thus, pre-service teacher A reflected on these suggestions in the deciding response he provided after the discussions in reflection paper 2 as follows:

> During the discussions, I got some good ideas that we can do to eliminate the student's misconception. One of them is to have the student cover the faces of the cube that are included in the surface area with sticky unit squares and to make the student notice the new faces that appear after removing the unit cubes. The other is to make the student, who realizes that the surface area is covered after dipping the cube into a container full of paint, realize that the

> faces that appear after removing the unit cubes are also included in the surface area. (Pre-service teacher A, Session 7, VC18, Reflection paper 2)

After the discussions in reflection paper 2, pre-service teacher A provided a detailed suggestion to make the student understand with the help of group and whole-class discussions. Thus, pre-service teacher A increased his level of deciding from substantial to robust at the end of the seventh session. Pre-service teacher A's expression in the interview also reveals how the implementations in the sessions affected his ideas regarding deciding how to respond as follows:

> Mathematics teaching should never be superficial. I realized that the idea of "If students understand how the operation is done, they understand the subject" is very wrong. I realized that direct instruction to eliminate students' misconceptions is not so effective. I learned various methods from my friends about how we can eliminate students' misconceptions in discussions. I became aware that we need to get rid of traditional teaching methods and use different techniques. (Pre-service teacher A, Interview)

This excerpt reveals the pre-service teacher's evolving perspective on mathematics teaching. He questions the efficacy of direct instruction for misconception elimination and advocates for a departure from traditional teaching methods in favor of more diverse and effective techniques. The pre-service teacher acknowledges learning various methods from his peers, particularly in the context of eliminating student misconceptions through discussions. This highlights the value of collaborative learning and the exchange of teaching strategies among peers.

In summary, the findings indicate that five pre-service teachers improved their deciding how to respond skills in the context of volume-surface area measurement. Two pre-service teachers, pre-service teacher A and pre-service teacher E, who provided limited evidence in their individual analysis at the beginning of the fourth session, provided substantial evidence for deciding how to respond in the sixth and seventh session. Pre-service teacher S, who provided limited evidence in her individual analysis at the beginning of the fourth session, was able to provide substantial or robust evidence in the other sessions. Pre-service teacher K, who provided medium evidence in her individual analysis at the beginning of the fourth session improved her attending skills and provided substantial evidence or robust evidence in the sixth and seventh sessions. Similarly, pre-service teacher G, who already provided substantial evidence in her individual analysis at the beginning of the fourth session, provided robust evidence for deciding in the sixth and seventh sessions when she individually analyzed the video clips.

The following graphs in Figure 4.55 show the pre-service teachers' individual deciding how to respond skills throughout the sessions.

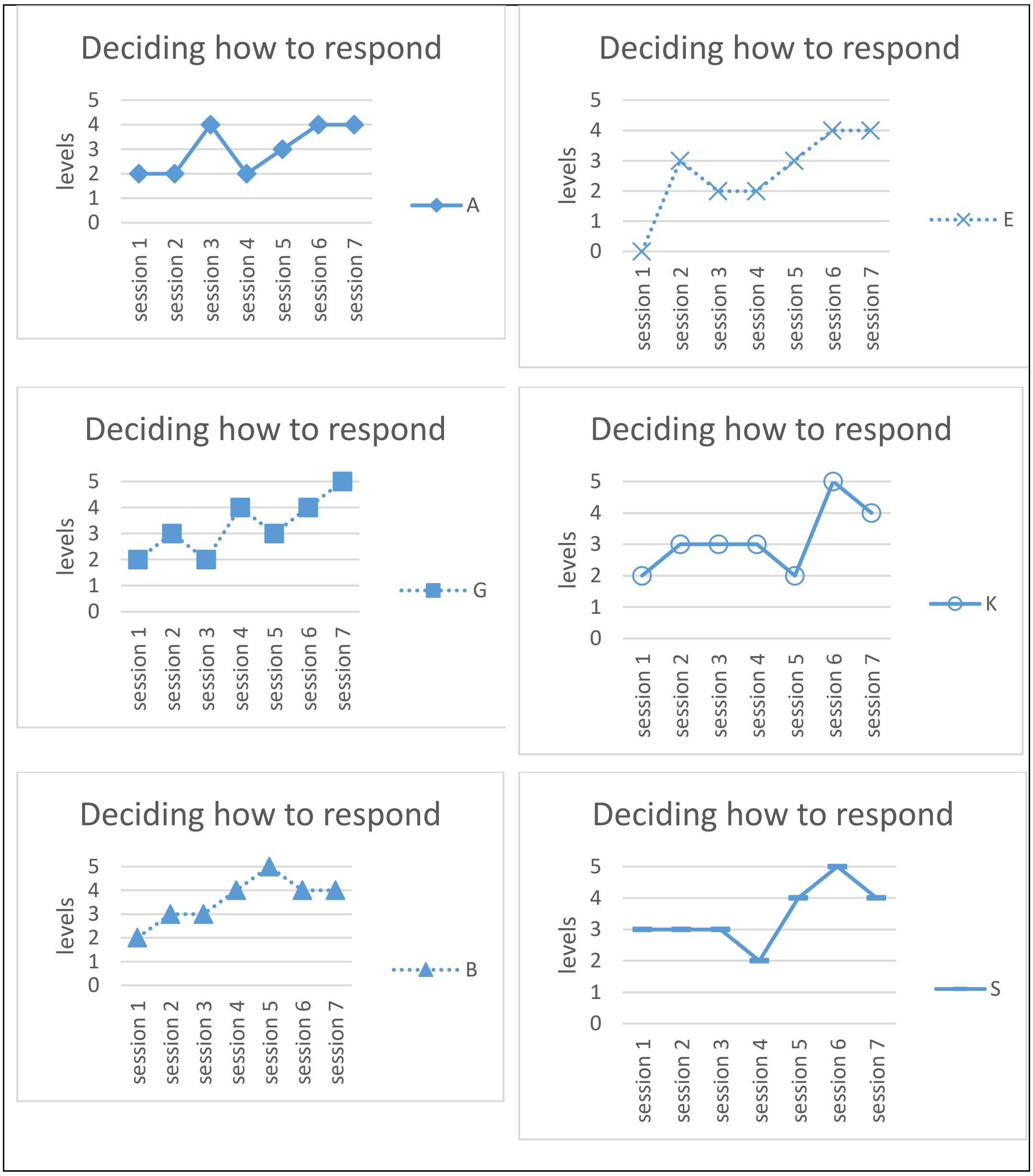


Figure 4. 55 Pre-service teachers' levels of deciding how to respond based on students' understanding in the video clips

All pre-service teachers increased their levels of deciding how to respond as the sessions progressed from early to late. Pre-service teacher E did not provide a response in the first session, and in the third and fourth sessions, he provided limited evidence. There was an increasing pattern in his deciding how to respond skills starting from the fourth session. Pre-service teacher A provided limited evidence in the first two sessions, but like pre-service teacher E, there was an increasing pattern in his deciding how to respond skills starting from the fourth session. In this way, pre-service teacher A and pre-service teacher E could provide substantial evidence in the last two sessions. Pre-service teacher K provided limited and medium evidence until the fifth session. In the last two sessions, she could provide a substantial and robust level of evidence. In the early sessions, pre-service teacher G provided limited and medium evidence. There was an increasing pattern in pre-service teacher G's deciding how to respond skills starting from the fifth session, and in the seventh session, she provided robust evidence. Pre-service teacher B provided limited and medium evidence in the early sessions, but starting from the fourth session, she could provide a substantial and robust level of evidence. Pre-service teacher S provided limited and medium evidence until the fifth session. Starting from the fifth session, she provided substantial and robust evidence.

4.2.2.2 The influence of approximation of practice on the development of pre-service teachers' professional noticing skills in the context of volume-surface area measurement

In this part, pre-service teachers' attending, interpreting and deciding how to respond skills at the approximation of practice stage of pedagogies of practice are presented to show how a video-based module situated in the pedagogies of practice framework supports pre-service teachers' professional noticing of students' mathematical thinking in volume-surface area measurement. As an approximation of practice, pre-service teachers conducted individual task-based interviews by using the tasks they had previously designed with middle school students of their choosing and video-recorded the interviews. Following the interview, they reflected on students' responses and actions by watching the videos and individually analyzed the students' mathematical thinking they elicited by using the professional noticing framework. Finally, they responded to three noticing prompts about attending to students' mathematical thinking, interpreting students' understanding and deciding how to respond, as well as to the questions about surprising/unexpected aspects of the student's solution and what they would change if they were to do the interview again in reflection paper 3. Three pre-service teachers, pre-service teacher A, pre-service teacher E and pre-service teacher G, designed tasks on volume and surface area measurement. For this reason, findings for these pre-service teachers are provided in this part. Table 4.24 presents the contents of the tasks the three pre-service teachers designed.

Table 4. 24 Content of the tasks the pre-service teachers designed in the context of volume and surface area measurement

PSTs	Content of the task	Concept
A	○ Asking the surface area and volume of a prism whose orthographic views from all directions are given ○ Constructing the prism with the unit cubes and asking the surface area and volume once again ○ Asking for the surface area and volume of a different prism whose orthographic views from three directions are given ○ Constructing the prism with the unit cubes and asking the surface area and volume once again	Volume and surface area
E	○ Showing two square prisms (green-2x2x9 and blue-3x3x4) and asking to compare the surface area of the prisms ○ Rotating the green prism and asking whether the surface area changes ○ Opening the prisms to get nets and asking to find the surface area of the prisms with the given unit squares ○ Removing a piece from the net of the green prism (2x2x9) and asking whether the surface area changes	Surface area
G	○ Giving soma cube pieces and asking for the volume and surface area of each piece ○ Asking to form the object with the smallest possible volume by using two pieces and the largest possible surface area by using two pieces ○ Providing T and L pieces with one face coincidence first, and then the same pieces with the three faces coincidence and asking whether volume and surface area change ○ Asking to construct a cube by using all pieces, removing the T piece from the cube, and asking how volume and surface area change	Volume and surface area

4.2.2.2.1 Attending to students' mathematical thinking in the approximation of practice in the context of volume-surface area measurement

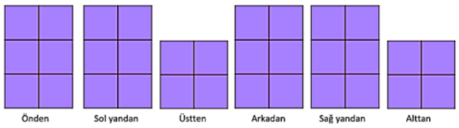
In order to investigate the extent to which pre-service teachers attend to students' mathematical thinking in the context of volume-surface area measurement, the mathematical details in the solutions of the interviewed students were first determined by the researcher. The mathematical details in the interviewed students' solutions are set out in Table 4.25.

Table 4. 25 Mathematical details in the interviewed students' solutions in the context of volume and surface area measurement

Students interviewed by the pre-service teachers	Students' misconceptions/difficulties in volume and surface area measurement
The student interviewed by preservice teacher A	Considering the surface area of a prism as the area of a top face/base area Finding the volume of a prism using the formula (l x w x h), Failing to realize that volume is the number of units cubes, using incorrect units (square units instead of cubic units)
The student interviewed by pre-service teacher E	Considering the surface area of a prism as the area of a face touching the floor Believing that surface area changes depending on the position of the prism
The student interviewed by pre-service teacher G	Incorrectly counting the surface area of the objects Believing that longer objects have a larger surface area

As shown in Table 4.25, the students interviewed by pre-service teacher A and pre-service teacher E had misconceptions similar to those in the video clips that the pre-service teachers viewed as representations of practice and analyzed as decompositions of practice in the sessions. All three pre-service teachers could provide specific evidence for the mathematical details in the students' solutions while analyzing the task-based interviews they conducted, and hence, they provided a robust level of evidence for attending in reflection paper 3. Figure 4.56 presents the response of pre-service teacher A to the attending prompt in reflection paper 3.

The student showed the correct approach to the problem by visualizing the prisms, whose 2D views were given from different directions, from two dimensions to three dimensions in his mind. He was aware of what the views corresponded to. Since he had a misconception about the surface area, he calculated the surface area incorrectly…He said the base area for the surface area. Since the base of the prism consists of 4 unit squares in the first case, I thought that he found the surface area by calculating the base area. When I asked him how he found it, he showed the top face of the prism and said that he calculated the area of it by multiplying 2 and 2.



He was able to calculate the volume of the prisms by looking at their appearance from different directions on paper. He seemed to have no misconceptions about the concept of volume, but he did not find the volume by counting unit cubes. I expected him to multiply the required lengths while finding the volume on paper, but after he built the prisms, he found the volume again by using a formula. When I asked him how he found it, he showed the width, length and height on the prism and said that he multiplied them. He calculated the volume of the prism in a procedural way both by looking at its appearance on paper and by looking at its 3D form. He found 12 cubic units from 2x2x3, but when expressing this, he said 12 square units.

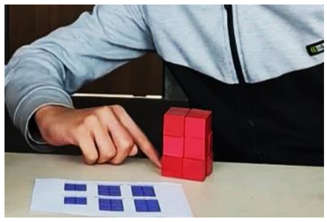


Figure 4. 56 Pre-service teacher A’s attending to interviewed student’s mathematical thinking

From the figure above we can see that pre-service teacher A provided the mathematical details in the student’s solution and mentioned the student’s misconception about surface area, which is considering the surface area of a prism as the area of a top face/base area, and approach to finding the volume of the prisms, which is finding the volume of a prism using the formula (l x w x h) and failing to realize that volume is the number of units cubes. The pre-service teachers also attended to students' work based on surprise by responding to the question in reflection paper 3, *"Were there any surprising/unexpected situations (that you could not predict) in the student's solution? Please explain."* As a response to this question, pre-service teacher A declared he was surprised at the student's finding the volume of the prism by looking at the given views easily and easily combining the views in his mind, imagining the prism and calculating the volume even with 2D views. However, she was also surprised that the student with such high spatial reasoning skills had a misconception about surface area. Furthermore, the pre-service teachers also responded to the question in reflection paper 3, "*If you were to do the interview again, what would you change (e.g., in the task you designed, in the questions you asked, etc.)? Please explain.*" In the responses, all three pre-service emphasized the importance of questioning. Pre-service teacher A wrote,

> If I were to do the interview again, I would ask different questions in addition to the ones I asked. I only tried to understand the student's thinking in his answers, and I didn't ask the questions necessary to elicit his mathematical ideas. (Pre-service teacher A, Reflection paper 3)

As an example, he asserted that after the student gave the correct answer for the volume of the prism, he asked the student how he found it. He realized that this questioning was not enough to find out how much he knew about the concept of volume. Therefore, he stated that he could try to reveal student's conceptual understanding by asking questions such as *"What is volume? Can you find the volume in a different way than the one you used? What did the result of multiplying the numbers show you? Can you find the volume without using the width x length x height formula?"* The response of pre-service teacher E to the attending prompt in reflection paper 3 is presented in Figure 4.57.

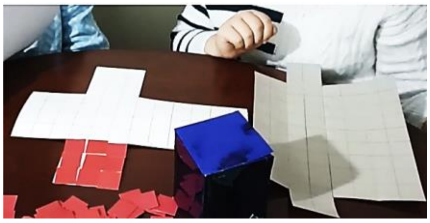
When asked to compare the surface areas of the prism, the student stated that the surface area of the blue prism was larger by saying, “If I name the sides at the base of the blue prism a, it will be smaller than a in the green prism because the sides at the base of the green prism are shorter. When I multiply the sides at the base of the blue prism, the value I find is larger, and when I multiply the sides at the base of the green prism, the value I find is smaller.” The student found the surface area of the prism by multiplying the sides on the base…She thinks that the surface area of the green prism increases when it is rotated. She said that one of the sides of the face touching the floor did not change, but the length of the other side increased. Again, she considered the surface area based on the area of the face touching the floor by saying that the surface area will increase when the sides are multiplied due to the increase seen on the length of one side.	 
When asked to find the surface areas of the prisms with the help of unit squares, she covered them in a way that was in line with her mathematical thinking: she covered only the faces touching the floor... When asked how she could find the surface area of the shape by covering it with unit squares in its net by changing its position, she said that this time, she would cover the rectangular face touching the floor. When a piece of two square units was cut from the net of the green prism, which stands on its square face... and when the object was rotated, she said that the surface area changed in both cases. She said that in the closed form of the object, there is a decrease in the length of the long side of the rectangular region at the base, so the surface area decreases.	 

Figure 4. 57 Pre-service teacher E’s attending to interviewed student’s mathematical thinking

It can be seen from the explanation of pre-service teacher E in Figure 4.57 that pre-service teacher E provided the mathematical details in the student’s solution and expressed the student’s misconception about surface area, which are considering the surface area of a prism as the area of a face touching the floor and believing that surface area changes depending on the position of the prism. While attending to students’ work based on surprise in reflection paper 3, pre-service teacher E stated that he was surprised at the student’s covering only the face touching the floor rather than all of the face when she was asked to find the surface area with unit squares. Moreover, the student made comments about the surface area based on the face touching the floor. Regarding the question of what they would change if they had the interview again, pre-service teacher E stated that he would design the material differently with more comprehensive content. He would create a prism mechanism that could be opened and closed from harder cardboard, and he would add questions about the volume using unit cubes in addition to the surface area. In this way, he believed that the misconception of the student about the concepts of volume and surface area could be identified. He also asserted that he would ask the student to answer mathematically how much change in surface area occurs after cutting a piece of two square units from the net of the prism. The response of pre-service teacher G to the attending prompt in reflection paper 3 is shown in Figure 4.58.

She found the volume of each piece by counting the unit cubes correctly. She followed the correct way when trying to get the smallest volume by combining two pieces. While trying to get the smallest volume object by choosing two pieces, she could think of choosing the piece with three unit cubes. She kept the one with the smallest volume fixed and stated that she could choose any of the other pieces next to it by realizing that any of the remaining pieces could come next to it. She stated that when we join the pieces T and L so that their one faces coincide instead of their three faces, the volume will not change because the number of unit cubes is the same. She said that when we remove a piece from the cube…, the volume will decrease since the number of unit cubes will decrease in the same way.

The student could not give the correct answer in some cases because she made counting errors while finding the surface areas of the pieces. She knows how to find the surface areas of the pieces, but she found the surface area of one piece wrong because she overcounted. While finding the largest surface area, she combined the two pieces, L and T pieces, that seemed the longest and largest to her eyes. She said that the surface area would increase when we combined the T and L pieces so that their one faces coincided, not their three faces because the object was longer and took up much more space. She said that an object with a larger surface area than this one cannot be obtained. She correctly reasoned that when we remove the T piece from the cube, the surface area will decrease…

Figure 4. 58 Pre-service teacher G's attending to interviewed student's mathematical thinking

As can be seen from the figure above, pre-service teacher G provided all the mathematical details regarding the volume and surface area measurement in the student's solution, i.e., finding the volume by counting the unit cubes correctly, choosing the piece with three unit cubes while building the object with the smallest volume, the volume does not change when the pieces are combined in different ways and the volume decreases when a piece is removed from the cube, as well as the student’s misconceptions and difficulties, which are counting the surface area of the objects incorrectly and believing that longer objects have larger surface area, by expressing the student’s actions and reactions. In addition, pre-service teacher G’s response to the question related to the student’s work based on surprise in reflection paper 3 revealed that she was surprised at the student’s answering the questions about volume easily. For instance, she did not expect that the student would be able to recognize the piece consisting of three unit cubes while obtaining the object with the smallest volume and to think about keeping it fixed and joining the other pieces next to it. Moreover, in reflection paper 3, she wrote:

> In the questions I asked about the surface area, when the student said, “I may have made a mistake while counting”, I would have reminded her that she could count again and make sure. Thus, I could understand the student’s comprehension more clearly. (Pre-service teacher G, Reflection paper 3)

That is, pre-service teacher G stated that she could do the interview again; she would have asked more questions by elaborating the questions more. In summary, all three pre-service teachers (pre-service teacher A, pre-service teacher E and pre-service teacher G) who designed tasks on volume-surface area measurement, identified all the mathematically significant details in the student’s solution by providing specific evidence for the mathematical details in the solutions while analyzing the task-based interviews they conducted. Accordingly, their responses to the attending prompt in reflection paper 3 showed a robust level of evidence.

4.2.2.2.2 Interpreting students’ understanding in the approximation of practice in the context of volume-surface area measurement

In this part, how pre-service teachers interpret the understanding of interviewed students in the approximation of practice stage of pedagogies of practice in the context of volume-surface area measurement is presented. All three pre-service teachers were able to interpret students' mathematical understanding by considering students' strategies, difficulties and misconceptions about volume and surface area measurement in detail in reflection paper 3. Accordingly, their responses to the interpreting prompt showed a robust level of evidence in the approximation of practice. The response of pre-service teacher A to the interpreting prompt in reflection paper 3 is presented in Figure 4.59.

The student had spatial reasoning skills because he was able to create an image in the mind, change and use this image, and visualize it in the mind from different directions. It was clear that the student had misconceptions about surface area, and he did not know the surface area. Since the base of the first prism consists of 4 unit squares, I thought that he found the surface area by calculating the base area. When I asked him how he found it, he said that he calculated the area of the prism by showing the top surface of the prism and said that 2x2 = 4 unit squares. Maybe he thinks that the area of the top surface is the surface area. Maybe he calculated the area of the top face because it was easily visible…He had a procedural understanding of the volume because he calculated the volume of the prism in a procedural way both by looking at its appearance and by looking at its 3-dimensional form. He calculated the volume correctly from the formula width x length x height and found 12 from 2x2x3, but when expressing this, he said 12 square units. He did not know that the measure of volume should be a cubic unit, or he was not careful in expressing it… He found the volume with the formula, but he may not know that the numerical value he found is related to the number of unit cubes.

Figure 4. 59 Pre-service teacher A's interpretation of the interviewed student's mathematical understanding

In response to the interpreting prompt, as shown in Figure 4.59, pre-service teacher A's explanation involved details from the student's solution to support evidence-based interpretation and the extent of evidence was categorized as robust. Using the details from the student's solution, the pre-service teacher made inferences about the student's incorrect response regarding the surface area that he might have conceptualized the surface area as either the area of the top face or the area of the base. Furthermore, pre-service teacher A drew attention to the student's procedural understanding of the volume based on the student's use of the volume formula both in the 2D view of the prism and in the 3D form rather than relating the number of unit cubes to the volume. Figure 4.60 shows the response of pre-service teacher E to the interpreting prompt in reflection paper 3.

> The student did not know what the surface area meant. Since she did not know that the surface area is a concept that concerns all faces of the object, she associated the surface area with the area of the face touching the floor. The student had a misconception that the surface area of objects with larger faces touching the floor would always be larger. The student had the misconception that the surface area of the given prism would change as its position changed. The reason underlying this idea was due to associating the surface area with the face touching the floor. While the student covered the prism with unit squares, she covered only the face touching the floor because she accepted the surface area as the area of the face touching the floor. The student covered only the face touching the floor by using unit squares depending on the change of the position of the square prism. The student may have this misconception due to the use of face and surfaces interchangeably in Turkish. In the process of cutting a square piece from the net, she thought that the decrease in the length of the side would decrease the surface area since she found the surface area by multiplying the sides of the face.

Figure 4. 60 Pre-service teacher E’s interpretation of the interviewed student’s mathematical understanding

As can be seen from the figure above, comment of pre-service teacher E regarding interpreting the student’s understanding of surface area shows that the pre-service teacher pointed out the student’s associating the surface area with the area of the face touching the floor and, as a result of this, believing that surface area of the prism changes as its position changes. The pre-service teacher also suggested that the student may have fallen into this misconception because of the interchangeable use of face and surface in Turkish. Thus, pre-service teacher E provided robust evidence since his interpretations of the student’s understanding included specific evidence to support the claims being made. The response of pre-service teacher G to the interpreting prompt in reflection paper 3 is set out in Figure 4.61.

She was able to calculate the volume of seven pieces correctly by counting the unit cubes. She knows how to find the volume. Similarly, she found the surface area by counting the outward-facing faces of the object. Although she gave wrong answers due to counting errors, she knows how to find the surface area. While obtaining the smallest volume by combining two pieces, she chose the piece with the smallest volume. Realizing this shows that the student knows the concept of volume. When obtaining the largest surface area, she joined the T and L pieces so that their one faces coincided and said that this object is longer and takes up more space... Here, she gave this answer by trying to get the object that looked the longest and largest to her eyes when she combined them. In fact, she could have obtained the same result by making different choices, but she said that the largest surface area would be as shown... It would have been more accurate if she had related the surface area to the number of the outward-facing faces of the object... She said that the volume of the object will not change when the pieces are combined in different ways since the number of unit cubes would not change… she knows that different combinations of two pieces will not affect the volume. She said that the surface area of the object will change when the pieces are combined in different ways…she knows that the surface area can change according to how we join the pieces. She said that when we remove the T piece from the cube we obtained, the volume will decrease as the number of unit cubes decreases. She thinks that the same situation will happen every time we remove a piece. She knows…how the volume will change by adding/removing pieces. She said that when piece T is removed from the cube, the surface area will decrease. Here, we see that when we remove a piece from the cube, the student can count the newly visible faces that appeared behind that piece and compare it with the first situation…

Figure 4. 61 Pre-service teacher G’s interpretation of the interviewed student’s mathematical understanding

From the figure above we can see that while interpreting the student’s understanding, pre-service teacher G provided specific evidence by describing the student’s actions and reactions. She gave details of exactly what the student said and did regarding the volume to show the student’s conceptual understanding of volume. Moreover, the pre-service teacher gave what she thinks the student knows about the surface area: finding the surface area, obtaining the largest surface and being aware of the change in surface area in different combinations of the pieces and removing a piece. She also stressed the student’s conceptualization that objects that are longer and take up more space have a greater surface area, which may not always be the case. Since the pre-service teacher used details from the student's solution to support evidence-based interpretations, she provided robust evidence of the student's understanding.

In summary, all three pre-service teachers (pre-service teacher A, pre-service teacher E and pre-service teacher G) who designed tasks on volume-surface area measurement were able to interpret the students' mathematical understanding by considering the students' strategies, difficulties and misconceptions about volume and surface area measurement in detail in reflection paper 3. Thus, they provided a robust level of evidence for interpreting students' understanding in the approximation of practice.

4.2.2.2.3 Deciding how to respond based on students' understanding in the approximation of practice in the context of volume-surface area measurement

In this part, pre-service teachers' deciding how to respond based on the understanding of interviewed students in the approximation of practice stage of pedagogies of practice in the context of volume-surface area measurement is presented. All three pre-service teachers suggested specific instructional actions based on the student's misconceptions and difficulties they identified to eliminate them and make students understand by providing rationale and detail in reflection paper 3. Hence, the responses they provided to the deciding how to respond prompt were categorized as a robust level of evidence. The suggestion of pre-service teacher A in reflection paper 3 is shown in Figure 4.62.

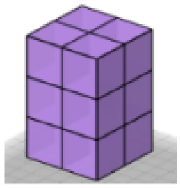
He had a misconception about the concept of surface area. He calculated the surface area over one face of the prism which was the base or top face. I would use a rectangular prism material that shows the net of the prism to eliminate this misconception. Firstly, I would show it in the closed form and ask, “Where do you think the faces of this prism are?”. After the student shows, I would open the prism and ask the student, “How many faces do you think this prism has now?” and make him see the faces more easily. After the concept of face is understood, I make a transition to the concept of surface and surface area…I bring a cardboard box and ask, “If I ask you to cover its surface with colored cardboard paper, where would you cover?” The concept of surface is reinforced with questions and then, the transition to surface area can be easier and more meaningful. I would give the cardboard paper to the student and ask him to cover the surface. Most probably, he will cover all the faces. Afterwards, I emphasize that all six faces together form the surface of the prism. Then I ask, “What is the surface area? Can you show me the faces whose areas we need to find to find the surface area?” After the questions, the first prism in the task is given to the student again. “So, what can you say about the surface area of this prism now? Let’s think about the surface area of this prism again, taking into account what you have learnt.” With such a process, I try to make the student understand the concepts of face, surface and surface area.	  
The student calculated the volume of the prisms using the formula. I would try to ensure that the student has a conceptual understanding of volume and to develop his mathematical understanding of volume. For this purpose, I would give an empty, openable prism material to the student with a ruler and unit cubes... I would ask the student, “How can the volume of this prism be found?” The student can measure the length of the dimensions with the help of a ruler…I would ask the student to write down the result and record the volume... Afterwards, I specifically would ask the student how we could find the volume without using a ruler and only with the help of unit cubes. I expect the student to find the volume by filling the prism with unit cubes. He also records the number of unit cubes required to fill the prism. Then, he sees that these two numbers are the same, and establishes a relationship. Before this, it can also be checked whether he recognizes unit cubes. He needs to know that the length of the unit cubes is one unit so that he can follow the correct path while establishing the relationship.	

Figure 4. 62 Pre-service teacher A’s deciding how to respond based on the interviewed student’s understanding

In reflection paper 3, as shown in Figure 4.62, pre-service teacher A offered an instructional suggestion about surface area to eliminate the student’s misconception.

Since the student calculated the area of one face of the prism while finding the surface area, the pre-service suggested showing the net of the prism to make the student realize the six faces of the prism. Then, pre-service teacher A proposed asking the student to cover the cardboard box using cardboard paper to help the student realize that all six faces together form the surface of the prism. In this way, the pre-service teacher believed that the student would understand the concepts of face, surface and surface area. To extend the understanding of the student about volume, pre-service teacher A suggested asking the student to find the volume of the empty openable prism by measuring the dimensions with a ruler. After the student records the volume, pre-service teacher A wants the student to find the volume without using a ruler and only with the help of unit cubes. The pre-service teacher considered that by comparing two results the student reached, he could establish a relationship between the number of unit cubes required to fill the prism and the volume of the prism. The suggestion of pre-service teacher E in reflection paper 3 is presented in Figure 4.63.

The student finds the surface area based on the face touching the floor. He thinks the surface areas of the objects with large faces touching the floor will be larger. In the student's mind, the surface area is related only to the face touching the floor, not to all faces. To make the student realize that the surface area is related to all faces of the object, "Has the surface area changed now, and if so, how did it change?" can be asked by disconnecting the object from the floor. Moreover, I would ask the student to find the surface area of the object by giving sticky unit squares and want her to cover the faces of the object. The student would probably cover the face touching the floor. The position of the object is changed each time, and "How can you find the surface area by covering it with unit squares now?" is asked. The student is ensured to cover all faces by covering the face touching the floor in each change of the position of the object. During this process, the student is made to feel that the surface area concerns all faces of the object, and the misconception is eliminated.

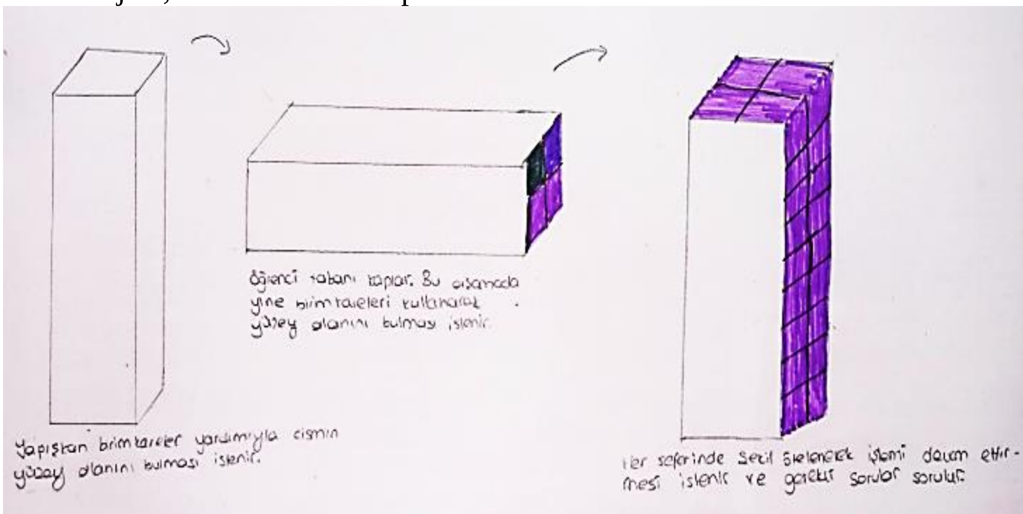
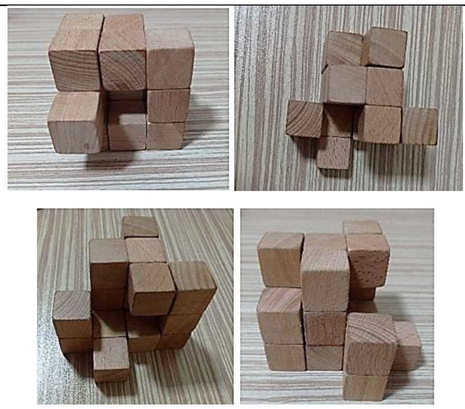


Figure 4. 63 Pre-service teacher E's deciding how to respond based on the interviewed student's understanding

The student interviewed by pre-service teacher E had a misconception about the surface area that the surface area of the prism is the area of the face touching the floor. As can be seen from the figure above, to help the student overcome her misconception, pre-service teacher E wants the student to find the surface area of the object by covering it with sticky unit squares. By changing the position of the object each time, the pre-service teacher asks the student to find the surface area by covering it with unit squares. Thus, after covering all faces by covering the face touching the floor in each change of the position of the object, pre-service teacher E believed that the student would understand that the surface area concerns all faces of the object, not only the face touching the floor. Figure 4.64 provides the suggestion of pre-service teacher G in reflection paper 3.

The student knows the concept of volume, what volume means and how it changes in which situations. In order to extend the student's understanding, I would obtain an irregular object from the pieces and ask the student to find its volume. I want to find out whether she will count the unit cubes that are not visible at first glance or whether she will count only what she sees without paying attention.



The students believes that longer objects have a larger surface area. Based on the student's comment while selecting two pieces to obtain the object with the largest surface area (I select T and L pieces because when I combine them, they become longer and take up more space), I would connect T and L pieces from different faces. I create two cases with the same surface area but different spreads and lengths. I want the student to find the surface area in both cases and compare the results she reaches.
I also try to eliminate the incorrect generalization in the student's mind by creating two objects having the same surface area but one with a smaller length. My aim here is to make the student realize that the surface area of the objects that are long in the horizontal position and small in height is not always larger. In other words, when the student sees an object with a bigger length, she should not immediately think that its surface area will be larger. I want the student to find the surface area of both objects and make an inference regarding the result she reaches.

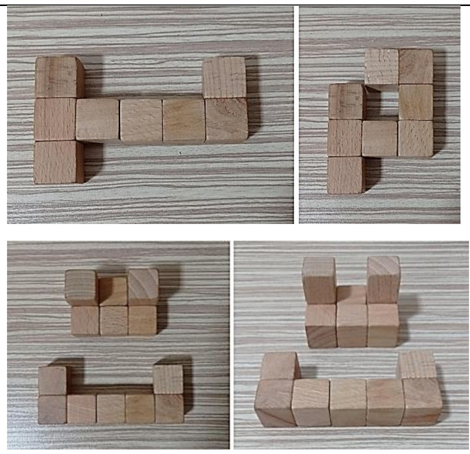


Figure 4. 64 Pre-service teacher G's deciding how to respond based on the interviewed student's understanding

From the figure above, we can see that since the student had already known the concept of volume, pre-service teacher G tried to extend the student's understanding by asking about the volume of the irregular object consisting of the soma pieces. To get rid of the student's conception about surface area, the objects that are longer and take up more space have a larger surface area, pre-service teacher G suggested providing the student with different cases based on the same surface area but different spreads, lengths and heights. By asking to find the surface area of both objects and comparing the results, pre-service teacher G believed that she could eliminate the incorrect generalization in the student's mind.

In summary, all three pre-service teachers (pre-service teacher A, pre-service teacher E and pre-service teacher G) who designed tasks on volume-surface area measurement, provided specific instructional suggestions based on the students' misconceptions and difficulties they identified in order to eliminate them and help the students understand, by providing rationale and details in reflection paper 3. Therefore, their responses to the deciding how to respond prompt in reflection paper 3 showed a robust level of evidence.

Summary

The aim of this study was to examine the extent to which pre-service mathematics teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement changed as they participated in the video-based module situated in the pedagogies of practice framework developed by Grossman et al. (2009). In addition, the study aimed to explore how the video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement. The findings indicated that pre-service teachers showed an improvement in all of the noticing components, i.e., attending to students' solutions, interpreting students' understanding, and deciding how to respond, and this change was statistically significant in both contexts. Furthermore, in the representation-decomposition of practice, among six pre-service teachers, the pre-service teachers who provided low level evidence for attending in reflection paper 1 in the early sessions were able to provide a high level of evidence in the late sessions. They were also more inclined to go beyond simply determining whether solutions were correct when assessing students' understanding. Moreover, the instructional actions proposed by all six pre-service teachers showed a substantial and robust level of evidence in reflection paper 1 in the late sessions. In the approximation of practice, all six pre-service teachers provided a robust level of evidence for attending to the mathematical details, interpreting students' understanding and deciding how to respond in the context of task-based interviews they conducted. More specifically, the following findings were obtained in the present study:

In the initial questionnaire, pre-service teachers' attending skills in volume-surface area measurement were lower than their attending skills in perimeter-area measurement.

Pre-service teachers experienced the most difficulty in interpreting students' understanding among the noticing components in both context. However, interpreting in the context of volume-surface area measurement was more challenging than perimeter-area measurement.

- Statistical significance was observed for each problem in the context of perimeter-area measurement for all three components in the noticing questionnaire. Therefore, there was a statistically significant change in attending to students' solutions, interpreting students' understanding, and deciding how to respond in the post-test in the context of perimeter-area measurement.
- Statistical significance was observed for each problem in the context of volume-surface area measurement for all three components in the noticing questionnaire. Therefore, there was a statistically significant change in attending to students' solutions, interpreting students' understanding, and deciding how to respond in the post-test in the context of volume-surface area measurement.
- The least improvement in the final questionnaire was observed in the attending component in both contexts.

- In the context of volume-surface area measurement, the strongest improvement was observed in the interpreting component.
- In the context of perimeter-area measurement, the greatest improvement was observed in the deciding how to respond component.
- Two pre-service teachers (pre-service teacher E and pre-service teacher S) in the group provided high levels of evidence for attending in reflection paper 1, even in the early sessions (first and second sessions) while other four pre-service teachers experienced difficulty.
- An increase was observed in pre-service teachers' levels of attending at the end of the sessions, and all six pre-service teachers provided substantial or robust evidence for attending in the late sessions.
- Whole-class discussions were more fruitful and richer compared to group discussions in terms of interpreting students' understanding because justifications about the possible reasoning behind students' solutions that were not provided in group discussions were provided in whole-class discussions.
- It was more difficult for pre-service teachers to extend the students' understanding in cases where the solution was correct in the early sessions while deciding how to respond.
- An increase was observed in pre-service teachers' levels of deciding how to respond at the end of the sessions, and all six pre-service teachers provided substantial or robust evidence for deciding how to respond in the late sessions.
- All six pre-service teachers' responses to the noticing prompts in the approximation of practice showed robust level of evidence.
- All six pre-service teachers were precise in supporting their ideas and providing descriptions of students' actions and using direct quotes of what

students said during the interviews as a way of communicating evidence in the approximation of practice.

CHAPTER 5

DISCUSSION, IMPLICATIONS AND RECOMMENDATIONS

The study aimed to examine the extent to which pre-service mathematics teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement changed as they participated in the video-based module situated in the pedagogies of practice framework developed by Grossman et al. (2009). In addition, this study aimed to explore how the video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement. More specifically, to answer the first research question, it was analyzed whether the intervention had an effect on pre-service teachers' professional noticing skills, and whether the effect was statistically significant. To answer the second research question, how this effect, i.e., the influence of the pedagogies of practice, occurred on pre-service teachers' professional noticing skills was investigated in a focus group. Based on the purposes and research questions, the findings of this study are discussed in light of the previous research studies in the literature. This chapter includes four parts. The first part presents a discussion of the findings related to the changes in pre-service teachers' professional noticing of students' mathematical thinking when their responses to the noticing questionnaire before and after participating in the module were compared. The second part provides a discussion of the findings related to the development of pre-service teachers' professional noticing skills through the video-based module situated in the pedagogies of practice framework. This part is presented under the headings of pre-service teachers' professional noticing skills in the representation-decomposition of practice and pre-service teachers' professional noticing skills in the approximation of practice since the pedagogies of practice framework consist of representation and decomposition of practice and approximation of practice components. The third and fourth parts provide educational implications and recommendations for future research studies, respectively.

5.1 Pre-post changes in pre-service teachers’ professional noticing skills

In the first research question, the purpose of the present study was to explore pre-service teachers’ professional noticing of students’ mathematical thinking before and after participating in the video-based module situated in the pedagogies of practice framework. With this aim, 32 pre-service teachers’ responses to the noticing prompts in the initial and final noticing questionnaire were compared to portray the change in their professional noticing skills. Findings revealed that pre-service teachers showed an improvement in all of the noticing components, i.e., attending to students’ solutions, interpreting students’ understanding, and deciding how to respond, and this change was statistically significant. The discussion of findings regarding the change in pre-service teachers’ professional noticing is presented under the headings of pre-post changes in pre-service teachers’ attending to students’ solutions, pre-post changes in pre-service teachers’ interpreting students’ understanding, and pre-post changes in pre-service teachers’ deciding how to respond based on students’ understanding. Moreover, as pre-service teachers’ professional noticing of students’ mathematical thinking was investigated separately in perimeter-area measurement and volume-surface area measurement, the findings are discussed separately for these two contexts.

5.1.1 Pre-post changes in pre-service teachers’ attending to students’ solutions

The findings showed that there was an improvement in the pre-service teachers’ attending to students’ solutions in the final noticing questionnaire, and the change was statistically significant. A discussion of the findings regarding the change in pre-service teachers’ attention to students’ solutions is provided for perimeter-area measurement and volume-surface area measurement, respectively.

5.1.1.1 Pre-post changes in pre-service teachers' attending to students' solutions in the context of perimeter-area measurement

Pre-service teachers could already attend to students' solutions by providing higher levels of evidence at the beginning of the study in the context of perimeter-area measurement compared to the other components. Therefore, they showed the least improvement in attending to students' solutions in the final questionnaire since the percentage of high levels of evidence (substantial and robust evidence) they provided in the attending component was higher than other components in the initial questionnaire. This finding supports the idea that pre-service teachers' attending skills were already more developed than their interpreting and deciding how to respond skills before participating in the video-based module situated in the pedagogies of practice framework. This finding corroborates the findings of the previous work on professional noticing of students' mathematical thinking (Gonzalez & Skultety, 2018; Jacobs et al., 2010; Luna & Sherin, 2017; Sanchez-Matamoros et al., 2015; Tekin-Sitrava et al., 2022). Pre-service teachers' higher attending skills than interpreting skills and deciding how to respond skills at the beginning of the study may be related to the pre-service teachers' previous experiences and prior knowledge (Casey et al., 2018; Star & Strickland, 2008). The pre-service teachers in the present study were the senior students (fourth grade) and had already completed most of the mathematics education courses. Therefore, their mathematical knowledge and knowledge of the content and students had an important role in their attending to students' solutions with higher levels of evidence than interpreting and deciding how to respond at the beginning of the study. In addition, the ability to attend is easier than the ability to interpret and decide how to respond (Barnhart & van Es, 2015; Sánchez-Matamoros et al., 2019), which could be attributed to more developed attending skills of the pre-service teachers in the initial questionnaire. Another possible explanation for this is that mathematics methods courses usually remain at the level of attending to students' solutions and do not include practices that focus on interpreting students' understanding and deciding how to respond to students based on their understanding.

Although the pre-service teachers' skills of attending to students' solutions were initially higher than the other components at the beginning of the study, in more than half of the responses to the attending prompt, pre-service teachers could not identify any mathematical properties in students' solutions in the initial questionnaire. However, in the final questionnaire, in more than half of the responses, the pre-service teachers could identify mathematical properties in students' solutions, at least to some extent. Therefore, the intervention had an effect, as the findings indicated an improvement in the pre-service teachers' attending skills in the final questionnaire compared to the initial questionnaire, but this effect was not as pronounced as for the other components. This finding of the study suggests that participating in the video-based module situated in the pedagogies of practice framework supported pre-service teachers' attending to students' solutions. Moreover, there was a statistically significant difference between each problem in the initial and final questionnaire for the attending component in favor of the final questionnaire in perimeter-area measurement. Thus, the statistical analysis of the pre-service teachers' attending to students' solutions given in the initial and final questionnaire provided insights into the effectiveness of the video-based module situated in the pedagogies of practice framework. The improvement in the attending component indicates the importance of providing pre-service teachers with opportunities in a supportive environment, which matches what the earlier studies found (McDuffie et al., 2014; Schack et al., 2013). For example, in the initial questionnaire, pre-service teachers faced the most difficulty with the perimeter comparison problem. However, a notable improvement in the context of perimeter-area measurement was observed concerning this specific problem, as evidenced by 29 out of 32 pre-service teachers providing higher levels of evidence in the final questionnaire. Particularly, during the second session, they discovered that the perimeter of a shape increases with the number of sides of pentomino pieces included and decreases when fewer sides are exposed. This insight enabled them to accurately discern the mathematical properties underlying Hazal's and Can's solutions when comparing the perimeters of two shapes constructed from tangram pieces, using the same rationale in the final questionnaire.

In this study, pre-service teachers were exposed to different students' solutions and reflected on students' mathematical thinking by having time to discuss mathematical details in the students' solutions with peers during the intervention sessions. In this way, their development was supported by the nature of video clips (representation of practice), discussions with peers (decomposition of practice), and reflections on students' mathematical thinking they elicited through task-based interviews (approximation of practice). These opportunities seemed useful, and this shows that support is important for the development of pre-service teachers' attending skills, as previous research studies showed (Star & Strickland, 2007; Vondrova & Zalaska, 2013). The details of the video-based module situated in the pedagogies of practice are discussed below in relation to the findings of the second research question.

5.1.1.2 Pre-post changes in pre-service teachers' attending to students' solutions in the context of volume-surface area measurement

In the initial questionnaire, pre-service teachers' attending skills in volume-surface area measurement were lower than their attending skills in perimeter-area measurement. The fact that pre-service teachers experienced more difficulty in attending to students' solutions in volume-surface area measurement can be explained by their lack of knowledge about volume measurement. Teachers need to have mathematical knowledge in order to describe students' solutions by using mathematical concepts with mathematical language (Ball et al., 2008). This highlights the relationship between professional noticing and mathematical knowledge (Thomas et al., 2017). Attending to mathematically significant details in students' solutions requires knowledge about mathematics and students' mathematics (Dreher & Kuntze, 2015; Stockero et al., 2017b; Yang et al., 2019), i.e., content-specific professional knowledge (Sánchez-Matamoros et al., 2015). Several reports have also shown pre-service teachers' difficulty in volume measurement, including difficulty in recognizing errors in students' solutions strategies and the reasons behind these errors regarding calculating volume with unit cubes (Esen & Çakıroğlu, 2012). In addition, pre-service teachers' tendency to solve volume problems using volume formula and through calculations (Tekin-Sitrava & Isiksal-Bostan, 2018) may have caused pre-service teachers to have difficulty in attending to students' solutions in concept-based problems different from those presented in schools and textbooks, as in this study. In the present study, among the four problems about surface area-volume measurement, pre-service teachers had the most difficulties with the surface area comparison problem, as three-quarters of them could not identify any of the mathematical properties in the students' solutions. This can be explained by their lack of knowledge about surface area measurement since the difficulties of pre-service teachers regarding the relationship between the dimensions and surface areas of prisms have also been reported in the literature (Tossavainen et al., 2017).

Although the pre-service teachers' initial attending skills in the context of volume-surface area measurement appeared to be low, they were higher than the other noticing components, as in perimeter-area measurement. As a result, pre-service teachers showed the least improvement in attending to students' solutions in the context of volume-surface area measurement. Even though pre-service teachers' attending skills were initially higher than for the other components at the beginning of the study, there was an improvement in the final questionnaire, with more than half of the responses showing high levels of evidence and a quarter of the responses in the category of robust evidence. There was also a statistically significant difference between each problem in the initial questionnaire and final questionnaire for the attending component in favor of the final questionnaire for volume-surface area measurement. This finding supports the idea that the inclusion of the video-based module situated in the pedagogies of practice framework in the current study may have helped pre-service teachers improve their attending to students' solutions in the context of volume-surface area measurement. As emphasized by Jacobs et al. (2010), paying attention to students' strategies requires knowledge of what is mathematically important and the ability to identify mathematically significant indicators in students' solutions. Therefore, mathematical knowledge is seen as essential for the ability to attend to students' solutions (Casey et al., 2018), and strong knowledge of content and students plays a crucial role in approaching students' strategies (Ballock et al., 2018). Thus, in the present study, the pre-service teachers' increased knowledge of content and students through the video-based module situated in the pedagogies of practice may have encouraged a more detailed analysis of students' solutions in the final questionnaire. For instance, when the pre-service teachers broke the video clips into parts, they identified the misconceptions related to the same area with the same volume and considering only one dimension while comparing the volumes of two prisms. This was particularly evident in the fourth session, through the decomposition of practice. Consequently, this process likely enhanced the attending skills of the pre-service teachers, enabling them to more accurately recognize the misconceptions in students' solutions to comparing the volumes of paper prisms in the final noticing questionnaire. The details of the video-based module situated in the pedagogies of practice are discussed below in relation to the findings of the second research question.

5.1.2 Pre-post changes in pre-service teachers' interpreting students' understanding

The findings revealed that an improvement was observed in the pre-service teachers' interpretation of students' understanding in the final noticing questionnaire, and the change was statistically significant. A discussion of the findings regarding the change in pre-service teachers' interpretation of students' understanding is provided for perimeter-area measurement and volume-surface area measurement, respectively.

5.1.2.1 Pre-post changes in pre-service teachers' interpreting students' understanding in the context of perimeter-area measurement

Pre-service teachers experienced the most difficulty interpreting students' understanding in the initial questionnaire in the context of perimeter-area measurement; in about three-quarters of the responses, pre-service teachers failed to provide a valid justification for any students' understanding. It can, therefore, be assumed that interpreting was more challenging than attending to students' solutions and deciding how to respond for the preservice teachers in the context of perimeter-area measurement. This finding is consistent with the findings of the studies conducted with in-service teachers (Dışbudak-Kuru et al., 2022; Melhuish et al., 2020), and it is difficult even for experienced teachers to engage in a process of interpretation and to draw inferences from what is observed (Little & Curry, 2008). This shows that pre-service teachers needed more support to develop their interpreting skills than to develop their attending and deciding how to respond skills. Moreover, in the present study, providing a high level of evidence for attending (substantial and robust evidence) in the initial questionnaire did not guarantee the interpretation of students' understanding by providing a high level of evidence. Recognizing the mathematical properties in students' solutions does not ensure that pre-service teachers take and use them as evidence while interpreting students' understanding (Barnhart & van Es, 2015). Thus, attending to students' solutions is necessary but not sufficient to interpret students' understanding (Sánchez-Matamoros et al., 2019; Tekin-Sitrava et al., 2022; Ulusoy & Çakıroğlu, 2021). This implies that pre-service teachers' ability to interpret students' understanding is not connected to the ability to attend to students' solutions.

The findings of the present study revealed that pre-service teachers were in a better position in the final questionnaire in terms of interpreting students' understanding in the final questionnaire because there was a decrease in the percentage of responses showing a low level of evidence, whereas there was an increase in the percentage of responses showing a high level of evidence. In addition, there was a statistically significant increase in each problem for the interpreting component in the final questionnaire in the context of perimeter-area measurement. Thus, this increase in the pre-service teachers' levels of interpreting students' understanding in the final questionnaire pointed to the effectiveness of the video-based module situated in the pedagogies of practice framework. This supports the idea that expertise in interpreting can be developed through repeated opportunities to reflect on students' mathematical thinking, as in the present study (Krupa et al., 2017). This study indicated that pre-service teachers could develop their ability to interpret if they are supported by an intervention. This finding matches those observed in earlier studies (Sherin & van Es, 2005; Vondrova & Zalaska, 2013). In the present study, the deliberately sequenced activities in the module, including viewing video clips of task-based interviews, discussing video clips, and interviewing a student, followed by a structured reflection, led to the shift in pre-service teachers' interpretations of students' understanding. Furthermore, the present study intentionally included three different problems on perimeter-area measurement about a fixed perimeter-changing area situation and a fixed area-changing perimeter situation in the noticing questionnaire to provide pre-service teachers with opportunities to perceive the differences between the problems. In addition, two or three students' responses to these problems that reflect different understandings were presented in order to enable pre-service teachers to make inferences about students' understanding and to provide instructional moves based on their interpretations (Sánchez-Matamoros et al., 2019). In this sense, using artifacts from real students representing sample student work and highlighting common misconceptions about perimeter and area measurement as well as students' correct reasoning enabled pre-service teachers to be better at interpreting students' understanding in detail by differentiating students' understanding in the final questionnaire. Accordingly, encountering different students' reasoning and misconceptions might have allowed the preservice teachers to interpret students' understanding in-depth at the end of the study.

5.1.2.2 Pre-post changes in pre-service teachers' interpreting students' understanding in the context of volume-surface area measurement

Pre-service teachers experienced the most difficulty interpreting students' understanding in the initial questionnaire in the context of volume-surface area measurement. In addition, interpreting in the context of volume-surface area measurement was more challenging for pre-service teachers than in the context of perimeter-area measurement because the percentage of responses where pre-service teachers failed to provide a valid justification for any students' understanding was higher. The process of interpreting students' understanding requires pre-service teachers to comprehend mathematical ideas (Fernandez et al., 2013), have mathematical content knowledge for interpretation (Casey et al., 2018), and have knowledge of students' mathematical thinking (Sánchez-Matamoros et al., 2019). Hence, pre-service teachers' lack of mathematical knowledge of volume and surface area measurement, which was also reported in several research studies (Hong & Runnals, 2021; Tekin-Sitrava & Isiksal-Bostan, 2016, 2018) can be a possible reason for pre-service teachers' difficulty interpreting students' understanding in the context of volume-surface area measurement.

Interestingly, among the noticing components, the strongest improvement was observed in the interpreting component in the context of volume-surface area measurement. In this regard, with lower pre-test scores, interpreting students' understanding had more room for improvement. Furthermore, there was a statistically significant increase in each problem for the interpreting component in the final questionnaire in the context of volume-surface area measurement. In the initial questionnaire, pre-service teachers tended to provide no interpretation or make comments about whether only the student understood the volume or surface area concept or not. In the final questionnaire, in most of the responses, they could provide a valid justification for at least some of the students' understanding. This finding suggests that the video-based module situated in the pedagogies of practice framework was effective in helping pre-service teachers support their statements with evidence and going beyond the correctness of solutions or students' errors in assessing students' understanding. To illustrate, in the context of volume and surface area measurement, the most notable improvement in interpreting students' understanding was observed in the surface area comparison problem. This was evidenced by 28 out of 32 pre-service teachers providing higher levels of evidence in the final noticing questionnaire. In the decomposition of practice, particularly in the sixth and seventh sessions, the pre-service teachers identified a common student misconception: considering the area of a single face as the surface area of the prism. They also realized that the surface area of a prism is determined by its faces rather than its height, i.e., a prism has a larger surface area when fewer faces are in contact with each other and more faces are visible. Following these insights, in the approximation of practice, they tested the surface area of prisms with students. As a result, in the final questionnaire, they showed an improved ability to interpret students' understanding of surface area by being aware of students' misconceptions and conceptualizations about surface area. The details of the intervention are discussed below in relation to the findings of the second research question.

However, even though there is an improvement in pre-service teachers' ability to interpret students' understanding in the final questionnaire, there were some pre-service teachers who still provided a lack of evidence for interpreting. This can be explained by the fact that the decomposition of practice lasting seven weeks was not sufficient for all pre-service teachers to develop expertise in interpreting. Jacobs et al. (2010) noted that interpreting students' understanding takes years to develop. Therefore, the inability of pre-service teachers to develop robust interpretations highlights that pre-service teachers need opportunities early in teaching education programs to relate their mathematical content knowledge to practices (Warshauer et al., 2021). In addition, research suggests that producing change in teachers' knowledge and beliefs is difficult (Franke et al., 1998), and hence, teacher learning should be maintained over long periods of time (Little, 1993). Therefore, the intervention in the present study may only be the beginning of pre-service teachers' need for extensive change. Moreover, it may not be realistic to expect all pre-service teachers to have a substantive change in the way of reflecting on students' understanding after just participating in the video-based module situated in the pedagogies of practice framework. Still, the impact of the intervention on the pre-service teachers' ability to interpret is encouraging. Focusing on professional noticing during pre-service teachers' undergraduate studies, as in the present study, may provide pre-service teachers with a foundation for learning interpreting students' understanding.

5.1.3 Pre-post changes in pre-service teachers' deciding how to respond based on students' understanding

The findings showed an improvement in the pre-service teachers' deciding how to respond based on students' understanding in the final noticing questionnaire, and the change was statistically significant. A discussion of the findings regarding the change in pre-service teachers' deciding skills is provided for perimeter-area measurement and volume-surface area measurement, respectively.

5.1.3.1 Pre-post changes in pre-service teachers' deciding how to respond based on students' understanding in the context of perimeter-area measurement

The skill of deciding how to respond to the student's work requires the teacher to provide differentiated tasks and rationales for each student in a manner consistent with the student's work. However, in the initial questionnaire, the smallest percentage of the instructional decisions demonstrated a robust level of evidence in perimeter-area measurement, and the highest percentage of the suggestions showed a limited level of evidence. In other words, most of the pre-service teachers' suggestions were not sufficient to eliminate the students' misconceptions or extend the students' understanding. This may suggest that pre-service teachers did not attend to the mathematical properties in the students' solutions and did not interpret students' understanding of concepts in the problems while making instructional decisions. Consequently, preservice teachers' instructional decisions appeared to be general, and they responded without building on the student's way of thinking. Furthermore, in the initial questionnaire, pre-service teachers generally tended to offer the same suggestions for two students when their solutions were incorrect (e.g., for Ada and Tuna in the change in perimeter problem), even if the reasoning behind the two solutions was different, regardless of students' understanding. It can, therefore, be assumed that at the beginning of the study, it was unlikely that pre-service teachers would base their responses on students' mathematical understanding without a deliberate intention to do so (Jacobs et al., 2010).

The findings of the present study revealed that in the final questionnaire, the percentage of suggested instructional actions showing a low level of evidence decreased, and the percentage of suggested instructional actions showing a high level of evidence increased in the context of perimeter-area measurement, which shows the improvement in pre-service teachers' skills of deciding how to respond at the end of the study. In the final questionnaire, the equal and highest percentage of suggestions demonstrated substantial and robust levels of evidence, and about two-thirds of the suggestions showed high levels of evidence in the final questionnaire. Accordingly, most of the instructional decisions provided by pre-service teachers in the final questionnaire were specific suggestions in the context of perimeter-area measurement. Furthermore, there was a statistically significant increase in each problem for the deciding how to respond component in the final questionnaire in the context of perimeter-area measurement. Thus, the increase in the quality of pre-service teachers' decisions on how to respond based on students' understanding in the final questionnaire could be an indication of the effectiveness of the video-based module situated in the pedagogies of practice framework. In addition, in the final questionnaire, pre-service teachers mostly gave student-centered responses in which students took responsibility for the activities rather than teacher-centered responses, and their suggestions to the students generally included the use of concrete models in the activities. A possible explanation for this might be related to students' use of concrete models in all the video clips while solving the tasks in the intervention sessions, which led the pre-service teachers to propose instructional actions involving concrete models in the final questionnaire.

Another finding that stands out from the findings reported earlier was the observation of the greatest improvement in the deciding how to respond component among the noticing components in the context of perimeter-area measurement. This finding was also reported by (Schack et al., 2013). However, this finding is also contrary to previous studies, which have emphasized that the skill of deciding how to respond is challenging to develop (Lesseig et al., 2016) and takes considerable time to develop, even in experienced teachers' deciding how to respond skills (Jacobs et al., 2010). Deciding how to respond based on students' understanding can be developed through purposeful experiences in which students' mathematical thinking and solutions, possible interpretations of students' understanding, and responses to students' mathematical thinking are analyzed and discussed (McDuffie et al., 2014; Schack et al., 2013). In particular, pre-service teachers should have the opportunity to see different types of students' errors and misconceptions and to consider a range of alternative responses (Son, 2013; Son & Sinclair, 2010). The present study included more in-class opportunities for pre-service teachers to examine different students' solutions and to collaboratively notice student thinking, as evidenced in the video clips, and these in-class opportunities were designed to teach pre-service teachers how to respond based on students' understanding. Thus, thinking about and learning alternative teaching methods and approaches may have facilitated pre-service teachers' ability to decide how to respond in this study. For example, the pre-service teachers realized that students often have the misconception that there is a constant relationship between perimeter and area measurement; specifically, the belief that an increase in area automatically means an increase in perimeter, and vice versa in the decomposition of practice. By applying their learning from the decomposition of practice to the approximation of practice, they honed their skills in formulating responses to students. This interactive decision-making practice on how to respond effectively allowed them to provide a higher level of evidence for deciding how to respond in the perimeter-area relationship in the final questionnaire.

5.1.3.2 Pre-post changes in pre-service teachers' deciding how to respond based on students' understanding in the context of volume-surface area measurement

In the initial questionnaire, a very small percentage of the instructional decisions demonstrated a robust level of evidence in the context of volume-surface area measurement, as in the perimeter-area measurement. The highest percentage of the suggestions showed a limited level of evidence, i.e., nearly half of the responses and more than one-third of the suggestions were in the no response category. This finding might be a result of pre-service teachers' lack of knowledge of the curriculum, knowledge of students' learning trajectories, and common students' difficulties or errors in volume and surface area measurement (Shin, 2020). In fact, it is expected that the pre-service teachers' ability to decide how to respond is low at the beginning of the study, as they generally respond with praise for correct answers or tell and explain the answer when they encounter an incorrect answer or a non-standard method (Crespo, 2002; Son & Crespo, 2009). Moreover, the focus on procedures when responding to students' errors is the case for pre-service teachers, although they interpret the errors conceptually (Son, 2013). Similarly, even teachers have difficulties in responding to students' written work, which is limited to affirmation, correction, or a series of guiding questions that direct students to the correct answer or procedure intended by the teacher (Herbal-Eisenmann & Breyfogle, 2005). Interestingly, analysis of instructional actions proposed by pre-service teachers for each problem revealed that pre-service teachers experienced the highest level of difficulty in the surface area comparison problem because none of the pre-service teachers provided a medium, substantial, and robust level of evidence for deciding how to respond in the initial questionnaire while providing such kinds of evidence in the other problems. This finding seemed to imply that each mathematical concept necessitates specific knowledge to suggest a suitable action (Kahan et al., 2003). Moreover, this indicates that various factors, such as mathematical pedagogical knowledge (Schoenfeld, 2011b), prior experience (Erickson, 2011), or context (Mitchell & Marin, 2015), affect teachers' decisions on how to respond to students. This is not surprising since several research studies reported Turkish pre-service teachers' difficulties in the concept of surface area (Çelik & Sağlam-Arslan, 2012; Gökkurt et al., 2015; Gökkurt & Soylu, 2016).

In the final questionnaire, the percentage of instructional decisions with a low level of evidence decreased, whereas the percentage of instructional decisions with a high level of evidence increased in the context of volume-surface area measurement. This shows that there was an improvement in the pre-service teachers' ability to decide how to respond at the end of the study. More than half of the suggestions showed a high level of evidence in the final questionnaire, with a fifth showing robust evidence. There was also a statistically significant increase in each problem for the deciding how to respond component of the final questionnaire in relation to the volume-surface area measurement. According to the statistically significant difference found in this study, the intervention regulated the way in which the pre-service teachers responded to students, which indicates the effectiveness of the video-based module situated in the pedagogy of practice framework. The details of the representation, decomposition, and approximation of practice components of the video-based module situated in the pedagogies of practice are discussed below in relation to the findings of the second research question. Besides, in the final questionnaire, as in the context of perimeter-area measurement, pre-service teachers also tended to give student-centered responses in which students took responsibility for the activities rather than teacher-centered suggestions, and their suggestions usually required students to use concrete models in the activities in the context of volume-surface area measurement. This may be due to the inclusion of concrete models in all of the video clips that pre-service teachers viewed in the intervention sessions, which in turn resulted in pre-service teachers' instructional actions involving concrete models in the final questionnaire.

5.2 Development in pre-service teachers' professional noticing skills through pedagogies of practice

The second purpose of the present study was to explore how the video-based module situated in the pedagogies of practice framework supported pre-service teachers' professional noticing of students' mathematical thinking in perimeter-area measurement and volume-surface area measurement. With this aim, six pre-service teachers' responses to the noticing prompts in the reflection papers with evidence from semi-structured interviews, group discussions, and whole-class discussions are provided to present the improvement in their professional noticing skills. The discussion of findings regarding the development in pre-service teachers' professional noticing is presented under the headings of pre-service teachers' professional noticing skills in the representation-decomposition of practice and pre-service teachers' professional noticing skills in the approximation of practice, which are the components of the pedagogies of practice framework.

5.2.1 Pre-service teachers' professional noticing skills in the representation-decomposition of practice

The following part is devoted to the discussion of the findings regarding how six pre-service teachers who were in the same small discussion group noticed students' mathematical thinking, i.e., how they attended to students' mathematical thinking, interpreted students' understanding and decided how to respond based on students' understanding in the representation-decomposition of practice.

5.2.1.1 Pre-service teachers' attending to students' mathematical thinking in the representation- decomposition of practice

In the present study, while the first three sessions of the seven intervention sessions were about perimeter-area measurement, the other four sessions were about volume-surface area measurement. The discussion of the findings regarding pre-service teachers' attention to students' mathematical thinking in the representation-decomposition of practice, which is a component of pedagogies of practice, is presented for perimeter-area measurement and volume-surface area measurement, respectively.

5.2.1.1.1 Pre-service teachers' attending to students' mathematical thinking in the representation-decomposition of practice in the context of perimeter-area measurement

In the present study, looking at what pre-service teachers identified on an individual basis in reflection paper 1 enabled the researcher to examine the mathematically significant details that pre-service teachers noticed while analyzing students' mathematical thinking on their own. A pre-service teacher may not have noticed a particular detail individually. However, when the detail is identified by another pre-service teacher, he/she can participate in the discussion about that detail. Since much of the work of teaching is done individually, pre-service teachers, as future teachers, need to be able to identify details for teaching on their own. Therefore, in addition to examining what pre-service teachers noticed supported by a group setting, it was also useful to examine what they noticed individually. Asking pre-service teachers to write reflection paper 1 in each session also enabled the researcher to explore how pre-service teachers' identifying mathematically significant details changed over time.

In this study, the first three sessions of the seven intervention sessions were about perimeter-area measurement. Two pre-service teachers (pre-service teacher E and pre-service teacher S) in the group provided high levels of evidence for attending in reflection paper 1, even in the early sessions (first and second sessions). The previous experiences and backgrounds of these pre-service teachers may have influenced how they engaged in students' mathematical thinking processes (Sánchez-Matamoros et al., 2019) and enabled them to provide high levels of evidence for attending. Representations of practice can be described as a window into practice (Grossman et al., 2009). In the present study, the structure of the video clips used as representations of practice made students' thinking visible and illustrated students' thinking, which focused directly on students' mathematical thinking rather than classroom videos that included complex classroom environments, may have enabled the pre-service teachers to attend to students' mathematical thinking even in the first session. Moreover, the intervention was designed as a tool to focus pre-service teachers' attention on the mathematically significant details in students' solutions about perimeter-area measurement, which may have enabled pre-service teachers to pay more attention to students' mathematical thinking using video clips, each of which showed only one student working on a task in an interview setting. Besides, the pre-service teachers were given the opportunity to focus on an individual student's thinking within a very specific content domain, i.e., perimeter and area measurement, which may have helped them zoom into students' mathematical thinking. This may have supported their analysis of students' mathematical thinking in the video clips from the very beginning of the intervention. On the other hand, four pre-service teachers found it difficult to identify mathematically significant details in the early sessions when they individually analyzed students' mathematical thinking. These pre-service teachers tended to provide a lack of evidence or limited evidence for attending in the first session, even though they viewed video clips, each depicting only one student's engagement in solving a task in an interview setting. This finding is consistent with the findings of the previous research studies (Jacobs et al., 2011; Ulusoy, 2020). Attending to students' mathematical thinking in a specific content domain requires the ability to focus on mathematically significant details as well as mathematical knowledge of teaching (Schlesinger et al., 2018). Accordingly, these four pre-service teachers' lack of knowledge about perimeter-area measurement might have led them to provide a low level of evidence for attending.

In the present study, pre-service teachers attended to students' misconceptions, strategies, and mathematical language in the representation-decomposition of practice. As a result of having incorrect conceptions and insufficient knowledge of perimeter-area measurement, in the first session, four pre-service teachers identified the student's solution incorrectly when individually analyzing the video clip, whereas two pre-service teachers (pre-service teacher E and pre-service teacher S) with higher initial attending skills could identify the incorrect solution of S2 in VC2. These four pre-service teachers thought that the perimeter did not change when the student cut a piece of paper from the inside of the rectangular paper. That is, they considered the perimeter to be related to the outer sides of the shape. Since these pre-service teachers did not seem to have a strong understanding of the perimeter, it may not be surprising that they failed to notice S2's limited understanding of the perimeter concept in VC2. Therefore, a possible explanation for this might be that teachers' content knowledge is a prerequisite for their professional noticing (Bartell et al., 2013). This can also be attributed to pre-service teachers' lack of familiarity with the perforated shapes/punctured squares. Fortunately, during the discussions, they became aware of students' incorrect answers that they could not notice in the individual analysis and enhanced their subject matter knowledge. In addition, pre-service teachers recognized their own existing misconceptions and errors during the discussions. They corrected these errors and misconceptions in reflection paper 2 after the discussions and internalized the meaning of mathematical concepts at the end of the intervention sessions. Pre-service teachers also highlighted the effects of analyzing and discussing students' mathematical thinking in the video clips on their own concept development, which is consistent with the previous research studies (Ulusoy, 2016). Accordingly, representation and decomposition of practice also provided an opportunity to eliminate pre-service teachers' misconception that perimeter is the distance around the figure and thus to improve their attending skills.

Decompositions of practice articulate and break down key components of the practice so that they can be taught to and practiced by pre-service teachers (Grossman et al., 2009). In the current study, pre-service teachers re-engaged with the students' solutions in discussions, in which they had a chance to break down the video clips provided as representations of practice. In this way, the decomposition of practice enabled the pre-service teachers to focus on the specific details of students' mathematical thinking by breaking it down into small pieces without burdening them. Accordingly, pre-service teachers had a chance to recognize students' incorrect answers that were not noticed in the individual analysis, to become aware of the new details of students' mathematical thinking in the video clips, and to deepen their knowledge during the discussions. Thus, the pre-service teachers not only identified mathematically significant details in the students' solutions and but also learned how to attend to students' mathematical thinking through the decomposition of practice. Furthermore, what pre-service teachers discussed as a whole class was also generally discussed as a group beforehand in the context of perimeter-area measurement. In group discussions, pre-service teachers with higher attending skills helped the other pre-service teachers become aware of incorrect solutions and mathematical details in the students' solutions that they had not individually identified. This shows that support for higher levels of noticing can come from peers (Bragelman et al., 2021). Accordingly, in the sessions on perimeter-area measurement, pre-service teachers with low attending skills in the individual analysis reflected what they learned from others in the discussions in reflection paper 2 and increased their level of attending at the end of each session. In addition, improvement was observed in five pre-service teachers' attending skills from the first session to the third session in the individual analysis. This finding suggests that breaking down students' mathematical thinking into parts allowed pre-service teachers to gain a better understanding of what to look for and how to describe mathematically significant details in the students' solutions.

5.2.1.1.2 Pre-service teachers' attending to students' mathematical thinking in the representation-decomposition of practice in the context of volume-surface area measurement

In the first session on volume-surface area measurement (fourth session), all pre-service teachers, except pre-service teacher K, provided high levels of evidence in reflection paper 1. Fortunately, after the fourth session, pre-service teacher K also provided high levels of evidence. Thus, the pre-service teachers who provided low level evidence for attending in reflection paper 1 in the early sessions that focused on perimeter-area measurement were able to provide a high level of evidence in the late sessions (sixth and seventh sessions) that focused on volume-surface area measurement individually, i.e., without support from discussions. Consequently, all six pre-service teachers provided substantial or robust evidence for attending to mathematical details in the late sessions. Accordingly, pre-service teachers considerably increased their attention to mathematically significant details in the students' thinking as they continued to analyze the video clips. The findings indicate that pre-service teachers developed stability in providing a high level of evidence in the late sessions and that the representation and decomposition of practice in the early sessions overwhelmingly allowed them to make sense of student thinking in the late sessions.

The improvement in pre-service teachers' attending skills shows the importance of focusing on a particular content domain in terms of attending to students' mathematical thinking in the representations of practice. Representations of practice make practice observable (Grossman et al., 2009). Research indicated that analysis of video clips, including students' mathematical thinking in particular content domains, may have supported pre-service teachers (Ulusoy & Çakıroğlu, 2021; Walkoe, 2015). While designing video-based professional development programs on a particular content domain, it is critical to include videos that show students engaging in a wide range of tasks in that domain. In this way, observing a variety of student thinking in a particular domain can help teachers discover the nuances between ways of thinking in that domain (Walkoe, 2013). In order for pre-service teachers to notice students' mathematical thinking deeply about the content domain, video clips should represent different kinds of student thinking that pre-service teachers are required to discuss. Accordingly, in the present study, pre-service teachers viewed 19 video clips as representations of practice, including eight video clips on perimeter-area measurement and 11 video clips on volume-surface area measurement; each session involved two or three video clips showing different students engaging in the same task, a total of seven tasks. This might have led to fruitful and rich discussions during the intervention sessions and enabled pre-service teachers to attend to students' mathematical thinking by comparing and contrasting different kinds of thinking in this particular content domain.

In the representation of practice used in the present study, the researcher interacted with middle school students, and hence, pre-service teachers had a chance to see how task-based interviews are conducted with lively sixth, seventh, and eighth graders. Thus, they viewed a video clip of the researcher interacting with a student in the context of volume-surface area measurement, observed how the students' mathematical thinking was elicited, and additional questions were asked based on the students' responses. Furthermore, concrete models were deliberately presented to students while solving the tasks. Since students could express themselves better with concrete models, this may have enabled pre-service teachers to attend to student thinking more. In addition, decomposition of practice means creating smaller components that pre-service teachers can master, enabling them to effectively comprehend the components of the practice (Grossman et al., 2009). In this study, through the decomposition of represented practice, pre-service teachers were able to recognize students' mathematical thinking with greater detail and focus, leading to an enhanced comprehension of what aspects to observe and how to describe their observations effectively. Thus, the pre-service teachers with low attending skills in the early sessions started to reflect on students' conceptions of measurement throughout the intervention sessions. Accordingly, during the intervention, pre-service teachers' attention shifted from providing none of the mathematically significant details in students' solutions to providing all mathematically significant details correctly. This finding suggests that representations and decompositions of practice provided pre-service teachers with opportunities to learn to attend to students' mathematical thinking by complementarily working. This is consistent with previous research studies in which science pre-service teachers learned to attend to students' thinking when they were shown how to do so (Barnhart & van Es, 2015) in a video-based course. Moreover, one pre-service teacher, E, consistently had higher expertise in attending to students' mathematical thinking throughout the intervention sessions because he could provide a substantial or robust level of evidence in each session. This finding suggests that the design of the video clips in the present study could support pre-service teachers' sustained attention to students' mathematical thinking, as Gonzalez and Skultety (2018) stated.

Interestingly, pre-service teacher S had the highest initial attending skill in the group in the first session and always provided a substantial or robust level of evidence in each session, except in the fifth session. The fifth session focused on the enumeration of cubes to measure the volume. S2 in VC13 found out the number of cubes that could be stacked along the height of each prism. The student then worked out the number of cubes that would fit into one face of the prism and multiplied this number by the number of faces. Pre-service teacher S identified this student's solution incorrectly, believing that the student did not have a misconception but only made a calculation error. As provided in the findings, pre-service teacher S had a chance to recognize the student's incorrect strategy based on the faces of the prisms, which is one of the critical misconceptions in volume measurement during the discussions.

Thus, discussing the student's incorrect reasoning in depth increased the attending level of the pre-service teacher after the discussions.

In some cases (discussions around VC11, VC13, and VC14), pre-service teachers realized mathematically significant details during the whole-class discussions that were not mentioned in group discussions. This situation reveals the importance of integrating whole-class discussions after the group discussions, as in this study. Moreover, pre-service teachers had an opportunity to share ideas with their peers and also compare and criticize them in group discussions and in whole-class discussions. In this way, pre-service teachers' attending to students' mathematical thinking varied as they interacted with their peers. The additions that pre-service teachers made in reflection paper 2 and the increase in their attention levels after the discussions indicate the influence of peers' ideas on identifying mathematically significant details in students' solutions in a positive manner, which can be explained by the social constructivist learning environment (Vygotsky, 1978). Accordingly, video-based discussions that focus on students' mathematical thinking in the context of volume-surface area measurement affected the development of attending skills of the pre-service teachers in this study. The decomposition of represented practice reduced the complexity of pre-service teachers' analysis by focusing their attention on mathematically significant details in the students' solutions and gave pre-service teachers an opportunity to analyze and reflect on salient features of students' mathematical thinking. Accordingly, each week's discussion around video clips facilitated by the researcher addressed noticing components, which may have provided pre-service teachers with opportunities to attend to specific aspects of students' work and to make sense of that work. In addition, the use of video clips produced by the researcher as a representation of practice through the task-based interviews with middle school students enabled the manageable focus for discussions by limiting the number of salient features. In this regard, observing students' mathematical thinking through video clips as representations of practice might allow pre-service teachers to prepare for in-the-moment situations in a real classroom environment, as Schack et al. (2013) suggested. Moreover, while designing video-based professional development programs, the selection of video clips to be viewed is crucial. Video clips should be of a quality that can stimulate productive discussions among participants (Sherin et al., 2009). In this sense, the present study provided the suggested dimensions of video clips that student thinking was visible (windows), student mathematical thinking was deep (depth), and students were clear while expressing their ideas (clarity).

5.2.1.2 Pre-service teachers' interpreting students' understanding in the representation-decomposition of practice

In the present study, while the first three sessions of the seven intervention sessions were about perimeter-area measurement, the other four sessions were about volume-surface area measurement. The discussion of the findings regarding pre-service teachers' interpretation of students' understanding in the representation-decomposition of practice, which is a component of pedagogies of practice, is presented for perimeter-area measurement and volume-surface area measurement, respectively.

5.2.1.2.1 Pre-service teachers' interpreting students' understanding in the representation-decomposition of practice in the context of perimeter-area measurement

The findings of this study show that four pre-service teachers, except pre-service teacher S, provided a lack of evidence or limited evidence for interpreting students' understanding in reflection paper 1 in the early sessions (first and second sessions) while analyzing the video clips individually in the context of perimeter and area measurement. That is, pre-service teachers were inclined to provide an incorrect interpretation of the student's understanding, make comments only about whether the student understood or not, make comments about the student's understanding without any mathematical properties in broad terms, or blame the student for lack of knowledge. This means that it was not easy for pre-service teachers to make use of evidence when interpreting students' solutions in the early sessions. This finding is consistent with that of Jacobs et al. (2010), who found that half of the pre-service teachers provided limited evidence of interpretation, and none of them provided robust evidence without intervention. This finding might be explained by the fact that interpreting students' understanding requires paying attention to students' strategies as well as having sufficient understanding to relate how these strategies reflect understanding of mathematical concepts, which makes the interpreting component more challenging than the attending component. On the other hand, pre-service teacher S, who provided high levels of evidence for attending in reflection paper 1 in the early sessions, provided high levels of evidence for interpreting in the same way. Since the identification of mathematical details has an important role in the interpretation of students' mathematical understanding (Callejo & Zapatera, 2017; Fernández et al., 2013; Magiera et al., 2013), pre-service teacher S' ability to identify the mathematically significant details in the students' solutions may have allowed her to provide valid justifications for these students' understanding.

Decomposition of practice is the unpacking of the complex practice into the smaller components that make up a practice (Grossman et al., 2009). In the present study, the decomposition of practice allowed pre-service teachers to examine and consider possible reasons behind the students' mathematical thinking. Consequently, an improvement was observed in three pre-service teachers' interpreting skills from the first session to the third session in the individual analysis. Yet, this improvement was less than the improvement that occurred in their attending skills in the context of perimeter and area measurement since, as Tekin-Sitrava et al. (2022) emphasized, this skill is more challenging than the attending skill. In addition, in terms of interpreting students' understanding, whole-class discussions were more fruitful and richer compared to group discussions because justifications about the possible reasoning behind students' solutions that were not provided in group discussions were provided in whole-class discussions. In this sense, in whole-class discussions, pre-service teachers put multiple interpretations on students' understanding, which highlights the importance of holding whole-class discussions in addition to group discussions in professional development programs. Indeed, pre-service teacher K's comment in the interview supports this: "Group discussion was also useful, but especially whole-class discussion was more effective. There were many different thoughts there." As a result, pre-service teachers made additions in reflection paper 2, and the increase observed in their levels of interpretations after the discussions showed the positive influence of peers' ideas on providing the possible reasons underlying students' understanding, which stressed the effect of collaboration on fostering professional noticing (Ulusoy & Çakıroğlu, 2021). For instance, pre-service teacher A's explanations during the interview, as indicated in the findings, show that he has become aware that possible reasons for students' misunderstanding of perimeter-area measurement are their procedural knowledge and rote memorization, which in turn result in establishing incorrect relationships between the concepts of perimeter and area.

As students have many misconceptions about perimeter and measurement, as reported in the literature, some of the video clips in this study were deliberately produced to include these misconceptions in order to determine the extent to which pre-service teachers were aware of these misconceptions. As presented in the findings, pre-service teachers became aware of the students' misconceptions and difficulties in perimeter and area measurement, such as believing that the perimeter is only related to the outer sides of the shape and believing that there is a linear relationship between perimeter and area. This finding was also reported by Çaylan Ergene and Işıksal Bostan (2022) and Girit Yildiz et al. (2022). Thus, in addition to the improvement of professional noticing skills, pre-service teachers' knowledge about students' difficulties and misconceptions was also enhanced through their involvement in this study (Lannin et al., 2013). Knowledge about students' difficulties and misconceptions is related to knowledge of content and students, and teachers use this kind of knowledge to "hear and interpret students' emerging and incomplete thinking as expressed in the ways that pupils use language" (Ball et al., 2008, p. 401). This implies a close and important link between interpreting students' understanding component of professional noticing of students' mathematical thinking and knowledge of content and students (Styers et al., 2020). In addition, the discussion environment in the intervention sessions enabled pre-service teachers to find out important concepts such as boundary for perimeter measurement in the first and second sessions and covering for area measurement in the second and third sessions, and to interpret students' understanding by using these concepts, which helped them elaborate on their interpretations of students' understanding after the discussions. Indeed, this was demonstrated by one of the statements written by pre-service teacher A's in reflection paper 2 after the discussions: "During the discussions, I learned that the perimeter is related to the boundaries so that the perimeter will increase."

5.2.1.2.2 Pre-service teachers' interpreting students' understanding in the representation-decomposition of practice in the context of volume-surface area measurement

While individually analyzing students' work in the late sessions on volume-surface area measurement, pre-service teachers were generally better able to identify mathematically significant details in students' solutions and were more inclined to go beyond simply determining whether solutions were correct when assessing students' understanding. In this way, in the late sessions, they could provide justifications about the possible reasoning behind the students' solutions. This shows that throughout the intervention, pre-service teachers gradually began to draw upon evidence when interpreting students' understanding. Thus, the findings of this study revealed that with a designed intervention built on students' mathematical thinking, pre-service teachers were able to provide valid justifications for students' mathematical understanding. Furthermore, what is surprising is that the level of evidence provided by three pre-service teachers in interpreting students' understanding decreased, while the level of evidence provided by three other pre-service teachers remained the same when moving from the fourth to the fifth session. As Warshauer et al. (2021) argued, this can be explained by the high cognitive demand for the mathematical task in the fifth session that focused on the enumeration of cubes to measure the volume. Several studies also reported students and pre-service teachers' difficulties in enumerating unit cubes in finding volume (Alstad et al., 2023; Esen & Çakıroğlu, 2012; Tekin-Sitrava & Işıksal-Bostan, 2014; Tekin-Sitrava & Isiksal-Bostan, 2018).

Bearing in mind the significance of involving more than one student's solution (Sánchez-Matamoros et al., 2019), in the present study, video clips were intentionally produced in a way that included two or three different students' solutions to the same task to present pre-service teachers with cases reflecting different student reasoning. Decomposition of practice is accomplished by breaking down the complex practice into smaller, more manageable practices (Grossman et al., 2009). Accordingly, in the present study, the decomposition of practice through small group and large group discussions that took place in the sessions provided pre-service teachers an environment to compare different solutions and recognize the different understanding characteristics of different students and the possible reasons underlying this understanding by reducing the complexity of interpreting students' understanding and by focusing pre-service teachers' attention on particular aspects of students' understanding. During the discussions, instead of focusing only on errors in students' solutions, they started to investigate the possible reasons behind the students' errors and misconceptions, elaborating their interpretations of students' understanding and developing different ideas about the possible causes of problematic situations in students' understanding. Indeed, this can also be seen in pre-service teacher K's one of the statements in the interview: "When the student makes a mistake, it is not only wrong, but it can go to different places as to why he/she did wrong. I realized this more." Representations of practice are different ways through which practice is portrayed in professional education (Grossman et al., 2009). Accordingly, in the present study, analyzing video clips of different students solving the same tasks provided as representations of practice, which includes those having different characteristics of understanding, might have enabled pre-service teachers to better interpret students' understanding in the late sessions. Pre-service teachers also generally increased their level of interpreting students' understanding at the end of the sessions. Additionally, previous research suggests that teacher education programs should focus on practice to improve pre-service teachers' professional noticing skills (Ivars et al., 2020), but this does not imply learning in real situations (Ball & Cohen, 1999). In the present study, this was achieved by using students' video clips, and it is considered that pre-service teachers learned from practice since they started to use evidence that showed students' understanding as the sessions progressed.

In the present study, pre-service teachers became aware of the students' misconceptions and difficulties in volume and surface area measurement, such as believing that volume is conserved when the area is conserved and believing that the surface area of objects changes as their positions change, consistent with the findings of Girit Yildiz et al. (2022). In addition, the discussion environment in the sessions enabled pre-service teachers to discover important concepts, such as the iteration of layers, and to interpret students' understanding by using these concepts, which helped them elaborate on their interpretations of students' understanding after the discussions. To illustrate, in the fifth session, S8 in VC14 built two prisms using unit cubes. The student first constructed the first layer, i.e., the base, and then she stacked this layer along the height. While finding the total number of unit cubes, she first found the number of cubes in a vertical layer, and then she multiplied this number by the number of layers. During the discussions about interpreting this student's understanding, as presented in the findings, the layering strategy that the student used was one of the important strategies that students use when calculating volume. This in-depth analysis of this strategy enabled the pre-service teachers to relate their comments to it in reflection paper 2, as can be seen in the comment of pre-service teacher G: "During the discussions, I realized that the student found the volume by dividing the cubes into layers, and her understanding was based on a layering approach."

5.2.1.3 Pre-service teachers' deciding how to respond in the representation-decomposition of practice

In the present study, while the first three sessions of the seven intervention sessions were about perimeter-area measurement, the other four sessions were about volume-surface area measurement. The discussion of the findings regarding pre-service teachers' deciding how to respond based on students' understanding in the representation-decomposition of practice, which is a component of pedagogies of practice, is presented for perimeter-area measurement and volume-surface area measurement, respectively.

5.2.1.3.1 Pre-service teachers' deciding how to respond in the representation-decomposition of practice in the context of perimeter-area measurement

In the present study, pre-service teachers experienced difficulties in deciding how to respond in the early sessions that focused on perimeter-area measurement. In this regard, their instructional suggestions in reflection paper 1 mostly demonstrated limited and medium evidence. To illustrate, in the first session, some of the pre-service teachers were distracted by certain elements in S1's solution in VC1, i.e., the student's way of cutting the paper. This led them to address peripheral issues in their suggestions, such as demonstrating that the perimeter increases or remains constant with different cutting techniques rather than key conceptual issues in the students' thinking, in this case, the relationship between perimeter and area. The difficulty for pre-service teachers to focus on relevant mathematics has been noted in several research studies (Anthony et al., 2015; Monson et al., 2020; Sleep, 2012). Particularly, it was more difficult for them to extend the students' understanding in cases where the solution was correct and there was no misconception rather than eliminating the students' misconceptions. Most of them could not provide any instructional suggestions in reflection paper 1 in the analysis of video clips involving correct student solutions. In other words, they had difficulties finding ways to deepen students' understanding or provide enriching experiences for students who have already grasped the concept. This finding may be related to the fact that the degree of complexity in these two forms (extending students' understanding and eliminating students' misconceptions) is different, and also, teachers have a stronger inclination to take over student thinking when they encounter incorrect solutions (Jacobs et al., 2022). This finding is in line with those of previous research studies in which pre-service teachers did not prefer to ask students for further explanations when problems were correctly solved by students (Çaylan Ergene & Işıksal Bostan, 2022; Kilic, 2018; Moyer & Milewich, 2002; Sun & van Es, 2015). In addition, even though pre-service teachers could attend to students' strategies, they struggled with deciding how to respond to students in order to extend students' understanding by building on their existing knowledge because identifying mathematically significant details did not guarantee that they could be used effectively (Barnhart & van Es, 2015; Jacobs et al., 2010). Decomposition of practice entails breaking down the practice into discrete components and directing learners' focus on these (Grossman et al., 2009). In the present study, the pre-service teachers' critical examination of students' mathematical thinking and constructing, defending, and discussing instructional suggestions based on students' mathematical understanding in small groups and as a whole class helped them to make suggestions in reflection paper 2 at the end of the sessions using these newly learned ideas. Thus, pre-service teachers increased their levels of deciding how to respond at the end of each session, which shows the effect of the decomposition of practice into smaller and more manageable parts that focused on instructional decisions.

5.2.1.3.2 Pre-service teachers' deciding how to respond in the representation-decomposition of practice in the context of volume-surface area measurement

One interesting finding is that between early and late sessions, there were shifts between low and high levels of evidence provided by some pre-service teachers in reflection paper 1. These shifts during the acquisition of a new skill may be due to movement within the zone of proximal development (Vygotsky, 1978) and self-scaffolding, which means learners' functioning at both high and low levels while constructing new knowledge (Granott et al., 2002). In the late sessions, on the other hand, the instructional actions proposed by all six pre-service teachers showed a substantial and robust level of evidence in reflection paper 1. That is, they could provide specific suggestions based on students' understanding. These six pre-service teachers also provided substantial or robust evidence for attending to mathematical details in the late sessions. One of the issues that emerges from these findings is that when teachers decide how to respond based on students' understanding, they are also likely to pay attention to students' mathematical thinking. However, the reverse is not always the case. When teachers attend to students' mathematical thinking, they may or may not decide how to respond based on students' understanding (Jacobs et al., 2011). This can be illustrated by pre-service teacher S, who provided a limited level of evidence for deciding how to respond, although she provided a robust level of evidence for attending in the fourth session in the present study.

Decomposition of practice is breaking down the practice into parts and identifying the component parts of practice (Grossman et al., 2009). In the present study, pre-service teachers had opportunities to observe different student solutions, talk about the mathematical details in these solutions with their peers, hear different ideas, and reflect on possible instructional decisions during the intervention sessions. Moreover, alternative instructional actions that the pre-service teachers did not think of in their individual and group analyses were brought up in whole-class discussions. In this way, pre-service teachers became aware of the other instructional actions and reflected these in reflection paper 2 after the discussions as suggestions for how to respond to students. As mentioned above, there was also a significant shift from the focus on direct instruction in the early sessions to the focus on suggestions for students to discover the concepts themselves and improve their conceptual understanding by building from students' thinking in the late sessions. Thus, the change in pre-service teachers' deciding how to respond skills in the late sessions in a positive manner showed that these opportunities were effective. Pre-service teachers' self-evaluations regarding how to respond to students in the interviews enlightened further how these opportunities promoted pre-service teachers' ability to decide how to respond. Thus, the decomposition of the practice of professional noticing was useful in supporting pre-service teachers to develop a particular component of this complex practice, i.e., deciding how to respond by engaging pre-service teachers in learning experiences that modeled the decision-making aspect of professional noticing.

Besides, pre-service teachers' expressions in the interviews regarding the evaluation of the implementations during the sessions reveal that representation and decomposition of practice in the present study provided the necessary structure to develop pre-service teachers' skills in deciding how to respond. These expressions also show how what the pre-service teachers learned during the sessions can be transferred to their future instructions. Considering the research that relates pre-service teachers' beliefs to their ability to center student thinking (Ding & Domínguez, 2016), instilling a disposition in pre-service teachers to notice students' mathematical thinking is crucial before entering the teaching profession. In this study, pre-service teachers learned how to attend to students' mathematical thinking and learned how to use mathematically significant details in deciding how to respond so that their suggestions kept students' thinking at the center. Indeed, pre-service teacher A's comment in the interview supports this: "I realized that direct instruction to eliminate students' misconceptions is not so effective. I learned various methods from my friends about how we can eliminate students' misconceptions in discussions. I became aware that we need to get rid of traditional teaching methods and use different techniques." Therefore, pre-service teachers should be taught how to take appropriate instructional actions based on students' mathematical understanding and given time to practice, as in this study, which in turn may also affect their classroom instruction. As Monson et al. (2020) asserted, giving preservice teachers a chance to reflect on and discuss alternative instructional moves in teacher education settings may assist them in recognizing effective practices that they can use in classrooms in the future.

5.2.2 Pre-service teachers' professional noticing skills in the approximation of practice

Pre-service teachers participated in an approximation of practice where they designed a task on perimeter-area measurement or volume-surface area measurement and conducted an interview with a middle school student around that task. This enabled pre-service teachers to try out the tasks with real students. Moreover, by video-recording the task-based interviews they conducted, they had a chance to view the video recording and analyze the students' mathematical thinking they elicited. Conducting task-based interviews as an approximation of practice might have provided pre-service teachers with an opportunity to practice noticing interactively with a method other than paper-pencil or verbal. In addition, by considering the wide use of the written report that follows video analysis to uncover pre-service teachers' noticing skills (Roller, 2015; Santagata et al., 2007), pre-service teachers in the present study reflected what they observed during their analysis to reflection paper 3, which in turn helped the researcher to determine their professional noticing skills in approximation of practice. The findings for attending to students' mathematical thinking, interpreting students' understanding, and deciding how to respond based on students' understanding in the approximation of practice, which is a component of pedagogies of practice, in both contexts are discussed together as they support each other.

In the tasks pre-service teachers designed, they included the use of concrete models like the tasks in the video clips (tangram and cardboards for perimeter-area measurement and unit cubes, paper prisms, and soma cubes for volume-surface area measurement), which shows the influence of the video-based module on pre-service teachers' preferences while deciding their own practices. In the context of both perimeter-area measurement and volume-surface area measurement, all six students who were interviewed by the pre-service teachers had misconceptions similar to those in the video clips that the pre-service teachers viewed as representations of practice and analyzed as decompositions of practice in the intervention sessions. Approximations of practice are opportunities to simulate certain aspects of professional practice (Grossman et al., 2009). In the stage of approximation of practice, attending to students' mathematical thinking involved eliciting students' thinking as well as listening to students carefully to identify the mathematically significant details in students' mathematical thinking based on what they do and say. This study found that all six pre-service teachers provided a robust level of evidence for attending to the mathematical details and interpreting students' understanding in the context of task-based interviews they conducted as an approximation of practice.

This means that they identified all of the mathematically significant details in the interviewed students' thinking and provided detailed explanations for students' mathematical thinking in the approximation of practice. The fact that the pre-service teachers were specific in their responses in reflection paper 3 and all of them had a robust categorization shows that these pre-service teachers had a strong ability to identify indicators of students' mathematical thinking. This can be attributed to the fact that pre-service teachers' discovering how to elicit and analyze students' mathematical thinking through the representation and decomposition of practice during the intervention sessions enabled them to attend to students' mathematical thinking and interpret students' understanding in the approximation of practice by providing the highest level of evidence in reflection paper 3. In this sense, task-based interviews conducted outside the classroom, serving as an approximation of practice, helped pre-service teachers develop a better understanding of how students think, which they gained in the representation-decomposition of practice.

Pre-service teachers make specific claims about how a student thinks or reasons mathematically that might be outside the knowledge base of people in other professions (Amador, 2020). In reflection paper 3, while responding to the noticing prompts, pre-service teachers used evidence that occurred during the task-based interviews, i.e., they conducted evidence-based analyses. Thus, pre-service teachers' being specific in evidence and effort to link that evidence to their claims in the present study can be related to professional vision (Goodwin, 1994) because in explaining what they noticed, they showed their ability to discuss the characteristics of their profession. Therefore, pre-service teachers' conducting task-based interviews is important to support them in understanding students' mathematical thinking deeply and providing evidence-based claims (Weiland et al., 2014). In the present study, pre-service teachers were precise in supporting their ideas, and they commonly provided descriptions of students' actions and used direct quotes of what students said during the interviews as a way to communicate evidence. This finding suggests that pre-service teachers' integration of concrete models into the tasks encouraged them to pay attention to students' actions, which in turn led them to attend to nonverbal student thinking while attending to the details in students' mathematical thinking while working with concrete models (Dominguez, 2019; Lam & Chan, 2020). In addition, students' use of concrete models to explain their solutions might have allowed the pre-service teachers to draw out the connections the students made between the actions and statements. This might have enabled the pre-service teachers to identify the mathematically significant details in students' mathematical thinking better.

Approximations of practice emphasize creating and executing instructional tasks where pre-service teachers engage in teaching-related activities (Zeichner, 212). In the present study, the pre-service teachers designed geometric measurement tasks and implemented these tasks with middle school students through task-based interviews. Pre-service teachers' paying attention to students' thinking and creating questions that address this thinking through the approximation of practice might have helped foster the development of pre-service teachers' responsive teaching. In this kind of teaching, teachers continuously modify their instructional decisions regarding what to emphasize and how to deliver it, adapting in real-time to the specific cognitive processes exhibited by their students (Jacobs & Empson, 2016). With the approximation of practice, the pre-service teachers had a chance to engage in practice and conduct interviews with the students through the tasks they designed, which are similar to those they would implement in school settings when they become teachers. Approximations that do not retain the complexity of actual practices are considered less authentic (Grossman et al., 2009). On the other hand, approximations that involve pre-service teachers in tasks similar to those carried out in school settings are considered to be more authentic. Therefore, in the present study, practicing task-based interviews with students as an approximation provided pre-service teachers with opportunities to anticipate and respond to student thinking. This led to more authentic practice for pre-service teachers by increasing complexity and interaction with students and closely approximating actual practice. In reflection paper 3, pre-service teachers offered some instructional actions as a response to interviewed students by adopting those that were suggested during the group discussions and whole-class discussions, i.e., by using newly learned ideas through representation and decomposition of practice. Thus, as a result of learning how to respond through the representation and decomposition of practice, pre-service teachers designed student-centered tasks, suggested detailed instructional moves, and established strong relationships between students' understanding and proper instructional moves. In addition, pre-service teachers viewed video clips of students being interviewed about perimeter-area measurement and volume-surface area measurement as representations of practice during the intervention sessions. Purposeful production and selection of the video clips seemed to direct the pre-service teachers' attention to mathematics and student thinking and the connection between student thinking and instructional actions because all the instructional suggestions provided by the pre-service teachers in reflection paper 3 were student-centered. Thus, the video-based module situated in the pedagogies of practice framework might have enabled pre-service teachers to adopt a student-centered approach in an approximation of practice while proposing instructional moves. In this sense, the last stage of the intervention by executing practice might have helped to understand how pre-service teachers developed their decision-making skills with the representation and decomposition of practice.

Moreover, the pre-service teachers tried to elicit students' mathematical thinking with the questions that they prepared beforehand. Hence, the extent to which they noticed student thinking was also related to the questions they asked during the interview and their clinical interview skills, i.e., whether they could elicit student thinking. As a response to the question in reflection paper 3, "If you were to do the interview again, what would you change?", all six pre-service emphasized the importance of questioning because they stated that they would ask different questions or more questions in addition to the ones they asked. In this regard, approximation of practice allowed pre-service teachers to experience task-based interviews and test the interview questions they prepared beforehand. Accordingly, they realized the importance of questioning, that is, asking students the right questions in the approximation of practice. In this study, the pre-service teachers engaged in observational listening, which is a type of listening (Empson & Jacobs, 2008) since they did not attempt to correct students or steer them towards a particular answer. They also engaged in responsive listening to a degree since they tried to make sense of students' thinking and probe specific aspects of students' thinking during the interview. This might have helped them attend to the mathematically significant details of what a student is doing and saying by providing robust evidence. This is encouraging since this was the pre-service teachers' first experience of conducting task-based interviews. Yet, they still struggled with questioning as they indicated in reflection paper 3 that they would ask more questions to better understand students' mathematical ideas if they were to do the interview again. Thus, the process of conducting task-based interviews and reflecting on their own practice was particularly useful in allowing pre-service teachers to determine where they struggled in eliciting and attending to students' mathematical thinking. However, in the present study, pre-service teachers attempted to make sense of students' mathematical thinking by probing specific aspects of that thinking, which contrasts with Jacobs and Ambrose's (2008) finding that teachers had difficulty in making sense of students' thinking and resorted to asking general questions. Pre-service teachers also learned how to elicit students' mathematical thinking by observing the researcher's questions in the task-based interviews in the video clips provided as representations of practice, in addition to the professional noticing of students' mathematical thinking through the decomposition of practice. This may explain why pre-service teachers were good at eliciting students' mathematical thinking by asking probing questions and making sense of students' mathematical thinking in the approximation of practice.

Furthermore, teachers are generally inclined to stress procedural understanding in schools in the context of perimeter-area measurement and volume-surface area measurement. Teachers' being less likely to attempt to understand student thinking in this area will result in students being at a disadvantage (Walkoe, 2015). Accordingly, one of the benefits of the video-based module situated in the pedagogies of practice framework was that pre-service teachers recognized in simulating the practice that the students with whom they conducted interviews lacked a conceptual understanding of geometric measurement. Thus, conducting task-based interviews with students is an effective approach for pre-service teachers to understand the significance of developing conceptual understanding in mathematics in addition to procedural understanding; which form of learning is more effective than simply telling or having them read in a methods course as Lesseig et al. (2016) suggested. In this regard, pre-service teachers might have benefitted from analyzing their own practice in approximations of practice in the present study. Indeed, two of the pre-service teachers were surprised at the students' correct responses (pre-service teacher B was surprised at the student's indicating the equality of the area of the shapes with the same pieces at the beginning of the interview, and pre-service teacher G was surprised at the student's answering the questions about volume easily. On the other hand, three pre-service teachers were surprised at the students' incorrect responses (pre-service teacher K was surprised at the student's confusing perimeter and area concepts, pre-service teacher S was surprised at the student's failure to try to build a relationship between the area of the tangram pieces and pre-service teacher E was surprised at the student's comments about the surface area based on the face touching the floor). One pre-service teacher, pre-service teacher A, was surprised at the student's both correct and incorrect responses, i.e., the student's finding the volume of the prism by looking at the given views easily and having a misconception about surface area with such high spatial reasoning skills. This diversity can be attributed to pre-service teachers' viewing various video clips involving correct and incorrect solutions in the intervention sessions. This might have led the pre-service teachers to have different expectations about the solutions of the students they would interview and to be surprised during the interviews.

5.3 Implications for Educational Practices

The ability to notice students' mathematical thinking is one of the teachers' professional practices that pre-service teachers need to have before starting teaching. Then, how teacher educators can help pre-service teachers develop this professional practice is an important question. Research revealed that there is an attempt to improve pre-service teachers' professional noticing skills among teacher educators in teacher education programs (Santagata et al., 2007; Schack et al., 2013; Star &

Strickland, 2008; van Es & Sherin, 2002). The findings of this study suggest practical implications for the design of learning environments for teacher educators.

The growth demonstrated and a significant increase in each of the three interrelated professional noticing skills suggests that the video-based module situated in the pedagogies of practice framework positively influenced pre-service teachers' ability to attend to students' solutions, interpret students' understanding and decide how to respond based on students' understanding in geometric measurement. Accordingly, this study opens up possibilities for considering how the pedagogies of practice framework in teacher education programs can support pre-service teachers' professional noticing skills. The designed intervention in this study can inform the future development of interventions to support preservice teachers' professional noticing skills. The present study revealed that in the late sessions, pre-service teachers were better at attending to students' mathematical thinking, interpreting students' understanding, and deciding how to respond based on students' understanding. Therefore, in mathematics methods courses, teacher educators can use video clips of individual students as representations of practice, which are produced through task-based interviews so that preservice teachers can develop expertise in noticing students' mathematical thinking. In this study, in particular, the use of concrete models while students were solving the tasks enabled the pre-service teachers to focus on students' actions as well as their responses, and also provided more information about students' mathematical thinking, allowing the pre-service teachers to better recognize the mathematically significant details in students' thinking. Therefore, this study showed that the use of video clips of students' responses and actions about the tasks was particularly helpful to attract the attention of pre-service teachers to students' mathematical thinking. Consequently, teacher educators can produce video clips of students solving tasks using concrete models to improve pre-service teachers' professional noticing of students' mathematical thinking. In addition, in the present study, tasks used in the intervention were designed with a specific purpose in mind, and the aim of the tasks was to develop pre-service teachers' mathematical knowledge for teaching and their professional noticing of children's mathematical thinking to better prepare future teachers to teach perimeter-area measurement and volume-surface area measurement based on students' understanding rather than focusing on procedures. Moreover, in addition to correct student reasoning, the video clips were selected to include the most common student misconceptions about perimeter-area measurement and volume-surface area measurement identified in the literature so that pre-service teachers got the chance to learn and discuss the most prominent misconceptions before starting teaching. In this manner, the video-based module situated in the pedagogies of practice framework enabled pre-service teachers to focus on an individual student's mathematical thinking in a specific content domain. For this reason, teacher educators can use students' common misconceptions in other mathematical content domains to develop pre-service teachers' knowledge of students as well as professional noticing skills. Additionally, the video-based module situated in the pedagogies of practice framework provided in-class opportunities for pre-service teachers to engage in individual and collaborative noticing of students' mathematical thinking (Mason, 2002; McDuffie et al., 2014) as decomposition of practice. The pre-service teachers worked individually and had small group and whole-class discussions about students' solutions in the video clips during the sessions. They also thought about alternative ways they could respond to students and shared their ideas. The findings showed there was an improvement in pre-service teachers' attending to students' mathematical thinking, interpreting students' understanding, and deciding how to respond based on students' understanding in the late sessions, which indicates that these opportunities were effective. Thus, the module can be adapted to other mathematical content domains by teacher educators to support pre-service teachers' professional noticing skills.

The video-based module situated in the pedagogies of practice framework designed in the present study prioritized professional noticing of students' mathematical thinking and learning from practice. In addition to in-class opportunities for pre-service teachers to engage in individual and collaborative noticing of students' mathematical thinking, pre-service teachers had active involvement in the practice by conducting task-based interviews with a student outside of class. Through task-based interviews, pre-service teachers elicited students' mathematical thinking and analyzed that thinking by reflecting on it. In this way, they were immersed in the content through the interview process. Accordingly, by integrating task-based interviews as an approximation of practice, pre-service teachers could learn from their own practice. Therefore, mathematics teacher educators may benefit from these findings. Teacher educators can help pre-service teachers develop their skills in designing tasks and implementing these tasks through engaging in practice, as well as helping pre-service teachers recognize how students think in a particular content domain. Thus, considering that pre-service teachers have not started teaching and have not been in close contact with students, presenting such an environment in teacher education programs might be useful in preparing them for the profession (Alsawaie & Alghazo, 2010). Consequently, in teaching practice courses, supervisors can provide pre-service teachers with opportunities to execute such kind of practice deliberately.

In conclusion, the findings of the present study provide evidence that although the pedagogies of practice framework is not a framework specific to professional noticing, the integration of pedagogies of practice to professional noticing enhanced pre-service teachers' professional noticing skills. Particularly, the sequence of activities in the video-based module situated in the pedagogies of practice framework, consisting of watching video clips of task-based interviews as representation of practice, analyzing students' mathematical thinking in the video clips both individually and collaboratively as decomposition of practice, conducting a task-based interview with a student, and analyzing and reflecting on pre-service teachers' own practices as an approximation of practice in a structured way, supported the development of pre-service teachers' professional noticing skills. Thus, although it cannot be claimed that the representation-decomposition of practice or approximation of practice alone led to changes in pre-service teachers' professional noticing of students' mathematical thinking, representation and decomposition of practice combined with approximation of practice led to valuable experience for pre-service teachers, which supported the development of their professional noticing skills. In this regard, the video-based module situated in the pedagogies of practice framework suggests a way for teacher educators to improve pre-service teachers' professional noticing skills. Thus, the module can be integrated into the courses of mathematics teacher education programs.

5.4 Recommendations for Further Research Studies

Little is known about how pre-service teachers notice students' mathematical thinking in geometric measurement. Thus, the present study extends prior research on teacher noticing of students' mathematical thinking and contributes to the field of mathematics education by attempting to characterize pre-service teachers' professional noticing and the potential of using pedagogies of practice framework to develop their expertise in the context of geometric measurement.

This study focused on the suggested hypothetical instructional actions for students based on students' understanding as deciding how to respond skill rather than how pre-service teachers actually reacted during teaching. By analyzing pre-service teachers' written responses to deciding how to respond prompt and verbal responses during the discussions, it was aimed to gain insight into pre-service teachers' deciding how to respond skills. For this reason, the present study is limited to the intended responses of pre-service teachers rather than the actual implementation of the responses. Thus, further research studies focusing on the analysis of pre-service teachers' written responses, combined with the instructional videos of their own teaching, may present a more comprehensive picture of what pre-service teachers pay attention to in teaching and how they respond to students.

In the present study, the duration of the intervention is limited to one semester. Hence, further research studies can be longer-term longitudinal studies and can explore pre-service teachers' learning to systematically analyze students' mathematical thinking in teacher education programs. In addition, the present study is limited to pre-service teachers' professional noticing skills in a teacher education program before they start the teaching profession. Therefore, further research studies can follow pre-service teachers in their first years of teaching, which can also give educators and researchers crucial insights into the impact of interventions on teachers' professional trajectories. Thus, the influence of the video-based module situated in the pedagogies of practice framework on pre-service teachers' classroom practice is an important issue for future research.

The present study is limited to pre-service teachers' professional noticing skills based on the video clips and one-on-one interview settings with students outside the classroom. However, noticing students' mathematical thinking in a classroom environment is not simple because there exist many distractors that attract pre-service teachers' attention (Jacobs et al., 2011) and hence, pre-service teachers' noticing from video clips and one-on-one interview setting may have been different from noticing in live classrooms. In this sense, there can be differences between viewing video clips and being in the classroom as a passive observer. Hence, future research can examine how pre-service teachers notice students' mathematical thinking in video clips, in one-on-one interview settings, and in a real classroom setting in the same content domain.

The present study is limited to one group of 32 pre-service teachers, which means that the development of pre-service teachers' professional noticing skills was explored in a single group through a pretest-posttest design. Future research can include a comparison group that receives no treatment or comparison groups by implementing the intervention at various sites. In this way, the professional noticing skills of different groups can be compared to reveal the effectiveness of the intervention named the video-based module situated in the pedagogies of practice.

The quantitative part of the present study is limited to pre-service teachers' written responses in the questionnaires for the intent of investigating their making sense of what they read in students' solutions. Future research can also conduct interviews to provide more information about pre-service teachers' professional noticing skills.

Lastly, in the present study, pre-service teachers' professional noticing skills were explored by integrating pedagogies of practice in perimeter-area measurement and volume-surface area measurement, i.e., geometric measurement. Since professional noticing is a domain-specific expertise (Jacobs & Empson, 2016), teachers can demonstrate different levels of professional noticing skills depending on the content. Further research studies can explore pre-service teachers' professional noticing skills in different content domains by adopting the pedagogies of practice framework, as in the present study.

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APPENDICES

A. Questions In Reflection Paper 1

Videoyu izledikten hemen sonra

1. Videoda öğrencinin matematiksel düşüncesiyle ilgili ne fark ettiniz? Öğrencinin düşünce sürecini ve problemi nasıl çözdüğünü videodan örneklerle destekleyerek açıklayınız.
2. Öğrencinin düşüncesine dayanarak, onun konuyu anlamasına (kavrayışına) ilişkin neler söyleyebilirsiniz? Nedenleriyle açıklayınız.
3. Bu öğrencinin öğretmeni olduğunuzu varsayın. Öğrencinin matematiksel düşünmesini/anlamasını geliştirmek ve (eğer varsa) kavram yanılgısını gidermek için ne(ler) yaparsınız? Detaylı olarak örneklerle açıklayınız.

B. Questions In Reflection Paper 2

Tartışmalardan sonra

1. Tartışma ortamı düşüncelerinizi nasıl etkiledi? Grup ve sınıf tartışmasından sonra düşüncelerinizde herhangi bir değişiklik olduysa ya da eklemek istedikleriniz varsa tartışmalardan örnekler vererek açıklayın.

C. Questions in Reflection Paper 3

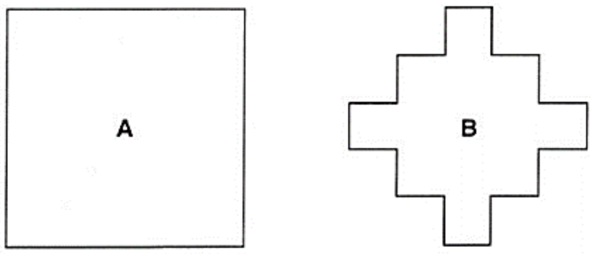
1. Yaptığınız klinik görüşmede öğrencinin matematiksel düşüncesiyle ilgili ne fark ettiniz? Öğrencinin düşünce sürecini ve hazırladığınız taskı/etkinliği nasıl çözdüğünü örneklerle destekleyerek açıklayınız.
2. Öğrencinin düşüncesine dayanarak, onun konuyu anlamasına (kavrayışına) ilişkin neler söyleyebilirsiniz? Nedenleriyle açıklayınız.
3. Bu öğrencinin öğretmeni olduğunuzu varsayın. Öğrencinin matematiksel düşünmesini/anlamasını geliştirmek ve (eğer varsa) kavram yanılgısını gidermek için ne(ler) yaparsınız? Detaylı olarak örneklerle açıklayınız.
4. Öğrencinin çözümünde size göre şaşırtıcı/beklenmedik durumlar (önceden tahmin edemediğiniz) oldu mu? Açıklayınız.
5. Görüşmeyi tekrar yapacak olsanız neleri (örneğin; hazırladığınız taskta/etkinlikte, sorduğunuz sorularda vb.) değiştirirdiniz? Aaçıklayınız.

D. Sample Interview Questions

- Şimdi, her hafta yazdıklarınıza daha detaylı baktığınızda, kendinizde bir gelişme görüyor musunuz? Cevabınız evet ise, lütfen açıklayınız.
- Bundan sonra ölçme konusunda farklı öğrenci çözümleri ile karşılaştığınızda, öğrencilerin düşüncelerini daha iyi dikkate alabileceğinizi, yorumlayabileceğinizi ve öğrencilere karşılık verebileceğinizi düşünüyor musunuz? Cevabınız evet ise lütfen açıklayınız.
- Tartışmalar faydalı oldu mu? Evet ise, tartışmaların faydaları nelerdi?

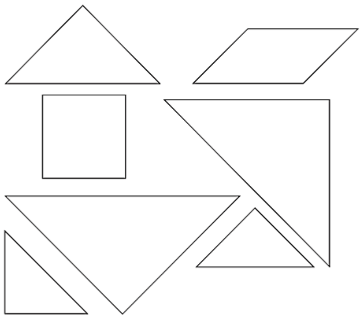
E. Problems in the Noticing Questionnaire

1. Aşağıda verilen A şeklindeki karton, genişlik ve uzunluğu değişmeyecek şekilde kesilerek B şekli elde ediliyor.



A ve B şekillerinin çevre uzunluklarını (büyüklük/küçüklük/eşitlik açısından) karşılaştırdığınızda ne söyleyebilirsiniz? Nedenleriyle açıklayınız.

2.



Yandaki 7 tangram parçasının tamamı kullanılarak aşağıdaki şekiller elde ediliyor.

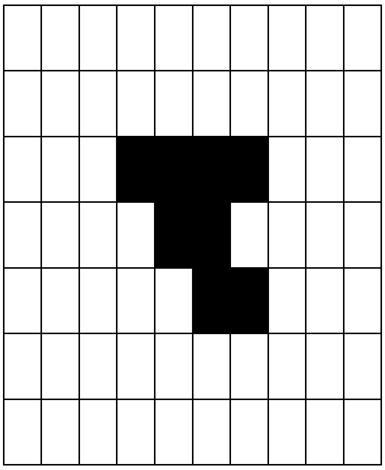
Şekil 1 Şekil 2

2a. Şekillerin çevre uzunluklarını (büyüklük/küçüklük/eşitlik açısından) karşılaştırdığınızda ne söyleyebilirsiniz? Nedenleriyle açıklayınız.

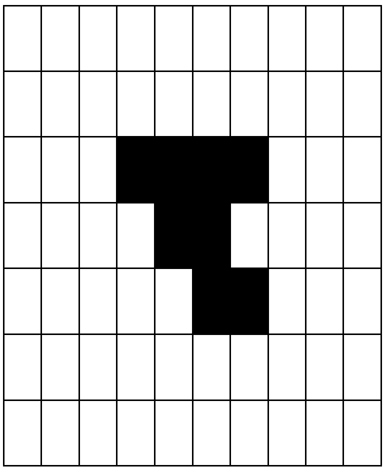
2b. Şekillerin alanlarını (büyüklük/küçüklük/eşitlik açısından) karşılaştırdığınızda ne söyleyebilirsiniz? Nedenleriyle açıklayınız.

3. Aşağıdaki eş birim karelerden oluşan zemindeki figüre **çevre uzunluğu değişmeyecek şekilde** kare(ler) ekleyiniz (Karelerin kenarlardan temas etmesi/kenarlarından birbirine bağlı olması gerekmektedir).

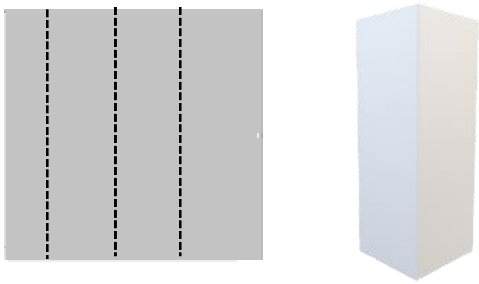
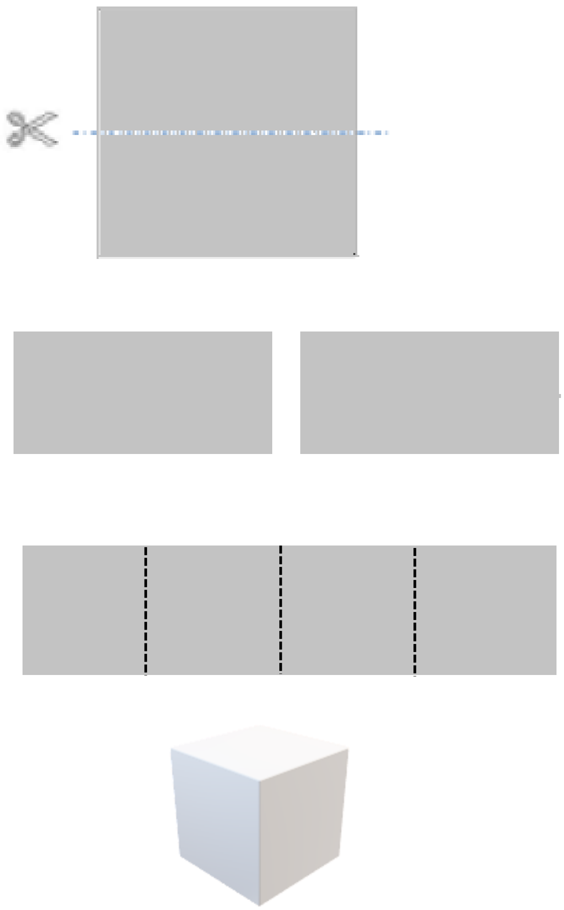
3a. Ekleyebileceğiniz en az kare sayısı ne olur? Kare(leri) nerelere eklediğinizi figür üzerinde boyayarak gösteriniz ve yeni şeklin alanını bulunuz.



3b. Ekleyebileceğiniz en çok kare sayısı ne olur? Kare(leri) nerelere eklediğinizi figür üzerinde boyayarak gösteriniz ve yeni şeklin alanını bulunuz.

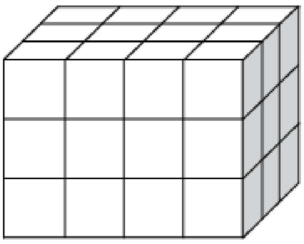
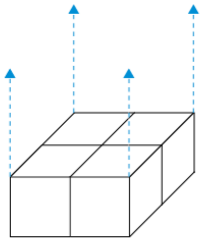
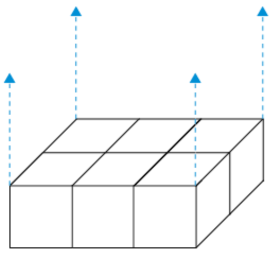


4.

Prizma Yapımı Yandaki kare şeklindeki kâğıt dört eş parçaya ayrılacak şekilde kesikli çizgiler boyunca katlanarak kat izleri oluşturuluyor. Daha sonra bu kâğıdın kenarları birleştirilerek üstü açık bir kare prizma elde ediliyor.	
Küp Yapımı Aynı ölçülerdeki başka bir kare şeklindeki kâğıt ise ortadan yatay olarak kesilerek iki eş parçaya ayrılıyor. Kesilen parçalar kısa kenarları çakışık olacak şekilde birleştirilerek yandaki dikdörtgen oluşturuluyor. Bu dikdörtgen şeklindeki kâğıt dört eş parçaya ayrılacak şekilde kesikli çizgiler boyunca katlanarak kat izleri oluşturuluyor. Daha sonra bu kâğıdın kenarları birleştirilerek üstü açık bir küp elde ediliyor.	

Prizma ve küpün hacimlerini (büyüklük/küçüklük/eşitlik açısından) karşılaştırdığınızda ne söyleyebilirsiniz? Nedenleriyle açıklayınız.

5.

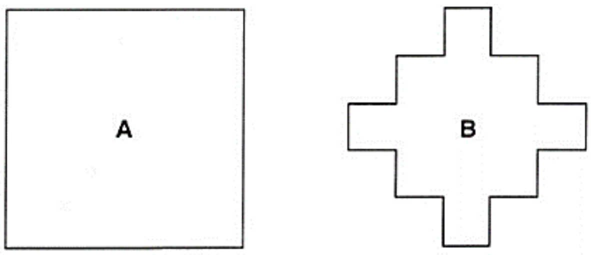
<table>
<tr>
<td rowspan="2">
(a)</td>
<td>5a. Yanda verilen (a) prizması birim küplere ayrılıyor ve küplerin tamamı Şekil 1’ de ilk katmanı verilen (b) prizmasını oluşturmak için tekrar kullanılıyor. Oluşan (b) prizmasının yüksekliği ne olur?

Şekil 1</td>
</tr>
<tr>
<td>5b. Eğer birim küplerin tamamı Şekil 2’de ilk katmanı verilen (c) prizmasını oluşturmak için kullanılırsa oluşan (c) prizmasının yüksekliği ne olur?

Şekil 2</td>
</tr>
</table>

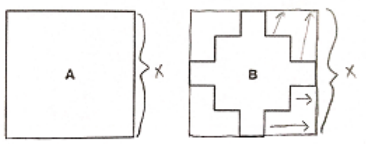
5c. Üç prizmadan (a-b-c) hangisinin hacmi daha büyüktür? Nedenleriyle açıklayınız

5d. Üç prizmadan (a-b-c) hangisinin yüzey alanı daha büyüktür? Nedenleriyle açıklayınız.

F. Students' Solutions to the Problems in the Noticing Questionnaire

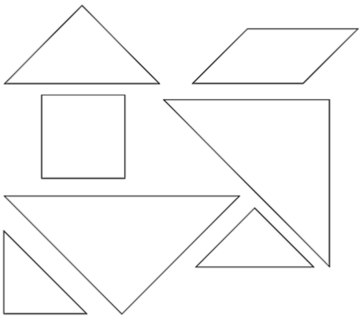
1. Aşağıda verilen A şeklindeki karton, genişlik ve uzunluğu değişmeyecek şekilde kesilerek B şekli elde ediliyor. A ve B şekillerinin çevre uzunluklarını (büyüklük/küçüklük/eşitlik açısından) karşılaştırdığınızda ne söyleyebilirsiniz? Nedenleriyle açıklayınız.



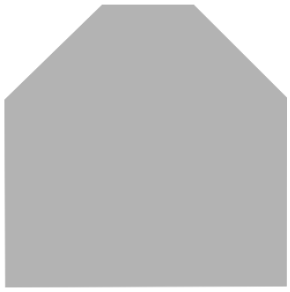
Ada	B'nin daha büyüktür. Çünkü Kıvrımlı bir yol Düz yoldan daha uzundur.
Tuna	B şekli A şeklinin kesilmiş hali olduğu için B şeklinin daha küçüktür
Sare	A ve B şekillerinin çevreleri eşittir B de sadece dıştaki çevre işte 

2.

Aşağıdaki 7 tangram parçasının tamamı kullanılarak yandaki şekiller elde ediliyor.



Şekil 1



Şekil 2

2a. Şekillerin çevre uzunluklarını (büyüklük/küçüklük/eşitlik açısından) karşılaştırdığınızda ne söyleyebilirsiniz? Nedenleriyle açıklayınız.

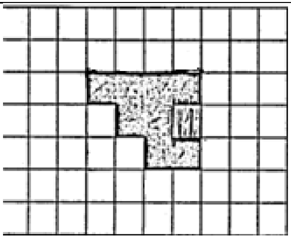
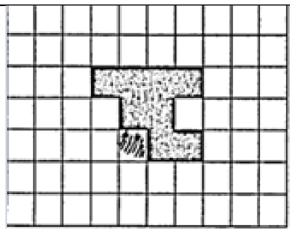
Can	1.şekildeki parçanın çoğu kenarı şeklin çevresini oluşturuyor. 2.şekilde daha fazla parçanın kenarı şeklin içinde kaldığı için 1. şeklin çevresi daha uzun
Hazal	Aslında Şekil 1 ve Şekil 2 eşittir. Sadece kullanılan tangram parçalarının yerleri farklıdır

2b. Şekillerin alanlarını (büyüklük/küçüklük/eşitlik açısından) karşılaştırdığınızda ne söyleyebilirsiniz? Nedenleriyle açıklayınız.

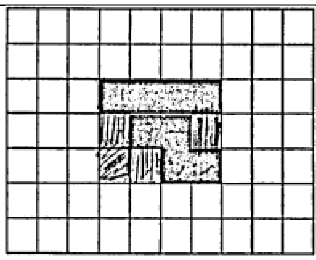
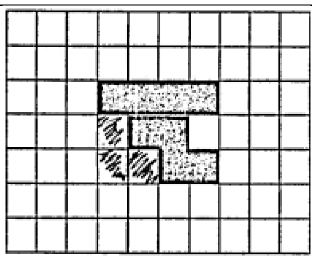
Can	2 şekil de aynı parcalardan oluştuğu için alanları eşitt
Hazal	Bence Şekil 2'nin alanı daha büyüktür. Çünkü; genişlik bakımından Şekil 2 daha büyüktür.

3. Aşağıdaki eş birim karelerden oluşan zemindeki figüre **<u>çevre uzunluğu değişmeyecek şekilde</u>** kare(ler) ekleyiniz (Karelerin kenarlardan temas etmesi/kenarlarından birbirine bağlı olması gerekmektedir).

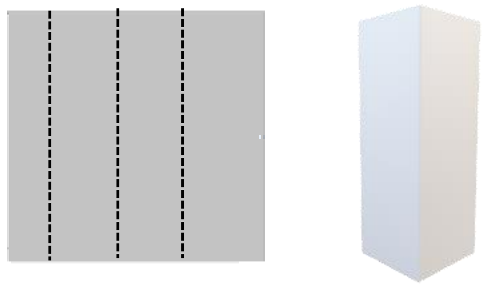
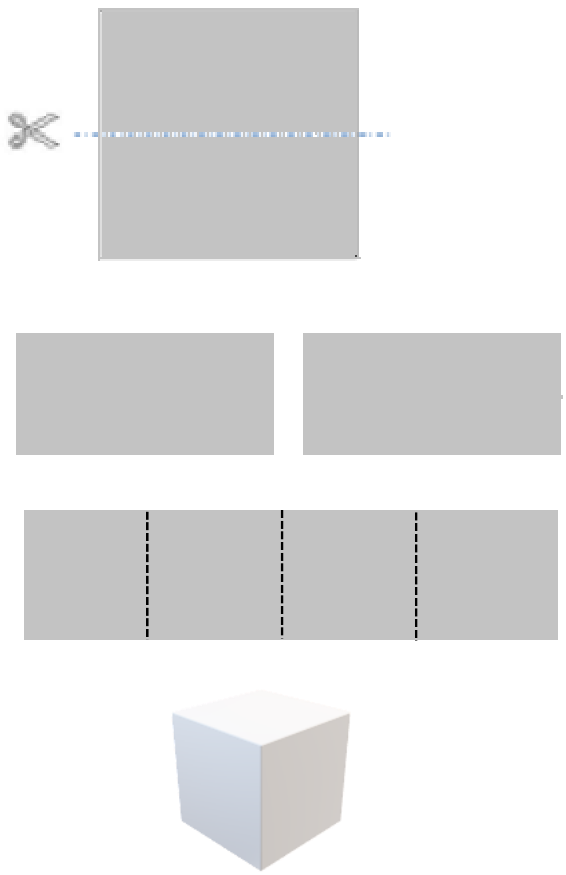
3a. Ekleyebileceğiniz en az kare sayısı ne olur? Kare(leri) nerelere eklediğinizi figür üzerinde boyayarak gösteriniz ve yeni şeklin alanını bulunuz.

Yelda		1 tane kare boyanabilir 9 birim kare olur alanı artar
Utku		1 adet kare kullanılabilir 3kere 4 =12 −3 9 br^2

3b. Ekleyebileceğiniz en çok kare sayısı ne olur? Kare(leri) nerelere eklediğinizi figür üzerinde boyayarak gösteriniz ve yeni şeklin alanını bulunuz.

Yelda		4 tane ekleyebiliriz 12 birim kare olur alanı arttı
Utku		3 adet kare kullanılabilir 3kere 4 =12 −1 11 br^2

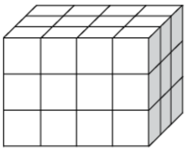
4.

Prizma Yapımı Yandaki kare şeklindeki kâğıt dört eş parçaya ayrılacak şekilde kesikli çizgiler boyunca katlanarak kat izleri oluşturuluyor. Daha sonra bu kâğıdın kenarları birleştirilerek üstü açık bir kare prizma elde ediliyor.	
Küp Yapımı Aynı ölçülerdeki başka bir kare şeklindeki kâğıt ise ortadan yatay olarak kesilerek iki eş parçaya ayrılıyor. Kesilen parçalar kısa kenarları çakışık olacak şekilde birleştirilerek yandaki dikdörtgen oluşturuluyor. Bu dikdörtgen şeklindeki kâğıt dört eş parçaya ayrılacak şekilde kesikli çizgiler boyunca katlanarak kat izleri oluşturuluyor. Daha sonra bu kâğıdın kenarları birleştirilerek üstü açık bir küp elde ediliyor.	

Prizma ve küpün hacimlerini (büyüklük/küçüklük/eşitlik açısından) karşılaştırdığınızda ne söyleyebilirsiniz? Nedenleriyle açıklayınız.

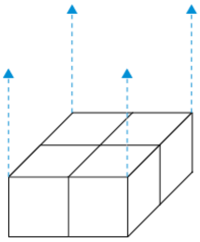
Mert	Alanları eşittir. Bu yüzden hacimleri de eşit olma durumundadır
Bade	bir küp pirinci düşündüğümde küpün tamamını kaplıyorsa, küpün de yüksekliği prizmanın yarısı olduğundan prizmanın da yarısı dolacaktır. (Prizmaya da aynı pirinci dökersek) Bu yüzden prizmanın daha büyük

5.



(a)

5a. Yanda verilen (a) prizması birim küplere ayrılıyor ve küplerin tamamı ilk katmanı yandaki şekil (1) de verilen (b) prizmasını oluşturmak için tekrar kullanılıyor. Oluşan (b) prizmasının yüksekliği ne olur?

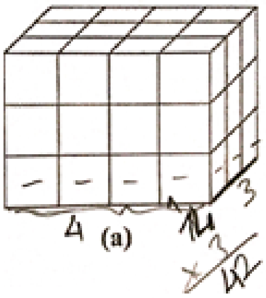


Eylem

9 birim yüksekliği

$$\begin{array}{r|l} 36 & 4 \\ -36 & 9 \\ \hline 00 & \end{array}$$

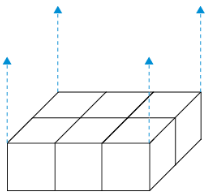
Kaan



$$\begin{array}{r|l} 42 & 4 \\ -4 & 10 \\ \hline 02 & \end{array}$$

10 kat yüksekliğinde

5b. Eğer birim küplerin tamamı yandaki şekil (2) de ilk katmanı verilen (c) prizmasını oluşturmak için kullanılırsa oluşan (c) prizmasının yüksekliği ne olur?

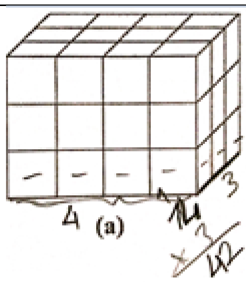


Eylem

6 birim yükseklik

$$\begin{array}{r|l} 36 & 6 \\ -36 & 6 \\ \hline 00 & \end{array}$$

Kaan



$$\begin{array}{r|l} 42 & 6 \\ -42 & 7 \\ \hline 00 & \end{array}$$

7 kat yüksekliğinde

5c. Üç prizmadan (a-b-c) hangisinin hacmi daha büyüktür? Nedenleriyle açıklayınız

Eylem	hacimleri eşittir çünkü hepsinde aynı sayıda küp kullanıldı
Kaan	"a" çünkü tabanı daha geniş.

5d. Üç prizmadan (a-b-c) hangisinin yüzey alanı daha büyüktür? Nedenleriyle açıklayınız.

Eylem	$A = 4 \times 3 = 12$ A çünkü üst yüzeyinde en çok küp bulunur $B = 2 \times 2 = 4$ $C = 3 \times 2 = 6$
Kaan	b prizması daha büyük olur. Çünkü daha çok katı vardır.

G. Noticing Prompts in the Noticing Questionnaire

… probleme ilişkin … öğrencinin çözümü yukarıda verilmiştir. Buna göre, her bir öğrenci için aşağıdaki soruları cevaplandırınız.

- Öğrencinin probleme yaklaşımını/çözümünü açıklayınız. Öğrenci çözümü sizce doğru mu? Neden?
- Öğrencinin yanıtına dayanarak konuyu anlaması (kavrayışı) hakkında neler söyleyebilirsiniz? Detaylı olarak açıklayınız.
- Bu öğrencinin öğretmeni olduğunuzu varsayın. Öğrenci çözümü doğruysa öğrencinin matematiksel düşünmesini/anlamasını geliştirmek için bir sonraki adımda bu öğrenciye hangi soruları sorarsınız? Neden? Öğrenci yanlış bir kavrayışa sahipse bunu gidermek için ne(ler) yaparsınız? Detaylı olarak örneklerle açıklayınız.

H. Approval of the Ethics Committee of Metu Research Center for Applied Ethics

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

DUMLUPINAR BULVARI 06800
ÇANKAYA ANKARA/TURKEY
T: +90 312 210 22 91
F: +90 312 210 79 59
ueam@metu.edu.tr
www.ueam.metu.edu.tr

Sayı: 28620816 / 369

28 EYLÜL 2021

Konu : Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi : İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Mine Işıksal BOSTAN

Danışmanlığını yürüttüğünüz Büşra ÇAYLAN ERGENE'nin "Matematik öğretmen adaylarının öğrencilerin geometrik ölçmeye ilişkin düşünmelerini fark etme becerilerinin uygulama pedagojileri aracılığıyla geliştirilmesi" başlıklı araştırması İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **369-ODTU-2021** protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

A. C.

Dr.Öğretim Üyesi Ali Emre TURGUT
İAEK Başkan Vekili

İ. Parent Approval Form

Sevgili Anne/Baba

Bu araştırma, Orta Doğu Teknik Üniversitesi Matematik ve Fen Bilimleri bölümü öğretim üyelerinden Prof. Dr. Mine Işıksal-Bostan'ın danışmanlığında; doktora tez öğrencisi Büşra Çaylan Ergene tarafından yürütülmektedir. Araştırmamızın amacı ortaokul öğrencilerinin çevre, alan, hacim ve yüzey alanı konularındaki bilgilerini ölçmektir. Bu amaç doğrultusunda çocuğunuzdan bazı soruları cevaplamasını isteyeceğiz ve cevaplarını/davranışlarını not ederek görüntü kaydı alacağız. Sizden çocuğunuzun katılımcı olmasıyla ilgili izin istediğimiz gibi, çalışmaya başlamadan çocuğunuzdan da sözlü olarak katılımıyla ilgili rızası alınacaktır.

Çocuğunuzdan alacağımız cevaplar tamamen gizli tutulacak ve sadece araştırmacılar tarafından değerlendirilecektir. Elde edilecek bilgiler sadece bilimsel amaçla (yayın, konferans sunumu, vb.) kullanılacak, çocuğunuzun ya da sizin isim ve kimlik bilgileriniz, hiçbir şekilde kimseyle paylaşılmayacaktır.

Katılım sırasında sorulan sorulardan ya da herhangi bir uygulama ile ilgili başka bir nedenden ötürü çocuğunuz kendisini rahatsız hissettiğini belirtirse, ya da kendi belirtmese de araştırmacı çocuğun rahatsız olduğunu öngörürse, çalışmaya sorular tamamlanmadan ve derhal son verilecektir.

Araştırma hakkında daha fazla bilgi almak için Büşra Çaylan Ergene ile (e-posta: [redacted]) iletişim kurabilirsiniz. Bu çalışmaya katılımınız için şimdiden teşekkür ederiz.

Yukarıdaki bilgileri okudum ve çocuğumun bu çalışmada yer almasını onaylıyorum (Lütfen alttaki iki seçenekten birini işaretleyiniz.

Evet onaylıyorum___ Hayır, onaylamıyorum___

Annenin/babanın adı-soyadı: _______________

Tarih: _________________

İmza: _________________

Çocuğun adı soyadı ve sınıf seviyesi: _________________

(Formu doldurup imzaladıktan sonra araştırmacıya ulaştırınız)

Curriculum Vitae

Surname, Name: Çaylan Ergene, Büşra

EDUCATION

Degree	Institution	Year of Graduation
MS	METU Elementary Science and Mathematics Education	2018
BS-Major	METU Elementary Mathematics Education	2015
BS-Minor	METU Business Administration-Entrepreneurship	2015
High School	Bozüyük Anatolian Teacher High School, Bilecik	2010

WORK EXPERIENCE

Years	Place	Enrollment
2016-Present	Department of Mathematics and Science Education, Sakarya University, Turkey	Research Assistant

FOREIGN LANGUAGES

Advanced English

PUBLICATIONS

Journal Papers

Ergene, Ö., & **Çaylan Ergene, B**. (2023). Posing problems and solving self-generated problems: the case of convergence and divergence of series. *International Journal of Mathematical Education in Science and Technology, 54*, 1-28.

Çaylan Ergene, B., & Isıksal Bostan, M. (2022). A case study on pre-service teachers' question types within the context of teaching practice course. *Muğla Sıtkı Koçman Üniversitesi Eğitim Fakültesi Dergisi, 9*(1), 1-20.

Çaylan Ergene, B., & Isıksal Bostan, M. (2022). Supporting pre-service mathematics teachers' professional noticing of students' reasoning about length. *European Journal of Science and Mathematics Education, 10*(1), 50-70.

Çaylan Ergene, B., & Haser, Ç. (2021). Students' algebra achievement, algebraic thinking and views in the case of using algebra tiles in groups. *Necatibey Faculty of Education Electronic Journal of Science and Mathematics Education, 15*(2), 254-280.

Çaylan Ergene, B., & Ergene, Ö. (2020). Repeating decimals and irrational numbers on the number line: through the lens of pre-service and in-service mathematics teachers. *Acta Didactica Napocensia, 13*(2), 215-232.

Ergene, Ö., **Çaylan Ergene, B**., & Yazıcı, E. Z. (2020). Ethnomathematics activities: reflections from the design and implementation process. *Turkish Journal of Computer and Mathematics Education (TURCOMAT), 11*(2), 402-437.

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