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Duality and norm completeness in the classes of limitedly Lwc and Dunford–Pettis Lwc operators

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Abstract: We study the duality and norm completeness in the new classes of limitedly L-weakly compact and Dunford–Pettis L-weakly compact operators from Banach spaces to Banach lattices.

Key words: Banach lattice, L-weakly compact set, Dunford–Pettis set, limited set

1. Introduction

The theory of L-weakly compact (briefly, Lwc) sets and operators was developed by P. Meyer-Nieberg in the beginning of seventies in order to diversify the concept of weakly compact operators via imposing the Banach lattice structure on the range of operators [18]. Dunford–Pettis sets appeared a decade later in the work [3] of K. T. Andrews. Shortly thereafter, J. Bourgain and J. Diestel introduced limited sets and operators [6]. Since then, L-weakly compact operators and limited operators have attracted permanent attention and inspiring researchers. Recently, further related classes of operators were introduced and studied by several authors (see, for example, [2, 5, 8, 9, 11–13, 17, 21–23] and references therein). Using the Meyer-Nieberg approach for the Dunford–Pettis and for limited (instead of bounded) sets in the domain, we introduce Dunford–Pettis Lwc and the limitedly Lwc operators. We study the duality and norm completeness in classes of these operators.

Throughout the text: vector spaces are real; operators are linear and bounded; X and Y stand for Banach spaces, E and F for Banach lattices; B_X for the closed unit ball of X; $\mathrm{L}(X,Y)$ for the space of all bounded operators from X to Y; X' for the norm-dual of X. E_+ for the positive cone of E; $\mathrm{sol}(A) := \bigcup\limits_{a\in A}[-|a|,|a|]$ for the solid hull of $A \subseteq E$; $E^a := \{x \in E : |x| \geq x_n \downarrow 0 \Rightarrow \|x_n\| \to 0\}$ for the order continuous part of E; and $\mathrm{L}_r(E,F)$ for the space $\mathrm{span}(\mathrm{L}_+(E,F))$ of regular operators in $\mathrm{L}(E,F)$. We identify X with its image $\widehat{X}$ in X'' under the canonical embedding $\hat{x}(f) = f(x)$. For further terminology and notations see [1, 16, 19].

The paper is organized as follows. In Section 2, we introduce Dunford–Pettis L weakly and limitedly L weakly compact operators, and investigate their basic properties. Among other things, we characterize Dunford–Pettis sets via limited sets in Theorem 1 and establish semiduality for arbitrary limitedly Lwc operators in Theorem 4. Section 3 is devoted to complete norms on spaces of such operators in Banach lattice setting.

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2. Main definitions and basic properties

In this section, we collect main definitions, introduce the Dunford–Pettis Lwc and limitedly Lwc operators, and study their basic properties. We begin with the following crucial definition belonging to P. Meyer-Nieberg [19].

Definition 1 A subset A of F is called an Lwc *set* whenever each disjoint sequence in $\mathrm{sol}(A)$ is norm-null. A bounded operator $T : X \to F$ is an Lwc *operator* (briefly, $T \in \mathrm{Lwc}(X,F)$) if $T(B_X)$ is an Lwc subset of F.

It can be easily seen that B_E is not Lwc unless $\dim(E) < \infty$. For every Lwc subset A of E, we have $A \subseteq E^a$. Indeed, otherwise there is $a \in A$ with $|a| \in E \setminus E^a$, and hence there exists a disjoint sequence (x_n) in $[0,|a|] \subseteq \mathrm{sol}(A)$ with $\|x_n\| \not\to 0$. The next important fact goes back to Meyer-Nieberg (cf. [1, Thm.5.63] and [7, Prop.2.2] for more general setting).

Proposition 1 *Let $A \subseteq E$ and $B \subseteq E'$ be nonempty bounded sets. Then every disjoint sequence in* $\mathrm{sol}(A)$ *is uniformly null on B iff every disjoint sequence in* $\mathrm{sol}(B)$ *is uniformly null on A.*

We include a proof of the following certainly well known fact.

Lemma 1 *Let L be a nonempty bounded subset of F'. TFAE.*

i) *L is an* Lwc *subset of F'.*

ii) *Each disjoint sequence in B_F is uniformly null on L.*

iii) *Each disjoint sequence in $B_{F''}$ is uniformly null on L.*

Proof Since $\|f\| = \sup\{|f(x)| : x \in B_X\} = \sup\{|y(f)| : y \in B_{X''}\}$ then (f_n) in X' converges uniformly on B_X iff it converges uniformly on $B_{X''}$ under the identification of $f \in X'$ with $\hat{f} \in X'''$. Now, applying Proposition 1 first to $A = B_F$ and $B = L$, and then to $A = L$ and $B = B_{F''}$, we obtain that both ii) and iii) are equivalent to the condition that every disjoint sequence in $\mathrm{sol}(L)$ is norm null, which means that L is an Lwc subset of F'. $\square$

For the equivalence $i) \Longleftrightarrow ii)$ of the next characterization of Lwc sets, we refer to [19, Prop.3.6.2], whereas the equivalence $i) \Longleftrightarrow iii)$ can be found in [4, Lem.4.2]

Proposition 2 *For a nonempty bounded subset A of E, TFAE.*

i) *A is an* Lwc *set.*

ii) *For every $\varepsilon > 0$, there exists $u_\varepsilon \in E^a_+$ such that $A \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_E$.*

iii) *For every $\varepsilon > 0$, there exists an* Lwc *subset A_ε of E with $A \subseteq A_\varepsilon + \varepsilon B_E$.*

A subset A of E is called *almost order bounded* whenever, for every $\varepsilon > 0$, there is $u_\varepsilon \in E_+$ such that $A \subseteq [-u_\varepsilon, u_\varepsilon] + \varepsilon B_E$. Every relatively compact subset of E is almost order bounded. By Proposition 2, each almost order bounded subset of E^a is an Lwc set.

The following notions are due to K. Andrews, J. Bourgain, and J. Diestel.

Definition 2 A bounded subset A of X is called:

a) a *Dunford–Pettis set* (briefly, A is DP) if (f_n) is uniformly null on A for each w-null (f_n) in X' (see [3, Thm.1]).

b) a *limited set* if (f_n) is uniformly null on A for each w*-null (f_n) in X' (see [6]).

In reflexive spaces, DP sets agree with limited sets. In general,

$$A \text{ is relatively compact } \Longrightarrow A \text{ is limited } \Longrightarrow A \text{ is DP.}$$

Limited sets are relatively compact in separable and in reflexive Banach spaces [6]. By the Josefson–Nissenzweig theorem [15, 20], B_X is not limited in X unless $\dim(X) < \infty$. However, $\widehat{B_{c_0}}$ is limited in $c_0'' = \ell^\infty$ by Phillip's lemma (cf., [1, Thm.4.67]). B_{c_0} is DP in c_0 because $c_0' = \ell^1$ has the Schur property. The DP sets turn to limited while embedded in the bi-dual.

Theorem 1 Let $A \subseteq X$. TFAE:

$i)$ A is a DP subset of X.

$ii)$ $\widehat{A}$ is limited in X''.

Proof $i) \Longrightarrow ii)$ Assume A is a DP subset of X. Let $f_n \xrightarrow{w^*} 0$ in X'''. Then $f_n|_X := f_n|_{\widehat{X}} \xrightarrow{w} 0$ in X', since $g(f_n|_X) = \hat{g}(\widehat{f_n|_X}) = \hat{g}(f_n) = f_n(g) \to 0$ for each $g \in X''$. By the assumption, $(f_n|_X)$ is uniformly null on A, and hence (f_n) is uniformly null on $\widehat{A}$ as desired. Therefore, $\widehat{A}$ is limited in X''.

$ii) \Longrightarrow i)$ Suppose $\widehat{A}$ is limited in X''. Let $f_n \xrightarrow{w} 0$ in X'. Then $\widehat{f_n} \xrightarrow{w^*} 0$ in X''' and, as $\widehat{A}$ is limited, $(\widehat{f_n})$ is uniformly null on $\widehat{A}$. Hence, $\sup_{a\in A} |f_n(a)| = \sup_{a\in A} |\hat{a}(f_n)| = \sup_{a\in A} |\widehat{f_n}(\hat{a})| = \sup_{b\in \widehat{A}} |\widehat{f_n}(b)| \to 0$. Therefore, (f_n) is uniformly null on A, which means that A is DP in X. $\square$

2.1. Recently, K. Bouras, D. Lhaimer, and M. Moussa introduced and studied a-Lwc *operators* from X to F carrying weakly compact sets to Lwc sets [5]. Here, we investigate operators carrying limited, or else the Dunford–Pettis subsets of X to Lwc sets of F.

Theorem 2 *Let $T \in \mathrm{L}(X, F)$. TFAE.*

$i)$ *T takes limited subsets of X onto* Lwc *subsets of F.*

$ii)$ *T takes compact subsets of X onto* Lwc *subsets of F.*

$iii)$ *$\{Tx\}$ is an* Lwc *subset of F for each $x \in X$.*

$iv)$ *$T(X) \subseteq F^a$.*

$v)$ *$T'f_n \xrightarrow{w^*} 0$ in X' for each disjoint bounded sequence (f_n) in F'.*

Proof The implications $i) \implies ii) \implies iii)$ are trivial, while $iii) \implies iv)$ yields because each Lwc subset of F lies in F^a.

$iv) \implies v)$: Let (f_n) be a disjoint bounded sequence in F' and $x \in X$. Since $T(X) \subseteq F^a$, $\{Tx\}$ is an Lwc set, and hence $T'f_n(x) = f_n(Tx) \to 0$ by Proposition 1. As $x \in X$ is arbitrary, $(T'f_n)$ is w*-null.

$v) \implies i)$: Assume in contrary $T(L)$ is not an Lwc set in F for some nonempty limited subset L of X. By Proposition 1, there exists a disjoint sequence (g_n) of $B_{F'}$ that is not uniformly null on $T(L)$. Therefore, $(T'g_n)$ is not uniformly null on L violating $T'g_n \xrightarrow{w^*} 0$ and the limitedness of L. The obtained contradiction completes the proof. $\square$

2.2. Because of Theorem 2 $i)$, we prefer to call operators satisfying the equivalent conditions of Theorem 2 by l-Lwc *operators* (they may equally deserve to be called *compactly* Lwc *operators* in view of Theorem 2 $ii)$). While preparing the paper, we have learned that the operators satisfying of Theorem 2 $v)$ have been already introduced and studied by F. Oughajji and M. Moussa under the name *weak* L-*weakly compact operators* [21] (this name looks more suitable for a-Lwc operators rather than for l-Lwc operators).

Definition 3 An operator $T : X \to F$ is called:

a) a *Dunford–Pettis* L-*weakly compact* (briefly, $T \in$ DP-Lwc(X, F)), if T carries DP subsets of X onto Lwc subsets of F.

b) *limitedly* L-*weakly compact* (briefly, $T \in$ l-Lwc(X, F)), if T carries limited subsets of X onto Lwc subsets of F.

Clearly, DP-Lwc(X, F) and l-Lwc(X, F) are vector spaces. Theorem 2 $ii)$ implies the second inclusion of the next formula, whereas the first one is trivial.

$$\mathrm{Lwc}(X, F) \subseteq \text{a-Lwc}(X, F) \subseteq \text{l-Lwc}(X, F). \tag{1}$$

A Banach lattice E has the *dual disjoint* w*-*property* (shortly, $E \in$ (DDw*P)) if each disjoint bounded sequence in E' is w*-null. We include the following consequence of Theorem 2.

Corollary 1 TFAE.

$i)$ $F \in$ (DDw*P).

$ii)$ $I_F \in$ l-Lwc(F).

$iii)$ Each limited subset of F is an Lwc-set.

$iv)$ l-Lwc$(F) =$ L(F).

$v)$ l-Lwc$(X, F) =$ L(X, F) for each Banach space X.

Proof The equivalence $i) \iff ii)$ follows from Theorem 2.

The implications $v) \implies iv) \implies ii) \iff iii)$ are trivial.

$iii) \implies v)$: Let $T \in$ L(X, F) and let L be limited subset of X. Then $T(L)$ is a limited subset of F, and hence $T(L)$ is an Lwc subset of F. Thus, $T \in$ l-Lwc(X, F), as desired. $\square$

The following result is a version of Theorem 2.

Theorem 3 *Let $T \in \mathrm{L}(X, F')$. TFAE.*

i) $T \in$ l-Lwc(X, F').

ii) T *takes compact subsets of* X *onto* Lwc *subsets of* F'.

iii) $\{Tx\}$ *is an* Lwc *subset of* F' *for each* $x \in X$.

iv) $T(X) \subseteq (F')^a$.

v) $T' f_n \stackrel{w^*}{\to} 0$ *in* X' *for each disjoint bounded sequence* (f_n) *in* F''.

vi) $T'\widehat{g_n} \stackrel{w^*}{\to} 0$ *in* X' *for each disjoint bounded sequence* (g_n) *in* F.

Proof The equivalence $i) \iff ii) \iff iii) \iff iv) \iff v)$ follows from Theorem 2, and the implication $v) \implies vi)$ is trivial.

$vi) \implies i)$: Assume in contrary $T(L)$ is not an Lwc-set in F' for some nonempty limited subset L of X. By Lemma 1, there exists a disjoint sequence (g_n) of B_F such that $(\widehat{g_n})$ is not uniformly null on $T(L)$. Therefore, $(T'\widehat{g_n})$ is not uniformly null on L, which is absurd because of $T'\widehat{g_n} \stackrel{w^*}{\to} 0$ in X' and L is limited in X. The obtained contradiction completes the proof. □

2.3. Following G. Emmanuele [10], a Banach space X is said to possess the *Bourgain–Diestel property* if each limited subset of X is relatively weakly compact, and an operator $T : X \to Y$ is called a *Bourgain–Diestel operator* (briefly, $T \in \mathrm{BD}(X, Y)$) if T carries limited sets onto relatively weakly compact sets. The weakly compactness of Lwc sets, Definitions 2, 3, and Theorem 2 together imply

$$\mathrm{Lwc}(X, F) \subseteq \text{DP-Lwc}(X, F) \subseteq \text{l-Lwc}(X, F) \subseteq \mathrm{BD}(X, F). \tag{2}$$

All inclusions in (2) are generally proper by items d)–f) of Example 1.

Example 1 a) $\mathrm{Id}_{\ell^1} \in$ a-Lwc$(\ell^1) \setminus$ Lwc(ℓ^1) (see [5, p.1435]).

b) $\mathrm{Id}_{\ell^2} \in$ l-Lwc$(\ell^2) \setminus$ a-Lwc(ℓ^2) since limited sets in ℓ^2 coincide with relatively compact sets that are in turn l-Lwc sets in ℓ^2, while B_{ℓ^2} is weakly compact but not an l-Lwc set.

c) It is easy to see that

$$T := \mathrm{Id}_{c_0} \in \text{l-Lwc}(c_0), \text{ yet } T'' = \mathrm{Id}''_{c_0} = \mathrm{Id}_{\ell^\infty} \notin \text{l-Lwc}(\ell^\infty) = \text{l-Lwc}(c''_0).$$

d) Since l-Lwc$(\ell^2) =$ DP-Lwc(ℓ^2) due to reflexivity of ℓ^2, item b) implies $\mathrm{Id}_{\ell^2} \in$ DP-Lwc$(\ell^2) \setminus$ a-Lwc(ℓ^2). We have no example of an operator $T \in$ a-Lwc$(X, F) \setminus$ DP-Lwc(X, F).

e) $\mathrm{Id}_{c_0} \in$ l-Lwc$(c_0) \setminus$ DP-Lwc(c_0) as B_{c_0} is not Lwc yet is a DP set in c_0.

f) $\mathrm{Id}_c \in \mathrm{BD}(c) \setminus$ l-Lwc(c) since limited sets in c coincide with relatively compact sets, while $c^a = c_0 \subsetneqq c$ implies $\mathrm{Id}_c \notin$ l-Lwc(c) by Theorem 2.

g) Combining examples d)–f) in one diagonal operator (3×3)-matrix:

$$\mathrm{Lwc}(\ell^2 \oplus c_0 \oplus c) \subsetneqq \text{DP-Lwc}(\ell^2 \oplus c_0 \oplus c) \subsetneqq \text{l-Lwc}(\ell^2 \oplus c_0 \oplus c) \subsetneqq \mathrm{BD}(\ell^2 \oplus c_0 \oplus c).$$

2.4. The next result is a consequence of Theorem 2.

Corollary 2 *Let $T \in \mathrm{L}(X,F)$. The following four conditions are equivalent:*

i) $T'' \in \text{l-Lwc}(X'',F'')$.

ii) T'' *takes compact subsets of* X'' *to* Lwc *subsets of* F''.

iii) $T''(X'') \subseteq (F'')^a$.

iv) $T'''f_n \overset{\mathrm{w}^*}{\to} 0$ *in* X''' *for each disjoint bounded sequence* (f_n) *in* F'''.

Each of above equivalent conditions implies:

v) $T \in \text{DP-Lwc}(X,F)$.

The condition v) in turn implies:

vi) $T \in \text{l-Lwc}(X,F)$.

Proof Theorem 2 implies $i) \iff ii) \iff iii) \iff iv)$.

$i) \implies v)$: Let A be a DP subset of X. By Theorem 1, $\widehat{A}$ is limited in X''. Then $\widehat{T(A)} = T''(\widehat{A})$ is an Lwc subset of F'', and hence $T(A)$ is an Lwc subset of F. The latter means $T \in \text{DP-Lwc}(X,F)$.

The implication $v) \implies vi)$ is trivial. □

Note that, $T \in \text{l-Lwc}(X,F)$ does not imply $T'' \in \text{l-Lwc}(X'',F'')$ in general (see Example 1 c)). If $T'' \in \text{DP-Lwc}(X'',F'')$ then $T'' \in \text{l-Lwc}(X'',F'')$ by (2), and hence $T \in \text{DP-Lwc}(X,F)$ by Corollary 2. We have no example of an operator $T \in \text{DP-Lwc}(X,F)$ such that $T'' \notin \text{DP-Lwc}(X'',F'')$.

2.5. The following definition [21, Def.3.1] was taken a starting point in [21]. In our approach, this definition is a derivation of Theorem 2 *iv)*, similarly to the classical approach to Mwc operators, which were introduced in [18] as a derivation of Lwc operators.

Definition 4 An operator $T : E \to Y$ is *limitedly* Mwc (briefly, $T \in \text{l-Mwc}(E,Y)$), if $Tx_n \overset{\mathrm{w}}{\to} 0$ for every disjoint bounded sequence (x_n) in E.

Recall that E' is a KB space iff every disjoint bounded sequence in E is w-null (cf. [1, Thm.4.59] and [19, 2.4.14]). Consequently, I_E is l-Mwc iff E' is a KB-space. Now, we discuss the semiduality for l-Lwc and l-Mwc operators. It was proved in [21] that $T \in \text{l-Mwc}_+(E,F)$ iff $T' \in \text{l-Lwc}_+(F',E')$. The next theorem is an extension of [21, Thm.4.13], where only the case of positive operators is considered.

Theorem 4 *The following statements hold:*

i) $T' \in \text{l-Mwc}(F',X') \implies T \in \text{l-Lwc}(X,F)$.

ii) $T' \in \text{l-Lwc}(Y',E') \iff T \in \text{l-Mwc}(E,Y)$.

Proof *i)* Let $T' \in$ l-Mwc(F', X'), and let (f_n) be disjoint bounded in F'. Then $T'f_n \xrightarrow{w} 0$, and hence $T'f_n \xrightarrow{w^*} 0$. Theorem 2 implies $T \in$ l-Lwc(X, F).

ii) ($\Longleftarrow$): Let $T \in$ l-Mwc(E, Y). According to Theorem 2, to demonstrate that $T' \in$ l-Lwc(Y', E'), we need to prove that $\{T'f\}$ is an l-Lwc subset of E' for each $f \in Y'$. Let $f \in Y'$. By Lemma 1, it suffices to show $f(Tx_n) \to 0$ for each disjoint sequence (x_n) in B_E. So, let (x_n) be disjoint in B_E. Since $T \in$ l-Mwc(E, Y) then $Tx_n \xrightarrow{w} 0$, and hence $f(Tx_n) \to 0$, as desired.

($\Longrightarrow$): Let $T' \in$ l-Lwc(Y', E'). Then $\{T'g\}$ is an Lwc subset of E' for each $g \in Y'$ by Theorem 4. It follows from Lemma 1 that $g(Tx_n) = T'g(x_n) \to 0$ for each disjoint bounded sequence (x_n) in E. Since $g \in Y'$ is arbitrary, $Tx_n \xrightarrow{w} 0$ for every disjoint bounded (x_n) in E, and therefore $T \in$ l-Mwc(E, Y). $\square$

Note that a similar semiduality was established in [5, Thm.2.5] for almost L-weakly compact operators, and in [17, Thm.2.3] for order L-weakly compact operators.

2.6. Although we have no sequential characterization of DP-Lwc operators like the characterization of l-Lwc operators given by Theorem 2 *v)*, we include the following result in this direction.

Theorem 5 *Let $T \in$ L(X, F). TFAE.*

i) $T'' \in$ DP-Lwc(X'', F'').

ii) $T'f_n \xrightarrow{w} 0$ *in* X' *for each disjoint* (f_n) *in* $B_{F'}$.

Proof *i)* $\Longrightarrow$ *ii)*: Let (f_n) be a disjoint sequence in $B_{F'}$. Suppose $(T'f_n)$ is not w-null in X'. Then $\widehat{f_n}(T''g) = T''g(f_n) = g(T'f_n) \not\to 0$ for some $g \in X''$, and hence $(\widehat{f_n})$ is not uniformly null on $\{T''g\}$. Lemma 1 implies that $\{T''g\}$ is not an Lwc subset of F''. However, $\{g\}$ is a DP subset of X'' and then $\{T''g\}$ must be Lwc in F'' by the condition *i)* The obtained contradiction proves the implication.

ii) $\Longrightarrow$ *i)*: Suppose in contrary $T'' \notin$ DP-Lwc(X'', F''). Then $T''(A)$ is not Lwc in F'' for some DP subset A of X'', and hence $(\widehat{f_n})$ is not uniformly null on $T''(A)$ for some disjoint sequence (f_n) in $B_{F'}$, by Lemma 1. Thus, $(\widehat{T'f_n}) = (T'''\widehat{f_n})$ is not uniformly null on A. By *ii)*, $T'f_n \xrightarrow{w} 0$ in X', and hence $\widehat{T'f_n} \xrightarrow{w} 0$ in X'''. Since A is a DP subset of X'' then $(\widehat{T'f_n})$ is uniformly null on A. The obtained contradiction completes the proof. $\square$

2.7. Clearly, DP-Lwc(X, F), l-Lwc(X, F), and l-Mwc(E, Y) are vector spaces. It is natural to ask whether or not DP-Lwc(X, F), l-Lwc(X, F), and l-Mwc(E, Y) are Banach spaces under the operator norm. The answer is affirmative.

Proposition 3 *Let $T \in$ L(X, F) and let (T_n) be a sequence in* DP-Lwc(X, F) *satisfying $T_n \xrightarrow{\|\cdot\|} T$. Then* $T \in$ DP-Lwc(X, F).

Proof Let D be a DP subset of X. WLOG $D \subseteq B_X$. Take an arbitrary $\varepsilon > 0$ and pick $k \in \mathbb{N}$ with $\|T - T_k\| \leq \varepsilon$. Since $T_k \in$ DP-Lwc(X, F) then $T_k(D)$ is an Lwc subset of F. As $T(D) \subseteq T_k(D) + \varepsilon B_F$, Proposition 2 implies that $T(D)$ is an Lwc subset of F, hence $T \in$ DP-Lwc(X, F). $\square$

Although the next proposition has a similar proof as the proof of Proposition 3, we give an alternative one.

Proposition 4 *Let* $T \in \mathrm{L}(X,F)$ *and let* (T_n) *be a sequence in* $\text{l-Lwc}(X,F)$ *satisfying* $T_n \overset{\|\cdot\|}{\to} T$. *Then* $T \in \text{l-Lwc}(X,F)$.

Proof Let (f_n) be disjoint bounded in F', and $x \in X$. By Theorem 2, we need to show $T'f_n(x) \to 0$. Let $\varepsilon > 0$. Pick any $k \in \mathbb{N}$ with $\|T - T_k\| \le \varepsilon$. Since $T_k \in \text{l-Lwc}(X,F)$ then $|T_k' f_n(x)| \le \varepsilon$ for $n \ge n_0$. As $\varepsilon > 0$ is arbitrary, it follows from

$$|T'f_n(x)| \le |T'f_n(x) - T_k'f_n(x)| + |T_k'f_n(x)| \le$$

$$\|T' - T_k'\|\|f_n\|\|x\| + |T_k'f_n(x)| \le (\|f_n\|\|x\| + 1)\varepsilon \qquad (\forall n \ge n_0)$$

that $T'f_n(x) \to 0$. □

Proposition 5 *Let* $T \in \mathrm{L}(E,Y)$ *and let* (T_n) *be a sequence in* $\text{l-Mwc}(E,Y)$ *satisfying* $T_n \overset{\|\cdot\|}{\to} T$. *Then* $T \in \text{l-Mwc}(E,Y)$.

Proof By Theorem 4, $\text{l-Lwc}(Y',E') \ni T_n' \overset{\|\cdot\|}{\to} T'$. By Proposition 4, we have $T' \in \text{l-Lwc}(Y',E')$. Then $T \in \text{l-Mwc}(E,Y)$ by Theorem 4. □

Corollary 3 *Let* E *be a Banach lattice. The following holds.*

i) $\text{l-Lwc}(E)$ *is a closed right ideal in* $\mathrm{L}(E)$ *(and hence a subalgebra of* $\mathrm{L}(E)$*), and it is unital iff* I_E *is* l-Lwc.

ii) $\text{l-Mwc}(E)$ *is a closed left ideal in* $\mathrm{L}(E)$ *(and hence a subalgebra of* $\mathrm{L}(E)$*), and it is unital iff* I_E *is* l-Mwc.

Proof *i)* $\text{l-Lwc}(E)$ is a closed subspace of $\mathrm{L}(E)$ by Proposition 4. As bounded operators carry limited sets onto limited sets, $\text{l-Lwc}(E)$ is a right ideal in $\mathrm{L}(E)$. The condition on I_E making algebra $\text{l-Lwc}(E)$ unital is trivial.

ii) By Proposition 5, $\text{l-Mwc}(E)$ is a closed subspace of $\mathrm{L}(E)$. It remains to show that $\text{l-Mwc}(E)$ is a left ideal in $\mathrm{L}(E)$. Let $T \in \text{l-Mwc}(E)$ and $S \in \mathrm{L}(E)$. Then $T' \in \text{l-Lwc}(E')$, and hence, *i)* implies $(ST)' = T'S' \in \text{l-Lwc}(E')$. Now, $ST \in \text{l-Mwc}(E)$ by Theorem 4. □

3. The Banach lattice case

In the Banach lattice setting, we investigate the completeness in the regular norm of linear spans of positive operators belonging to the classes introduced in Section 2. We begin with some technical notions.

3.1. Let $\emptyset \ne \mathcal{P} \subseteq \mathrm{L}(E,F)$. The set $\mathcal{P}$ is said to satisfy the *domination property* whenever $0 \le S \le T$ and $T \in \mathcal{P}$ imply $S \in \mathcal{P}$. We say that $T \in \mathrm{L}(E,F)$ is $\mathcal{P}$*-dominated*, if there exists an $U \in \mathcal{P}$ such that $\pm T \le U$, and denote $\mathcal{P}_+ := \mathcal{P} \cap \mathrm{L}_+(E,F)$.

Proposition 6 *Let* $\mathcal{P} \subseteq \mathrm{L}(E,F)$, $\mathcal{P} \pm \mathcal{P} \subseteq \mathcal{P} \ne \emptyset$, *and* $T \in \mathrm{L}(E,F)$. *Then the following holds.*

i) $T \in \mathrm{span}(\mathcal{P}_+) \Longleftrightarrow T$ *is a* $\mathcal{P}$*-dominated operator from* $\mathcal{P}$.

ii) Assume the modulus $|T|$ *of* T *exists in* $\mathrm{L}(E,F)$ *and* $\mathcal{P}$ *possesses the domination property. Then* $T \in \mathrm{span}(\mathcal{P}_+) \Longleftrightarrow |T| \in \mathcal{P}$.

Proof $i)$ Let $T \in \operatorname{span}(\mathcal{P}_+)$. WLOG, $T = T_1 - T_2$, where $T_1, T_2 \in \mathcal{P}_+$. $\mathcal{P} \pm \mathcal{P} \subseteq \mathcal{P}$ implies $T \in \mathcal{P}$ and $U = T_1 + T_2 \in \mathcal{P}$. From $\pm T \leq U$, we obtain that T is $\mathcal{P}$-dominated.

Now, let $T \in \mathcal{P}$ be $\mathcal{P}$-dominated. Take $U \in \mathcal{P}$ such that $\pm T \leq U$. Since $T = U - (U - T)$, and both U and $U - T$ are in $\mathcal{P}_+$ then $T \in \operatorname{span}(\mathcal{P}_+)$.

$ii)$ First suppose $|T| \in \mathcal{P}$. Since $T = T_+ - T_-$, $0 \leq T_\pm \leq |T| \in \mathcal{P}$, the domination property implies that T_+ and T_- are positive operators in $\mathcal{P}$, and hence $T = T_+ - T_- \in \operatorname{span}(\mathcal{P}_+)$.

Now, suppose $T \in \operatorname{span}(\mathcal{P}_+)$. Then there exist $T_1, T_2 \in \mathcal{P}_+$ satisfying $T = T_1 - T_2$. Since $0 \leq T_+ \leq T_1$ and $0 \leq T_- \leq T_2$, the domination property implies $T_+, T_- \in \mathcal{P}$. Hence, $|T| = T_+ + T_- \in \mathcal{P}$. □

Proposition 7 *Let F be Dedekind complete, and let $\mathcal{P}$ be a subspace in $\mathrm{L}(E,F)$ possessing the domination property. Then $\operatorname{span}(\mathcal{P}_+)$ is an order ideal in the Dedekind complete vector lattice $\mathrm{L}_r(E,F)$.*

Proof Since F is Dedekind complete, $\mathrm{L}_r(E,F)$ is a Dedekind complete vector lattice. By Proposition 6 $ii)$, $T \in \operatorname{span}(\mathcal{P}_+) \Longrightarrow |T| \in \operatorname{span}(\mathcal{P}_+)$, and hence $\operatorname{span}(\mathcal{P}_+)$ is a vector sublattice of $\mathrm{L}_r(E,F)$. Since $\mathcal{P}$ has the domination property, $\operatorname{span}(\mathcal{P}_+)$ is an order ideal in $\mathrm{L}_r(E,F)$. □

3.2. The following proposition is contained in [21, Thm.3.6 and Thm.4.6].

Proposition 8 *Let operators $S, T \in \mathrm{L}(E,F)$ satisfy $\pm S \leq T$. Then:*

$i)$ $T \in \text{l-Lwc}(E,F) \Longrightarrow S \in \text{l-Lwc}(E,F)$.

$ii)$ $T \in \text{l-Mwc}(E,F) \Longrightarrow S \in \text{l-Mwc}(E,F)$.

We include here the following domination result for DP-Lwc operators.

Proposition 9 *Let operators $S, T \in \mathrm{L}(E,F)$ satisfy $0 \leq S \leq T$. If $T'' \in \text{DP-Lwc}(E'', F'')$ then $S'' \in \text{DP-Lwc}(E'', F'')$.*

Proof Let (f_n) be a disjoint sequence in $B_{F'}$. Then $(|f_n|)$ is also disjoint in $B_{F'}$, and hence $T'|f_n| \xrightarrow{\mathrm{w}} 0$ by Theorem 5. It follows from

$$|g(S'f_n)| \leq |g|(|S'f_n|) \leq |g|(S'|f_n|) \leq |g|(T'|f_n|) \to 0 \quad (\forall g \in E'')$$

that $S'f_n \xrightarrow{\mathrm{w}} 0$. Using Theorem 5 again, we get $S'' \in \text{DP-Lwc}(E'', F'')$. □

The next proposition follows from Proposition 6 $ii)$ by Proposition 8.

Proposition 10 Let an operator $T \in \mathrm{L}(E,F)$ possess the modulus. Then

$i)$ $T \in \operatorname{span}(\text{l-Lwc}_+(E,F))$ iff $|T| \in \text{l-Lwc}(E,F)$.

$ii)$ $T \in \operatorname{span}(\text{l-Mwc}_+(E,F))$ iff $|T| \in \text{l-Mwc}(E,F)$.

3.3. We conclude the paper with some algebraic aspects concerning the l-Lwc and l-Mwc operators. We need the following lemma.

Lemma 2 Let $\mathcal{P}$ be closed in the operator norm subspace of $\mathrm{L}(E)$. Then $\|T\|_{\text{r-}\mathcal{P}} := \inf\{\|S\| : \pm T \le S \in \mathcal{P}\}$ defines a norm on $\mathrm{span}(\mathcal{P}_+)$ such that $\|T\|_{\text{r-}\mathcal{P}} \ge \|T\|_r \ge \|T\|$ for $T \in \mathrm{span}(\mathcal{P}_+)$, where $\|T\|_r = \inf\{\|S\| : \pm T \le S \in \mathrm{L}(E)\}$ is the regular norm. Moreover, $\big(\mathrm{span}(\mathcal{P}_+), \|\cdot\|_{\text{r-}\mathcal{P}}\big)$ is a Banach space.

Proof It should be clear that $\|\cdot\|_{\text{r-}\mathcal{P}}$ is a norm satisfying $\|\cdot\|_{\text{r-}\mathcal{P}} \ge \|\cdot\|_r \ge \|\cdot\|$. Take a sequence (T_n) in $\mathrm{span}(\mathcal{P}_+)$ that is Cauchy in $\|\cdot\|_{\text{r-}\mathcal{P}}$, say $T_n = G_n - R_n$ for some $G_n, R_n \in \mathcal{P}_+$. WLOG, $\|T_{n+1} - T_n\|_{\text{r-}\mathcal{P}} < 2^{-n}$ for all $n \in \mathbb{N}$. As $\|\cdot\|_{\text{r-}\mathcal{P}} \ge \|\cdot\|$, there exists $T \in \mathrm{L}(E)$ such that $\|T - T_n\| \to 0$. Since $\mathcal{P}$ is closed in the operator norm, $T \in \mathcal{P}$. Pick $S_n \in \mathcal{P}$ with $\|S_n\| < 2^{-n}$ and $\pm(T_{n+1} - T_n) \le S_n$. Then

$$T_{n+1}(x^+) - T_n(x^+) \le S(x^+) \quad \text{and} \quad -T_{n+1}(x^-) + T_n(x^-) \le S(x^-) \tag{3}$$

for each $x \in E$. Summing up the inequalities in (3) gives $T_{n+1}x - T_n x \le S_n|x|$. Replacing x by $-x$ gives $T_n x - T_{n+1}x \le S_n|x|$, and hence

$$|(T_{n+1} - T_n)x| \le S_n|x| \quad (\forall x \in E). \tag{4}$$

As $\mathcal{P}$ is closed in the operator norm, $Q_n := \sum\limits_{k=n}^{\infty} S_k \in \mathcal{P}$ for all n. By (4),

$$|(T - T_n)x| = \lim_{k\to\infty} |(T_k - T_n)x| \le \sum_{k=n}^{\infty} |(T_{k+1} - T_n)x| \le Q_n|x| \quad (x \in E),$$

and hence $\pm(T - T_n) \le Q_n$. Then

$$-Q_n \le (T - T_n) \le Q_n \text{ and } 0 \le (T - T_n) + Q_n$$

for all $n \in \mathbb{N}$. Therefore,

$$T = [(T - T_n) + Q_n] + [T_n - Q_n] = [(T - T_n) + Q_n + G_n] - [R_n + Q_n] \in \mathrm{span}(\mathcal{P}_+),$$

and hence $(T - T_n) \in \mathrm{span}(\mathcal{P}_+)$ for all $n \in \mathbb{N}$. As $\|T - T_n\|_{\text{r-}\mathcal{P}} \le \|Q_n\| < 2^{1-n}$, we conclude $T_n \xrightarrow{\|\cdot\|_{\text{r-}\mathcal{P}}} T$. $\square$

Theorem 6 *The following statements hold:*

i) $\mathrm{span}(\text{l-Lwc}_+(E))$ (*resp.,* $\mathrm{span}(\text{l-Mwc}_+(E))$) *is a subalgebra of* $\mathrm{L}_r(E)$.

Moreover,

$$\mathrm{span}(\text{l-Lwc}_+(E)) = \mathrm{L}_r(E) \iff I_E \in \text{l-Lwc}(E), \tag{5}$$

$$\mathrm{span}(\text{l-Mwc}_+(E)) = \mathrm{L}_r(E) \iff I_E \in \text{l-Mwc}(E). \tag{6}$$

ii) *If* E *is Dedekind complete then* $\mathrm{span}(\text{l-Lwc}_+(E))$ (*resp.,* $\mathrm{span}(\text{l-Mwc}_+(E))$) *is a closed order ideal of the Banach lattice* $(\mathrm{L}_r(E), \|\cdot\|_r)$.

Proof We give the proof in the case of span(l-Lwc$_+(E)$). Arguments for span(l-Mwc$_+(E)$) are analogues.

i) It follows from Corollary 3 that span(l-Lwc$_+(E)$) is a right ideal and hence is a subalgebra of $\mathrm{L}_r(E)$. Formulas (5) and (6) follow from Corollary 3.

ii) By Proposition 10, span(l-Lwc$_+(E)$) is a Riesz subalgebra of $\mathrm{L}_r(E)$. Proposition 7 implies that span(l-Lwc$_+(E)$) is an order ideal of $\mathrm{L}_r(E)$. Since $\|T\|_r = \|\ |T|\ \| = \|T\|_{\text{r-l-Lwc}(E)}$ for $T \in$ span(l-Lwc$_+(E)$), we obtain that span(l-Lwc$_+(E)$) is a norm closed order ideal in $(\mathrm{L}_r(E), \|\cdot\|_r)$ by Proposition 4 and Lemma 2. □

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