GENERATING THE LEVEL 2 SUBGROUP BY INVOLUTIONS

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ABSTRACT. We obtain a minimal generating set of involutions for the level 2 subgroup of the mapping class group of a closed nonorientable surface.

1. INTRODUCTION

Let N_g be a closed nonorientable surface of genus $g \geq 2$. The mapping class group $\mathrm{Mod}(N_g)$ is defined to be the group of isotopy classes of all diffeomorphisms of N_g. The first homology group $H_1(N_g;\mathbb{Z})$ is generated by $\{x_1, x_2, \dots, x_g\}$, where x_i for $1 \leq i \leq g$ are the homology classes of one-sided curves as depicted in Figure 1.

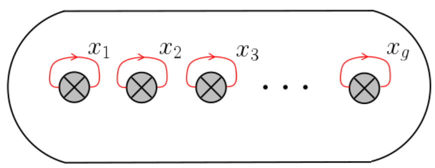

FIGURE 1. Generators of $H_1(N_g;\mathbb{Z})$.

The $\mathbb{Z}_2$-homology classes $\overline{x}_i$ of these curves form a basis for $H_1(N_g;\mathbb{Z}/2\mathbb{Z})$. The $\mathbb{Z}_2$-valued intersection pairing is a symmetric bilinear form $\langle\,,\rangle$ on $H_1(N_g;\mathbb{Z}/2\mathbb{Z})$ satisfying $\langle \overline{x}_i, \overline{x}_j\rangle = \delta_{ij}$ for $1 \leq i, j \leq g$. For more on automorphisms of $H_1(N_g;\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{Z}_2$-valued intersection pairings we refer the reader to [2]. Let $\mathrm{Iso(H_1(N_g;\mathbb{Z}/2\mathbb{Z})}$ be the group of automorphisms of $H_1(N_g;\mathbb{Z}/2\mathbb{Z})$ which preserve $\langle\,,\rangle$. *The level* 2 *subgroup* $\Gamma_2(N_g)$ of $\mathrm{Mod}(N_g)$ is the group of isotopy classes of diffeomorphisms which act trivially on $H_1(N_g;\mathbb{Z}/2\mathbb{Z})$. It fits into the following short exact sequence:

$$1 \longrightarrow \Gamma_2(N_g)) \longrightarrow \mathrm{Mod}(N_g) \longrightarrow \mathrm{Iso(H_1(N_g;\mathbb{Z}/2\mathbb{Z})} \longrightarrow 1.$$

For a two-sided simple closed curve α and a one-sided simple closed curve μ which intersect in one point, let K denote a regular neighborhood of $\mu \cup \alpha$ that is homeomorphic to the Klein bottle with a hole. Let $M \subset K$ be a regular neighbourhood of μ, which is a Möbius strip. We define the *crosscap slide* (also called *Y-homeomorphism*) $Y_{\mu,\alpha}$ as the self-diffeomorphism of N_g obtained from sliding M once along α and fixing each point of the boundary of K (Figure 2).

For $I = \{i_1, i_2, \dots, i_k\}$ a subset of $\{1, 2, \dots, g\}$, let α_I be the simple closed curve shown in Figure 3. Throughout the paper, we introduce the following notations:

- $Y_{i_1;i_2,\dots,i_k} = Y_{\alpha_{i_1};\alpha_{\{i_1,i_2,\dots,i_k\}}}$,

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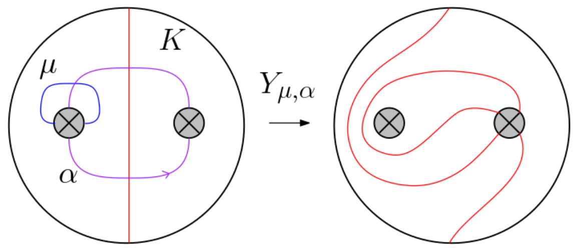

FIGURE 2. The crosscap slide $Y_{\mu,\alpha}$.

- $T_{i_1,i_2,\dots,i_k} = T_{\alpha_{\{i_1,i_2,\dots,i_k\}}}$,
- $\alpha_i = \alpha_{\{i,i\}}$.

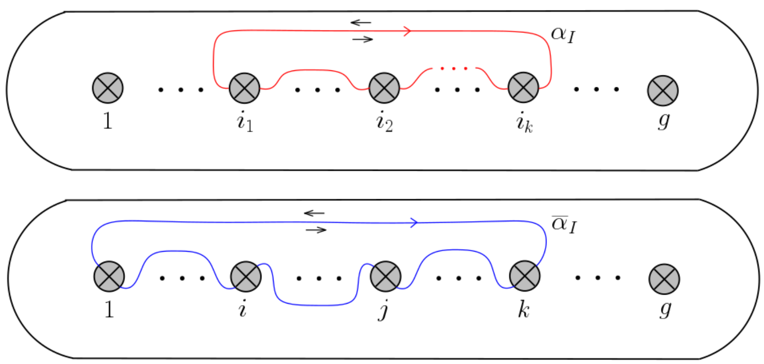

FIGURE 3. The curves α_I and $\overline{\alpha}_I$ for $I = \{i_1, i_2, \dots, i_k\}$.

Szepietowski proved that $\Gamma_2(N_g)$ is equal to the subgroup of $\mathrm{Mod}(N_g)$ generated by all crosscap slides [3, Theorem 5.5]. Moreover, he proved that $\Gamma_2(N_g)$ can be generated by (infinitely many) involutions [3, Theorem 3.7]. In [4], Szepietowski also gave a finite set of generators for $\Gamma_2(N_g)$. Later, Hirose and Sato reduced the number of generators of $\Gamma_2(N_g)$, their generating set is as follows [1, Theorem 1.2].

Theorem 1.1. *For $g \geq 4$, the level 2 subgroup $\Gamma_2(N_g)$ is generated by the following two types of elements:*

(1) *$Y_{i;j}$ for $i \in \{1, 2, \dots, g-1\}$, $j \in \{1, 2, \dots, g\}$ and $i \neq j$;*
(2) *$T^2_{1,j,k,l}$ for $1 < j < k < l$.*

Note that when $g = 3$, the group $\Gamma_2(N_3)$ is generated only by the elements of type (1). Hirose and Sato [1, Theorem 1.4] also showed that for $g \geq 4$

$$H_1(\Gamma_2(N_g); \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{\binom{g}{2}+\binom{g}{3}},$$

which in turn implies that the above generating set is minimal.

In this paper, our purpose is to give a minimal generating set of involutions for the level 2 subgroup $\Gamma_2(N_g)$.

2. A generating set for $\Gamma_2(N_g)$

Let us start this section by introducing bar notation for two-sided simple closed curves. In the remainder of this paper, let $\overline{\alpha}_{1,i,j,k}$ and $\overline{\alpha}_{i,j}$ be two sided simple closed curves depicted in Figure 4. Observe that when we put a bar over a two-sided simple closed curve it passes below the in-between crosscaps. For the ease of notation, we also use the following notations:

- $\overline{Y}_{i,j} = Y_{\alpha_i;\overline{\alpha}_{i,j}}$,
- $\overline{T}_{1,i,j,k} = T_{\overline{\alpha}_{1,i,j,k}}$.

Recall that $\Gamma_2(N_g)$ is generated by all crosscap slides [3, Theorem 5.5]. Let $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ be the subgroups of $\Gamma_2(N_g)$ generated by elements of the form $Y_{i,j}$ and $\overline{Y}_{i,j}$, for $i \in \{1, 2, \ldots, g-1\}$, $j \in \{1, 2, \ldots, g\}$ and $i \neq j$, respectively.

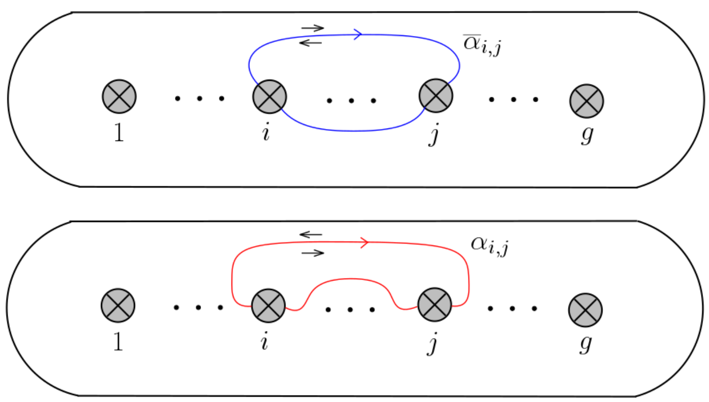

Figure 4. The curves $\overline{\alpha}_{i,j}$ and $\alpha_{i,j}$ for $1 < j < k < l$.

Lemma 2.1. *The subgroups $\mathcal{Y}$ and $\overline{\mathcal{Y}}$ are equal to each other.*

Proof. Let us first show that $\overline{\mathcal{Y}} \subseteq \mathcal{Y}$. For $\overline{Y}_{i,j} \in \overline{\mathcal{Y}}$, if we assume that $\mid i-j \mid = 1$, since

$$\overline{Y}_{i,i+1} = Y_{i,i+1} \text{ and } \overline{Y}_{i+1,i} = Y_{i+1,i}$$

for all $i = 1, 2, \ldots, g-1$, we have $\overline{Y}_{i,j} \in \mathcal{Y}$. Assume now that $\mid i-j \mid > 1$: For $i < j$, let us first consider the case $j - i = 2$. It is easy to verify that

$$\overline{Y}^{-1}_{i+1,i+2}(\alpha_i, \alpha_{i,i+2}) = (\alpha_i, \overline{\alpha}_{i,i+2}),$$

for all $i = 1, \ldots, g-2$ (see Figure 5). Using $\overline{Y}_{i+1,i+2} = Y_{i+1,i+2} \in \mathcal{Y}$, we have

$$\overline{Y}_{i,i+2} = \overline{Y}^{-1}_{i+1,i+2} Y_{i,i+2} \overline{Y}_{i+1,i+2} \in \mathcal{Y}.$$

For the case $j - i = 3$, one can see that (see Figure 6)

$$\overline{Y}^{-1}_{i+1,i+3}\overline{Y}^{-1}_{i+2,i+3}(\alpha_i, \alpha_{i,i+3}) = (\alpha_i, \overline{\alpha}_{i,i+3}).$$

Now, since $\overline{Y}_{i+2,i+3}$ and $\overline{Y}_{i+1,i+3}$ are all contained in $\mathcal{Y}$ we have

$$\overline{Y}_{i,i+3} = (\overline{Y}^{-1}_{i+1,i+3}\overline{Y}^{-1}_{i+2,i+3}) Y_{i,i+2} (\overline{Y}^{-1}_{i+1,i+3}\overline{Y}^{-1}_{i+2,i+3})^{-1} \in \mathcal{Y}$$

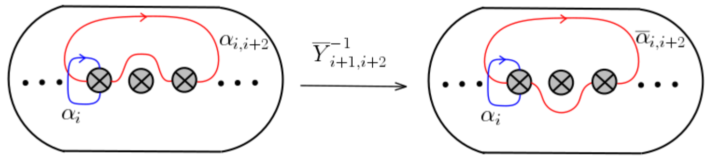

FIGURE 5. $\overline{Y}^{-1}_{i+1,i+2}(\alpha_i, \alpha_{i,i+2}) = (\alpha_i, \overline{\alpha}_{i,i+2})$.

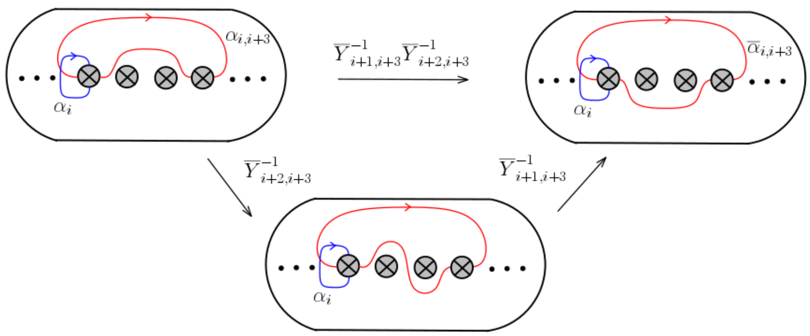

FIGURE 6. $\overline{Y}^{-1}_{i+1,i+3}\overline{Y}^{-1}_{i+2,i+3}(\alpha_i, \alpha_{i,i+3}) = (\alpha_i, \overline{\alpha}_{i,i+3})$.

for $i = 1, \dots, g-3$. For the remaining $i < j$ cases, one can see that

$$\overline{Y}_{i,j} = (\overline{Y}^{-1}_{i+1,j}\overline{Y}^{-1}_{i+2,j}\cdots\overline{Y}^{-1}_{j-1,j})Y_{i,j}(\overline{Y}^{-1}_{i+1,j}\overline{Y}^{-1}_{i+2,j}\cdots\overline{Y}^{-1}_{j-1,j})^{-1} \in \mathcal{Y}$$

for all $i = 1, \dots, g-1$, $j = 1, \dots, g$.

Now, we consider the cases where $i > j$. For $i - j > 2$, we have (see Figure 7)

$$\overline{Y}_{i+1,i}(\alpha_{i+2}, \alpha_{i,i+2}) = (\alpha_{i+2}, \overline{\alpha}_{i,i+2}).$$

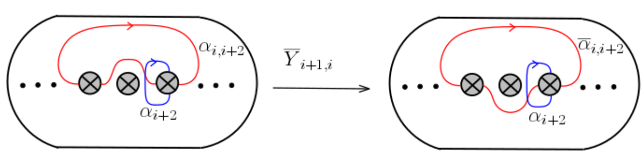

FIGURE 7. $\overline{Y}_{i+1,i}(\alpha_{i+2}, \alpha_{i,i+2}) = (\alpha_{i+2}, \overline{\alpha}_{i,i+2})$.

Using $\overline{Y}_{i+1,i} \in \mathcal{Y}$ for $i = 1, \dots, g-3$, we get

$$\overline{Y}_{i+2,i} = \overline{Y}_{i+1,i}Y_{i+2,i}\overline{Y}^{-1}_{i+1,i} \in \mathcal{Y}.$$

As before, for all $i = 1, \dots, g-1$ and $j = 1, \dots, g-2$, we have

$$\overline{Y}_{i,j} = (\overline{Y}_{i,j-1}\cdots\overline{Y}_{i,i+1})Y_{i,j}(\overline{Y}_{i,j-1}\cdots\overline{Y}_{i,i+1})^{-1} \in \mathcal{Y}.$$

Thus, $\overline{Y}_{i,j} \in \mathcal{Y}$ for $1 \leq i < j \leq g$. Since we cover all the cases, we have shown $\overline{\mathcal{Y}} \subseteq \mathcal{Y}$. For the reverse inclusion, note that we have the following equalities

$$Y_{i,j} = \begin{cases} (\overline{Y}_{i+1,j}^{-1} \cdots \overline{Y}_{j-1,j}^{-1})^{-1} \overline{Y}_{i,j} (\overline{Y}_{i+1,j}^{-1} \cdots \overline{Y}_{j-1,j}^{-1}), & \text{if } i < j, \\ (\overline{Y}_{i,j-1}^{-1} \cdots \overline{Y}_{i,i+1}^{-1})^{-1} \overline{Y}_{i,j} (\overline{Y}_{i,j-1}^{-1} \cdots \overline{Y}_{i,i+1}^{-1}), & \text{if } i > j, \end{cases} \tag{1}$$

which immediately imply that $\mathcal{Y} \subseteq \overline{\mathcal{Y}}$. □

Next, we present a minimal generating set for the level 2 subgroup $\Gamma_2(N_g)$ (cf. [1, Theorem 1.2]).

Theorem 2.2. *For $g \geq 4$, the level 2 subgroup $\Gamma_2(N_g)$ can be generated by*

1. $\overline{Y}_{i,j}$ *for* $i \in \{1, \ldots, g-1\}$, $j \in \{1, \ldots, g\}$ *and* $i \neq j$,
2. $\overline{T}_{1,i,j,k}^2$ *for* $1 < i < j < k$.

Proof. Let G be the subgroup of $\Gamma_2(N_g)$ generated by the elements given in (1) and (2). Since by Lemma 2.1 we have $\mathcal{Y} = \overline{\mathcal{Y}}$, it is enough to prove that $T_{1,i,j,k}^2$ is contained in the subgroup G for $1 < i < j < k$.

It is easy to check that

$$\overline{Y}_{i+1,j}^{-1} \cdots \overline{Y}_{j-2,j}^{-1} \overline{Y}_{j-1,j}^{-1} (\alpha_{1,i,j,k}) = \overline{\alpha}_{1,i,j,k}.$$

Thus

$$\overline{T}_{1,i,j,k}^2 = (\overline{Y}_{i+1,j}^{-1} \cdots \overline{Y}_{j-2,j}^{-1} \overline{Y}_{j-1,j}^{-1}) T_{1,i,j,k}^2 (\overline{Y}_{i+1,j}^{-1} \cdots \overline{Y}_{j-2,j}^{-1} \overline{Y}_{j-1,j}^{-1})^{-1},$$

which implies that

$$T_{1,i,j,k}^2 = (\overline{Y}_{i+1,j}^{-1} \cdots \overline{Y}_{j-2,j}^{-1} \overline{Y}_{j-1,j}^{-1})^{-1} \overline{T}_{1,i,j,k}^2 (\overline{Y}_{i+1,j}^{-1} \cdots \overline{Y}_{j-2,j}^{-1} \overline{Y}_{j-1,j}^{-1}) \in G$$

for $1 < i < j < k$. This completes the proof. □

3. Involution generators for $\Gamma_2(N_g)$

In this section, we give a generating set of involutions for $\Gamma_2(N_g)$. Throughout this section, consider the surface N_g as shown in Figure 8 so that it is invariant under the reflection R about the indicated plane. Note that, R acts trivially on $H_1(N_g; \mathbb{Z}/2\mathbb{Z})$, which implies that it is an element of the subgroup $\Gamma_2(N_g)$.

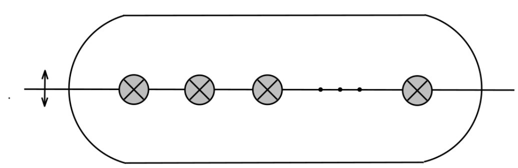

Figure 8. The reflection R.

Proposition 3.1. *For $g \geq 4$, the group $\Gamma_2(N_g)$ can be generated by*

1. R,
2. $R\overline{Y}_{i,j}$ *for* $i \in \{1, \ldots, g-1\}$, $j \in \{1, \ldots, g\}$ *and* $i \neq j$,
3. $R\overline{Y}_{1,i} Y_{\alpha_k, \overline{\alpha}_{j,k}} \overline{T}_{1,i,j,k}^2$ *for* $1 < i < j < k$.

Proof. Let G be the subgroup generated by the elements listed in the statement of the proposition. Since the subgroup G contains R and $R\overline{Y}_{i,j}$, it also contains

$$\overline{Y}_{i,j} = R(R\overline{Y}_{i,j})$$

for $i \in \{1, \ldots, g-1\}$, $j \in \{1, \ldots, g\}$ and $i \neq j$. Recall that $\overline{\mathcal{Y}}$ is generated by such elements, hence $\overline{\mathcal{Y}} \subseteq G$. By Theorem 2.2, it remains to prove that $\overline{T}^2_{1,i,j,k}$ also belongs to G. Now, it is easy to see that G contains

$$\overline{Y}_{1,i}Y_{\alpha_k,\overline{\alpha}_{j,k}}\overline{T}^2_{1,i,j,k} = R(R\overline{Y}_{1,i}Y_{\alpha_k,\overline{\alpha}_{j,k}}\overline{T}^2_{1,i,j,k}).$$

The elements $Y_{\alpha_k,\overline{\alpha}_{j,k}}$ are contained in $\mathcal{Y} = \overline{\mathcal{Y}}$ by [4, Lemma 3.5] and Lemma 2.1. Since the elements $\overline{Y}_{1,i}$ are also contained in G, one can conclude that $\overline{T}^2_{1,i,j,k} \in G$ for $1 < i < j < k$, which finishes the proof. $\square$

Lemma 3.2. *The reflection R can be expressed as a product of finitely Y-homeomorphisms. In particular*

$$R = \overline{Y}_{g-1,g}\overline{Y}_{g-2,g}\cdots\overline{Y}_{1,g}.$$

Proof. It follows from the proof of [3, Lemma 3.4] that R can be written as

$$\begin{aligned} R \;=\; & Y_{g-1,g}T^{-1}_{\alpha_{g-1,g}}Y_{g-2,g-1}T_{\alpha_{g-1,g}}(T_{\alpha_{i+1,i+2}}T_{\alpha_{i+2,i+3}}\cdots T_{\alpha_{g-1,g}})^{-1} \\ & Y_{i,i+1}(T_{\alpha_{i+1,i+2}}T_{\alpha_{i+2,i+3}}\cdots T_{\alpha_{g-1,g}})\cdots(T_{\alpha_{2,3}}\cdots T_{\alpha_{g-1,g}})^{-1}Y_{1,2}(T_{\alpha_{2,3}}\cdots T_{\alpha_{g-1,g}}). \end{aligned}$$

It is easy to see that

$$(T_{\alpha_{i+1,i+2}}T_{\alpha_{i+2,i+3}}\cdots T_{\alpha_{g-1,g}})^{-1}(\alpha_i, \alpha_{i,i+1}) = (\alpha_i, \overline{\alpha}_{1,g}),$$

from which we obtain

$$\overline{Y}_{i,g} = (T_{\alpha_{i+1,i+2}}T_{\alpha_{i+2,i+3}}\cdots T_{\alpha_{g-1,g}})^{-1}Y_{i,i+1}(T_{\alpha_{i+1,i+2}}T_{\alpha_{i+2,i+3}}\cdots T_{\alpha_{g-1,g}}),$$

for $i \in \{1, \ldots, g-1\}$. This completes the proof. $\square$

Next, we show that the elements mentioned in Theorem 3.1 are all involutions. We already know that the reflection R is an involution.

Lemma 3.3. *If $g \geq 4$, then the elements $R\overline{Y}^{\pm 1}_{i,j}$ are all involutions for $i \in \{1, \ldots, g-1\}$, $j \in \{1, \ldots, g\}$ and $i \neq j$.*

Proof. It is enough to see that $R(\alpha_i, \overline{\alpha}_{i,j}) = (\alpha_i^{-1}, \overline{\alpha}^{-1}_{i,j})$. $\square$

Lemma 3.4. *If $g \geq 4$, then the elements $R\overline{Y}_{1,i}Y_{\alpha_k,\overline{\alpha}_{j,k}}\overline{T}^2_{1,i,j,k}$ are all involutions for $1 < i < j < k$.*

Proof. First of all, it is easy verify that

$$R(\overline{\alpha}_{1,i}, \overline{\alpha}_{j,k}) = (\overline{\alpha}^{-1}_{1,i}, \overline{\alpha}^{-1}_{j,k}) \text{ and } R(\alpha_i, \alpha_k) = (\alpha_i^{-1}, \alpha_k^{-1}).$$

Then we have the following:

$$\begin{aligned} R\overline{Y}_{1,i}Y^{-1}_{\alpha_k,\overline{\alpha}_{j,k}}R^{-1} &= \overline{Y}^{-1}_{1,i}Y_{\alpha_k,\overline{\alpha}_{j,k}} \\ &= Y_{\alpha_k,\overline{\alpha}_{j,k}}\overline{Y}^{-1}_{1,i}, \end{aligned}$$

where the last identity follows from the commutativity of crosscap slides $\overline{Y}_{1,i}$ and $Y_{\alpha_k,\overline{\alpha}_{j,k}}$. Observe that, this implies $R\overline{Y}_{1,i}Y^{-1}_{\alpha_k,\overline{\alpha}_{j,k}}$ is an involution. Moreover, since

$$R\overline{Y}_{1,i}Y^{-1}_{\alpha_k,\overline{\alpha}_{j,k}}(\overline{\alpha}_{1,i,j,k}) = \overline{\alpha}^{-1}_{1,i,j,k}$$

it follows that $R\overline{Y}_{1,i}Y_{\alpha_k,\overline{\alpha}_{j,k}}\overline{T}^2_{1,i,j,k}$ is also an involution. □

Finally, we present our involution generators. Note that in the following, the number of involution generators is equal to $\binom{g}{2}+\binom{g}{3}$ which is the minimal possible number of generators for $\Gamma_2(N_g)$.

Theorem 3.5. *For $g \geq 5$ and odd, $\Gamma_2(N_g)$ is generated by the following involutions:*

1. $R\overline{Y}_{1,g}, R\overline{Y}^{-1}_{2,g}, \ldots, R\overline{Y}_{g-2,g}, R\overline{Y}^{-1}_{g-1,g}$,
2. $R\overline{Y}_{i,j}$ *for* $i, j \in \{1, 2, \ldots, g-1\}$ *and* $i \neq j$,
3. $R\overline{Y}_{1,i}Y^{-1}_{\alpha_k,\overline{\alpha}_{j,k}}\overline{T}^2_{1,i,j,k}$ *for* $1 < i < j < k$.

For $g \geq 4$ and even, $\Gamma_2(N_g)$ is generated by the following involutions:

1. $R, R\overline{Y}_{1,g}, R\overline{Y}^{-1}_{2,g}, \ldots, R\overline{Y}_{g-2,g}$,
2. $R\overline{Y}_{i,j}$ *for* $i, j \in \{1, 2, \ldots, g-1\}$ *and* $i \neq j$,
3. $R\overline{Y}_{1,i}Y^{-1}_{\alpha_k,\overline{\alpha}_{j,k}}\overline{T}^2_{1,i,j,k}$ *for* $1 < i < j < k$.

Proof. Let G denote the subgroup of $\Gamma_2(N_g)$ generated by the elements listed in Theorem 3.5. It follows from lemmata 3.3 and 3.4 that the generators of the group G are involutions.

Let us first assume that $g \geq 5$ and odd. By Proposition 3.1, it is enough to prove that R is contained in the subgroup G. It follows from Lemma 3.2 the reflection R can be expresses as

$$\begin{aligned} R &= \overline{Y}_{g-1,g}\overline{Y}_{g-2,g}\cdots\overline{Y}_{1,g} \\ &= R^2\overline{Y}_{g-1,g}\overline{Y}_{g-2,g}R^2\overline{Y}^{-1}_{g-3,g}\cdots R^2\overline{Y}_{2,g}\overline{Y}_{1,g} \\ &= R\overline{Y}^{-1}_{g-1,g}R\overline{Y}_{g-2,g}R\overline{Y}^{-1}_{g-3,g}R\cdots R\overline{Y}^{-1}_{2,g}R\overline{Y}_{1,g}, \end{aligned}$$

which is contained in the subgroup G using $R\overline{Y}^{-1}_{i,g}R = \overline{Y}_{i,g}$.

Assume now that $g \geq 4$ and even. In this case, by Proposition 3.1, it suffices to show that the subgroup G contains the element $\overline{Y}_{g-1,g}$. The following element is contained in the subgroup G:

$$\begin{aligned} &R(R\overline{Y}_{g-2,g}R\overline{Y}^{-1}_{g-3,g}R\overline{Y}_{g-4,g}R\cdots R\overline{Y}^{-1}_{2,g}R\overline{Y}_{1,g}) \\ &= R(R\overline{Y}_{g-2,g}R^2\overline{Y}_{g-3,g}\overline{Y}_{g-4,g}\cdots R^2\overline{Y}_{2,g}\overline{Y}_{1,g}) \\ &= \overline{Y}_{g-2,g}\overline{Y}_{g-3,g}\overline{Y}_{g-4,g}\cdots\overline{Y}_{2,g}\overline{Y}_{1,g}, \end{aligned}$$

using again $R\overline{Y}^{-1}_{i,g}R = \overline{Y}_{i,g}$. One can conclude that $\overline{Y}_{g-1,g} \in G$ since $R \in G$ by Lemma 3.2, which finishes the proof. □

References

[1] S. Hirose, M. Sato: *A minimal generating set of the level 2 mapping class group of a non-orientable surface*, Math. Proc. Cambridge Philos. Soc. **157**, (2) (2014), 345–355.

[2] J. D. McCarthy, U. Pinkall: *Representing homology automorphisms of nonorientable surfaces*, Max Planc Inst. preprint MPI/SFB 85–11, revised version written in 2004. Available at http://www.math.msu. edu/ mccarthy (2004)

[3] B. Szepietowski: *Crosscap slides and the level 2 mapping class group of a nonorientable surface*, Geom. Dedicata **160** (2012), 169–183.

[4] B. Szepietowski: *A finite generating set for the level 2 mapping class group of a nonorientable surface*, Kodai Mathematical Journal **36** (2013), 1–14.

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