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# QCD sum rule study of $J^P = 1^{\pm}$ exotic states with two heavy quarks in the molecular picture

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**Abstract:** In this study, the spectroscopic parameters of exotic molecular states composed of mesons containing two heavy quarks (scalar - axial and pseudoscalar - axial meson combinations) are investigated within the QCD sum rules. Our findings reveal that molecular states containing charm quarks do not form bound states, whereas states with $b$-quarks can form exotic molecular states. This observation has significant implications for understanding the structure of these exotic states.

**Keywords:** QCD sum rules, exotic mesons, molecular states

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## I. INTRODUCTION

Numerous exotic hadronic states have been observed [1–5], among which the exotic doubly charmed $T_{cc}$ state is of particular interest. This state was observed at LHCb near the $D^0D^{*+}$ threshold in the $D^0D^0\pi^+$ mass distribution with a mass difference $\delta m = m_{T_{cc}} - (m_{D^{*+}} + m_{D^0}) = (-273 \pm 65 \pm 5^{+11}_{-14})$ keV and decay width $\Gamma = (410 \pm 165 \pm 43^{+18}_{-98})$ keV [6]. Further analysis revealed that this state has quantum numbers $J^P = 1^+$. Although this state had previously been predicted theoretically in [7, 8], this was the first observation of a tetraquark with doubly charmed quarks. This discovery triggered intensive theoretical studies [3–22]. See the review articles [23–28] for further details on the topic.

In addition to the exotic state with doubly $c$-quarks, the quark model also predicts the existence of states with two $bb$ and $bc$ heavy quarks. States with doubly heavy quarks are usually described as tetraquark or molecular states. In the molecular picture, two quarks and two antiquarks can form two color singlet mesons by exchanging light mesons. With this prediction, we can further explore the characteristics of these states. To better understand these characteristics, high-precision experimental data and refined theoretical calculations are needed near the $DD^*(s)$ and $BB^*(s)$ thresholds.

In this study, we investigate exotic molecular states composed of scalar - axial mesons and pseudoscalar - axial mesons containing two heavy quarks with quantum numbers $J^P = 1^{\pm}$ in the framework of the QCD sum rules [19]. The paper is organized as follows. In Section II, we calculate the correlation functions of the exotic states with quantum numbers $J^P = 1^{\pm}$ within the molecular picture and derive formulas for the mass and residues of these states. Section III presents a numerical analysis of the sum rules obtained for the masses and residues, and Section IV contains our conclusion.

## II. SUM RULES FOR EXOTIC $J^P = 1^{\pm}$ STATES IN THE MOLECULAR PICTURE

In this section, we derive mass sum rules for exotic states with quantum numbers $J^P = 1^{\pm}$ within the molecular picture. The interpolating current for these states can be written as

$$j_\mu = \left(\bar{Q}_1^a \Gamma_1 q^a\right)\left(\bar{Q}_2^b \Gamma_{2\mu} q^b\right) , \quad (1)$$

where $q_1$, $q_2$ , and $Q$ represent light $u, d, s$ and heavy $b$ or $c$ quarks. The sum rules for the interpolating current with $\Gamma_1 = i\gamma_5$ and $\Gamma_{2\mu} = \gamma_\mu$ were studied in [11]. However, the interpolating current with $J^P = 1^+$ can also be chosen as $\Gamma_1 = 1$, $\Gamma_{2\mu} = \gamma_\mu\gamma_5$ , and the state with $J^P = 1^-$ can be represented by $\Gamma_1 = \gamma_5$ and $\Gamma_{2\mu} = \gamma_\mu\gamma_5$ or $\Gamma_1 = 1$. These cases are denoted as I and II, respectively. In this study, we at-

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tempt to answer the following question: If these molecular structures were realized in nature, what would be their masses and residues?

To determine the mass sum rules, the following correlation function is considered:

$$\Pi^{(i)}_{\mu\nu} = \mathrm{i}\int \mathrm{d}^4x\, \mathrm{e}^{\mathrm{i}px}\langle 0|T\left\{j_\mu(x)j^+_\nu(0)\right\}|0\rangle$$
$$= \left(-g_{\mu\nu}+\frac{p_\mu p_\nu}{p^2}\right)\Pi^{(i)}_1(p^2)+\frac{p_\mu p_\nu}{p^2}\Pi^{(i)}_0(p^2), \quad (2)$$

where $i$ corresponds to the relevant currents.

The polarization functions $\Pi^{(i)}_0(p^2)$ and $\Pi^{(i)}_1(p^2)$ correspond to the spin-0 and spin-1 intermediate states, respectively. In further discussions, we focus on the structure $-g_{\mu\nu}+\frac{p_\mu p_p\nu}{p^2}$ because it contains only the contribution of the spin-1 particles.

The sum rules for the relevant physical quantity can be obtained by calculating the correlation function in terms of the hadrons and quark-gluon degrees of freedom in the deep Euclidean region $p^2\to-\infty$ using operator product expansion (OPE). Subsequently, by matching both representations and performing Borel transformation over the variable $-p^2$, the desired sum rules are obtained.

At the hadronic level, the correlation function can be calculated using the dispersion relation for the invariant function $\Pi^{(i)}_1$,

$$\Pi^{(i)}_1 = \frac{(p^2)^\alpha}{\pi}\int_{(2m_Q+m_{q_1}+m_{q_2})^2}^{\infty}\frac{\mathrm{Im}\Pi^{(i)}_1(s)}{(s^2)^\alpha(s-p^2-\mathrm{i}\epsilon)}$$
$$+\,\text{subtraction terms}. \quad (3)$$

In the sum rules method, $\mathrm{Im}\Pi^{(i)}_1(s)$ is defined as the spectral density function,

$$\rho^{(i)}(s) = \frac{1}{\pi}\mathrm{Im}\Pi^{(i)}_1 = \lambda_i^2\delta(s-m_i^2)$$
$$+\,\text{continuum}\,+\,\text{higher states contributions}, \quad (4)$$

where $m_i$ is the lowest-lying resonance. The decay constant $\lambda_i$ is defined in the standard way as

$$\langle 0|\, j^{(i)}_\mu\left|T_{QQ}(p)\right\rangle = \lambda_i\epsilon_\mu, \quad (5)$$

with the polarization vector $\epsilon_\mu$ of the $J^P=1^-(1^+)$ states. Using Eq. (4), we obtain the following expression for the correlation function from the hadronic side for the structure $\left(-g_{\mu\nu}+\frac{p_\mu p_\nu}{p^2}\right)$,

$$\Pi^{(i)}_1 = \frac{\lambda_i^2}{m_i^2-p^2}. \quad (6)$$

Moreover, the correlation function $\Pi^{(i)}_1$ can be calculated from the QCD side using OPE in the deep Euclidean domain $p^2\to-\infty$, and we obtain

$$\Pi_{\mu\nu} = \mathrm{i}\int \mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}px}\Big\{\mathrm{Tr}\left[\Gamma_1 S^{aa_1}_Q(x)\widetilde{\Gamma}_1 S^{a_1a}_{q_1}(-x)\right]$$
$$\times\,\mathrm{Tr}\left[\Gamma_{2\mu}S^{bb_1}_Q(x)\widetilde{\Gamma}_{2\nu}S^{b_1b}_{q_2}(-x)\right]$$
$$-\,\mathrm{Tr}\left[S^{ba_1}_Q(x)\widetilde{\Gamma}_1 S^{a_1a}_{q_1}(-x)\Gamma_1 S^{ab_1}_Q(x)\widetilde{\Gamma}_{2\nu}S^{b_1b}_{q_2}(-x)\Gamma_{2\mu}\right]\Big\}, \quad (7)$$

in terms of the light $S_{q_i}$ and heavy $S_Q$ quark propagators. Hence, to calculate the correlation function from the QCD side, we need the explicit expressions of the light and heavy quark propagators.

The light quark propagator in the $x$-representation up to linear order in the light quark mass is given as

$$S^{ab}_q(x) = \frac{\mathrm{i}\not{x}}{2\pi^2x^4}\delta^{ab}-\frac{m_q}{4\pi^2x^2}\delta^{ab}-\frac{\langle\bar{q}q\rangle}{12}\left(1-\mathrm{i}\frac{m_q}{4}\not{x}\right)\delta^{ab}$$
$$-\frac{x^2}{192}m_0^2\langle\bar{q}q\rangle\left(1-\mathrm{i}\frac{m_q}{6}\not{x}\right)\delta^{ab}$$
$$+\frac{\mathrm{i}}{32\pi^2x^2}g_sG^{ab}_{\mu\nu}(\sigma^{\mu\nu}\not{x}+\not{x}\sigma^{\mu\nu})$$
$$-\frac{1}{3^3\,2^{10}}\langle\bar{q}q\rangle\langle g_s^2G^2\rangle x^4\delta^{ab}+\cdots, \quad (8)$$

where $\langle\bar{q}q\rangle$ is the light quark condensate, and $G_{\mu\nu}$ is the gluon field strange tensor. Furthermore, the heavy quark propagator in the $x$–representation is given as (see, for example, [11])

$$S_Q(x)^{ab} = \frac{m_Q^2\delta^{ab}}{(2\pi)^2}\left[\mathrm{i}\not{x}\frac{K_2(m_Q\sqrt{-x^2})}{(\sqrt{-x^2})^2}+\frac{K_1(m_Q\sqrt{-x^2})}{\sqrt{-x^2}}\right]$$
$$-\frac{m_Qg_sG^{ab}_{\mu\nu}}{8(2\pi)^2}\left[\mathrm{i}(\sigma^{\mu\nu}\not{x}+\not{x}\sigma^{\mu\nu})\frac{K_1(m_Q\sqrt{-x^2})}{\sqrt{-x^2}}\right.$$
$$\left.+2\sigma^{\mu\nu}K_0(m_Q\sqrt{-x^2})\right]$$
$$-\frac{\langle g_s^2G^2\rangle\delta^{ab}}{(3^2\,2^8\pi)^2}\left[(\mathrm{i}m_Q\not{x}-6)(-x^2)\frac{K_1(m_Q\sqrt{-x^2})}{\sqrt{-x^2}}\right.$$
$$\left.+m_Qx^4\frac{K_2(m_Q\sqrt{-x^2})}{(\sqrt{-x^2})^2}\right], \quad (9)$$

where $K_0$, $K_1$, and $K_2$ are the modified Bessel functions of the second kind. Using the explicit expressions of the light and heavy quark propagators, we obtain the spectral density $\rho^{(i)}$ after standard but tedious calculations, as shown in Appendix A.

By equating the coefficients of the corresponding Lorentz structures of the correlation function from both representations, we obtain the sum rules of the relevant physical quantities. As a final step, to suppress the higher and continuum state contributions, we perform Borel transformation with respect to $-p^2$

$$\lambda^2 e^{-m^2/M^2} = \int_{s_{\min}}^{s_0} ds\, e^{-s/M^2} \rho(s)\,, \tag{10}$$

where $s_0$ is the continuum threshold, $M^2$ is the Borel mass parameter, and $s_{\min} = (m_{Q_1} + m_{Q_2} + m_{q_1} + m_{q_2})^2$. Note that the mass sum rules for $T_{QQ}$ can easily be obtained by taking the derivative with respect to the inverse Borel mass parameter, from which we obtain

$$m^2 = \frac{\int_{s_{\min}}^{s_0} ds\, s e^{-s/M^2} \rho(s)}{\int_{s_{\min}}^{s_0} ds\, e^{-s/M^2} \rho(s)}\,. \tag{11}$$

Having determined the mass of the considered states, the residues can be found using Eq. (10).

## III. NUMERICAL ANALYSIS

In this section, we perform a numerical analysis of the sum rules for the tetraquark states with doubly heavy quarks with $J^P = 1^{\pm}$ obtained in the previous section. The sum rules contain many input parameters, such as quark and gluon condensates of appropriate dimensions and quark masses. The numerical values of these input parameters are listed in Table 1. For the heavy quark masses, we use their values in the $\overline{\text{MS}}$ scheme.

In addition to these input parameters, the sum rules also contain two auxiliary parameters, namely, the Borel mass $M^2$ and continuum threshold $s_0$. To decide on the working regions of $s_0$ and $M^2$, it is important to examine how the mass of the tetraquark state will vary with changes in the parameter values. The dependency should be minimal. Moreover, the following requirements should be met for an acceptable Borel mass parameter region:

a) The upper bound of $M^2$ is obtained from the requirement that the continuum and higher state contributions constitute a maximum of 40% of the total result, that is, the pole contribution (PC) should dominate. This is determined using the following ratio:

$$PC = \frac{\int_{s_{\min}}^{s_{\max}} ds \rho(s) e^{-s/M^2}}{\int_{s_{\min}}^{\infty} ds \rho(s) e^{-s/M^2}}\,.$$

b) The minimum value of $M^2$ is established using the requirement that OPE should be convergent. We must ensure that any condensate with the highest number of dimensions contributes less than 5% to the total result.

c) The continuum threshold parameter $s_0$ can be determined by requiring minimum variation in the dependency of the mass of the considered state for the Borel mass window that satisfies the conditions mentioned above.

To ensure that all the requirements are met, numerical analyses are performed. The working regions of the parameters $M^2$ and $s_0$ are also identified, as shown in Table 2.

As an example, Figs. 1 and 2 show the dependency of the masses and residues of the $BB_1$ and $B_0B_1$ states on $M^2$ at fixed values of $s_0$ for the $J^P = 1^-$ and $J^P = 1^+$ states, respectively. These figures indicate that when $M^2$ varies in the domain $10\ \text{GeV}^2 \leq M^2 \leq 12\ \text{GeV}^2$ (after $M^2 = 12$

**Table 1.** Numerical values of the constant parameters used in our calculations.

| $\Pi_i$ | Structures |
|---|---|
| $\overline{m}_s(2\ \text{GeV})$ | $93.4^{+8.6}_{-3.4}$ MeV [29] |
| $\overline{m}_b(\overline{m}_b)$ | $4.18^{+0.03}_{-0.02}$ GeV [29] |
| $\overline{m}_c(\overline{m}_c)$ | $(1.27 \pm 0.02)$ GeV [29] |
| $\langle \bar{q}q \rangle (1\ \text{GeV})$ | $(-1.65 \pm 0.15) \times 10^{-2}\ \text{GeV}^3$ [30] |
| $\langle \bar{s}s \rangle$ | $(0.8 \pm 0.2) \langle \bar{q}q \rangle\ \text{GeV}^3$ [30] |
| $m_0^2$ | $(0.8 \pm 0.2)\ \text{GeV}^2$ [30] |
| $\langle \frac{\alpha_s G^2}{\pi} \rangle$ | $(0.012 \pm 0.006)\ \text{GeV}^4$ [31] |

**Table 2.** Working regions of $M^2$ and $s_0$ for the considered molecular states.

| | | $M^2/\text{GeV}^2$ | $s_0/\text{GeV}^2$ |
|---|---|---|---|
| $J^P = 1^-$(I) | $BB_1$ | $9-12$ | $130 \pm 2$ |
| | $BB_{1s}$ | $9-12$ | $132 \pm 2$ |
| | $B_sB_1$ | $9-12$ | $132 \pm 2$ |
| | $B_sB_{1s}$ | $9-12$ | $134 \pm 2$ |
| $J^P = 1^-$(II) | $B_0B^*$ | $9-12$ | $130 \pm 2$ |
| | $B_0B_s^*$ | $9-12$ | $132 \pm 2$ |
| | $B_{0s}B^*$ | $9-12$ | $132 \pm 2$ |
| | $B_{0s}B_s^*$ | $9-12$ | $134 \pm 2$ |
| $J^P = 1^+$ | $B_0B_1$ | $9-12$ | $134 \pm 2$ |
| | $B_0B_{1s}$ | $9-12$ | $136 \pm 2$ |
| | $B_{0s}B_1$ | $9-12$ | $136 \pm 2$ |
| | $B_{0s}B_{1s}$ | $9-12$ | $140 \pm 2$ |

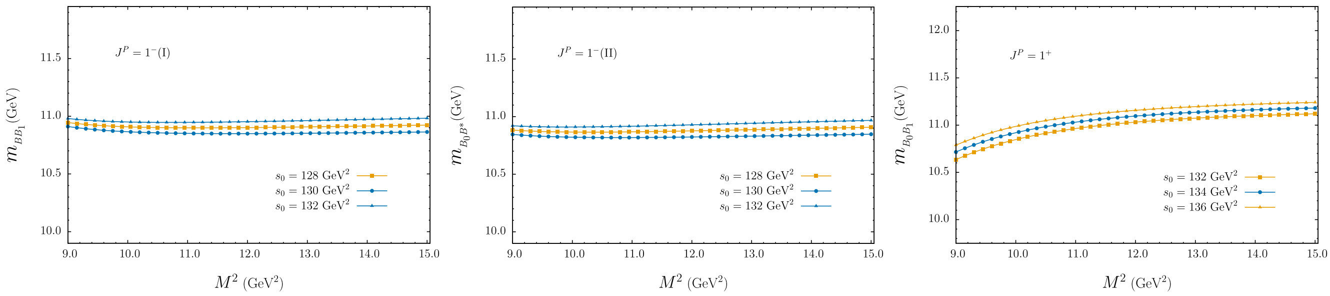


**Fig. 1.** (color online) Dependence of the mass of the $BB_1$ $B_0B^*$, and $B_0B_1$ molecular states on $M^2$ at fixed values of the continuum threshold $s_0$.

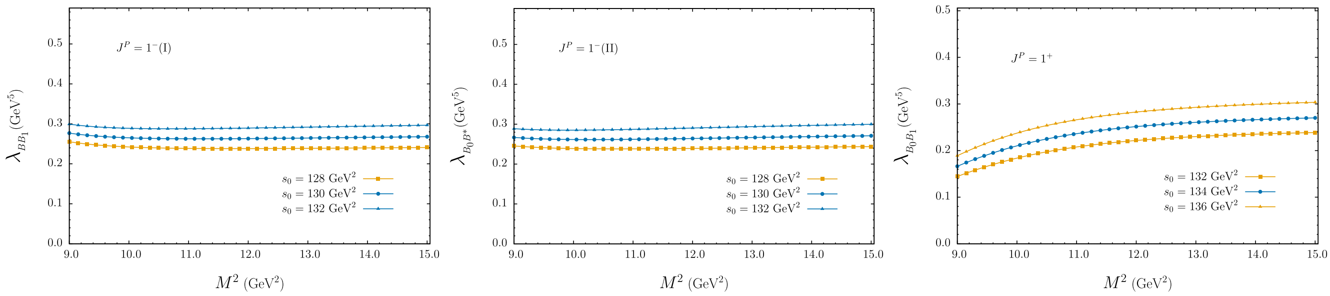


**Fig. 2.** (color online) Dependence of the residue of the $BB_1$, $B_0B^*$, and $B_0B_1$ molecular states on $M^2$ at fixed values of the continuum threshold $s_0$.

GeV$^2$, the results essentially do not change) the masses of the $BB_1$ and $B_0B_1$ states exhibit good stability.

Performing a similar analysis, we obtain the masses and residues of the other possible molecular states, the results of which are given in Table 3.

Similar analyses are also performed for doubly charmed systems. Our numerical calculations show that for the Borel mass parameter region in which the pole contribution is greater than 1/2, we cannot find a stable plateau for the mass of the doubly charmed states. Therefore, we infer that mesons containing two charm quarks do not form bound states in the molecular picture considered. This phenomenon has also been noted in several previous studies [32–37]. The large difference in the QQ binding energy for the $c$ and $b$ quarks can explain this outcome. It should be noted that doubly charmed tetraquark states with $J^P = 1^+$ (scalar-axial current) were considered in [9] with a slightly different current, and the masses of these states were predicted to be different from our results. The main difference is that in [9], the input parameters of the QCD parameters were used at a different $\mu$ scale; however, in our case, the traditional $\mu = 1$ GeV scale is used. These studies can help put together the pieces of the puzzle regarding the composition of these tetraquark states.

As a final remark, we would like to note that the intermediate two meson states can also contribute to the correlation functions under consideration. However, as shown in [38, 39], contributions to the correlation functions originating from these two meson states are small and therefore can safely be neglected. For this reason, these contributions are not considered in this study.

**Table 3.** Numerical results for the masses and residues of the $B_{(s)}B_{1(s)}$ and $B_{0(s)}B_{1(s)}$ molecular systems.

| | | Mass/GeV | Residue/GeV$^5$ |
|---|---|---|---|
| $J^P = 1^-$(I) | $BB_1$ | $10.90 \pm 0.03$ | $0.26 \pm 0.02$ |
| | $BB_{1s}$ | $10.94 \pm 0.03$ | $0.27 \pm 0.04$ |
| | $B_sB_1$ | $10.94 \pm 0.03$ | $0.27 \pm 0.04$ |
| | $B_sB_{1s}$ | $11.10 \pm 0.02$ | $0.30 \pm 0.02$ |
| $J^P = 1^-$(II) | $B_0B^*$ | $10.92 \pm 0.03$ | $0.27 \pm 0.02$ |
| | $B_sB_s^*$ | $10.94 \pm 0.03$ | $0.28 \pm 0.02$ |
| | $B_{0s}B^*$ | $10.96 \pm 0.03$ | $0.28 \pm 0.02$ |
| | $B_{0s}B_s^*$ | $11.10 \pm 0.04$ | $0.32 \pm 0.02$ |
| $J^P = 1^+$ | $B_0B_1$ | $11.00 \pm 0.10$ | $0.20 \pm 0.02$ |
| | $B_0B_{1s}$ | $11.15 \pm 0.05$ | $0.24 \pm 0.03$ |
| | $B_{0s}B_1$ | $11.15 \pm 0.05$ | $0.24 \pm 0.03$ |
| | $B_{0s}B_1$ | $11.25 \pm 0.05$ | $0.28 \pm 0.02$ |

## IV. CONCLUSION

In this study, we investigate the spectroscopic parameters, namely, the masses and residues of potential exotic states with quantum numbers $J^P = 1^\pm$, containing two heavy quarks in the molecular picture within the QCD sum rules. Our findings reveal that no bound states can be formed for the $T_{cc}$ states with quantum numbers $J^P = 1^\pm$ for the considered currents. However, for the $T_{bb}$

states, our calculations demonstrate that bound states can indeed exist within both considered molecular frameworks. This result significantly contributes to the ongoing discussions in the literature regarding the inner nature of exotic states. Furthermore, the obtained mass values for the $J^P=1^+$ and $J^P=1^-$ exotic states within the molecular picture exhibit a difference of $\sim 200\,\text{MeV}$. This result can be verified in future experiments to establish the "correct" picture.

Even though it is challenging to determine the states near the $BB^*(s)$ and $DD^*(s)$ thresholds, the results obtained for the masses of the tetraquark states can be considered in future experiments searching for exotic hadrons.


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## APPENDIX A

In this appendix, we present the expressions of the spectral density $\rho(s)$ for the $J^P=1^-$(I), $J^P=1^-$(II), and $J^P=1^+$ tetraquark systems. The upper (lower) sign corresponds to the $1^-$(I) ($1^+$) quantum number. With the replacement of $m_u\to -m_u$ and $\langle\bar{u}u\rangle=-\langle\bar{u}u\rangle$ for the $J^P=1^-$(I) case, we can obtain the spectral density for the $J^P=1^-$(II) case.

$$
\begin{aligned}
\rho^{(pert)}(s) = &\pm\frac{1}{(3\times 2^{14})\pi^6}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha^3}\int_{\beta_{\min}}^{\beta_{\max}}\frac{\mathrm{d}\beta}{\beta^3}\left[(\alpha+\beta)m_b^2-\alpha\beta s\right]^2\\
&\times\Big\{(1-\beta)\beta m_b^3\Big([8+29\beta(1+\beta)]m_b\mp 120\beta[(1+\beta)m_d\mp 2m_u]\Big)\\
&-\alpha^4(29m_b^2-33\beta s)(m_b^2-\beta s)+2\alpha m_b\Big\{m_b\left[(4+21\beta-58\beta^3)m_b^2\mp 12\beta(33-32\beta)m_d m_u\right.\\
&\left.\mp 60\beta m_b(m_d-3\beta^2 m_d\mp 2m_u\pm 4\beta m_u)\right]-(1-\beta)\beta\left[(4+31\beta(1+\beta))m_b\mp 60\beta(m_d+\beta m_d\mp 2m_u)\right]s\Big\}\\
&-2\alpha^3\beta(58m_b^4\mp 60m_b^3m_d-93\beta m_b^2 s\pm 60\beta m_b m_d s+33\beta^2 s^2)\\
&+3\alpha^2\Big\{(7-58\beta^2)m_b^4+40\beta m_b^3(\pm 3\beta m_d-2m_u)\mp 80\beta^2 m_b(\beta m_d\mp m_u)s\\
&\pm 2\beta m_b^2(128m_d m_u\mp 9s\pm 31\beta^2 s)+\beta^2 s\left[\pm 8m_d m_u+11(1-\beta^2)s\right]\Big\}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\rho^{(\langle\bar{d}d\rangle\langle\bar{u}u\rangle)}(s) = &-\frac{\langle\bar{d}d\rangle\langle\bar{u}u\rangle}{(3\times 2^8)\pi^2}\int_{\alpha_{\min}}^{\alpha_{\max}}\mathrm{d}\alpha\Big\{4m_b\left[13m_b-5(\pm 2m_d-m_u)\right]+\alpha\left[12m_0^2+8m_b(m_b\pm 5m_d)\right.\\
&\left.-m_u(20m_b\pm 33m_d)-12s\right]-3\alpha^2(4m_0^2\mp 11m_d m_u-4s)\Big\},
\end{aligned}
$$

$$
\rho^{(m_0^2\langle\bar{d}d\rangle\langle g^2G^2\rangle)}(s) = \frac{5(m_0^2\langle\bar{d}d\rangle\langle g^2G^2\rangle)}{(3\times 2^{12})m_b\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\mathrm{d}\alpha(1-\alpha)^2,
$$

$$
\rho^{(m_0^2\langle\bar{u}u\rangle\langle g^2G^2\rangle)}(s) = \mp\frac{5(m_0^2\langle\bar{u}u\rangle\langle g^2G^2\rangle)}{(3\times 2^{11})m_b\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\mathrm{d}\alpha(1-\alpha)^2,
$$

$$
\begin{aligned}
\rho^{(m_0^2\langle\bar{d}d\rangle)}(s) = &\mp\frac{m_0^2\langle\bar{d}d\rangle}{(3\times 2^{10})\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\Big\{m_b\left[\pm 30m_b^2-20m_d m_u(1-\alpha)\alpha-\alpha m_b(12m_d+22\alpha m_d\mp 39m_u\mp 6\alpha m_u)\right]\\
&-(1-\alpha)\alpha(\pm 30m_b-22\alpha m_d\pm 9\alpha m_u)s\Big\}+\frac{m_0^2\langle\bar{d}d\rangle}{(3\times 2^{10})\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\int_{\beta_{\min}}^{\beta_{\max}}\mathrm{d}\beta\Big\{m_b^2\left[30(\alpha+\beta)m_b\mp\alpha m_d\right]-30m_b\alpha\beta s\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\rho^{(m_0^2\langle\bar{u}u\rangle)}(s) = &\pm\frac{m_0^2\langle\bar{u}u\rangle}{(3\times 2^{10})\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\Big\{m_b\left[30m_b^2\mp 10m_d m_u(1-\alpha)\alpha+\alpha m_b(12m_u+22\alpha m_u\mp 39m_d\mp 6\alpha m_d)\right]\\
&-(1-\alpha)\alpha(30m_b\mp 9\alpha m_d+22\alpha m_u)s\Big\}\mp\frac{m_b^2 m_u m_0^2\langle\bar{u}u\rangle}{(3\times 2^{10})\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\mathrm{d}\alpha\int_{\beta_{\min}}^{\beta_{\max}}\mathrm{d}\beta,
\end{aligned}
$$

$$\rho^{(\langle\bar{d}d\rangle\langle g^2G^2\rangle)}(s) = -\frac{\langle\bar{d}d\rangle\langle g^2G^2\rangle}{(3^2\times 2^{13})m_b\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\Big\{4\big[3-\alpha(193-15\alpha)\big]m_b^2+\alpha m_b\big[(\pm763\mp760\alpha)m_d-2(61+74\alpha+6\alpha^2)m_u\big]$$
$$\pm40(1-\alpha)\alpha\big[3(1-\alpha)m_dm_u+(\pm3\mp2\alpha)s\big]\Big\}+\frac{\langle\bar{d}d\rangle\langle g^2G^2\rangle}{(3^2\times 2^{13})m_b\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\int_{\beta_{\min}}^{\beta_{\max}}\mathrm{d}\beta\Big\{4\big[8\alpha-9(3+2\beta)\big]m_b^2$$
$$\pm3\alpha m_bm_d+120\alpha\beta s\Big\},$$

$$\rho^{(\langle\bar{u}u\rangle\langle g^2G^2\rangle)}(s) = -\frac{\langle\bar{u}u\rangle\langle g^2G^2\rangle}{(3^2\times 2^{13})m_b\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\Big\{\pm4\big[21+2\alpha(17-5\alpha)\big]m_b^2-\alpha m_b\big[2(61+74\alpha+6\alpha^2)m_d\mp(133-130\alpha)m_u\big]$$
$$-60(1-\alpha)^2\alpha(m_dm_u\pm2s)\Big\}\pm\frac{\langle\bar{u}u\rangle\langle g^2G^2\rangle}{(3\times 2^{13})\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\int_{\beta_{\min}}^{\beta_{\max}}\mathrm{d}\beta(12m_b+\alpha m_u),$$

$$\rho^{(\langle g^2G^2\rangle^2)}(s) = \mp\frac{\langle g^2G^2\rangle^2}{(3^3\times 2^{18})m_b^2\pi^6}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\Big\{\big[192-\alpha(4143+584\alpha)\big]m_b^2$$
$$\mp132\alpha\big[1-2(3-\alpha)\alpha\big]m_dm_u\pm20\alpha m_b\big[(31+74\alpha)m_d\mp11(m_u+2\alpha m_u)\big]-8(3-\alpha)(1-\alpha)\alpha s\Big\}$$
$$+\frac{\langle g^2G^2\rangle^2}{(3^3\times 2^{17})m_b^2\pi^6}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha}\int_{\beta_{\min}}^{\beta_{\max}}\frac{\mathrm{d}\beta}{\beta}\Big\{(\pm)m_b\big[18\alpha^2m_b+18(22-9\beta)\beta m_b$$
$$+5\alpha\beta(37m_b\mp24m_d)\big]\mp6\alpha\beta(3\alpha+5\beta)s\Big\},$$

$$\rho^{(\langle g^2G^2\rangle)}(s) = \pm\frac{\langle g^2G^2\rangle}{(3\times 2^{15})m_b\pi^6}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha(1-\alpha)}\Big\{\big[m_b^2-(1-\alpha)\alpha s\big]$$
$$\times\Big(m_b\big[33m_b^2\mp80(1-\alpha)m_dm_u-20m_b(m_d\mp m_u)\big]-(1-\alpha)\alpha\big[33m_b\mp20(m_d\mp m_u)\big]s\Big)\Big\}$$
$$-\frac{\langle g^2G^2\rangle}{(3\times 2^{16})m_b\pi^6}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha^2}\int_{\beta_{\min}}^{\beta_{\max}}\frac{\mathrm{d}\beta}{\beta^2}\Big\{\pm\beta m_b^4\Big(\big\{108+\beta\big[219+\beta(360+233\beta)\big]\big\}m_b$$
$$+24\beta\big\{\big[6+4(1-\beta)\beta\big]m_d\mp(3-5\beta)m_u\big\}\Big)\mp3\alpha^4m_b(m_b^2-\beta s)^2$$
$$\pm2\alpha^3(m_b^2-\beta s)\big\{2m_b^2\big[(27+47\beta)m_b\pm18\beta m_d\big]\mp\beta^2(\pm61m_b-80m_d)s\big\}$$
$$+2m_b^2\alpha\Big(m_b\Big[\pm3\big\{18+\beta\big[1+2\beta(82+55\beta)\big]\big\}m_b^2-6\beta(12+35\beta)m_dm_u$$
$$+4\beta m_b\big\{3\big[6-5\beta(14+\beta)\big]m_d\mp(9-94\beta)m_u\big\}\Big]\mp\beta\Big[\big\{54+\beta\big[111+5\beta(114+37\beta)\big]\big\}m_b$$
$$-12\beta\big\{\big[6+\beta(4-9\beta)\big]m_d\mp(3-10\beta)m_u\big\}\Big]s\Big)$$
$$-\alpha^2\Big[m_b^3\Big((\pm)3(71-244\beta-206\beta^2)m_b^2+8\beta m_b\big[6(37-\beta)m_d\mp79m_u\big]+420\beta m_dm_u\Big)$$
$$\pm2\beta m_b\Big(m_b\big\{\big[105-\beta(702+343\beta)\big]m_b\pm8\beta(111+19\beta)m_d\big\}$$
$$-94\beta(4m_b\mp3m_d)m_u\Big)s\pm\beta^2\big\{\big[3+\beta(780+137\beta)\big]m_b-120\beta(\pm\beta m_d-m_u)\big\}s^2\Big]\Big\},$$

$$\rho^{(\langle\bar{d}d\rangle)}(s) = \mp\frac{\langle\bar{d}d\rangle}{2^{10}\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha(1-\alpha)}\big[m_b^2-(1-\alpha)\alpha s\big]\Big\{m_b\big\{11m_bm_d\mp2\big[m_b\mp10(1-\alpha)m_d\big]m_u\big\}$$
$$-(1-\alpha)\alpha(11m_d\mp2m_u)s\Big\}-\frac{\langle\bar{d}d\rangle}{2^{10}\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha^2}\int_{\beta_{\min}}^{\beta_{\max}}\frac{\mathrm{d}\beta}{\beta}\big[(\alpha+\beta)m_b^2-\alpha\beta s\big]\Big\{m_b^2\big\{(\alpha+\beta)$$
$$\times\big[20(\alpha+\beta)m_b\pm9\alpha m_d\big]+44\alpha m_u\big\}\mp\alpha\beta\big[(\pm)20(\alpha+\beta)m_b+11\alpha m_d\big]s\Big\},$$

$$
\begin{aligned}
\rho^{\langle\langle\bar{u}u\rangle\rangle}(s) = & -\frac{\langle\bar{u}u\rangle}{2^{10}\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha(1-\alpha)}\left[m_b^2-(1-\alpha)\alpha s\right] \\
& \times\left[20(1-\alpha)m_b m_d m_u + m_b^2(2m_d \mp 11m_u) - (1-\alpha)\alpha(2m_d \mp 11m_u)s\right] \\
& \pm\frac{\langle\bar{u}u\rangle}{2^{10}\pi^4}\int_{\alpha_{\min}}^{\alpha_{\max}}\frac{\mathrm{d}\alpha}{\alpha^2}\int_{\beta_{\min}}^{\beta_{\max}}\frac{\mathrm{d}\beta}{\beta}\left[(\alpha+\beta)m_b^2-\alpha\beta s\right] \\
& \times\left\{m_b\left\{\pm 4m_b\left[\pm 5(\alpha+\beta)m_b - 11\alpha m_d\right] - \alpha\left[9(\alpha+\beta)m_b \mp 20\beta m_d\right]m_u\right\} - \alpha\beta(20m_b - 11\alpha m_u)s\right\},
\end{aligned}
$$

$$
\alpha_{\min} = \frac{s-\sqrt{s(s-4m_b^2)}}{2s}, \quad
\alpha_{\max} = \frac{s+\sqrt{s(s-4m_b^2)}}{2s}, \quad
\beta_{\min} = \frac{m_b^2\alpha}{s\alpha-m_b^2}, \quad
\beta_{\max} = 1-\alpha\,.
$$

The results for the spectral densities containing strange quarks can also be obtained from the presented results by replacing the mass(es) and condensate(s) of the appropriate light quark(s) with those of the $s$-quark.

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