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The Morse Index for Manifolds with Constant Sectional Curvature

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Abstract. We compute the Morse index of a critical submanifold of the energy functional on the loop space of a manifold with constant sectional curvature. The case of constant non-positive sectional curvature is a known result and the case of a sphere has been proved by Klingenberg. We adapt Klingenberg's proof of the case of a sphere to the case of constant sectional curvature, to obtain the possible Morse indices of critical submanifolds of the energy functional.

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1. Introduction

The Morse index of a critical submanifold of the energy functional on the loop space of a Riemannian manifold has been computed in the literature. It has been computed by Klingenberg for spheres in [7] and for complex projective spaces, quaternionic projective spaces, and Cayley projective planes in [6]. The indices for compact rank one symmetric spaces (which are spheres, real projective spaces, complex projective spaces, quaternionic projective spaces, and the Cayley projective plane) have been computed by Ziller [16]. The indices for simply connected compact rank one symmetric spaces have also been computed by Hingston [3].[1]

The Morse indices of the critical submanifolds of the energy functional have been used in the literature to obtain obstruction results for the embeddings of Lagrangian tori (see [13]), spheres and other compact rank one symmetric spaces (see [11]) utilizing an index relation (see [1,5,15]) involving the Morse index. They also are used to obtain restrictions on the Maslov class of Lagrangian submanifolds [4]. The result obtained in this paper modifies the possible Morse indices listed in [12] where the use of Morse indices

[1] There are also studies extending the classical Morse index theorem to semi-Riemannian manifolds [9,10,14].

provided an alternative proof of a nonexistence result for certain displaceable constant sectional curvature Lagrangian submanifolds in symplectically aspherical symplectic manifolds (M^{2n}, ω) with $n > 4$ [12].[2]

In this paper, we adapt the proof of [7] for the case of spheres to manifolds with constant sectional curvature to obtain Theorem 1.1. In Theorem 1.1, the case $A \leq 0$ is a classical result, a consequence of the known more general statement that the Morse index I is zero if M^n is a manifold of nonpositive sectional curvature. From the remaining cases, the ones not overlapping with Klingenberg's, Hingston's and Ziller's results are new to the best of my knowledge.

Theorem 1.1. *Let M^n be a complete Riemannian manifold of constant sectional curvature A. If $c(t) \in \Lambda(M^n)$ is a critical point of the energy functional $\mathcal{E}_g(q) = \int_0^1 \frac{1}{2}\|\dot{q}(t)\|^2 dt$ where $q(t) \in \Lambda(M^n)$, then the Morse index I of the nondegenerate critical submanifold S_c (containing c) of the energy functional is as follows:*

If $A \leq 0$: $I = 0$.

If $0 < A$: $I = 0$ or $I = (2B + 1)(n - 1)$ or $I = 2(D + 1)(n - 1)$,

where

$$B = \begin{cases} \left\lfloor \frac{\sqrt{A}\|\dot{c}\|}{2\pi} \right\rfloor & if \quad \frac{\sqrt{A}\|\dot{c}\|}{2\pi} \notin \mathbb{Z} \\ \frac{\sqrt{A}\|\dot{c}\|}{2\pi} - 1 & if \quad \frac{\sqrt{A}\|\dot{c}\|}{2\pi} \in \mathbb{Z} \end{cases}$$

and

$$D = \begin{cases} \left\lfloor \frac{\sqrt{A}\|\dot{c}\| - \pi}{2\pi} \right\rfloor & if \quad \frac{\sqrt{A}\|\dot{c}\| - \pi}{2\pi} \notin \mathbb{Z} \\ \frac{\sqrt{A}\|\dot{c}\| - \pi}{2\pi} - 1 & if \quad \frac{\sqrt{A}\|\dot{c}\| - \pi}{2\pi} \in \mathbb{Z} \end{cases}$$

with $\|\dot{c}\|$ denoting the length of the closed geodesic c.

Furthermore, for $M = \mathbb{R}P^n$, if $\sqrt{A}\|\dot{c}\|$ is a nonzero even multiple of π, then $I = (2B + 1)(n - 1)$; if $\sqrt{A}\|\dot{c}\|$ is an odd multiple of π, then $I = 2(D + 1)(n - 1)$; and if $\|\dot{c}\|=0$, then $I = 0$.

For $M = S^n$, if $\|\dot{c}\|=0$, then $I = 0$; and if $\sqrt{A}\|\dot{c}\|$ is nonzero, then $I = (2B + 1)(n - 1)$.

2. Preliminaries

The *loop space* $\Lambda(M^n)$ is defined as the set of H^1-maps of S^1 into M^n (see [3,7,16]).

The following definitions agree with their corresponding ones in [7]:

[2] The main theorem of [12], Theorem 1.1, still holds and its method of proof still applies with the indices listed in this paper.

Let $c \in \Lambda(M^n)$ be a critical point of the energy functional $\mathcal{E}_g(q) = \int_0^1 \frac{1}{2}\|\dot{q}(t)\|^2 dt$, $q(t) \in \Lambda(M^n)$. Let A_c denote the self-adjoint operator $A_c : T_c\Lambda(M^n) \longrightarrow T_c\Lambda(M^n)$ defined by the identity

$$< A_c\xi, \xi' >_1 = < \xi, A_c^T \xi' >_1 = D^2\mathcal{E}_g(c)(\xi, \xi'),$$

where one defines

$$< \xi, \xi >_1 = < \xi, \xi >_0 + < \nabla\xi, \nabla\xi >_0 \quad ,$$

$$< \xi, \xi >_0 = \int_S < \xi(t), \xi(t) > dt \quad ,$$

and $\nabla\xi$ is the covariant derivative $\nabla_{\dot{c}}\xi$.

$T_c\Lambda(M^n)$ has an orthogonal decomposition into subspaces $T_c^-\Lambda(M^n)$, $T_c^0\Lambda(M^n)$ and $T_c^+\Lambda(M^n)$, spanned by the eigenvectors of A_c having positive, zero and negative eigenvalue, respectively. The dimension of $T_c^-\Lambda(M^n)$ is called the $(Morse)$ *index* of c and the dimension of $T_c^0\Lambda(M^n)$ is called the *nullity* of c.

Define the map $\chi : S^1 \times \Lambda(M^n) \to \Lambda(M^n)$ by $\chi(z, c) = z.c$ with $z.c(t) = c(t + r)$ where $z = e^{2\pi i r} \in S^1$.

A closed submanifold S of $\Lambda(M^n)$ is called *critical* if it is closed under the S^1-action χ, if $\mathcal{E}_g$ restricted to S is constant and if S consists entirely of critical points of $\mathcal{E}_g$.

If the index of c is the same constant for all c in a critical submanifold S, this constant is called the $(Morse)$ *index* of S. If the nullity of $D^2\mathcal{E}_g(c)$ is constant for all $c \in S$, then this number is called the *nullity* of S.

S is called a $nondegenerate\ \ critical\ \ submanifold$ if the critical submanifold S has an index and if the nullity of S is equal to the dimension of S.

3. Known Results

Klingenberg computed the index of a critical submanifold of the energy functional on the loop space of the sphere as given in the following theorem:

Theorem 3.1 ([6], p.71). *The critical set $Cr\Lambda S^n$ of the energy functional $\mathcal{E}_g(q) = \int_0^1 \frac{1}{2}\|\dot{q}(t)\|^2 dt$ on the loop space $\Lambda(S^n)$ decomposes into the non-degenerate critical submanifolds $\Lambda^0 S^n$ which is isomorphic to S^n and B_q consisting of the q-fold covered great circles $q = 1, 2, \ldots$. B_q is isomorphic to the Stiefel manifold $V(2, n+1)$ of orthonormal 2-frames in $\mathbb{R}^{n+1}$ and the index of B_q is $(2q-1)(n-1)$.*

As stated in the introduction, the index calculated for the sphere case in Theorem 3.1 has also been calculated for the complex projective space, the quaternionic projective space and the Cayley projective plane by Klingenberg in [6] and by Hingston in [3]. This is given in Theorem 3.2. The indices for compact rank one symmetric spaces are given by Ziller in [16].

Theorem 3.2 (p.103 of [2], p.19 of [5]). *The standard metric on M is normalized so that the maximal sectional curvature is 1. Then the critical set of the energy function on ΛM consists of the critical manifolds A_m of geodesics of length $2\pi m$, $m \geq 0$. We have $A_0 \cong M$ and $A_m \cong STM$ the unit tangent bundle of M for $m \geq 1$. By counting zeros of the Jacobi fields, one can see that A_m $(m > 0)$ is a nondegenerate critical submanifold of index*

$$I(A_m) = (2m-1)(a-1) + (m-1)(s-1)a,$$

where $a = n$, 2, 4, or 8, respectively, for $M = S^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$, CaP^2 and $s = \frac{dimM}{a}$ (= 1, n, n, n = 2, respectively).

In addition, for $M = \mathbb{R}P^n$, the index of a nondegenerate critical submanifold A_m of m-fold covered geodesics for $m \geq 1$ is $(m-1)(n-1)$, by Ziller [16].

4. Proof of Theorem 1.1

Let $c(t) \in \Lambda(M^n)$ be a critical point of $\mathcal{E}_g$. For a Riemannian manifold with constant sectional curvature A, the formula of the Riemann curvature endomorphism for $X = \eta(t)$ in the tangent space $T_c\Lambda(M^n)$, $Y = \dot{c}$ and $Z = \dot{c}$ is given by

$$R(\eta(t), \dot{c}(t), \dot{c}(t)) = A\Big(< \dot{c}(t), \dot{c}(t) > \eta(t) - < \eta(t), \dot{c}(t) > \dot{c}(t)\Big) \tag{4.1}$$

[8]. To find the Morse index, we will count the periodic solutions of

$$(\lambda - 1)(\nabla^2 - 1)\eta - (R+1)\eta = 0 \tag{4.2}$$

for $\lambda < 0$ [7].[3]

Here, we use the notation $R\eta = R(\eta(t), \dot{c}(t), \dot{c}(t))$ [4]and ∇ is the covariant derivative on tangent bundle TM, derived from the Levi-Civita connection [7].

When $A = 0$, we have $R = 0$, and Eq. (4.2) reduces to

$$(\lambda - 1)(\nabla^2 - 1)\eta - \eta = 0. \tag{4.3}$$

Since $\lambda = 1$ is not an eigenvalue, the equation can be written as

$$\nabla^2\eta + \frac{\lambda}{1-\lambda}\eta = 0. \tag{4.4}$$

This equation has no periodic solutions when $\lambda < 0$ (p.59 of [7]); so the dimension of the subspace of the tangent space $T_c\Lambda M$ spanned by the eigenvectors having negative eigenvalue, dim $T_c^-\Lambda M$, for c in a nondegenerate critical submanifold is zero.

Next, we consider $A \neq 0$:

[3]The periodic solutions of this differential equation are the eigenvectors of the self-adjoint operator A_c (for an eigenvalue $\lambda \in \mathbb{R}$).This is 2.4.4 Corollary 2 on p.58 of [5]. For A_c and the notation used for the terms in A_c, see p.56–58 in [5].

[4]What we have denoted with R is denoted with $K_{\tilde{c}}$ in [7].

We insert Eq. (4.1) in Eq. (4.2). When $\lambda < 0$, Eq. (4.2) becomes

$$\nabla^2\eta + \left(\frac{A\|\dot{c}\|^2}{1-\lambda} + \frac{\lambda}{1-\lambda}\right)\eta - \frac{A < \eta, \dot{c} >}{1-\lambda}\dot{c} = 0. \tag{4.5}$$

We decompose η into the tangential part, $\alpha(t)\dot{c}(t)$, and the vertical part, $\xi(t)\perp\dot{c}$.

Then, the last formula decomposes into

$$(tan) \qquad \ddot{\alpha}(t) + \left(\frac{\lambda}{1-\lambda}\right)\alpha(t) = 0, \tag{4.6}$$

$$(ver) \qquad \nabla^2\xi + \left(\frac{A\|\dot{c}\|^2 + \lambda}{1-\lambda}\right)\xi = 0. \tag{4.7}$$

Solutions of (tan) for $\lambda < 0$:

- If $\lambda < 0$, then there are no periodic solutions. *Solutions of (ver) for* $\lambda < 0$:
- If $\lambda < -A\|\dot{c}\|^2$, then there are no periodic solutions.[5]
- If $-A\|\dot{c}\|^2 \le \lambda < 0$, then we consider Eq. (4.6) on the universal cover of M. The universal cover of a constant sectional curvature manifold with positive A is S^n by the Killing–Hopf theorem. The general solution of (4.6) considered on the universal cover is of the form[6,7]

$$\xi(t) = \xi_0 \cos\left(\sqrt{\frac{(A\|\dot{c}\|^2 + \lambda)}{(1-\lambda)}}t\right) + \xi_1 sin\left(\sqrt{\frac{((A\|\dot{c}\|^2 + \lambda)}{(1-\lambda)}}t\right). \tag{4.8}$$

Here, if $\tilde{c}$ denotes a lift of c (with $\|\dot{c}\| = \|\dot{\tilde{c}}\|$) and $\Omega\tilde{M}$ denotes the space of paths on the universal cover $\tilde{M}$, as it is the case with ξ, ξ_0 and ξ_1 also belong to $T_{\tilde{c}}\Omega\tilde{M}$ and are vector fields along $\tilde{c}$ which are orthogonal to $\dot{\tilde{c}}$ and are induced by parallel transport.

Counting the periodic solutions to find $dimT_c^-\Lambda M$

For $\lambda < 0$:

Case 1: When $-A\|\dot{c}\|^2 \le \lambda < 0$ is not satisfied, then there are no periodic solutions. (This will always be the case for $A < 0$ and may be possible for $A > 0$.)

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[5] Equation (4.6) is of the form $\nabla^2\xi + E\xi = 0$ where $E = \left(\frac{A\|\dot{c}\|^2+\lambda}{1-\lambda}\right)$. If $\lambda < -A\|\dot{c}\|^2$, then $E < 0$. Let $\dot{c}, X_1, ..., X_{n-1}$ be an orthonormal frame invariant by parallelism along a geodesic c. If $E < 0$, $Y_i = sinh(\sqrt{-E}s)X_i$'s and $Z_i = cosh(\sqrt{-E}s)X_i$'s form a basis for the solutions of $\nabla^2\xi + E\xi = 0$ that are orthogonal to $\dot{c}$. (s denotes the arclength). (see p.485–486 of [2].) Noting that these solutions are **not** periodic, we conclude that there are no periodic solutions orthogonal to $\dot{c}$.

[6] With the notation in Footnote 5, if $-A\|\dot{c}\|^2 \le \lambda < 0$, then $E \ge 0$.If $E > 0$, $Y_i = sin(\sqrt{E}s)X_i$'s and $Z_i = \cos(\sqrt{E}s)X_i$'s form a basis for the solutions of $\nabla^2\xi + E\xi = 0$ that are orthogonal to $\dot{c}$ [2]. If $E = 0$, then X_i's form a basis for the solutions [2].

[7] For S^n with $A = 1$, letting $A = 1$ in Eq. 4.8, we obtain the same $\xi(t)$ obtained by Klingenberg in [7].

Case 2: If $-A\|\dot{c}\|^2 \leq \lambda < 0$ is satisfied (which is possible only if $A > 0$), the general solution of (4.5) considered on the universal cover is

$$\eta = \left[\cos\left(\sqrt{\frac{(A\|\dot{c}\|^2+\lambda)}{(1-\lambda)}}t\right)\right]\xi_0 + \left[sin\left(\sqrt{\frac{(A\|\dot{c}\|^2+\lambda)}{(1-\lambda)}}t\right)\right]\xi_1, \tag{4.9}$$

where $\xi_0(t), \xi_1(t) \perp \dot{\tilde{c}}(t)$ and $\xi_0(t), \xi_1(t)$ are induced by parallel transport.

From considering Eq. (4.1) on the universal cover, it is seen that η is an eigenvector of R. Since R is invariant under parallel transport, parallel transport leaves the eigenspaces of R invariant, so $\eta(0) = \pm\eta(1)$ should be satisfied (p.7 of [16]).[8]

Subcase 2.1: If $\eta(0) = \eta(1)$, then

$$\sqrt{\frac{(A\|\dot{c}\|^2+\lambda)}{(1-\lambda)}} = 2\pi p, \tag{4.10}$$

where $p \in \mathbb{Z}$ should be satisfied.

Solving Eq. (4.10) for λ gives us

$$\lambda = \frac{4\pi^2p^2 - A\|\dot{c}\|^2}{1+4\pi^2p^2}. \tag{4.11}$$

We get a bound for p coming from imposing the condition that $\lambda < 0$ in (4.11). We obtain

$$p < \frac{\sqrt{A}\|\dot{c}\|}{2\pi}. \tag{4.12}$$

Equivalently,

$$p \leq B, \tag{4.13}$$

where

$$B = \begin{cases} \left\lfloor \frac{\sqrt{A}\|\dot{c}\|}{2\pi} \right\rfloor & if \quad \frac{\sqrt{A}\|\dot{c}\|}{2\pi} \notin \mathbb{Z} \\ \frac{\sqrt{A}\|\dot{c}\|}{2\pi} - 1 & if \quad \frac{\sqrt{A}\|\dot{c}\|}{2\pi} \in \mathbb{Z}. \end{cases}$$

Using (4.10) and the condition on p of (4.13), (4.9) becomes

$$\eta = cos(2\pi pt)\xi_0 + sin(2\pi pt)\xi_1, \tag{4.14}$$

where $\xi_0(t), \xi_1(t) \perp \dot{\tilde{c}}(t)$ and $p \leq B$.

We will count the number of possible choices of directions for fixed p: It is n-1 for $p = 0$ since $\eta = \xi_0$. For $p \geq 1$, it is $2(n-1)$. Hence,

$$\begin{aligned} dimT_c^-\Lambda M &= \dim T_{\tilde{c}}^-\Omega M = (n-1) + \sum_{p=1}^{B} 2(n-1) \\ &= (n-1) + B2(n-1) = (n-1)(2B+1). \end{aligned} \tag{4.15}$$

[8]Since $c(t)$ is a closed loop, $c(0) = c(1)$ and the projection of $\eta(0)$ and $\eta(1)$ to M^n are the same point.

Subcase 2.2: If $\eta(0) = -\eta(1)$, then

$$\sqrt{\frac{(A\|\dot{c}\|^2 + \lambda)}{(1-\lambda)}} = (2p+1)\pi, \tag{4.16}$$

where $p \in \mathbb{Z}$ should be satisfied.

Solving Eq. (4.16) for λ gives us

$$\lambda = \frac{(2p+1)^2\pi^2 - A\|\dot{c}\|^2}{1 + (2p+1)^2\pi^2}. \tag{4.17}$$

We get a bound for p coming from imposing the condition that $\lambda < 0$ in (4.17). We obtain

$$p < \frac{\sqrt{A}\|\dot{c}\| - \pi}{2\pi}. \tag{4.18}$$

Equivalently,

$$p \leq D, \tag{4.19}$$

where

$$D = \begin{cases} \left\lfloor \frac{\sqrt{A}\|\dot{c}\|-\pi}{2\pi} \right\rfloor & if \quad \frac{\sqrt{A}\|\dot{c}\|-\pi}{2\pi} \notin \mathbb{Z} \\ \frac{\sqrt{A}\|\dot{c}\|-\pi}{2\pi} - 1 & if \quad \frac{\sqrt{A}\|\dot{c}\|-\pi}{2\pi} \in \mathbb{Z}. \end{cases}$$

Using (4.16) and the condition on p of (4.19), (4.9) becomes

$$\eta = \cos\big((2p+1)\pi t\big)\xi_0 + \sin\big((2p+1)\pi t\big)\xi_1, \tag{4.20}$$

where $\xi_0(t), \xi_1(t) \perp \dot{c}(t)$ and $p \leq D$.

We will count the number of possible choices of directions for fixed p: It is $2(n-1)$ for $p \geq 0$. Hence,

$$\dim T_c^- \Lambda M = \sum_{p=0}^{D} 2(n-1) = 2(n-1)(D+1). \tag{4.21}$$

Considering both cases (with all subcases): When $A < 0$, only Case 1 is possible. When $0 < A$, both Case 1 and Case 2 are possible.

Also, for $M = \mathbb{R}P^n$, if $\sqrt{A}\|\dot{c}\|$ is an even multiple of π, c lifts to a closed loop $\tilde{c}$ in the universal cover S^n, and $\eta(0) = \eta(1)$, and if $\sqrt{A}\|\dot{c}\|$ is an odd multiple of π, c lifts to a non-closed path in the universal cover and $\eta(0) = -\eta(1)$. For $M = S^n$, for a closed geodesic, $\sqrt{A}\|\dot{c}\|$ is an even multiple of π, so $\eta(0) = \eta(1)$. These lead to the statement. □

Example: Let M be the n-dimensional sphere with radius r in $\mathbb{R}^{n+1}$ with the metric induced by $\mathbb{R}^{n+1}$. Then M has constant sectional curvature $A = \frac{1}{r^2}$. The closed geodesics (with positive length) have length $2\pi r m$, where m is a positive integer. We have $\frac{\sqrt{A}\|\dot{c}\|}{2\pi} = m \in \mathbb{Z}$, so $B = m - 1$. Hence the Morse index of a nondegenerate critical submanifold of M is $I = 0$ (for the critical submanifold consisting of the constant loops) or $I = (n-1)(2B+1) = (n-1)(2m-1)$, which is consistent with the known results.

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Declarations

Conflict of interest The author declares to have no conflict of interest.

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