

INVOLUTION GENERATORS OF THE BIG MAPPING CLASS GROUP

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ABSTRACT. Let $S = S(n)$ denote the infinite surface with n ends, $n \in \mathbb{N}$, accumulated by genus. For $n \geq 6$, we show that the mapping class group of S is topologically generated by five involutions. When $n \geq 3$, it is topologically generated by six involutions.

1. INTRODUCTION

Let S be a second countable, connected, orientable surface with compact (possibly empty) boundary. We say that S is of *finite type* if its fundamental group is finitely generated and *infinite type* otherwise. The *mapping class group* of S , denoted by $\text{Mod}(S)$, is defined as the group of isotopy classes of orientation preserving self-homeomorphisms of S .

Let us first consider finite type surfaces. The mapping class group of a surface of finite type has been studied in details for many years. Many sets of generators are known. It is now a classical result that $\text{Mod}(S)$ is generated by finitely many Dehn twists about non-separating simple closed curves [3, 5, 10]. The study of algebraic properties of mapping class group, finding small generating sets, generating sets with particular properties, has been an active one leading to interesting developments. Wajnryb [16] showed that $\text{Mod}(S)$ can be generated by two elements given as a product of Dehn twists. As the group is not abelian, this is the smallest possible. Later, Korkmaz [9] showed that one of these generators can be taken as a Dehn twist, he also proved that $\text{Mod}(S)$ can be generated by two torsion elements. The third author showed that $\text{Mod}(S)$ is generated by two torsion elements of small orders [17].

Let g denote the genus of S . Generating $\text{Mod}(S)$ by involutions was first considered by McCarthy and Papadopoulos [13]. They showed that for $g \geq 3$, $\text{Mod}(S)$ can be generated by infinitely many conjugates of a single involution. In terms of generating by finitely many involutions, Luo [11] showed that any Dehn twist about a non-separating simple closed curve can be written as a product six involutions, which in turn implies that $\text{Mod}(S)$ can be generated by $12g + 6$ involutions. Brendle and Farb [2] obtained a generating set of six involutions for $g \geq 3$. Following their work, Kassabov [7] showed that $\text{Mod}(S)$ can be generated by four involutions if $g \geq 7$. Korkmaz [8] showed that $\text{Mod}(S)$ is generated by three involutions if $g \geq 8$ and four involutions if $g \geq 3$. Also, the third author improved his result showing that it is generated by three involutions if $g \geq 6$ [18].

Infinite-type surfaces and their mapping class groups, also called big mapping class groups, have generated great interest in the last several years. These big mapping class groups can be seen as limit objects of the mapping class groups of finite type surfaces. While work has been done by many authors to show that mapping class groups of finite-type surfaces are generated by torsion elements, not much has been done for the infinite-type case. The goal of this note is to investigate involution generators for big mapping class groups.

It is now well-known that the homeomorphism type of a finite-type surface is determined by the triple (g, p, b) , where $g \geq 0$ is the genus, and $p \geq 0$ is the number of punctures and $b \geq 0$ is the number of boundary components of the surface. To give a similar classification result for infinite type surfaces we should first define the space of ends of a surface.

An *end* of a surface S is the equivalence class of a nested sequence of connected subsurfaces $U_1 \supset U_2 \supset \dots$ of S with compact boundary and with the property that for any compact subsurface $K \subset S$, $K \cap U_r = \emptyset$ for high enough r . Two such sequences $U_1 \supset U_2 \supset \dots$ and $V_1 \supset V_2 \supset \dots$ are equivalent if for every $r \in \mathbb{N}$ there exists $s \in \mathbb{N}$

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such that $V_s \subset U_r$ and vice versa. An end given by a sequence $U_1 \supset U_2 \supset \dots$ is said to be *accumulated by genus (nonplanar)* if every U_r has positive genus. Otherwise, it is said to be a *planar end*.

The *space of ends* of S , denoted by $\text{Ends}(S)$, is a topological space whose points are the ends of S and whose basic open sets correspond to subsurfaces $U \subset S$ with compact boundary. An end $U_1 \supset U_2 \supset \dots$ of S is contained in the basic open set corresponding to U if $U_n \subset U$ for high enough n . By construction, $\text{Ends}(S)$ is compact, separable, and totally disconnected—in other words, it is homeomorphic to a closed subspace of a Cantor space. The set of ends accumulated by genus is denoted by $\text{Ends}_\infty(S)$ and is always a closed subspace of $\text{Ends}(S)$. By work of Richards [15], an orientable, boundaryless, infinite-type surface, S , is completely classified by its (possibly infinite) genus, its space of ends, $\text{Ends}(S)$, and the closed subset of ends which are accumulated by genus, $\text{Ends}_\infty(S)$.

Throughout the paper, we consider surfaces with infinite genus and n ends, $n \in \mathbb{N}$, and all ends are accumulated by genus. Let us denote such a surface by $S(n)$. The *pure mapping class group*, denoted by $\text{PMod}(S(n))$, is the subgroup of $\text{Mod}(S(n))$ fixing $\text{Ends}(S)$ pointwise. For the involution generators of big mapping class groups of some other infinite type surfaces we refer the reader to [6, Theorem 1.2, Theorem 1.3] where the author obtained generating sets with the minimum possible number of generators.

In the case of surfaces of infinite type, $\text{Mod}(S(n))$ is not countably generated. On the other hand, being a quotient of the group of orientation self-homeomorphism of $S(n)$ (that is equipped with the compact open topology), $\text{Mod}(S(n))$ inherits a topology. This makes $\text{Mod}(S(n))$ a Polish group [1, Proposition 4.1], in particular $\text{Mod}(S(n))$ is separable. Hence, $\text{Mod}(S(n))$ is topologically generated by a countable set i.e., there is a countable set that generates a dense subgroup (see [1] for more details).

The pure mapping class group $\text{PMod}(S(n))$ is a normal subgroup of $\text{Mod}(S(n))$ of index $n!$. For $n \geq 2$, we have the following exact sequence:

$$1 \longrightarrow \text{PMod}(S(n)) \longrightarrow \text{Mod}(S(n)) \longrightarrow \text{Sym}_n \longrightarrow 1,$$

where Sym_n is the symmetric group on n letters and the last projection is given by the restriction of the isotopy class of a diffeomorphism to its action on ends.

Results on mapping class groups of finite-type surfaces do not immediately extend for infinite-type surfaces. However, when it comes to generating the mapping class group, Patel and Vlamis [14, Theorem 4] showed that the pure mapping class group of a surface is topologically generated by Dehn twists if the surface has at most one end accumulated by genus, and by Dehn twists and maps called *handle shifts* otherwise.

Now, let us immediately define handle shifts. Consider the surface S obtained by taking $\mathbb{R} \times [0, 1]$, removing the interior of each disk of radius $\frac{1}{4}$ for each $n \in \mathbb{Z}$, and gluing a torus with one boundary component to the boundary of each disk. Define a homeomorphism σ that acts like $(x, y) \mapsto (x + 1, y)$ on the interior of S and extends as the identity homeomorphism in a neighbourhood of ∂S . Roughly speaking, the homeomorphism σ slides the n^{th} -handle (in the disk centered at $(n, \frac{1}{2})$) horizontally until it comes to the position of $(n + 1)^{\text{th}}$ -handle (in the disk centered at $(n + 1, \frac{1}{2})$) (see Figure 1).

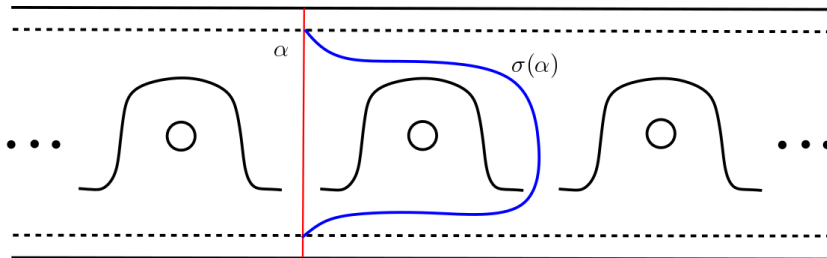


FIGURE 1. The homeomorphism σ on the surface S .

Before we state the main theorem of the paper, we want to note here that by the results of Malestein and Tao [12, Theorem A] for a uniformly self-similar surface (surface that has a self-similar ends space with infinitely

many maximal ends and zero or infinite genus) $\text{Mod}(S)$ is generated by involutions, normally generated by a single involution. Also in [4], the authors prove that the closure of the compactly supported mapping class group of an infinite-type surface is not generated by the collection of multitwists.

In this paper, we obtain the following main result:

Main Theorem. *For $n \geq 6$, $\text{Mod}(S(n))$ is topologically generated by five involutions and for $n \geq 3$, it is topologically generated by six involutions.*

Before we finish this section, let us fix our notation. Throughout the paper we do not distinguish a diffeomorphism from its isotopy class. For the composition of two diffeomorphisms, we use the functional notation; if f and g are two diffeomorphisms, then the composition fg means that g is applied first and then f .

For a simple closed curve a on $\text{Mod}(S(n))$, we denote the right-handed Dehn twist t_a about a by the corresponding capital letter A . We denote inverse of any mapping class X by \overline{X} .

Finally, let us recall the following basic facts of Dehn twists that we use frequently in the rest of the paper. Let a and b be two simple closed curves on $S(n)$ and $f \in \text{Mod}(S(n))$.

- If a and b are disjoint, then $AB = BA$ (*Commutativity*).
- If $f(a) = b$, then $fA\overline{f} = B$. (*Conjugation*).

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2. PROOF OF THE MAIN THEOREM

Let us start with reminding the following basic fact from group theory.

Lemma 2.1. *Let G and K be groups. Suppose that the following short exact sequence holds,*

$$1 \longrightarrow N \xrightarrow{i} G \xrightarrow{\pi} K \longrightarrow 1.$$

Then a subgroup Γ of G satisfies $i(N) \subseteq \Gamma$ and $\pi(\Gamma) = K$ if and only if $\Gamma = G$.

In our case where $G = \text{Mod}(S(n))$ and $N = \text{PMod}(S(n))$, the following short exact sequence holds for $n \geq 2$:

$$1 \longrightarrow \text{PMod}(S(n)) \xrightarrow{i} \text{Mod}(S(n)) \xrightarrow{\pi} \text{Sym}_n \longrightarrow 1.$$

Hence, we get the following useful result which follows immediately from Lemma 2.1: If Γ is a subgroup of $\text{Mod}(S(n))$ with $\text{PMod}(S(n)) \subseteq \Gamma$ and $\pi(\Gamma) = \text{Sym}_n$, then $\Gamma = \text{Mod}(S(n))$.

Now, consider the model for $S(n)$ depicted in Figure 2. If $n \geq 2$, there is a handle shift $h_{i,i+1}$, whose action can be described as

$$\begin{aligned} h_{i,i+1}(b_1^i) &= b_1^{i+1}, & h_{i,i+1}(a_1^i) &= (a_1^{i+1})', & h_{i,i+1}(c_0^i) &= c_1^{i+1}, \\ h_{i,i+1}(b_{j \neq 1}^i) &= b_{j-1}^i, & h_{i,i+1}(a_{j \neq 1}^i) &= a_{j-1}^i, & h_{i,i+1}(c_{j \neq 0}^i) &= c_{j-1}^i, \\ h_{i,i+1}(b_j^{i+1}) &= b_{j+1}^{i+1}, & h_{i,i+1}(a_j^{i+1}) &= a_{j+1}^{i+1}, & h_{i,i+1}(c_j^{i+1}) &= c_{j+1}^{i+1}. \end{aligned}$$

Note that the surface $S(n)$ is invariant under the rotations ρ_1 and ρ_2 which are the π rotations about the indicated lines shown in Figure 2. Moreover, the homeomorphism $R = \rho_1\rho_2$ is the rotation by $\frac{2\pi}{n}$, which satisfies

$$R(\alpha^i) = \alpha^{i+1}, \text{ where } \alpha \in \{a_k, b_k, c_{k-1}\} \text{ for } k = 1, 2, \dots$$

Lemma 2.2. *For $n \geq 3$, the group topologically generated by the elements*

$$\{\rho_1, \rho_2, A_1^1\overline{A_1^2}, B_1^1\overline{B_1^2}, C_0^1\overline{C_0^2}, h_{1,2}\}$$

contains the Dehn twists $A_1^2\overline{A_2^2}$, $B_1^2\overline{B_2^2}$ and $C_1^2\overline{C_2^2}$.

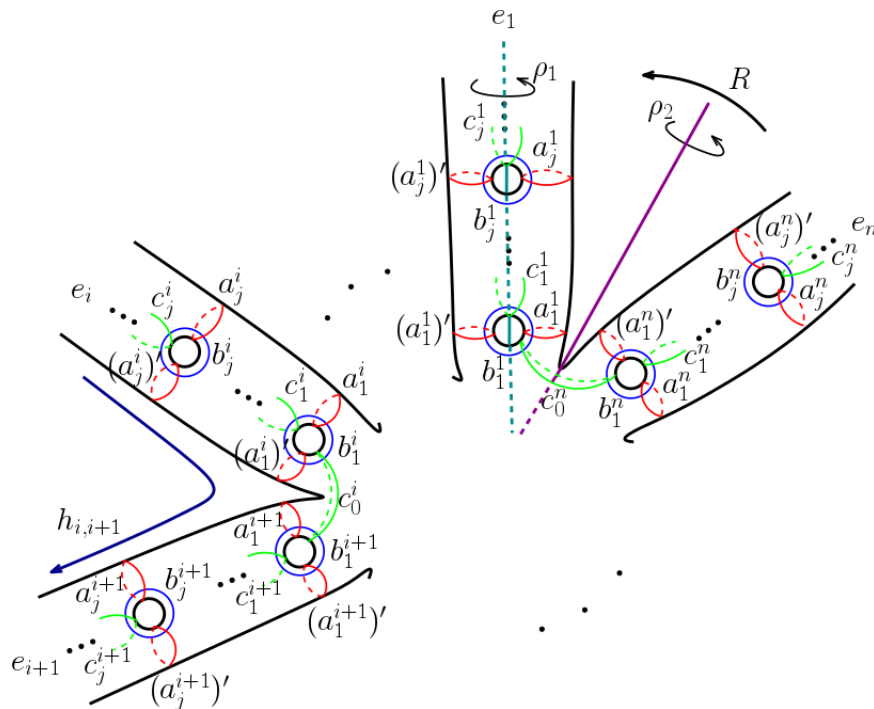


FIGURE 2. The curves a_j^i, b_j^i, c_j^i , and c_0^i , the rotations ρ_1, ρ_2, R and the handle shift $h_{i,i+1}$ for $i = 1, 2, \dots, n$ on $S(n)$.

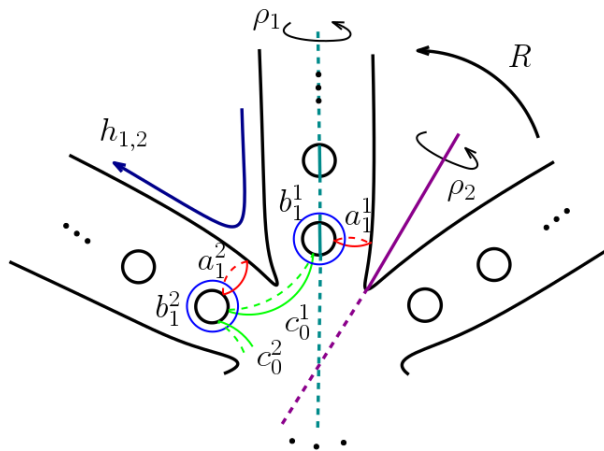


FIGURE 3. The curves $a_1^1, a_1^2, b_1^1, b_1^2, c_0^1$ and c_0^2 on $S(n)$.

Proof. Let G denote the subgroup topologically generated by the elements

$$\{\rho_1, \rho_2, A_1^1 \overline{A_1^2}, B_1^1 \overline{B_1^2}, C_0^1 \overline{C_0^2}, h_{1,2}\}.$$

Since the rotation ρ_1 sends the curves (a_1^1, a_1^2) to the curves $((a_1^1)', (a_1^n)')$, we have

$$(A_1^1)' \overline{(A_1^n)'} = (A_1^1 \overline{A_1^2})^{\rho_1} \in G.$$

Then it follows from $h_{1,2}((a_1^1)', (a_1^n)') = (a_1^2, (a_1^n)')$ that we have

$$A_1^2 \overline{(A_1^n)'} = ((A_1^1)' \overline{(A_1^n)'})^{h_{1,2}} \in G.$$

Moreover, since $h_{1,2}(a_1^2, (a_1^n)') = (a_2^2, (a_1^n)')$, we obtain

$$A_2^2 \overline{(A_1^n)'} = (A_1^2 \overline{(A_1^n)'})^{h_{1,2}} \in G.$$

Hence, we have

$$A_1^2 \overline{A_2^2} = (A_1^2 \overline{(A_1^n)'})(A_1^1 \overline{A_2^2}) \in G.$$

It follows from $h_{1,2}(b_1^1, b_1^2) = (b_1^2, b_2^2)$ that we get

$$B_1^2 \overline{B_2^2} = (B_1^1 \overline{B_1^2})^{h_{1,2}} \in G.$$

Note that the rotation $R = \rho_1 \rho_2 \in G$. Since $R(c_0^1, c_0^2) = (c_0^2, c_0^3)$, the following element

$$C_0^2 \overline{C_0^3} = (C_0^1 \overline{C_0^2})^R \in G$$

From this, we have

$$C_0^1 \overline{C_0^3} = (C_0^1 \overline{C_0^2})(C_0^2 \overline{C_0^3}) \in G.$$

Moreover, since

$$\begin{aligned} h_{1,2}(c_0^1, c_0^3) &= (c_1^2, c_0^3) \text{ if } n \neq 3 \\ (h_{1,2}(c_0^1, c_0^3) &= (c_1^2, c_0^2) \text{ if } n = 3), \end{aligned}$$

we obtain that the element

$$\begin{aligned} C_1^2 \overline{C_0^3} &= (C_0^1 \overline{C_0^3})^{h_{1,2}} \in G \text{ if } n \neq 3 \\ (C_1^2 \overline{C_0^3} &= (C_0^1 \overline{C_0^3})^{h_{1,2}}(C_0^2 \overline{C_0^3}) \in G \text{ if } n = 3) \end{aligned}$$

Then the element

$$C_0^1 \overline{C_1^2} = (C_0^1 \overline{C_0^3})(C_0^3 \overline{C_1^2}) \in G.$$

Finally, since $h_{1,2}(c_0^1, c_1^2) = (c_1^2, c_2^2)$, we have

$$C_1^2 \overline{C_2^2} = (C_0^1 \overline{C_1^2})^{h_{1,2}} \in G.$$

This completes the proof. □

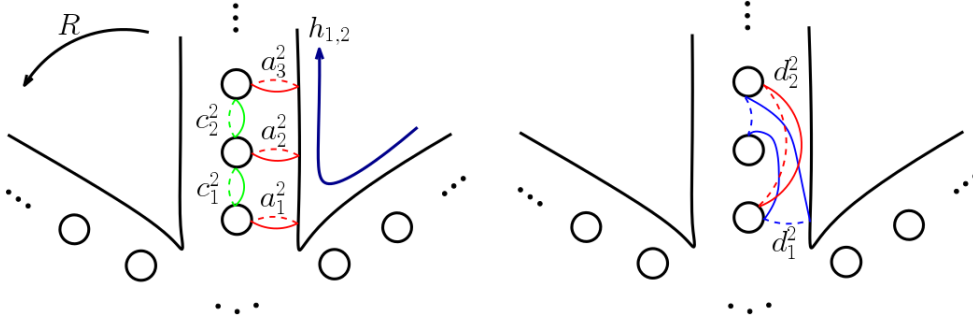


FIGURE 4. The curves of the embedded lantern relation $A_1^2 C_1^2 C_2^2 A_3^2 = A_2^2 D_1^2 D_2^2$ on $S(n)$.

Lemma 2.3. *For $n \geq 3$, the group topologically generated by the elements*

$$\{\rho_1, \rho_2, A_1^2 \overline{A_2^2}, B_1^2 \overline{B_2^2}, C_1^2 \overline{C_2^2}, h_{1,2}\}$$

contains the Dehn twists A_i^j, B_i^j, C_{i-1}^j for all $j = 1, 2, \dots, n$ and for all $i = 1, 2, 3, \dots$

Proof. Let G be the subgroup generated by the elements

$$\{\rho_1, \rho_2, \rho_1, \rho_2, A_1^2 \overline{A_2^2}, B_1^2 \overline{B_2^2}, C_1^2 \overline{C_2^2}, h_{1,2}\}.$$

Let us now denote the curves a_i^2, b_i^2 and c_i^2 by a_i, b_i and c_i , and also corresponding Dehn twists by A_i, B_i and C_i , respectively, so that the subgroup G contains the elements $A_1 \overline{A_2}, B_1 \overline{B_2}$ and $C_1 \overline{C_2}$.

Since $h_{1,2}(a_1, a_2) = (a_2, a_3)$, the element $A_2 \overline{A_3}$ is in the subgroup G . This also implies that

$$A_1 \overline{A_3} = (A_1 \overline{A_2})(A_2 \overline{A_3}) \in G.$$

It can be verified that

$$(A_1 \overline{A_2})(B_1 \overline{B_2})(a_1, a_3) = (b_1, a_3)$$

so we have $B_1 \overline{A_3} \in G$ and since

$$(B_1 \overline{B_2})(C_1 \overline{C_2})(b_1, a_3) = (c_1, a_3)$$

we also have $C_1 \overline{A_3} \in G$. Moreover, we have the following elements

$$\begin{aligned} B_2 \overline{A_1} &= (B_2 \overline{B_1})(B_1 \overline{A_3})(A_3 \overline{A_2})(A_2 \overline{A_1}), \\ C_1 \overline{A_1} &= (C_1 \overline{A_3})(A_3 \overline{A_2})(A_2 \overline{A_1}), \\ C_2 \overline{A_1} &= (C_2 \overline{C_1})(C_1 \overline{A_1}) \text{ and} \\ A_2 \overline{C_2} &= (A_2 \overline{A_1})(A_1 \overline{C_2}), \end{aligned}$$

which are all contained in the subgroup G . It can be checked that

$$(B_2 \overline{A_1})(C_1 \overline{A_1})(A_1 \overline{A_2})(C_2 \overline{A_1})(b_2, a_1) = (d_1, a_1),$$

where the curve d_1 is as in Figure 4. Since each factor on the left hand side and $B_2 \overline{A_1}$ are contained in G , we get $D_1 \overline{A_1} \in G$. Since $h_{1,2}(b_1, b_2) = (b_2, b_3)$, the element $B_2 \overline{B_3} \in G$. Hence, we have the element

$$B_3 \overline{A_1} = (B_3 \overline{B_2})(B_2 \overline{A_1}) \in G.$$

It can also be verified that

$$(B_3 \overline{A_1})(C_2 \overline{A_1})(A_3 \overline{A_1})(B_3 \overline{A_1})(d_1, a_1) = (d_2, a_1),$$

where the curve d_1 is as in Figure 4. Again, since G contains each factors and $D_1 \overline{A_1}$, it also contains $D_2 \overline{A_1}$. This implies that

$$D_2 \overline{C_1} = (D_2 \overline{A_1})(A_1 \overline{C_1}) \in G.$$

Now, using the following lantern relation (see Figure 4),

$$A_1 C_1 C_2 A_3 = A_2 D_1 D_2$$

we obtain

$$A_3 = (A_2 \overline{C_2})(D_1 \overline{A_1})(D_2 \overline{C_1}) \in G,$$

since each factor is contained in G . Thus using the actions of the elements $R = \rho_1 \rho_2$ and $h_{1,2}$, we get all Dehn twists $A_i^j \in G$ for all $j = 1, 2, \dots, n$. Moreover, the subgroup G contains

$$\begin{aligned} C_1 &= (C_1 \overline{A_1}) A_1 \in G \text{ and} \\ B_1 &= (B_1 \overline{A_3}) A_3 \in G. \end{aligned}$$

By conjugating these elements with $R = \rho_1 \rho_2$ and $h_{1,2}$, we conclude that B_i^j and C_i^j are all contained in the subgroup G for all $j = 1, 2, \dots, n$. Also it follows from $h_{1,2}(c_1^2) = C_0^1$ that we get

$$C_0^1 = (C_1^2)^{h_{1,2}} \in G,$$

which implies C_0^j is contained in G for each $j = 1, 2, \dots, n$ by the action of R . This finishes the proof. \square

Using lemmata 2.2 and 2.3, we immediately conclude the following theorem.

Theorem 2.4. *For $n \geq 3$, the group topologically generated by the elements*

$$\{\rho_1, \rho_2, A_1^1 \overline{A_1^2}, B_1^1 \overline{B_1^2}, C_0^1 \overline{C_0^2}, h_{1,2}\}$$

contains the Dehn twists A_i^j, B_i^j, C_{i-1}^j for all $j = 1, 2, \dots, n$ and for all $i = 1, 2, 3, \dots$

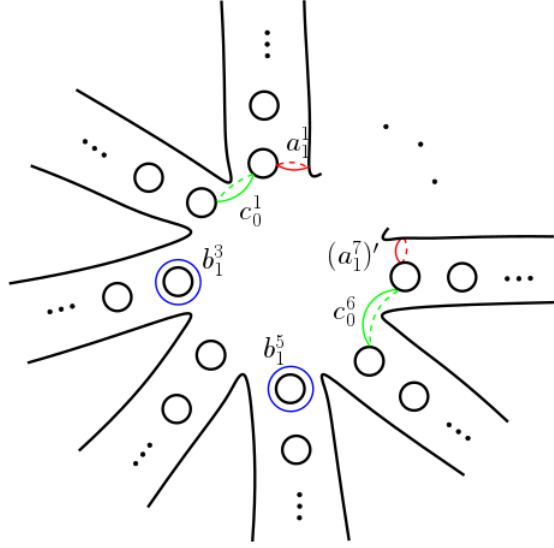


FIGURE 5. The curves $a_1^1, c_0^1, b_1^3, b_1^5, c_0^6, a_1^7$ on $S(n)$ for $n \geq 7$.

Lemma 2.5. *For $n \geq 7$, the group topologically generated by the elements*

$$\{\rho_1, \rho_2, A_1^1 C_0^1 B_1^3 \overline{B_1^5 C_0^6 A_1^7}, h_{1,2}\}$$

contains the Dehn twists A_i^j, B_i^j, C_i^j and C_0^j for all $j = 1, 2, \dots, n$.

Proof. Let F_1 denote the element $A_1^1 C_0^1 B_1^3 \overline{B_1^5 C_0^6 A_1^7}$ and H be the subgroup topologically generated by the elements $\{\rho_1, \rho_2, F_1, h_{1,2}\}$. Note that the rotation $R = \rho_1 \rho_2$ belongs to the subgroup H . By Theorem 2.4, it suffices to prove that the elements $A_1^1 \overline{A_1^2}, B_1^1 \overline{B_1^2}$ and $C_0^1 \overline{C_0^2}$ are contained in H .

For simplicity, let us denote the Dehn twists $A_1^i, A_1^{i'}, B_1^i, C_0^i$ by A_i, A_i', B_i, C_i , respectively. Let F_2 be the element obtained by conjugation of F_1 with R . Hence

$$\begin{aligned} F_2 &= F_1^R = A_2 C_2 B_4 \overline{B_6 C_7 A_8'} \in H \text{ if } n > 7 \\ (F_2 &= F_1^R = A_2 C_2 B_4 \overline{B_6 C_7 A_1'} \in H \text{ if } n = 7). \end{aligned}$$

It follows from

$$\begin{aligned} F_2 F_1(a_2, c_2, b_4, b_6, c_7, a_8') &= (a_2, b_3, b_4, c_6, c_7, a_8') \text{ if } n > 7 \\ (F_2 F_1(a_2, c_2, b_4, b_6, c_7, a_1') &= (a_2, b_3, b_4, c_6, c_7, a_1') \text{ if } n = 7) \end{aligned}$$

that

$$\begin{aligned} F_3 &= F_2^{F_2 F_1} = A_2 B_3 B_4 \overline{C_6 C_7 A'_8} \in H \text{ if } n > 7 \\ (F_3 &= F_2^{F_2 F_1} = A_2 B_3 B_4 \overline{C_6 C_7 A'_1} \in H \text{ if } n = 7). \end{aligned}$$

The subgroup H contains the element

$$F_4 = F_3^{\overline{R}} = A_1 B_2 B_3 \overline{C_5 C_6 A'_7}.$$

Since $F_4 F_3(a_1, b_2, b_3, c_5, c_6, a'_7) = (a_1, a_2, b_3, c_5, c_6, a'_7)$, we get the following element:

$$F_5 = F_4^{F_4 F_3} = A_1 A_2 B_3 \overline{C_5 C_6 A'_7} \in H.$$

Thus, the element $F_5 \overline{F_4} = A_2 \overline{B_2} \in H$, which implies that $A_1 \overline{B_1} = (A_2 \overline{B_2})^{\overline{R}} \in H$ and $A_3 \overline{B_3} = (A_2 \overline{B_2})^R \in H$. The elements

$$\begin{aligned} F_6 &= (A_3 \overline{B_3}) F_1 = (A_3 \overline{B_3})(A_1 C_1 B_3 \overline{B_5 C_6 A'_7}) = (A_1 C_1 A_3 \overline{B_5 C_6 A'_7}) \in H \text{ and} \\ F_7 &= F_6^R = A_2 C_2 A_4 \overline{B_6 C_7 A'_8} \in H \text{ if } n > 7 \\ (F_7 &= F_6^R = A_2 C_2 A_4 \overline{B_6 C_7 A'_1} \in H \text{ if } n = 7). \end{aligned}$$

Since

$$\begin{aligned} F_7 F_6(a_2, c_2, a_4, b_6, c_7, a'_8) &= (a_2, c_2, a_4, c_6, c_7, a'_8) \text{ if } n > 7 \\ (F_7 F_6(a_2, c_2, a_4, b_6, c_7, a'_1) &= (a_2, c_2, a_4, c_6, c_7, a'_1) \text{ if } n = 7), \end{aligned}$$

we obtain the element

$$\begin{aligned} F_8 &= F_7^{F_7 F_6} = A_2 C_2 A_4 \overline{C_6 C_7 A'_8} \in H \text{ if } n > 7 \\ (F_8 &= F_7^{F_7 F_6} = A_2 C_2 A_4 \overline{C_6 C_7 A'_1} \in H \text{ if } n = 7). \end{aligned}$$

Then we have $\overline{F_7} F_8 = B_6 \overline{C_6} \in H$, which leads to $B_i \overline{C_i} \in H$ for $i = 1, 2, \dots, n$ by the action of R . Moreover, it follows from $\rho_1(b_1, c_1) = (b_1, c_n)$ and $R(b_1, c_1) = (b_2, c_1)$ that we get $B_1 \overline{C_n} \in H$ and $B_2 \overline{C_1} \in H$. We conclude that the subgroup H contains the following elements:

$$\begin{aligned} C_1 \overline{C_2} &= (C_1 \overline{B_2})(B_2 \overline{C_2}), \\ B_1 \overline{B_2} &= (B_1 \overline{C_1})(C_1 \overline{B_2}) \text{ and} \\ A_1 \overline{A_2} &= (A_1 \overline{B_1})(B_1 \overline{B_2})(B_2 \overline{A_2}), \end{aligned}$$

which completes the proof. \square

Lemma 2.6. *For $n = 6$, the group topologically generated by the elements*

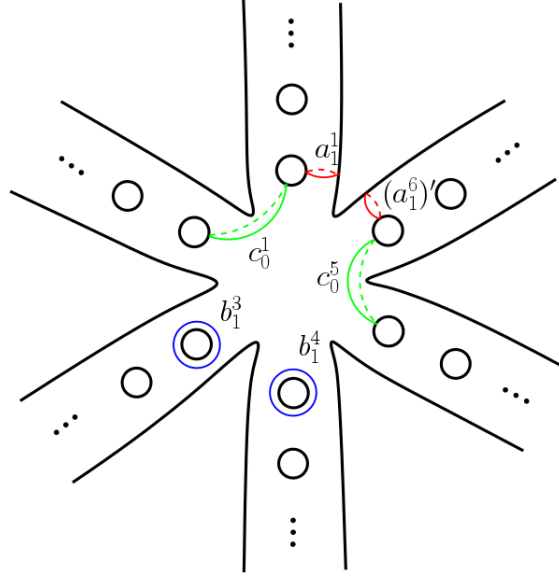
$$\{\rho_1, \rho_2, A_1^1 C_0^1 B_1^3 B_1^4 C_0^5 A_1^{6'}\},$$

contains the Dehn twists A_i^j, B_i^j, C_i^j and C_0^j for all $j = 1, 2, \dots, n$.

Proof. Let K_1 denote the element $A_1^1 C_0^1 B_1^3 B_1^4 C_0^5 A_1^{6'}$. Let K denote the subgroup topologically generated by the elements $\{\rho_1, \rho_2, K_1, h_{1,2}\}$. Note that the rotation $R = \rho_1 \rho_2 \in K$. By Theorem 2.4, it is enough to show that the subgroup K contains the elements $A_1^1 \overline{A_1^2}, B_1^1 \overline{B_1^2}$ and $C_0^1 \overline{C_0^2}$.

Again for simplicity, let us denote the Dehn twists $A_1^i, A_1^{i'}, B_1^i, C_0^i$ by A_i, A_i', B_i, C_i , respectively. The subgroup K contains the element

$$K_2 = K_1^R = A_2 C_2 B_4 \overline{B_5 C_6 A'_1}.$$


 FIGURE 6. The curves $a_1^1, c_0^1, b_1^3, b_1^4, c_0^5, a_1^6'$ on $S(6)$.

It can be verified that

$$\begin{aligned} K_2 K_1(a_2, c_2, b_4, b_5, c_6, a_1') &= (a_2, b_3, b_4, c_5, c_6, a_1') \text{ and} \\ K_1 K_2(a_1, c_1, b_3, b_4, c_5, a_6') &= (a_1, c_1, c_2, b_4, b_5, a_6'). \end{aligned}$$

We then have the following elements:

$$\begin{aligned} K_3 &= K_2^{K_2 K_1} = A_2 B_3 B_4 \overline{C_5 C_6 A_1'} \in K \text{ and} \\ K_4 &= K_1^{K_1 K_2} = A_1 C_1 C_2 \overline{B_4 B_5 A_6'} \in K. \end{aligned}$$

The following elements are also contained in K :

$$\begin{aligned} K_5 &= K_3^{R^2} = A_4 B_5 B_6 \overline{C_1 C_2 A_3'} \\ K_6 &= \overline{K_5} = A_3' C_2 C_1 \overline{B_6 B_5 A_4}. \end{aligned}$$

It is easy to see that

$$K_6 K_4(a_3', c_2, c_1, b_6, b_5, a_4) = (a_3', c_2, c_1, a_6', b_5, b_4),$$

which implies that

$$K_7 = K_6^{K_6 K_4} = A_3' C_2 C_1 \overline{A_6' B_5 B_4} \in K.$$

From this, we get

$$\begin{aligned} K_8 &= K_4 \overline{K_7} = A_1 \overline{A_3'} \in K \text{ and} \\ K_9 &= \overline{K_8} = A_3' \overline{A_1} \in K \end{aligned}$$

Since $K_9 K_3(a_3', a_1) = (b_3, a_1)$, we obtain the element

$$K_{10} = B_3 \overline{A_1} \in K.$$

Moreover, we have the following elements:

$$\begin{aligned} K_{11} &= K_{10}^{R^2} = B_5 \overline{A_3} \in K \text{ and} \\ K_{12} &= \overline{K_{11}} = A_3 \overline{B_5} \in K. \end{aligned}$$

It can be checked that

$$K_{12} K_{10}(a_3, b_5) = (b_3, b_5),$$

which implies that

$$K_{13} = B_3 \overline{B_5} \in K.$$

Similarly, it follows from

$$K_{13} K_3(b_3, b_5) = (b_3, c_5)$$

that we get

$$K_{14} = B_3 \overline{C_5} \in K.$$

The subgroup K also contains the element

$$K_{15} = K_{14}^{\rho_1} = B_5 \overline{C_2}$$

since $\rho_1(b_3, c_5) = (b_5, c_2)$. The following elements are also contained in K :

$$\begin{aligned} B_1 \overline{C_4} &= K_{15}^{R^2} = (B_5 \overline{C_2})^{R^2} \text{ and} \\ C_4 \overline{B_2} &= \overline{K_{14}^R} = (C_5 \overline{B_3})^{\overline{R}}. \end{aligned}$$

This implies that

$$B_1 \overline{B_2} = (B_1 \overline{C_4})(C_4 \overline{B_2}) \in K.$$

Also, the subgroup K contains the element

$$A_1 \overline{A_2} = \overline{K_{10}}(B_1 \overline{B_2})^{R^2} K_{10}^R = (A_1 \overline{B_3})(B_3 \overline{B_4})(B_4 \overline{A_2}).$$

It remains to prove that $C_1 \overline{C_2} \in K$. Consider the following elements:

$$\begin{aligned} B_2 \overline{B_3} &= (B_1 \overline{B_2})^R, \\ B_1 \overline{B_3} &= (B_1 \overline{B_2})(B_2 \overline{B_3}) \text{ and} \\ C_4 \overline{C_5} &= \overline{(B_1 \overline{C_4})(B_1 \overline{B_3})} K_{14} = (C_4 \overline{B_1})(B_1 \overline{B_3})(B_3 \overline{C_5}), \end{aligned}$$

which are all contained in K . This finishes the proof since we get

$$C_1 \overline{C_2} = (C_4 \overline{C_5})^{R^3} \in K.$$

□

Lemma 2.7. *For $n \geq 3$, the group topologically generated by the elements*

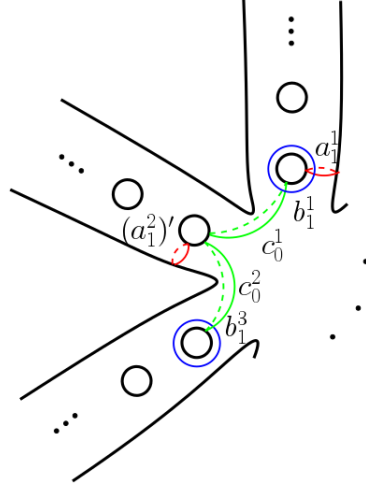
$$\{\rho_1, \rho_2, B_1^1 C_0^1 \overline{C_0^2 B_1^3}, A_1^1 \overline{A_1^{2'}}, h_{1,2}\}$$

contains the Dehn twists A_i^j, B_i^j, C_i^j and C_0^j for all $j = 1, 2, \dots, n$.

Proof. Let L_1 and L_2 denote the elements $B_1^1 C_0^1 \overline{C_0^2 B_1^3}$ and $A_1^1 \overline{A_1^{2'}}$, respectively. Let L denote the subgroup topologically generated by the elements $\{\rho_1, \rho_2, L_1, L_2, h_{1,2}\}$. It is clear that the rotation $R = \rho_1 \rho_2 \in L$. It follows from Theorem 2.4 that we need to show that the subgroup L contains the elements $A_1^1 \overline{A_1^2}, B_1^1 \overline{B_1^2}$ and $C_0^1 \overline{C_0^2}$.

Let us again denote the Dehn twists $A_1^i, A_1^{i'}, B_1^i, C_0^i$ by A_i, A_i', B_i, C_i , respectively. The subgroup L contains the elements

$$\begin{aligned} L_3 &= \overline{L_1} = B_3 C_2 \overline{C_1 B_1} \text{ and} \\ L_4 &= L_2^R = A_2 \overline{A_3'}. \end{aligned}$$

FIGURE 7. The curves $a_1^1, b_1^1, c_0^1, a_1^{2'}, c_0^2, b_1^3$ on $S(n)$ for $n \geq 3$

It can be shown that $L_2 L_1(a_1, a_2') = (b_1, a_2')$, which implies that

$$L_5 = L_2^{L_2 L_1} = B_1 \overline{A_2'} \in L.$$

Also we get the following elements

$$\begin{aligned} L_6 &= \overline{L_4} = A_3' \overline{A_2} \in L \text{ and} \\ L_7 &= L_5^R = B_2 \overline{A_3'} \in L. \end{aligned}$$

It is easy to see that $L_1(b_1, a_2') = (c_1, a_2')$. This implies that

$$L_8 = L_5^{L_1} = C_1 \overline{A_2'} \in L.$$

Also, the subgroup L contains

$$L_9 = L_8^R = C_2 \overline{A_3'}.$$

Since $L_6 L_3(a_3', a_2) = (b_3, a_2)$, we obtain

$$L_{10} = L_6^{L_6 L_3} = B_3 \overline{A_2} \in L.$$

Moreover, the subgroup L contains the following elements:

$$\begin{aligned} L_{11} &= L_{10}^R = B_2 \overline{A_1} \text{ and} \\ L_{12} &= L_{11} L_2 = (B_2 \overline{A_1})(A_1 \overline{A_2'}) = B_2 \overline{A_2'}. \end{aligned}$$

From these we have

$$B_1 \overline{B_2} = L_5 \overline{L_{12}} = (B_1 \overline{A_2'})(A_2' \overline{B_2}) \in L.$$

We also obtain

$$L_{13} = \overline{L_5}(B_1 \overline{B_2}) L_7 = (A_2' \overline{B_1})(B_1 \overline{B_2})(B_2 \overline{A_3'}) = A_2' \overline{A_3'} \in L.$$

Hence, we have

$$C_1 \overline{C_2} = L_8 L_{13} \overline{L_9} = (C_1 \overline{A_2'})(A_2' \overline{A_3'})(A_3' \overline{C_2}) \in L.$$

Finally, the element

$$A_1 \overline{A_2} = \overline{L_{11}}(B_1 \overline{B_2})^R L_{11}^R = (A_1 \overline{B_2})(B_2 \overline{B_3})(B_3 \overline{A_2}) \in L.$$

This completes the proof. \square

Involution generators. Now, we present our involution generators of $\text{Mod}(S(n))$ for $n \geq 3$. We first express the handle shift $h_{1,2}$ as a product of two involutions (here we use the involutions that were introduced in [6]).

Consider the models of $S(n)$ which are obtained by decomposing $S(n)$ so that $S(n) = A \cup D_1 \cup D_2 \cup \dots \cup D_{n-2}$, where A is an infinite surface with two ends accumulated by genus and $n - 2$ boundary components and each D_i is an infinite genus surface with one end accumulated by genus and one boundary component (see Figures 8 and 9). Note that the surface $S(n)$ is invariant under the two rotations τ_1 and τ_2 , where they are rotations by π about indicated lines shown in Figures 8 and 9, respectively. Observe that $h_{1,2} = \tau_1\tau_2$.

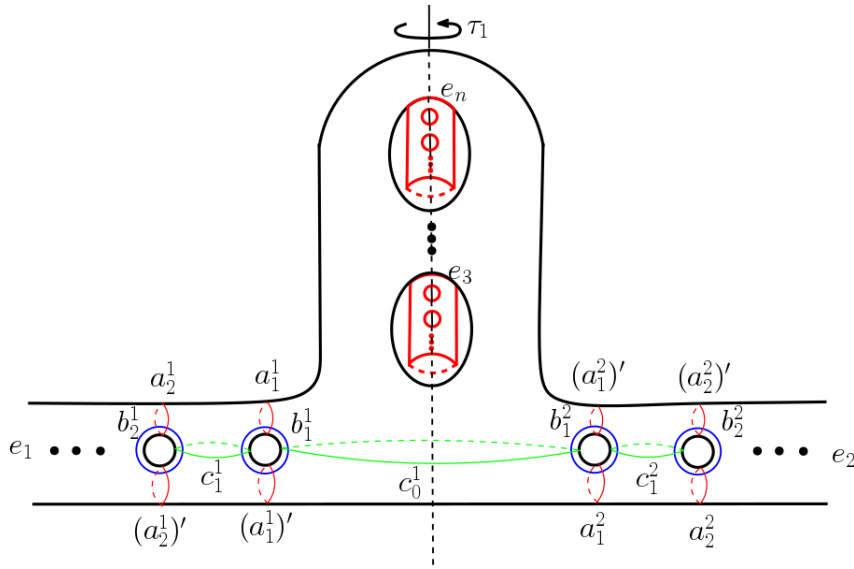


FIGURE 8. The involution τ_1 on $S(n)$.

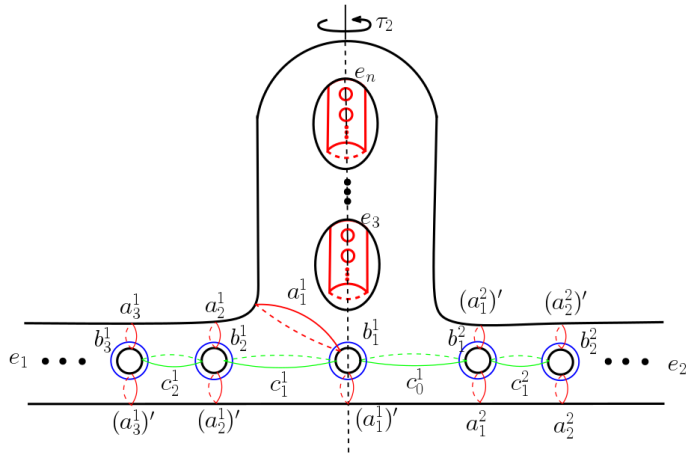


FIGURE 9. The involution τ_2 on $S(n)$.

Consider now the surface $S(n)$ for $n \geq 7$ (see Figure 5). Since $\rho_3(a_1^1) = (a_1^7)'$, $\rho_3(c_0^1) = c_0^6$ and $\rho_3(b_1^3) = b_1^5$, where $R = \rho_1\rho_2$ and $\rho_3 = R^3\rho_1R^{-3}$, we have

$$\rho_3 A_1^1 C_0^1 B_1^3 \overline{B_1^5 C_0^6 A_1^{7'}}$$

is an involution.

For the surface $S(6)$, it is easy to see that $\rho_2(a_1^1) = (a_1^6)'$, $\rho_2(c_0^1) = c_0^5$ and $\rho_2(b_1^3) = b_1^4$ (see Figure 6), which implies that

$$\rho_2 A_1^1 C_0^1 B_1^3 \overline{B_1^4 C_0^5 A_1^{6'}}$$

is also an involution.

Let $\rho_4 = R\rho_1R^{-1}$ and $\rho_5 = R\rho_2R^{-1}$. For $n \geq 3$, it can be observed that $\rho_4(b_1^3) = b_1^3$, $\rho_4(c_0^1) = c_0^2$, $\rho_5(a_1^1) = (a_2^2)'$ (see Figure 7). Hence, we get

$$\rho_4 B_1^1 C_0^1 \overline{C_0^2 B_1^3} \text{ and } \rho_5 A_1^1 \overline{A_1^{2'}}$$

are involutions.

The proof of the main theorem is now immediate: Let Γ be the subgroup of $\text{Mod}(S(n))$ generated by the following involutions:

$$\begin{cases} \rho_1, \rho_2, \rho_3 A_1^1 C_0^1 B_1^3 \overline{B_1^5 C_0^6 A_1^{7'}}, \tau_1, \tau_2 & \text{if } n \geq 7, \\ \rho_1, \rho_2, \rho_2 A_1^1 C_0^1 B_1^3 \overline{B_1^4 C_0^5 A_1^{6'}}, \tau_1, \tau_2 & \text{if } n = 6, \\ \rho_1, \rho_2, B_1^1 C_0^1 \overline{C_0^2 B_1^3}, A_1^1 \overline{A_1^{2'}}, \tau_1, \tau_2 & \text{if } n \geq 3. \end{cases}$$

Note that $h_{1,2} = \tau_1\tau_2$ and $R = \rho_1\rho_1$ belong to Γ . It follows from lemmata 2.5-2.7 that Γ contains the Dehn twists A_i^j , B_i^j , C_i^j and C_0^j for all $j = 1, 2, \dots, n$. Also, handle shifts are contained in Γ by the conjugation of R . Hence, $\text{PMod}(S(n))$ is contained in Γ [14, Theorem 4]. We finish the proof by showing that Γ is mapped surjectively onto Sym_n by Lemma 2.1. The images of R and τ_1 are the n -cycle $(1, 2, \dots, n)$ and the 2-cycle $(1, 2)$, respectively. These elements generate Sym_n , which completes the proof.

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