AN APPROACH TO DECOMPOSABILITY OF A CLASS OF ALMOST COMPLETELY DECOMPOSABLE GROUPS

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

JUNE 2024

Approval of the thesis:

AN APPROACH TO DECOMPOSABILITY OF A CLASS OF ALMOST COMPLETELY DECOMPOSABLE GROUPS

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ABSTRACT

AN APPROACH TO DECOMPOSABILITY OF A CLASS OF ALMOST COMPLETELY DECOMPOSABLE GROUPS

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June 2024, 52 pages

A torsion-free abelian group G is completely decomposable of finite rank if G is isomorphic to a finite direct sum of subgroups of $\mathbb{Q}$ and almost completely decomposable if G contains a completely decomposable subgroup R with G/R a finite group.The regulator $R(G)$ is intersection of all regulating subgroups of G and is a completely decomposable subgroup of finite index in G. The isomorphism types of the regulator $R(G)$ and the regulator quotient $G/R(G)$ are near-isomorphism invariants of an almost completely decomposable group G. In this thesis we consider a special case. Let p be a prime, $(1,2) = (t_1, t_2, t_3)$ be a set of types, partially ordered as t_1 is not comparable with t_2 and t_3 and $t_2 < t_3$. An almost completely decomposable G with critical typeset $(1,2)$ and a regulating index a p-power is called a p-local $(1,2)$-group. For p-local $(1,2)$-groups, the main question is to determine the near isomorphism classes of indecomposable $(1,2)$-groups.

Keywords: Almost completely decomposable groups, Torsion free groups, Decom-

posability of almost completely decomposable groups.

ÖZ

HEMEN HEMEN AYRIŞAN GRUPLARIN BİR SINIFININ AYRIŞMASINA BİR YAKLAŞIM

Kocabıyık, Makbule
Yüksek Lisans, Matematik Bölümü
Tez Yöneticisi: Doç. Dr. Ebru Solak

Haziran 2024 , 52 sayfa

Sonlu ranklı torsiyonsuz değişmeli bir grup G, $\mathbb{Q}$ grubunun altgruplarının direkt toplamına izormorf ise, o gruba tamamen ayrışan grup denir. Bu grup G tamamen ayrışan bir altküme R içeriyorsa öyleki G/R bir sonlu grup olsun o zaman G grubuna hemen hemen ayrışan grup denir. Regulatör $R(G)$, G grubunun regule altgruplarının kesişimidir ve G grubunun sonlu indeksli tamamen ayrışan bir altgrubudur. Regulatör $R(G)$ grubunun ve bölüm regulatörü $G/R(G)$ grubunun izomorfizma tipleri G grubunun yakın-izomorfizma değişmezleridir. Bu tezde biz özel bir durumu düşüneceğiz. p asal bir sayı olsun, $(1, 2) = (t_1, t_2, t_3)$ bir tipler kümesi olsun öyle ki t_1 , t_2 ve t_3'den bağımsız, $t_2 < t_3$ şeklinde düzenlensin. Kritik tip kümesi $(1, 2)$ olan ve regulator indeksi bir p-kuvveti olan hemen hemen ayrışan bir grup G'ye p-lokal $(1, 2)$-group denir. p-lokal$(1, 2)$ gruplar için asıl soru ayrışamayan $(1, 2)$ grupların yakın izomorfizma sınıflarını belirlemektir.

Anahtar Kelimeler: Hemen hemen ayrışan gruplar, Torsiyonsuz gruplar, Hemen he-

men ayrışan grupların parçalanması.

To my family

ACKNOWLEDGMENTS

I would like to thank my supervisor Assoc. Prof. Dr. Ebru Solak for the valuable help and the guidance. I also thank my mother Havvana Kocabıyık, my father Alim Kocabıyık and my friend Tunahan Erçoğul for their supports during my study. I also want to thank TÜBİTAK for their valuable supports.

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CHAPTER 1

INTRODUCTION

Torsion-free abelian groups are the additive subgroups of rational vector spaces and it is appealing to attack this groups computationally. However, this has some difficulties and hence may be applied only to some special classes of torsion-free abelian groups.

Completely decomposable groups are direct sums of rank one groups and almost completely decomposable groups are torsion-free abelian groups which contain a completely decomposable subgroup of finite index. Although the class of almost completely decomposable groups have been used for many examples and counterexamples, it is quite hard to develop a general theory.

The first basic concepts for almost completely decomposable groups, like regulating subgroups, nearly isomorphism are developed by E.Lee Lady. Burkhardt defined a new concept called regulator as the intersection of regulating subgroups and he showed that regulator is also completely decomposable, see [3]. An almost completely decomposable group G is an extension of its regulator R and a finite group G/R called the regulator quotient. Arnold and Faticoni and Schultz showed that the decomposition of p-local almost completely decomposable groups can be classified up to near isomorphism if the indecomposable groups are determined, see [5]. Therefore indecomposable groups play an important role in the decomposition of almost completely decomposable groups.

We can describe an almost completely decomposable group G by a representing matrix called coordinate matrix relative to the regulator R and the regulator quotient G/R. A p-local, p-reduced almost completely decomposable group is decomposable if and only if it has a decomposable coordinate matrix. An almost completely decomposable group is decomposable if and only if it is nearly isomorphic to a decomposable group. Hence we can establish a matrix equivalence between coordinate matrices of nearly isomorphic groups.

In this thesis we deal with the decomposability of almost completely groups of type $(1, 2)$, called $(1, 2)$-groups. Arnold and Dugas showed that local $(1, 2)$-groups with regulator quotient of exponent $\geq p^7$ has infinitely many isomorphism types of indecomposable groups, see [7] and [8]. The nearly isomorphism classes of indecomposable $(1, 2)$-groups with regulator quotient of exponent $\leq p^4$ and the nearly isomorphism classes of indecomposable $(1, 2)$-groups with regulator quotient of exponent p^6 are already determined, see [11]. But the decomposability problem of $(1, 2)$ -groups with regulator quotient of exponent p^5 is not resolved.

It is still an open question whether the class of $(1, 2)$-groups with regulator quotient of exponent p^5 has bounded representation type or not, i.e., there are up to near isomorphism finitely many indecomposable groups or not.

In this thesis we give a partial answer for this open question. We list the near isomorphism classes of indecomposable $(1, 2)$-groups G with regulator quotient of exponent p^5 and isomorphic to $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_p$, $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^2}$, $(\mathbb{Z}_{p^5})^2 \bigoplus \mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^3}$.

CHAPTER 2

PRELIMINARIES

2.1 Basic Definitions

A torsion-free group is a group whose elements except the identity have infinite order. In this thesis we only work with torsion-free abelian groups. This chapter presents definitions and basic properties of torsion-free abelian groups, like characteristics and types. The books of L. Fuchs and A. Mader are good sources for the definitions, see [1] and [6].

Let G be a torsion-free abelian group. Every torsion-free abelian group G is a subgroup of a $\mathbb{Q}$-vector space V such that a maximal independent set in G serves as a basis of V. Let $\{u_i \mid i \in I\}$ be a maximal independent set in G. Then every element $g \in G$ can be written uniquely as

$$g = r_1 u_1 + \cdots + r_n u_n$$

where $r_i \in \mathbb{Q}$.

Definition 1 *Let $p_1, p_2, \cdots, p_n$ denote a sequence of prime numbers in an increasing order. Let G be a torsion-free abelian group and let g be a an arbitrary element in G. For a prime p, the largest integer k with p^k divides g, i.e., the largest integer k for which the equation $p^k y = g$ is solvable in G, is called the p-**height** of g denoted $h_p(g)$. If there is no such maximal integer k, then $h_p(g) = \infty$. The sequence of p-heights,*

$$\chi_G(g) = (h_{p_1}(g), h_{p_2}(g), \cdots, h_{p_n}(g), \dots)$$

*is called the **characteristic** of g.*

Definition 2 *Let $k = (k_1, \dots, k_n, \dots)$ and $l = (l_1, \dots, l_n, \dots)$ be two characteristics of a torsion-free abelian group G. The characteristics k and l are called **equivalent** if $k_n = l_n$ for almost all n such that k_n and l_n are finite and they have exactly the same ∞-components. The equivalence classes of characteristics are said to be **types**. The books L.Fuchs and A.Mader are good sources for the definitions, see [1] and [6].*

If $\chi_G(g)$ belongs to the type t, then we write $t(g) = t$. It is clear that $t(g) = t(ng)$ for all natural numbers n. For the types t_1 and t_2, $t_1 \leq t_2$ means that there are characteristics $(k_1, \dots, k_n, \dots)$ in t_1 and $(l_1, \dots, l_n, \dots)$ in t_2 such that $(k_1, \dots, k_n, \dots) \leq (l_1, \dots, l_n, \dots)$.

Definition 3 *Let G be a torsion-free abelian group. G is called **homogenous** if all its nonzero elements have the same type t.*

Definition 4 *Let G a torsion-free abelian group and let H be a subgroup of G. The subgroup H is said to be* **pure** *if the equation $nx = h \in H$ for $n \in \mathbb{N}$ is solvable for x in H whenever it is solvable in G and is denoted $H \subset_* G$.*

Remark 1 *Let G be an abelian group and let H be a pure subgroup of G. Then by definition of the pure subgroup, the divisibility properties of the elements in H by integers are the same in G or in H.*

Definition 5 *Let G be a torsion-free abelian group and let t be a type in G. Define*

$$G(t) := \{g \in G \mid \mathrm{t}(g) \geq t\},$$

$$G^*(t) := \langle g \in G \mid t(g) > t \rangle$$

and

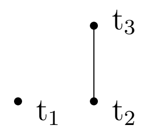
$$G^\#(t) := G^*(t)_*^G$$

The type t is a **critical type** of G if $\frac{G(t)}{G^\#(t)} \neq 0$.

Let T be a set of critical types of elements of a torsion-free abelian group G. Let $(T, \leq)$ be a poset. Two elements t_1 and t_2 of T are **comparable** if either $t_1 \leq t_2$ or

$t_2 \leq t_1$. Otherwise they are incomparable. A poset T is called V**-free** if T is disjoint union of inverted trees.

Posets can be represented by Hasse diagrams. For example, assume that T is a poset of the critical types of elements of a torsion-free abelian group G with three elements t_1, t_2 and t_3. If two of the critical types are comparable in such a way that t_1 is incomparable with t_2, t_3 and $t_2 < t_3$ then there are two minimal and two maximal elements. In this case the corresponding Hasse diagram of T is



CHAPTER 3

ALMOST COMPLETELY DECOMPOSABLE GROUPS

3.1 Basic Definitions

Let G be a torsion-free abelian group. The inclusion map $\mathbb{Z} \to \mathbb{Q}$ induces an embedding $G \simeq \mathbb{Z} \otimes G \to \mathbb{Q} \otimes G$. The group $\mathbb{Q} \otimes G$ is torsion-free abelian and also a $\mathbb{Q}$-vector space. Hence we can simply say that G is isomorphic to an additive subgroup of a $\mathbb{Q}$-vector space. The group G spans a subspace $\mathbb{Q}G$, called the **divisible hull** of G. The dimension of $\mathbb{Q}G$ is called the **rank** of G denoted $\operatorname{rank} G$. In this thesis we only consider groups of finite rank, i.e., those groups which have a finite dimensional divisible hull.

Definition 6 *The nonzero subgroups of $\mathbb{Q}$ are called **rational groups**.*

Rational groups are of rank 1. In a rational group G, all nonzero elements have the same type, denoted $t(G)$.

Definition 7 *Let G be a torsion-free abelian group. G is called **completely decomposable** if it is direct sum of rational groups. If G has only trivial direct summands then G is called **indecomposable.***

The following Lemma is the Lemma $2.4.8$ in [6].

Lemma 3.1.1 *Let G be a completely decomposable group of finite rank n. Then G can be written as*

$$G = R_1 v_1 \oplus \cdots R_n v_n$$

such that R_i's are rational groups with the property that $R_i \subset R_j$ if and only if $t(r_i v_i) \leq t(r_j v_j)$.

Proof 1 *Since G is completely decomposable,*

$$G = G_1 \oplus G_2 \oplus \cdots \oplus G_n$$

where G_i's are rank one groups. Take a nonzero $u_i \in G_i$. Then $G_i = A_i u_i$ where $A_i = \{r \in \mathbb{Q} : ru_i \in G_i\}$. The rest of the proof follows by induction on n, see the proof of Lemma 2.4.8 in [6].

Definition 8 *Let G be a torsion-free abelian group. G is called **almost completely decomposable** if G contains a completely decomposable subgroup of finite index.*

3.2 Isomorphism Invariants of almost completely decomposable groups

In this section we define two isomorphism invariants of an almost completely decomposable group which play an important role by the construction of the corresponding coordinate matrix. Let G be an almost completely decomposable group of finite index. By definition of almost completely decomposable groups, they have completely decomposable subgroups. The completely decomposable subgroups of minimal index are called the **regulating subgroups**. The intersection on all regulating subgroups is a completely decomposable subgroup of finite index, called the **regulator**. The regulator has a structure that influences the structure of the group.

If R is the regulator of an almost completely decomposable group then the quotient group G/R is called the **regulator quotient** of G. The isomorphism types of the regulator R and the regulator quotient G/R are isomorphism invariants of G. It is possible that an almost completely decomposable group has a unique regulating subgroup. In this case, we say that G has a **regulating regulator**.

Theorem 3.2.1 ***(Mutzbauer)*** *Let G be an almost completely decomposable group with critical typeset T which is V-free. Then G has a regulating regulator.*

Remark 2 *Let G be an almost completely decomposable group and let R be its regulator. Then the group G can be considered as an extension of R by its regulator quotient G/R, i.e., almost completely decomposable groups are torsion-free finite extensions of completely decomposable groups of finite rank.*

Definition 9 *Let G be an almost completely decomposable group and let p be a prime number. Then G is a called* ***p-reduced*** *if G contains only trivial p-divisible subgroups. G is called* ***p-local*** *if the regulator quotient G/R is of exponent p^k for some integer k.*

CHAPTER 4

GAUSS ELIMINATION AND BASIS TRANSFORMATION

This chapter is about the modified Gauss eliminations and basis transformations that we use on torsion-free abelian groups. The theorems and proofs are due to C.Teichert. For the details, you can see [9].

Theorem 4.0.1 *Let G be a torsion-free abelian group and let H be a subgroup of G. Let*

$$G = H + \mathbb{Z}p^{-k_1}\left(p^{l_1}x + y_1\right) + \mathbb{Z}p^{-k_2}\left(p^{l_2}x + y_2\right),$$

where

- $x, y_1, y_2 \in H - pH$,
- $k_1, k_2 \in \mathbb{N}$ *with* $k_1 \geq k_2$ *and*
- $l_1, l_2 \in \mathbb{N}$.

1. *Gauss elimination downwards:*
   If $l_2 \geq l_1$, *then*

   $$G = H + \mathbb{Z}p^{-k_1}\left(p^{l_1}x + y_1\right) + \mathbb{Z}p^{-k_2}\left(y_2 - p^{l_2-l_1}y_1\right).$$

2. *Gauss elimination upwards:*
   If $k_2 - l_2 \geq k_1 - l_1$, *then*

   $$G = H + \mathbb{Z}p^{-k_1}\left(y_1 - p^{l_1-l_2}y_2\right) + \mathbb{Z}p^{-k_2}\left(p^{l_2}x + y_2\right),$$

   where $p^{l_1-l_2} \in \mathbb{Z}$.

Proof 2 1. *By assumption* $l_2 \geq l_1$ *and* $k_1 \geq k_2$, *so* $l_2 - l_1 \geq 0 \geq k_2 - k_1$ *which implies* $l_2 - l_1 - k_2 \geq -k_1$. *Hence* $-p^{l_2-l_1-k_2}\mathbb{Z} \subseteq p^{-k_1}\mathbb{Z}$. *Then the assumptions of Lemma* 3.1.1 *in [9] are satisfied. By setting* $p^{l_2}x+y_2+(-p^{l_2-l_1})(p^{l_1}x+y_1) = y_2 - p^{l_2-l_1}y_1$, *the desired result is followed.*

2. *Let* $\lambda := -p^{l_1-l_2}$, $S := \mathbb{Z}p^{-k_1}$, $a := p^{l_1}x + y_1$, $S' := \mathbb{Z}p^{-k_2}$, $b := p^{l_2}x + y_2$. *By assumption* $l_1 - l_2 - k_1 \geq -k_2$,

$$\lambda S = -p^{l_1-l_2-k_1}\mathbb{Z} \subseteq p^{-k_2}\mathbb{Z} = S'.$$

Then the hypotheses of Lemma 3.1.1 *in [9] are satisfied and hence by setting* $a + \lambda b = y_1 - p^{l_1-l_2}y_2$ *we obtain the desired result.*

4.1 Basis Transformation

Lemma 4.1.1 *Let* $R = \bigoplus_{i=1}^{n} R_i x_i$ *be a torsion-free, abelian group and let* R_i *be rational groups. Assume that*

- $R_j \subseteq R_i$ *for all* $i \in \{1, \ldots, n\}$ *and*
- $x'_j = x_j + \sum_{\substack{i=1 \\ i\neq j}}^{n} a_i x_i$ *with* $a_i \in \mathbb{Z}$

where $j \in \{1, \ldots, n\}$.
Then

$$R = \bigoplus_{\substack{i=1 \\ i\neq j}}^{n} R_i x_i \oplus R_j x'_j.$$

Proof 3 *Let* $R' := \bigoplus_{\substack{i=1 \\ i\neq j}}^{n} R_i x_i \oplus R_j x'_j$.
First we will show that $R' \subseteq R$. *It is enough to show that* $R_j x'_j \subseteq R$:
Let $x := r_j x'_j \in R_j x'_j$, *where* $r_j \in R_j$. *Then*

$$x = r_j x_j + \sum_{\substack{i=1 \\ i\neq j}}^{n} r_j a_i x_i \in R,$$

because by assumption $r_j \in R_j \subseteq R_i$ *and hence* $r_j a_i \in R_i$ *for all* $1 \leq i \leq n$.
Similarly we can show that $R \subseteq R'$.

CHAPTER 5

COORDINATE MATRICES OF ALMOST COMPLETELY DECOMPOSABLE GROUPS AND SOME MATRIX RESULTS

A matrix M called **decomposed** if it is of the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. It is possible in a decomposed form, A and B have no rows or no columns.

A matrix M is called **decomposable** if there are row and column permutations that transform M to a decomposed form, i.e., there are permutation matrices U, V such that UMV is decomposed.

Definition 10 *Let G be a p-reduced, p-local almost completely decomposable group and let R be a completely decomposable subgroup of G of finite index. A matrix $M = [m_{ij}]$ is called a* ***coordinate matrix*** *of G if there is a basis $(g_1 + R, \cdots, g_r + R)$ of G/R and there is a p-basis $(x_1, \cdots, x_n)$ of R such that*

$$g_i = p^{-k_i}\Big(\sum_{j=1}^{n} m_{ij}x_j\Big),$$

where $\langle g_i + R\rangle \simeq \mathbb{Z}_{p^{k_i}}$. The definition of p-basis will be given later.

This thesis deals with the decomposition of almost completely decomposable groups. In this chapter we discuss how the decomposability of an almost completely decomposable group is related to its coordinate matrix.

Lemma 5.0.1 *Let G be a p-reduced, p-local almost completely decomposable group with regulating regulator R and with a coordinate matrix M. The group G is decomposable if and only if it has a decomposable coordinate matrix M.*

Proof 4 *Suppose that G is decomposable. Then we can write $G = G_1 \oplus G_2$, with $G_1 \neq 0 \neq G_2$. Then the regulating regulator R of G can be written as $R = R_1 \oplus R_2$ where R_1 is the regulator of G_1 and R_2 is the regulator of G_2. Furthermore, the regulator quotient G/R can be written as*

$$G/R = (G_1 \oplus G_2)/R = (G_1 + R)/R \oplus (G_2 + R)/R \simeq G_1/R_1 \oplus G_2/R_2$$

Let $(g_1+R, \cdots, g_r+R)$ be a basis of G/R such that $g_1, \ldots, g_{r_1} \in G_1$ and $g_{r_1+1}, \ldots, g_r \in G_2$ and a p-basis $(x_1, \ldots, x_n)$ of R such that $(x_1, \ldots, x_m)$ is a p-basis of R_1 and $(x_{m+1}, \ldots, x_n)$ is a p-basis of R_2. Therefore, the coordinate matrix M can be written in the form $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, i.e., M is decomposable.
Next assume that M is decomposable. Then by definition there exist permutation matrices X and Y such that $XMY = M_1 \oplus M_2$. Let B be a basis of R. Each column of B corresponds to a basis element and the columns of M_1 and M_2 divides the p-basis as $B_1 \cup B_2$. Hence we can write $R = R_1 \oplus R_2$ and it implies that G can be written as $G = G_1 \oplus G_2$ where G_i is the purification of R_i in G and so G is decomposable.

CHAPTER 6

(1,2)-GROUPS

Let G be a p-local, p-reduced almost completely decomposable group where p is prime. Let $(1,2) = (t_1, t_2 < t_3)$ be a set of types of G partially ordered as given with $t_i(p) \neq 0$. Then G is called a $(1,2)$-group.

Let G be a $(1,2)$-group with regulator $R = R_1 \oplus R_2 \oplus R_3$, where R_i is homogeneous of rank r_i, type t_i and $n = r_1 + r_2 + r_3$ is the rank of G. The regulator quotient $G/R \cong \bigoplus_{j=1}^{h} \big(\mathbb{Z}_{p^{k_j}}\big)^{l_j}$, where $k = k_1 > \cdots > k_j \geq 1$, is of exponent p^k and rank $r = l_1 + l_2 + \ldots + l_h$. Then $\big(\mathbb{Z}_{p^{k_j}}\big)^{l_j}$ is called the jth *step* of G/R. If $\{g_f + R \mid \sum_{i=1}^{j-1} l_i < f \leq \sum_{i=1}^{j} l_i\}$ is a basis of the jth step of G/R then the union of the bases of these steps forms a basis of the regulator quotient G/R. Let $(g_j + R \mid 1 \leq j \leq r)$ be a basis of the regulator quotient G/R where

$$g_j = p^{-k_f}\Big(\sum_{i=1}^{r_1} \alpha_{ji} x_i + \sum_{i=1}^{r_2} \beta_{ji} y_i + \sum_{i=1}^{r_3} \gamma_{ji} z_i\Big), \tag{61}$$

where the p-power in front is p^{-k_f} if $\sum_{i=1}^{f-1} l_i < j \leq \sum_{i=1}^{f} l_i$ for $1 \leq f \leq h$, according to the given decomposition of G/R. For the details see [10].

Then $\alpha = (\alpha_{ji}), \beta = (\beta_{ji}), \gamma = (\gamma_{ji})$ are of size $r \times r_1$, $r \times r_2$, $r \times r_3$, respectively. Let $D = \operatorname{diag}\big(p^{-k_j} I_{l_j} \mid 1 \leq j \leq h\big)$, where I_{l_j} are unit matrices of size l_j. This diagonal matrix D is called the structure matrix. Then the matrix $M = D[\alpha \mid \beta \mid \gamma]$ represents the group G and M is the coordinate matrix of G. The details about the coordinate matrices will be given in the next chapter.

The isomorphism types of the regulator $R = R_1 \oplus R_2 \oplus R_3$ of a $(1,2)$-group G is given by the sequence $\big((r_1, t_1), (r_2, t_2), (r_3, t_3)\big)$, where r_i's are the ranks of R_i's and t_i's are the types of R_i's for $i = 1, 2, 3$ and the isomorphism type of G/R is given by the sequence $\big((k_h, l_h) \mid h = 1, \ldots, f\big)$ where k_h and l_h are defined as above.

6.1 Construction of a Coordinate matrix of a $(1,2)$-group

Let G be a p-reduced, p-local $(1,2)$-group. The coordinate matrix is obtained by means of bases of R and G/R. Here our aim is to show how to form the coordinate matrix of a $(1,2)$ group. A $(1,2)$-group has three critical types t_1, t_2, t_3 such that t_1 is not comparable with t_2, t_3 and $t_2 < t_3$. Let R be the regulator of G. We can write $R = R_1 \oplus R_2 \oplus R_3$, where R_i is homogeneous of rank r_i, and the types of the R_i are t_i, respectively. Let $R_1 = \sum_{i=1}^{r_1} S_i x_i, R_2 = \sum_{i=1}^{r_2} S_i y_i, R_3 = \sum_{i=1}^{r_3} S_i z_i$ where $\mathbb{Z} \subset S_i \subset \mathbb{Q}$ and $p^{-1} \notin S_i$. The ordered set $\{x_1, \ldots, x_{r_1}, y_1, \ldots, y_{r_2}, z_1, \ldots, z_{r_3}\}$ is called a p-**basis** of R. Let $\{x_1, \ldots, x_{r_1}\}$ be a p-basis of R_1, $\{y_1, \ldots, y_{r_2}\}$ be a p-basis of R_2 and $\{z_1, \ldots, z_{r_3}\}$ be a p-basis of of R_3. Since G is a p-local group, G/R is of exponent p^k for some $k \in \mathbb{N}$. Assume that $G/R \cong \bigoplus_{i=1}^{r} \mathbb{Z}_{p^{k_i}}$ where $k_i \in \mathbb{N}$ and $k = k_1 \geq k_2 \geq \ldots \geq k_r$. Let $(g_j + R \mid 1 \leq i \leq r)$ be a basis of the regulator quotient G/R and assume $\operatorname{ord}(g_j + R) = p^{k_j}$. Then

$$g_j = p^{-k_j}\Big(\sum_{i=1}^{r_1} \alpha_{ji} x_i + \sum_{i=1}^{r_2} \beta_{ji} y_i + \sum_{i=1}^{r_3} \gamma_{ji} z_i\Big),$$

with respect to the given decomposition of G/R and

$$G = R + \sum_{j=1}^{r} \mathbb{Z} p^{-k_j}\Big(\sum_{i=1}^{r_1} \alpha_{ji} x_i + \sum_{i=1}^{r_2} \beta_{ji} y_i + \sum_{i=1}^{r_3} \gamma_{ji} z_i\Big).$$

Then we can write the coordinate matrix M of G as $M = D[\alpha \mid \beta \mid \gamma]$ where the structure matrix $D = \operatorname{diag}\big(p^{k_1}, \ldots, p^{k_r}\big)$. The diagonal matrix D is determined by the isomorphism type of the regulator quotient and is unique for the given group G.

CHAPTER 7

DECOMPOSABILITY OF $(1,2)$-GROUPS

In this chapter we state and prove some theorems that play an important role in the decomposability problem of $(1,2)$-groups.

The following Lemma is the **regulator criterion** for $(1,2)$-groups.

Lemma 7.0.1 *If $M = D[\alpha \mid \beta \mid \gamma]$ is a coordinate matrix of a $(1,2)$-group G, then in each row of α there is a unit and there is a unit in each row of (β, γ).*

Proof 5 *Let G be a $(1,2)$-group and let $R = R_1 \oplus R_2 \oplus R_3$ be the regulator of G. The result follows by the regulator criterion for R, see Lemma 13 in [13]. If there is no unit in each row of α, then $R_2 \oplus R_3$ is not pure in G and if there is a row in (β, γ) without unit, then R_1 is not pure in G and this contradicts the regulator criterion for R.*

Definition 11 *A $(1,2)$-group G is called* ***clipped*** *if it has no completely decomposable direct summand.*

Lemma 7.0.2 *Let G be a clipped $(1,2)$-group and let $M = D[\alpha \mid \beta \mid \gamma]$ be the coordinate matrix of G. Then there is a p-basis of R_1 and a basis of G/R such that $M' = D[I \mid \beta \mid \gamma]$ is the corresponding representing matrix.*

Proof 6 *By Lemma 14 in [13] there exists a matrix V and a matrix Y_1 resulted by the change of bases of R and G/R such that α is translated to $\alpha' = V'\alpha Y_1$, where the pair (V, V') satisfies $VD = DV'$. By regulator criterion, there is a unit in each row of α. By column permutation the identity in the first row of α can be moved to the position*

$(1,1)$ *in* α. *Then by Gauss elimination downwards, all the entries below this unit can be annihilated. We repeat this process with the second row, third row etc. This is done by multiplying* α *by* V' *defined above. Hence we get* $\alpha' = V'\alpha Y_1 = [I \mid 0]$, *where* Y_1 *is invertible. Since* G *is clipped there is no* 0*-column in* α, *hence* α *is a square matrix and is* p*-invertible. The matrix* Y_1 *in the above equation can be chosen as* $Y_1 = \alpha^{-1}$ *which changes the* p*-basis of* R_1 *but no changes in* R_2 *and* R_3. *Thus, the resulting matrix changes to* $M' = D[I \mid \beta \mid \gamma]$.

Lemma 7.0.3 ([13, Lemma 19]) *Let* G *be a clipped,* p*-reduced,* p*-local* $(1,2)$*-group and let* $M = D[\alpha \mid \beta \mid \gamma]$ *be a coordinate matrix of* G. *If* β*-part of* M *is decomposable then* G *is decomposable. Conversely, if* G *is decomposable then it has a coordinate matrix* M *with decomposable* β*-part.*

Proof 7 *Let* G *be a* $(1,2)$*-group and let* $M = D[\alpha \mid \beta \mid \gamma]$ *be a coordinate matrix of* G. *Assume that* β *is indecomposable. Then by Lemma* 14 *in [13] there are matrices* V *and* Y *such that* $V[\alpha|\beta|\gamma]Y = [V\alpha|\beta'|V\gamma]$ *where* $Y = \mathrm{diag}(I, Y', I)$ *with a upper triangular matrix* Y'. *By Lemma 7.0.2,* $V\alpha$ *can be changed to identity matrix and* $V\gamma = \begin{bmatrix} I \\ 0 \end{bmatrix}$. *These transformations do not affect* β'. *By Corollary* 17 *in [13],* $\beta' = V\beta Y'$ *is decomposable since* β *is decomposable. Hence* M *changes to* $M' = D[I|\beta'|V\gamma]$ *and so decomposability of* β*-part determines the decomposability of* G. *For the proof of the converse of the theorem see Lemma* 12 *in [12].*

Definition 12 *Let* G *be a* $(1,2)$*-group with a coordinate matrix* $M = D[\alpha \mid \beta \mid \gamma]$. *If*

$$(\beta,\gamma) = \left[\begin{array}{ccccccc|ccc} & & & 0 & & & & & & \\ & & & \vdots & & & & & & \\ & & & 0 & & & & & & \\ 0 & \cdots & 0 & a & 0 & \cdots & 0 & 0 & \cdots & 0 \\ & & & 0 & & & & & & \\ & & & \vdots & & & & & & \\ & & & 0 & & & & & & \end{array}\right]$$

where a *is at position* (i,j) *in* β. *Then we say that there is a* ***cross*** *in* β *with a cross point* a.

PROPOSITION 7.0.4 *Let* G *be a* $(1,2)$*-group with* $M = D[I \mid \beta \mid \gamma]$ *as a coordinate matrix.*

1. *If*

$$(\beta,\gamma)=\left[\begin{array}{ccccccc||ccc}
 & & & 0 & & & & & & \\
 & & & \vdots & & & & & & \\
 & & & 0 & & & & & & \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
 & & & 0 & & & & & & \\
 & & & \vdots & & & & & & \\
 & & & 0 & & & & & &
\end{array}\right]$$

where 1 *is at position* (i,j) *in* β*. Then there is a cross in* β *with a cross point* 1 *and* $\langle x_i, y_j\rangle_*$ *is a direct summand of rank* 2*.*

If

$$(\beta,\gamma)=\left[\begin{array}{ccc||ccccccc}
 & & & & & & 0 & & & \\
 & & & & & & \vdots & & & \\
 & & & & & & 0 & & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 & & & & & & 0 & & & \\
 & & & & & & \vdots & & & \\
 & & & & & & 0 & & &
\end{array}\right]$$

where 1 *is at position* (i,j) *in* γ*. Then there is a zero row in* β *and* $\langle x_i, z_j\rangle_*$ *is a direct summand of rank* 2*.*

2. *If*

$$(\beta,\gamma)=\left[\begin{array}{ccccccc||ccccccc}
 & & & 0 & & & & & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & \\
0 & \cdots & 0 & p^l & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & &
\end{array}\right]$$

where $p^l \neq 1$ *is at position* (i,j) *in* β*. Then there is a cross in* β *with a cross point* p^l*. Then* $\langle x_i, y_j, z_i\rangle_*$ *is a direct summand of rank* 3*.*

3. *If*

$$(\beta,\gamma)=\left[\begin{array}{ccccccc||ccccccc}
 & & & 0 & & & & & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & \\
0 & \cdots & 0 & p^l & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & &
\end{array}\right]$$

where $p^l \neq 1$ *is at position* (i_1,j) *and* 1 *at position* (i_2,j)*, both in* β*, then* $\langle x_{i_1}, x_{i_2}, y_j, z_{i_1}\rangle_*$ *is a direct summand of rank* 4*.*

4. *If*

$$
(\beta,\gamma) = \left[\begin{array}{ccccccc||cccccccccc}
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
0 & \cdots & 0 & p^m & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
0 & \cdots & 0 & p^n & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
 & & & 0 & & & & & & & 0 & & & 0 & & & \\
 & & & \vdots & & & & & & & \vdots & & & \vdots & & & \\
 & & & 0 & & & & & & & 0 & & & 0 & & &
\end{array}\right]
$$

where $p^m \neq 1$ *is at position* (i_1, j)*, and* $p^n \neq 1$*, at position* (i_2, j) *in* β*, then* $\langle x_{i_1}, x_{i_2}, y_j, z_{i_1}, z_{i_2} \rangle_*$ *is a direct summand of rank* 5*.*

Lemma 7.0.5 *Let* G *be a* $(1,2)$*-group with coordinate matrix* $M = D[I \mid \beta \mid \gamma]$*. If there is a zero row in* β*, then* G *is decomposable.*

Proof 8 *This follows directly by* (1) *in Proposition 7.0.4.*

CHAPTER 8

NEARLY ISOMORPHIC GROUPS

Definition 13 *Let G and H be two torsion-free groups of finite rank. If for every integer $n \in \mathbb{Z}^+$, there is a monomorphism $\alpha_n : G_1 \to G_2$ such that $[H : \alpha_n G]$ is finite, n and $[H : \alpha_n G]$ are relatively prime, then G and H are called* ***nearly isomorphic****, denoted $G \cong_{nr} H$.*

Note that near isomorphism is a weaker form of isomorphism and isomorphic groups are also nearly isomorphic.

By Arnold's Theorem two near-isomorphic torsion-free groups of finite rank have (up to near-isomorphism of summands) the same decomposition properties. Let G be a $(1,2)$-group and let M be a coordinate matrix of G. By allowed row and column transformations M is transformed to M' that is the coordinate matrix of the same group or of a nearly isomorphic group G'. If we arrive at a matrix that shows that the group to which it belongs decomposes or not, then the original group is decomposable or not.

Due to the required structure of the matrices in Corollary 17 in [13] , only certain row and column transformations are allowed. We list below the allowed row and column transformations.

Lemma 8.0.1 ([13, Lemma 21 and 24]) *Let G be a $(1,2)$-group and let $M = D[\alpha|\beta|\gamma]$ be the coordinate matrix where $D = \operatorname{diag}(p^{k_1}, \ldots, p^{k_r})$. Then the following row and column operations on the coordinate matrix results in a coordinate matrix of a group G' which is nearly isomorphic to G.*

1. *Any multiple of a row may be added to any row below it.*

2. *For $i_1 < i_2$, the $p^{k_{i_1}-k_{i_2}}$-times of row i_2 may be added to a row r_1 .*

3. *Any line may be multiplied by an integer relatively prime to p.*

4. *All elementary column operations can be applied to α and γ.*

5. *Any multiple of a column of β may be added to another column of $[\beta|\gamma]$.*

The transformations in Lemma 8.0.2 are called **allowed transformations**. While annihilating an entry, other entries that were zero may change to nonzero entries, then those entries are called **fill-ins**.

Theorem 8.0.2 ([12, Theorem 1]) *Let G be a $(1,2)$-group and let M be the coordinate relative to the p-basis $(x_1,\dots,x_n)$, and the basis $(g_1+R,\dots,g_r+R)$ of G/R. Let $D = \operatorname{diag}\big(p^{k_1},\dots,p^{k_r}\big)$.*

(i) *Let M' be the coordinate matrix of G relative to the p-basis $(y_1,\dots,y_n)$ and basis $(h_1+R,\dots,h_r+R)$ of G/R such that $t(x_i) = t(y_i)$ for $i = 1,\dots,n$. Then there is a pair (V,V') and a matrix*

$$\begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix}$$

where Y_{ij} is an $r_i \times r_j$ integer matrix and the diagonal blocks Y_{ii} are p-invertible such that $M' = VMY$.

(ii) *Assume that a pair (V,V') and a matrix Y of the form $\begin{bmatrix} Y_{11} & 0 & 0 \\ 0 & Y_{22} & Y_{23} \\ 0 & 0 & Y_{33} \end{bmatrix}$ where Y_{ij} is an $r_i \times r_j$ integer matrix and the diagonal blocks Y_{ii} are p-invertible are given. Then there is a group G' nearly isomorphic to G with regulator R , a basis $(h'_1,\dots,h'_r)$ of H/R and a p-basis $(y_1,\dots,y_n)$ of R such that H has the structure matrix S, x_i and y_i have equal types for $i = 1,\dots,n$, and VMV' is a coordinate matrix of H.*

8.1 Some Examples

Example 1 Let G be a $(1,2)$-group with regulator quotient of exponent p^5 and let $M = [I \mid \beta \mid \gamma]$ be its coordinate matrix.

1. Let $R = \mathbb{Z}[q^{-1}]x_1 \oplus \mathbb{Z}[q^{-1}]x_2 \oplus \mathbb{Z}[r^{-1}]x_3 \oplus \mathbb{Z}[(rs)^{-1}]x_4 \oplus \mathbb{Z}[(rs)^{-1}]x_5$ be the regulator of G, and $R \subset G = \langle R, g_1, g_2 \rangle$, $g_1 = p^{-5}(x_1 + x_3)$, $g_2 = p^{-5}(x_2 + x_4)$. Then the coordinate matrix

$$M = \begin{bmatrix} 1 & 0 & \mid & 1 & \mid & 0 & 0 \\ 0 & 1 & \mid & 0 & \mid & 1 & 0 \end{bmatrix}$$

   The last column of M is 0, and $\langle x_5 \rangle_*$ is a direct summand of rank 1.

2. Let $R = \mathbb{Z}[q^{-1}]x_1 \oplus \mathbb{Z}[q^{-1}]x_2 \oplus \mathbb{Z}[r^{-1}]x_3 \oplus \mathbb{Z}[(rs)^{-1}]x_4$ be the regulator of G, and $R \subset G = \langle R, g_1, g_2 \rangle$, $g_1 = p^{-5}(x_1 + px_3 + x_4)$, $g_2 = p^{-5}(x_2 + x_5)$. Then the coordinate matrix

$$M = \begin{bmatrix} 1 & 0 & \mid & p & \mid & 1 & 0 \\ 0 & 1 & \mid & 0 & \mid & 0 & 1 \end{bmatrix}$$

   The β-part of the coordinate matrix M has a 0-row and $\langle x_2, x_5 \rangle_*$ is a direct summand of rank 2. Moreover, $\langle x_1, x_3, x_4 \rangle_*$ is another direct summand of rank 3.

3. Let $R = \mathbb{Z}[q^{-1}]x_1 \oplus \mathbb{Z}[q^{-1}]x_2 \oplus \mathbb{Z}[r^{-1}]x_3 \oplus \mathbb{Z}[r^{-1}]x_4 \oplus \mathbb{Z}[(rs)^{-1}]x_5$ be the regulator of G, and $R \subset G = \langle R, g_1, g_2 \rangle$, $g_1 = p^{-5}(x_1 + x_3)$, $g_2 = p^{-5}(x_2 + p^3 x_4 + x_5)$. Then the coordinate matrix

$$M = \begin{bmatrix} 1 & 0 & \mid & 1 & 0 & \mid & 0 \\ 0 & 1 & \mid & 0 & p^3 & \mid & 1 \end{bmatrix}$$

   has two crosses, one with cross entry a unit, the other not, and $\langle x_1, x_3 \rangle_*$ and $\langle x_2, x_4, x_5 \rangle_*$ are direct summands of rank 2 and 3, respectively.

CHAPTER 9

INDECOMPOSABLE $(1,2)$-GROUPS WITH REGULATOR QUOTIENT OF EXPONENT p^5

PROPOSITION 9.0.1 *The following three $(1,2)$-groups with regulator quotient of exponent p^5 given by the isomorphism types of their regulator with fixed types, their regulator quotient and their coordinate matrix $M = \begin{bmatrix} \alpha & | & \beta & | & \gamma \end{bmatrix}$ are indecomposable and pairwise not near-isomorphic.*

(i) $M = \left[\begin{array}{cc|c|c} 1 & 0 & p^3 & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_p$ and* $\operatorname{rank} G = 4$.

(ii) $M = \left[\begin{array}{cc|c|c} 1 & 0 & p^2 & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_p$ and* $\operatorname{rank} G = 4$.

(iii) $M = \left[\begin{array}{cc|c|c} 1 & 0 & p & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_p$ and* $\operatorname{rank} G = 4$.

Proof 9 *(i) Consider the following matrix:*

$$\begin{bmatrix} 1 & ap^4 \\ c & 1 \end{bmatrix} \begin{bmatrix} p^3 \\ 1 \end{bmatrix} = \begin{bmatrix} p^3 + ap^4 \\ cp^3 + 1 \end{bmatrix} = \begin{bmatrix} p^3 + ap^4 \\ 1 \end{bmatrix}$$

Since the entry at position $(1,1)$ is not 0 modulo p^5, the column is not 0 which shows that $\begin{bmatrix} p^3 \\ 1 \end{bmatrix}$ can not be decomposed. Hence G is indecomposable.

(ii) Consider the following matrix:

$$\begin{bmatrix} 1 & ap^4 \\ c & 1 \end{bmatrix} \begin{bmatrix} p^2 \\ 1 \end{bmatrix} = \begin{bmatrix} p^2 + ap^4 \\ cp^2 + 1 \end{bmatrix} = \begin{bmatrix} p^2 + ap^4 \\ 1 \end{bmatrix}$$

Since the entry at position $(1,1)$ *is not* 0 *modulo* p^5*, the column is not* 0 *which shows that* $\begin{bmatrix} p^2 \\ 1 \end{bmatrix}$ *can not be decomposed. Hence G is indecomposable.*

(iii) Consider the following matrix:

$$\begin{bmatrix} 1 & ap^4 \\ c & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} p + ap^4 \\ cp + 1 \end{bmatrix} = \begin{bmatrix} p + ap^4 \\ 1 \end{bmatrix}$$

Since the entry at position $(1,1)$ *is not* 0 *modulo* p^5*, the column is not* 0 *which shows that* $\begin{bmatrix} p \\ 1 \end{bmatrix}$ *can not be decomposed. Hence G is indecomposable.*

PROPOSITION 9.0.2 *The following six* $(1,2)$*-groups with regulator quotient of exponent* p^5 *given by the isomorphism types of their regulator with fixed types, their regulator quotient and their coordinate matrix* $M = \begin{bmatrix} \alpha & | & \beta & | & \gamma \end{bmatrix}$ *are indecomposable and pairwise not near-isomorphic.*

(i) $M = \left[\begin{array}{cc|c|c} 1 & 0 & p & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 4$.

(ii) $M = \left[\begin{array}{cc|c|c} 1 & 0 & p^2 & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 4$.

(iii) $M = \left[\begin{array}{cc|c|cc} 1 & 0 & p^3 & 1 & 0 \\ \hline 0 & 1 & p & 0 & 1 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 4$.

(iv) $M = \left[\begin{array}{cc|cc|c} 1 & 0 & p^2 & 0 & 1 \\ \hline 0 & 1 & 1 & p & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 5$.

(v) $M = \left[\begin{array}{cc|cc|c} 1 & 0 & p & 0 & 1 \\ \hline 0 & 1 & 1 & p & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 5$.

(vi) $M = \left[\begin{array}{ccc|cc|cc} 1 & 0 & 0 & p^3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & p & 0 & 1 \\ \hline 0 & 0 & 1 & p & 1 & 0 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^2}$ *and* $\operatorname{rank} G = 7$.

Proof 10 *The decomposability proofs of the cases of (i), (ii), (iii) are similar to the proofs of Proposition 9.0.1.*

(v) The indecomposability proofs of cases (iv) and (v) are similar. We will prove the case (v). Consider the following matrix:

$$\begin{bmatrix} 1 & ap^3 \\ c & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ 1 & p^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p + ap^3 & ap^4 \\ cp + 1 & p \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p + ap^3 & b(p + ap^3) + ap^4 \\ cp + 1 & b(cp + 1) + p \end{bmatrix}$$

In the first column the entries are not both zero. If $b \equiv 0 \pmod{p^2}$ *then the entry at position* $(1, 2)$ *may be done* 0*. But in this case the entry at position* $(2, 2)$ *is not* 0 *modulo* p^2*.*

(vi) The proof follows by [1, Lemma 9.3].

PROPOSITION 9.0.3 *The following six* $(1, 2)$*-groups with regulator quotient of exponent* p^5 *given by the isomorphism types of their regulator with fixed types, their regulator quotient and their coordinate matrix* $M = \left[\alpha \mid \beta \mid \gamma\right]$ *are indecomposable and pairwise not near-isomorphic.*

(i) $M = \left[\begin{array}{cc|c|c} 1 & 0 & p & 1 \\ \hline 0 & 1 & 1 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 4$.

(ii) $M = \left[\begin{array}{cc|c|cc} 1 & 0 & p^2 & 1 & 0 \\ \hline 0 & 1 & p & 0 & 1 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 5$.

(iii) $M = \left[\begin{array}{cc|c|cc} 1 & 0 & p^3 & 1 & 0 \\ \hline 0 & 1 & p^2 & 0 & 1 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 5$.

(iv) $M = \left[\begin{array}{cc|cc|c} 1 & 0 & p & 0 & 1 \\ \hline 0 & 1 & 1 & p & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 5$.

(v) $M = \left[\begin{array}{cc|cc|c} 1 & 0 & p & 0 & 1 \\ \hline 0 & 1 & 1 & p^2 & 0 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 5$.

(vi) $M = \left[\begin{array}{cc|cc|cc} 1 & 0 & p^2 & 0 & 1 & 0 \\ \hline 0 & 1 & p & p^2 & 0 & 1 \end{array}\right]$ *with regulator quotient isomorphic to* $\mathbb{Z}_{p^5} \bigoplus \mathbb{Z}_{p^3}$ *and* $\operatorname{rank} G = 6$.

Proof 11 *(iv) Consider the following matrix:*

$$\begin{bmatrix} 1 & ap^2 \\ c & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ 1 & p \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p + ap^2 & ap^3 \\ cp + 1 & p \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p + ap^2 & b(p + ap^2) + ap^3 \\ cp + 1 & b(cp + 1) + p \end{bmatrix}$$

In the first column the entries are not both zero. If $b \equiv 0 \pmod{p^3}$ *then the entry at position* $(1, 2)$ *may be done* 0*. But in this case the entry at position* $(2, 2)$ *is not* 0 *modulo* p^3.

(v) Consider the following matrix:

$$\begin{bmatrix} 1 & ap^2 \\ c & 1 \end{bmatrix} \begin{bmatrix} p & 0 \\ 1 & p^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p + ap^2 & ap^4 \\ cp + 1 & p^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p + ap^2 & b(p + ap^2) + ap^4 \\ cp + 1 & b(cp + 1) + p^2 \end{bmatrix}$$

In the first column the entries are not both zero. If $b \equiv 0 \pmod{p^3}$ *then the entry at position* $(1,2)$ *may be done* 0*. But in this case the entry at position* $(2,2)$ *is not* 0 *modulo* p^3*.*

(vi) Consider the following matrix:

$$\begin{bmatrix} 1 & ap^2 \\ c & 1 \end{bmatrix} \begin{bmatrix} p^2 & 0 \\ p & p^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p^2 + ap^3 & ap^4 \\ cp^2 + p & p^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} p^2 + ap^3 & b(p^2 + ap^3) + ap^4 \\ cp^2 + p & b(cp^2 + p) + p^2 \end{bmatrix}$$

In the first column the entries are not both zero. If $b \equiv 0 \pmod{p^2}$ *then the entry at position* $(1,2)$ *may be done* 0*. But in this case the entry at position* $(2,2)$ *is not* 0 *modulo* p^3*. The proof of the other cases are very similar to the proof of 9.0.1.*

CHAPTER 10

SMITH NORMAL FORMS

Let G be a $(1,2)$-group and let M be a coordinate matrix of G. If we write "We form the Smith Normal form" of M we mean that there are p-invertible matrices U and V such that UMV is a p-diagonal matrix. These transformations affect the matrices in the subblocks called submatrices. But it is possible to reestablish submatrices that were 0 or of the form $p^t I$.

We denote $(1,2)$-groups with regulator quotient of exponent p^5 by $((1,2),p^5)$.

Theorem 10.0.1 *There are three near-isomorphism classes of indecomposable* $((1,2),p^5)$ *groups with regulator quotient isomorphic to* $(Z_{p^5})^{l_1} \bigoplus (Z_p)^{l_2}$ *where* $l_1 \geq 1$ *and* $l_2 \geq 1$ *as in Proposition 9.0.1.*

Proof 12 *Let G be a $((1,2),p^5)$-group with regulator R and regulator quotient G/R is isomorphic to $(\mathbb{Z}_{p^5})^{l_1} \bigoplus (\mathbb{Z}_p)^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$. Suppose that G is indecomposable. Let $M = \begin{bmatrix} I & \mid & \beta & \mid & \gamma \end{bmatrix}$ be the coordinate matrix of G. We first form the Smith Normal forms of the subblocks of β. If β is indecomposable, then by Proposition 7.0.3 the group G is indecomposable. Since we assumed that G is indecomposable then if a summand is displayed, then it leads to a contradiction or we check its class in the list given in Proposition 9.0.1.*

By this method we find all indecomposable $((1,2),p^5)$ groups with regulator quotient isomorphic to $(\mathbb{Z}_{p^5})^{l_1} \bigoplus (\mathbb{Z}_p)^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$. Since we assumed that G is indecomposable, β does not contain 0-rows, there can not be any 0-column in M, and there can not be a cross in β by Proposition 7.0.4. Set $\beta = \begin{bmatrix} X \\ \hline Y \end{bmatrix}$. There is no *unit in X to avoid a cross. Hence we write pX instead. Since the matrix Y is in the p-block, the entries of Y are either units or zero. We successively form Smith Normal form of the sub-block X to split out the parts p^4I , p^3I , p^2I and pI. Note that there is no zero column and no zero row in β to avoid direct summands. Then we can write β as follows*

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline A_1 & A_2 & A_3 & A_4 & A_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^5 \\ p \end{array}$$

There can not be a zero column in A_5 since β contains no 0-columns. With a unit in A_5 we can annihilate the entries in A_1 , A_2 , A_3 and A_4 this will lead to a direct summand of rank 2. Hence we omit the units in A_5. Therefore the last block column of β does not exist and hence we get

$$[\beta] = \left[\begin{array}{cccc} p^4I & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 \\ 0 & 0 & p^2I & 0 \\ 0 & 0 & 0 & pI \\ \hline A_1 & A_2 & A_3 & A_4 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p \end{array}$$

There will be no zero column in A_1, A_2, A_3 and A_4 otherwise this would lead to a cross in β. If there is a unit in A_1 by Gauss elimination upwards we can annihilate in p^4I and then by basis transformation we annihilate the entries in the same row as this unit in A_2, A_3 and A_4 but so we get a direct summand of rank 2. Hence $A_1 = 0$ and the first column of β is not present. Thus β is of the form

$$[\beta] = \left[\begin{array}{ccc} p^3I & 0 & 0 \\ 0 & p^2I & 0 \\ 0 & 0 & pI \\ \hline A_2 & A_3 & A_4 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p \end{array}$$

If there is a unit in A_2 then the corresponding rows in A_3 and A_4 can be annihilated with this unit. There will be fill-ins in the first block row of β which can be removed *by p^2I and pI in p^5-block, respectively. This leads to a direct summand (i) of rank 4 listed in Proposition 9.0.1. Omitting this summand we may assume that $A_2 = 0$. But then the first block column containing p^3I leads to a cross. Hence the first block row and the first block column of β do not exist. β changes to Hence we get*

$$[\beta] = \left[\begin{array}{cc} p^2I & 0 \\ 0 & pI \\ \hline A_3 & A_4 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p \end{array}$$

Similarly with a unit in A_3 we obtain a direct summand (ii) of rank 4 listed in Proposition 9.0.1 . Omitting this summand we may assume that $A_3 = 0$ Then

$$[\beta] = \left[\begin{array}{c} pI \\ \hline A_4 \end{array}\right] \begin{array}{c} p^5 \\ p \end{array}$$

A unit in A_4 leads to a direct summand (iii) of rank 4 listed in Proposition 9.0.1 . This finishes the proof.

Theorem 10.0.2 *There are six near-isomorphism classes of indecomposable $((1,2), p^5)$ groups with regulator quotient isomorphic to $(\mathbb{Z}_{p^5})^{l_1} \bigoplus (\mathbb{Z}_{p^2})^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$ as in Proposition 9.0.2.*

Proof 13 *Assume that G is an indecomposable $((1,2), p^5)$-group with regulator R and regulator quotient G/R is isomorphic to $(\mathbb{Z}_{p^5})^{l_1} \bigoplus (\mathbb{Z}_{p^2})^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$. Let $M = \begin{bmatrix} I & | & \beta & | & \gamma \end{bmatrix}$ be the coordinate matrix of G. Our method consists of forming the Smith Normal forms of the subblocks of β since by Proposition 7.0.3 it follows that if β is decomposable then G is decomposable. If a summand is displayed, then it leads to a contradiction or we check its class in the list given in Proposition 9.0.2.*

By using this method we find all indecomposable $((1,2), p^5)$ groups with regulator quotient isomorphic to $(\mathbb{Z}_{p^5})^{l_1} \bigoplus (\mathbb{Z}_{p^2})^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$. Due to our assumption that G is indecomposable, β can not contain 0-rows, there can not be any 0-column in M, and there can not be a cross in β by Proposition 7.0.4. Write $\beta = \left[\begin{array}{c} X \\ \hline Y \end{array}\right]$. There is no unit in X to avoid a cross. Hence set pX instead. Since the *matrix Y is in the p^2-block, the entries of Y are units,zero or in $p\mathbb{Z}$. We successively form Smith Normal form of the sub-block X to split out the parts p^4I , p^3I , p^2I and pI. Then we can write β as follows*

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline A_1 & A_2 & A_3 & A_4 & A_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^2 \end{array}$$

There is no zero column in A_5 due to the reason that there is no zero column in β. With a unit in A_5 we can annihilate the entries in A_1 , A_2 , A_3 and A_4 without causing any fill-ins in the other subblocks and this will lead to a direct summand of rank 2. Hence we execute the units in A_5. Therefore A_5 is of the form pA_5 and hence we get

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline A_1 & A_2 & A_3 & A_4 & pA_5 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^2 \end{array}$$

If there is a unit in A_1, then by Gauss elimination upwards we can annihilate in p^4I and afterwards by basis transformation we annihilate the entries in the same row as this unit in A_2, A_3 and A_4. After these eliminations we get a direct summand of rank 2. Hence we set pA_1. The same holds for A_2 and we set pA_2. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_1 & pA_2 & A_3 & A_4 & pA_5 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^2 \end{array}$$

There is no zero column in pA_5 and thus the Smith Normal form of pA_5 is $pA_5 =$

$\left[\begin{array}{c} pI \\ \hline 0 \end{array}\right]$. *The pI in this Smith Normal form of pA_5 allows us to annihilate the entries in pA_2 and pA_1. This operation do not cause any fill-ins in the other blocks. Therefore β is transformed to*

$$\beta = \left[\begin{array}{ccccc} p^4 I & 0 & 0 & 0 & 0 \\ 0 & p^3 I & 0 & 0 & 0 \\ 0 & 0 & p^2 I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & A_{31} & A_{41} & pI \\ pA_1 & pA_2 & A_{32} & A_{42} & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \end{array}$$

There is no 0-column in pA_1 to avoid a cross. Hence the Smith Normal form of pA_1 is $pA_1 = \left[\begin{array}{c} pI \\ \hline 0 \end{array}\right]$. With this pI we can annihilate in pA_2. The fill-ins in the first block row are $p^4\mathbb{Z}$ and can be annihilated by $p^3 I$ in the second block row and so we get

$$\beta = \left[\begin{array}{ccccc} p^4 I & 0 & 0 & 0 & 0 \\ 0 & p^3 I & 0 & 0 & 0 \\ 0 & 0 & p^2 I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & A_{31} & A_{41} & pI \\ pI & 0 & A_{32} & A_{42} & 0 \\ 0 & pA_{22} & A_{33} & A_{43} & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \end{array}$$

With pI in the sixth block row we can annihilate $p^4 I$ in the first block row. The fill-ins in the first row are in $p^3\mathbb{Z}$ and can be removed by $p^3 I$, $p^2 I$ and pI respectively. Hence the first block row is 0 and so it is not present. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} 0 & p^3 I & 0 & 0 & 0 \\ 0 & 0 & p^2 I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & A_{31} & A_{41} & pI \\ pI & 0 & A_{32} & A_{42} & 0 \\ 0 & pA_{22} & A_{33} & A_{43} & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \end{array}$$

A unit in A_{33} allows us to annihilate in pA_{22} and in A_{43}. The fill-ins in p^5-block can be removed by p^3I and pI respectively. But also we can annihilate with this unit in A_{32} and A_{31}. Thus we get a direct summand (ii) listed in Proposition 9.0.2. Omitting this summand we can assume that pA_{33}. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & A_{31} & A_{41} & pI \\ pI & 0 & A_{32} & A_{42} & 0 \\ 0 & pA_{22} & pA_{33} & A_{43} & 0 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \end{array}$$

With a unit in A_{31} we first annihilate in A_{41}. The fill-ins in the second block row are in $p^2\mathbb{Z}$ and can be annihilated by pI below in the third block row. Then with this unit we annihilate in A_{32} and in pA_{33}. These cause fill-ins in the last block column in p^2-block and they can be removed by pI in the fifth block row or there are $p^2\mathbb{Z}$ and hence can be ignored. This leads to a summand (iv) listed in Proposition 9.0.2. We omit this summand and we assume pA_{31}. By pI in the last block column, the submatrix pA_{31} is annihilated without causing any fill-ins. Hence set $pA_{31} = 0$. Therefore β is transformed to

$$\beta = \left[\begin{array}{ccccc} 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & 0 & A_{41} & pI \\ pI & 0 & A_{32} & A_{42} & 0 \\ 0 & pA_{22} & pA_{33} & A_{43} & 0 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \end{array}$$

A unit in A_{32} leads to a summand (iv) listed in Proposition 9.0.2. Since by assumption G is indecomposable, we omit this summand and so we may assume that pA_{32}. But then by pI in the fifth block row we annihilate the submatrix pA_{32}. So $pA_{32} = 0$ and

β changes to

$$\beta = \left[\begin{array}{ccccc} 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & 0 & A_{41} & pI \\ pI & 0 & 0 & A_{42} & 0 \\ 0 & pA_{22} & pA_{33} & A_{43} & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \end{array}$$

There is no zero column in pA_{22} to avoid a cross. Hence the entries of A_{22} are in $p\mathbb{Z}$ and so the Smith Normal form of pA_{22} is $pA_{22} = \begin{bmatrix} pI \\ 0 \end{bmatrix}$ We annihilate the corresponding entries of pA_{33} with pI in the Smith Normal form of pA_{22}. The fill-ins in the first block row are in $p^3\mathbb{Z}$ and can be violated by p^2I in the second block row. Thus,

$$\beta = \left[\begin{array}{ccccc} 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & 0 & A_{41} & pI \\ pI & 0 & 0 & A_{42} & 0 \\ 0 & pI & 0 & A_{43} & 0 \\ 0 & 0 & pA_{33} & A_{44} & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

A unit in A_{44} leads to a summand (i) listed in Proposition 9.0.2. We omit this summand and assume pA_{44}. But then by pI above in the third block row, the submatrix pA_{44} is annihilated. Thus,

$$\beta = \left[\begin{array}{ccccc} 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & 0 & A_{41} & pI \\ pI & 0 & 0 & A_{42} & 0 \\ 0 & pI & 0 & A_{43} & 0 \\ 0 & 0 & pA_{33} & 0 & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \\ p^2 \end{array}$$

The entries of pA_{33} are in $p\mathbb{Z}$. A p in pA_{33} leads to a summand (ii) listed in Proposition

9.0.2. Omitting this summand we set $pA_{33} = 0$ since it is in the p^2-block. This leads to a 0-row in β and thereafter a cross with a cross point p^2 in the second block row. Hence $p^2 I$ in the second block row and the corresponding rows and columns do not exist. Hence β is transformed to

$$\beta = \left[\begin{array}{cccc} 0 & p^3 I & 0 & 0 \\ 0 & 0 & pI & 0 \\ \hline 0 & 0 & A_{41} & pI \\ pI & 0 & A_{42} & 0 \\ 0 & pI & A_{43} & 0 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \end{array}$$

A zero row in A_{43} results to a summand (iii) listed in Proposition 9.0.2. Hence the Smith Normal form of A_{43} is $A_{43} = \begin{bmatrix} I & 0 \end{bmatrix}$. With this I in the Smith Normal form we annihilate in A_{42} and in A_{41}. The resulting fill-ins are in $p\mathbb{Z}$ and can be annihilated by pI in the fifth and fourth block rows, respectively. Thus β is transformed to

$$\beta = \left[\begin{array}{ccccc} 0 & p^3 I & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline 0 & 0 & 0 & A_{41} & pI \\ pI & 0 & 0 & A_{42} & 0 \\ 0 & pI & I & 0 & 0 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \\ p^2 \end{array}$$

A unit in A_{41} leads to a summand (v) listed in Proposition 9.0.2. Omitting this summand we may assume that pA_{41}. But then pA_{41} can be annihilated by pI on the right in the same block row, so pA_{41}=0. This causes to a cross with a cross point p in the last block column. Hence the last block column and the fourth block row are not present. Therefore β changes to

$$\beta = \left[\begin{array}{cccc} 0 & p^3 I & 0 & 0 \\ 0 & 0 & pI & 0 \\ 0 & 0 & 0 & pI \\ \hline pI & 0 & 0 & A_{42} \\ 0 & pI & I & 0 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^2 \\ p^2 \end{array}$$

The entries of A_{42} that are in $p\mathbb{Z}$ can be annihilated by pI on the left in the same block row. Hence the entries of A_{42} are either units or zeros. There is no zero row and no zero column in A_{42} otherwise there will be crosses in β. Hence the Smith Normal form of A_{42} changes to the identity matrix. But then a direct summand (v) listed in Proposition 9.0.2 is obtained. We omit this summand and get β as

$$\beta = \left[\begin{array}{cc} p^3I & 0 \\ 0 & pI \\ \hline pI & I \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ \hline p^2 \end{array}$$

From this form of β we can read another direct summand (vi) listed in Proposition 9.0.2. This finishes the proof.

Theorem 10.0.3 *There are six near-isomorphism classes of indecomposable $((1,2),p^5)$ groups with regulator quotient isomorphic to $(\mathbb{Z}_{p^5})^{l_1}\bigoplus(\mathbb{Z}_{p^3})^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$ as in Proposition 9.0.3.*

Let G is an indecomposable $((1,2),p^5)$-group with regulator R and regulator quotient G/R is isomorphic to $(\mathbb{Z}_{p^5})^{l_1}\bigoplus(\mathbb{Z}_{p^3})^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$. Let $M = \left[I \mid \beta \mid \gamma\right]$ be the coordinate matrix of G. We want to find a complete list of indecomposable $(1,2)$-groups with the given regulator quotient. By Proposition 7.0.3 we know that if β-part of M is decomposable then G is decomposable. Hence we deal only with the section matrix β and check its decomposability. We will form the Smith Normal forms of the subblocks of β and while doing this if we get a direct then we say that either it leads to a contradiction or we check its class in the list given in Proposition. In this way we will find all indecomposable $((1,2),p^5)$ groups with regulator quotient isomorphic to $\mathbb{Z}_{p^5}\bigoplus\mathbb{Z}_{p^3}$.

Since we supposed that G is indecomposable, β not contain 0-rows, there can not be any 0-column in δ, and there can not be a cross in β by Proposition 7.0.4. Let $\beta = \left[\begin{array}{c} X \\ \hline Y \end{array}\right]$. There is no unit in X to avoid a cross. Hence we can write pX instead. Since the matrix Y is in the p-block, the entries of Y are units,zero pI or p^2I. We successively form Smith Normal form of the sub-block X to split out the parts p^4I

, p^3I , p^2I and pI. Note that there is no zero column and no zero row in β to avoid direct summands. Then we can write β as follows

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline A_1 & A_2 & A_3 & A_4 & A_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^3 \end{array}$$

There is no zero column in A_5 due to the reason that there is no zero column in β. With a unit in A_5 we can annihilate the entries in A_1 , A_2 , A_3 and A_4 this will lead to a direct summand of rank 2. Hence we execute the units in A_5. Therefore A_5 is of the form pA_5 and hence we get β as

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline A_1 & A_2 & A_3 & A_4 & pA_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \end{array}$$

If there is a unit in A_1 by Gauss elimination upwards we can annihilate in p^4I and then by basis transformation we annihilate the entries in the same row as this unit in A_2, A_3 and A_4. But so we get a cross and so a direct summand. Hence we may write pA_1.

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_1 & A_2 & A_3 & A_4 & pA_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \end{array}$$

With a unit in A_2 we first apply Gauss elimination upwards to annihilate p^3I in the second block row. Then by basis transformation we can annihilate the corresponding entries in A_1, A_2, A_3, A_4 and A_5. This causes to a cross in β. Hence A_2 is assumed to have the form pA_2.

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_1 & pA_2 & A_3 & A_4 & pA_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \end{array}$$

If there is a unit in A_3, then we get a cross. The same holds for the block matrices A_1 and A_2. So we write pA_1 and pA_3, instead. Hence β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_1 & pA_2 & pA_3 & A_4 & pA_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \end{array}$$

In the block marix A_4 there are only units and zeros due to pI above in p^5-Block. Otherwise we can apply Gauss elimination upwards and get a cross. There is no zero column in A_4 to avoid a cross with a cross point in p in p^5-Block. Hence the Smith Normal form of A_4 is $A_4 = \begin{bmatrix} I \\ 0 \end{bmatrix}$.

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_{11} & pA_{21} & pA_{31} & I & pA_{51} \\ pA_{12} & pA_{22} & pA_{32} & 0 & pA_{52} \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

The submatrix pA_{31} can be annihilated by I on the right. The fill-ins left to pI in the p^5-block can be annihilated by p^2 above in p^5-Block.Hence $pA_{31} = 0$.

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_{11} & pA_{21} & 0 & I & pA_{51} \\ pA_{12} & pA_{22} & pA_3 & 0 & pA_{52} \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

With a p in pA_{12} we can annihilate in pA_{11} and the corresponding entries of p^4I above. Then we can annihilate in the block row of pA_{12}-block. This operations will cause a cross and so to a direct summand. Hence we conclude that the entries of pA_{12} are in $p^2\mathbb{Z}$ and we write p^2A_{12}. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_{11} & pA_{21} & 0 & I & pA_{51} \\ p^2A_{12} & pA_{22} & pA_3 & 0 & pA_{52} \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

If there is a p in pA_{22}, then this leads to cross. So we assume p^2A_{22}. Similarly a p in pA_{52} leads to a cross and so to a direct summand. Hence we assume p^2A_{52}.

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_{11} & pA_{21} & 0 & I & pA_{51} \\ p^2A_{12} & p^2A_{22} & pA_3 & 0 & p^2A_{52} \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

A p in pA_{51} leads to a summand (iv) listed in Proposition 9.0.3. With a p in pA_{51} we first annihilate in p^2A_{52}. This results to fill-ins in the fourth block column below I which are in $p\mathbb{Z}$. However these fill-ins can be removed by pI above in p^5-block.

Omitting this summand we may assume that p^2A_{51}. So we β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline pA_{11} & pA_{21} & 0 & I & p^2A_{51} \\ p^2A_{12} & p^2A_{22} & pA_3 & 0 & p^2A_{52} \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

With a p in pA_{11} we first annihilate in p^2A_{12}. This results to fill-ins below I in p^3-Block but this fill-ins can be removed by pI above in p^5-Block. Then we annihilate with this p in pA_{11} in pA_{21} and in p^2A_{51}. This causes fill-ins in the first block row right to p^4I which are in $p^4\mathbb{Z}$ and $p^5\mathbb{Z}$ respectively. The first fill-in can be annihilated by p^3I below and the other are in $p^5\mathbb{Z}$ and hence be ignored. Next we annihilate with this p in pA_{11} above in p^4I. The fill-ins in the first block row can be removed by pI in the fourth block row. Hence we get a direct summand of (iv) listed in the Proposition 9.0.3. Omitting this summand we may assume that p^2A_{11}. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline p^2A_{11} & pA_{21} & 0 & I & p^2A_{51} \\ p^2A_{12} & p^2A_{22} & pA_3 & 0 & p^2A_{52} \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

We know that pA_3 has no entries in $p^2\mathbb{Z}$ due to the p^2I above. Moreover, there is no 0 column in pA_3 otherwise we get a cross. Hence the Smith Normal form of pA_3 is

pA_3= $\begin{bmatrix} pI \\ 0 \end{bmatrix}$. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline p^2A_{11} & pA_{21} & 0 & I & p^2A_{51} \\ p^2A_{12} & p^2A_{22} & pI & 0 & p^2A_{52} \\ p^2A_{13} & p^2A_{23} & 0 & 0 & p^2A_{53} \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \\ p^3 \end{array}$$

With a p^2 in p^2A_{53} we first annihilate in its row in p^2A_{23} and p^2A_{13}. Then we annihilate above in p^2A_{52} and in p^2A_{51}. This leads to a cross hence we assume p^2A_{53}=0 since it is located in p^3-Block.

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline p^2A_{11} & pA_{21} & 0 & I & p^2A_{51} \\ p^2A_{12} & p^2A_{22} & pI & 0 & p^2A_{52} \\ p^2A_{13} & p^2A_{23} & 0 & 0 & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \\ p^3 \end{array}$$

Thereafter with a p^2 in p^2A_{13} we annihilate in its row and then in its whole column and this leads to a cross and hence $p^2A_{13} = 0$ since A_{13} are in p^3-block. The block matrix p^2A_{22} can be annihilate by pI on its right but then the resulting fill-ins in the third block row are in $p^3\mathbb{Z}$ and can be removed by p^3I above it. Hence $p^2A_{22} = 0$. Thus we get

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline p^2A_{11} & pA_{21} & 0 & I & p^2A_{51} \\ p^2A_{12} & 0 & pI & 0 & p^2A_{52} \\ 0 & p^2A_{23} & 0 & 0 & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \\ p^3 \end{array}$$

With a p^2 in p^2A_{52} we annihilate in p^2A_{12} and then we annihilate in p^2A_{51} but this results to fill-ins in $p\mathbb{Z}$ in the fifth block row below p^2I in p^5-block. We annihilate this fill-ins by I on the right to it and then this result again fill-ins in the fourth block row left to pI which are in $p^2\mathbb{Z}$. This fill-in can be removed by p^2I in the third block row. Hence we get a direct summand (vi) listed in Proposition 9.0.3. Omitting this summand we may assume p^2A_{52}=0. Thus β is transformed to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline p^2A_{11} & pA_{21} & 0 & I & p^2A_{51} \\ p^2A_{12} & 0 & pI & 0 & 0 \\ 0 & p^2A_{23} & 0 & 0 & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \\ p^3 \end{array}$$

With a p^2 in p^2A_{12} we annihilate first in p^2A_{11}. The fill-ins can be removed by the same reasons like in the above paragraph. Then we can annihilate in p^4I in the first block row. The resulted fill-ins can be removed by p^2I in the third block row. This leads to (vi) listed in Proposition 9.0.3. Omitting this summand we may assume that $p^2A_{12} = 0$ since A_{12} is in the p^3-block. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline p^2A_{11} & pA_{21} & 0 & I & p^2A_{51} \\ 0 & 0 & pI & 0 & 0 \\ 0 & p^2A_{23} & 0 & 0 & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \\ p^3 \end{array}$$

Now we can read another summand (ii) listed in Proposition 9.0.3 with pivots in the third and in the sixth block rows. Omitting this summand we can also omit the third block column and third block row and sixth block row. Thus we get

$$\beta = \left[\begin{array}{cccc} p^4I & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 \\ 0 & 0 & pI & 0 \\ \hline p^2A_{11} & pA_{21} & I & p^2A_{51} \\ 0 & p^2A_{23} & 0 & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

There is no zero column in p^2A_{51} to avoid a direct summand. The Smith Normal form of A_{51} is $A_{51} = \begin{bmatrix} p^2I \\ 0 \end{bmatrix}$. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & pI & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ \hline p^2A_{11} & pA_{21} & I & 0 & p^2I \\ p^2A_{12} & pA_{22} & 0 & I & 0 \\ 0 & p^2A_{23} & 0 & 0 & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \\ p^3 \end{array}$$

By p^2I in the fifth block row the submatrix p^2A_{11} can be annihilated. Assume that there is a p in pA_{22}. With this p we first annihilate in p^2A_{23}. The fill-ins in the last block row are in $p^3\mathbb{Z}$ and in $p\mathbb{Z}$, respectively. They can be removed since we are in p^3-block and pI above in the fourth block row, respectively. Next we annihilate with the p in pA_{22} in p^2A_{11}. The fill-ins in the second block row and in the fifth block row are $p^4\mathbb{Z}$ and $p^2\mathbb{Z}$, respectively and they can be removed be p^4I in the first block row in the fifth row on the right, respectively. Then we annihilate with this pivot p in pA_{21}. The resulted fill-ins in the fifth block row can be removed by I on the left. This causes again to fill-ins the third block row and they can be removed by pI in the fourth block row. Therefore with a p in pA_{22} we annihilate the corresponding entry in p^3I in the second block row. The fill-ins in the second block row are $p^2\mathbb{Z}$ and can be removed by pI in the fourth block row. This leads to a summand (iv) listed in Proposition 9.0.3. Omitting this summand we may assume that p^2A_{22} but then this can be annihilated by I on the right hence $pA_{22} = 0$. Then a p^2 in p^2A_{11} leads to a summand (v) listed in Proposition 9.0.3. Omitting this summand we may assume that p^2A_{11}. This p^2A_{11} can be annihilated by pI in the same row on the right. Hence set $p^2A_{11} = 0$. So we get a direct summand (i) listed in Proposition 9.0.3. Thus β changes to

$$\beta = \left[\begin{array}{cccc} p^4I & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 \\ 0 & 0 & pI & 0 \\ \hline 0 & pA_{21} & I & p^2I \\ 0 & p^2A_{23} & 0 & 0 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

With a p in pA_{21} we first annihilate in p^2A_{23}. The resulting fill-ins are in $p\mathbb{Z}$ and in $p^3\mathbb{Z}$ can be removed by pI in third block row and can be neglected since we are in the p^3-block. This leads to a direct summand (iv) listed in Proposition 9.0.3. Omitting this summands we may assume that p^2A_{21}. Hence β is changed to

$$\beta = \left[\begin{array}{cccc} p^4I & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 \\ 0 & 0 & pI & 0 \\ \hline 0 & p^2A_{21} & I & p^2I \\ 0 & p^2A_{23} & 0 & 0 \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ \hline p^3 \\ p^3 \end{array}$$

The submatrix p^2A_{21} can be annihilated by p^2I on the right in p^3-block. Then we get a summand (v) listed in Proposition 9.0.3. Omitting this summand the third block column, the third and and the fourth block rows do not exist. Hence β changes to

$$\beta = \left[\begin{array}{cc} p^4I & 0 \\ 0 & p^3I \\ \hline 0 & p^2A_{23} \end{array}\right] \quad \begin{array}{c} p^5 \\ p^5 \\ \hline p^3 \end{array}$$

The first block row and the first block column are not present to avoid a cross. Moreover, there is neither a zero column nor zero row in p^2A_{23} to avoid direct summands. Hence the Smith Normal form of p^2A_{23} is p^2I. This leads to a direct summand (iii) listed in Proposition 9.0.3.

Theorem 10.0.4 *There is no indecomposable $(1, 2)$-group with regulator quotient isomorphic to $(\mathbb{Z}_{p^5})^{l_1} \bigoplus (\mathbb{Z}_{p^4})^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$.*

Proof 14 *Let G be a $((1,2),p^5)$-group with regulator R and regulator quotient G/R is isomorphic to $(\mathbb{Z}_{p^5})^{l_1} \bigoplus (\mathbb{Z}_{p^4})^{l_2}$ where $l_1 \geq 1$ and $l_2 \geq 1$. Let $M = \begin{bmatrix} I & \mid & \beta & \mid & \gamma \end{bmatrix}$ be the coordinate matrix of G. Assume that G is indecomposable. By Proposition 7.0.3 for the decomposability of M it is sufficient to check the decomposability of the β-section M. Since we supposed that G is indecomposable, β not contain 0-rows, there can not be any 0-column in M, and there can not be a cross in β. Let $\beta = \begin{bmatrix} X \\ \hline Y \end{bmatrix}$. There is no unit in X to avoid a cross. Hence we can write pX instead. Since the matrix Y is in the p-block, the entries of Y are units,zero pI, p^2I or p^3I. We form Smith Normal form of the sub-block X to split out the parts p^4I , p^3I , p^2I and pI. Note that there is no zero column and no zero row in β to avoid direct summands. Thus we can write β as follows*

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline A_1 & A_2 & A_3 & A_4 & A_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^4 \end{array}$$

There is no unit in A_1, A_2, A_3, A_4 and A_5 to avoid crosses.Hence we write pA_1, pA_2, pA_3, pA_4 and pA_5.

The submatrix pA_4 can be annihilated by pI above in the p^5-block. A p in pA_1 leads to a cross. Hence we set p^2A_1 instead. Similarly a p in pA_5 leads to a cross.So we write p^2A_5. Hence β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline p^2A_1 & pA_2 & pA_3 & 0 & p^2A_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^4 \end{array}$$

If there is a p in pA_2 or in pA_3 then we obtain a cross. Hence we assume p^2A_2 and

p^2A_3. But then p^2A_3 can be annihilated by p^2I above in p^5-block.Thus we get

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline p^2A_1 & p^2A_2 & 0 & 0 & p^2A_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^4 \end{array}$$

A p^2 in p^2A_5 or a p^2 in p^2A_2 leads to a crosses so we set p^3A_2 and p^3A_5. The same holds for p^2A_1. Set p^3A_1. With a p^3 in p^3A_1 we annihilate in the block row and in the block column and get a cross. Hence we can assume that $A_1 = 0$. Thus β changes to

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & p^3A_2 & 0 & 0 & p^3A_5 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^4 \end{array}$$

The submatrix p^3A_2 can be annihilated by p^3I above in p^5-block. This results to no fill-ins. Hence p^3A_2=0. With a p^3 in p^3A_5 we get a cross. So we may assume that p^3A_5=0.

$$\beta = \left[\begin{array}{ccccc} p^4I & 0 & 0 & 0 & 0 \\ 0 & p^3I & 0 & 0 & 0 \\ 0 & 0 & p^2I & 0 & 0 \\ 0 & 0 & 0 & pI & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^5 \\ p^4 \end{array}$$

There is a 0-row in p^4-block and this shows that there is no indecomposable group with the given regulator quotient.

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