Nash bargaining solution under pre-donation and collusion in a duopoly*

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Abstract
In this paper, the manipulability of the Nash bargaining solution through pre-donation is investigated focusing on the two-person bargaining problems with triangular bargaining sets and arbitrary threat points. When pre-donation is given from total payoffs and the greater ideal payoff is high enough, then the lucky bargainer (the bargainer with the greater ideal payoff) has an incentive to make a donation to her opponent and, as long as her threat payoff is different from zero, both bargainers obtain higher payoffs compared to the outcome without pre-donation. When pre-donation is given from the excess of the threat payoffs, there is always an incentive for pre-donation, however, only the lucky bargainer gains from pre-donation. For the duopolistic collusion problem, the efficient firm (with lower marginal cost) has an incentive to give a share of its profit to its rival only if the share is given from its excess threat profit under all Cournot, Bertrand and Stackelberg (under the leadership of the efficient firm) threats.

1. Introduction
Sertel (1992) shows that the Nash solution of the two-person bargaining problem is manipulable by instituting a pre-bargaining stage in which the lucky bargainer gives a unilateral donation of payoffs to accrue in the next stage, thereby changing the bargaining set. He also shows that the Nash bargaining solution of the new set distributes the payoffs according to the Talmudic division rule when the threat point is at the origin and the utility possibility frontier is linear. He calls this type of bargaining problem 'simple'. In this study, the pre-donationally manipulated Nash bargaining solution is examined in a more general setting of T-simple bargaining problems where the threat point is no longer confined to the origin. The results show that with pre-donations from

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total payoffs, both bargainers become happier when the bargaining set is pre-
donationally manipulated, barring the case where the lucky bargainer has zero
payoff at the threat point, in which case only this player gains due to pre-
donation. However, under this scenario, pre-donation is possible only when the
ideal payoff of the lucky player is sufficiently high. According to the second
scenario, pre-donation is assumed to be given from the excess of threat payoffs.
Results show that while there is always an incentive for pre-donation, only the
lucky player who makes the donation gains from it. The other player is
indifferent between the outcome of the Nash bargaining solution, with and
without pre-donation. It is important to note here that although the nature of the
environment seems to be cooperative, it is, in fact, not. Consequently, instead of
considering all feasible bargaining problems, we restrict the analysis to
bargaining problems with incentive compatible transfers. In other words,
although the payoffs are assumed to be physically transferable, the transfers are
voluntary and subject to the approval of the giver. For this reason, they are kept
out of the bargaining game and are considered in the pre-donation stage. Players
bargain over the pie, and payoffs, which are obtained by utilizing individual
skills, are transferred only when there is an incentive to do this. We believe this
model of payoff transfers better represents many real life situations than the
models in which a division somehow happens without any reference to property
rights.

Collusion in a duopolistic market is taken as one of the real life situations
that constitutes an example for our bargaining problem. There are a number of
studies constructing the collusion among firms as a bargaining problem
(Osborne and Pitchik, 1983; Osborne and Pitchik, 1987; Schmalensee, 1987;
and Harrington, 1989 and 1991)

1 Osborne and Pitchik (1983) examined collusion among symmetric duopolists with different
capacities, applying the Nash bargaining solution. Osborne and Pitchik’s (1987) analysis is similar to
their 1983 study, but in their later paper they also examined the stage at which firms choose their
capacities. They applied the Nash bargaining solution to their collusion problem in a duopoly
assuming that threats are to cut prices rather than to increase output (the latter is the form of threat
found in Osborne and Pitchik (1983)). Harrington (1989) investigated collusion among asymmetric
firms (with asymmetry due to different discount factors for the firms), arguing that the bargaining set
should consist of the outcomes which are self-enforceably implementable. He applied the Nash
bargaining solution to select the collusive outcome in the bargaining set restricted to the self-
enforceingly implementable outcomes. Using the same approach, Harrington (1991) applied the Nash
bargaining solution as a selection criterion for the collusion game between duopolists with different
cost functions instead of different discount factors.

2 Schmalensee (1987) showed that when firms share the market without geographically dividing it, the
bargaining set, which is defined as the set of feasible profits, is not convex.
the market can be divided by duopolists, assigning each consumer to a single
firm as in one of Schmalensee’s (1987) scenarios. We apply the pre-donationally
manipulated Nash bargaining solution to the collusion problem of our duopolists
under both pre-donation scenarios mentioned above. When pre-donation is
assumed to be a share given from the total profit, there is no incentive for the
efficient firm to make such a donation for any of the threat points examined
here, which are specified as Cournot, Bertrand and Stackelberg equilibria (under
the leadership of the efficient firm). However, when the share is given from
excess of the threat profits, there is always an incentive for pre-donation, but
only the efficient firm gains from pre-donation. The percentage gain for the
efficient firm is highest under the Cournot threat.

This paper is organized as follows. T-simple bargaining problems are defined
in Section 2 and their pre-donationally manipulated Nash bargaining solutions
are obtained under two different pre-donation scenarios in Section 3. In Section
4, collusion in a duopoly is analyzed using this concept. Specifying the threat
point as the Cournot, Bertrand and Stackelberg equilibria, Nash bargaining
solutions of this collusion problem are examined for both pre-donation scenarios
described in Section 3. Concluding remarks are presented in Section 5.

2. Simple bargaining problems with arbitrary threat points

A simple bargaining problem (b.p.) is defined formally in Sertel (1992) as
\[ S_a = \{ (u_1, u_2) \in \mathbb{R}_+^2 \mid u_2 \leq a(1 - u_1) \} \] for some real number \( a \geq 1 \). Here we
extend the problem to cases where the threat point need no longer be fixed at the
origin. In what follows, the class of simple bargaining problems is redefined to
allow for a threat point through a shift of origin.

**Definition 1**: A b.p. \( S \) is called T-simple iff \( S = S_{a,b} \) for some \( a \geq 1 \) and \( b \geq 0 \), where
\[ S_{a,b} = \{ (u_1, u_2) \in \mathbb{R}_+^2 \mid u_2 \leq a(1 - u_1) - b \} \]

**Definition 2**: For a T-simple b.p. the Nash bargaining solution, \( N \), is the
function which picks the point maximizing the Nash social welfare function
\( W = u_1 u_2 \). Thus, the Nash bargaining solution for \( S_{a,b} \) is
\[ N(S_{a,b}) = (\frac{(a - b)}{2a}, \frac{(a - b)}{2}) \]

A b.p. with a linear utility possibility frontier and an arbitrary threat point can
be transformed into a T-simple b.p. by shifting the origin to the threat point.
Consider a b.p. where two bargainers divide among themselves an item for
which they may have different monetary valuations. The one with higher
valuation is called the lucky player and the other one is the unlucky player. In case of disagreement, they obtain exogenously given threat payoffs. Let us denote the share of the item received by player 1 as \( \rho \). The utilities of the bargainers can be written as

\[
\begin{align*}
  u_1 &= \rho \\
  u_2 &= \alpha(1 - \rho).
\end{align*}
\]

The corresponding b.p. is

\[
S = \{ u \in \mathbb{R}_+^2 | u_2 \leq \alpha(1 - u_1) \text{ and } u \geq d \}
\]

(2)

where \( d = (d_1, d_2) \in \mathbb{R}_+^2 \) is the threat point and \( u = (u_1, u_2) \geq d \) means that \( u_i \geq d_i \) for \( i \in \{1, 2\} \). Transforming the origin to the threat point, this b.p. can be written in the form of a \( T \)-simple b.p. as follows

\[
S_{\alpha,b} = \{ u \in \mathbb{R}_+^2 | u_2 \leq \alpha(1 - u_1) - b \}
\]

(3)

where \( a = \alpha \) and \( b = \alpha d_1 + d_2 \).

3. Pre-donationally manipulated bargaining problems

Sertel (1992) showed that for any simple b.p. \( S_\alpha \), there is an incentive for the lucky bargainer to make a pre-donation under the Nash bargaining solution. He also proved that the Nash bargaining solution manipulated via the lucky bargainer’s optimal pre-donation results in a Talmudic division. According to the Talmudic division, the disputed portion of the pie, which corresponds to the entire claim of unlucky bargainer, is split equally between the two bargainers, while the undisputed portion is given to a single claimant: the lucky bargainer. In the present paper, we extend the b.p. with pre-donation possibilities to one with an arbitrary threat point. A pre-donation is understood to be the share of the lucky player’s payoff given to the unlucky player.

3.1. Pre-donation as a share from the total payoff

According to our first scenario, a pre-donation is assumed to be a share of the lucky player’s total payoff, rather than the excess of this payoff above \( d_2 \). We denote \([0,1]\) as the share in \( u_2 \) which is unilaterally donated to Player 1 (the unlucky player) before bargaining begins. The pre-donation transforms each payoff pair \((u_1, u_2)\) into \((u_1', u_2') = (u_1 + \lambda u_2, (1 - \lambda)u_2')\), which is equal to \((\rho + \lambda(1 - \rho)\alpha, (1 - \lambda)(1 - \rho)\alpha)\). The corresponding b.p. can be written as \((S^\lambda, d^\lambda)\) where
\[ S^A = \{ (u_i^A, u_j^A) \in \mathbb{R}_+^2 \mid u_i^A \leq \frac{(1-\lambda)\alpha}{1-\lambda\alpha} (1-u_i^A), u_j^A \leq \alpha(1-\lambda) \text{ and } d^A \leq u_j^A \}. \] (4)
\[ d^A = (d_1^A, d_2^A) \in \mathbb{R}_+^2, d_1^A = d_1 + \lambda d_2, \text{ and } d_2^A = (1-\lambda)d_2. \] (The set given in (4) is the comprehensive closure \( f_A^*(S) \) of the pre-donationally manipulated form \( f^A(S) \) of the bargaining set \( S \) (see Sertel, 1992) where \( f_A^* : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+^2 \) is the transformation defined through \( f_A^*(u_i, u_j) = (u_i + \lambda u_i, (1-\lambda)u_i) (\lambda \in [0,1]) \).

Shifting the origin to the threat point, the b.p. \((S^A, d^A)\) becomes the b.p. \((S^A_{\star,0}, 0)\) as determined by
\[ S^A_{\star,0} = \{ (u_i^\star, u_j^\star) \in \mathbb{R}_+^2 \mid u_i^\star \leq a'(1-u_i') - b', u_j^\star \leq (1-\lambda)\alpha - d_j^\star \} \] (5) where
\[ u_i' = u_i^\star - d_i^\star, u_j' = u_j^\star - d_j^\star, a' = \frac{(1-\lambda)\alpha}{1-\lambda\alpha}, \text{ and } b' = d_j^\star \frac{(1-\lambda)\alpha + d_j^\star (1-\lambda\alpha)}{1-\lambda\alpha}. \]
(Note that this b.p., without the feasibility constraint \( u_j' \leq (1-\lambda)\alpha - d_j^\star \), can be considered to be a T-simple b.p. \( S^A_{\alpha,0} \). The Nash bargaining solution for this b.p. is
\[ N(S^A_{\alpha,0}) = \frac{\alpha(1-d_i') - d_j'}{2\alpha}, (1-\lambda)(\alpha(1-d_i') - d_j') \] if \( \lambda \leq \lambda_\alpha \)
\[ (\lambda(\alpha - d_i') - d_j'(1-\lambda)(\alpha - d_i')) \] if \( \lambda < \lambda \leq \frac{\alpha-d_i}{\alpha} \) (6)
where \( \lambda_\alpha = \frac{\alpha(1+d_i') - d_j'}{2\alpha(\alpha - d_i')} \). Note that as long as \( \lambda \leq \lambda_\alpha \).

\( S^A_{\alpha,0} \subset S^A_{\star,0} \) and \( N(S^A_{\alpha,0}) \subset S^A_{\star,0} \). Since the Nash bargaining solution satisfies what Nash called the “independence of irrelevant alternatives” axiom, we thus have \( N(S^A_{\alpha,0}) = N(S^A_{\star,0}) = (a'(1-2b') - b')/2(a'(1-2b') - b') / 2 \) for all \( \lambda \leq \lambda_\alpha \).

For the remaining case where \( \lambda > \lambda_\alpha \), the slope of the Nash social welfare function \( a'(1-2u_i') - b' \) w.r.t. \( u_i' \) is negative, so its maximum in this region of \( \lambda \) is attained at \( u_i' = \lambda(\alpha - d_i') - d_i' \), which is the smallest value of \( u_i' \) in the region. (It should also be noted that for the values of \( \lambda \) greater than \( (\alpha - d_i')/\alpha \), \( u_i' \) is less than \( d_i' \) and, thus, the bargaining set is empty.)
Knowing that the b.p. will result in the above payoffs, Player 2 (the lucky player) chooses the share, $\lambda$, to be given to Player 1. The following Proposition gives the optimal share from the viewpoint of Player 2 and the outcome of the Nash bargaining solution.

**Proposition 1**

Given a $T$-simple b.p. $S_{a,b}$, where $b = ad_1 + d_2$ and $d = (d_1, d_2)$ is the threat point, Player 2’s optimal pre-donation from her total payoff is

$$\lambda^*(\alpha, d) = \begin{cases} \frac{\alpha(1 + d_1) - d_2}{2\alpha(d_1 - d_2)} & \text{if } \alpha \geq 1 + d_2 \\ 0 & \text{otherwise} \end{cases}$$

and the resulting Nash bargaining solution of the pre-donationally manipulated $T$-simple b.p. is

$$N(S_{a,b}^\lambda) = \begin{cases} \frac{\alpha(1-d_1)-d_2+2d_1d_2}{2(\alpha-d_2)} & \text{if } \alpha \geq 1 + d_2 \\ \frac{\alpha(1-d_1)-d_2}{2\alpha} & \text{otherwise} \end{cases}$$

**Proof:** Player 2 chooses $\lambda$ to maximize her own payoff. Since the threat point also moves by pre-donational manipulation of the bargaining set, the scale is different for each different $\lambda$. Thus, in order to obtain the optimal $\lambda$, we need to transform the payoffs given in (6) into the original scale. As a result of this transformation we obtain the following payoff for Player 2

$$N_2(S_{a,b}^\lambda) + d_2^\lambda = \tilde{u}_2^\lambda = \begin{cases} (1-\lambda)(\alpha(1-d_1)+d_2(1-2\alpha\lambda)) & \lambda < \lambda_1 \\ \frac{2(1-\alpha\lambda)}{\alpha(1-\lambda)} & \text{otherwise} \end{cases}$$

For $\lambda \leq \lambda_1$, the derivative of $\tilde{u}_2^\lambda$ w.r.t. $\lambda$ is

$$\frac{\partial \tilde{u}_2^\lambda}{\partial \lambda} = \frac{(\alpha^2 - \alpha)(1-d_1) - d_2(1+\alpha) + 4\alpha d_2 \lambda - 2\alpha^2 d_2 \lambda^2}{2(\lambda\alpha - 1)^2}.$$

This derivative vanishes at

$$\lambda_1 = \frac{1}{\alpha} - \frac{\sqrt{(\alpha-1)(\alpha(1-d_1)-d_2)}}{\alpha\sqrt{2d_2}}$$

and $\lambda_2 = \frac{1}{\alpha} + \frac{\sqrt{(\alpha-1)(\alpha(1-d_1)-d_2)}}{\alpha\sqrt{2d_2}}.$
The second-order derivative of $\hat{u}^2_2$ w.r.t. $\lambda$ is positive at $\lambda_1$ and negative at $\lambda_2$. It can easily be seen that $\lambda_2$ is greater than $\lambda_1$ so that it is out of the region. Since the second-order derivative is positive, there is a minimum at $\lambda_1$. Thus, in order to obtain the local maximum of the region, we only need to compare the values of $\hat{u}^2_2$ at $\lambda = 0$ and $\lambda = \lambda_1$. This comparison shows that the maximum is attained at $\lambda_1$ as long as $\alpha \geq 1 + d_2$ and $\lambda = 0$ otherwise. Since $\frac{\partial \hat{u}^2_2}{\partial \lambda}$ is always negative for the region where $\lambda > \lambda_1$, $\hat{u}^2_2$ has again a maximum at $\lambda_1$. Thus, the global maximum is at $\lambda_1$ for $\alpha \geq 1 + d_2$ and at $\lambda = 0$ otherwise. Substituting $\lambda_1$ into $N(S^2_{a,b})$ we obtain $N(S^2_{a,b})$. This completes the proof.

(It must be noted that the Nash bargaining solution under pre-donation is no longer Talmudic when an arbitrary threat point is introduced and the pre-donation share in Player 2’s payoff is taken as a share in her total payoff $u_2$ rather than in her excess payoff $u_2 - d_2$ above $d_2$.) The following proposition gives a comparison between pre-donationally manipulated and unmanipulated Nash bargaining solutions.

**Proposition 2**

Given a $T$-simple b.p. $S^b_a$, where $b = ad_1 + d_2$ and $d = (d_1,d_2)$ is the threat point with $d_2 \neq 0$ as long as there is an incentive for a pre-donation from the total payoff (i.e. $\alpha \geq 1 + d_2$), the pre-donationally manipulated Nash bargaining solution yields a higher payoff for both bargainers than the Nash bargaining solution of the unmanipulated b.p.

**Proof:** Given that $\lambda > 0$ is an optimal share for Player 2 to pre-donate, for any $\alpha \geq 1 + d_2$ her payoff is no less than at $\lambda_1$ than at $\lambda = 0$. Thus, we only need to compare the pre-donationally manipulated Nash bargaining solution payoff of Player 1, $N^1(S^{\lambda}_a)$, with his payoff under no pre-donation. This comparison (see equation (8)) is given by

$$\frac{\alpha(1 + d_1) - d_2}{2\alpha} < \frac{\alpha(1 + d_1) - d_2}{2(\alpha - d_2)}.$$
where the left hand side is the unmanipulated payoff and the right hand side is the pre-donationally manipulated payoff at $\lambda = \lambda_c$. Since $2\alpha > 2(\alpha - d_2)$ and $d_2 \neq 0$, the above inequality is always satisfied. This completes the proof.

3.2 Pre-donation as a share from the excess of the threat payoff:

In this section, pre-donations are assumed to be given from the excess of the payoff of Player 2 above her threat payoff. Thus, each payoff pair of $(u_1, u_2)$ is transformed into $(\bar{u}_1^\lambda, \bar{u}_2^\lambda) = (u_1 + \lambda (u_2 - d_2), d_2 + (1 - \lambda)(u_2 - d_2))$, which corresponds to $(\rho + ((1 - \rho)\alpha - d_2)\lambda, d_2 + ((1 - \rho)\alpha - d_2)(1 - \lambda))$. The corresponding b.p., after carrying the origin to the threat point, can be written as $\bar{S}_{\alpha, b}^\lambda$ where

$$\bar{S}_{\alpha, b}^\lambda = \{(u_1^\lambda, u_2^\lambda) \in \mathbb{R}^2_+ | \bar{u}_1^\lambda - \bar{u}_2^\lambda \leq (1 - \bar{u}_1^\lambda) - \bar{b}, \bar{u}_2^\lambda \leq (\alpha - d_2)(1 - \lambda)\} \quad (10)$$

Note that when the feasibility constraint $\bar{u}_2^\lambda \leq (\alpha - d_2)(1 - \lambda)$ is not binding, the above bargaining problem turns out to be a $T$-simple b.p. and it can also be easily transformed into a simple b.p. by changing the scale. Nevertheless, since the definition of $\lambda$ is different, it is not possible to apply the results obtained in the previous section directly to the b.p. $\bar{S}_{\alpha, b}^\lambda$. Moreover, when this b.p. is expressed in the form of a simple b.p., the feasibility constraint is not the same as the case where the threat point is at the origin, so, Sertel (1992)'s results are not directly applicable either. Therefore, computation of the Nash bargaining solution of $(\bar{S}_{\alpha, b}^\lambda, 0)$ requires some additional calculations, rendering

$$N(\bar{S}_{\alpha, b}^\lambda) = \begin{cases} \left( \frac{\alpha(1 - d_1) - d_2}{2\alpha}, \frac{(1 - \lambda)(\alpha(1 - d_1) - d_2)}{2(1 - \alpha \lambda)} \right) & \text{if } \lambda \leq \lambda_c \\ \left( \frac{\lambda(\alpha - d_2) - d_1, (1 - \lambda)(\alpha - d_2)}{2(1 - \alpha \lambda)} \right) & \text{if } \lambda_c \leq \lambda \end{cases} \quad (11)$$

It must be noted here that Player 1's payoff does not depend on $\lambda$; so he stands to gain nothing and lose nothing from pre-donation.
Proposition 3

Given a T-simple b.p. \( S_{ab} \), where \( b = ad_1 + d_2 \) and \( d = (d_1, d_2) \) is the threat point, Player 2's optimal pre-donation from the excess of her threat payoff is

\[
\lambda^*(\alpha, d) = \frac{\alpha(1 + d_1) - d_2}{2\alpha(\alpha - d_2)}
\]

and the resulting Nash bargaining solution of the pre-donationally manipulated T-simple b.p. is

\[
N(\overline{S}_{ab}) = \left( \frac{\alpha(1 - d_1) - d_2}{2\alpha}, \frac{2\alpha^2 - \alpha(1 + d_1) - d_2(2\alpha - 1)}{2\alpha} \right)
\]

Proof: It can easily be seen from Equation (11) that

\[
\frac{\partial N_1(\overline{S}_{ab})}{\partial \lambda} > 0 \text{ when } \lambda < \lambda_t \quad \text{and} \quad \frac{\partial N_2(\overline{S}_{ab})}{\partial \lambda} \leq 0 \text{ otherwise.}
\]

Therefore, the payoff-maximizing share for Player 2 from his excess threat payoff is \( \lambda_t \).

Substituting \( \lambda^* \) into \( N(\overline{S}_{ab}) \) we obtain \( N(\overline{S}_{ab}) \). This completes the proof.

The above proposition establishes that when pre-donations are shares given from excess of the threat payoff, there is always an incentive for Player 2 to manipulate the b.p. by pre-donation and the resulting outcome is always efficient.

3.3. On the generalization of the two-person bargaining games under pre-donation

The model considered above presents the manipulability of the Nash bargaining solution in a linear setting as in Sertel (1992), thus enabling a direct comparison of the results. It should, however, be noted that manipulability is not restricted to this specific case. In this section, our purpose is to provide an intuition for extending these results to a general bargaining problem.

Without loss of generality, assume that bargaining is still represented as a linear pie division problem, but the payoffs (utilities) can be arbitrary. Let \( f(\cdot) \) and \( g(\cdot) \) be the respective payoff functions for players 1 and 2. The feasible set is defined as follows.

\[
B = \{(u_1, u_2) | (u_1, u_2) = (f(x_1), g(x_2)) \text{ for some } (x_1, x_2) \in X\}
\]

where, \( X \) is the pie to be divided.
\( X = \{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 + x_2 \leq 1\} \). (15)

Letting \( d = (d_1, d_2) \) be the threat point, \((B, d)\) defines a general bargaining problem. We will also assume that \( f(\cdot) \) and \( g(\cdot) \) are continuous, increasing and concave over \([0, 1]\). These assumptions ensure that \( B \subset \mathbb{R}_+^2 \) is a compact and convex set. Hence there is a unique solution to the bargaining problem. Furthermore, since both functions are increasing, each player prefers increasing shares of the pie: i.e., \( x_i \) is preferred to \( x_i' \) iff \( x_i \geq x_i' \).

The possibility of a unilateral payoff transfer from one party to another obviously requires an asymmetry of the payoffs for a given share of the pie. Consider the trade where Player 1 gives a portion of his share of the pie to Player 2 in return for monetary compensation. Such a transaction is possible only if the gain of Player 2 is higher than Player 1 from the transfer of the share. In other words, Player 2 can offer a pre-donation if and only if marginal payoff of Player 2 is higher than the marginal payoff of Player 1. Since the bargaining solution is efficient,

\[ f'(x) \leq g'(1 - x), \text{ for all } x \in [0, 1] \] (16)

is a sufficient condition for pre-donation. It is clear that this condition is both necessary and sufficient for enlarging the bargaining set by a pre-donation from player 2, thus making payoffs to both parties at least as good as without pre-donation. This is, however, a global condition. Pre-donation will still be possible even if the above holds only locally at the bargaining solution.

Extending the above bargaining problem to the n-person case might also be considered. Intuitively, there will be room for Pareto improvement under pre-donation, if the marginal payoff of one party is sufficiently large and all other parties' marginal payoffs are equal. In general, pre-donations from multiple parties are possible and obviously present a complex game among the parties.

4. Collusion in a duopolistic market under pre-donation

We consider a duopolistic market where a homogeneous good is produced by two asymmetric firms, one with high cost and the other with low cost. The inverse demand is assumed to be affine with

\[ p(x) = a - x \] (17)

where \( a > 0 \) and \( x = x_1 + x_2, \ x_i \geq 0 \) is the output of Firm i, \( i \in \{1, 2\} \). The costs of Firm 1 and Firm 2 are, respectively,

\[ c_1(x_i) = c_1x_i, \] (18a)
\[ c_2(x_2) = c_2 x_2, \quad (18b) \]
where \( c_1 \) and \( c_2 \) are constant marginal costs and \( c_1 > c_2 \geq 0 \). In order to convexify the bargaining set, we assume that firms can divide the market\(^3\). For example, each firm can sell the product at its monopoly price in its own market regardless of the price in the other market. Therefore, the profits of Firm 1 and Firm 2 can be written as, respectively,

\[
P_1(\theta_1, \rho) = \rho \theta_1^2 / 4 \quad (19a)
\]
\[
P_2(\theta_2, \rho) = (1 - \rho) \theta_2^2 / 4 \quad (19b)
\]
where \( \theta_1 = a - c_1 \), \( \theta_2 = a - c_2 \) and \( \rho \) is a market dividing constant between zero and one\(^4\). For simplicity, equations \((19a)\) and \((19b)\) are normalized by taking \( \theta_i^2 / 4 \) as unit. Setting \( \eta_i = \theta_i^2 / \theta_i^2 \), the normalized profits are

\[
\pi_1(\alpha, \rho) = \rho, \quad (20a)
\]
\[
\pi_2(\alpha, \rho) = (1 - \rho) \eta_2^2, \quad (20b)
\]
Clearly, after specifying the threat point and carrying the origin to this point, the above collusion problem turns out to be a T-simple b.p., as defined above. We specify the possible threat points as the Cournot \((d^{c})\), Bertrand \((d^{b})\) and Stackelberg \((d^{s})\) equilibria (under the leadership of Firm 2). Formally, this b.p. can be defined as

\[
\Pi^d = \{ (\pi_1', \pi_2') \in \mathbb{R}_+^2 | \pi_2' \leq e(1 - \pi_1') - f \} \quad (21)
\]
where
\[
\pi_1' = \pi_1 - d_1, \quad \pi_2' = \pi_2 - d_2, \quad e = \eta_2^2 \text{ and } f = \eta_2^2 d_1 + d_2 \text{ and }
\]
\[
d = (d_1, d_2) \in \{ d^{c}, d^{b}, d^{s} \}. \text{ Table 1 gives the equilibrium outcomes at these threat points.}
\]

\(^3\) As noted by Schmalensee (1987), it is not unusual to observe this type of collusive behavior among firms. The main difficulty in dividing markets is the resulting arbitrage opportunity among the markets; i.e., the buyers in one market may switch to the other market if the price is lower there. However, the existence of such behavior suggests that it is possible to avoid arbitrage by certain division rules or other mechanisms, such as when markets can be divided geographically.

\(^4\) To continue from the previous footnote, the division of the market is governed by specific circumstances of the market. In this sense, all values may not be feasible. We ignore this difficulty in favor of preserving the convexity of the bargaining set.
Table 1

Profits \((\pi_1, \pi_2)\) A Different Threat Points

<table>
<thead>
<tr>
<th>Threat</th>
<th>((\pi_1, \pi_2))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cournot Threat</td>
<td>(\left(\frac{4(\eta^2 - 4\eta + 4)}{9}, \frac{4(4\eta^2 - 4\eta + 1)}{9}\right))</td>
</tr>
<tr>
<td></td>
<td>If (\eta &lt; 2)</td>
</tr>
<tr>
<td></td>
<td>Otherwise ((0, \eta^2))</td>
</tr>
<tr>
<td>Bertrand Threat</td>
<td>((0, 4(\eta - 1)))</td>
</tr>
<tr>
<td></td>
<td>If (\eta &lt; 2)</td>
</tr>
<tr>
<td></td>
<td>Otherwise ((0, \eta^2))</td>
</tr>
<tr>
<td>Stackelberg Threat</td>
<td>(\left(\frac{4\eta^2 - 12\eta + 9}{4}, \frac{4\eta^2 - 4\eta + 1}{2}\right))</td>
</tr>
<tr>
<td></td>
<td>If (\eta &lt; \frac{9}{4})</td>
</tr>
<tr>
<td></td>
<td>Otherwise ((0, \eta^2))</td>
</tr>
</tbody>
</table>

It can easily be seen from Table 1 that Firm 1 cannot obtain a positive profit unless \(\eta < 2\) at the Cournot and the Bertrand equilibria and unless \(\eta < \frac{9}{4}\) at the Stackelberg equilibria. Thus, Firm 2 is a monopoly above these levels of \(\eta\).

According to the above model, Firm 2 has an advantage since its marginal cost is lower than Firm 1’s marginal cost. Interpreting this advantage in a b.p. with pre-donation possibilities, it is possible to say that there may be an incentive for Firm 2 to offer a share of its future profit to its rival, knowing that the Nash bargaining solution will be the operant. This share offer can be considered under two different scenarios described in the previous section, one from the total profit and the other from the excess of the threat profit. When the pre-donation share is assumed to be given from the total profit, the incentive constraint \((\eta^2 \geq 1 + d_2)\) is never satisfied under any of the specified threat points. Thus, pre-donation will not occur in this case. However, when the pre-donation is taken as a share from excess of the threat profit, Firm 2 always offers a share to its rival. The outcomes of pre-donationally manipulated Nash bargaining solution for duopolists, under Cournot, Bertrand and Stackelberg threats, are given in Table 2.
Table 2
The Optimal Excess Threat Profit Pre-Donation Shares and the Collusive Outcomes at Different Threat Points

<table>
<thead>
<tr>
<th>Threat</th>
<th>( \text{Cournot} )</th>
<th>( \text{Bertrand} )</th>
<th>( \text{Stackelberg} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda )</td>
<td>( \frac{-4\eta^2 + 8\eta + 7\eta - 2}{2\eta(7\eta - 2)} )</td>
<td>( \frac{1}{2\eta^2} )</td>
<td>( \frac{4\eta^2 - 12\eta + 5\eta^2 + 8\eta - 2}{4\eta(2\eta - 4\eta - 1)} )</td>
</tr>
<tr>
<td>( \bar{\pi}_1, \bar{\pi}_2 )</td>
<td>( \frac{4\eta^2 - 16\eta + 9\eta^2 + 16\eta - 4}{18\eta^2} ), ( \frac{14\eta^2 - 30\eta + 29\eta^2 - 10\eta + 4}{18\eta^2} ) if ( \eta &lt; 2 )</td>
<td>( \frac{(\eta - 2)^2}{2\eta^2}, \frac{2\eta^2 - \eta^2 + 4\eta - 4}{2\eta} ) if ( \eta &lt; 2 )</td>
<td>( \frac{4\eta^2 - 12\eta + 5\eta^2 + 8\eta - 2}{8\eta^2} ), ( \frac{4\eta^2 + 12\eta^2 - 5\eta^2 - 8\eta + 2}{8\eta^2} ) if ( \eta &lt; \frac{3}{2} ), ( (0, \eta^*) ) otherwise</td>
</tr>
</tbody>
</table>

Table 2 shows that as long as Firm 1 is able to compete, there is always an incentive for Firm 2 to give a share of its excess threat profit to Firm 1. The resulting outcome is efficient: i.e., only the efficient firm, Firm 2, serves the market. We know from equation (11) that Player 1’s payoff is independent of the pre-donation share \( \lambda \), so that Firm 1’s gain is equal to its Nash bargaining solution profit without pre-donation. In other words, Firm 1 has no reason to reject the pre-donation. Table 3 shows some numerical results which help us compare the outcomes at different threat points.

Intuitively, as the relative efficiency of Firm 2 increases, both the optimal pre-donation share and the percentage gain of Firm 2 through pre-donation decreases for all threat points. In comparison, percentage gains for Firm 2 are highest under the Cournot threat and lowest under the Stackelberg threat. The reason is obvious: compared with the Cournot equilibrium, competition is less intense at the Stackelberg and Bertrand equilibria. As such, Firm 2’s profit is already high at these equilibria and the potential gain by pre-donation is not as much as the potential gain from the Cournot threat. (Note that under the Nash bargaining solution Firm 1’s profit does not change by pre-donation.)
Table 3
Numerical Results*

<table>
<thead>
<tr>
<th>η</th>
<th>λ^*</th>
<th>π_1^*</th>
<th>π_2^*</th>
<th>(\frac{π_2^* - π_2^0}{π_2^0} \times 100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cournot</td>
<td>1.2</td>
<td>0.5972</td>
<td>0.3397</td>
<td>1.1003</td>
</tr>
<tr>
<td></td>
<td>1.4</td>
<td>0.4089</td>
<td>0.2126</td>
<td>1.7474</td>
</tr>
<tr>
<td>Threat</td>
<td>1.6</td>
<td>0.2823</td>
<td>0.1154</td>
<td>2.4446</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.1920</td>
<td>0.0452</td>
<td>3.1948</td>
</tr>
<tr>
<td>Bertrand</td>
<td>1.2</td>
<td>0.3472</td>
<td>0.2222</td>
<td>1.2178</td>
</tr>
<tr>
<td></td>
<td>1.4</td>
<td>0.2551</td>
<td>0.0918</td>
<td>1.8682</td>
</tr>
<tr>
<td>Threat</td>
<td>1.6</td>
<td>0.1953</td>
<td>0.0312</td>
<td>2.5288</td>
</tr>
<tr>
<td></td>
<td>1.8</td>
<td>0.1543</td>
<td>0.0062</td>
<td>3.2338</td>
</tr>
<tr>
<td>Stackelberg</td>
<td>1.2</td>
<td>0.4450</td>
<td>0.2047</td>
<td>1.2353</td>
</tr>
<tr>
<td>Threat</td>
<td>1.4</td>
<td>0.2698</td>
<td>0.0917</td>
<td>1.8683</td>
</tr>
</tbody>
</table>

*π_1^0 and π_2^0 represents profits obtained under Nash bargaining solution without pre-donation.

5. Conclusions

This paper examined the pre-donational manipulation of the Nash bargaining solution under two different pre-donation scenarios for a bargaining problem with linear utility possibility frontier and an arbitrary threat point. It was shown that the Nash bargaining solution is pre-donationally manipulable under both scenarios. When pre-donations are shares from the total payoff of the "lucky" player and she has an incentive to pre-donate, both players become better off under the resulting outcome. However, only Player 2 gains from pre-donation when the pre-donated shares are constrained to be in excess of her threat payoff.

For the collusion problem described in an asymmetric duopoly, the pre-donation stage is considered to be a stage where the efficient firm makes a profit-sharing contract with its rival concerning future profits. In the second stage, firms bargain on their market shares. Our results show that the efficient firm has an incentive to donate a share of its profit to its rival under the Nash bargaining solution only when the share is given from its excess threat profit. Since the resulting outcome is efficient, only the efficient firm, Firm 2, produces for the market, giving a share which makes Firm 1 indifferent between the outcomes with and without pre-donation.
References


Özet

Önceden bağış durumunda Nash pazarlık çözümü ve bir duopol piyasasında kartelleşme