DISTANCES FOR MULTIPARAMETER PERSISTENCE MODULES

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ABSTRACT

DISTANCES FOR MULTIPARAMETER PERSISTENCE MODULES

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Persistent homology is an algebraic method to capture the essential topological features of an object. These objects are sometimes a data set called a point cloud or a topological space. After applying filtrations to the data set or topological space, we get persistence modules. One generally computes the interleaving distance between persistence modules to understand the algebraic similarities of these persistence modules. In addition to the interleaving distance, the bottleneck distance can be computed between the barcodes of these persistence modules. For one-parameter persistence modules, the interleaving distance equals the bottleneck distance. This fact is known as the isometry theorem. There is no isometry theorem for multiparameter persistence modules, even for special ones.

Furthermore, unlike the one-parameter case, interleaving and bottleneck distance computation is not easy, even for special persistence modules. Therefore, we define a new distance called steady matching distance and show it is an extended metric for finitely presented interval decomposable persistence modules.

Moreover, we investigate the relations between other distances. For interval persistence modules, we show that the matching distance is equal to the steady matching distance, and the interleaving distance is equal to the bottleneck distance. Moreover, by using the geometry of the underlying intervals of interval persistence modules, we can compute the interleaving distance, which is an upper bound for the steady matching distance. Additionally, we show that the steady matching distance is an upper bound for the matching distance and a lower bound for the bottleneck distance for interval decomposable persistence modules.

Keywords: Persistent, Interleaving, Bottleneck, Matching, Steady matching

ÖZ

ÇOK PARAMETRELİ KALICILIK MODÜLLERİ İÇİN MESAFELER

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Kalıcı homoloji, bir nesnenin temel topolojik özelliklerini yakalamak için kullanılan cebirsel bir yöntemdir. Bu nesneler bazen nokta bulutu olarak adlandırılan bir veri kümesi veya topolojik bir uzaydır. Veri kümesine veya topolojik uzaya filtreleme uyguladıktan sonra, kalıcılık modülü elde ederiz. Bu kalıcılık modüllerinin cebirsel benzerliklerini anlamak için genellikle kalıcılık modülleri arasındaki serpiştirme mesafesi hesaplanır. Serpiştirme mesafesine ek olarak, bu kalıcılık modüllerinin barkodları arasında darboğaz mesafesi de hesaplanabilir. Tek parametreli kalıcılık modülleri için, serpiştirme mesafesi darboğaz mesafesine eşittir. Bu sonuç izometri teoremi olarak bilinir. Çok parametreli kalıcılık modülleri için, hatta özel kalıcılık modülü türleri için bile izometri teoremi yoktur.

Ayrıca, tek parametreli durumdan farklı olarak, özel tipteki kalıcılık modülleri için bile serpiştirme ve darboğaz mesafesinin hesaplanması kolay değildir. Bu nedenle, sabit eşleşme mesafesi adı verilen yeni bir mesafe tanımladık ve sonlu olarak sunulan aralık ayrıştırılabilir kalıcılık modülleri için genişletilmiş bir metrik olduğunu gösterdik.

Ayrıca, diğer mesafeler arasındaki ilişkileri de araştırıyoruz. Aralık kalıcı modülleri için, eşleştirme mesafesinin sabit eşleştirme mesafesine eşit olduğunu ve serpiştirme mesafesinin darboğaz mesafesine eşit olduğunu gösterdik. Dahası, aralık kalıcı modüllerinin temel aralıklarının geometrisini kullanarak, serpiştirme mesafesini doğru bir şekilde hesaplayabiliriz ve bu, sabit eşleşme mesafesi için bir üst sınırdır. Ayrıca, sabit eşleşme mesafesinin aralık ayrıştırılabilir kalıcı modülleri için eşleşme mesafesine bir üst sınır ve darboğaz mesafesine bir alt sınır olduğunu gösterdik.

Anahtar Kelimeler: Kalıcılık, Serpiştirme, Darboğaz, Eşleşme, Sabit eşleşme

To My Lovely Son, MERT

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CHAPTER 1

INTRODUCTION

Topological Data Analysis (TDA) has become an indispensable tool for analyzing complex datasets across various domains, providing insights into their underlying structures. Central to TDA is the concept of persistence, which captures the evolution of topological features across different scales. With the increasing complexity of modern data, there has been a growing interest in extending TDA to multiparameter settings, where data vary across multiple dimensions. In the following section, we will give a brief literature review about multiparameter persistence modules and the distances for these persistence modules.

1.1 Literature Review

In this literature review, we survey the advancements in understanding distances on multiparameter persistence modules, focusing on the interleaving distance, the bottleneck distance, and the matching distance.

The interleaving distance, introduced by Chazal et al. [10], measures the similarity between persistence modules by capturing the extent to which their topological features are preserved under continuous transformations of the input data. This distance has been instrumental in assessing the stability of TDA algorithms and the robustness of topological invariants in the presence of noise and perturbations. Extensions of the interleaving distance to multiparameter settings have been explored by Bauer et al. [2], providing a framework for comparing persistence modules across multiple dimensions.

Building upon the interleaving distance, the bottleneck distance, introduced by Cohen-Steiner et al. [12], offers a refined measure of dissimilarity between persistence modules. By identifying the optimal matching between their essential features and quantifying the maximum discrepancy between corresponding features, the bottleneck distance provides a robust metric for comparing persistence diagrams. Extensions of the bottleneck distance to multiparameter persistence modules have been investigated by Kerber et al. [16], facilitating precise analyses of topological changes in multiparameter data.

In addition to the interleaving and bottleneck distances, the matching distance has emerged as a novel metric for comparing multiparameter persistence modules. Introduced by Bauer et al. [3], the matching distance extends the notion of distance to multiparameter settings, accounting for the complexity of data that vary across multiple dimensions. By capturing the interrelationships between different aspects of variation, the matching distance offers a comprehensive framework for comparing and analyzing multiparameter persistence modules.

Recent advancements in computational techniques have enabled efficient computation of distances on multiparameter persistence modules, facilitating their applications in various domains, including computational biology, materials science, and network analysis. However, challenges remain in developing robust distance metrics that account for the inherent complexity and heterogeneity of multiparameter data. Future research directions may focus on refining existing distance measures, developing new computational techniques, and exploring applications of distances on multiparameter persistence modules in real-world datasets.

In conclusion, distances on multiparameter persistence modules play a crucial role in quantifying the similarity and dissimilarity between topological features across multiple dimensions. The interleaving distance, bottleneck distance, and matching distance offer versatile frameworks for comparing and analyzing multiparameter persistence modules, contributing to the advancement of TDA and its applications in diverse fields.

1.2 Thesis Outline

In Chapter 2 of this thesis, we provide some background information on persistence modules. In Chapter 3, we define persistence modules and discuss various special types, including interval modules, interval decomposable persistence modules, as well as more specific types such as rectangle persistence modules and rectangle decomposable persistence modules.

In Chapter 4 of this thesis, we introduce the most commonly used distances: the interleaving distance and the bottleneck distance. These distances are applicable not only in one-parameter cases but also in multiparameter cases for evaluating the distance between persistence modules. In the last section of this chapter, Section 4.3, we give a brief comparison of these distances.

In Chapter 5, we introduce the matching distance and explore its properties. In Chapter 6, we define a new distance called the steady matching distance to address some limitations of the matching distance. We demonstrate that the steady matching distance is a metric for finitely presented persistence modules, unlike the matching distance. Additionally, in Chapter 7, we provide a brief comparison between the matching distance and the steady matching distance.

In Chapter 8, we provide the exact computation of the interleaving distance for rectangle persistence modules and for interval persistence modules, under certain assumptions. In the last chapter, Chapter 9, we use this exact computation and combine it with previous findings from earlier chapters to present a general comparison between these four distances.

CHAPTER 2

PRELIMINARIES

2.1 Notions and Conventions

For $n \geq 1$, let us start with defining a partial order on $\mathbb{R}^n$ by taking

$$u = (u_1, \ldots, u_n) \preceq v = (v_1, \ldots, v_n)$$

if and only if $u_i \leq v_i$ for all i, and $u \prec v$ if and only $u_i < v_i$ for all i. Note that $u \succ v$ is not the negation of $u \preceq v$.

Also, let us endow $\mathbb{R}^n$ with the max-norm, that is

$$\|u\|_\infty \doteq \max_{i=1,2,\ldots,n} \{|u_i|\}$$

for all $u \in \mathbb{R}^n$ and the metric induced by the max-norm is

$$d_\infty(u, v) \doteq \max\{|u_1 - v_1|, |u_2 - v_2|, \ldots, |u_n - v_n|\}$$

for all $u, v \in \mathbb{R}^n$.

Let us define the **extended real line** or more general the **extended space** as $\overline{\mathbb{R}} \doteq \mathbb{R} \cup \{-\infty, +\infty\}$ and $\overline{\mathbb{R}}^n \doteq \prod_{i=1}^{n} \overline{\mathbb{R}}$.

Throughout the thesis, for any $a \in \mathbb{R}$, we suppose that

$$\begin{aligned} &a + (\pm\infty) = (\pm\infty) + a = \pm\infty, \\ &+\infty > a \text{ and} \\ &-\infty < a, \end{aligned} \tag{2.1}$$

and for any $b \in \overline{\mathbb{R}}$, we suppose that

$$\begin{aligned} &+\infty \geq b \text{ and} \\ &-\infty \leq b. \end{aligned} \tag{2.2}$$

Moreover, we suppose that

$$\begin{aligned} &(\pm\infty) - (\pm\infty) = 0, \\ &(\pm\infty) + (\pm\infty) = \pm\infty \text{ and} \\ &|\pm\infty| = +\infty. \end{aligned} \tag{2.3}$$

2.2 Multisets and Partial Multibijections

Definition 2.2.1. *A* ***multiset*** *in the extended space* $\overline{\mathbb{R}}^n$ *is a subset* A *of* $\overline{\mathbb{R}}^n$ *such that each point* $a \in A$ *is assigned a multiplicity* $\mathrm{mult}_A(a) \in \mathbb{N} \cup \{+\infty\}$.

Roughly, a multiset is a generalization of the concept of a set where the multiplicity of elements matters.

Definition 2.2.2. *A* ***multibijection*** σ *between two multisets* (A, mult_A) *and* (B, mult_B) *is a bijection*

$$\sigma\colon \bigcup_{a\in A} \coprod_{i=1}^{\mathrm{mult}_A(a)} \{a\} \to \bigcup_{b\in B} \coprod_{i=1}^{\mathrm{mult}_B(b)} \{b\}.$$

We will use different symbols to denote different copies of the same element in a multiset.

Definition 2.2.3. *A* ***partial multibijection*** $\sigma : A \nrightarrow B$ *between multisets* A *and* B *is a multibijection* $\sigma\colon \tilde{A} \to \tilde{B}$ *where* $\tilde{A} \doteq \operatorname{coim}\sigma$ *is a subset of* A *and* $\tilde{B} \doteq \operatorname{im}\sigma$ *is a subset of* B.

CHAPTER 3

PERSISTENCE MODULES

3.1 Persistence Modules

Definition 3.1.1. *An n-parameter **persistence module** $\mathcal{M}$ over a field k is a family $\{\mathcal{M}_u\}_{u\in\mathbb{R}^n}$ of vector spaces over k together with a family of linear maps called **transition maps***

$$\{\varphi_{\mathcal{M}}(u,v)\colon \mathcal{M}_u \to \mathcal{M}_v,\ u \preceq v \in \mathbb{R}^n\}$$

such that for every $u \preceq v \preceq w \in \mathbb{R}^n$, we have

(i) $\varphi_{\mathcal{M}}(v,w) \circ \varphi_{\mathcal{M}}(u,v) = \varphi_{\mathcal{M}}(u,w)$,

(ii) $\varphi_{\mathcal{M}}(u,u) = \mathrm{id}_{\mathcal{M}_u}$.

We say that the n-parameter persistence module $\mathcal{M}$ is the **zero persistence module** if $\mathcal{M}_u$ is the zero vector space for all $u \in \mathbb{R}^n$.

Definition 3.1.2. *A **morphism** $f\colon \mathcal{M} \to \mathcal{N}$ between two persistence modules $\mathcal{M}$ and $\mathcal{N}$ is a collection of linear maps $\{f_u\colon \mathcal{M}_u \to \mathcal{N}_u\}$ such that the following diagram commutes for all $u \preceq v \in \mathbb{R}^n$:*

$$\begin{array}{ccc} \mathcal{M}_u & \xrightarrow{\varphi_{\mathcal{M}}(u,v)} & \mathcal{M}_v \\ {\scriptstyle f_u}\downarrow & & \downarrow{\scriptstyle f_v} \\ \mathcal{N}_u & \xrightarrow{\varphi_{\mathcal{N}}(u,v)} & \mathcal{N}_v \end{array}$$

Definition 3.1.3. *We say that the persistence modules $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, denoted by $\mathcal{M} \cong \mathcal{N}$, if there exist two morphisms $f\colon \mathcal{M} \to \mathcal{N}$ and $g\colon \mathcal{N} \to \mathcal{M}$ such that $f \circ g$ and $g \circ f$ are identity maps.*

Definition 3.1.4. *Let $\mathcal{M}$ be an n-parameter persistence module defined as above. Then, for any $\vec{\epsilon} = (\epsilon, \ldots, \epsilon) \in \mathbb{R}^n$ with $\epsilon \geq 0$, $\mathcal{M}(\vec{\epsilon})$ is an n-parameter persistence module defined as follows:*

- *$\mathcal{M}(\vec{\epsilon})_u \doteq \mathcal{M}_{u+\vec{\epsilon}}$ for all $u \in \mathbb{R}^n$ and*
- *$\varphi_{\mathcal{M}(\vec{\epsilon})}(u, v) \doteq \varphi_{\mathcal{M}}(u + \vec{\epsilon}, v + \vec{\epsilon})$ for any $u \preceq v \in \mathbb{R}^n$.*

*The n-parameter persistence module $\mathcal{M}(\vec{\epsilon})$ is called **ϵ-shifting** of the persistence module $\mathcal{M}$.*

Definition 3.1.5. *Let $\mathcal{M}$ be a persistence module, we call $\mathcal{M} = \bigoplus \mathcal{M}_i$ a **direct sum** of the persistence module for some collection of persistence modules $\{\mathcal{M}_i\}$. We say that $\mathcal{M}$ is an **indecomposable persistence module** if it is a non-zero persistence module and cannot be written as a direct sum of two non-zero persistence modules.*

3.2 Interval Persistence Modules

Definition 3.2.1 (Bjerkevik, [4])**.** *We say that $I \subseteq \mathbb{R}^n$ is an n-parameter connected **interval** if*

- *I is non-empty.*
- *If $u \preceq v \in I$ and $u \preceq w \preceq v$, then $w \in I$.*
- *If $u \preceq v \in I$, then for some $m \in \mathbb{N}$ there exist $u_1, u_2, \ldots, u_{2m}$ such that $u \preceq u_1 \succeq u_2 \preceq \ldots \succeq u_{2m} \preceq v$.*

We will denote the closure of an interval I as $\bar{I}$ and the interior of an interval I as I° in the standard topology of extended space $\bar{\mathbb{R}}^n$. Bear in mind that a closed interval must contain all boundary points, whereas an open interval cannot contain any boundary point.

Definition 3.2.2 (Dey and Xin, [14])**.** *The **lower boundary** of an interval I is defined as*

$$L(I) \doteq \{u \in \bar{I} \subset \mathbb{R}^n : \text{for all } v \in \mathbb{R}^n \text{ with } v_i < u_i \text{ for all i implies } v \notin I\}$$

*and the **upper boundary** of an interval I is defined as*

$$U(I) \doteq \{u \in \bar{I} \subset \mathbb{R}^n : \text{for all } v \in \mathbb{R}^n \text{ with } v_i > u_i \text{ for all } i \text{ implies } v \notin I\}.$$

*The **boundary** of an interval is defined as*

$$B(I) \doteq L(I) \cup U(I).$$

Definition 3.2.3 (Bjerkevik, [4]). *We say that a persistence module $\mathcal{I}$ is an **interval persistence module** if*

- *for some interval $I \subseteq \mathbb{R}^n$, $\mathcal{I}_u = k$ for every $u \in I$ and $\mathcal{I}_u = 0$ for $u \notin I$.*
- *$\varphi_{\mathcal{I}}(u, v) = \mathrm{id}_k$ for points $u \preceq v \in I$.*

We denote an interval persistence module $\mathcal{I}$ with underlying interval I as $\mathcal{I}^I$ or $I_{\mathcal{I}}$. In other words, we will always use math calligraphy letters for the interval persistence modules and regular letters for the underlying intervals of these interval persistence modules. However, if the underlying interval is not of interest, we sometimes use $\mathcal{M}$ (or $\mathcal{M}_j$ for some index element j in the index set **J** if we have more than one interval persistence module).

3.3 Barcode and Interval Decomposable Persistence Modules

Definition 3.3.1 (Bjerkevik, [4]). *If $\mathcal{M} = \bigoplus_{I \in B} \mathcal{I}^I$ for a multiset B of intervals (B may have some intervals with multiplicity more than one), we say that $\mathcal{M}$ is an **interval decomposable persistence module** and $B \doteq B(\mathcal{M})$ is the **barcode** of $\mathcal{M}$.*

If $\mathcal{M} = \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^I$ is an interval decomposable persistence module, then a **closed interval decomposable persistence module** $\overline{\mathcal{M}} \doteq \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^{\bar{I}}$ is an interval decomposable persistence module where each interval persistence module has closed underlying interval defined as above. Similarly, an **open interval decomposable persistence module** $\mathcal{M}^\circ \doteq \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^{I^\circ}$ is an interval decomposable persistence module where each interval persistence module has an open underlying interval.

Remark 3.3.2. *Any interval persistence module $\mathcal{M}$ is indecomposable. For details, see [5]. Hence, any interval persistence module is an interval decomposable persistence module with only one indecomposable summand.*

Now, let us define a rectangle, a rectangle persistence module, and a rectangle decomposable persistence module.

Definition 3.3.3. *We say that $R \subseteq \mathbb{R}^n$ is an n-parameter **rectangle** if $R = I_1 \times I_2 \times \ldots \times I_n$ where each I_i is a one-parameter interval. We say that $\mathcal{R}$ is an n-parameter **rectangle persistence module** if it is an interval persistence module with underlying n-parameter rectangle R, and a **rectangle decomposable persistence module** if it decomposes into a direct sum of only rectangle persistence modules.*

Note that each one-parameter open interval is one of the following 4 forms:

- (s, t) where both endpoints are finite numbers and $s < t$,
- $(-\infty, t)$ where t is a finite number,
- $(s, +\infty)$ where s is a finite number,
- $(-\infty, +\infty)$.

Definition 3.3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two n-parameter rectangle persistence modules with underlying rectangles $R_{\mathcal{M}} = I_1 \times I_2 \times \ldots \times I_n$ and $R_{\mathcal{N}} = J_1 \times J_2 \times \ldots \times J_n$, respectively.*

*We say that rectangle persistence modules $\mathcal{M}$ and $\mathcal{N}$ are of the **same type** if $I_i \setminus J_i$ and $J_i \setminus I_i$ are bounded sets for every $i \in \{1, 2, \ldots, n\}$.*

*We say that rectangle persistence modules $\mathcal{M}$ and $\mathcal{N}$ are of the **quasi-same type** if $I_i \setminus J_i$ and $J_i \setminus I_i$ are bounded sets for some $i \in \{1, 2, \ldots, n\}$.*

Example 3.3.5. *There are 16 different types of open rectangle bipersistence modules, and their underlying rectangles are listed in the following table, with s, t, u, and v being real numbers:*

Remark 3.3.6. *If rectangle persistence modules $\mathcal{M}$ and $\mathcal{N}$ are of the same type, then they are of the quasi-same type, but the converse is not valid. For instance, the*

Table 3.1: Different types of rectangle bipersistence modules.

$R_1 = (-\infty, +\infty) \times (-\infty, +\infty)$	$R_2 = (-\infty, +\infty) \times (-\infty, v)$
$R_3 = (-\infty, +\infty) \times (u, +\infty)$	$R_4 = (-\infty, +\infty) \times (u, v)$
$R_5 = (-\infty, t) \times (-\infty, +\infty)$	$R_6 = (-\infty, t) \times (-\infty, v)$
$R_7 = (-\infty, t) \times (u, +\infty)$	$R_8 = (-\infty, t) \times (u, v)$
$R_9 = (s, +\infty) \times (-\infty, +\infty)$	$R_{10} = (s, +\infty) \times (-\infty, v)$
$R_{11} = (s, +\infty) \times (u, +\infty)$	$R_{12} = (s, +\infty) \times (u, v)$
$R_{13} = (s, t) \times (-\infty, +\infty)$	$R_{14} = (s, t) \times (-\infty, v)$
$R_{15} = (s, t) \times (u, +\infty)$	$R_{16} = (s, t) \times (u, v)$

rectangle persistence modules with underlying rectangles R_1, R_2, R_3 and R_4 are of the quasi-same type; on the other hand, any two of these are not of the same type.

Remark 3.3.7. *The relation of being quasi-same type persistence modules is not transitive since the persistence modules with underlying rectangles R_5 and R_6, and the persistence modules with underlying rectangles R_6 and R_{10} are of the quasi-same type, but the persistence modules with underlying rectangles R_5 and R_{10} are not of the quasi-same type.*

3.4 Finitely Presented Persistence Modules

Definition 3.4.1. *A **free interval** generated by $u \in \mathbb{R}^n$ is an interval of the form $\langle u \rangle \doteq \{v \in \mathbb{R}^n : u \preceq v\}$ and a **free interval persistence module** or shortly **free persistence module** is an interval persistence module, denoted by $\mathcal{F}^{\langle u \rangle}$ with $\langle u \rangle$ as its underlying interval. A **free decomposable persistence module** $\mathcal{F}$ is a persistence module whose indecomposable summands are all free persistence modules. Note that every free persistence module is an interval persistence module, but the converse may not be correct.*

Definition 3.4.2. *A persistence module $\mathcal{M}$ is said to be a **finitely generated** persistence module if and only if there exists an epimorphism $\phi : \bigoplus_{i=1}^{m} \mathcal{F}_i \to \mathcal{M}$ where $\mathcal{F}_1, \ldots, \mathcal{F}_m$ are free persistence modules. Furthermore, a persistence module $\mathcal{M}$ is* *called a* ***finitely presented*** *persistence module if it is finitely generated and* $\ker\phi$ *is also finitely generated. Equivalently,* $\mathcal{M}$ *is a finitely presented persistence module if and only if there exists an exact sequence*

$$\bigoplus_{j=1}^{n}\mathcal{G}_j \to \bigoplus_{i=1}^{m}\mathcal{F}_i \xrightarrow{\phi} \mathcal{M} \to 0$$

where $\mathcal{G}_1, \ldots, \mathcal{G}_n$ *are also free persistence modules [13].*

Remark 3.4.3. *Any finitely presented one-parameter persistence module is an interval decomposable, but this is not true for* n*-parameter persistence modules if* $n > 1$*.*

In one-parameter case, we have the structure theorem [20], which states that every finitely presented persistence module can be expressed as a finite direct sum of the two types of indecomposable persistence modules roughly called free indecomposable and torsion indecomposable persistence modules. However, we do not have such a theorem for multiparameter persistence modules. On the other hand, although it is known that a finitely presented n-parameter persistence module can be written in an essentially unique way as a direct sum of indecomposables, the downside is that the set of isomorphism classes of finitely presented, indecomposable, n-parameter persistence modules is extremely complicated.

We know that if an n-parameter persistence module $\mathcal{M}$ is finitely presented, then it admits a unique decomposition as a direct sum of indecomposables thanks to the standard formulation of the Krull-Schmidt Theorem [1]. Moreover, if an n-parameter persistence module $\mathcal{M}$ is finitely presented, then it has finitely many summands in the decomposition. The following example shows that the converse is not always true.

Example 3.4.4. *Consider two interval persistence modules as below.*

Figure 3.1: Two non-isomorphic rectangle bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ with underlying rectangles $R_{\mathcal{M}} = [a_1, b_1] \times [a_2, b_2]$ and $R_{\mathcal{N}} = [a_1, b_1) \times [a_2, b_2)$, respectively.

The rectangle persistence module $\mathcal{M}$ is indecomposable, so it has one summand in the decomposition, but it is not finitely presented. Unlike the rectangle persistence module $\mathcal{M}$, the rectangle persistence module $\mathcal{N}$ is finitely presented since the underlying interval of $\mathcal{N}$ contains all lower boundary points and contains no upper boundary point (see Remark 3.4.5).

Remark 3.4.5. *An interval persistence module is finitely presented if its underlying interval contains all lower boundary points and no upper boundary point.*

CHAPTER 4

INTERLEAVING AND BOTTLENECK DISTANCE

4.1 Interleaving Distance

In this subsection, we will give the definition of the interleaving distance for multiparameter persistence modules.

Definition 4.1.1. *Let ϵ be a non-negative real number and let $\vec{\epsilon} \doteq (\epsilon, \ldots, \epsilon) \in \mathbb{R}^n$. An ϵ-**interleaving** between n-parameter persistence modules $\mathcal{M}$ and $\mathcal{N}$ is a collection of morphisms $f_u \colon \mathcal{M}_u \to \mathcal{N}_{u+\vec{\epsilon}}$ and $g_u \colon \mathcal{N}_u \to \mathcal{M}_{u+\vec{\epsilon}}$ such that the following four diagrams commute for all $u \preceq v \in \mathbb{R}^n$.*

$$\begin{CD} \mathcal{M}_u @>>> \mathcal{M}_v \\ @V{f_u}VV @VV{f_v}V \\ \mathcal{N}_{u+\vec{\epsilon}} @>>> \mathcal{N}_{v+\vec{\epsilon}} \end{CD} \qquad \begin{CD} \mathcal{N}_u @>>> \mathcal{N}_v \\ @V{g_u}VV @VV{g_v}V \\ \mathcal{M}_{u+\vec{\epsilon}} @>>> \mathcal{M}_{v+\vec{\epsilon}} \end{CD} \tag{4.1}$$

$$\begin{array}{ccc} & & \mathcal{N}_{u+\vec{\epsilon}} \\ & \overset{f_u}{\nearrow} & \Big\downarrow{\scriptstyle g_{u+\vec{\epsilon}}} \\ \mathcal{M}_u & & \\ & \searrow & \\ & & \mathcal{M}_{u+2\vec{\epsilon}} \end{array} \qquad \begin{array}{ccc} & & \mathcal{M}_{u+\vec{\epsilon}} \\ & \overset{g_u}{\nearrow} & \Big\downarrow{\scriptstyle f_{u+\vec{\epsilon}}} \\ \mathcal{N}_u & & \\ & \searrow & \\ & & \mathcal{N}_{u+2\vec{\epsilon}} \end{array} \tag{4.2}$$

The non-labelled maps are transition maps mentioned in Definition 3.1.1.

Note that, if $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved, then they are δ-interleaved for any $\epsilon \leq \delta$.

Therefore, one can define the **interleaving distance** by

$$d_I(\mathcal{M},\mathcal{N}) \doteq \inf\{\epsilon \in [0,+\infty) : \mathcal{M} \text{ and } \mathcal{N} \text{ are } \epsilon\text{-interleaved}\}.$$

If no such ϵ exists, then we put $d_I(\mathcal{M},\mathcal{N}) = +\infty$.

Proposition 4.1.2. *Two n-parameter persistence modules $\mathcal{M}$ and $\mathcal{N}$ are isomorphic if and only if they are 0-interleaved.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be two isomorphic n-parameter persistence modules, then by Definition 3.1.3, there exist two morphisms $f\colon \mathcal{M} \to \mathcal{N}$ and $g\colon \mathcal{N} \to \mathcal{M}$ such that $f \circ g$ and $g \circ f$ are identity maps. Now, consider the following four diagrams:

$$\begin{array}{ccccccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v & & \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \\ {\scriptstyle f_u}\downarrow & & \downarrow{\scriptstyle f_v} & & {\scriptstyle g_u}\downarrow & & \downarrow{\scriptstyle g_v} \\ \mathcal{N}_u & \longrightarrow & \mathcal{N}_v & & \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \end{array} \tag{4.3}$$

$$\begin{array}{ccccccc} & & \mathcal{N}_u & & & & \mathcal{M}_u \\ & \overset{f_u}{\nearrow} & \downarrow{\scriptstyle g_u} & & & \overset{g_u}{\nearrow} & \downarrow{\scriptstyle f_u} \\ \mathcal{M}_u & & & & \mathcal{N}_u & & \\ & \searrow & \downarrow & & & \searrow & \downarrow \\ & & \mathcal{M}_u & & & & \mathcal{N}_u \end{array} \tag{4.4}$$

Note that square diagrams in 4.3 are commutative for all $u \preceq v \in \mathbb{R}^n$ since $f\colon \mathcal{M} \to \mathcal{N}$ and $g\colon \mathcal{N} \to \mathcal{M}$ are morphisms given in the Definition 3.1.2. Moreover, the triangle diagrams in 4.4 are also commutative for all $u \in \mathbb{R}^n$ since by assumption $f \circ g$ and $g \circ f$ are identity maps, and $\varphi_{\mathcal{M}}(u,u) = \mathrm{id}_{\mathcal{M}_u}$ and $\varphi_{\mathcal{N}}(u,u) = \mathrm{id}_{\mathcal{N}_u}$ by Definition 3.1.1. Hence, by Definition 4.1.1, there exists 0-interleaving between n-parameter persistence modules $\mathcal{M}$ and $\mathcal{N}$, that is, they are 0-interleaved.

Conversely, suppose that the persistence modules $\mathcal{M}$ and $\mathcal{N}$ are 0-interleaved, then there exists 0-interleaving between n-parameter persistence modules $\mathcal{M}$ and $\mathcal{N}$. Hence, by Definition 4.1.1, we have the same four commutative diagrams as in 4.3 for all $u \preceq v \in \mathbb{R}^n$ and 4.4 for all $u \in \mathbb{R}^n$. The square diagrams in 4.3 implies that $f\colon \mathcal{M} \to \mathcal{N}$ and $g\colon \mathcal{N} \to \mathcal{M}$ are morphisms and the triangle diagrams in 4.4 implies that $f_u \circ g_u = \mathrm{id}_{\mathcal{N}_u}$ and $g_u \circ f_u = \mathrm{id}_{\mathcal{M}_u}$ for all $u \in \mathbb{R}^n$. Therefore, by Definition 3.1.3 the n-parameter persistence modules $\mathcal{M}$ and $\mathcal{N}$ are isomorphic since there exist two morphisms $f\colon \mathcal{M} \to \mathcal{N}$ and $g\colon \mathcal{N} \to \mathcal{M}$ such that $f \circ g$ and $g \circ f$ are identity maps. □

Remark 4.1.3. *Note that if the persistence modules $\mathcal{M}$ and $\mathcal{N}$ are 0-interleaved, then $d_I(\mathcal{M}, \mathcal{N}) = 0$. However, the following example shows that the converse is not always true.*

Example 4.1.4 (Lesnick, [18]). *Let $\mathcal{M}$ be a one-parameter persistence module with $\mathcal{M}_0 = k$ and $\mathcal{M}_u = 0$ if $u \neq 0$. Let $\mathcal{N}$ be one-parameter zero persistence module, that is, $\mathcal{N}_u = 0$ for all $u \in \mathbb{R}$. Then, one-parameter persistence modules $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic, and so by Proposition 4.1.2, they are not 0-interleaved, but it is easy to check that one-parameter persistence modules $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved for any $\epsilon > 0$. Thus, because of the infimum in the Definition 4.1.1, the interleaving distance $d_I(M, N) = 0$.*

Thanks to the following theorem, the converse statement of the remark above will also be true for finitely presented persistence modules.

Theorem 4.1.5 (**Closure Theorem**, Lesnick, [18]). *If $\mathcal{M}$ and $\mathcal{N}$ are finitely presented multidimensional persistence modules and $d_I(\mathcal{M}, \mathcal{N}) = \epsilon$, then $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved.*

Considering the case $\epsilon = 0$, it follows from the closure theorem that d_I restricts to a metric on isomorphism classes of finitely presented multidimensional persistence modules.

Remember that we say n-parameter persistence module $\mathcal{M}$ is a **zero persistence module** if each $\mathcal{M}_u$ is zero vector space for all $u \in \mathbb{R}^n$.

Definition 4.1.6. *An n-parameter persistence module $\mathcal{M}$ is called **ϵ-significant** if $\varphi_{\mathcal{M}}(u, u + \epsilon) \neq 0$ for some $u \in \mathbb{R}^n$, and **ϵ-trivial** otherwise.*

Proposition 4.1.7. *An n-parameter persistence module $\mathcal{M}$ is 2ϵ-trivial if and only if it is ϵ-interleaved with the zero persistence module.*

Proof. Let $\mathcal{M}$ be an n-parameter persistence module and let $\mathcal{N}$ be the zero persistence module, that is $\mathcal{N}_u = 0$ for all $u \in \mathbb{R}^n$. Consider the following diagrams:

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \\ {\scriptstyle f_u}\downarrow & & \downarrow{\scriptstyle f_v} \\ 0 = \mathcal{N}_{u+\vec{\epsilon}} & \longrightarrow & \mathcal{N}_{v+\vec{\epsilon}} = 0 \end{array} \qquad \begin{array}{ccc} 0 = \mathcal{N}_u & \longrightarrow & \mathcal{N}_v = 0 \\ {\scriptstyle g_u}\downarrow & & \downarrow{\scriptstyle g_v} \\ \mathcal{M}_{u+\vec{\epsilon}} & \longrightarrow & \mathcal{M}_{v+\vec{\epsilon}} \end{array} \tag{4.5}$$

$$\begin{array}{ccc} & \overset{f_u}{\nearrow} & \mathcal{N}_{u+\vec{\epsilon}} = 0 \\ \mathcal{M}_u & & \downarrow{\scriptstyle g_{u+\vec{\epsilon}}} \\ & \searrow & \mathcal{M}_{u+2\vec{\epsilon}} \end{array} \qquad \begin{array}{ccc} & \overset{g_u}{\nearrow} & \mathcal{M}_{u+\vec{\epsilon}} \\ 0 = \mathcal{N}_u & & \downarrow{\scriptstyle f_{u+\vec{\epsilon}}} \\ & \searrow & \mathcal{N}_{u+2\vec{\epsilon}} = 0 \end{array} \tag{4.6}$$

Suppose that the persistence module $\mathcal{M}$ is 2ϵ-trivial. So, by Definition 4.1.6 the transitions maps $\varphi_{\mathcal{M}}(u, u + 2\epsilon) = 0$ for all $u \in \mathbb{R}^n$. This implies that all diagrams in (4.5) commute for all $u \preceq v \in \mathbb{R}^n$ and all diagrams in (4.6) commute for all $u \in \mathbb{R}^n$. Hence, the persistence module $\mathcal{M}$ is ϵ-interleaved with zero persistence module. Conversely, if the persistence module $\mathcal{M}$ is ϵ-interleaved with zero persistence module, then all diagrams in (4.5) commute for all $u \preceq v \in \mathbb{R}^n$ and all diagrams in (4.6) commute for all $u \in \mathbb{R}^n$. Thus, by commutativity of left triangle diagrams in (4.6) for all $u \in \mathbb{R}^n$, the transition maps $\varphi_{\mathcal{M}}(u, u + 2\epsilon) = 0$ for all $u \in \mathbb{R}^n$. Hence, by Definition 4.1.6 the persistence module $\mathcal{M}$ is 2ϵ-trivial.

□

The following fact given by Dey, T. K., and Xin, C. will be used later.

Proposition 4.1.8 (Dey and Xin, [14]). *Let $\mathcal{M} = \mathcal{I}^I$ and $\mathcal{N} = \mathcal{I}^J$ be two interval persistence modules with underlying intervals I and J, respectively. Then, $d_I(\mathcal{M}, \mathcal{N}) = d_I(\overline{\mathcal{M}}, \overline{\mathcal{N}})$ where $\overline{\mathcal{M}} \doteq \mathcal{I}^{\bar{I}}$ and $\overline{\mathcal{N}} \doteq \mathcal{I}^{\bar{J}}$.*

Thanks to proposition above, if $\mathcal{M}$ and $\mathcal{N}$ are rectangle persistence modules in Example 3.4.4 then $d_I(\mathcal{M}, \mathcal{N}) = 0$ although $\mathcal{M}$ and $\mathcal{N}$ are non-isomorphic persistence modules. In general, if $\mathcal{M}$ is an any interval persistence module, then

$d_I(\mathcal{M}, \overline{\mathcal{M}}) = 0$. Hence, the interleaving distance is not a metric for non-finitely presented interval persistence modules.

Corollary 4.1.9. *Let $\mathcal{M} = \mathcal{I}^I$ and $\mathcal{N} = \mathcal{I}^J$ be two interval persistence modules. Then, $d_I(\mathcal{M}, \mathcal{N}) = d_I(\mathcal{M}^\circ, \mathcal{N}^\circ)$ where $\mathcal{M}^\circ \doteq \mathcal{I}^{I^\circ}$ and $\mathcal{N}^\circ \doteq \mathcal{I}^{J^\circ}$.*

In the next section, we will define the bottleneck distance between interval decomposable persistence modules, but to do so, we will first define some basic notions.

4.2 Bottleneck Distance

Definition 4.2.1 (Bjerkevik, [4]). *Let A and B be multisets of intervals. An **ϵ-matching** between A and B is a partial multibijection $\sigma : A \nrightarrow B$ such that*

(i) for all $I \in A - \operatorname{coim} \sigma$, $\mathcal{I}^I$ is 2ϵ-trivial,

(ii) for all $I \in B - \operatorname{im} \sigma$, $\mathcal{I}^I$ is 2ϵ-trivial,

(iii) for all $I \in \operatorname{coim} \sigma$, $\mathcal{I}^I$ and $\mathcal{I}^{\sigma(I)}$ are ϵ-interleaved.

If there is an ϵ-matching σ between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$ of persistence modules $\mathcal{M}$ and $\mathcal{N}$, then we say that $\mathcal{M}$ and $\mathcal{N}$ are **ϵ-matched** and the infimum of ϵ for which σ is an ϵ-matching is said to be the **cost** of $\sigma : B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ denoted by $\operatorname{cost}(\sigma)$, that is

$$\operatorname{cost}(\sigma) \doteq \inf\{\epsilon \geq 0 : \sigma \text{ is an } \epsilon\text{-matching}\}$$

if such an ϵ exists, otherwise $\operatorname{cost}(\sigma) = +\infty$.

Now, we can define the bottleneck distance between the barcodes of multiparameter interval decomposable persistence modules.

Definition 4.2.2 (Bjerkevik, [4]). *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval decomposable persistence modules and S be the set of all partial multibijections between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$. Then, the **bottleneck distance** between interval decomposable persistence modules is defined as*

$$d_B(\mathcal{M}, \mathcal{N}) \doteq \inf\{\epsilon \in [0, +\infty) : \mathcal{M} \textit{ and } \mathcal{N} \textit{ are } \epsilon\textit{-matched}\}.$$

If there is no such an ϵ, we put $d_B(\mathcal{M},\mathcal{N}) = +\infty$.

Note that, this is also equivalent to setting $d_B(\mathcal{M},\mathcal{N}) = \inf_{\sigma \in S} \mathrm{cost}(\sigma)$. Also, bear in mind that if $\mathcal{M}$ and $\mathcal{N}$ are finitely presented persistence modules, then one can use minimum in the definition of the bottleneck distance instead of infimum since there always exists a partial multibijection $\overline{\sigma} : B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ such that $d_B(\mathcal{M},\mathcal{N}) = \mathrm{cost}(\overline{\sigma})$. The partial multibijection $\overline{\sigma}$ is called the **optimal** multibijection, see [8].

Due to the following proposition, we can compute the bottleneck distance between interval decomposable persistence modules concerning pairwise interleaving distances between interval persistence modules.

Proposition 4.2.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two finitely presented interval decomposable persistence modules with given decompositions*

$$\mathcal{M} = \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^I \text{ and } \mathcal{N} = \bigoplus_{J \in B(\mathcal{N})} \mathcal{I}^J$$

where each summand $\mathcal{I}^I$ and $\mathcal{I}^J$ are interval persistence modules with underlying intervals I and J, respectively. Let S be the set of all partial multibijections between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$. Let $\sigma \in S$ and let $\mathbf{I}' = B(\mathcal{M}) - \mathrm{coim}\,\sigma$, $\mathbf{J}' = B(\mathcal{N}) - \mathrm{im}\,\sigma$. Then, the bottleneck distance is equal to

$$\min_{\sigma \in S} \big(\max \big\{ \max_{I \in \mathrm{coim}\,\sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \max_{I \in \mathbf{I}'} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'} \{d_I(0, \mathcal{I}^J)\} \big\} \big) \tag{4.7}$$

Proof. Let

$$\epsilon = d_B(\mathcal{M},\mathcal{N}) = \inf\{\epsilon \in [0,+\infty) : \mathcal{M} \text{ and } \mathcal{N} \text{ are } \epsilon\text{-matched}\} = \inf_{\sigma \in S} \mathrm{cost}(\sigma).$$

Then, since $\mathcal{M}$ and $\mathcal{N}$ are finitely presented interval decomposable persistence modules, S is finite. So, the infimum is minimum and there exists a partial multibijection $\bar{\sigma} \colon B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$ such that $\bar{\sigma}$ is an ϵ-matching. Therefore,

(i) for all $I \in B(\mathcal{M}) - \mathrm{coim}\,\bar{\sigma}$, $\mathcal{I}^I$ is 2ϵ-trivial,

(ii) for all $J \in B(\mathcal{N}) - \mathrm{im}\,\bar{\sigma}$, $\mathcal{I}^J$ is 2ϵ-trivial,

(iii) for all $I \in \mathrm{coim}\,\bar{\sigma}$, $\mathcal{I}^I$ and $\mathcal{I}^{\bar{\sigma}(I)}$ are ϵ-interleaved.

Now, by Proposition 4.1.7, the interval persistence module $\mathcal{I}^I$ is 2ϵ-trivial if and only if $\mathcal{I}^I$ is ϵ-interleaved with the zero persistence module for all $I \in B(\mathcal{M}) - \operatorname{coim} \bar{\sigma}$. Thus, $\epsilon \geq d_I(\mathcal{I}^I, 0)$ for all $I \in B(\mathcal{M}) - \operatorname{coim} \bar{\sigma}$.

Similarly, by Proposition 4.1.7, the interval persistence module $\mathcal{I}^J$ is 2ϵ-trivial if and only if $\mathcal{I}^J$ is ϵ-interleaved with the zero persistence module for all $J \in B(\mathcal{N}) - \operatorname{im} \bar{\sigma}$. Thus, $\epsilon \geq d_I(0, \mathcal{I}^J)$ for all $J \in B(\mathcal{N}) - \operatorname{im} \bar{\sigma}$.

Also, it is given that $\mathcal{I}^I$ and $\mathcal{I}^{\bar{\sigma}(I)}$ are ϵ-interleaved for all $I \in \operatorname{coim} \bar{\sigma}$. Thus, $\epsilon \geq d_I(\mathcal{I}^I, \mathcal{I}^{\bar{\sigma}(I)})$ for all $I \in \operatorname{coim} \bar{\sigma}$.

Therefore, $\epsilon \geq \max\big\{ \max_{I \in \operatorname{coim} \bar{\sigma}} \{d_I(\mathcal{I}^I, \mathcal{I}^{\bar{\sigma}(I)})\}, \max_{I \in \mathbf{I}'} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'} \{d_I(0, \mathcal{I}^J)\}\big\}$. It follows that

$$\epsilon \geq \min_{\sigma \in S} \big(\max\big\{ \max_{I \in \operatorname{coim} \sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \max_{I \in \mathbf{I}'} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'} \{d_I(0, \mathcal{I}^J)\}\big\}\big).$$

Hence,

$$d_B(\mathcal{M}, \mathcal{N}) \geq \min_{\sigma \in S} \big(\max\big\{ \max_{I \in \operatorname{coim} \sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \max_{I \in \mathbf{I}'} \{d_I(\mathcal{I}^I, 0)\}, \max_{J \in \mathbf{J}'} \{d_I(0, \mathcal{I}^J)\}\big\}\big).$$

Conversely, assume now that

$$\epsilon = \inf_{\sigma \in S} \big(\max\big\{ \sup_{I \in \operatorname{coim} \sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \sup_{I \in \mathbf{I}'} \{d_I(\mathcal{I}^I, 0)\}, \sup_{J \in \mathbf{J}'} \{d_I(0, \mathcal{I}^J)\}\big\}\big).$$

Since $\mathcal{M}$ and $\mathcal{N}$ are finitely presented interval decomposable persistence modules, there exists a partial multibijection $\bar{\sigma}$ such that

$$\epsilon = \max\big\{ \sup_{I \in \operatorname{coim} \bar{\sigma}} \{d_I(\mathcal{I}^I, \mathcal{I}^{\bar{\sigma}(I)})\}, \sup_{I \in \mathbf{I}'} \{d_I(\mathcal{I}^I, 0)\}, \sup_{J \in \mathbf{J}'} \{d_I(0, \mathcal{I}^J)\}\big\}.$$

It follows that $\epsilon \geq d_I(\mathcal{I}^I, \mathcal{I}^{\bar{\sigma}(I)})$ for all $I \in \operatorname{coim} \bar{\sigma}$. Similarly, $\epsilon \geq d_I(\mathcal{I}^I, 0)$ for all $I \in \mathbf{I}' = B(\mathcal{M}) - \operatorname{coim} \bar{\sigma}$ and $\epsilon \geq d_I(0, \mathcal{I}^J)$ for all $J \in \mathbf{J}' = B(\mathcal{N}) - \operatorname{im} \bar{\sigma}$. Clearly, $\mathcal{I}^I$ is ϵ-interleaved with the zero persistence module for each $I \in B(\mathcal{M}) - \operatorname{coim} \bar{\sigma}$ and $\mathcal{I}^J$ is ϵ-interleaved with the zero persistence module for each $J \in B(\mathcal{N}) - \operatorname{im} \bar{\sigma}$. So, by Proposition 4.1.7, $\mathcal{I}^I$ is 2ϵ-trivial for each $I \in B(\mathcal{M}) - \operatorname{coim} \bar{\sigma}$ and $\mathcal{I}^J$ is 2ϵ-trivial for each $J \in B(\mathcal{N}) - \operatorname{im} \bar{\sigma}$. So we have

(i) for all $I \in B(\mathcal{M}) - \operatorname{coim} \bar{\sigma}$, $\mathcal{I}^I$ is 2ϵ-trivial,

(ii) for all $J \in B(\mathcal{N}) - \operatorname{im}\bar{\sigma}$, $\mathcal{I}^J$ is 2ϵ-trivial,

(iii) for all $I \in \operatorname{coim}\bar{\sigma}$, $\mathcal{I}^I$ and $\mathcal{I}^{\bar{\sigma}(I)}$ are ϵ-interleaved.

Thus, $\bar{\sigma}$ is an ϵ-matching. So, by Definition 4.2.2, $\epsilon \geq d_B(\mathcal{M},\mathcal{N}) = \inf_{\sigma\in S} \operatorname{cost}(\sigma)$.

Hence,

$$d_B(\mathcal{M},\mathcal{N}) = \inf_{\sigma\in S} \big(\max\big\{ \sup_{I\in\operatorname{coim}\sigma} \{d_I(\mathcal{I}^I,\mathcal{I}^{\sigma(I)})\}, \sup_{I\in\mathbf{I}'}\{d_I(\mathcal{I}^I,0)\}, \sup_{J\in\mathbf{J}'}\{d_I(0,\mathcal{I}^J)\}\big\}\big).$$

□

The value $\max\big\{ \sup_{I\in\operatorname{coim}\sigma} \{d_I(\mathcal{I}^I,\mathcal{I}^{\sigma(I)})\}, \sup_{I\in\mathbf{I}'}\{d_I(\mathcal{I}^I,0)\}, \sup_{J\in\mathbf{J}'}\{d_I(0,\mathcal{I}^J)\}\big\}$ appears on the above proposition will be also called cost of the partial multibijection σ and denoted by $\operatorname{cost}(\sigma)$. Hence, the bottleneck distance between interval decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$ can be also defined as $d_B(\mathcal{M},\mathcal{N}) = \inf_{\sigma\in S} \operatorname{cost}(\sigma)$ where

$$\operatorname{cost}(\sigma) \doteq \max\big\{ \sup_{I\in\operatorname{coim}\sigma} \{d_I(\mathcal{I}^I,\mathcal{I}^{\sigma(I)})\}, \sup_{I\in\mathbf{I}'}\{d_I(\mathcal{I}^I,0)\}, \sup_{J\in\mathbf{J}'}\{d_I(0,\mathcal{I}^J)\}\big\}.$$

Therefore, we can alternatively compute the bottleneck distance between interval decomposable persistence modules as in Proposition 4.2.3 above. We will also refer to that distance function as the bottleneck distance.

A persistence module $\mathcal{M}$ is called **point-wise finite-dimensional** if $\mathcal{M}_u$ is a finite-dimensional vector space for each $u \in \mathbb{R}^n$ and observe that every finitely presented persistence module is point-wise finite-dimensional persistence module.

Now, one can observe that for one-parameter persistence modules, the bottleneck distance definition is given in the Definition 4.2.2 or computed in Proposition 4.2.3 coincides with the definition of bottleneck distance between barcodes of point-wise finite-dimensional one-parameter persistence modules defined by Lesnick, M. (for details, see [18]). In other words, the bottleneck distance definition given above is a generalization of the bottleneck distance defined in [18].

Moreover, one can generalize the computation in Proposition 4.2.3 to any type of decomposable n-parameter persistence modules as follows:

Definition 4.2.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be finitely presented decomposable persistence modules with given decomposition $\mathcal{M} = \bigoplus_{i\in\mathbf{I}} \mathcal{M}_i$ and $\mathcal{N} = \bigoplus_{j\in\mathbf{J}} \mathcal{N}_j$ where each $\mathcal{M}_i$ and $\mathcal{N}_j$ is an indecomposable persistence module.*

Let S be the set of all partial multibijections between multisets $\mathbf{I}$ and $\mathbf{J}$. For $\sigma \in S$ let $\mathbf{I}' \doteq \mathbf{I} - \operatorname{coim}\sigma$, $\mathbf{J}' \doteq \mathbf{J} - \operatorname{im}\sigma$. Then, the bottleneck distance can be computed as

$$\begin{aligned} d_B(\mathcal{M},\mathcal{N}) &= \inf_{\sigma\in S} \big(\max\big\{ \sup_{i\in\operatorname{coim}\sigma} \{d_I(\mathcal{M}_i, \mathcal{N}_{\sigma(i)})\}, \sup_{i\in\mathbf{I}'}\{d_I(\mathcal{M}_i, 0)\}, \sup_{j\in\mathbf{J}'}\{d_I(0, \mathcal{N}_j)\}\big\}\big) \\ &= \inf_{\sigma\in S} \operatorname{cost}(\sigma). \end{aligned}$$

Proposition 4.2.5. *Let $\mathcal{M} = \bigoplus_{I\in B(\mathcal{M})} \mathcal{I}^I$ and $\mathcal{N} = \bigoplus_{J\in B(\mathcal{N})} \mathcal{I}^J$ be two finitely presented interval decomposable persistence modules. Then $d_B(\mathcal{M},\mathcal{N}) = d_B(\overline{\mathcal{M}},\overline{\mathcal{N}})$ where $\overline{\mathcal{M}} \doteq \bigoplus_{I\in B(\mathcal{M})} \mathcal{I}^{\bar{I}}$ and $\overline{\mathcal{N}} \doteq \bigoplus_{J\in B(\mathcal{N})} \mathcal{I}^{\bar{J}}$. Likewise, $d_B(\mathcal{M},\mathcal{N}) = d_B(\mathcal{M}^{\circ},\mathcal{N}^{\circ})$ where $\mathcal{M}^{\circ} \doteq \bigoplus_{I\in B(\mathcal{M})} \mathcal{I}^{I^{\circ}}$ and $\mathcal{N}^{\circ} \doteq \bigoplus_{J\in B(\mathcal{N})} \mathcal{I}^{J^{\circ}}$.*

Proof. Straightforwardly from Proposition 4.1.8 and Proposition 4.2.3, we get

$$\begin{aligned} d_B(\mathcal{M},\mathcal{N}) &= \min_{\sigma\in S} \big(\max\big\{ \max_{I\in\operatorname{coim}\sigma} \{d_I(\mathcal{I}^I, \mathcal{I}^{\sigma(I)})\}, \max_{I\in\mathbf{I}'}\{d_I(\mathcal{I}^I, 0)\}, \max_{J\in\mathbf{J}'}\{d_I(0, \mathcal{I}^J)\}\big\}\big) \\ &= \min_{\sigma\in S} \big(\max\big\{ \max_{\bar{I}\in\operatorname{coim}\sigma} \{d_I(\mathcal{I}^{\bar{I}}, \mathcal{I}^{\sigma(\bar{I})})\}, \max_{\bar{I}\in\mathbf{I}'}\{d_I(\mathcal{I}^{\bar{I}}, 0)\}, \max_{\bar{J}\in\mathbf{J}'}\{d_I(0, \mathcal{I}^{\bar{J}})\}\big\}\big) \\ &= d_B(\overline{\mathcal{M}},\overline{\mathcal{N}}). \end{aligned}$$

In a similar way, from Corollary 4.1.9 and Proposition 4.2.3, we get

$$d_B(\mathcal{M},\mathcal{N}) = d_B(\mathcal{M}^{\circ},\mathcal{N}^{\circ}).$$

□

4.3 Comparison of Interleaving Distance and Bottleneck Distance

Since the bottleneck distance is the maximum of the pairwise interleaving distances between indecomposable persistence modules, the interleaving distance is a lower bound for the bottleneck distance for any n-parameter persistence modules. In one-parameter, by algebraic stability theorem [10, 11], the interleaving distance between finitely presented persistence modules $\mathcal{M}$ and $\mathcal{N}$ is bigger than or equal to the bottleneck distance between the barcodes of $B(\mathcal{M})$ and $B(\mathcal{N})$. Hence, in the one-parameter case, it is known that the interleaving distance between finitely presented persistence modules is equal to the bottleneck distance between the barcodes of those persistence modules. This important fact is known as the isometry theorem [18].

However, there is no isometry theorem for finitely presented n-parameter persistence modules when $n > 1$, see [4, Example 5.2]. Instead, we have the following facts for multiparameter persistence modules.

Proposition 4.3.1. *If $\mathcal{M} = \mathcal{I}^I$ and $\mathcal{N} = \mathcal{I}^J$ are two n-parameter interval persistence modules, then $d_B(\mathcal{M}, \mathcal{N}) = d_I(\mathcal{M}, \mathcal{N})$.*

Proof. Let us show that $d_B(\mathcal{M}, \mathcal{N}) \leq d_I(\mathcal{M}, \mathcal{N})$. Suppose that the interval persistence modules $\mathcal{M} = \mathcal{I}^I$ and $\mathcal{N} = \mathcal{I}^J$ are ϵ-interleaved. Now, consider partial multibijection $\sigma\colon \{I\} \nrightarrow \{J\}$ with $\operatorname{im}\sigma = \{J\}$ and $\operatorname{coim}\sigma = \{I\}$. Therefore, $B(\mathcal{M}) - \operatorname{coim}\sigma = \emptyset$ and $B(\mathcal{N}) - \operatorname{im}\sigma = \emptyset$.

Now, by Definition 4.2.1 the partial multibijection $\sigma\colon \{I\} \nrightarrow \{J\}$ is an ϵ-matching since $\mathcal{I}^I$ and $\mathcal{I}^J$ are ϵ-interleaved, and $B(\mathcal{M}) - \operatorname{coim}\sigma = \emptyset$ and $B(\mathcal{N}) - \operatorname{im}\sigma = \emptyset$.

Therefore, if there exists an ϵ-interleaving between interval persistence modules $\mathcal{M}$ and $\mathcal{N}$, then there exists ϵ-matching between the barcodes of $B(\mathcal{M}) = \{I\}$ and $B(\mathcal{N}) = \{J\}$. In particular, if $\mathcal{M}$ and $\mathcal{N}$ are two n-parameter interval persistence modules, then $d_B(\mathcal{M}, \mathcal{N}) \leq d_I(\mathcal{M}, \mathcal{N})$.

Let us now show that $d_I(\mathcal{M}, \mathcal{N}) \leq d_B(\mathcal{M}, \mathcal{N})$. Since $\mathcal{M} = \mathcal{I}^I$ and $\mathcal{N} = \mathcal{I}^J$ are two n-parameter interval persistence modules, there are exactly two partial multibijections $\sigma_1\colon B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ with $B(\mathcal{M}) - \operatorname{coim}(\sigma_1) = \emptyset$ and $B(\mathcal{N}) - \operatorname{im}(\sigma_1) = \emptyset$, and $\sigma_2\colon B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ with $B(\mathcal{M}) - \operatorname{coim}(\sigma_2) = I$ and $B(\mathcal{N}) - \operatorname{im}(\sigma_2) = J$. In other words, there are exactly two partial multibijections. One matches the interval persistence module $\mathcal{M}$ to thinterval persistence module $\mathcal{N}$, and the other matches each interval persistence module with the zero persistence module.

Suppose now, $d_B(\mathcal{M}, \mathcal{N}) = \epsilon$. Then, either σ_1 or σ_2 is an ϵ-matching.

If σ_1 is an ϵ-matching, then by Definition 4.2.1, the interval persistence modules $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved. Hence, $d_I(\mathcal{M}, \mathcal{N}) \leq d_B(\mathcal{M}, \mathcal{N}) = \epsilon$.

If σ_2 is an ϵ-matching, then $\mathcal{M}$ and $\mathcal{N}$ are both 2ϵ-trivial. Let morphisms $f_u \colon \mathcal{M}_u \to \mathcal{N}_{u+\vec{\epsilon}}$ and $g_u \colon \mathcal{N}_u \to \mathcal{M}_{u+\vec{\epsilon}}$ be zero maps for all $u \in \mathbb{R}^n$. Now, observe that all diagrams in the Definition 4.1.1 are commutative for all $u, v \in \mathbb{R}^n$ since transition maps $\varphi_{\mathcal{M}}(u, u+2\epsilon) = 0$ and $\varphi_{\mathcal{N}}(u, u+2\epsilon) = 0$ for all $u \in \mathbb{R}^n$. Therefore, the interval persistence modules $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved. Hence, $d_I(\mathcal{M}, \mathcal{N}) \leq d_B(\mathcal{M}, \mathcal{N}) = \epsilon$. Therefore, for any case $d_I(\mathcal{M}, \mathcal{N}) \leq d_B(\mathcal{M}, \mathcal{N})$.

Hence, if $\mathcal{M}$ and $\mathcal{N}$ are two n-parameter interval persistence modules, then $d_B(\mathcal{M}, \mathcal{N}) = d_I(\mathcal{M}, \mathcal{N})$. □

The previous result can be generalized to any indecomposable persistence module, and it can be proven by using the same analogy given in the proof of Proposition 4.3.1 above.

Remark 4.3.2. *If $\mathcal{M}$ and $\mathcal{N}$ are two n-parameter indecomposable persistence modules, then $d_B(\mathcal{M}, \mathcal{N}) = d_I(\mathcal{M}, \mathcal{N})$.*

CHAPTER 5

MATCHING DISTANCE

In this chapter, we discuss another frequently used distance for multiparameter persistence modules which is called the matching distance. The use of this distance allows us to examine multiparameter persistence modules in one-parameter. Roughly, this can be achieved by restricting the underlying intervals of these persistence modules by intersecting them with lines. For details, please see [6].

5.1 Admissible Lines

Let us start with some notation. Let Λ be the set of lines that can be parameterized by $(m, b) \in (0, 1]^n \times \mathbb{R}^n$, i.e., $L\colon u = tm + b$ for $t \in \mathbb{R}$ where m is the unit direction vector in the max norm. We refer to such lines as **admissible lines**. Moreover, for $m = (m_1, \ldots, m_n)$ we set

$$m^L \doteq \min_{i=1,\ldots,n} \{m_i\}$$

where $m_i > 0$ for all $i = 1, \ldots, n$.

The admissible line with unit direction vector $m = (1, \ldots, 1)$ will be called **diagonal line** and $m^L = 1$ for any diagonal line L.

Line parameterization will be used in the calculations, as the results are expected to generalize for the n-parameter persistence modules.

5.2 Restriction of Persistence Modules to Lines

Let $\mathcal{M}$ be an n-parameter persistence module and L be an admissible line with direction vector $m \in (0,1]^n$. We can obtain a one-parameter persistence module, denoted as $\mathcal{M}^L$, by restricting persistence module $\mathcal{M}$ to the line L such that for all $t \in \mathbb{R}$ we have $(M^L)_t = \mathcal{M}_u$ and for $t \leq t' \in \mathbb{R}$, $\varphi_{\mathcal{M}^L}(t,t') = \varphi_{\mathcal{M}}(u,u')$ where $u = tm + b$ and $u' = t'm + b$.

For an n-parameter persistence module $\mathcal{M}$, we denote the one-parameter persistence module obtained by restricting $\mathcal{M}$ to L by $\mathcal{M}^L$. That is, to every point on L, $\mathcal{M}^L$ assigns the vector space determined by $\mathcal{M}$ at that point in $\mathbb{R}^n$ with transition maps induced by $\mathcal{M}$.

Lemma 5.2.1. *Let $\mathcal{I}^I$ be an n-parameter finitely presented interval persistence module with underlying interval I in $\mathbb{R}^n$. Then $\mathcal{I}^{I^L}$ is either a one-parameter interval persistence module with underlying interval $I^L \doteq I \cap L$ if $I^L \neq \emptyset$ or zero persistence module if $I^L = \emptyset$ for any admissible line.*

Proof. Let $\mathcal{I}^I$ be an n-parameter interval persistence module with underlying interval I in $\mathbb{R}^n$ and let L be an admissible line. By restricting $\mathcal{I}^I$ to the admissible line we have either $I^L \doteq I \cap L \neq \emptyset$ or $I^L \doteq I \cap L = \emptyset$. If $I^L \doteq I \cap L \neq \emptyset$, then $\mathcal{I}^{I^L}$ is a one-parameter interval persistence module with underlying interval $I^L \doteq I \cap L \in \mathbb{R}$ as follows:

- $\mathcal{I}^L_u = k$ for every $u \in I^L$ and $\mathcal{I}^L_u = 0$ for every $u \notin I^L$.
- $\varphi_{\mathcal{I}^L}(u,v) = \mathrm{id}_k$ for points $u \preceq v \in I^L$.

If $I^L \doteq I \cap L = \emptyset$, by Definition 3.2.1 and Definition 3.2.3, $\mathcal{I}^{I^L}$ is the zero persistence module. □

Later, we will show that $\mathcal{I}^{I^L}$ is a finitely presented one-parameter interval persistence module if $\mathcal{I}^I$ is a finitely presented interval persistence module.

Corollary 5.2.2. *If $\mathcal{F}$ is an n-parameter indecomposable free interval persistence* *module, then the one-parameter persistence module $\mathcal{F}^L$ is also a free interval persistence module for any admissible line L.*

Proof. Suppose that $\mathcal{F}$ is an n-parameter free interval persistence module with underlying free interval $I = \langle v \rangle = \{v' \in \mathbb{R}^n : v \preceq v'\}$. Let L be a line with direction vector $m \in (0, +\infty)^n$ and passing through $b \in \mathbb{R}^n$. Then by restricting $\mathcal{F}$ to the line L, we have $(\mathcal{F}^L)_t = \mathcal{F}_u$ and $\varphi_{\mathcal{F}^L}(t, t') = \varphi_{\mathcal{F}}(u, u')$ for all $t \leq t' \in I^L = I \cap L = \langle \overline{s} \rangle = \{s' \in \mathbb{R} : \overline{s} \leq s'\}$ where $u = tm + b \preceq u' = t'm + b \in I$, $\overline{v} = \overline{s}m + b$ for a unique $\overline{v} \in \partial(I) \cap L$ and I^L is the free interval derived from the free interval I. Hence, $\mathcal{F}^L$ is a one-parameter free interval persistence module with underlying free interval I^L. □

Remark 5.2.3. *Note that the restriction of a direct sum of persistence modules is the direct sum of the restriction of persistence modules. In particular, for any n-parameter free decomposable persistence module $\mathcal{F}$, the one-parameter persistence module $\mathcal{F}^L$ is also a free decomposable persistence module.*

Proposition 5.2.4. *If $\mathcal{M}$ is an n-parameter finitely presented persistence module, then the one-parameter persistence module $\mathcal{M}^L$ is also finitely presented.*

Proof. Assume that $\mathcal{M}$ is an n-parameter finitely presented persistence module. Equivalently, we have an exact sequence

$$\ker(\phi) \hookrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{M} \to 0$$

with $\mathcal{F} = \bigoplus_{i=1}^{m} \mathcal{F}_i, \ker(\phi) = \bigoplus_{j=1}^{k} \mathcal{G}_j$ where $\mathcal{F}_1, \ldots, \mathcal{F}_m$ and $\mathcal{G}_1, \ldots, \mathcal{G}_k$ are n-parameter free interval persistence modules. It is enough to show that

$$\ker(\phi^L) \hookrightarrow \mathcal{F}^L \xrightarrow{\phi^L} \mathcal{M}^L \to 0$$

is an exact sequence where F^L and $\ker(\phi^L)$ are one-parameter free persistence modules by Proposition 5.2.2 and Remark 5.2.3 obtained by restricting free persistence modules $\mathcal{F}$ and $\ker(\phi)$ to the line L .

Now, we know that persistence modules $\mathcal{F}$ and $\mathcal{M}$ are functors from the category $\mathbb{R}^n$ to the category of vector spaces **Vect**. By assumption, we have surjective natural transformation $\phi : \mathcal{F} \to \mathcal{M}$. So, by definition of the natural transformation, we have the following commutative diagram for all $u, u' \in \mathbb{R}^n$:

$$\begin{array}{ccc} \mathcal{F}_u & \xrightarrow{\phi_u} & \mathcal{M}_u \\ {\scriptstyle \varphi_{\mathcal{F}}(u,u')} \downarrow & & \downarrow {\scriptstyle \varphi_{\mathcal{M}}(u,u')} \\ \mathcal{F}_{u'} & \xrightarrow{\phi_{u'}} & M_{u'} . \end{array}$$

Moreover, for all $u \in \mathbb{R}^n$, the linear map $\phi_u : \mathcal{F}_u \to \mathcal{M}_u$ is surjective since we have surjective natural transformation between functors. Therefore, by the line restriction construction given at the beginning of Chapter 5.2, we have the following commutative diagram for all $t, t' \in \mathbb{R}$ where $(\phi^L)_t := \phi_u$ with $u = mt + b$:

$$\begin{array}{ccc} (\mathcal{F}^L)_t & \xrightarrow{(\phi^L)_t} & (M^L)_t \\ {\scriptstyle \varphi_{(\mathcal{F}^L)(t,t')}} \downarrow & & \downarrow {\scriptstyle \varphi_{(\mathcal{M}^L)(t,t')}} \\ (\mathcal{F}^L)_{t'} & \xrightarrow{(\phi^L)_{t'}} & (M^L)_{t'} . \end{array}$$

Now, since, for all $u \in \mathbb{R}^n$, ϕ_u is surjective, we have surjective natural transformation $\phi^L : \mathcal{F}^L \to \mathcal{M}^L$.

Now, consider other natural transformations, which are inclusion maps, $\iota : \ker(\phi) \hookrightarrow \mathcal{F}$. Then, we have following commutative diagram for all $u, u' \in \mathbb{R}^n$ where each $\iota_u : \ker(\phi)_u \hookrightarrow \mathcal{F}_u$ is inclusion since $\iota : \ker(\phi) \hookrightarrow \mathcal{F}$ inclusion natural transformation:

$$\begin{array}{ccc} \ker(\phi)_u & \xrightarrow{(\iota)_u} & \mathcal{F}_u \\ {\scriptstyle \varphi_{\ker(\phi)}(u,u')} \downarrow & & \downarrow {\scriptstyle \varphi_{\mathcal{F}}(u,u')} \\ \ker(\phi)_{u'} & \xrightarrow{(\iota)_{u'}} & \mathcal{F}_{u'} . \end{array}$$

Again, by line restriction construction above we have the following commutative diagrams for all $t, t' \in \mathbb{R}$ where $(\iota^L)_t := \iota_u$ with $u = mt + b$:

$$\begin{array}{ccc} \ker(\phi^L)_t & \xrightarrow{(\iota^L)_t} & (\mathcal{F}^L)_t \\ {\scriptstyle \varphi_{\ker(\phi^L)}(t,t')} \downarrow & & \downarrow {\scriptstyle \varphi_{\mathcal{F}^L}(t,t')} \\ \ker(\phi^L)_{t'} & \xrightarrow{(\iota^L)_{t'}} & (\mathcal{F}^L)_{t'} . \end{array}$$

Since, for all $t \in \mathbb{R}$, $(\iota^L)_t$ is inclusion, we have inclusion natural transformation $\iota^L : \ker(\phi^L) \to \mathcal{F}^L$. Therefore, the sequence $\ker(\phi^L) \hookrightarrow \mathcal{F}^L \xrightarrow{\phi^L} \mathcal{M}^L \to 0$ is exact since $\mathrm{im}(\iota^L) = \ker(\phi^L)$. Hence, the one-parameter persistence module $\mathcal{M}^L$ is finitely presented.

□

5.3 Matching Distance

We are ready to give the definition of the matching distance between persistence modules.

Definition 5.3.1 (Landi, [17])**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be n-parameter persistence modules. The **matching distance** between persistence modules $\mathcal{M}$ and $\mathcal{N}$ of is defined by*

$$d_{\mathrm{match}}(\mathcal{M},\mathcal{N}) \doteq \sup_{L\in\Lambda} m^L d_B(\mathcal{M}^L,\mathcal{N}^L)$$

where $m^L \doteq \min_{i=1,\ldots,n}\{m_i\}$ and $m_i > 0$ for all $i = 1,\ldots,n$.

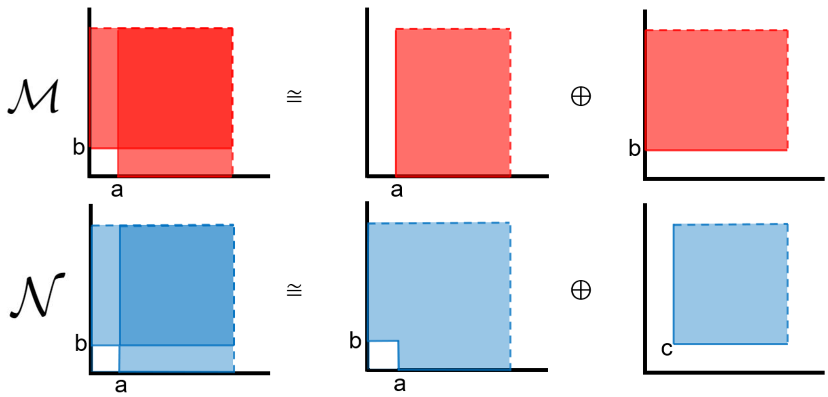


Figure 5.1: Two non-isomorphic interval decomposable bipersistence modules.

The example above shows that interval decomposable bipersistence modules are non-isomorphic as their decompositions differ. However, after restricting the persistence modules for any admissible line, we get isomorphic one-parameter persistence modules since we get an isomorphic copy of each summand in one-parameter. Therefore, the bottleneck distance between one-parameter persistence modules is zero. Hence, the matching distance between interval decomposable bipersistence modules is zero.

By the universality of the interleaving distance [18], we know that any distance that is a lower bound of the interleaving distance is stable. Thus, the matching distance is stable since the matching distance is a lower bound for the interleaving distance [17].

Proposition 5.3.2. *Let* $\mathcal{M} = \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^I$ *and* $\mathcal{N} = \bigoplus_{J \in B(\mathcal{N})} \mathcal{I}^J$ *be two finitely presented interval decomposable persistence modules. Then*

$$d_{\text{match}}(\mathcal{M}, \mathcal{N}) = d_{\text{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}})$$

where $\overline{\mathcal{M}} \doteq \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^{\bar{I}}$ *and* $\overline{\mathcal{N}} \doteq \bigoplus_{J \in B(\mathcal{N})} \mathcal{I}^{\bar{J}}$*. Similarly, we have*

$$d_{\text{match}}(\mathcal{M}, \mathcal{N}) = d_{\text{match}}(\mathcal{M}^{\circ}, \mathcal{N}^{\circ})$$

where $\mathcal{M}^{\circ} \doteq \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^{I^{\circ}}$ *and* $\mathcal{N}^{\circ} \doteq \bigoplus_{J \in B(\mathcal{N})} \mathcal{I}^{J^{\circ}}$*.*

Proof. Straightforwardly from Proposition 4.2.5 for any admissible line L, we have

$$d_B(\mathcal{M}^L, \mathcal{N}^L) = d_B(\overline{\mathcal{M}}^L, \overline{\mathcal{N}}^L) = d_B(\mathcal{M}^{\circ L}, \mathcal{N}^{\circ L}).$$

Hence, we have

$d_{\text{match}}(\mathcal{M}, \mathcal{N}) = \sup_{L \in \Lambda} m^L d_B(\mathcal{M}^L, \mathcal{N}^L) = \sup_{L \in \Lambda} m^L d_B(\overline{\mathcal{M}}^L, \overline{\mathcal{N}}^L) = d_{\text{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}})$ and

$d_{\text{match}}(\mathcal{M}, \mathcal{N}) = \sup_{L \in \Lambda} m^L d_B(\mathcal{M}^L, \mathcal{N}^L) = \sup_{L \in \Lambda} m^L d_B(\mathcal{M}^{\circ L}, \mathcal{N}^{\circ L}) = d_{\text{match}}(\mathcal{M}^{\circ}, \mathcal{N}^{\circ}).$

□

The proposition above shows that the matching distance is not a metric for non-finitely presented persistence modules. Moreover, as the previous example shows, the matching distance is still not a metric, even for finitely presented modules. In other words, two non-isomorphic finitely presented persistence modules may have zero matching distance [19].

CHAPTER 6

STEADY MATCHING DISTANCE

While the matching distance is relatively simple to define, it is not easy to make actual computations. The main reason is that one needs to consider optimal matchings for infinitely many admissible lines. Furthermore, it is an incomplete invariant even for finitely presented persistence modules. Because of these reasons, we define a new distance called steady matching distance and study its properties. We start this chapter by explaining the restriction of partial multibijections to admissible lines, after which we define steady matching distance and investigate its properties.

6.1 Restriction of Partial Multibijections to Lines

Let Λ be the set of admissible lines defined in Section 5.1 and let $\mathcal{M}$ be an n-parameter persistence module. We denote the one-parameter persistence module obtained by restricting $\mathcal{M}$ to L by $\mathcal{M}^L$ (for details, see Section 5.2).

Note that if $\mathcal{M} = \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^I$ where each summand $\mathcal{I}^I$ is an interval persistence module with an underlying interval I, then by Lemma 5.2.1, for any admissible line L, we have $\mathcal{M}^L \doteq \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^{I^L}$ where each summand $\mathcal{I}^{I^L}$ is the interval persistence module with underlying interval $I^L \doteq I \cap L$. However, keep in mind that $\mathcal{I}^{I^L}$ can be the zero persistence module if the admissible line L does not intersect the underlying interval I of the interval persistence module $\mathcal{I}^I$. If this is the case, since an interval cannot be the empty set because of the Definition 3.2.1, it will not be an element of the barcode $B(\mathcal{M}^L)$ and hence the zero persistence module $\mathcal{I}^{I^L}$ will not be in the decomposition. In other words, $B(\mathcal{M}^L)$ is a one-parameter barcode, and its elements come from the non-trivial intersection of the elements of the barcode $B(\mathcal{M})$ and the admissible line L. Hence each summand $\mathcal{I}^{I^L}$ that appears in the decomposition will be a non-zero interval persistence module.

Let $\mathcal{M}$ and $\mathcal{N}$ be two finitely presented interval decomposable persistence modules with given decompositions

$$\mathcal{M} = \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^I \text{ and } \mathcal{N} = \bigoplus_{J \in B(\mathcal{N})} \mathcal{I}^J$$

where each summand $\mathcal{I}^I$ and $\mathcal{I}^J$ are interval persistence modules with underlying intervals I and J, and $B(\mathcal{M})$ and $B(\mathcal{N})$ are the barcodes for interval decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$, respectively. Now, we can define the partial multibijection σ^L induced by the partial multibijection σ after the restriction of line L.

Definition 6.1.1. *Let $\sigma\colon B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ be a partial multibijection between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$ with $\sigma\colon \tilde{B}(\mathcal{M}) \to \tilde{B}(\mathcal{N})$ being the multibijection between*

$$\tilde{B}(\mathcal{M}) \doteq \operatorname{coim}\sigma \subseteq B(\mathcal{M}) \text{ and } \tilde{B}(\mathcal{N}) \doteq \operatorname{im}\sigma \subseteq B(\mathcal{N}).$$

Then, for any admissible line L, define $\sigma^L\colon B(\mathcal{M}^L) \nrightarrow B(\mathcal{N}^L)$, as follows:

- *If $I \notin \operatorname{coim}\sigma$, then $I^L \notin \operatorname{coim}\sigma^L$.*
- *If $J \notin \operatorname{im}\sigma$, then $J^L \notin \operatorname{im}\sigma^L$.*
- *If $I \in \operatorname{coim}\sigma$ but $I \cap L = \emptyset$, then $I^L \notin \operatorname{coim}\sigma^L$.*
- *If $J \in \operatorname{im}\sigma$ but $J \cap L = \emptyset$, then $J^L \notin \operatorname{im}\sigma^L$.*
- *If $I \in \operatorname{coim}\sigma$ with $I \cap L \neq \emptyset$ and $\sigma(I) = J \in \operatorname{im}\sigma$ with $J \cap L \neq \emptyset$, then $\sigma^L(I^L) \doteq J^L$.*

Proposition 6.1.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two n-parameter finitely presented interval decomposable persistence modules with barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$, respectively. Let σ be a partial multibijection between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$ with $\sigma\colon \tilde{B}(\mathcal{M}) \to \tilde{B}(\mathcal{N})$ being the multibijection between $\tilde{B}(\mathcal{M})$ and $\tilde{B}(\mathcal{N})$. Then for any admissible line $L \in \Lambda$, σ^L is a partial multibijection between one-parameter barcodes $B(\mathcal{M}^L)$ and $B(\mathcal{N}^L)$.*

Proof. Let $\sigma\colon B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ be a partial multibijection between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$ with $\sigma\colon \tilde{B}(\mathcal{M}) \to \tilde{B}(\mathcal{N})$ being the multibijection between $\tilde{B}(\mathcal{M}) = \operatorname{coim}\sigma \subseteq B(\mathcal{M})$ and $\tilde{B}(\mathcal{N}) = \operatorname{im}\sigma \subseteq B(\mathcal{N})$.

Let L be an admissible line. After restricting the persistence modules $\mathcal{M}$ and $\mathcal{N}$ to the admissible line L, by Lemma 5.2.1 and Remark 5.2.3, we have one-parameter barcodes $B(\mathcal{M}^L)$ and $B(\mathcal{N}^L)$ of one-parameter persistence modules $\mathcal{M}^L$ and $\mathcal{N}^L$, respectively. Now, by Definition 6.1.1, $\sigma^L\colon B(\mathcal{M}^L) \nrightarrow B(\mathcal{N}^L)$ is a partial multibijection between the barcodes $B(\mathcal{M}^L)$ and $B(\mathcal{N}^L)$ with $\sigma^L\colon \tilde{B}(\mathcal{M}^L) \to \tilde{B}(\mathcal{N}^L)$ being the multibijection between $\tilde{B}(\mathcal{M}^L) = \operatorname{coim}\sigma^L \subseteq B(\mathcal{M}^L)$ and $\tilde{B}(\mathcal{N}^L) = \operatorname{im}\sigma^L \subseteq B(\mathcal{N}^L)$. □

Remark 6.1.3. *Note that if $I \in B(\mathcal{M})$ and $I \cap L \neq \emptyset$, then necessarily $I \cap L = I^L \in B(\mathcal{M}^L)$ for any admissible line L. However, if $I \in \tilde{B}(\mathcal{M})$ and $I \cap L \neq \emptyset$, then it is not necessarily true that $I \cap L = I^L \in \tilde{B}(\mathcal{M}^L)$ for any admissible line L.*

6.2 Steady Matching Distance

In this section, we define the steady matching distance between two n-parameter interval decomposable persistence modules.

Definition 6.2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two n-parameter interval decomposable persistence modules with given decompositions*

$$\mathcal{M} = \bigoplus_{I \in B(\mathcal{M})} \mathcal{I}^I \text{ and } \mathcal{N} = \bigoplus_{J \in B(\mathcal{N})} \mathcal{I}^J.$$

*The **steady matching distance** is defined as*

$$SD_{\text{match}}(\mathcal{M}, \mathcal{N}) \doteq \inf_{\sigma \in S} \sup_{L \in \Lambda} m^L\big(\operatorname{cost}(\sigma^L)\big)$$

where S is the set of all partial multibijections $\sigma\colon B(\mathcal{M}) \nrightarrow B(\mathcal{N})$, σ^L is the partial multibijection induced by σ defined as above, and $\operatorname{cost}(\sigma^L)$ is as defined in the Definition 4.2.1.

Note that the definition of steady matching distance is defined for n-parameter interval decomposable persistence modules with given decomposition since we need n-parameter barcodes of the persistence modules and the set of all partial multibijections between the barcodes. Furthermore, we need to define steady matching distance for interval decomposable persistence modules since after restricting the persistence module to an admissible line, we need to be able to determine one-parameter barcodes of persistence modules and the corresponding partial multibijection between one-parameter barcodes.

Recall that a pseudometric is a generalization of a metric for which the distance between two distinct points can be zero. An extended pseudometric is a pseudometric that can assume the value of infinity.

Proposition 6.2.2. *The steady matching distance $SD_{\mathrm{match}}(\mathcal{M}, \mathcal{N})$ is an extended pseudometric for interval decomposable persistence modules.*

Proof. Note that if the number of free interval persistence modules is different in the decompositions of interval decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$, then by Corollary 5.2.2 the number of one-parameter free interval persistence modules are different in the decomposition of one-parameter persistence modules $\mathcal{M}^L$ and $\mathcal{N}^L$ since restricting each free interval persistence module to any admissible line will give us a one-parameter free interval persistence module. Now, since after restricting, we have a different number of one-parameter free persistence modules, the cost of each partial multibijection between barcodes $B(\mathcal{M}^L)$ and $B(\mathcal{N}^L)$ is infinite for at least one admissible line L since there will be at least one unmatched one-parameter free interval persistence module. Hence the steady matching distance is infinite. Therefore, the steady matching distance is an extended distance.

Now let us show that the steady matching distance is a pseudometric. We know that if $\mathcal{M}$ and $\mathcal{N}$ are isomorphic persistence modules, then their decomposition is the same thanks to the standard formulation of the Krull-Schmidt Theorem [1]. Moreover, $\mathcal{M}^L$ and $\mathcal{N}^L$ will be isomorphic persistence modules. So, there exists a partial multibijection $\bar{\sigma}$ (optimal partial multibijection between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$) such that $\bar{\sigma}^L$ has a zero cost between the barcodes $B(\mathcal{M}^L)$ and $B(\mathcal{N}^L)$ for any admissible line L, and hence the steady matching distance will be zero straightforwardly from the definition of the steady matching distance.

Note that for every interval decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$,
$SD_{\text{match}}(\mathcal{M},\mathcal{N}) = \inf_{\sigma\in S} \sup_{L\in\Lambda} m^L\big(\operatorname{cost}(\sigma^L)\big) = \inf_{\beta\in S} \sup_{L\in\Lambda} m^L\big(\operatorname{cost}(\beta^L)\big) = SD_{\text{match}}(\mathcal{N},\mathcal{M})$
where β^L is the inverse partial multibijection of σ^L for any admissible line L.

Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ be two persistence modules, and $\sigma^L\colon B(\mathcal{M}^L) \nrightarrow B(\mathcal{N}^L)$ and $\tau^L : B(\mathcal{N}^L) \nrightarrow B(\mathcal{P}^L)$ be the partial multibijections for an admissible line L induced from the partial multibijections σ and τ, respectively. Note that $(\tau\circ\sigma)^L = \tau^L \circ \sigma^L$ is the partial multibijection between the barcodes $B(\mathcal{M}^L)$ and $B(\mathcal{P}^L)$ for any admissible line L induced from the partial multibijection $\tau\circ\sigma$ between the barcodes $B(\mathcal{M})$ and $B(\mathcal{P})$. Note that, $\operatorname{cost}\big((\tau\circ\sigma)^L\big) \le \operatorname{cost}(\tau^L)+\operatorname{cost}(\sigma^L)$ for any admissible line L. Therefore,

$$\sup_L \operatorname{cost}\big((\tau\circ\sigma)^L\big) \le \sup_L \big(\operatorname{cost}(\tau^L)+\operatorname{cost}(\sigma^L)\big) \le \sup_L \big(\operatorname{cost}(\tau^L)\big)+\sup_L \big(\operatorname{cost}(\sigma^L)\big).$$

Thus, $SD_{\text{match}}(\mathcal{M},\mathcal{P}) \le SD_{\text{match}}(\mathcal{M},\mathcal{N}) + SD_{\text{match}}(\mathcal{N},\mathcal{P})$.

Hence, the steady matching distance is an extended pseudometric.

□

The following example shows that the steady matching distance is not a metric.

Example 6.2.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval bipersistence modules with given underlying intervals as below where $a = (a_1, a_2)$ and $b = (b_1, b_2)$. In particular, they are rectangle bipersistence modules since underlying intervals are rectangles (for details, see Definition 3.3.3). Note that, they are non-isomorphic bipersistence modules since they are not* 0*-interleaved.*

Now let $\sigma \in S$ be the partial multibijection matching the underlying rectangles

$$R_{\mathcal{M}} = [a_1, b_1] \times [a_2, b_2] \text{ and } R_{\mathcal{N}} = [a_1, b_1) \times [a_2, b_2).$$

Then, for any admissible line L, σ^L is an ϵ-matching for any $\epsilon > 0$. Since $\operatorname{cost}(\sigma^L) = \inf\{\epsilon \ge 0 : \sigma^L$ *is an ϵ-matching*$\}$, *we have* $\operatorname{cost}(\sigma^L) = 0$ *and so*

$$SD_{\text{match}}(\mathcal{M},\mathcal{N}) = \inf_{\sigma\in S} \sup_{L\in\Lambda} m^L\big(\operatorname{cost}(\sigma^L)\big) = 0.$$

Hence, the steady matching distance is not a metric for interval persistence modules.

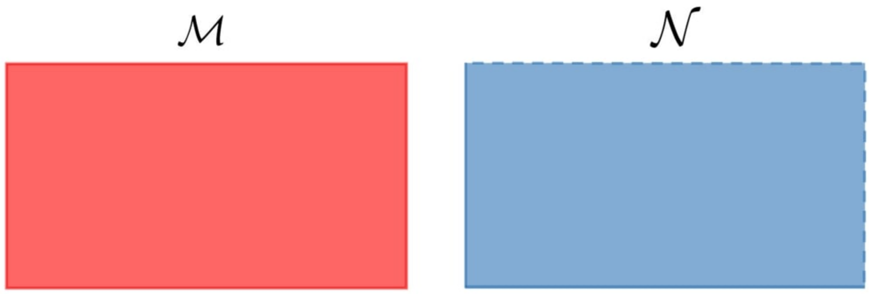


Figure 6.1: Two non-isomorphic rectangle bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ with $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = 0$.

As we observed above, steady matching distance is not a metric for arbitrary types of persistence modules, like the matching distance, the interleaving distance, and the bottleneck distance. Fortunately, unlike the matching distance, steady matching distance is a metric when restricted to finitely presented interval decomposable persistence modules.

Proposition 6.2.4. *The steady matching distance is a metric for finitely presented interval decomposable persistence modules.*

Proof. We already know that the steady matching distance is a pseudometric. Thus it suffices to show having $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = 0$ implies that the finitely presented interval decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$ are isomorphic. We will get a contradiction by assuming $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = 0$, and $\mathcal{M}$ and $\mathcal{N}$ are non-isomorphic finitely presented persistence modules. Since $\mathcal{M}$ and $\mathcal{N}$ are finitely presented persistence modules and $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = 0$, there exists a partial multibijection $\bar{\sigma}\colon B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ such that $\sup\limits_{L \in \Lambda} m^L \operatorname{cost}(\bar{\sigma}^L) = 0$. Thus, we have $m^L \operatorname{cost}(\bar{\sigma}^L) = 0$, and so, $\operatorname{cost}(\bar{\sigma}^L) = 0$ for every admissible line L. Moreover, we must have $\operatorname{coim}(\bar{\sigma}) = B(\mathcal{M})$ and $\operatorname{im}(\bar{\sigma}) = B(\mathcal{N})$, in particular $|B(\mathcal{M})| = |B(\mathcal{N})|$. Otherwise, there exists a line $\bar{L}$ such that $|B(\mathcal{M}^{\bar{L}})| \neq |B(\mathcal{N}^{\bar{L}})|$. In that case, $\operatorname{cost}(\bar{\sigma}^{\bar{L}}) \neq 0$ since there exists at least one non-trivial one-parameter persistence module unmatched which is 2ϵ-trivial for some $\epsilon > 0$. Contradiction. Therefore, let $|B(\mathcal{M})| = |B(\mathcal{N})| = n$ for some $n \in \mathbb{N}^+$. However, by assumption, it is given that they are non-isomorphic persistence modules. Thus, there exits at least one interval persistence module in the decomposition of the persistence module $\mathcal{M}$, say $\mathcal{M}_i$ where $i \in \{1, 2, \ldots, n\}$, which is not isomorphic to any interval persistence module in the decomposition of the persistence module $\mathcal{N}$. Let $\bar{\sigma}(\mathcal{M}_i) = \mathcal{N}_j$ where $j \in \{1, 2, \ldots, n\}$. We can find a line L' such that $\mathcal{M}_i^{L'}$ and $\mathcal{N}_j^{L'}$ are not isomorphic persistence modules since $\mathcal{M}_i$ and $\mathcal{N}_j$ are not isomorphic persistence modules. If either $\mathcal{M}_i^{L'}$ or $\mathcal{N}_j^{L'}$ is the zero persistence module, then $\operatorname{cost}(\bar{\sigma}^{L'}) \neq 0$ for the same reason above. Again, it is a contradiction. Suppose neither $\mathcal{M}_i^{L'}$ nor $\mathcal{N}_j^{L'}$ is the zero persistence module. By Proposition 5.2.4, we know that both $\mathcal{M}_i^{L'}$ and $\mathcal{N}_j^{L'}$ finitely presented persistence modules. Thus, again $\operatorname{cost}(\bar{\sigma}^{L'}) \neq 0$ since they are non-isomorphic finitely presented persistence modules. Thus, it is again a contradiction. Thus, any interval persistence module in the decomposition of $\mathcal{M}$ is isomorphic to one of the interval persistence modules in the decomposition of $\mathcal{N}$. Hence, if $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = 0$, then the finitely presented interval decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$ must be isomorphic. □

We will now show that if $\mathcal{M}$ and $\mathcal{N}$ are two interval persistence modules, then $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = SD_{\text{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}}) = SD_{\text{match}}(\mathcal{M}^{\circ}, \mathcal{N}^{\circ})$.

Proposition 6.2.5. *Let $\mathcal{M} = \mathcal{I}^I$ and $\mathcal{N} = \mathcal{I}^J$ be two interval bipersistence modules with underlying intervals I and J, respectively. Then, $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = SD_{\text{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}})$ where $\overline{\mathcal{M}} \doteq \mathcal{I}^{\bar{I}}$, and $\overline{\mathcal{N}} \doteq \mathcal{I}^{\bar{J}}$. Likewise, $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = SD_{\text{match}}(\mathcal{M}^{\circ}, \mathcal{N}^{\circ})$ where $\mathcal{M}^{\circ} \doteq \mathcal{I}^{I^{\circ}}$ and $\mathcal{N}^{\circ} \doteq \mathcal{I}^{J^{\circ}}$.*

Proof. Let us show that $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = SD_{\text{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}})$. By using the same analogy one can prove that $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = SD_{\text{match}}(\mathcal{M}^{\circ}, \mathcal{N}^{\circ})$.

Now, we will first show that $SD_{\text{match}}(\mathcal{M}, \overline{\mathcal{M}}) = 0$ for any interval persistence module $\mathcal{M} = \mathcal{I}^I$ with underlying interval I where $\overline{\mathcal{M}} = \mathcal{I}^{\bar{I}}$ is also an interval persistence module with underlying interval $\bar{I}$. Let σ be a partial multibijection between the barcodes $B(\mathcal{M}) = \{I\}$ and $B(\mathcal{N}) = \{\bar{I}\}$ with $\sigma\colon \tilde{B}(\mathcal{M}) \to \tilde{B}(\mathcal{N})$ being the multibijection between $\tilde{B}(\mathcal{M}) = \operatorname{coim} \sigma = \{I\}$ and $\tilde{B}(\mathcal{N}) = \operatorname{im} \sigma = \{\bar{I}\}$. By way of explanation, the partial multibijection σ matches the interval persistence module $\mathcal{M}$ to the interval persistence module $\overline{\mathcal{M}}$. Let L be an any admissible line, then observe that σ^L is an ϵ-matching for any $\epsilon > 0$ since σ^L is a partial multibijection between the sets $\{I^L\}$ and $\{\bar{I}^L\}$ where $I^L = I \cap L$ and $\bar{I}^L = \bar{I} \cap L$. Thus, $\mathrm{cost}(\sigma^L) = 0$ for any admissible line L and so, $\sup\limits_{L\in\Lambda} m^L \mathrm{cost}(\sigma^L) = 0$. Then, clearly $SD_{\mathrm{match}}(\mathcal{M}, \overline{\mathcal{M}}) = \inf\limits_{\sigma\in S} \sup\limits_{L\in\Lambda} m^L \mathrm{cost}(\sigma^L) = 0$ for any interval persistence module $\mathcal{M}$. By symmetry, also $SD_{\mathrm{match}}(\overline{\mathcal{M}}, \mathcal{M}) = 0$.

Now, we have the following inequalities by the triangle inequality property of the steady matching distance.

- $SD_{\mathrm{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}}) \leq SD_{\mathrm{match}}(\overline{\mathcal{M}}, \mathcal{M}) + SD_{\mathrm{match}}(\mathcal{M}, \mathcal{N}) + SD_{\mathrm{match}}(\mathcal{N}, \overline{\mathcal{N}})$ and
- $SD_{\mathrm{match}}(\mathcal{M}, \mathcal{N}) \leq SD_{\mathrm{match}}(\mathcal{M}, \overline{\mathcal{M}}) + SD_{\mathrm{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}}) + SD_{\mathrm{match}}(\overline{\mathcal{N}}, \mathcal{N})$.

Now, by above we have shown that $SD_{\mathrm{match}}(\mathcal{M}, \overline{\mathcal{M}}) = 0$ for any interval persistence module $\mathcal{M}$. Therefore, we have $SD_{\mathrm{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}}) \leq SD_{\mathrm{match}}(\mathcal{M}, \mathcal{N})$ and $SD_{\mathrm{match}}(\mathcal{M}, \mathcal{N}) \leq SD_{\mathrm{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}})$ from the inequalities above. Hence, we can conclude that $SD_{\mathrm{match}}(\mathcal{M}, \mathcal{N}) = SD_{\mathrm{match}}(\overline{\mathcal{M}}, \overline{\mathcal{N}})$ for any interval persistence modules $\mathcal{M}$ and $\mathcal{N}$. □

We want to finish this chapter with a technical result to be used in the Chapter 9.

Lemma 6.2.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. If*

$$\min \Big\{ \max \Big\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \Big\}, \max\{\|c - a\|_\infty, \|d - b\|_\infty\} \Big\} = +\infty,$$

then $SD_{\mathrm{match}}(\mathcal{M}, \mathcal{N}) = +\infty$.

Proof. By contradiction, suppose that the steady matching distance between rectangle bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ is finite. Remember that

$$SD_{\mathrm{match}}(\mathcal{M}, \mathcal{N}) = \min\{\sup_{L\in\Lambda} m^L \mathrm{cost}(\sigma_1^L), \sup_{L\in\Lambda} m^L \mathrm{cost}(\sigma_2^L)\}$$

where σ_1 and σ_2 are two partial multibijections such that one matches bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ with the zero bipersistence modules and the other one matches the bipersistence module $\mathcal{M}$ with the bipersistence module $\mathcal{N}$, respectively. It is clear that $\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_1^L) < +\infty$ or $\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L) < +\infty$ since $SD_{\text{match}}(\mathcal{M},\mathcal{N}) < +\infty$.

By assumption, we know that

$$\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2}, \min_{i=1,2}\frac{d_i-c_i}{2}\Big\} = +\infty$$

and

$$\max\{\|c-a\|_\infty, \|d-b\|_\infty\} = +\infty.$$

This implies that $\min_{i=1,2}\frac{b_i-a_i}{2} = +\infty$ or $\min_{i=1,2}\frac{d_i-c_i}{2} = +\infty$, and $\|c-a\|_\infty = +\infty$ or $\|d-b\|_\infty = +\infty$.

Without loss of generality, suppose that $\min_{i=1,2}\frac{b_i-a_i}{2} = +\infty$ otherwise, just change the role of the given persistence modules. Suppose for the moment that $\|c-a\|_\infty = +\infty$. If not, then $\|d-b\|_\infty = +\infty$, and one can prove the argument by using the same idea by replacing the point c with d and a with b. The former assumption implies that $\frac{b_i-a_i}{2} = +\infty$ for every $i \in \{1,2\}$, and the latter assumption implies that $|c_i - a_i| = +\infty$ for some $i \in \{1,2\}$. Observe that because of the first assumption $\mathcal{M}$ can only be of one of the types $R_1, R_2, R_3, R_5, R_6, R_7, R_9, R_{10}$ or R_{11} in Example 3.3.5.

Case 1: Suppose that $\mathcal{M}$ is one of the types R_1, R_2, R_5, or R_6. Then, $\mathcal{M}$ has the underlying rectangle $R_{\mathcal{M}} = (-\infty, b_1) \times (-\infty, b_2)$ where $b_1, b_2 \in \mathbb{R} \cup \{+\infty\}$. Let $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$ where $c_1, c_2 \in \mathbb{R} \cup \{-\infty\}$ and $d_1, d_2 \in \mathbb{R} \cup \{+\infty\}$. It is assumed that $\|c-a\|_\infty = +\infty$, which is why c_1 or c_2 must be a finite number. Suppose first that c_1 is finite and consider the diagonal line L passing through the point (c_1, c_1) parameterized by $L : u = t \cdot (1,1) + (c_1, c_1)$ where $u \in L$ and $t \in \mathbb{R}$ (for details see Subsection 5.1). After the line restriction, $\mathcal{M}^L$ is a one-parameter persistence module with underlying interval $I_{\mathcal{M}^L} = (-\infty, \lambda)$ where $\lambda \in \mathbb{R} \cup \{+\infty\}$. On the other hand, $\mathcal{N}^L$ is either one-parameter persistence module with underlying interval $I_{\mathcal{N}^L} = (0, \mu)$ with $\mu \in \mathbb{R} \cup \{+\infty\}$ or the zero persistence module if $c_1 \geq c_2$, or it is either one-parameter persistence module with underlying interval $I_{\mathcal{N}^L} = (c_2 - c_1, \mu)$ with $\mu \in \mathbb{R} \cup \{+\infty\}$ or the zero persistence module if $c_2 > c_1$. In all cases, either the birth point of one-parameter persistence module $\mathcal{N}^L$ is a finite number while that of

$\mathcal{M}^L$ is not or $\mathcal{N}^L$ is the zero persistence module, we have $\sup\limits_{L\in\Lambda} m^L \operatorname{cost}(\sigma_1^L) = +\infty$ and $\sup\limits_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L) = +\infty$, which contradicts with our assumption. A similar argument can be used to get a contradiction if c_2 is a finite number by just considering the diagonal line passing through (c_2, c_2) and imitating the argument above.

Case 2: Suppose that $\mathcal{M}$ is one of the types R_3 or R_7 with underlying rectangle being $R_{\mathcal{M}} = (-\infty, b_1) \times (a_2, +\infty)$ where $a_2 > -\infty$ and $b_1 \in \mathbb{R} \cup \{+\infty\}$. Let $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$ where $c_1, c_2 \in \mathbb{R} \cup \{-\infty\}$ and $d_1, d_2 \in \mathbb{R} \cup \{+\infty\}$. Since $\|c - a\|_\infty = +\infty$, c_1 is a finite number or $c_2 = -\infty$. Suppose first $c_1 > -\infty$, and consider the sequence of diagonal lines L_k passing through the point $(c_1 - k, a_2)$ parameterized by $L : u = t \cdot (1, 1) + (c_1 - k, a_2)$ where $k > 0$, $u \in L$ and $t \in \mathbb{R}$. After the line restriction, if k is so large that $c_1 - k < b_1$, then $\mathcal{M}^{L_k}$ is a one-parameter persistence module with underlying interval $I_{\mathcal{M}^{L_k}} = (0, \lambda)$ where $\lambda \in \mathbb{R} \cup \{+\infty\}$, whereas $\mathcal{N}^{L_k}$ is a one-parameter persistence module with underlying interval $I_{\mathcal{N}^{L_k}} = (k, \mu)$ where $\mu \in \mathbb{R} \cup \{+\infty\}$, or $I_{\mathcal{N}^{L_k}} = \emptyset$. Now, if one and only one of the values λ or μ is equal to $+\infty$, then $m^{L_k} \operatorname{cost}(\sigma_1^{L_k}) = +\infty$ and $m^{L_k} \operatorname{cost}(\sigma_2^{L_k}) = +\infty$, which is a contradiction, so suppose both of them are equal to plus infinity or finite numbers. If both of them are infinite, then $m^{L_k} \operatorname{cost}(\sigma_1^{L_k}) = +\infty$ and $m^{L_k} \operatorname{cost}(\sigma_2^{L_k}) = k$, and moreover $\lim\limits_{k\to+\infty} m^{L_k} \operatorname{cost}(\sigma_2^{L_k}) = +\infty$. Hence, $\sup\limits_{L\in\Lambda} m^L \operatorname{cost}(\sigma_1^L) = +\infty$ and $\sup\limits_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L) = +\infty$ which is again contradiction. So, suppose that both λ and μ are finite numbers and in this case $m^{L_k} \operatorname{cost}(\sigma_1^{L_k}) = \max\{\frac{|\lambda|}{2}, \frac{|\mu - k|}{2}\}$ and $m^{L_k} \operatorname{cost}(\sigma_2^{L_k}) = \min\{\max\{\frac{|\lambda|}{2}, \frac{|\mu - k|}{2}\}, \max\{k, |\lambda - \mu|\}\}$. Observe that $\lim\limits_{k\to+\infty} m^{L_k} \operatorname{cost}(\sigma_1^{L_k}) = +\infty$ and $\lim\limits_{k\to+\infty} m^{L_k} \operatorname{cost}(\sigma_2^{L_k})$. Hence, again $\sup\limits_{L\in\Lambda} m^L \operatorname{cost}(\sigma_1^L) = +\infty$ and $\sup\limits_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L) = +\infty$. Thus, in any case, we get a contradiction under the assumption that c_1 is a finite number.

Suppose now that $c_1 = c_2 = -\infty$ and consider the diagonal line L passing through the point (a_2, a_2) parameterized by $L : u = t \cdot (1, 1) + (a_2, a_2)$ where $u \in L$ and $t \in \mathbb{R}$. After line restriction, $\mathcal{M}^L$ is a one-parameter persistence module with underlying interval $I_{\mathcal{M}^L} = (0, \mu)$ if $b_1 > a_2$, or $I_{\mathcal{M}^L} = \emptyset$ if $a_2 \geq b_1$, whereas $\mathcal{N}^L$ is a one-parameter persistence module with underlying interval $I_{\mathcal{N}^L} = (-\infty, \lambda)$ where $\lambda \in \mathbb{R} \cup \{+\infty\}$. In both cases, $\sup\limits_{L\in\Lambda} m^L \operatorname{cost}(\sigma_1^L) = +\infty$ and $\sup\limits_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L) = +\infty$. Again contradiction.

Case 3: Suppose $\mathcal{M}$ is one of the types R_9 or R_{10} with underlying rectangle $R_{\mathcal{M}} = (a_1, +\infty) \times (-\infty, b_2)$ where $a_1 > -\infty$ and $b_2 \in \mathbb{R} \cup \{+\infty\}$. Let $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$ where $c_1, c_2 \in \mathbb{R} \cup \{-\infty\}$ and $d_1, d_2 \in \mathbb{R} \cup \{+\infty\}$. In this case, we can get a contradiction analogous to the case 2.

Case 4: Suppose $\mathcal{M}$ is of the type R_{11} with underlying rectangle $R_{\mathcal{M}} = (a_1, +\infty) \times (a_2, +\infty)$ where $a_1, a_2 > -\infty$. Let $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$ where $c_1, c_2 \in \mathbb{R} \cup \{-\infty\}$ and $d_1, d_2 \in \mathbb{R} \cup \{+\infty\}$. Since $\|c - a\|_\infty = +\infty$, $c_1 = -\infty$ or $c_2 = -\infty$. If both $c_1 = c_2 = -\infty$, then consider the diagonal line L passing through the point (a_1, a_2) parameterized by $L : u = t \cdot (1, 1) + (a_1, a_2)$ where $u \in L$ and $t \in \mathbb{R}$. After the line restriction, $\mathcal{M}^L$ is a one-parameter persistence module with underlying interval $I_{\mathcal{M}^L} = (0, +\infty)$, whereas $\mathcal{N}^L$ is a one-parameter persistence module with underlying interval $I_{\mathcal{N}^L} = (-\infty, \lambda)$ where $\lambda \in \mathbb{R} \cup \{+\infty\}$. In this case, it is obvious that $\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_1^L) = +\infty$ and $\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_2^L) = +\infty$. Suppose now only $c_1 = -\infty$ and c_2 is a finite number, and consider the sequence of diagonal lines L_k passing through the point $(a_1 - k, a_2)$ parameterized by $L : u = t \cdot (1, 1) + (a_1 - k, a_2)$ where $k > 0$, and $u \in L$, $t \in \mathbb{R}$. After line restriction, $\mathcal{M}^{L_k}$ is a one-parameter persistence module with underlying interval $I_{\mathcal{M}^{L_k}} = (k, +\infty)$, whereas $\mathcal{N}^{L_k}$ is a one-parameter persistence module with underlying interval $I_{\mathcal{N}^{L_k}} = (c_2 - a_2, \mu)$ where $\mu \in \mathbb{R} \cup \{+\infty\}$. If μ is finite, then $\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_1^L) = +\infty$ and $\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_2^L) = +\infty$, contradiction. So, suppose that $\mu = +\infty$, this implies that $m^{L_k} \operatorname{cost}(\sigma_1^{L_k}) = +\infty$ and $m^{L_k} \operatorname{cost}(\sigma_2^{L_k}) = |k - c_2 + a_2|$, and hence $\lim_{k \to +\infty} m^{L_k} \operatorname{cost}(\sigma_2^{L_k}) = +\infty$. Hence, $\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_1^L) = +\infty$ and $\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_2^L) = +\infty$, again contradiction. The case when c_1 is a finite number and $c_2 = -\infty$ is quite similar.

□

Remark 6.2.7. *The previous result verifies that the steady matching distance can attain the infinity. Thus, the steady matching distance is an extended (pseudo)metric.*

CHAPTER 7

COMPARISON OF THE MATCHING AND THE STEADY MATCHING DISTANCE

7.1 Comparison of the Matching and the Steady Matching Distance

First, we show that the steady matching distance is an upper bound for the matching distance for interval decomposable persistence modules.

Proposition 7.1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be finitely presented interval decomposable persistence modules. Then, the steady matching distance is an upper bound for the matching distance, that is, $d_{\text{match}}(\mathcal{M},\mathcal{N}) \leq SD_{\text{match}}(\mathcal{M},\mathcal{N})$.*

Proof. Let S be the set of all partial multibijections $\sigma : B(\mathcal{M}) \nrightarrow B(\mathcal{N})$ between the barcodes of the interval decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$. Since $\mathcal{M}$ and $\mathcal{N}$ are finitely presented interval decomposable persistence modules, there are finitely many partial multibijections between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$. Suppose that

$$\epsilon = SD_{\text{match}}(\mathcal{M},\mathcal{N}) = \min_{\sigma\in S} \sup_{L\in\Lambda} m^L\big(\operatorname{cost}(\sigma^L)\big).$$

Then, there exists an optimal partial multibijection $\bar{\sigma} \in S$, that is $\epsilon = \sup_{L\in\Lambda} m^L\big(\operatorname{cost}(\bar{\sigma}^L)\big)$. It follows that $\epsilon \geq m^L(\operatorname{cost}(\bar{\sigma}^L))$ for any admissible line L. Now, it is clear that

$$\frac{\epsilon}{m^L} \geq \operatorname{cost}(\bar{\sigma}^L) \geq \inf_{\sigma} \operatorname{cost}(\sigma^L) = d_B(\mathcal{M}^L,\mathcal{N}^L)$$

for any admissible line L. So, it implies that $\epsilon \geq m^L d_B(\mathcal{M}^L,\mathcal{N}^L)$ for any admissible line L. Therefore, we have

$$\epsilon \geq \sup_{L\in\Lambda} m^L d_B(\mathcal{M}^L,\mathcal{N}^L) = d_{\text{match}}(\mathcal{M},\mathcal{N}).$$

Hence, $d_{\text{match}}(\mathcal{M}, \mathcal{N}) \leq SD_{\text{match}}(\mathcal{M}, \mathcal{N})$ for every finitely presented interval decomposable persistence modules. □

Now, thanks to Proposition 5.3.2 and Proposition 6.2.5, and the previous result, we have the following fact:

Corollary 7.1.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be any two interval decomposable persistence modules. Then, the steady matching distance is an upper bound for the matching distance, that is, $d_{\text{match}}(\mathcal{M}, \mathcal{N}) \leq SD_{\text{match}}(\mathcal{M}, \mathcal{N})$.*

Since being rectangle decomposable persistence module is a special type of interval decomposable persistence module, we can also say that the steady matching distance is an upper bound for the matching distance for rectangle decomposable persistence modules.

We will now show that if $\mathcal{M}$ and $\mathcal{N}$ are two interval persistence modules, that is, each has only summand in the decomposition, then the matching distance becomes equal to the steady matching distance.

Proposition 7.1.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules. Then, the steady matching distance is equal to the matching distance.*

Proof. It suffices to show that $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) \leq d_{\text{match}}(\mathcal{M}, \mathcal{N})$. This is because the converse inequality is already known to hold for interval decomposable persistence modules, in particular for interval persistence modules, as stated previously. Since there are only two partial multibijections, we can rewrite the steady matching distance as follows:

$$SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = \min\{\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_1^L), \sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_2^L)\}$$

where σ_1 and σ_2 are two partial multibijections such that one matches bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ with the zero bipersistence modules and the other one matches the bipersistence module $\mathcal{M}$ with the bipersistence module $\mathcal{N}$, respectively. Let $\epsilon = SD_{\text{match}}(\mathcal{M}, \mathcal{N})$, then it is clear that $\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_1^L) \geq \epsilon$ and $\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_2^L) \geq \epsilon$. By Definition 4.2.1 and the isometry theorem, we know that $\operatorname{cost}(\sigma_2^L) = d_I(\mathcal{M}^L, \mathcal{N}^L)$ and $d_I(\mathcal{M}^L, \mathcal{N}^L) = d_B(\mathcal{M}^L, \mathcal{N}^L)$ for any admissible line L, respectively. Thus, it is obvious that $m^L \operatorname{cost}(\sigma_2^L) = m^L d_B(\mathcal{M}^L, \mathcal{N}^L)$ for any admissible line L. Therefore, we have

$$\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L) = \sup_{L\in\Lambda} m^L d_B(\mathcal{M}^L, \mathcal{N}^L) = d_{\text{match}}(\mathcal{M}, \mathcal{N}) \geq \epsilon.$$

Hence, the claim is proved since $\epsilon = SD_{\text{match}}(\mathcal{M}, \mathcal{N})$. □

CHAPTER 8

EXACT COMPUTATION OF THE INTERLEAVING DISTANCE

This chapter shows that the interleaving distance on rectangle persistence modules can be computed using the geometry of underlying rectangles. Then, we show that this can be further generalized on a larger class of persistence modules, namely interval persistence modules under some assumptions.

We want to point out that from now on, we use the conventions stated in Subsection 2.1 for the calculations (for details, see (2.1), (2.2) and (2.3)).

In the following section, we will have results for open interval (particularly rectangle) bipersistence modules. Fortunately, by Proposition 4.1.8 and by Corollary 4.1.9, we know that, for any interval persistence modules $\mathcal{M} = \mathcal{I}^I$ and $\mathcal{N} = \mathcal{I}^J$, $d_I(\mathcal{M}, \mathcal{N}) = d_I(\overline{\mathcal{M}}, \overline{\mathcal{N}}) = d_I(\mathcal{M}^\circ, \mathcal{N}^\circ)$ where $\overline{\mathcal{M}} \doteq \mathcal{I}^{\bar{I}}$ and $\overline{\mathcal{N}} \doteq \mathcal{I}^{\bar{J}}$, and $\mathcal{M}^\circ \doteq \mathcal{I}^{I^\circ}$ and $\mathcal{N}^\circ \doteq \mathcal{I}^{J^\circ}$. Thus, our results are also valid for closed, non-open, non-closed interval bipersistence modules or, more importantly, finitely presented interval bipersistence modules.

8.1 Computing the Interleaving Distance for Rectangle Persistence Modules

Throughout this section, whenever $R_\mathcal{M} = (a_1, b_1) \times (a_2, b_2)$ is an underlying rectangle of a rectangle bipersistence module $\mathcal{M}$, for points $a = (a_1, a_2)$ and $b = (b_1, b_2)$, we always assume that $a_i \in \mathbb{R} \cup \{-\infty\}$, $b_i \in \mathbb{R} \cup \{+\infty\}$, and $a_i < b_i$ for every $i \in \{1, 2\}$.

Lemma 8.1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_\mathcal{M} = (a_1, b_1) \times (a_2, b_2)$ and $R_\mathcal{N} = (c_1, d_1) \times (c_2, d_2)$, respectively.*

There is a non-trivial morphism $f\colon \mathcal{M} \to \mathcal{N}$ if and only if $c = (c_1, c_2) \preceq a =$

$(a_1, a_2) \prec d = (d_1, d_2) \preceq b = (b_1, b_2)$. *Moreover, when such* $f : \mathcal{M} \to \mathcal{N}$ *exists,* f_u *can be considered the identity map for all* $u \in R_{\mathcal{M}} \cap R_{\mathcal{N}}$*, and the zero map for all* $u \notin R_{\mathcal{M}} \cap R_{\mathcal{N}}$.

Proof. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a non-trivial morphism. We start showing that $c \preceq a$. Suppose on the contrary that $c \npreceq a$, i.e. there exists at least one $i \in \{1, 2\}$ for which $c_i > a_i$. Without loss of generality, let $c_1 > a_1$ so that $c_1 > -\infty$. Now, if $u = (u_1, u_2) \notin R_{\mathcal{M}}$, then $f_u = 0$. If $u = (u_1, u_2) \in R_{\mathcal{M}}$, then $a_2 < u_2$. Thus, there exists $a_2^+ \in \mathbb{R}$ such that $a_2 < a_2^+ \leq u_2$. Now, if $u_1 < c_1$, then $u \notin R_{\mathcal{N}}$, and so $f_u = 0$. Otherwise, if $u_1 \geq c_1$, since we are assuming that $a_1 < c_1$, there exits $a_1^+ \in \mathbb{R}$ such that $a_1 < a_1^+ \leq c_1 \leq u_1$. This implies that $a^+ = (a_1^+, a_2^+) \in R_{\mathcal{M}} \setminus R_{\mathcal{N}}$ and $a \prec a^+ \preceq u$. Now, consider the following diagram, where the horizontal maps are the transition morphisms of $\mathcal{M}$ and $\mathcal{N}$, respectively

$$\begin{array}{ccc} k = \mathcal{M}_{a^+} & \xrightarrow{\mathrm{id}_k} & \mathcal{M}_u \\ \big\downarrow{\scriptstyle 0 = f_{a^+}} & & \big\downarrow{\scriptstyle f_u} \\ 0 = \mathcal{N}_{a^+} & \longrightarrow & \mathcal{N}_u. \end{array}$$

Since $f\colon \mathcal{M} \to \mathcal{N}$ is a morphism, the diagram must be commutative. This implies that $f_u = 0$ also for the case $u_1 \geq c_1$. Hence, $f_u = 0$ for all $u \in \mathbb{R}^2$. So, it contradicts $f\colon \mathcal{M} \to \mathcal{N}$ being a non-trivial morphism. Hence, $c \preceq a$.

Now, still letting $f\colon \mathcal{M} \to \mathcal{N}$ be a non-trivial morphism, we will show that $a \prec d$. Suppose on the contrary that $a = (a_1, a_2) \nprec (d_1, d_2) = d$ where $a_1, a_2 \in \mathbb{R} \cup \{-\infty\}$ and $d_1, d_2 \in \mathbb{R} \cup \{+\infty\}$. So, $R_{\mathcal{M}} \cap R_{\mathcal{N}} = \emptyset$. Thus, $f_u\colon \mathcal{M}_u \to \mathcal{N}_u$ is the trivial morphism for all $u \in \mathbb{R}^2$ since if $u \in R_{\mathcal{M}}$, then $u \notin R_{\mathcal{N}}$ and hence $\mathcal{N}_u = 0$; similarly, if $u \in R_{\mathcal{N}}$, then $u \notin R_{\mathcal{M}}$ and hence $\mathcal{M}_u = 0$. So, it contradicts $f\colon \mathcal{M} \to \mathcal{N}$ being a non-trivial morphism. Hence, $a \prec d$.

Still assuming that $f\colon \mathcal{M} \to \mathcal{N}$ is a non-trivial morphism, we will now show that $d \preceq b$. Suppose on the contrary that $d \npreceq b$, i.e. there exists at least one $i \in \{1, 2\}$ for which $d_i > b_i$. Without loss of generality, let $d_1 > b_1$ so that $b_1 < +\infty$. Now, if $u = (u_1, u_2) \notin R_{\mathcal{N}}$, then $f_u = 0$. If $u = (u_1, u_2) \in R_{\mathcal{N}}$, then $u_2 < d_2$. Thus, there exists $d_2^- \in \mathbb{R}$ such that $u_2 < d_2^- \leq d_2$. Now, if $b_1 < u_1$, then $u \notin R_{\mathcal{M}}$, and so $f_u = 0$. Otherwise, if $b_1 \geq u_1$, since we are assuming that $b_1 < d_1$, there exits $d_1^- \in \mathbb{R}$ such that $u_1 < b_1 \leq d_1^- < d_1$. This implies that there exists $d^- = (d_1^-, d_2^-) \in R_{\mathcal{N}} \setminus R_{\mathcal{M}}$ is such that $u \preceq d^- \prec d$. Now, consider the following diagram, where the horizontal maps are the interval morphisms of $\mathcal{M}$ and $\mathcal{N}$, respectively

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_{d^-} = 0 \\ \big\downarrow{\scriptstyle f_u} & & \big\downarrow{\scriptstyle 0 = f_{d^-}} \\ k = \mathcal{N}_u & \xrightarrow{\mathrm{id}_k} & \mathcal{N}_{d^-} = k. \end{array}$$

The diagram must be commutative since $f\colon \mathcal{M} \to \mathcal{N}$ is a morphism. This implies that $f_u = 0$ also for the case $b_1 \geq u_1$. Hence, $f_u = 0$ for all $u \in \mathbb{R}^2$. So, it contradicts $f\colon \mathcal{M} \to \mathcal{N}$ being a non-trivial morphism. Hence, $d \preceq b$.

For the proof of the converse statement, assuming $c \preceq a \prec d \preceq b$, we can define a non-trivial map $f\colon \mathcal{M} \to \mathcal{N}$ as follows:

$$f_u = \begin{cases} \mathrm{id}_k & \text{if } a \prec u \prec d, \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

Let us now check that it is a valid morphism. For $a \prec u \preceq v \prec d$, we have the following commutative diagram:

$$\begin{array}{ccc} k = \mathcal{M}_u & \xrightarrow{\mathrm{id}_k} & \mathcal{M}_v = k \\ \big\downarrow{\scriptstyle \mathrm{id}_k} & & \big\downarrow{\scriptstyle \mathrm{id}_k} \\ k = \mathcal{N}_u & \xrightarrow{\mathrm{id}_k} & \mathcal{N}_v = k \end{array}$$

For $a \not\prec u$ and $u \preceq v$, observe that the following diagram commutes:

$$\begin{array}{ccc} 0 = \mathcal{M}_u & \xrightarrow{0} & \mathcal{M}_v \\ \big\downarrow{\scriptstyle 0} & & \big\downarrow \\ \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \end{array}$$

For $a \prec u \prec d$, $u \preceq v$ and $v \not\prec d$, observe that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \\ \downarrow & & \downarrow 0 \\ \mathcal{N}_u & \xrightarrow{0} & \mathcal{N}_v = 0 \end{array}$$

Therefore, the following diagram commutes for all $u \preceq v \in \mathbb{R}^2$:

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \\ \downarrow f_u & & \downarrow f_v \\ \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \end{array}$$

Hence, $f\colon \mathcal{M} \to \mathcal{N}$ as in (8.1) is the desired non-trivial morphism. □

Corollary 8.1.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively.*

Let $\vec{\epsilon} = (\epsilon, \epsilon)$ such that $\epsilon \geq 0$ and let $\mathcal{N}(\vec{\epsilon})$ is the ϵ-shifting of the persistence module $\mathcal{N}$ defined as in the Definition 3.1.4. Then, there exists a non-trivial morphism $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ if and only if

$$\max\left\{ \max_{i=1,2}\{c_i - a_i\}, \max_{i=1,2}\{d_i - b_i\}\right\} \leq \epsilon < \min_{i=1,2}\{d_i - a_i\}.$$

Proof. Note that, by Lemma 8.1.1, there is a non-trivial morphism $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ if and only if $c - \vec{\epsilon} \preceq a \prec d - \vec{\epsilon} \preceq b$. We note that:

- $c - \vec{\epsilon} \preceq a$ if and only if $c_i - \epsilon \leq a_i$ for $i = 1, 2$, or equivalently $\epsilon \geq c_i - a_i$ for $i = 1, 2$;
- $a \prec d - \vec{\epsilon}$ if and only if $a_i < d_i - \epsilon$ for $i = 1, 2$, or equivalently $\epsilon < d_i - a_i$ for $i = 1, 2$;
- $d - \vec{\epsilon} \preceq b$ if and only if $d_i - \epsilon \leq b_i$, for $i = 1, 2$, or equivalently $\epsilon \geq d_i - b_i$ for $i = 1, 2$.

Hence, there is a non-trivial morphism $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ if and only if

$$\max\left\{ \max_{i=1,2}\{c_i - a_i\}, \max_{i=1,2}\{d_i - b_i\}\right\} \leq \epsilon < \min_{i=1,2}\{d_i - a_i\}.$$

□

Lemma 8.1.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. Then,*

$$d_I(\mathcal{M}, \mathcal{N}) \leq \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}.$$

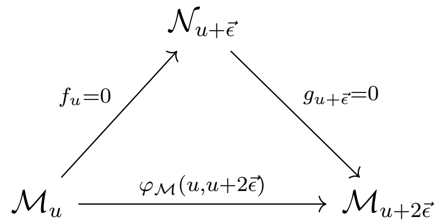
Proof. If

$$\max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\} = +\infty,$$

then there is nothing to prove. Thus, let

$$\epsilon \doteq \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\} < +\infty.$$

We note that $\epsilon > 0$. So, we can consider $\mathcal{M}(\vec{\epsilon})$ and $\mathcal{N}(\vec{\epsilon})$. Let us take the maps $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ to be trivial. Thus, all square diagrams (4.1) in the Definition 4.1.1 are commutative. Now consider the following diagram:



Note that, for all points $u \in \mathbb{R}^2$, the diagram above will be commutative if we have $\varphi_{\mathcal{M}}(u, u + 2\vec{\epsilon}) = 0$. If $a \not\prec u$, then it is obvious because $\mathcal{M}_u = 0$. Suppose now that $a \prec u$. By assumption, we have

$$\epsilon = \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\},$$

thus

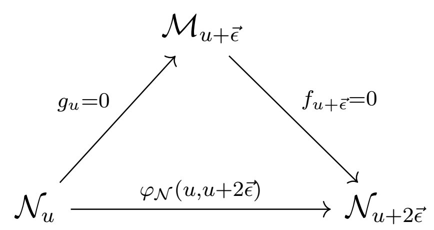
$$\epsilon \geq \min_{i=1,2} \frac{b_i - a_i}{2}.$$

Without loss of generality, let

$$\min_{i=1,2} \frac{b_i - a_i}{2} = \frac{b_1 - a_1}{2}.$$

So, $a_1 + 2\epsilon \geq a_1 + b_1 - a_1 = b_1$, which implies $a + 2\vec{\epsilon} \not\prec b$ so that $\mathcal{M}_{u+2\epsilon} = 0$ for every $a \prec u$. Thus, $\varphi_{\mathcal{M}}(u, u + 2\vec{\epsilon}) = 0$ for all points $u \in \mathbb{R}^2$.

Similarly, one can show that the following diagram is also commutative for all points $u \in \mathbb{R}^2$:



Therefore, the trivial morphisms $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ are ϵ-interleaving morphisms such that all diagrams in the Definition 4.1.1 are commutative for every point in $\mathbb{R}^2$. Hence, the rectangle bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved, and thus, we can conclude that

$$d_I(\mathcal{M},\mathcal{N}) \leq \epsilon = \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}.$$

□

Lemma 8.1.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. If*

$$\max\left\{ \|c-a\|_\infty, \|d-b\|_\infty \right\} < \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}$$

then

$$\max\left\{ \|c-a\|_\infty, \|d-b\|_\infty \right\} < \min\left\{ \min_{i=1,2} \{d_i - a_i\}, \min_{i=1,2} \{b_i - c_i\} \right\}.$$

Proof. Without loss of generality, suppose that

$$\min\left\{ \min_{i=1,2} \{d_i - a_i\}, \min_{i=1,2} \{b_i - c_i\} \right\} = d_1 - a_1.$$

Now, let $\epsilon \doteq \max\left\{ \|c-a\|_\infty, \|d-b\|_\infty \right\}$, and suppose on the contrary that $d_1 - a_1 \leq \epsilon$. Note that $\frac{d_1 - c_1}{2} \leq \frac{d_1 - a_1 + \epsilon}{2}$ since $\|c-a\|_\infty \leq \epsilon$ by definition of ϵ, thus we have $\frac{d_1 - c_1}{2} \leq \epsilon$ as $d_1 - a_1 \leq \epsilon$ by assumption. Similarly, note that $\frac{b_1 - a_1}{2} \leq \frac{d_1 - a_1 + \epsilon}{2}$ since $\|d-b\|_\infty \leq \epsilon$ by definition of ϵ, thus we have $\frac{b_1 - a_1}{2} \leq \epsilon$ as $d_1 - a_1 \leq \epsilon$ by assumption. Therefore,

$$\min_{i=1,2} \frac{b_i - a_i}{2} \leq \epsilon \text{ and } \min_{i=1,2} \frac{d_i - c_i}{2} \leq \epsilon.$$

Hence,

$$\max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\} \leq \epsilon$$

which contradicts the assumption. □

Lemma 8.1.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. If*

$$\max\left\{\|c-a\|_\infty, \|d-b\|_\infty\right\} < \min\left\{\min_{i=1,2}\{d_i - a_i\}, \min_{i=1,2}\{b_i - c_i\}\right\},$$

then

$$d_I(\mathcal{M},\mathcal{N}) \leq \max\left\{\|c-a\|_\infty, \|d-b\|_\infty\right\}.$$

Proof. If

$$\max\left\{\|c-a\|_\infty, \|d-b\|_\infty\right\} = +\infty,$$

then there is nothing to prove. Thus, let

$$0 \leq \epsilon \doteq \max\left\{\|c-a\|_\infty, \|d-b\|_\infty\right\} < +\infty.$$

Hence,

$$\epsilon \geq \max\left\{\max_{i=1,2}\{c_i - a_i\}, \max_{i=1,2}\{d_i - b_i\}\right\}$$

and, by assumption,

$$\epsilon < \min\left\{\min_{i=1,2}\{d_i - a_i\}, \min_{i=1,2}\{b_i - c_i\}\right\} \leq \min_{i=1,2}\{d_i - a_i\}.$$

Analogously,

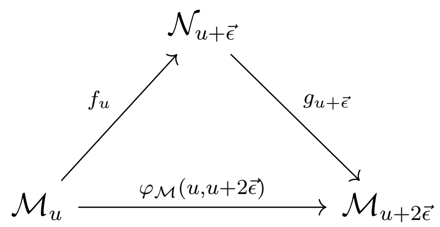
$$\epsilon \geq \max\left\{\max_{i=1,2}\{a_i - c_i\}, \max_{i=1,2}\{b_i - d_i\}\right\}$$

and, by assumption,

$$\epsilon < \min\left\{\min_{i=1,2}\{d_i - a_i\}, \min_{i=1,2}\{b_i - c_i\}\right\} \leq \min_{i=1,2}\{b_i - c_i\}.$$

In this case, by Corollary 8.1.2, we can take morphisms $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ to be non-trivial and precisely $\mathrm{id}_k : k \to k$ at all points $u \in \mathbb{R}^2$ such that domain and codomain of f_u (respectively g_u) are both non-trivial.

Let us show that they make the triangle diagrams



commute.

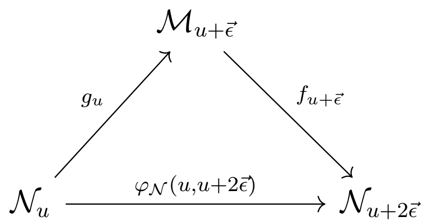
If $\mathcal{M}_u = 0$ or $\mathcal{M}_{u+2\vec{\epsilon}} = 0$, then the triangle diagram is clearly commutative. So, suppose that $\mathcal{M}_u = k$ and $\mathcal{M}_{u+2\vec{\epsilon}} = k$. Thus, $\varphi_{\mathcal{M}}(u, u+2\vec{\epsilon}) = \mathrm{id}_k$ and $a \prec u \prec b - 2\vec{\epsilon}$. Thus, we have

$$u + \vec{\epsilon} \prec b - \vec{\epsilon} \preceq b - \|d - b\|_\infty \cdot (1,1) \preceq d$$

since $\|d - b\|_\infty \leq \epsilon$ by definition of ϵ. Also, we have

$$c \prec a + \|c - a\|_\infty \cdot (1,1) \preceq a + \vec{\epsilon} \preceq u + \vec{\epsilon}$$

since $\|c-a\|_\infty \leq \epsilon$ by definition of ϵ. These two inequalities imply that $c \prec u+\vec{\epsilon} \prec d$. It follows that $\mathcal{N}_{u+\vec{\epsilon}} = k$, and hence $f_u = \mathrm{id}_k$ and $g_{u+\vec{\epsilon}} = \mathrm{id}_k$ since we are assuming that $\mathcal{M}_u = k$ and $\mathcal{M}_{u+2\vec{\epsilon}} = k$. Hence, the triangle diagram above commutes for all points $u \in \mathbb{R}^2$. Analogously, the triangle diagram below is commutative for all points $u \in \mathbb{R}^2$:



Therefore, f, g form an ϵ-interleaving pair, implying that

$$d_I(\mathcal{M},\mathcal{N}) \leq \epsilon = \max\Big\{\|c - a\|_\infty, \|d - b\|_\infty\Big\}.$$

□

We are ready to give our first main result in this section.

Theorem 8.1.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. It holds that:*

$$d_I(\mathcal{M},\mathcal{N}) \leq \min\left\{ \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}, \max\Big\{\|c-a\|_\infty, \|d-b\|_\infty\Big\} \right\}.$$

Proof. First, suppose that

$$\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\Big\}\leq\max\Big\{\|c-a\|_\infty,\|d-b\|_\infty\Big\}.$$

By Lemma 8.1.3, we know that

$$d_I(\mathcal{M},\mathcal{N})\leq\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\Big\}.$$

Hence, the result is obvious.

Now, suppose that

$$\max\Big\{\|c-a\|_\infty,\|d-b\|_\infty\Big\}<\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\Big\}.$$

Then, by Lemma 8.1.4, we know that

$$\max\Big\{\|c-a\|_\infty,\|d-b\|_\infty\Big\}<\min\Big\{\min_{i=1,2}\{d_i-a_i\},\min_{i=1,2}\{b_i-c_i\}\Big\}.$$

Therefore, by Lemma 8.1.5,

$$d_I(\mathcal{M},\mathcal{N})\leq\max\Big\{\|c-a\|_\infty,\|d-b\|_\infty\Big\}.$$

Hence, in any case

$$d_I(\mathcal{M},\mathcal{N})\leq\min\left\{\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\Big\},\max\Big\{\|c-a\|_\infty,\|d-b\|_\infty\Big\}\right\}.$$

□

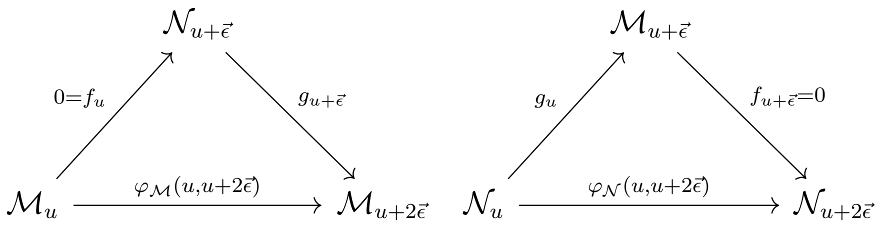
We will now prove the converse of the inequality. Specifically, if we possess two rectangle bipersistence modules, $\mathcal{M}$ and $\mathcal{N}$, with underlying rectangles $R_{\mathcal{M}}=(a_1,b_1)\times(a_2,b_2)$ and $R_{\mathcal{N}}=(c_1,d_1)\times(c_2,d_2)$, respectively, then

$$d_I(\mathcal{M},\mathcal{N})\geq\min\left\{\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\Big\},\max\Big\{\|c-a\|_\infty,\|d-b\|_\infty\Big\}\right\}.$$

Lemma 8.1.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be ϵ-interleaved persistence modules with two interleaving morphisms $f\colon\mathcal{M}\to\mathcal{N}(\vec{\epsilon})$ and $g\colon\mathcal{N}\to\mathcal{M}(\vec{\epsilon})$. If f or g is a trivial morphism, then*

$$\epsilon\geq\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\Big\}.$$

Proof. Without loss of generality, suppose that $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ is a trivial morphism. Thus, $f_u\colon \mathcal{M}_u \to \mathcal{N}_{u+\vec{\epsilon}}$ is a zero map for every $u \in \mathbb{R}^2$. By assumption, we know that $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved. Hence, both diagrams below must be commutative for every $u \in \mathbb{R}^2$.



As a result of this fact, both transition maps $\varphi_{\mathcal{M}}(u, u + 2\vec{\epsilon})$ and $\varphi_{\mathcal{N}}(u, u + 2\vec{\epsilon})$ are zero linear maps for every $u \in \mathbb{R}^2$. Hence, both $\mathcal{M}$ and $\mathcal{N}$ are 2ϵ-trivial persistence modules. Respectively, this implies that $2\epsilon \geq \min\limits_{i=1,2}\{b_i - a_i\}$ and $2\epsilon \geq \min\limits_{i=1,2}\{d_i - c_i\}$. Hence, we can conclude that

$$\epsilon \geq \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}.$$

Analogously, if $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ is a trivial morphism, then we have the same result.

□

Lemma 8.1.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be ϵ-interleaved persistence modules with two interleaving morphisms $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$. If f and g are non-trivial morphisms, then*

$$\epsilon \geq \max\Big\{\|c - a\|_\infty, \|d - b\|_\infty\Big\}.$$

Proof. By Corollary 8.1.2, it is known that if $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ is a non-trivial morphism, then

$$\epsilon \geq \max\left\{ \max_{i=1,2}\{c_i - a_i\}, \max_{i=1,2}\{d_i - b_i\} \right\}.$$

By assumption, we know that both $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ are non-trivial morphisms. Thus, respectively we have

$$\epsilon \geq \max\left\{ \max_{i=1,2}\{c_i - a_i\}, \max_{i=1,2}\{d_i - b_i\} \right\}$$

and

$$\epsilon \geq \max\left\{ \max_{i=1,2}\{a_i - c_i\}, \max_{i=1,2}\{b_i - d_i\} \right\}.$$

Hence, we can conclude that

$$\epsilon \geq \max\Big\{\|c-a\|_\infty, \|d-b\|_\infty\Big\}.$$

□

Theorem 8.1.9. *Let $\mathcal{M}$ and $\mathcal{N}$ be rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. It holds that:*

$$d_I(\mathcal{M}, \mathcal{N}) \geq \min\left\{ \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}, \max\Big\{\|c-a\|_\infty, \|d-b\|_\infty\Big\} \right\}.$$

Proof. Suppose $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved persistence modules with interleaving morphisms $f \colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g \colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$. Suppose that f or g is a trivial morphism. By Lemma 8.1.7, we know that

$$\epsilon \geq \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}.$$

Thus, we have

$$\epsilon \geq \min\left\{ \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}, \max\Big\{\|c-a\|_\infty, \|d-b\|_\infty\Big\} \right\}.$$

Now, suppose that f and g are non-trivial morphisms. By Lemma 8.1.8, we know that

$$\epsilon \geq \max\Big\{\|c-a\|_\infty, \|d-b\|_\infty\Big\}.$$

Thus, again, we have

$$\epsilon \geq \min\left\{ \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}, \max\Big\{\|c-a\|_\infty, \|d-b\|_\infty\Big\} \right\}.$$

Since ϵ is arbitrary where $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved and in any case, whether at least one of the morphisms is trivial, or both are non-trivial morphisms, we have

$$\epsilon \geq \min\left\{ \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}, \max\Big\{\|c-a\|_\infty, \|d-b\|_\infty\Big\} \right\}.$$

Hence, we can conclude that

$$d_I(\mathcal{M}, \mathcal{N}) \geq \min\left\{ \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}, \max\Big\{\|c-a\|_\infty, \|d-b\|_\infty\Big\} \right\}.$$

□

Corollary 8.1.10. *Let $\mathcal{M}$ and $\mathcal{N}$ be rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. It holds that:*

$$d_I(\mathcal{M}, \mathcal{N}) = \min \left\{ \max\left\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \right\}, \max\left\{ \|c-a\|_\infty, \|d-b\|_\infty \right\} \right\}.$$

Proof. The result is straightforwardly from Theorem 8.1.6 and Theorem 8.1.9. □

Remark 8.1.11. *Thanks to the facts in Proposition 4.1.8 and in Corollary 4.1.9, the above result is independent of whether the underlying rectangles are open, closed or neither.*

By imitating the previous results, we can generalize Corollary 8.1.10 to any n-paramater rectangle persistence modules as follows.

Remark 8.1.12. *Let $\mathcal{M}$ and $\mathcal{N}$ be n-parameter rectangle persistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2) \times \ldots \times (a_n, b_n)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2) \times \ldots \times (c_n, d_n)$ where $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n), c = (c_1, c_2, \ldots, c_n)$ and $d = (d_1, d_2, \ldots, d_n)$, respectively. It holds that:*

$$d_I(\mathcal{M}, \mathcal{N}) = \min \left\{ \max\left\{ \min_{i=1,\ldots,n} \frac{b_i - a_i}{2}, \min_{i=1,\ldots,n} \frac{d_i - c_i}{2} \right\}, \max\left\{ \|c-a\|_\infty, \|d-b\|_\infty \right\} \right\}.$$

8.2 Computing the Interleaving Distance for Interval Persistence Modules

In this section, we extend our previous results to interval persistence modules. However, we first need to introduce notions and make observations that differ from what we used in the previous section.

Throughout this section, unless otherwise stated, we assume that both interval persistence modules $\mathcal{M}$ and $\mathcal{N}$ are finitely presented and have bounded underlying intervals $I_{\mathcal{M}}$ and $I_{\mathcal{N}}$, respectively.

Let us start by defining the most important notion of this section, namely, the definition of a minimal and a maximal element of an underlying interval of a persistence module.

Definition 8.2.1. *An element a of the underlying interval $I_{\mathcal{M}}$ of an interval persistence module $\mathcal{M}$ is said to be **minimal** if $a \preceq u$ for any comparable point $u \in I_{\mathcal{M}}$, and an element **b** of the underlying interval $I_{\mathcal{M}}$ of the persistence module $\mathcal{M}$ is said to be **maximal** if $v \preceq b$ for any comparable point $v \in I_{\mathcal{M}}$.*

Remark 8.2.2. *Any two minimal or maximal elements of the underlying interval of the same persistence module are non-comparable. In other words, if a^1 and a^2 are two minimal or maximal elements of $I_{\mathcal{M}}$, then neither $a^1 \preceq a^2$ nor $a^2 \preceq a^1$.*

Recall that a finitely presented interval persistence module contains all lower boundary points and no upper boundary point. We denote the set of all minimal elements of an underlying interval $I_{\mathcal{M}}$ by $S_{\min}(\mathcal{M})$ and the set of all maximal elements of the closure of the same underlying interval by $S_{\max}(\overline{\mathcal{M}})$.

In this section, we are going to prove the next conjecture.

Conjecture 8.2.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules. Suppose $\mathcal{M}$ and $\mathcal{N}(\vec{\epsilon})$, and $\mathcal{M}(\vec{\epsilon})$ and $\mathcal{N}$ have at most one intersection component. Then,*

$$d_I(\mathcal{M}, \mathcal{N}) = \min\{\dagger^1, \dagger^2\} = \epsilon$$

where

$$\dagger^1 = \max \left\{ \begin{array}{l} \max\limits_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\}, \\ \max\limits_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a \in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d \in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\} \right\} \end{array} \right\} \end{array} \right\}$$

and

$$\dagger^2 = \max \left\{ \max\limits_{\substack{a \in S_{\min}(\mathcal{M}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min\limits_{i=1,2} \frac{b_i - a_i}{2} \right\}, \max\limits_{\substack{c \in S_{\min}(\mathcal{N}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min\limits_{i=1,2} \frac{d_i - c_i}{2} \right\} \right\}.$$

Remark 8.2.4. *For the computation of $\dagger^2$, considering only comparable pairs (a, b) where $a \in S_{\min}(\mathcal{M})$ and $b \in S_{\max}(\overline{\mathcal{M}})$ is enough since $\min\limits_{i=1,2}\{b_i - a_i\} < 0$ for any non-comparable pair (a, b).*

The following proposition guarantees that having a non-trivial morphism between interval persistence modules implies that there exists a non-trivial intersection component between interval modules.

Proposition 8.2.5. *Let $f : \mathcal{M} \to \mathcal{N}$ be a non-trivial morphism. Then, there exists a connected intersection component Q with underlying interval $I_Q \subseteq I_{\mathcal{M}} \cap I_{\mathcal{N}}$ such that $f_u \neq 0$ for every $u \in I_Q$.*

Proof. Suppose that there exists a non-trivial morphism $f : \mathcal{M} \to \mathcal{N}$, so $f_u \neq 0$ for some $u \in \mathbb{R}^2$. Thus, $\mathcal{M}_u = \mathcal{N}_u = k$, that is $u \in I_{\mathcal{M}} \cap I_{\mathcal{N}}$. Consequently, there exists an intersection connected component Q with underlying interval I_Q such that $u \in I_Q \subseteq I_{\mathcal{M}} \cap I_{\mathcal{N}}$. Next, using the fact that $f_u \neq 0$ for some $u \in I_Q$, we will show that $f_v \neq 0$ for any $v \in I_Q$.

Suppose first that (u, v) is a comparable pair in I_Q and $u \preceq v$. Consider the following diagram:

$$\begin{array}{ccc} k = \mathcal{M}_u & \xrightarrow{\mathrm{id}_k} & \mathcal{M}_v = k \\ \big\downarrow {\scriptstyle f_u \neq 0} & & \big\downarrow {\scriptstyle f_v} \\ k = \mathcal{N}_u & \xrightarrow{\mathrm{id}_k} & \mathcal{N}_v = k. \end{array}$$

Since $u, v \in I_Q$, we know that $\mathcal{M}_u = \mathcal{M}_v = \mathcal{N}_u = \mathcal{N}_v = k$, and $\varphi_{\mathcal{M}}(u,v) = \mathrm{id}_k$ and $\varphi_{\mathcal{N}}(u,v) = \mathrm{id}_k$. By assumption, it is assumed that $f_u \neq 0$ and $f\colon \mathcal{M} \to \mathcal{N}$ is a non-trivial morphism. Thus, the diagram must be commutative. Hence, we can conclude that $f_v \neq 0$.

For $v \preceq u$, one can show $f_v \neq 0$ by using the argument above and considering the following diagram.

$$\begin{array}{ccc} k = \mathcal{M}_v & \xrightarrow{\mathrm{id}_k} & \mathcal{M}_u = k \\ \big\downarrow f_v & & \big\downarrow f_u \neq 0 \\ k = \mathcal{N}_v & \xrightarrow{\mathrm{id}_k} & \mathcal{N}_u = k. \end{array}$$

Now, suppose that u and v are non-comparable points. Since Q is a connected component and both $u, v \in I_Q$, there exist $q^1, q^2, \dots, q^r \in I_Q$ such that $u \succeq q^1 \preceq q^2 \succeq \dots \preceq q^r \succeq v$ or $v \succeq q^1 \preceq q^2 \succeq \dots \preceq q^r \succeq u$. Without loss of generality, say $u \succeq q^1 \preceq q^2 \succeq \dots \preceq q^r \succeq v$. By assumption, $f_u \neq 0$. This implies that $f_{q^1} \neq 0$ and the reason is analogous to the case when $v \preceq u$ in the above. Observe that $f_{q^2} \neq 0$ since $f_{q^1} \neq 0$, which is analogous to the case when $u \preceq v$ in the above. Ultimately, one can show that $f_v \neq 0$ by applying the same idea.

Hence, in any case, $f_v \neq 0$ for any $v \in I_Q$ if $f_u \neq 0$ for some $u \in I_Q$.

□

Lemma 8.2.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules given as above. Then, there is a non-trivial morphism $f\colon \mathcal{M} \to \mathcal{N}$ if and only if there exists an intersection component Q of $\mathcal{M}$ and $\mathcal{N}$ with underlying interval $I_Q \subseteq I_{\mathcal{M}} \cap I_{\mathcal{N}}$ such that for any $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$ satisfying $a \preceq u \prec d$ for some $u \in I_Q$, there exist $c \in S_{\min}(\mathcal{N})$ and $b \in S_{\max}(\overline{\mathcal{M}})$ such that $c \preceq a \prec d \preceq b$.*

Proof. Suppose a non-trivial morphism $f\colon \mathcal{M} \to \mathcal{N}$ exists. Thus, by Proposition 8.2.5, there exists an intersection component Q with underlying interval $I_Q \subseteq I_{\mathcal{M}} \cap I_{\mathcal{N}}$ such that $f_v \neq 0$ for every $v \in I_Q$. To prove that the condition is satisfied, consider a pair (a, d) where $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$ satisfying $a \preceq u \prec d$ for some $u \in I_Q$. Suppose by contradiction that there exists no $c \in S_{\min}(\mathcal{N})$ such that $c \preceq a \preceq u \prec d$. Thus, $a \notin I_{\mathcal{N}}$, equivalently $\mathcal{N}_a = 0$. Consider the following diagram:

$$\begin{array}{ccc} k = \mathcal{M}_a & \xrightarrow{\mathrm{id}_k} & \mathcal{M}_u = k \\ \downarrow & & \downarrow{\scriptstyle f_u \neq 0} \\ 0 = \mathcal{N}_a & \longrightarrow & \mathcal{N}_u = k \end{array}$$

Observe that the diagram is not commutative since $\varphi_{\mathcal{M}}(a, u) = \mathrm{id}_k$ and $f_u \neq 0$, which contradicts $f\colon \mathcal{M} \to \mathcal{N}$ being a morphism. By a similar idea, one can show that there exists $b \in S_{\max}(\overline{\mathcal{M}})$ such that $a \preceq u \prec d \preceq b$.

Hence, one direction of the claim is proven.

For the proof of the converse statement, we can define a non-trivial map $f^Q\colon \mathcal{M} \to \mathcal{N}$ as follows:

$$f_u^Q = \begin{cases} \mathrm{id}_k & \text{if } u \in I_Q, \\ 0 & \text{otherwise.} \end{cases} \tag{8.2}$$

Let us now check that it is a valid morphism. Let $u \preceq v \in \mathbb{R}^2$.

For $u, v \in I_Q$, the following diagram is commutative since $f_u^Q = \mathrm{id}_k$ and $f_v^Q = \mathrm{id}_k$ by definition of $f^Q\colon \mathcal{M} \to \mathcal{N}$, and $\mathcal{M}_u = \mathcal{M}_v = \mathcal{N}_u = \mathcal{N}_v = k$:

$$\begin{array}{ccc} k = \mathcal{M}_u & \xrightarrow{\mathrm{id}_k} & \mathcal{M}_v = k \\ \downarrow{\scriptstyle \mathrm{id}_k} & & \downarrow{\scriptstyle \mathrm{id}_k} \\ k = \mathcal{N}_u & \xrightarrow{\mathrm{id}_k} & \mathcal{N}_v = k \end{array}$$

For $u \in I_Q$ and $v \notin I_Q$, by definition of $f^Q\colon \mathcal{M} \to \mathcal{N}$, we know that $f_u^Q = \mathrm{id}_k$ and $f_v^Q = 0$. Consider the following diagram:

$$\begin{array}{ccc} k = \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \\ \downarrow{\scriptstyle \mathrm{id}_k} & & \downarrow{\scriptstyle 0} \\ k = \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \end{array}$$

It is sufficient to show that $v \notin I_{\mathcal{N}}$ for achieving commutativity. Suppose otherwise, $v \in I_{\mathcal{N}}$. Since $u \in I_Q$ and $u \preceq v$ there exist $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$ such that $a \preceq u \preceq v \prec d$. By assumption, $v \notin I_Q$, thus $v \notin I_{\mathcal{M}}$ as $v \in I_{\mathcal{N}}$. Since $a \preceq u \preceq v \prec d$ where $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$, and $u \in I_Q$, there exist $c \in S_{\min}(\mathcal{N})$ and $b \in S_{\max}(\overline{\mathcal{M}})$ such that $c \preceq a \preceq u \preceq v \prec d \preceq b$ because of the assumption of the statement. This implies $v \in I_{\mathcal{M}}$ since $a \preceq u \preceq v \prec b$ where $a \in S_{\min}(\mathcal{M})$ and $b \in S_{\max}(\overline{\mathcal{M}})$. Contradiction. Thus, $v \notin I_{\mathcal{N}}$ as desired.

For $u \notin I_Q$ and $v \in I_Q$, we can again show that the following diagram commutes as $\mathcal{M}_u = 0$ by showing $u \notin I_{\mathcal{M}}$ because of the assumption, similarly.

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v = k \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \mathrm{id}_k} \\ \mathcal{N}_u & \longrightarrow & \mathcal{N}_v = k \end{array}$$

For $u \notin I_Q$ and $v \notin I_Q$, observe that the diagram is again commutative since $f^Q_u = 0$ and $f^Q_v = 0$ by definition of $f^Q \colon \mathcal{M} \to \mathcal{N}$:

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \end{array}$$

Therefore, the following diagram commutes for all $u \preceq v \in \mathbb{R}^2$:

$$\begin{array}{ccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v \\ \downarrow{\scriptstyle f^Q_u} & & \downarrow{\scriptstyle f^Q_v} \\ \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \end{array}$$

Hence, $f^Q \colon \mathcal{M} \to \mathcal{N}$ as in (8.2) is the desired non-trivial morphism. $\square$

Remark 8.2.7. *If for any pair (a, d) where $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$, we have $c \preceq a \prec d \preceq b$ for some $c \in S_{\min}(\mathcal{N})$ and $b \in S_{\max}(\overline{\mathcal{M}})$, then there is a non-trivial morphism $f \colon \mathcal{M} \to \mathcal{N}$.*

Corollary 8.2.8. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules given as above. Then, there is a non-trivial morphism $f \colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ if and only if there exists an intersection component Q of $\mathcal{M}$ and $\mathcal{N}(\vec{\epsilon})$ with underlying interval $I_Q \subseteq I_{\mathcal{M}} \cap I_{\mathcal{N}(\vec{\epsilon})}$ such that for any $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$ satisfying $a \preceq u \prec d - \vec{\epsilon}$ for some $u \in I_Q$, there exist $c \in S_{\min}(\mathcal{N})$ and $b \in S_{\max}(\overline{\mathcal{M}})$ such that $c - \vec{\epsilon} \preceq a \prec d - \vec{\epsilon} \preceq b$.*

Proof. The proof is analogous to Lemma 8.2.6 after replacing $\mathcal{N}$ with $\mathcal{N}(\vec{\epsilon})$.

$\square$

Lemma 8.2.9. *For any $a, d \in \mathbb{R}^2$, either $a \prec d$ or $\min_{i=1,2}\{d_i - a_i\} \leq 0$.*

Proof. If $a \prec d$, then there is nothing to prove. Let $a \nprec d$ and assume on the contrary that $\min_{i=1,2}\{d_i - a_i\} > 0$. The second assumption implies $0 < d_i - a_i$ for every $i \in \{1, 2\}$. Thus, $a_i < d_i$ for every $i \in \{1, 2\}$, which implies $a \prec d$. Contradiction.

Hence, given any points $a, d \in \mathbb{R}^2$, either $a \prec d$ or $\min_{i=1,2}\{d_i - a_i\} \leq 0$. $\square$

Lemma 8.2.10. *If $I_\mathcal{M} \cap I_\mathcal{N} \neq \emptyset$, then $\max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \min_{i=1,2}\{d_i - a_i\} > 0$.*

Proof. Suppose on the contrary that $\max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \min_{i=1,2}\{d_i - a_i\} \leq 0$. This implies that $\min_{i=1,2}\{d_i - a_i\} \leq 0$ for any pair (a, d) where $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$. Thus, by Lemma 8.2.9, we get $a \nprec d$ for any pair (a, d). This will let $I_\mathcal{M} \cap I_\mathcal{N} = \emptyset$, which is a contradiction. $\square$

Remark 8.2.11. *If $I_\mathcal{M} \cap I_\mathcal{N} \neq \emptyset$, then $\max_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \min_{i=1,2}\{b_i - c_i\} > 0$ can be shown similarly.*

Theorem 8.2.12. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules with at most one intersection component. There exists a non-trivial morphism $f\colon \mathcal{M} \to \mathcal{N}$ if and only if*

$$\max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min\left\{ \min_{i=1,2}\{d_i - a_i\}, \min_{c \in S_{\min}(\mathcal{N})} \max_{i=1,2}\{c_i - a_i\} \right\}, \\ \min\left\{ \min_{i=1,2}\{d_i - a_i\}, \min_{b \in S_{\max}(\overline{\mathcal{M}})} \max_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\} \leq 0$$

and

$$0 < \max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \min_{i=1,2}\{d_i - a_i\}.$$

Proof. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a non-trivial morphism. By Proposition 8.2.5, it is known that there exists an intersection component Q with underlying interval $I_Q \subseteq I_\mathcal{M} \cap I_\mathcal{N}$

such that $f_u \neq 0$ for every $u \in I_Q$. Moreover, since $I_{\mathcal{M}} \cap I_{\mathcal{N}} \neq \emptyset$, we have

$$0 < \max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \min_{i=1,2} \{d_i - a_i\}$$

thanks to Lemma 8.2.10.

Now, it is left to show that

$$\max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2} \{d_i - a_i\}, \min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2} \{c_i - a_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2} \{d_i - a_i\}, \min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2} \{d_i - b_i\} \right\} \end{array} \right\} \leq 0\,.$$

Now, consider any pair (a, d) where $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$. If $a \not\prec d$, then by Lemma 8.2.9, we know that $\min\limits_{i=1,2} \{d_i - a_i\} \leq 0$ and this implies

$$\min \left\{ \min_{i=1,2} \{d_i - a_i\}, \min_{c \in S_{\min}(\mathcal{N})} \max_{i=1,2} \{c_i - a_i\} \right\} \leq 0$$

and

$$\min \left\{ \min_{i=1,2} \{d_i - a_i\}, \min_{b \in S_{\max}(\overline{\mathcal{M}})} \max_{i=1,2} \{d_i - b_i\} \right\} \leq 0\,.$$

Consequently, we have

$$\max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2} \{d_i - a_i\}, \min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2} \{c_i - a_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2} \{d_i - a_i\}, \min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2} \{d_i - b_i\} \right\} \end{array} \right\} \leq 0\,.$$

Suppose now $a \prec d$. By Lemma 8.2.5, since given $f\colon \mathcal{M} \to \mathcal{N}$ is a non-trivial morphism if there exists $u \in I_Q$ for some intersection component Q satisfying $a \preceq u \prec d$, then there exist $c \in S_{\min}(\mathcal{N})$ and $b \in S_{\max}(\overline{\mathcal{M}})$ such that $c \preceq a \prec d \preceq b$. The reason is we have at most intersection component, more precisely, exactly one intersection component by assumption. Thus, having $c \preceq a \prec d \preceq b$ for some $c \in S_{\min}(\mathcal{N})$ and $b \in S_{\max}(\overline{\mathcal{M}})$ implies that

$$\min \left\{ \min_{i=1,2} \{d_i - a_i\}, \min_{c \in S_{\min}(\mathcal{N})} \max_{i=1,2} \{c_i - a_i\} \right\} \leq 0$$

and

$$\min \left\{ \min_{i=1,2} \{d_i - a_i\}, \min_{b \in S_{\max}(\overline{\mathcal{M}})} \max_{i=1,2} \{d_i - b_i\} \right\} \leq 0$$

since $\min_{c \in S_{\min}(\mathcal{N})} \max_{i=1,2}\{c_i - a_i\} \leq 0$ and $\min_{b \in S_{\max}(\overline{\mathcal{M}})} \max_{i=1,2}\{d_i - b_i\} \leq 0$, respectively. Consequently, we have

$$\max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\} \leq 0\,.$$

Suppose now $a \prec d$, but $u \notin I_Q$ satisfying $a \preceq u \prec d$. This implies that there exit no $c \in S_{\min}(\mathcal{N})$ or $b \in S_{\max}(\overline{\mathcal{M}})$ such that $c \preceq a \preceq u \prec d \preceq b$. Otherwise, observe that $u \in I_{\mathcal{M}}$ and $u \in I_{\mathcal{N}}$, that is, $u \in I_{\mathcal{M}} \cap I_{\mathcal{N}}$, which implies $u \in I_Q$ since we have exactly one intersection component. Suppose now there exists no $c \in S_{\min}(\mathcal{N})$ such that $c \preceq a \preceq u \prec d$ and take any point $v \in I_Q$. Therefore, we can find $w \in I_{\mathcal{M}} \setminus I_{\mathcal{N}}$ such that $w \prec v$. Now, consider the following diagram:

$$\begin{array}{ccc} k = \mathcal{M}_w & \xrightarrow{\mathrm{id}_k} & \mathcal{M}_v = k \\ \downarrow{\scriptstyle f_w} & & \downarrow{\scriptstyle \mathrm{id}_k} \\ 0 = \mathcal{N}_w & \xrightarrow{\mathrm{id}_k} & \mathcal{N}_v = k \end{array}$$

Since the diagram is not commutative, $f \colon \mathcal{M} \to \mathcal{N}$ cannot be a non-trivial morphism. Thus, we get a contradiction.

Suppose now there exists no $b \in S_{\max}(\overline{\mathcal{M}})$ such that $a \preceq u \prec d \preceq b$ and take any point $v' \in I_Q$. Therefore, we can find $w' \in I_{\mathcal{N}} \setminus I_{\mathcal{M}}$ such that $v' \prec w'$. Now, consider the following diagram:

$$\begin{array}{ccc} k = \mathcal{M}_{v'} & \xrightarrow{\mathrm{id}_k} & \mathcal{M}_{w'} = 0 \\ \downarrow{\scriptstyle \mathrm{id}_k} & & \downarrow{\scriptstyle f_{w'}} \\ k = \mathcal{N}_{v'} & \xrightarrow{\mathrm{id}_k} & \mathcal{N}_{w'} = k \end{array}$$

Since the diagram is not commutative, $f \colon \mathcal{M} \to \mathcal{N}$ cannot be a non-trivial morphism. Thus, again we get a contradiction.

Hence, if $a \prec d$, then there must exist $u \in I_Q$ satisfying $a \preceq u \prec d$ where Q is the intersection component and the results follow as discussed above.

Conversely, suppose we have

$$\max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\} \end{array} \right\} \leq 0$$

and

$$0 < \max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \min_{i=1,2}\{d_i-a_i\}.$$

By the second inequality, there exists at least one pair (a,d) where $a\in S_{\min}(\mathcal{M})$ and $d\in S_{\max}(\overline{\mathcal{N}})$ such that $a\prec d$. Moreover, observe that the first inequality implies

$$\min\left\{\min_{i=1,2}\{d_i-a_i\},\ \min_{c\in S_{\min}(\mathcal{N})}\max_{i=1,2}\{c_i-a_i\}\right\} \leq 0 \tag{A}$$

and

$$\min\left\{\min_{i=1,2}\{d_i-a_i\},\ \min_{b\in S_{\max}(\overline{\mathcal{M}})}\max_{i=1,2}\{d_i-b_i\}\right\} \leq 0 \tag{B}$$

for any pair (a,d) where $a\in S_{\min}(\mathcal{M})$ and $d\in S_{\max}(\overline{\mathcal{N}})$.

Now, consider any pair (a,d) where $a\in S_{\min}(\mathcal{M})$ and $d\in S_{\max}(\overline{\mathcal{N}})$ such that $a\prec d$, we know it exists by above. Respectively, by the Inequality (A) and Inequality (B), we have $\min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\leq 0$ and $\min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\leq 0$ since $\min\limits_{i=1,2}\{d_i-a_i\}>0$. Thus, respectively there exist $c\in S_{\min}(\mathcal{N})$ such that $c\preceq a$ and $b\in S_{\max}(\overline{\mathcal{M}})$ such that $d\preceq b$. Therefore, for any pair (a,d) where $a\in S_{\min}(\mathcal{M})$ and $d\in S_{\max}(\overline{\mathcal{N}})$ such that $a\prec d$, there exist $c\in S_{\min}(\mathcal{N})$ and $b\in S_{\max}(\overline{\mathcal{M}})$ such that $c\preceq a\prec d\preceq b$. Hence, by Lemma 8.2.6, there exists a non-trivial morphism $f\colon \mathcal{M}\to\mathcal{N}$. □

Corollary 8.2.13 (Shifted Version)**.** *Let $\mathcal{M}$ and $\mathcal{N}(\vec{\epsilon})$ be two interval persistence modules with at most one intersection component where $\mathcal{N}(\vec{\epsilon})$ is ϵ-shifting of the persistence module $\mathcal{N}$. There exists a non-trivial morphism $f\colon\mathcal{M}\to\mathcal{N}(\vec{\epsilon})$ if and only if*

$$\max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\} \end{array} \right\} \leq \epsilon$$

and

$$\epsilon < \max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \min_{i=1,2}\{d_i-a_i\}.$$

Proof. In the previous proof, replace $\mathcal{N}$ with $\mathcal{N}(\vec{\epsilon})$. $\square$

Corollary 8.2.14 (Symmetric Version)**.** *Let $\mathcal{M}(\vec{\epsilon})$ and $\mathcal{N}$ be two interval persistence modules with at most one intersection component where $\mathcal{M}(\vec{\epsilon})$ is ϵ-shifting of the persistence module $\mathcal{M}$. There exists a non-trivial morphism $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ if and only if*

$$\max_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min\left\{\min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a\in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\}\right\} \end{array} \right\} \leq \epsilon$$

and

$$\epsilon < \max_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \min_{i=1,2}\{b_i - c_i\}.$$

Proof. In the previous proof, switch the roles of the persistence modules $\mathcal{M}$ and $\mathcal{N}$. $\square$

Lemma 8.2.15. *Let $\mathcal{M}$ be an interval persistence module. Then, $\mathcal{M}$ is ϵ-trivial where*

$$\epsilon = \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2} \{b_i - a_i\} \right\}.$$

Proof. By Definition 4.1.6, it is enough to show that $\varphi_{\mathcal{M}}(u, u+\vec{\epsilon}) = 0$ for every $u \in \mathbb{R}^2$. Consider $u \notin I_{\mathcal{M}}$, then $\mathcal{M}_u = 0$, and so $\varphi_{\mathcal{M}}(u, u+\vec{\epsilon}) = 0$. Thus, let $u \in I_{\mathcal{M}}$. Observe that there exist a minimal element $a^q \in S_{\min}(M)$ and a maximal element $b^r \in S_{\max}(M)$ such that $a^q \preceq u \preceq b^r$. By assumption, we know that

$$\epsilon = \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2} \{b_i - a_i\} \right\}.$$

Thanks to the maximality, we have

$$\epsilon \geq \min_{i=1,2} \{b_i^r - a_i^q\}.$$

Alternatively, we can say $a_s^q + \epsilon \geq b_s^r$ for some $s \in \{1,2\}$. It is known that $u_s \geq a_s^q$ for every $s \in \{1,2\}$, or equivalently, $u_s + \epsilon \geq a_s^q + \epsilon$ for every $s \in \{1,2\}$. Thus, $u_s + \epsilon \geq b_s^r$ for some $s \in \{1,2\}$. However, b^r is a maximal element of $I_{\mathcal{M}}$. This implies that $u + \vec{\epsilon} \notin I_{\mathcal{M}}$. Therefore, $\mathcal{M}_{u+\vec{\epsilon}} = 0$, and so $\varphi_{\mathcal{M}}(u, u+\vec{\epsilon}) = 0$, again.

Hence, $\varphi_{\mathcal{M}}(u, u+\vec{\epsilon}) = 0$ for every $u \in \mathbb{R}^2$ as desired.

$\square$

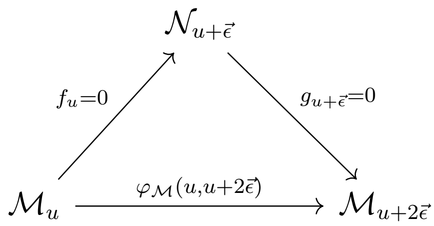
Lemma 8.2.16. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules. Then,*

$$d_I(\mathcal{M},\mathcal{N}) \leq \dagger^2 = \max\left\{ \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{\min_{i=1,2} \frac{b_i - a_i}{2}\right\}, \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{\min_{i=1,2} \frac{d_i - c_i}{2}\right\} \right\}.$$

Proof. On condition that

$$\max\left\{ \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{\min_{i=1,2} \frac{b_i - a_i}{2}\right\}, \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{\min_{i=1,2} \frac{d_i - c_i}{2}\right\} \right\} = +\infty,$$

there is nothing to prove. Thus, assume that the quantity above is finite and equal to $\epsilon \in \mathbb{R}^+$. Let us take the maps $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ to be trivial where $\vec{\epsilon} = (\epsilon, \epsilon)$. Thus, all square diagrams (4.1) in the Definition 4.1.1 are commutative. Now consider the third diagram in the Definition 4.1.1:



Note that the diagram above will be commutative for every point $u \in \mathbb{R}^2$ if we can show that $\varphi_{\mathcal{M}}(u, u+2\vec{\epsilon}) = 0$ for every point $u \in \mathbb{R}^2$.

By assumption, we know that

$$\epsilon = \max\left\{ \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{\min_{i=1,2} \frac{b_i - a_i}{2}\right\}, \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{\min_{i=1,2} \frac{d_i - c_i}{2}\right\} \right\}$$

thus, because of the maximality, this gives us the condition

$$\epsilon \geq \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{\min_{i=1,2} \frac{b_i - a_i}{2}\right\}$$

and equivalently, we have

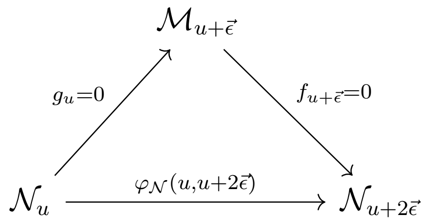
$$2\epsilon \geq \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2} \{b_i - a_i\} \right\}.$$

By Lemma 8.2.15, it is known that $\mathcal{M}$ is δ-trivial where

$$\delta = \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2} \{b_i - a_i\} \right\}.$$

Thus, $\mathcal{M}$ is also 2ϵ-trivial as $2\epsilon \geq \delta$. Hence, $\varphi_{\mathcal{M}}(u, u+2\vec{\epsilon}) = 0$ for every point $u \in \mathbb{R}^2$ as desired.

Similarly, one can show that the following diagram is also commutative for every point $u \in \mathbb{R}^2$:



Therefore, the trivial morphisms $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ are ϵ-interleaving morphisms such that all diagrams in the Definition 4.1.1 are commutative for every point in $\mathbb{R}^2$. Hence, interval bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved, and thus, we can conclude that

$$d_I(\mathcal{M},\mathcal{N}) \leq \epsilon = \max\left\{ \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2} \frac{b_i - a_i}{2} \right\}, \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2} \frac{d_i - c_i}{2} \right\} \right\}.$$

□

Lemma 8.2.17. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules. Suppose $\mathcal{M}$ and $\mathcal{N}(\vec{\epsilon})$, and $\mathcal{M}(\vec{\epsilon})$ and $\mathcal{N}$ have at most one intersection component. If*

$$\dagger^1 = \max\left\{\begin{array}{l}\max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\max\left\{\begin{array}{l}\min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\}\end{array}\right\},\\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\max\left\{\begin{array}{l}\min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})}\max\limits_{i=1,2}\{a_i-c_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})}\max\limits_{i=1,2}\{b_i-d_i\}\right\}\end{array}\right\}\end{array}\right\}$$

$$< \dagger^2 = \max\left\{\max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\left\{\min\limits_{i=1,2}\frac{b_i-a_i}{2}\right\},\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\left\{\min\limits_{i=1,2}\frac{d_i-c_i}{2}\right\}\right\},$$

then

$$\dagger^1 = \max\left\{\begin{array}{l}\max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\max\left\{\begin{array}{l}\min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\}\end{array}\right\},\\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\max\left\{\begin{array}{l}\min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})}\max\limits_{i=1,2}\{a_i-c_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})}\max\limits_{i=1,2}\{b_i-d_i\}\right\}\end{array}\right\}\end{array}\right\}$$

$$< \min\left\{\max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\left\{\min\limits_{i=1,2}\{d_i-a_i\}\right\},\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\left\{\min\limits_{i=1,2}\{b_i-c_i\}\right\}\right\}.$$

Proof. Let

$$\epsilon = \max\left\{\begin{array}{l}\max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\max\left\{\begin{array}{l}\min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\}\end{array}\right\},\\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\max\left\{\begin{array}{l}\min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})}\max\limits_{i=1,2}\{a_i-c_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})}\max\limits_{i=1,2}\{b_i-d_i\}\right\}\end{array}\right\}\end{array}\right\}.$$

This implies that

$$\epsilon \geq \max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\max\left\{\begin{array}{l}\min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\}\end{array}\right\}$$

and

$$\epsilon \geq \max_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\} \right\} \end{array} \right\}.$$

Suppose on the contrary that

$$\dagger^1 = \max \left\{ \begin{array}{l} \max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{d_i - a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{d_i - a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\}, \\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\} \right\} \end{array} \right\} \end{array} \right\}$$

$$= \epsilon \geq \min \left\{ \max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2}\{d_i - a_i\} \right\},\ \max_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2}\{b_i - c_i\} \right\} \right\}.$$

We will get a contradiction. Without loss of generality, suppose that

$$\min \left\{ \max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2}\{d_i - a_i\} \right\},\ \max_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2}\{b_i - c_i\} \right\} \right\} = \max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2}\{d_i - a_i\} \right\}$$

Thus, by above, we have

$$\dagger^1 = \epsilon \geq \max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2}\{d_i - a_i\} \right\}.$$

It follows that $\epsilon \geq \min\limits_{i=1,2}\{d_i - a_i\}$ for any $a \in S_{\min}(\mathcal{M})$ and $d \in S_{\max}(\overline{\mathcal{N}})$. Consider any pair (c, d) where $c \in S_{\min}(\mathcal{N})$ and $d \in S_{\max}(\overline{\mathcal{N}})$. We will show that $\epsilon \geq \min\limits_{i=1,2} \dfrac{d_i - c_i}{2}$. Now, observe that

$$\min_{i=1,2}\{d_i - c_i\} \leq \min_{i=1,2}\{d_i - a_i\} + \min_{i=1,2}\{a_i - c_i\} \leq \min_{i=1,2}\{d_i - a_i\} + \max_{i=1,2}\{a_i - c_i\}.$$

The first inequality is a form of the triangle inequality and the second inequality is straightforward.

By above, we know that $\min_{i=1,2}\{d_i - a_i\} \leq \epsilon$ and $\max_{i=1,2}\{a_i - c_i\} \leq \epsilon$.

Thus,

$$\min_{i=1,2}\{d_i - c_i\} \leq \epsilon + \epsilon,$$

or equivalently,

$$\epsilon \geq \min_{i=1,2} \frac{d_i - c_i}{2}.$$

Since it is true for any pair (c, d) where $c \in S_{\min}(\mathcal{N})$ and $d \in S_{\max}(\overline{\mathcal{N}})$, we get

$$\epsilon \geq \max_{\substack{c \in S_{\min}(\mathcal{N}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2} \frac{d_i - c_i}{2} \right\}.$$

By a similar argument, we can show that

$$\epsilon \geq \max_{\substack{a \in S_{\min}(\mathcal{M}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2} \frac{b_i - a_i}{2} \right\}.$$

Hence,

$$\epsilon = \dagger^1 \geq \dagger^2 = \max \left\{ \max_{\substack{a \in S_{\min}(\mathcal{M}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2} \frac{b_i - a_i}{2} \right\}, \max_{\substack{c \in S_{\min}(\mathcal{N}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2} \frac{d_i - c_i}{2} \right\} \right\}$$

which is against the main assumption. □

Lemma 8.2.18. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules. Suppose $\mathcal{M}$ and $\mathcal{N}(\vec{\epsilon})$, and $\mathcal{M}(\vec{\epsilon})$ and $\mathcal{N}$ have at most one intersection component. If*

$$\dagger^1 = \max \left\{ \begin{array}{l} \max\limits_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\}, \\ \max\limits_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a \in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d \in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\} \right\} \end{array} \right\} \end{array} \right\}$$

$$< \min \left\{ \max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2}\{d_i - a_i\} \right\}, \max_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2}\{b_i - c_i\} \right\} \right\},$$

then

$$d_I(\mathcal{M},\mathcal{N}) \leq \max \left\{ \begin{array}{l} \max\limits_{\substack{a\in S_{\min}(\mathcal{M}) \\ d\in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c\in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\}, \\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N}) \\ b\in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a\in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\} \right\} \end{array} \right\} \end{array} \right\}.$$

Proof. Let

$$\epsilon = \dagger^1 = \max \left\{ \begin{array}{l} \max\limits_{\substack{a\in S_{\min}(\mathcal{M}) \\ d\in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c\in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\}, \\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N}) \\ b\in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a\in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\} \right\} \end{array} \right\} \end{array} \right\}.$$

This implies that

$$\epsilon \geq \max\limits_{\substack{a\in S_{\min}(\mathcal{M}) \\ d\in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c\in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\}$$

and

$$\epsilon \geq \max\limits_{\substack{c\in S_{\min}(\mathcal{N}) \\ b\in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a\in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\} \right\}, \\ \min\left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\} \right\} \end{array} \right\}.$$

Moreover, by main assumption, we know

$$\epsilon < \min \left\{ \max\limits_{\substack{a\in S_{\min}(\mathcal{M}) \\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min\limits_{i=1,2}\{d_i - a_i\} \right\}, \max\limits_{\substack{c\in S_{\min}(\mathcal{N}) \\ b\in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min\limits_{i=1,2}\{b_i - c_i\} \right\} \right\}.$$

Thus,

$$\epsilon < \max\limits_{\substack{a\in S_{\min}(\mathcal{M}) \\ d\in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min\limits_{i=1,2}\{d_i - a_i\} \right\}$$

and

$$\epsilon < \max_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2}\{b_i - c_i\} \right\}.$$

Therefore, respectively, we get

$$\max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \right\} \end{array} \right\} \leq \epsilon$$

and

$$\epsilon < \max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \min_{i=1,2}\{d_i - a_i\},$$

and also,

$$\max_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a \in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d \in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\} \right\} \end{array} \right\} \leq \epsilon$$

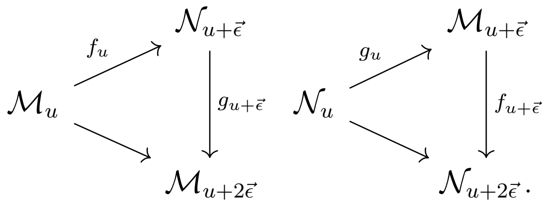
and

$$\epsilon < \max_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \min_{i=1,2}\{b_i - c_i\}.$$

Hence, respectively by Corollary 8.2.13 and Corollary 8.2.14, we have non-trivial morphisms $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$. Thus, the following square diagrams are commutative for all $u \preceq v \in \mathbb{R}^2$:

$$\begin{array}{ccccccc} \mathcal{M}_u & \longrightarrow & \mathcal{M}_v & \quad & \mathcal{N}_u & \longrightarrow & \mathcal{N}_v \\ \downarrow{\scriptstyle f_u} & & \downarrow{\scriptstyle f_v} & & \downarrow{\scriptstyle g_u} & & \downarrow{\scriptstyle g_v} \\ \mathcal{N}_{u+\vec{\epsilon}} & \longrightarrow & \mathcal{N}_{v+\vec{\epsilon}} & & \mathcal{M}_{u+\vec{\epsilon}} & \longrightarrow & \mathcal{M}_{v+\vec{\epsilon}}\,. \end{array}$$

Let us now show that these morphims make the following triangle diagrams commutative for all $u \in \mathbb{R}^2$:



We will only show that the first triangle diagram commutes for all $u \in \mathbb{R}^2$. The commutativity of other triangle diagram can be proven in a similar way.

If $\mathcal{M}_u = 0$ or $\mathcal{M}_{u+2\vec{\epsilon}} = 0$, then the first triangle diagram is clearly commutative. So, suppose that $\mathcal{M}_u = k$ and $\mathcal{M}_{u+2\vec{\epsilon}} = k$. Thus, $\varphi_{\mathcal{M}}(u, u+2\vec{\epsilon}) = \mathrm{id}_k$ and $a \prec u \prec b-2\vec{\epsilon}$ for some $a \in S_{\min}(\mathcal{M})$ and $b \in S_{\max}(\overline{\mathcal{M}})$. Thus, we have

$$u + \vec{\epsilon} \prec b - \vec{\epsilon} \preceq b - \max_{i=1,2}\{d_i - b_i\} \cdot (1,1) \preceq d$$

where $d \in S_{\max}(\overline{\mathcal{N}})$ since $\min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\} \leq \epsilon$. So, $u + \vec{\epsilon} \prec d$ for some $d \in S_{\max}(\overline{\mathcal{N}})$. Also, we have

$$c \prec a + \max_{i=1,2}\{c_i - a_i\} \cdot (1,1) \preceq a + \vec{\epsilon} \preceq u + \vec{\epsilon}$$

for some $c \in S_{\min}(\mathcal{N})$ since $\min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\} \leq \epsilon$. So, $c \prec u + \vec{\epsilon}$ for some $c \in S_{\min}(\mathcal{N})$. These two inequalities imply that $c \prec u+\vec{\epsilon} \prec d$ for some $c \in S_{\min}(\mathcal{N})$ and $d \in S_{\max}(\overline{\mathcal{N}})$. It follows that $\mathcal{N}_{u+\vec{\epsilon}} = k$, and hence $f_u = \mathrm{id}_k$ and $g_{u+\vec{\epsilon}} = \mathrm{id}_k$ since we are assuming that $\mathcal{M}_u = k$ and $\mathcal{M}_{u+2\vec{\epsilon}} = k$. Therefore, the first triangle diagram commutes for all points $u \in \mathbb{R}^2$.

Hence, $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ form a non-trivial ϵ-interleaving morphism pair, implying that $d_I(\mathcal{M},\mathcal{N}) \leq \epsilon = \dagger^1$ as desired.

$\square$

We are ready to give our first main result in this section.

Theorem 8.2.19. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules given as above. Then,*

$$d_I(\mathcal{M},\mathcal{N}) \leq \min\{\dagger^1, \dagger^2\}$$

where

$$\dagger^1 = \max\left\{\begin{array}{l} \max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max\left\{\begin{array}{l} \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\}\end{array}\right\},\\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max\left\{\begin{array}{l} \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})}\max\limits_{i=1,2}\{a_i-c_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})}\max\limits_{i=1,2}\{b_i-d_i\}\right\}\end{array}\right\}\end{array}\right\}$$

and

$$\dagger^2 = \max\left\{\max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\left\{\min_{i=1,2}\frac{b_i-a_i}{2}\right\},\ \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\left\{\min_{i=1,2}\frac{d_i-c_i}{2}\right\}\right\}.$$

Proof. First, suppose that

$$\dagger^2 \leq \dagger^1.$$

By Lemma 8.2.16, we know that

$$d_I(\mathcal{M},\mathcal{N}) \leq \dagger^2.$$

Hence, the result is obvious.

Now, suppose that

$$\dagger^1 \leq \dagger^2.$$

Then, by Lemma 8.2.17, we know that

$$\dagger^1 = \max\left\{\begin{array}{l} \max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max\left\{\begin{array}{l} \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\}\end{array}\right\},\\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max\left\{\begin{array}{l} \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})}\max\limits_{i=1,2}\{a_i-c_i\}\right\},\\ \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})}\max\limits_{i=1,2}\{b_i-d_i\}\right\}\end{array}\right\}\end{array}\right\}$$

$$< \min\left\{\max_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\left\{\min_{i=1,2}\{d_i-a_i\}\right\},\ \max_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\left\{\min_{i=1,2}\{b_i-c_i\}\right\}\right\}.$$

Then, by Lemma 8.2.18, we have

$$d_I(\mathcal{M},\mathcal{N}) \leq \dagger^1.$$

Hence, in any case

$$d_I(\mathcal{M},\mathcal{N}) \leq \min\{\dagger^1, \dagger^2\}$$

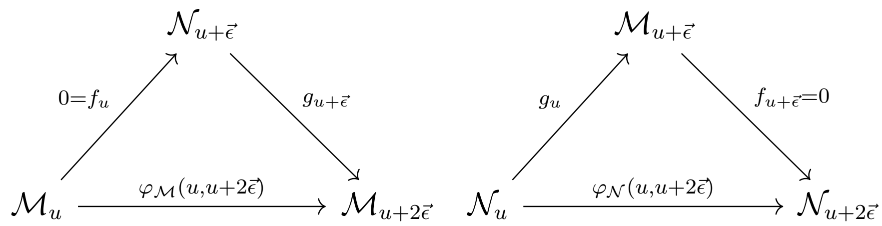
as desired.

$\square$

Lemma 8.2.20. *Let $\mathcal{M}$ and $\mathcal{N}$ be ϵ-interleaved persistence modules with two interleaving morphisms $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$. If f or g is a trivial morphism, then*

$$\epsilon \geq \max\left\{ \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\mathcal{M})}} \left\{\min_{i=1,2} \frac{b_i - a_i}{2}\right\}, \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\mathcal{N})}} \left\{\min_{i=1,2} \frac{d_i - c_i}{2}\right\} \right\}.$$

Proof. Without loss of generality, suppose that $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ is a trivial morphism. Thus, $f_u\colon \mathcal{M}_u \to \mathcal{N}_{u+\vec{\epsilon}}$ is a zero map for every $u \in \mathbb{R}^2$. By assumption, we know that $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved. Hence, both diagrams below must be commutative for every $u \in \mathbb{R}^2$.



As a result of this fact, both transition maps $\varphi_{\mathcal{M}}(u, u + 2\vec{\epsilon})$ and $\varphi_{\mathcal{N}}(u, u + 2\vec{\epsilon})$ are zero linear maps for every $u \in \mathbb{R}^2$. Hence, both $\mathcal{M}$ and $\mathcal{N}$ are 2ϵ-trivial persistence modules. Respectively, this implies that

$$2\epsilon \geq \max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\mathcal{M})}} \left\{\min_{i=1,2}\{b_i - a_i\}\right\}$$

and

$$2\epsilon \geq \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\mathcal{N})}} \left\{\min_{i=1,2}\{d_i - c_i\}\right\}.$$

Hence, we can conclude that

$$\epsilon \geq \max \left\{ \max_{\substack{a \in S_{\min}(\mathcal{M}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \left\{ \min_{i=1,2} \frac{b_i - a_i}{2} \right\}, \max_{\substack{c \in S_{\min}(\mathcal{N}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \left\{ \min_{i=1,2} \frac{d_i - c_i}{2} \right\} \right\}.$$

Analogously, if $g \colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ is a trivial morphism, then we have the same conclusion.

$\square$

Lemma 8.2.21. *Let $\mathcal{M}$ and $\mathcal{N}$ with the condition $\mathcal{M}$ and $\mathcal{N}(\vec{\epsilon})$, and $\mathcal{M}(\vec{\epsilon})$ and $\mathcal{N}$ have at most one intersection component be ϵ-interleaved persistence modules and suppose that the two interleaving morphisms $f \colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g \colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ are given. If f and g are non-trivial morphisms, then*

$$\epsilon \geq \max \left\{ \begin{array}{l} \max\limits_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2} \{d_i - a_i\}, \min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2} \{c_i - a_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2} \{d_i - a_i\}, \min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2} \{d_i - b_i\} \right\} \end{array} \right\}, \\ \max\limits_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2} \{b_i - c_i\}, \min\limits_{a \in S_{\min}(\mathcal{M})} \max\limits_{i=1,2} \{a_i - c_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2} \{b_i - c_i\}, \min\limits_{d \in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2} \{b_i - d_i\} \right\} \end{array} \right\} \end{array} \right\}.$$

Proof. By assumption, we know that both $f \colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g \colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$ are non-trivial morphisms. Thus, respectively by Corollary 8.2.13 and Corollary 8.2.14, we have

$$\epsilon \geq \max_{\substack{a \in S_{\min}(\mathcal{M}) \\ d \in S_{\max}(\overline{\mathcal{N}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2} \{d_i - a_i\}, \min\limits_{c \in S_{\min}(\mathcal{N})} \max\limits_{i=1,2} \{c_i - a_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2} \{d_i - a_i\}, \min\limits_{b \in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2} \{d_i - b_i\} \right\} \end{array} \right\}$$

and

$$\epsilon \geq \max_{\substack{c \in S_{\min}(\mathcal{N}) \\ b \in S_{\max}(\overline{\mathcal{M}})}} \max \left\{ \begin{array}{l} \min \left\{ \min\limits_{i=1,2} \{b_i - c_i\}, \min\limits_{a \in S_{\min}(\mathcal{M})} \max\limits_{i=1,2} \{a_i - c_i\} \right\}, \\ \min \left\{ \min\limits_{i=1,2} \{b_i - c_i\}, \min\limits_{d \in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2} \{b_i - d_i\} \right\} \end{array} \right\}.$$

Hence, we can conclude that

$$\epsilon \geq \dagger^1 = \max\left\{\begin{array}{c} \max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max\left\{\begin{array}{c} \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\} \end{array}\right\}, \\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max\left\{\begin{array}{c} \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})}\max\limits_{i=1,2}\{a_i-c_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})}\max\limits_{i=1,2}\{b_i-d_i\}\right\} \end{array}\right\} \end{array}\right\}.$$

□

We are ready to give our second main result in this section.

Theorem 8.2.22. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules given as above. Then,*

$$d_I(\mathcal{M},\mathcal{N}) \geq \min\left\{\dagger^1, \dagger^2\right\}$$

where

$$\dagger^1 = \max\left\{\begin{array}{c} \max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max\left\{\begin{array}{c} \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{c\in S_{\min}(\mathcal{N})}\max\limits_{i=1,2}\{c_i-a_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{d_i-a_i\},\ \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})}\max\limits_{i=1,2}\{d_i-b_i\}\right\} \end{array}\right\}, \\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max\left\{\begin{array}{c} \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{a\in S_{\min}(\mathcal{M})}\max\limits_{i=1,2}\{a_i-c_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{b_i-c_i\},\ \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})}\max\limits_{i=1,2}\{b_i-d_i\}\right\} \end{array}\right\} \end{array}\right\}$$

and

$$\dagger^2 = \max\left\{\max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\left\{\min_{i=1,2}\frac{b_i-a_i}{2}\right\},\ \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\left\{\min_{i=1,2}\frac{d_i-c_i}{2}\right\}\right\}.$$

Proof. Suppose $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved persistence modules with interleaving morphisms $f\colon \mathcal{M} \to \mathcal{N}(\vec{\epsilon})$ and $g\colon \mathcal{N} \to \mathcal{M}(\vec{\epsilon})$. Suppose that f or g is a trivial morphism. By Lemma 8.2.20, we know that

$$\epsilon \geq \dagger^2 = \max\left\{\max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\left\{\min_{i=1,2}\frac{b_i-a_i}{2}\right\},\ \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\left\{\min_{i=1,2}\frac{d_i-c_i}{2}\right\}\right\}.$$

Thus, we have

$$\epsilon \geq \min\left\{\dagger^1, \dagger^2\right\}.$$

Now, suppose that f and g are non-trivial morphisms. By Lemma 8.2.21, we know that

$$\epsilon \geq \dagger^1 = \max\left\{\begin{array}{l} \max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max\left\{\begin{array}{l} \min\left\{\min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c\in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\}\right\} \end{array}\right\}, \\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max\left\{\begin{array}{l} \min\left\{\min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a\in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\}\right\} \end{array}\right\} \end{array}\right\}.$$

Thus, again, we have

$$\epsilon \geq \min\left\{\dagger^1, \dagger^2\right\}.$$

Since ϵ is arbitrary where $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved and in any case, whether at least one of the morphisms is trivial, or both are non-trivial morphisms, we have

$$\epsilon \geq \min\left\{\dagger^1, \dagger^2\right\}.$$

Hence, we can conclude that

$$d_I(\mathcal{M}, \mathcal{N}) \geq \min\left\{\dagger^1, \dagger^2\right\}.$$

□

Corollary 8.2.23. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules given as above. It holds that:*

$$d_I(\mathcal{M}, \mathcal{N}) = \min\left\{\dagger^1, \dagger^2\right\}$$

where

$$\dagger^1 = \max\left\{\begin{array}{l} \max\limits_{\substack{a\in S_{\min}(\mathcal{M})\\ d\in S_{\max}(\overline{\mathcal{N}})}} \max\left\{\begin{array}{l} \min\left\{\min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{c\in S_{\min}(\mathcal{N})} \max\limits_{i=1,2}\{c_i - a_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{d_i - a_i\}, \min\limits_{b\in S_{\max}(\overline{\mathcal{M}})} \max\limits_{i=1,2}\{d_i - b_i\}\right\} \end{array}\right\}, \\ \max\limits_{\substack{c\in S_{\min}(\mathcal{N})\\ b\in S_{\max}(\overline{\mathcal{M}})}} \max\left\{\begin{array}{l} \min\left\{\min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{a\in S_{\min}(\mathcal{M})} \max\limits_{i=1,2}\{a_i - c_i\}\right\}, \\ \min\left\{\min\limits_{i=1,2}\{b_i - c_i\}, \min\limits_{d\in S_{\max}(\overline{\mathcal{N}})} \max\limits_{i=1,2}\{b_i - d_i\}\right\} \end{array}\right\} \end{array}\right\}$$

and

$$\dagger^2 = \max\left\{\max_{\substack{a\in S_{\min}(\mathcal{M})\\ b\in S_{\max}(\overline{\mathcal{M}})}}\left\{\min_{i=1,2}\frac{b_i - a_i}{2}\right\}, \max_{\substack{c\in S_{\min}(\mathcal{N})\\ d\in S_{\max}(\overline{\mathcal{N}})}}\left\{\min_{i=1,2}\frac{d_i - c_i}{2}\right\}\right\}.$$

Proof. The result is straightforwardly from Theorem 8.2.19 and Theorem 8.2.22. □

CHAPTER 9

GENERAL COMPARISON OF DISTANCES

In this chapter, we will compare all distances including the interleaving distance, the bottleneck distance, the matching distance and the steady matching distance on different types of persistence modules such as interval decomposable persistence modules and rectangle decomposable persistemce modules, or more specifically interval persistence modules and rectangle persistence modules.

9.1 Comparison of Distances with the Steady Matching Distance

Proposition 9.1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be finitely presented interval decomposable persistence modules. Then, the steady matching distance $SD_{\text{match}}(\mathcal{M},\mathcal{N})$ is a lower bound for the bottleneck distance $d_B(\mathcal{M},\mathcal{N})$.*

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be finitely presented interval decomposable persistence modules with given decomposition $\mathcal{M} = \bigoplus_{i\in\mathbf{I}} \mathcal{M}_i$ and $\mathcal{N} = \bigoplus_{j\in\mathbf{J}} \mathcal{N}_j$ where each $\mathcal{M}_i$ and $\mathcal{N}_j$ is an interval persistence module. Let S be the set of all partial multibijections between finite multisets $\mathbf{I}$ and $\mathbf{J}$. Let $\sigma \in S$ and let $\mathbf{I}' = \mathbf{I} - \operatorname{coim}\sigma$, $\mathbf{J}' = \mathbf{J} - \operatorname{im}\sigma$. If $d_B(\mathcal{M},\mathcal{N}) = +\infty$, then the claim is obvious, so suppose that $d_B(\mathcal{M},\mathcal{N}) = \epsilon$ for some $\epsilon \geq 0$. Thus, we have

$$\epsilon = \min_{\sigma\in S} \big(\max\big\{\{\max_{i\in\operatorname{coim}\sigma}\{d_I(\mathcal{M}_i,\mathcal{N}_{\sigma(i)})\}, \max_{i\in\mathbf{I}'}\{d_I(\mathcal{M}_i,0)\}, \max_{j\in\mathbf{J}'}\{d_I(0,\mathcal{N}_j)\}\big\}\big).$$

It follows that there exists a partial multibijection $\bar\sigma$ such that $\max_{i\in\operatorname{coim}\sigma}\{d_I(\mathcal{M}_i,\mathcal{N}_{\bar\sigma(i)})\} \leq \epsilon$, $\max_{i\in\mathbf{I}'}\{d_I(\mathcal{M}_i,0)\} \leq \epsilon$, and $\max_{j\in\mathbf{J}'}\{d_I(0,\mathcal{N}_j)\} \leq \epsilon$. Then, we have

- $d_I(\mathcal{M}_i,\mathcal{N}_{\bar\sigma(i)}) \leq \epsilon$ for all $i \in \operatorname{coim}\bar\sigma$,

- $d_I(\mathcal{M}_i, 0) \leq \epsilon$ for all $i \in \mathbf{I}' = \mathbf{I} - \operatorname{coim} \bar{\sigma} \subseteq \mathbf{I} - \operatorname{coim} \bar{\sigma}^L$,
- $d_I(0, \mathcal{N}_j) \leq \epsilon$ for all $j \in \mathbf{J}' = \mathbf{J} - \operatorname{im} \bar{\sigma} \subseteq \mathbf{J} - \operatorname{im} \bar{\sigma}^L$.

Now, we know that if $\mathcal{M}$ and $\mathcal{N}$ are ϵ-interleaved, then $\mathcal{M}^L$ and $\mathcal{N}^L$ are $\frac{\epsilon}{m^L}$-interleaved [17]. So, we have

- $d_I(\mathcal{M}_i, \mathcal{N}_{\bar{\sigma}(i)}) \leq \epsilon$ implies $d_I(\mathcal{M}_i^L, \mathcal{N}^L_{\bar{\sigma}^L(i)}) \leq \frac{\epsilon}{m^L}$ for all $i \in \operatorname{coim} \bar{\sigma}^L$,
- $d_I(\mathcal{M}_i, 0) \leq \epsilon$ implies $d_I(\mathcal{M}_i^L, 0) \leq \frac{\epsilon}{m^L}$ for all $i \in \mathbf{I}'$,
- $d_I(0, \mathcal{N}_j) \leq \epsilon$ implies $d_I(0, \mathcal{N}_j^L) \leq \frac{\epsilon}{m^L}$ for all $j \in \mathbf{J}'$.

So, for any admissible line L, we have

$$\operatorname{cost}(\bar{\sigma}^L) \leq \max\left\{\frac{\epsilon}{m^L}, \frac{\epsilon}{m^L}, \frac{\epsilon}{m^L}\right\} = \frac{\epsilon}{m^L}.$$

Then, for any admissible line L, $m^L \operatorname{cost}(\bar{\sigma}^L) \leq \epsilon$. It follows that $\sup_{L \in \Lambda} m^L \operatorname{cost}(\bar{\sigma}^L) \leq \epsilon$. Hence,

$$SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = \min_{\sigma \in S} \sup_{L \in \Lambda} m^L \big(\operatorname{cost}(\sigma^L)\big) \leq \epsilon.$$

Therefore, $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) \leq d_B(\mathcal{M}, \mathcal{N})$ for all finitely presented interval decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$. □

Now, thanks to Proposition 4.2.5 and Proposition 6.2.5, and the previous result, we have the following fact:

Corollary 9.1.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be any two interval decomposable persistence modules. Then, the steady matching distance $SD_{\text{match}}(\mathcal{M}, \mathcal{N})$ is a lower bound for the bottleneck distance $d_B(\mathcal{M}, \mathcal{N})$.*

Unfortunately, we do not have a similar relation between the steady matching distance and the interleaving distance for interval decomposable persistence modules. Instead, we have the following result.

Corollary 9.1.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two n-parameter rectangle decomposable persistence modules. Then, it holds that:*

$$SD_{\text{match}}(\mathcal{M}, \mathcal{N}) \leq (2n-1) d_I(\mathcal{M}, \mathcal{N}).$$

Proof. In [4], it is shown that for ϵ-interleaved n-parameter rectangle decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$, there exists a $(2n-1)\epsilon$ matching between the barcodes $B(\mathcal{M})$ and $B(\mathcal{N})$. In particular, we have $d_B(\mathcal{M},\mathcal{N}) \leq (2n-1)d_I(\mathcal{M},\mathcal{N})$ for n-parameter rectangle decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$. By Corollary 9.1.2, we know that $SD_{\text{match}}(\mathcal{M},\mathcal{N}) \leq d_B(\mathcal{M},\mathcal{N})$ for interval decomposable persistence modules, in particular for rectangle decomposable persistence modules. Therefore, we can conclude that $SD_{\text{match}}(\mathcal{M},\mathcal{N}) \leq (2n-1)d_I(\mathcal{M},\mathcal{N})$. □

The next example, provided by Bjerkevik [4], shows that the upper bound $2n-1$ is the best possible value for $n=2$.

Example 9.1.4. *Let* $B(\mathcal{M}) = \{I_1, I_2, I_3\}$ *and* $B(\mathcal{N}) = \{J_1, J_2, J_3\}$, *where*

$$I_1 = (0,10)\times(1,11),\ I_2 = (0,12)\times(-1,11),\ I_3 = (2,10)\times(1,9)$$

$$J_1 = (1,11)\times(0,10),\ J_2 = (1,9)\times(0,12),\ J_3 = (-1,11)\times(2,10).$$

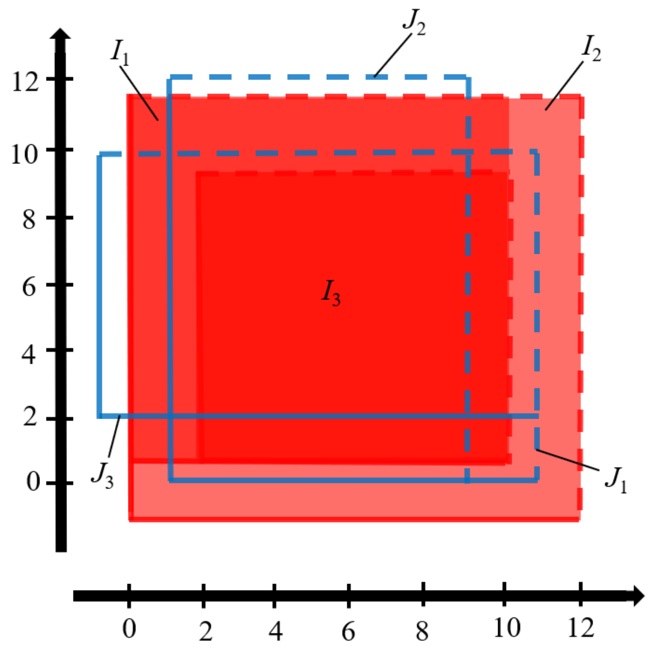


Figure 9.1: Two rectangle decomposable persistence modules $\mathcal{M}$ and $\mathcal{N}$ with $d_I(\mathcal{M},\mathcal{N}) = 1$ and $d_B(\mathcal{M},\mathcal{N}) = 3$.

9.2 Comparison of Distances on Interval Persistence Modules

Proposition 9.2.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules. Then,*

$$d_{\text{match}}(\mathcal{M},\mathcal{N}) = SD_{\text{match}}(\mathcal{M},\mathcal{N}) \leq d_I(\mathcal{M},\mathcal{N}) = d_B(\mathcal{M},\mathcal{N}).$$

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be two interval persistence modules. Thanks to Proposition 7.1.3 and Proposition 4.3.1, we know that $d_{\text{match}}(\mathcal{M},\mathcal{N}) = SD_{\text{match}}(\mathcal{M},\mathcal{N})$ and $d_I(\mathcal{M},\mathcal{N}) = d_B(\mathcal{M},\mathcal{N})$, respectively. From the previous fact, we also know that $SD_{\text{match}}(\mathcal{M},\mathcal{N}) \leq d_B(\mathcal{M},\mathcal{N})$ for interval decomposable persistence modules, and so it is also true for interval persistence modules. Hence, the result is obvious. □

Corollary 9.2.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. Then,*

$$d_{\text{match}}(\mathcal{M},\mathcal{N}) = SD_{\text{match}}(\mathcal{M},\mathcal{N}) \leq d_I(\mathcal{M},\mathcal{N}) = d_B(\mathcal{M},\mathcal{N})$$

where

$$d_I(\mathcal{M},\mathcal{N}) = \min\left\{\max\left\{\min_{i=1,2}\frac{b_i - a_i}{2}, \min_{i=1,2}\frac{d_i - c_i}{2}\right\}, \max\left\{\|c-a\|_\infty, \|d-b\|_\infty\right\}\right\}.$$

Proof. The result is straightforwardly from Proposition9.2.1 and Corollary 8.1.10. □

Remark 9.2.3. *We have the same previous result for any rectangle persistence modules $\mathcal{M}$ and $\mathcal{N}$ whether their underlying rectangles are closed, open or neither since it is known that*

- $d_{\text{match}}(\mathcal{M},\mathcal{N}) = d_{\text{match}}(\overline{\mathcal{M}},\overline{\mathcal{N}}) = d_{\text{match}}(\mathcal{M}^{\circ},\mathcal{N}^{\circ})$,
- $SD_{\text{match}}(\mathcal{M},\mathcal{N}) = SD_{\text{match}}(\overline{\mathcal{M}},\overline{\mathcal{N}}) = SD_{\text{match}}(\mathcal{M}^{\circ},\mathcal{N}^{\circ})$,
- $d_I(\mathcal{M},\mathcal{N}) = d_I(\overline{\mathcal{M}},\overline{\mathcal{N}}) = d_I(\mathcal{M}^{\circ},\mathcal{N}^{\circ})$,
- $d_B(\mathcal{M},\mathcal{N}) = d_B(\overline{\mathcal{M}},\overline{\mathcal{N}}) = d_B(\mathcal{M}^{\circ},\mathcal{N}^{\circ})$.

In certain cases, it is possible to show that the matching distance, the steady matching distance, the interleaving distance, and the bottleneck distance between rectangle persistence modules $\mathcal{M}$ and $\mathcal{N}$ are all equivalent to each other and all equal to

$$\min\big\{\max\big\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\big\},\max\{\|c-a\|_\infty,\|d-b\|_\infty\}\big\}.$$

where $R_{\mathcal{M}}=(a_1,b_1)\times(a_2,b_2)$ and $R_{\mathcal{N}}=(c_1,d_1)\times(c_2,d_2)$ are underlying rectangles of the rectangle persistence modules $\mathcal{M}$ and $\mathcal{N}$, respectively. For all details, see the proposition below.

Proposition 9.2.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules of one of the same types $R_1,R_2,R_3,R_4,R_5,R_6,R_7,R_9,R_{10},R_{11}$ or R_{13}. Then,*

$$d_{\text{match}}(\mathcal{M},\mathcal{N})=SD_{\text{match}}(\mathcal{M},\mathcal{N})=d_I(\mathcal{M},\mathcal{N})=d_B(\mathcal{M},\mathcal{N}).$$

Proof. Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules of the same type with underlying rectangles $R_{\mathcal{M}}=(a_1,b_1)\times(a_2,b_2)$ and $R_{\mathcal{N}}=(c_1,d_1)\times(c_2,d_2)$, respectively. By Corollary 9.2.2, we know that

$$d_{\text{match}}(\mathcal{M},\mathcal{N})=SD_{\text{match}}(\mathcal{M},\mathcal{N})\le d_I(\mathcal{M},\mathcal{N})=d_B(\mathcal{M},\mathcal{N})$$

where

$$d_I(\mathcal{M},\mathcal{N})=\min\left\{\max\left\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\right\},\max\left\{\|c-a\|_\infty,\|d-b\|_\infty\right\}\right\}.$$

Thus, to prove the statement, it is enough to show that

$$SD_{\text{match}}(\mathcal{M},\mathcal{N})\ge\min\left\{\max\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\},\max\{\|c-a\|_\infty,\|d-b\|_\infty\}\right\}.$$

Equivalently, it is enough to show that

$$\sup_{L\in\Lambda}m^L\operatorname{cost}(\sigma_1^L)\ge\min\left\{\max\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\},\max\{\|c-a\|_\infty,\|d-b\|_\infty\}\right\}$$

and

$$\sup_{L\in\Lambda}m^L\operatorname{cost}(\sigma_2^L)\ge\min\left\{\max\{\min_{i=1,2}\frac{b_i-a_i}{2},\min_{i=1,2}\frac{d_i-c_i}{2}\},\max\{\|c-a\|_\infty,\|d-b\|_\infty\}\right\}$$

since

$$SD_{\text{match}}(\mathcal{M},\mathcal{N})=\min\{\sup_{L\in\Lambda}m^L\operatorname{cost}(\sigma_1^L),\sup_{L\in\Lambda}m^L\operatorname{cost}(\sigma_2^L)\}$$

where σ_1 and σ_2 are two partial multibijections such that σ_1 matches bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ with the zero bipersistence module, and σ_2 matches the bipersistence module $\mathcal{M}$ with the bipersistence module $\mathcal{N}$, respectively.

To prove the claim, we will present it in 4 cases as follows:

Case 1: Suppose that $\mathcal{M}$ and $\mathcal{N}$ are of the same type R_1. Thus, the underlying rectangles are $R_{\mathcal{M}} = (-\infty, +\infty) \times (-\infty, +\infty)$ and $R_{\mathcal{N}} = (-\infty, +\infty) \times (-\infty, +\infty)$, respectively. Observe that

$$\min \big\{ \max \big\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \big\}, \max\{\|c - a\|_\infty, \|d - b\|_\infty\} \big\} = 0$$

since $\max\{\|c - a\|_\infty, \|d - b\|_\infty\} = 0$. Thus, there is nothing to prove in this case.

Case 2: Suppose that $\mathcal{M}$ and $\mathcal{N}$ are of the same type R_2. Thus, the underlying rectangles are $R_{\mathcal{M}} = (-\infty, +\infty) \times (-\infty, b_2)$ and $R_{\mathcal{N}} = (-\infty, +\infty) \times (-\infty, d_2)$, respectively. Observe that

$$\min \big\{ \max \big\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \big\}, \max\{\|c - a\|_\infty, \|d - b\|_\infty\} \big\} = |d_2 - b_2|.$$

To prove the claim, we can consider any diagonal line, so let us consider the diagonal line that passes through the point $(0, 0)$ parameterized by $L : u = t \cdot (1, 1)$ where $u \in L$ and $t \in \mathbb{R}$. Then, $I_{\mathcal{M}^L} = (-\infty, b_2)$ and $I_{\mathcal{N}^L} = (-\infty, d_2)$. So, we have $m^L \operatorname{cost}(\sigma_1^L) = +\infty$ and $m^L \operatorname{cost}(\sigma_2^L) = |d_2 - b_2|$. Hence, the claim is proved.

We skip proving cases when both rectangle persistence modules of the same type R_3, R_5, and R_9 as they are similar to case R_2.

Case 3: Suppose that $\mathcal{M}$ and $\mathcal{N}$ are of the same type R_4. Thus, the underlying rectangles are $R_{\mathcal{M}} = (-\infty, +\infty) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (-\infty, +\infty) \times (c_2, d_2)$, respectively. Observe that

$$\begin{aligned} &\min \big\{ \max \big\{ \min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2} \big\}, \max\{\|c - a\|_\infty, \|d - b\|_\infty\} \big\} \\ &= \min \big\{ \max \big\{ \frac{b_2 - a_2}{2}, \frac{d_2 - c_2}{2} \big\}, \max\{|c_2 - a_2|, |d_2 - b_2|\} \big\}. \end{aligned}$$

To prove the claim, we can consider any diagonal line. Let's focus on the diagonal that passes through the point $(0, 0)$ parameterized by $L : u = t \cdot (1, 1)$ where $u \in L$ and $t \in \mathbb{R}$. Then, one-parameter underlying intervals are $I_{\mathcal{M}^L} = (a_2, b_2)$ and

$I_{\mathcal{N}^L} = (c_2, d_2)$. So, we have $m^L \operatorname{cost}(\sigma_1^L) = \max\big\{\frac{b_2-a_2}{2}, \frac{d_2-c_2}{2}\big\}$ and $m^L \operatorname{cost}(\sigma_2^L) = \min\big\{\max\big\{\frac{b_2-a_2}{2}, \frac{d_2-c_2}{2}\big\}, \max\{|c_2-a_2|, |d_2-b_2|\}\big\}$. Therefore,

$$\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_1^L) \geq \min\Big\{\max\Big\{\frac{b_2-a_2}{2}, \frac{d_2-c_2}{2}\Big\}, \max\{|c_2-a_2|, |d_2-b_2|\}\Big\},$$

and

$$\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L) \geq \min\Big\{\max\Big\{\frac{b_2-a_2}{2}, \frac{d_2-c_2}{2}\Big\}, \max\{|c_2-a_2|, |d_2-b_2|\}\Big\}.$$

Hence, we have proved the claim. We can skip the proof of the case when both rectangle persistence modules of the same type R_{13} as it is similar to case R_4.

Case 4: Suppose that $\mathcal{M}$ and $\mathcal{N}$ are of the same type R_6. Thus, the underlying rectangles are $R_{\mathcal{M}} = (-\infty, b_1) \times (-\infty, b_2)$ and $R_{\mathcal{N}} = (-\infty, d_1) \times (-\infty, d_2)$, respectively. Observe that

$$\min\Big\{\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2}, \min_{i=1,2}\frac{d_i-c_i}{2}\Big\}, \max\{\|c-a\|_\infty, \|d-b\|_\infty\}\Big\} = \|d-b\|_\infty.$$

Now, if $\|d-b\|_\infty = d_i - b_i$ for some $i \in \{1,2\}$, then consider the diagonal line L_1 passing through the point (d_1, d_2), otherwise consider the diagonal line L_2 passing through the point (b_1, b_2). Without loss of generality, let us suppose that $\|d-b\|_\infty = d_i - b_i$, with $i \in \{1,2\}$, and consider the line L_1 parametrized by $L_1 : u = t \cdot (1,1) + (d_1, d_2)$ where $u \in L_1$ and $t \in \mathbb{R}$. Thus, $I_{\mathcal{M}^{L_1}} = (-\infty, b_i - d_i)$ and $I_{\mathcal{N}^{L_1}} = (-\infty, 0)$. Therefore,

$$\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_1^L) = +\infty > d_i - b_i,$$

and

$$\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L) \geq d_i - b_i.$$

Hence, the claim is proved.

Notice that cases when both rectangle persistence modules of the same type R_7, R_{10}, and R_{11} are quite similar to case R_6. Thus, we will skip proving these cases.

□

As we observed, a quadruple equality exists for some rectangle persistence modules of the same type. Alongside this, we also have quadruple equality when

$$\min\Big\{\max\Big\{\min_{i=1,2}\frac{b_i-a_i}{2}, \min_{i=1,2}\frac{d_i-c_i}{2}\Big\}, \max\{\|c-a\|_\infty, \|d-b\|_\infty\}\Big\} = +\infty.$$

Proposition 9.2.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$, respectively. If*

$$\min\big\{\max\big\{\min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2}\big\}, \max\{\|c-a\|_\infty, \|d-b\|_\infty\}\big\} = +\infty,$$

then,

$$d_{\text{match}}(\mathcal{M}, \mathcal{N}) = SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = d_I(\mathcal{M}, \mathcal{N}) = d_B(\mathcal{M}, \mathcal{N}) = +\infty.$$

Proof. Thanks to Corollary 9.2.2 and by assumption, we have

$$d_{\text{match}}(\mathcal{M}, \mathcal{N}) = SD_{\text{match}}(\mathcal{M}, \mathcal{N}) \leq d_I(\mathcal{M}, \mathcal{N}) = d_B(\mathcal{M}, \mathcal{N})$$

such that

$$d_I(\mathcal{M}, \mathcal{N}) = d_B(\mathcal{M}, \mathcal{N}) = +\infty.$$

Now, by Lemma 6.2.6, we know that $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = +\infty$. Hence, the result is obvious. □

Unlike the previous two results, the distances are not always equal to each other. The next example confirms that we do not always have quadruple equality even for rectangle bipersistence modules.

Example 9.2.6. *Let $\mathcal{M}$ and $\mathcal{N}$ be two rectangle bipersistence modules with underlying rectangles $R_{\mathcal{M}} = (4, 10) \times (4, +\infty)$ and $R_{\mathcal{N}} = (6, 10) \times (1, +\infty)$, respectively. Then, by Corollary 8.1.10, we know that*

$$d_I(\mathcal{M}, \mathcal{N}) = \min\big\{\max\big\{\min_{i=1,2} \frac{b_i - a_i}{2}, \min_{i=1,2} \frac{d_i - c_i}{2}\big\}, \max\{\|c-a\|_\infty, \|d-b\|_\infty\}\big\}.$$

where $R_{\mathcal{M}} = (a_1, b_1) \times (a_2, b_2)$ and $R_{\mathcal{N}} = (c_1, d_1) \times (c_2, d_2)$ are underlying rectangles of the persistence modules $\mathcal{M}$ and $\mathcal{N}$, respectively. Hence, we can find the interleaving distance effortlessly as follows:

$$d_I(\mathcal{M}, \mathcal{N}) = \min\big\{\max\big\{3, 2\big\}, \max\{3, 0\}\big\} = 3.$$

Now, let us compute the steady matching distance between these persistence modules. Remember that

$$SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = \min\{\sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_1^L), \sup_{L \in \Lambda} m^L \operatorname{cost}(\sigma_2^L)\}$$

where σ_1 and σ_2 are two partial multibijections such that one matches bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ with the zero bipersistence modules and the other one matches the bipersistence module $\mathcal{M}$ with the bipersistence module $\mathcal{N}$, respectively.

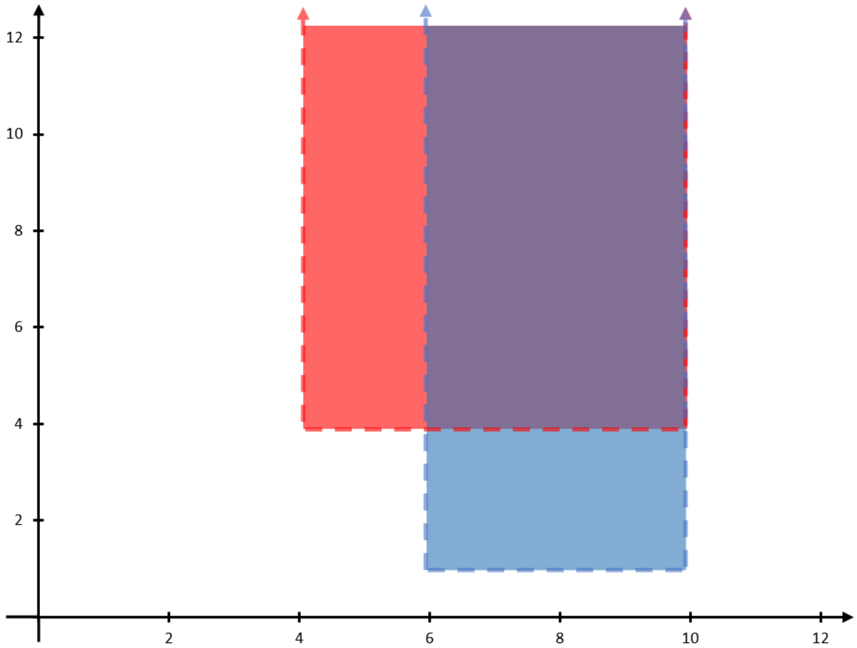


Figure 9.2: Two rectangle bipersistence modules $\mathcal{M}$ and $\mathcal{N}$ with underlying rectangles $R_{\mathcal{M}} = (4, 10) \times (4, +\infty)$ and $R_{\mathcal{N}} = (6, 10) \times (1, +\infty)$, respectively.

Now, consider the diagonal line $\bar{L} : y = x$, parameterized by $u = t(1,1)$ where $u \in \bar{L}$ and $t \in \mathbb{R}$. Observe that $\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_1^L) = 3$ since after the line restriction, we have the real intervals $I_{\mathcal{M}^{\bar{L}}} = (4, 10)$ and $I_{\mathcal{N}^{\bar{L}}} = (6, 10)$. Next, let us compute $\sup_{L\in\Lambda} m^L \operatorname{cost}(\sigma_2^L)$. Observe that $\mathcal{M}^L$ and $\mathcal{N}^L$ are not ϵ-interleaved for any $\epsilon < \frac{2}{m^L}$ for some admissible lines such as the diagonal line above. On the other hand, for any admissible lines, $\mathcal{M}^L$ and $\mathcal{N}^L$ are ϵ-interleaved for any $\epsilon \geq \frac{2}{m^L}$. Thus, $\sup_{L\in\Lambda} m^L d_I(\mathcal{M}^L, \mathcal{N}^L) = \sup_{L\in\Lambda} \operatorname{cost}(\sigma_2^L) = 2$ which results in $SD_{\text{match}}(\mathcal{M}, \mathcal{N}) = \min\{3, 2\} = 2 \neq 3 = d_I(\mathcal{M}, \mathcal{N})$. Hence, we can conclude that the steady matching distance is not always equal to the interleaving distance, even for rectangle bipersistence modules. .

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