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Research Article

Middle school mathematics teachers' knowledge of integers

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ABSTRACT

The current research aimed to unpack teachers' knowledge of integers by investigating how they used the number line and counter models to represent the two meanings of division (i.e., partitive and measurement). The participants were three middle school mathematics teachers working in different cities in Türkiye. Data consisted of teachers' written responses to an open-ended questionnaire consisting of four division operations and interviews conducted thereafter. Findings revealed that although two teachers could accurately model all division operations with the number line model, one teacher could neither provide a problem context nor a model displaying one of the division operations. For the counter model, only one teacher could accurately model all division operations in the questionnaire.

Keywords: integers, division, number line, counters, mathematics teachers

INTRODUCTION

Negative numbers are fundamental for students' mathematical learning (Whitacre et al., 2011). A sense of magnitude, direction, and an understanding of how operations with integers work are emphasized for 6-8 grade middle school students by curricular standards (e.g., National Council of Teachers of Mathematics, [NCTM], 2000). In this sense, students are expected to identify and locate integers on the number line, compare and order them, and identify and make sense of the absolute value of any integer by the time they reach sixth grade (Common Core State Standards Initiative [CCSSI], 2010; Ministry of Education [MoNE], 2018). In seventh grade, students are expected to operate successfully with integers and solve related problems (MoNE, 2018). Nevertheless, students encounter obstacles in making sense of integers and integer arithmetic (Lewis et al., 2020; Özdemir, 2021), partly because they find integers abstract and cannot relate the concept to their real-life experiences (Kilhamn, 2008) and also because the representations used for modeling the concept generate certain cognitive impediments (Goldin & Shteingold, 2001).

Researchers have spent quite a time finding an ideal model to demonstrate operations with integers (Wessman-Enzinger, 2019). However, there has not been a consensus on the ideal model for integer arithmetic (Lee, 2013) since each model has limitations and affordances (e.g., Wessman-Enzinger, 2019). Not having a unique model that accounts for all operations poses difficulties in acknowledging the meaning of negative integers (Fischbein, 2002). The number line and counters are widely used models in teaching integer arithmetic (Sari et al., 2020). Yet, their use leads to difficulties for students, especially regarding division operations (Özdemir, 2021). Even teachers have been found to demonstrate a limited understanding of the models they used for teaching integer concepts (Bozkurt & Polat, 2011; Çevik & Cihangir, 2020; Durmaz, 2017) and they do not favor using the number line and counters to model multiplication and division of integers (Bozkurt & Polat, 2011). Although these models fall short of modeling multiplication and division, they could help learners develop an insight into the conceptual underpinnings of these two operations (Beswick, 2011). Therefore, given the widespread use of the number line and counters along with the challenges that students and teachers face in their use, how in-service teachers practice these two models should be of importance and should be given explicit attention. As it is on the part of the teachers to exploit different sorts of representations to enhance the quality of students' learning (Dreher & Kuntze, 2015), they need to be aware of which representations are more effective and helpful in rendering the subject understandable for all learners (Shulman, 1986). Otherwise, during their teaching practice, they may transfer their lack of knowledge to their students (Reeder & Bateiha, 2016).

Studies over the past two decades have offered significant insights into teaching integers by studying with pre-service mathematics teachers (Almeida & Bruno, 2014; Işık, 2018; Reeder & Bateiha, 2016; Steiner, 2009; Wessman-Enzinger & Tobias, 2015, 2022), and in-service mathematics teachers (Bozkurt & Polat, 2011; Koç-Şanlı & Işık, 2019; Kumar et al., 2017) to understand their reasoning of integers and modeling of integer operations better. More recently, researchers have given weight to conducting studies involving both pre- and in-service teachers as participants (e.g., Durmaz, 2017; Erdem, 2022). Despite the large number of research that has been conducted on this subject in general, very little is known about how in-service mathematics teachers interpret the division operation using these two models (i.e., the number line and counters) that have predominantly been introduced to the students. What seems to be lacking, in particular, is how these models are interpreted by in-service middle school mathematics teachers in modeling the division of integers for the different versions of the signs of the dividend and the divisor. We believe that uncovering in-service middle school mathematics teachers' knowledge of integers, the division operation, and models (i.e., using the number line and counters to model the division operations with integers), and how the interplay of these constituents manifests itself in the modeling process could draw attention to the complexities derived from models themselves and that our work could provide suggestions on how to facilitate teaching of this concept.

Based on this background, this study intended to unpack in-service middle school mathematics teachers' knowledge of integers by examining their use of the number line and counter models to interpret and model the two meanings of division (i.e., partitive and measurement) (Ball et al., 2008). We considered these two models as *external representations* (Goldin & Shteingold, 2001) and drew on Ball et al.'s (2008) *specialized content knowledge* (SCK) to articulate teachers' knowledge of integers through their understanding of division and model use. The following question was addressed for this purpose:

- How do teachers use number line and counter models to interpret division operations with integers?

THEORETICAL BACKGROUND

Domains of Teacher Knowledge

Much groundwork has been done to identify and define the dimensions of the two core domains of teacher knowledge, i.e., *subject matter knowledge* (SMK) and *pedagogical content knowledge* (PCK) (Borko et al., 1992). Different conceptualizations have been provided for both domains to articulate their nature and relation. By subsuming it under content knowledge, Shulman (1986, p. 9) defined SMK as "the amount and organization of knowledge per se in the mind of the teacher," which not only includes the knowledge of the facts pertaining to a field but also requires comprehension of its structure. In the mathematics domain, Ball (1988) identified two categories for SMK–*knowledge of mathematics* (i.e., understanding ideas and procedures and their relationships) and *knowledge about mathematics* (i.e., understanding the epistemology of the discipline) and regarded these two dimensions of SMK as critical for teachers to have so they could teach effectively. SMK plays a key role in how the students learn the subject; when teachers' conceptions are flawed, they are more likely to convey these conceptions to their students (Ball & McDiarmid, 1989). The ability to see the connections of mathematical ideas and make sense of why mathematical procedures work is just as crucial for students as being familiar with those concepts and procedures (Ball, 1990). Therefore, teachers should have a strong command of the subject themselves to teach it to the pupils, as their understanding will be influential in their practice (Ball, 1988). In addition to this knowledge, teachers should be able to explain and justify for which conditions mathematical statements hold true and for which conditions they are not (Shulman, 1986). However, teachers' tacit knowledge might remain insufficient per se; as they should have an explicit knowledge of mathematics that transcends beyond communicating procedures to be taken, including developing a vocabulary that can articulate the necessary conceptual foundations of those processes (Ball, 1988). This calls for a special kind of knowledge peculiar to teaching, namely *specialized content knowledge* (SCK), coined by Ball et al. (2008). SCK refers to a form of knowledge distinct from pedagogy and students yet demanding a repertoire of skills needed for teaching mathematics (Ball et al., 2008). It includes but is not limited to the ability to differentiate different meanings of division (i.e., partitive and measurement) or of subtraction (i.e., take-away and comparison) (Ball et al., 2008).

In this study, following Ball et al.'s (2008) definition, as part of SMK, we are particularly interested in studying SCK of teachers. In this research, we conceptualized teachers' SCK as knowledge and understanding of the two meanings of the division operation (i.e., partitive and measurement) and the models (i.e., the number line and counters) to represent those meanings when working with integers.

Models Used in Integer Operations

Researchers regard representations as useful in understanding mathematical notions and, in turn, enhancing learning (Mitchell et al., 2014). Among the wide range of representations in teaching the concept of integers, models are the ones that are commonly resorted to (Kumar et al., 2017). Two models that have been widely used in modeling integer operations are the number line and counters (Işık, 2018), both of which have their benefits and challenges. Many researchers recognize the number line (e.g., Lamberg et al., 2020; Pratiwi et al., 2013) and counter models (e.g., Beswick, 2011) as helpful in carrying out operations with integers. Although the value of these models has been appreciated, some researchers raise objections to existing models used to embody integer operations in many respects. For instance, though he found it helpful, Battista (1983) considered the number line model inadequate to demonstrate four operations with integers. Likewise, Heeffer (2011) argued against using the number line in early attempts to teach integer arithmetic, particularly with negative numbers, even though he appreciated its importance in teaching basic arithmetic concepts. In addition, the number line model may cause difficulties when a negative number is subtracted from a positive number since it contradicts students' conceptualization of subtraction with positives as a takeaway (Liebeck, 1990). Neutralization models, which point out cardinality (Steiner, 2009) and involve the cancellation of integers with opposite signs (Verzosa et al., 2018), have also come under criticism by researchers. For instance, the two-color tiles model has shortcomings as it does not sufficiently embody the notion that the subtraction of a negative amount corresponds to the addition of its multiplicative inverse, even though it clarifies the meaning of the negative number and subtraction operation (Çemen, 1993). Moreover, as argued by Bofferding (2014), the cancellation model may pose challenges to fully understanding the binary meaning of the minus sign, although it is partly helpful in explaining its unary meaning.

Consequently, the models that are anticipated to offer logical explanations for mathematical ideas, in fact, are not effective enough to arouse curiosity in students (Reeves & Webb, 2004). Commonly used traditional models, such as the number line or charged particles, often carry complexities and appear uncompelling for students (Petrella, 2001). Overall, finding a model that could be suited to all operations with integers is a challenging endeavor (Lee, 2013). Considering the challenges that many of the models pose, it is of utmost importance for teachers to be able to select and correctly use the appropriate models that would be mathematically meaningful illustrations for the operations they represent. Teachers' use of models to interpret the division of integers may provide further insight into how and in which contexts these and other models are preferred and used. Thus, an investigation of teachers' interpretation of division on the number line and counter models, and their awareness of the models' limitations and affordances might contribute to research in identifying, if any, a lack of knowledge and awareness demonstrated by teachers and may suggest new ways of overcoming the challenges faced by teachers and students in teaching and learning of integers.

METHODOLOGY

Research Design & Participants

A qualitative exploratory design (Stebbins, 2001) was used to carry out the current study. The participants were three middle school mathematics teachers, two of whom were female and one of whom was male. All teachers, conveniently selected based on their willingness, teach mathematics to middle school students at public schools in different cities of Türkiye and have at least one and a half years of teaching experience. They were the same cohort of students who graduated from one of the largest public universities in Türkiye. All teachers hold a bachelor's degree in elementary mathematics education and are pursuing a master's degree in the mathematics education program at the university from which they graduated. **Table 1** presents information about the participants of this study.

Table 1. Information about participants

	Gender	Experience	Degree
T1	Female	1.5 years	Bachelors and masters (ongoing)
T2	Male	1.5 years	Bachelors and masters (ongoing)
T3	Female	1.5 years	Bachelors and masters (ongoing)

Data Collection

Questionnaire

A four-item questionnaire containing four open-ended questions was prepared to gather data from teachers. **Table 2** illustrates the items included in the data collection instrument.

Table 2. Items in data collection instrument

Item	Operation
1	20÷4
2	(-20)÷4
3	20÷(-4)
4	(-20)÷(-4)

As demonstrated in **Table 2**, all items required teachers to perform a division operation, and each item differed with respect to the sign of the dividend and divisor. Moreover, all items further involved three options, as presented below:

a. Write a real-life problem that could be solved by the given operation.

b. Solve your problem by modeling it with the number line model.

c. Solve your problem by modeling it with the counter model.

As seen above, the first option required teachers to create a real-life problem that entailed applying the corresponding operation. We did not give teachers problems at hand to provide them flexibility in selecting a problem context. The second and the third options expected participants to solve the problems they posed through the number line and counters. We particularly selected these models because they are the most common models presented in Turkish mathematics textbooks, and students are expected to be competent in using them.

Procedure

Each questionnaire was emailed to teachers, and teachers were expected to send it over within a week. Our rationale for providing teachers with some time was to allow them to ponder the questions thoroughly and ensure they were able to complete the questionnaires. Upon completing the questionnaires, teachers sent their responses back to us by email. Following this process, we arranged 40-50 minutes meetings with each teacher to have them expand on their ideas on each question, which, in turn, provided us with richer data on which we could ground our inferences.

Data Analysis

To investigate teachers' knowledge of the division of integers with models, we analyzed how appropriately they used the models to interpret different meanings of division operations and which interpretations they provided for their use. Appropriateness criteria required teachers' models to be mathematically meaningful representations of the relevant operations and coherent with the meaning(s) of the division operation being used. If the teacher could not provide any models to demonstrate the given operation, his/her response was labeled as *no model use*, regardless of the problem created and the meaning of the division underlying the created problem. If the teacher could provide a model, but his/her model did not correctly represent the corresponding operation, we described it as an *inadequate model use* regardless of the problem created and the meaning of the division underlying the created problem. If the teacher could accurately use the model to demonstrate the operation and his/her model was coherent with the meaning (i.e., measurement or partitive division) being used, his/her response was regarded as an *appropriate model use*, and a checkmark was put on the table as an indicator of appropriateness. Although we did not ignore reporting the coherency of problems that teachers posed concerning given operations to operations' underlying structure, nor did we overlook when they make, if any, inaccurate interpretations of the meanings of division, the prominent role was given to the model used when evaluating the appropriates of the models. Therefore, our appropriateness criteria for the model use only looked for if the teachers used the meanings of the operation of division appropriately when modeling the given division operations with number lines and counters, regardless of the meaning of the operation of division that is more appropriate to underlying structure of problem they created.

We used data and investigator triangulation methods to address the trustworthiness issue. As stated earlier, once the teachers' written responses were obtained, the researcher conducted a 40-50-minute one-to-one interview with each participant to have them elaborate more on their responses to the questions on the questionnaire. Therefore, the interview transcripts and the written responses together constituted the data of this study. Moreover, the researcher and a second coder coded the entire data set together until they reached a complete consensus.

FINDINGS

In this section, to unpack their knowledge of the division of integers and how they used models in relation to that, we provided brief interview episodes that demonstrate teachers' use of each model to perform division operations presented in the questionnaire.

Teachers' Use of Number Line Model

The number line was one of the models that teachers were expected to use to carry out the operations on the questionnaire. **Table 3** presents how teachers used the number line model to perform division operations with integers.

Table 3. Teachers' use of number line model

Number line				
Item	Operation	T1	T2	T3
1	20÷4	√	√	√
2	(-20)÷4	√	√	√
3	20÷(-4)	√	No model use	√
4	(-20)÷(-4)	√	√	√

Note. Check marks indicate appropriate use of model with chosen meaning of division, regardless of underlying structure of problem created by teachers

As seen in **Table 3**, item 1 and item 4 contain a dividend and a divisor with the same sign, while item 2 and item 3 require working with numbers with opposite signs. Below, teachers' use of the number line in each item will be presented in order.

Positive dividend, positive divisor

Item 1 was the first operation for which the teachers were asked to write a problem through which they could model the given operation. As seen from **Table 3**, it contained a positive dividend and a divisor. All teachers could appropriately model the operation with the number line. All teachers drew on the measurement meaning of the division to model the given operation on the number line. For instance, T1 proposed the following problem for modeling 20÷4 operation: "I have 20 walnuts. I want to make a pack of four. How many packs can I make?" Her solution to this problem is demonstrated in **Figure 1**.

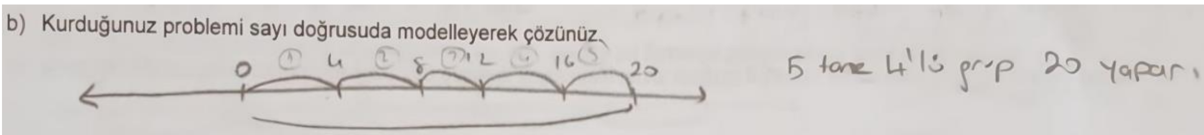

Figure 1. T1's number line model for 20÷4 operation (Source: Authors)

As illustrated above, T1 drew five equal-length arrows from zero to 20, representing the number of walnuts each pack should contain. Compared to the other items, modeling this operation was the most simplistic one for all teachers. Neither providing a problem context that could be modeled by a number line nor interpreting the meanings of division become an obstacle for any of them.

Negative dividend, positive divisor

Item 2 contained the same numbers given in item 1, except, in this operation, the sign of the dividend was negative. All teachers could appropriately model the operation in item 2 with the number line. Interestingly, all teachers proposed a problem with a debt context. While T1 and T2 applied the partitive meaning of the division, T3 employed the measurement meaning, as she performed the operation.

For instance, T3 proposed the following problem for item 2: "Mehmet owes 20 Turkish Liras. If he divides the debt equally among his four friends, how many Turkish Liras will each person owe?" It is clear that T3 proposed a problem whose underlying operation might be more appropriate to the application of partitive division. Nevertheless, she preferred applying measurement division, as she modeled the operation on the number line. **Figure 2** represents T3's number line model on (-20)÷4 operation.

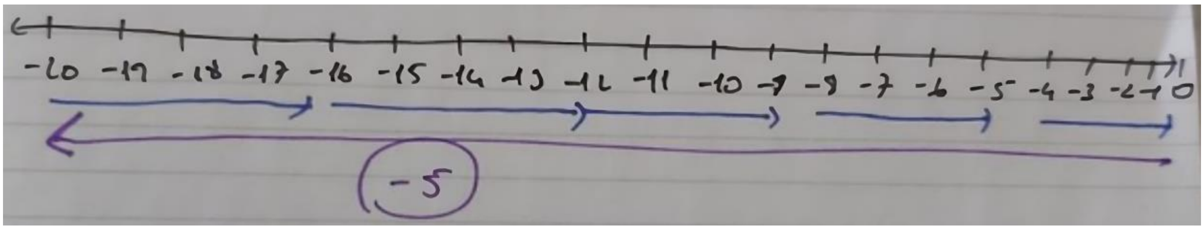

Figure 2. T3's number line model for (-20)÷4 operation (Source: Authors)

As seen in **Figure 2**, T3 drew a 20-unit long arrow in the left direction by locating its starting point on the zero to represent 20 lira debts. Then, she repeatedly subtracted four-unit long arrows and counted how many times she could subtract this number from 20 lira debts. During the interview, she explained her model as the following:

> T3: What I said here … Since we subtracted five times -4. It is -5, probably because we did subtraction. I mean, I related [the minus sign with the subtraction]. I mean, since I subtracted, the result became negative.

To model this operation on the number line, one could either add five times four to -20, (i.e., -20+5×4)., or one could subtract -5 times four from -20 (i.e., -20-(-5)×4=0). **Figure 2** shows that T3 added five times four to -20 to reach point zero (i.e., -20+5×4), which could be considered mathematically appropriate. Yet, her explanation differed slightly from what she drew, as it represented the following operation: -20-5×(-4). In other words, even though she drew five arrows in the positive direction to denote 4, she said she would subtract -4 (from -20), which was neither evident in her model nor the operation provided in item 2. Therefore, as **Figure 2** and the above episode demonstrate, T3 could appropriately model the operation, yet her explanation did not align much with what she drew.

Positive dividend, negative divisor

The operation given in item 3 was the most compelling one for the teachers in terms of explaining how they modeled it on the number line. Even though T1 and T3 could demonstrate the operation appropriately by drawing on the measurement meaning of the division, they were challenged to explain how they used the model. In fact, T3 could not provide a problem context at all. T2 also could not propose a problem that could be solved with the 20÷(-4) operation, nor could he provide a model. He pondered on a context for a while, yet he could not be entirely sure about it.

> T2: 20 over -4. I thought about something here. For example, I received a refund of four Turkish Liras from Ada's credit cards and received a total of 20 Turkish Liras. Accordingly, if we say how many credit cards Ada has received, we cannot find -5 because it is 20 divided by four. There is no such thing as -5 credit cards. It would be ridiculous.

As the above excerpt demonstrates, having a negative divisor in this problem context did not make sense to T2 because, from his point of view, having a negative amount of credit cards was unreasonable. Compared to item 2, this operation was less straightforward for teachers, as the teachers struggled to model the operation and provide a problem context that could make sense with the given structure. Moreover, teachers preferred the measurement meaning of division to partitive meaning to model this operation on the number line.

Negative dividend, negative divisor

The last item on the questionnaire was the operation including a negative dividend and a negative divisor. All teachers could appropriately model this operation with the number line. In this operation, all teachers utilized the measurement meaning of the division. While T1 and T2 used a debt context, T3 proposed the following problem: "A diver who wants to dive 20 meters below sea level has to press the safety button every four meters. How many times does the diver press the safety button when he reaches 20 meters below sea level?" Her solution to this problem on the number line is illustrated in **Figure 3**.

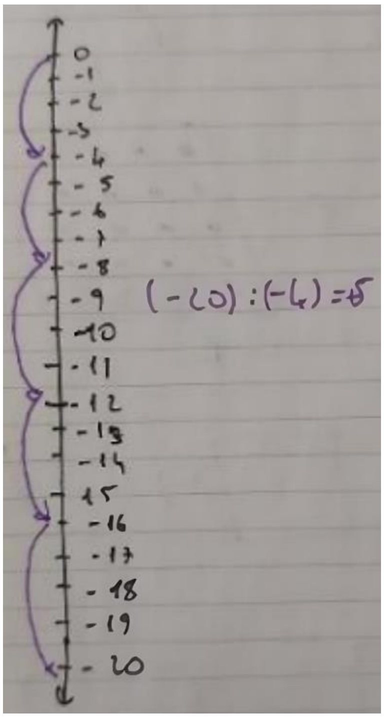

Figure 3. T3's number line model for (-20)÷(-4) operation (Source: Authors)

Figure 3 shows that T3 drew a 20-unit-long vertical number line to indicate the total distance the diver drove under sea level. She drew four unit-long arrows in the negative direction to show the points, where the diver was supposed to press the safety button and found out the number of times the diver should have pushed the safety button until reaching 20 meters under sea level by repeatedly adding the four unit-long arrows.

Moreover, as in the case of item 3, teachers merely preferred using the measurement meaningful of division when they modeled this operation on the number line.

Teachers' Use of Counter Model

Teachers' performance on the counter model displayed a quite resemblance to that of on the number line. **Table 4** demonstrates how teachers used the counter model in each operation.

Table 4. Teachers' use of counters

Counters				
Item	Operation	T1	T2	T3
1	20÷4	√	√	√
2	(-20)÷4	√	√	√
3	20÷(-4)	Inadequate model	No model	√
4	(-20)÷(-4)	√	√	√

Note. Check marks indicate appropriate use of model with chosen meaning of division, regardless of underlying structure of problem created by teachers

Positive dividend, positive divisor

As mentioned earlier, the first item had a dividend and a divisor, both with a positive sign. All teachers could appropriately use the counters to model the operation in item 1, as shown in **Table 3**. Again, T1 and T3 used the measurement meaning of the division, while T2 utilized partitive division in his model. **Figure 4** demonstrates T2's counter model for the 20÷4 operation.

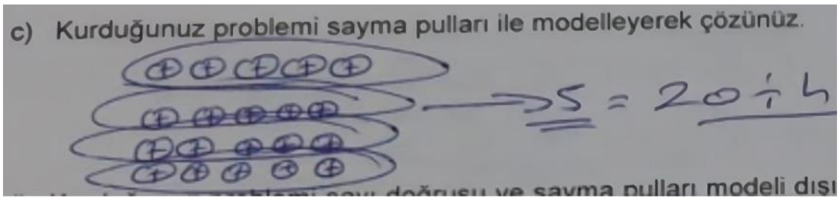

Figure 4. T2's counter model for 20÷4 operation (Source: Authors)

T2's problem required finding out the number of walnuts each person should have taken after the fair-sharing of 20 walnuts among four friends. As his model demonstrates, T2 partitioned 20 walnuts equally among four people and circled each group to indicate the number of walnuts that each person should have taken. As in the case of the number line, none of the teachers struggled to model this operation with counters, and they could use both meanings of division accurately in creating a problem and modeling the operation.

Negative dividend, positive divisor

Item 2, with a negative dividend and a positive divisor, was not challenging either for the teachers to model with counters. All teachers could correctly use the counter model to demonstrate the operation in item 2. While T1 and T2 used the partitive meaning of the division, T3 used measurement meaning. **Figure 5** represents T3's use of the counter model to the (-20)÷4 operation.

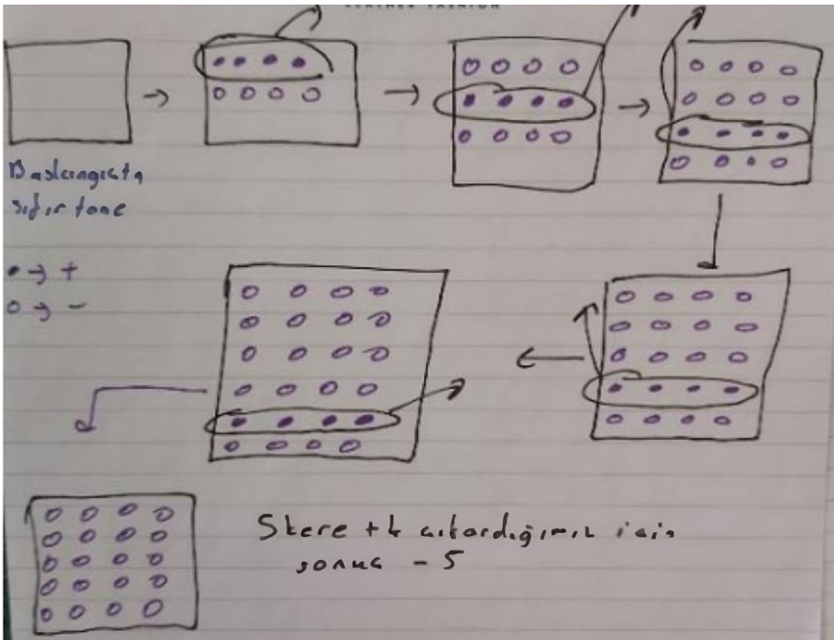

Figure 5. T3's counter model for (-20)÷4 operation (Source: Authors)

As demonstrated in **Figure 5**, T3 started with an empty set, and in each step, she added zero pairs consisting of four positive and four negative counters to her initial set so that she could have positive counters that could be taken away. After that, she repeatedly subtracted four positive counters until she obtained a set of counters representing -20. In the end, she attributed the minus sign of the result to the subtraction operation and related the obtained quantity to the number of actions that she carried out.

Positive dividend, negative divisor

As in the case of the number line, having a negative divisor challenged teachers more than having a negative dividend when using counters to interpret both meanings of the division. T2, for instance, could not carry out the 20÷(-4) operation with counters. He left the question unanswered in his written response to the questionnaire. During the interview, T2 thought about finding a context for quite a long time. However, he could neither produce a problem context nor model the operation. T1 as well could not provide an appropriate model representing the 20÷(-4) operation. Instead, she drew a group of 20 positive counters and wrote "I could not model" in her written response to the questionnaire, as displayed in **Figure 6**.

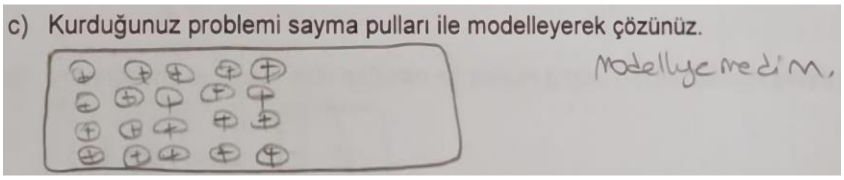

Figure 6. T1's counter model for (20)÷(-4) operation (Source: Authors)

However, during the interview, after thinking about the context, she understood how she obtained that model in the first place. She proposed the following explanation:

> T1: Yes, I think I am drawing right now. I put it inside. My aim is to lend four Turkish Liras. I have groups of four. If I remove my +4 groups because I loaned four Turkish Liras, I have five -4 groups left. I would have given something five times. I have made up my own mind, but I still prefer … My negatives are left. Each of my groups has one negative, two negatives, three negatives, four negatives, and five negatives. So, I reached -5. I know I must do something five times. Debt … What should I do? It happens -5 times so that I do not have money in my pocket, so what I need to do is my friend …

T1's reasoning shows that she thought of the 20 as a loan that needs to be paid in several payments. She made groups of four, considered them loans to be paid each time, and counted the number of groups in 20, representing the number of payments that she needed to make. She found she needed to make five payments, and she assigned a negative value to her action because she thought paying a loan should be represented as a negative number.

Negative dividend, negative divisor

In modeling the operation given in item 4, all teachers could effectively use the counter model. As for the number line, all teachers drew on the measurement meaning of the division when they modeled the operation in item 4 with the counters. They repeatedly added a group of -4 counters until they obtained a group of -20 counters. For instance, T1 wrote the following problem: "I owe my friend 20 Turkish Liras. If I pay four Turkish Liras every week, how many weeks later will I have paid my debt?" Her solution to this problem can be illustrated in **Figure 7**, as the following:

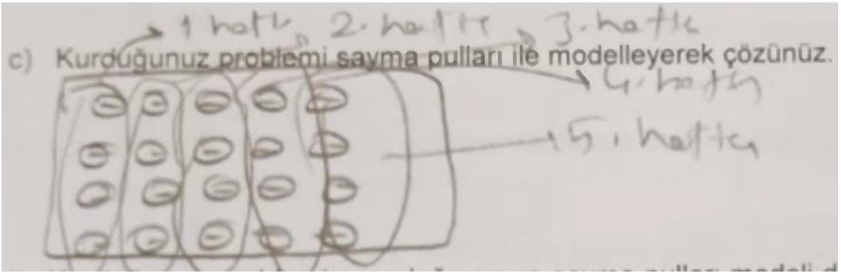

Figure 7. T1's counter model for (-20)÷(-4) operation (Source: Authors)

As illustrated above, T1 drew 20 negative counters to represent the debt of 20 Turkish Liras. Then, she circled each group of negative four counters and found five groups of negative four groups in a total of 20 negative counters.

DISCUSSION

SMK is at the core of teaching (Ball, 2000), and even though it is not the exclusive factor, teachers' SMK influences their pedagogical decisions (Even, 1990). Building on this notion, the current study sought to unpack teachers' SCK of the division of integers by investigating their use of the number line and counter models to perform division operations. Literature has demonstrated that teachers are less inclined to model multiplication and division with counters than addition and subtraction either because they find the modeling difficult (Bozkurt & Polat, 2011) or they believe that the counters are ineffective in performing multiplication and division (Durmaz, 2017). They resort to the rules when dividing negative integers (Crabtree, 2017), and they are even less likely to include conceptually based questions in mathematics exams when it comes to multiplication and division partly because they lack the necessary conceptual understanding of these two operations (Avcu & Avcu, 2018).

Regarding division with counters, our research offers supportive evidence to these arguments since teachers in our study occasionally encountered difficulties in modeling, even though they were largely capable of effectively using both representations. In particular, item 3 (see **Table 3**) challenged teachers more than the other versions of the division operation, especially in the counter model. While T1 struggled to model this particular operation (i.e., item 3) with the number line, T2 could not illustrate either model. This might have happened because teachers did not find it reasonable to look for a negative number of groups in a set of positive counters. The case was not much different for the number line, either. Regardless of the model in question, having a negative operator exacerbated difficulties for teachers in our study, not only for modeling the operation but also for creating problem contexts suitable to its structure. The challenges posed by the negative operator have also been discussed by Fischbein (2002) in the context of multiplication. Fischbein (2002) emphasized the need to develop novel modeling techniques that would seem more plausible in the presence of a negative operator and even agreed with Freudenthal's (1973) idea of using formal methods for teaching negative integers.

Another important issue that needs to be highlighted here is the key distinction between using two different meanings of division with the number line and counters, especially modeling item 3 and item 4, wherein most of the modeling challenges were experienced with a negative divisor. It is known that number line and counter models were challenging to model operations with negative integers (Bossé, 2016). On the number line, carrying out a division operation with a negative divisor through repeated subtraction of negative fours from 20 until reaching zero (i.e., item 3) made sense to the teachers since one could show the negative four on the number line by specifying the direction of the arrow. In that case, by using the measurement meaning of the division, one could count how many times the quantity should be added to or subtracted from the dividend, which resulted in obtaining a positive answer for addition, a negative answer for subtraction, and a quantity that indicated the number of actions that one could operate on the number line. However, partitive meaning was not as practical as the measurement meaning of division for teachers to model item 3 with the number line since equally sharing among the negative four groups did not make sense to them, so it led teachers to use measurement meaning instead. Consistent with the work of Erdem et al. (2015), we observed that teachers were having difficulties in using counters, as neither meaning of division make sense to two teachers when the divisor was negative (i.e., item 3 and item 4), even though one teacher (i.e., T3) could represent the division operation with counters by using the takeaway meaning. This could be accounted for by the fact that division operation requires repeated subtraction, wherein the purpose is to identify the number of groups when the number of elements in each group is given (Fischbein et al., 1985). Apparently, the definition of division operation did not lend itself to be modeled meaningfully when the divisor was negative.

CONCLUSIONS

Our study demonstrated that both the number line and counters have their challenges to model division operation, especially with the presence of a negative divisor. Even though teachers in our study were aware of each meaning of the division, the models were not practical in some cases and became frustrating for them to use. For that matter, our study raises an important implication that teachers' knowledge of integers and their understanding of division may not be sufficient to use the models effectively. They should also be able to acknowledge the pitfalls and strengths of each representation they use and accommodate them to their teaching to render learning more effective for their students (Ball et al., 2008). They must exercise caution when incorporating various representations into their practices since some of these representations serve to promote misconceptions (Mitchell et al., 2014). To address this issue further, future studies might be centered on exploring alternative models used by teachers and their students that can exploit the different meanings of division, which in turn may uncover the potentialities of other models that might be suited to demonstrate both meanings of the division operation with integers.

There is also room for further research that might illuminate the cognitive affordances of alternative models, which may have the power to unravel the complexities stemming from an insufficient understanding of the existing models in modeling integer operations. From the affective standpoint, we recommend that instead of solely introducing existing models or using the algorithm as a fallback strategy, teachers may allocate time to encouraging their students to produce models for dividing integers that may eventually motivate them to bring their ideas into the open, which, in turn, may enrich classroom discussion and facilitate their learning. For this reason, further research conducted with students is also recommended to learn their views and understanding of the models to model integer operations. We believe that taking students' views into account might offer teachers a guide to identifying their students' needs better and tailoring their instructions accordingly.

Finally, our main intention was to analyze whether the teachers used the number line and counter models appropriately with the meaning of the division they preferred. However, we noticed that the teachers sometimes created problems that were inconsistent with the meaning of the division they preferred to use when modeling the operation on the number line and counters. Although their models were correct representations of the chosen meaning of the division operation, in a classroom environment, this conflict might create confusion among students. Since the focus of our study was not on how the meaning of the division operation underlying the problems written by the teachers was reflected in the models used, we did not provide a detailed account of this issue, but we suggest that future studies provide a fine-grained analysis of the teachers' model use for division operation with integers in relation to the problems they generate.

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