ON A GROUP UNDER WHICH SYMMETRIC REED-MULLER CODES ARE INVARIANT

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ABSTRACT. The Reed-Muller codes are a family of error-correcting codes that have been widely studied in coding theory. In 2020, Wei Yan and Sian-Jheng Lin introduced a variant of Reed-Muller codes so called symmetric Reed-Muller codes. We investigate linear maps of the automorphism group of symmetric Reed-Muller codes and show that the set of these linear maps forms a subgroup of the general linear group, which is the automorphism group of punctured Reed-Muller codes. We provide a method to determine all the automorphisms in this subgroup explicitly for some special cases.

1. Introduction and Preliminaries

Working with automorphisms of the codes is important for various reasons in the field of coding theory. The study of automorphisms can contribute to developing efficient decoding algorithms. By exploiting the symmetries within a code, it may be possible to design more computationally efficient algorithms in order to correct errors. Thus automorphisms of the codes play a crucial role in understanding and characterizing the error-correction properties of codes. To sum up by studying the symmetries of a code, one can gain insights into how errors affect the encoded information and how the code can be designed to correct or detect these errors.

Reed-Muller codes (RM codes, for short) are a family of error-correcting codes that were first introduced by Irving S. Reed and Gustave Solomon Muller in 1954. A large number of RM code variations and generalizations were introduced in the literature, for instance, see [8, 12, 13]. They have a large minimum distance, which makes them good at correcting errors. They also have simple encoding and decoding algorithms, which make them useful and efficient to implement. There are some variants of RM codes, such as the binary RM codes defined on the prime field $\mathbb{F}_2$ and the p-ary RM codes defined on the prime field $\mathbb{F}_p$, where p is a prime number. Furthermore, RM codes have been generalized in many ways, such as the generalized Reed-Muller codes (GRM, for short), which are defined over an arbitrary finite field, and the symmetric Reed-Muller codes (SRM, for short) which have a certain symmetry property.

In coding theory, the majority of classical codes have a sizable automorphism group that is connected to a linear group. For example, GRM codes are invariant under the general

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affine group $GA(m,q)$ [8]. A requirement for a code to be invariant under the affine group is provided by Kasami et al. in [10]. Delsarte describes the codes as invariant under certain linear groups in [7]. He shows that under $GA(m,q)$, only p-ary RM codes can be invariant; nonetheless, the issue of fully determining the automorphism group of affine invariant codes has not yet been resolved. Determining the complete automorphism group of a code is a challenging topic that is frequently connected to the categorization of simple groups. Dür is credited with the research of the automorphism groups of Reed-Solomon codes and their extensions [7]. In [2], Berger has proved the complete automorphism groups of GRM codes. In 1996, Berger has demonstrated that $GA(m,q)$ contains the permutation group of any affine-invariant code [3] and then he demonstrates how to create a formal expression for every affine-invariant code's permutation group in [5]. The complete automorphism groups of the projective and homogeneous RM codes are found in [4]. These groups are associated with the projective linear group and the general linear group, respectively. In this work, we aim to investigate a subgroup of the automorphism group SRM codes whose elements are linear maps.

SRM codes are first introduced by using bivariate polynomials over $\mathbb{F}_q$ in [14]. The local correctability of the bivariate SRM codes is discussed in [15]. The authors begin by outlining the advantages of the symmetric structure and offering intuitions to indicate the superiority of SRM codes over GRM codes in terms of local correctability. The tolerance of error ratios of SRM codes and GRM codes over the same finite field, as well as their code rate, are then demonstrated. They establish that a class of locally-correctable codes is composed of multivariable SRM codes $SRM_q[n,r]$ in [15]. Furthermore, the dual of $SRM_q[n,r]$ is also presented. However, transformations preserving these codes, specifically belonging to the automorphism group, have not been studied yet. Taking inspiration from this gap in this research, we have directed our focus toward addressing this problem.

First, we will recall some basic notions and some results on RM and GRM codes which will be useful throughout the paper.

1.1. **Equivalence of Linear Codes.** Let $\mathbb{F}_q$ be the finite field of order $q = p^m$ for a prime p and $\mathbb{F}_q^n$ be the n-dimensional vector space over $\mathbb{F}_q$. The measure of dissimilarity between two vectors is established by counting the coordinates in which they differ. A linear $[n,k,d]$-code is k-dimensional linear subspace of $\mathbb{F}_q^n$ with the minimum distance d. A generator matrix G of a linear $[n,k]$-code C is any matrix of row rank k, whose rows come from the code C. We will give the definitions of some types of code-equivalence. We refer to [9] for further details. We need the following definition for simplicity.

Definition 1.1. [6] *Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ be a permutation of $\{1, 2, \ldots, n\}$. The permutation π can be given in the equivalent form as an $n \times n$ permutation matrix P with 1's in positions (i, π_i) for $i = 1, 2, \ldots, n$ and 0's elsewhere. $\mathcal{P}_n$ denotes the corresponding set of all $n \times n$ permutation matrices P.*

Two linear codes C_1 and C_2 are called *permutation equivalent* if there is a permutation of coordinates which sends C_1 to C_2. Hence, two linear codes C_1 and C_2 of the same length are permutation equivalent if there exists a permutation matrix $P \in \mathcal{P}_n$ such that

$$G_2 = G_1 P,$$

where G_1 and G_2 are the generator matrices of the codes C_1 and C_2, respectively.

If we work on a finite field except $\mathbb{F}_2$, then we may need a more general form of the equivalence. Recall that a square matrix is called *monomial* if it has exactly one nonzero entry from that finite field in each column and row. Hence, every monomial matrix over $\mathbb{F}_2$ is a permutation matrix. A monomial matrix M can be expressed as either DP or PD_1, where D and D_1 are non-singular diagonal matrices over the corresponding finite field and P is a permutation matrix. Two linear codes C_1 and C_2 of the same length over $\mathbb{F}_q$ are said to be *monomially equivalent* when there exists a monomial matrix M such that

$$G_2 = G_1 M,$$

where G_1 and G_2 are the generator matrices of the codes C_1 and C_2, respectively. Note that monomial equivalence and permutation equivalence coincide for binary codes.

Let γ be a field automorphism of $\mathbb{F}_q$ and $M = DP$ be a monomial matrix over $\mathbb{F}_q$, where P is a permutation matrix and D is a non-singular diagonal matrix over $\mathbb{F}_q$. Applying the map $M\gamma$ to any codeword is described by the following process: Firstly, the ith component of code is multiplied by the ith diagonal entry of D for all i's. Then the corresponding permutation associated with the permutation matrix P is applied to codeword. Finally, the automorphism γ is applied to all components. Two linear codes C_1 and C_2 of the same length over $\mathbb{F}_q$ are said to be *equivalent* when there is a monomial matrix M and a field automorphism γ of F_q such that

$$C_2 = C_1 M\gamma,$$

where $C_1 M\gamma$ is obtained by applying $M\gamma$ to each codeword of C_1. This is the most general form of the equivalence.

Note that, all equivalence definitions are the same for the binary codes. Furthermore, monomial equivalence and general equivalence coincide for p-ary codes, where p is a prime.

Since we have three types of equivalence, there exist three possible definitions of the automorphism groups of the code families by considering $C_1 = C_2$ in the above definitions.

Now consider a code C of length n over the field $\mathbb{F}_q$. The set of coordinate permutations that map the code C to itself forms a group, called the *permutation automorphism group of C* and denoted by $PAut(C)$. Obviously, $PAut(C)$ is a subgroup of the symmetric group S_n. The set of monomial matrices, by which C is monomially equivalent to itself, forms the group $MAut(C)$, which is called the *monomial automorphism group of C*. The set of maps of the form $M\gamma$, where M is a monomial matrix and γ is a field automorphism, that map C to itself forms the group $Aut(C)$, called *automorphism group of C*.

In general, we have that $PAut(C) \subseteq MAut(C) \subseteq Aut(C)$. If $q = 2$, then $PAut(C) = MAut(C) = Aut(C)$. If q is prime, then $MAut(C) = Aut(C)$.

All these definitions and conclusions are well-known in the literature and can be found in any basic coding theory book, for example in [9].

1.2. **The Automorphisms of Reed-Muller Codes.**
There exist some equivalent definitions of the RM codes. Following [12], RM codes can be defined in terms of multivariable polynomials as follows: Let $x = (x_1, \ldots, x_m)$ range over $\mathbb{F}_2{}^m$. Any function $f(x) = f(x_1, \ldots, x_m)$ which takes the values 0 and 1 is called *a binary multivariable function*. We recall the following definitions.

Definition 1.2. [12] *The rth order binary RM code $R(r, m)$ of length $n = 2^m$ for $0 \leqslant r \leqslant m$, is the set of all vectors f, where $f(x_1, \ldots, x_m)$ is a binary multivariable polynomial of degree*

at most r, i.e.,

$$R(r, m) = \left\{ (f(\alpha))_{\alpha \in \mathbb{F}_2^m} \mid f \in \mathbb{F}_2[x_1, \ldots, x_m], \ deg(f) \leqslant r \right\}.$$

Definition 1.3. [12] *For $0 \leqslant r \leqslant m - 1$, the RM code which is obtained by puncturing (or deleting) the coordinate corresponding to $x_1 = \cdots = x_m = 0$ from all the codewords of $R(r, m)$ is called* the punctured RM code, *and it is denoted by $R(r, m)^*$.*

Now we will mention the notion of affine invariance which is crucial for our discussion on automorphism groups. Before going further, we need some basic notions.

Let $A = [a_{ij}]$ be an invertible $m \times m$ binary matrix and $\mathbf{b}$ be a binary $m \times 1$ vector. Consider the transformation T from binary m-tuples to binary m-tuples defined by

$$T : \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \mapsto A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} + \mathbf{b},$$

which permutes binary m-tuples. T can be also considered as a permutation of multivariate polynomials as follows:

$$(1) \qquad T_f(A, \mathbf{b}) : f(x_1, \ldots, x_m) \mapsto f\left(\sum a_{1j} x_1 + b_1, \ldots, \sum a_{mj} x_j + b_j \right).$$

The set of all such transformations formed by T is a group, which is known as the *general affine group* over $\mathbb{F}_2$ and is denoted by $GA(m, 2)$ (see [12]). It is obvious that if f is a polynomial of degree r, so is $T_f(A, \mathbf{b})$.

As we mentioned above, for a binary code it is known that $PAut(C) = Aut(C)$, and hence it is a subgroup of the symmetric group S_n. Following [1], a code C is said to be *affine invariant* if $Aut(C)$ includes a subgroup that is isomorphic to the affine linear group.

The following example demonstrates the RM code families are one of the examples of affine invariant codes. This example is important to see the equivalence between transformations applied to variables of the function to evaluate codeword and transformations applied to the codeword itself.

Example 1.4. [1] *$R(r, m)$ codes are affine invariant.*

Proof. Let A be an $m \times m$ invertible matrix over $\mathbb{F}_2$ and $\mathbf{b} \in \mathbb{F}_2^m$. The affine linear transform $T : x \mapsto Ax + \mathbf{b}$ yields a permutation on the coordinates of the codeword since the codewords of RM codes are evaluation vectors and are indexed by the vectors $x \in \mathbb{F}_2^m$. Then such a permutation belongs to $Aut(R(r, m))$. Let c be a codeword in $R(r, m)$. Then there exists a polynomial $f \in \mathbb{F}_2[x_1, \ldots, x_n]$ with $deg(f) \leqslant r$ such that $c = (f(\alpha))_{\alpha \in \mathbb{F}_{2^m}}$. Since the outcome of the transformation $(f \circ T)(x)$ is another polynomial of degree less than or equal to r, we have $c' = (f \circ T)(\alpha)_{\alpha \in \mathbb{F}_2^m} \in R(r, m)$. Thus, $R(r, m)$ codes are affine invariant. $\qquad \square$

Thus, the general affine group $GA(m, 2)$ permutes the codewords of the rth order $R(r, m)$ and $GA(m, 2) \subset \text{Aut} R(r, m)$ (see [12]). The subgroup of $GA(m, 2)$ consisting of all transformations

$$T : \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \mapsto A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

(i.e., for which $\mathbf{b} = 0$) is known as the *general linear group* and is denoted by $GL(m,2)$. We can consider the transformation T described above in the following way, too:

(2) $$T(A) : (x_1, \ldots, x_m) \mapsto \left(\sum a_{1j} x_1, \ldots, \sum a_{mj} x_j \right).$$

For the sake of convenience in usage, the function $T_f(A, b)$ will be denoted as $T_f(A)$ when $\mathbf{b} = 0$. Since the transformation $T(A)$ in (2) fixes the zero m-tuple, the group $GL(m, 2)$ permutes the codewords of the punctured RM code $R(r, m)^*$, i.e., $GL(m, 2) \subset \mathrm{Aut} R(r, m)^*$ (see [12]).

We know from [12, p. 400] that

$$Aut(R(r, m)^*) = S_{2^m - 1} \quad \text{for} \quad r = 0 \quad \text{and} \quad m - 1,$$
$$Aut(R(r, m)) = S_{2^m} \quad \text{for} \quad r = 0 \quad \text{and} \quad m.$$

Furthermore, the following result is also given in [12, p. 400] which determines the exact automorphism group of RM codes of rth order with length 2^m over $\mathbb{F}_2$ completely.

Theorem 1.5. [12, p. 400] *For* $1 \leqslant r \leqslant m - 2$:
- $Aut(R(r, m)^*) = GL(m, 2)$,
- $Aut(R(r, m)) = GA(m, 2)$.

Note that one may easily adopt the definitions of the general affine group and general linear group on $\mathbb{F}_q$ for any prime q. These groups are denoted by $GA(m, q)$ and $GL(m, q)$, respectively.

1.3. **Generalized Reed-Muller Codes.** GRM codes are a generalization of RM codes. They are obtained by constructing the codes over any finite field $\mathbb{F}_q$, where q is a prime power. The following is a formal definition of GRM codes:

Definition 1.6. [8] *Let m and r be positive integers. The GRM code of order r with block length q^m over $\mathbb{F}_q$ is defined by*

$$GRM_q(m, r]) = \left\{ (f(\alpha))_{\alpha \in \mathbb{F}_q^m} \mid f \in \mathbb{F}_q[x_1, \ldots, x_m], \ deg(f) \leqslant r \right\}.$$

The GRM codes have the following property:

Theorem 1.7. [8] *For $0 \leqslant r \leqslant m(q - 1)$, the automorphism group of $GRM_q(m, r)$ codes contains the general affine group $GA(m, q)$ under the natural action on $V = \mathbb{F}_{q^m}$.*

Knorr and Willems in [11] give a complete description of the automorphism group of the p-ary RM codes for any prime p. The automorphism group of the p-ary RM-codes equals the general affine group $GA(m, p)$.

In [2], Berger and Charpin provide a complete description of the automorphism group of a GRM code. They show that the automorphism group of GRM codes is the affine linear group, i.e.,

$$Aut(GRM_q(m, r)) = GA(m, q).$$

1.4. **Outline of the Paper.** Unless otherwise stated, throughout the article we will work on the field $\mathbb{F}_q$, where q is a prime. Main part of the presented work consists of demonstrating the set of transformations under the general linear group that leaves the symmetric Reed-Muller (SRM) codes invariant.

Transformations that remain SRM codes invariant have not been studied yet. Motivated by this research gap, we have shifted our attention to tackling with this unsolved issue. In Section 2, we provide some preliminaries about SRM codes. In Section 3, we investigate transformations that leave SRM codes invariant. We determine all the linear transformations under which the SRM codes are invariant for the parameters $n = 2$ and $n = 3$. Also, the set of these linear transformations forms a subgroup of the general linear group. We notice a relationship between the transformations that remain the SRM codes invariant for $n = 2$ and those for $n = 3$. Therefore, we believe that we can form this group for a generic n, despite the challenge of determining all transformations. For any given n, we anticipate the group that leaves the SRM codes invariant, akin to previous cases. This group involving affine transformations is a subgroup of the general linear group $GA(n, q)$. However, establishing whether this group forms a complete set leaving the SRM codes invariant remains an open problem for the future. Therefore, in Section 4, we state a conjecture on this problem.

2. Symmetric Reed-Muller Codes

This section is devoted to some basic properties of Symmetric Reed-Muller (SRM) codes. These codes were first introduced by using bivariate polynomials over $\mathbb{F}_q$ in [14]. SRM codes which exhibit specific symmetry properties within their codewords may be considered as subcodes of GRM codes. The same authors generalized SRM codes in another work [15]. Before presenting the formal definition in [15], we recall some notions.

The set $E_q(n, r) \subseteq \mathbb{F}_q[x_1, x_2, \ldots, x_n]$ is defined by

$$E_q(n, r) := \left\{ f(x_1, x_2, \ldots, x_n) = \sum_{\substack{0 \leqslant i_1 < i_2 < \ldots < i_n \leqslant q-1 \\ i_1 + i_2 + \ldots + i_n \leqslant r}} a_{i_1 i_2 \ldots i_n} det(x, i) \mid a_{i_1 i_2 \ldots i_n} \in \mathbb{F}_q \right\},$$

where $x = (x_1, x_2, \ldots, x_n)$, $i = (i_1, i_2, \ldots, i_n)$ and

$$det(x, i) := \begin{vmatrix} x_1^{i_1} & x_2^{i_1} & \ldots & x_n^{i_1} \\ x_1^{i_2} & x_2^{i_2} & \ldots & x_n^{i_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{i_n} & x_2^{i_n} & \ldots & x_n^{i_n} \end{vmatrix}.$$

Consider the set

$$\Delta = \{(a_1, a_2, \ldots, a_n) \in \mathbb{F}_q^n \mid a_j \neq a_i, 1 \leqslant j < i \leqslant n\}$$

and an equivalence relation $\sim$ on Δ defined as follows:

$$c \sim d \iff \exists \sigma \in S_n \text{ such that } \sigma(c) = d, \text{ for } c, d \in \mathbb{F}_q^n.$$

Then define the set $\Omega_q(n) := \Delta / \sim = \{[\alpha] \mid \alpha \in \Delta\}$, where $[\alpha]$ denotes the equivalent class of α. In order to get rid of dublications, the definition of SRM codes is given by using this quotient set as follows:

Definition 2.1. [15] *Let n and r be positive integers, where n denotes the number of variables. The SRM code of degree r over $\mathbb{F}_q$ is defined by*

$$SRM_q[n, r] = \{(f(\alpha))_{[\alpha] \in \Omega_q(n)} \mid f \in E_q(n, r)\}.$$

Remark 2.2. *According to the definition, when $q < r$, the possible $\{i_1, i_2, \ldots, i_n\}$ sequence may not cover a partition of r. On the other hand, for the sequence $\{i_1, i_2, \ldots, i_n\} = \{0, 1, 2, \ldots, n-1\}$ the smallest value of r should be $\frac{n(n-1)}{2}$. Thus, the definition of SRM code is meaningful under the condition $q \geqslant r \geqslant \frac{n(n-1)}{2}$, where q is chosen to be large enough.*

Note that under the condition mentioned in the remark above, when $n = 1$, $SRM_q[1, r]$ codes are exactly generalized Reed-Solomon codes with degree parameter r.

When $n = 2$ we have,

$$E_q(2, r) := \left\{ \sum_{\substack{0 \leqslant i < j \leqslant q-1 \\ i+j \leqslant r}} a_{ij}(x_1^i x_2^j - x_1^j x_2^i) \,\middle|\, a_{ij} \in \mathbb{F}_q \right\} \subseteq \mathbb{F}_q[x_1, x_2].$$

The evaluation of $f(x_1, x_2)$ at $(x_1, x_2) \in \mathbb{F}_q^2$ forms as the following matrix

$$\begin{bmatrix} f(\alpha_0, \alpha_0) & f(\alpha_0, \alpha_1) & \ldots & f(\alpha_0, \alpha_{q-1}) \\ f(\alpha_1, \alpha_0) & f(\alpha_1, \alpha_1) & \ldots & f(\alpha_1, \alpha_{q-1}) \\ \vdots & \vdots & \ddots & \vdots \\ f(\alpha_{q-1}, \alpha_0) & f(\alpha_{q-1}, \alpha_1) & \ldots & f(\alpha_{q-1}, \alpha_{q-1}) \end{bmatrix}.$$

Since $f(x_1, x_1) = 0$ and $f(x_1, x_2) = -f(x_2, x_1)$, the above matrix is a skew-symmetric matrix. Thus it is entirely determined by the entries in the strictly upper triangular part. Therefore, the codeword of bivariate SRM, i.e., when $n = 2$, codes are defined as the strictly upper triangular part of this matrix. Furthermore, $SRM_q[2, r]$ codes are of length $\frac{q(q-1)}{2}$.

3. Linear Transformations Under Which SRM Codes are Invariant

In this section, we will derive the invaraint groups of SRM for $n = 2$ and $n = 3$ over the field $\mathbb{F}_p$, where p is any prime number. For $n = 2$ and $n = 3$, we determine the exact set formed by transformations that leave SRM codes invariant under a subgroup of the affine linear group. Different methods were employed to determine this set for the values of $n = 2, 3$, separately. The reason for using different methods is that one approach may not be highly suitable for the other. Our main theorems, which identify the group formed by transformations that leave the SRM code invariant, and the necessary lemmas for these theorems are provided for each $n = 2, 3$.

3.1. **The case** $n = 2$. Recall $SRM_q[2, r]$ is an evaluated code family whose evaluation polynomials come from the set $E_q(2, r)$,

$$(3) \qquad E_q(2, r) = \left\{ \sum_{\substack{0 \leqslant i < j \leqslant q-1 \\ i+j \leqslant r}} a_{ij} \begin{vmatrix} x_1^i & x_2^i \\ x_1^j & x_2^j \end{vmatrix} \,\middle|\, a_{ij} \in \mathbb{F}_q \right\}.$$

Let $f(x_1, x_2) = \begin{vmatrix} x_1{}^i & x_2{}^i \\ x_1{}^j & x_2{}^j \end{vmatrix} \in E_q(2, r)$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Under the $T_f(A)$ transformation in (1), we get

$$T_f(A) = \begin{vmatrix} (ax_1 + bx_2)^i & (cx_1 + dx_2)^i \\ (ax_1 + bx_2)^j & (cx_1 + dx_2)^j \end{vmatrix}$$
$$= (ax_1 + bx_2)^i \cdot (cx_1 + dx_2)^j - (ax_1 + bx_2)^j \cdot (cx_1 + dx_2)^i.$$

We investigate the coefficients $a, b, c, d \in \mathbb{F}_q$ to find the transformations that keep the set $E_q(2, r)$ invariant. For this purpose, we need the following auxiliary lemmas:

Lemma 3.1. *Let $f \in E_q(2, r)$ and $\alpha, \beta \in \mathbb{F}_q$ and t be a positive integer. Then we have*

$$(\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2)^t \cdot f \in E_q(2, r).$$

Proof. Without loss of generality, we shall assume that

$$f = \begin{vmatrix} x_1^i & x_2^i \\ x_1^j & x_2^j \end{vmatrix} = x_1^i x_2^j - x_1^j x_2^i,$$

where $0 \leqslant i < j \leqslant (q - 1)$ such $i + j \leqslant r$. We use the induction on t to prove the claim. When $t = 1$, then

$$(\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2) \cdot f = \alpha(x_1^2 + x_2^2)(x_1^i x_2^j - x_1^j x_2^i) + \beta(x_1 x_2)(x_1^i x_2^j - x_1^j x_2^i)$$
$$= \alpha(f_1 + f_2) + \beta f_3,$$

where

$$f_1 = x_1^{i+2} x_2^j - x_1^j x_2^{i+2}, \quad f_2 = x_1^i x_2^{j+2} - x_1^{j+2} x_2^i \quad \text{and} \quad f_3 = x_1^{i+1} x_2^{j+1} - x_1^{j+1} x_2^{i+1}.$$

Thus $f_1, f_2, f_3 \in E_q(2, r)$, which implies that $(\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2) \cdot f \in E_q(2, r)$. For the next step, suppose that the claim holds for $t > 1$, i.e. we have

$$(4) \qquad (\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2)^t \cdot f \in E_q(2, r).$$

Then it is sufficient to show that the statement holds for $(t + 1)$.

$$(5) \qquad (\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2)^{t+1} \cdot f = (\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2)((\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2)^t \cdot f).$$

By the equation (4), we have

$$g = (\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2)^t \cdot f \in E_q(2, r).$$

Since $g \in E_q(2, r)$, we may consider the equation (5) as follows:

$$(\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2) \cdot g = (\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2) \cdot \sum_{\substack{0 \leqslant i < j \leqslant q-1 \\ i+j \leqslant r}} a_{ij} g_{ij},$$

where $a_{ij} \in \mathbb{F}_q$ and

$$g_{ij} = \begin{vmatrix} x_1^i & x_2^i \\ x_1^j & x_2^j \end{vmatrix} = x_1^i x_2^j - x_1^j x_2^i.$$

By the case of $t = 1$, we have all $(\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2) \cdot g_{ij} \in E_q(2, r)$. Thus $(\alpha x_1^2 + \beta x_1 x_2 + \alpha x_2^2) \cdot g \in E_q(2, r)$. In the same manner, this is generalized for any $f \in E_q(2, r)$, which completes the proof. □

Lemma 3.2. *Let $a, b \in \mathbb{F}_q$ and t be a positive integer, then*

$$\begin{vmatrix} 1 & 1 \\ (ax_1 + bx_2)^t & (bx_1 + ax_2)^t \end{vmatrix} \in E_q(2, r).$$

Proof. When t is odd, consider

$$\begin{vmatrix} 1 & 1 \\ (ax_1 + bx_2)^t & (bx_1 + ax_2)^t \end{vmatrix} = (bx_1 + ax_2)^t - (ax_1 + bx_2)^t$$

$$= \sum_{k=0}^{t} \binom{t}{k} \left((bx_1 + 1)^k (ax_2)^{t-k} - (ax_1)^k (bx_2)^{t-k} \right)$$

$$= \sum_{k=0}^{t} \binom{t}{k} \left(b^k a^{t-k} - a^k b^{t-k} \right) x_1^k x_2^{t-k}$$

$$= \sum_{k=0}^{(t-1)/2} \binom{t}{k} \left(b^k a^{t-k} - a^k b^{t-k} \right) \left(x_1^k x_2^{t-k} - x_1^{t-k} x_2^k \right)$$

$$= \sum_{k=0}^{(t-1)/2} \binom{t}{k} \left(b^k a^{t-k} - a^k b^{t-k} \right) \begin{vmatrix} x_1^k & x_2^k \\ x_1^{t-k} & x_2^{t-k} \end{vmatrix}.$$

Clearly, the corresponding determinant is an element of $E_q(2, r)$.

When t is even, similarly we have

$$\begin{vmatrix} 1 & 1 \\ (ax_1 + bx_2)^t & (bx_1 + ax_2)^t \end{vmatrix} = \sum_{k=0}^{t} \binom{t}{k} \left(b^k a^{t-k} - a^k b^{t-k} \right) x_1^k x_2^{t-k}$$

$$= \sum_{k=0}^{t/2-1} \binom{t}{k} \left(b^k a^{t-k} - a^k b^{t-k} \right) \left(x_1^k x_2^{t-k} - x_1^{t-k} x_2^k \right).$$

In the above equation when $k = t/2$, the coefficient of the term $x_1^{t/2} x_2^{t/2}$ is zero. Finally, as in the odd case, the corresponding determinant is an element of $E_q(2, r)$. Thus, the proof is completed. $\square$

Afterward, we have the following proposition.

Proposition 3.3. *Let $a, b \in \mathbb{F}_q$ and i, j be positive integers such that $i < j$, then*

$$\begin{vmatrix} (ax_1 + bx_2)^i & (bx_1 + ax_2)^i \\ (ax_1 + bx_2)^j & (bx_1 + ax_2)^j \end{vmatrix} \in E_q(2, r).$$

Proof. The determinant can be written as:

$$\begin{vmatrix} (ax_1 + bx_2)^i & (bx_1 + ax_2)^i \\ (ax_1 + bx_2)^j & (bx_1 + ax_2)^j \end{vmatrix} = (ax_1 + bx_2)^i (bx_1 + ax_2)^i \begin{vmatrix} 1 & 1 \\ (ax_1 + bx_2)^{j-i} & (bx_1 + ax_2)^{j-i} \end{vmatrix}$$

$$= \left(abx_1^2 + (a^2 + b^2)x_1 x_2 + abx_2^2 \right)^i \begin{vmatrix} 1 & 1 \\ (ax_1 + bx_2)^t & (bx_1 + ax_2)^t \end{vmatrix}$$

where $t = j - i$. By utilizing Lemmas 3.2 and 3.1, the proof follows.

$\square$

The following theorem gives a necessary and sufficient condition for $SRM_q[2,r]$ to be invariant under which subgroup of $GL(2,q)$.

Theorem 3.4. *Let M be a set defined as*

$$M = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \mid a,b \in \mathbb{F}_q,\ a \neq \pm b \right\} \subset GL(2,q).$$

The automorphism group of the $SRM_q[2,r]$, where $q \geqslant r > 2$ except $q = r = 3$, code family contains a subgroup isomorphic to M, i.e. $SRM_q[2,r]$ is invariant under the transformations come from M.

Proof. Let $A \in GL(2,q)$. Then take the transform T:

$$T : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

In the set $E_q(2,r)$, there exist unique polynomials of degree 1 and 2, which are

$$f(x_1,x_2) = \begin{vmatrix} 1 & 1 \\ x_1 & x_2 \end{vmatrix} = x_2 - x_1$$

and

$$g(x_1,x_2) = \begin{vmatrix} 1 & 1 \\ x_1^2 & x_2^2 \end{vmatrix} = x_2^2 - x_1^2,$$

respectively. If $T_f(A)$ and $T_g(A)$ are elements of $E_q(2,r)$, it is easy to see that $T_f(A)$ and $T_g(A)$ must be scalar multiples of $f(x_1,x_2)$ and $g(x_1,x_2)$, respectively. In the light of this fact, we have

$$T_f(A) = f(ax_1 + bx_2, cx_1 + dx_2) = \begin{vmatrix} 1 & 1 \\ (ax_1 + bx_2) & (cx_1 + dx_2) \end{vmatrix}$$
$$= (d - b)x_2 - (a - c)x_1$$

and

$$T_g(A) = g(ax_1 + bx_2, cx_1 + dx_2) = \begin{vmatrix} 1 & 1 \\ (ax_1 + bx_2)^2 & (cx_1 + dx_2)^2 \end{vmatrix}$$
$$= (cx_1 + dx_2)^2 - (ax_1 + bx_2)^2$$
$$= (d^2 - b^2)x_2^2 - (a^2 - c^2)x_1^2 + (2cd - 2ab)x_1x_2.$$

From the above, we obtain the following equations

$$a - c = d - b,$$
$$(a^2 - c^2) = (b^2 - d^2),$$
$$(2cd - 2ab) = 0.$$

If we solve the equations above together with the fact $ad - bc \neq 0$, then we get $a = d$ and $b = c$. Combining this with Proposition 3.3, $SRM_q[2,r]$ is invariant under the transformations that come from the set M, which completes the proof. $\qquad\square$

Note that when $q = r = 3$, the set $SRM_3[2,3]$ contains all vectors of length 3 so that $Aut(SRM_3[2,3]) = S_3$.

In the following subsection, we will focus on the $SRM_q(3,r)$ in the same manner.

3.2. **The case** $n = 3$. Recall that the code family $SRM_q[3, r]$ with the length $\sum_{i=1}^{q-2} \frac{(i)(i+1)}{2}$ is an evaluated code family whose evaluation polynomials come from the set $E_q(3, r)$,

$$(6) \qquad E_q(3, r) = \left\{ \sum_{\substack{0 \leqslant i < j < k \leqslant q-1 \\ i+j+k \leqslant r}} a_{ijk} \begin{vmatrix} x_1{}^i & x_2{}^i & x_3{}^i \\ x_1{}^j & x_2{}^j & x_3{}^j \\ x_1{}^k & x_2{}^k & x_3{}^k \end{vmatrix} \;\middle|\; a_{ijk} \in \mathbb{F}_q \right\}.$$

Under the $T_f(A)$ transformation in (1), where

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

we investigate the coefficients a, b, c, d, e, f, g, h, $i \in \mathbb{F}_q$ to find that keep the set $E_q(3, r)$ invariant. We require the auxiliary lemmas for this aim.

The following lemma gives us a different interpretation of the set $E_q(3, r)$.

Lemma 3.5. *Let $f(x_1, x_2, x_3) \in \mathbb{F}_q[x_1, x_2, x_3]$ with a degree less than or equal to r, where q is odd, such that for any $\pi \in S_3$,*

$$(7) \qquad f(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) = \begin{cases} -f(x_1, x_2, x_3), & \pi \text{ is an odd permutation,} \\ f(x_1, x_2, x_3), & \pi \text{ is an even permutation.} \end{cases}$$

Then $f(x_1, x_2, x_3) \in E_q(3, r)$.

Proof. Firstly, we may assume that we have a homogeneous nonzero multivariate polynomial $f(x_1, x_2, x_3)$ of degree $t \leqslant r$, with the property (7).

Since f is nonzero, we have a monomial term $A_0 x_1^i x_2^j x_3^k$ in f. Without loss of generality, we may choose the powers $0 \leqslant i \leqslant j \leqslant k$, where i, j, k are integers such that $i + j + k = t$. So we write f as follows

$$(8) \qquad f(x_1, x_2, x_3) = A_0 x_1^i x_2^j x_3^k + g_0(x_1, x_2, x_3)$$

where $A_0 \in \mathbb{F}_q$ and $g_0(x_1, x_2, x_3)$ is a homogeneous polynomial of degree t such that the coefficient of the $x_1^i x_2^j x_3^k$ in $g_0(x_1, x_2, x_3)$ is zero.

Consider the case $i = j$. Then by the property (7), we get

$$f(x_2, x_1, x_3) = A_0 x_2^i x_1^i x_3^k + g_0(x_2, x_1, x_3)$$

$$= -f(x_1, x_2, x_3) = -A_0 x_1^i x_2^i x_3^k - g_0(x_1, x_2, x_3).$$

Equivalently, we have

$$2 A_0 x_2^i x_1^i x_3^k + g_0(x_2, x_1, x_3) + g_0(x_1, x_2, x_3) = 0.$$

Since the coefficient of the monomial $x_1^i x_2^i x_3^k$ in $g_0(x_1, x_2, x_3)$ is zero, $x_2^i x_1^i x_3^k$ must be a monomial term of $g_0(x_2, x_1, x_3)$, whose coefficient is $-2A_0$. This is the contradiction. Similarly, we get a contradiction for the cases $i = k$ and $j = k$. Thus there is not a monomial term $x_1^i x_2^j x_3^k$ such that at least two of i, j and k values are the same.

By the cases mentioned above, we may assume that in the monomial term $x_1^i x_2^j x_3^k$, i, j, k are distinct, i.e. $0 \leqslant i < j < k$. Consider the relation (8). When $\pi = (12)$, we have

$$f(x_2, x_1, x_3) = A_0 x_2^i x_1^j x_3^k + g_0(x_2, x_1, x_3)$$
$$= -f(x_1, x_2, x_3) = -A_0 x_1^i x_2^j x_3^k - g_0(x_1, x_2, x_3).$$

By the above equation, the monomial term $-A_0 x_2^i x_1^j x_3^k$ must be in $g_0(x_1, x_2, x_3)$. Thus the relation (8) rewrite as

$$f(x_1, x_2, x_3) = A_0 x_1^i x_2^j x_3^k - A_0 x_1^j x_2^i x_3^k + g_1(x_1, x_2, x_3).$$

Applying similar steps for the permutations $\pi = (13)$ and $\pi = (23)$, we get

$$f(x_1, x_2, x_3) = A_0 x_1^i x_2^j x_3^k - A_0 x_1^j x_2^i x_3^k - A_0 x_1^k x_2^j x_3^i - A_0 x_1^i x_2^k x_3^j + g_3(x_1, x_2, x_3)$$

When applying the permutation $\pi = (123)$ to $f(x_1, x_2, x_3)$ in above relation, we get

$$f(x_2, x_3, x_1) = A_0 x_2^i x_3^j x_1^k - A_0 x_1^j x_2^i x_3^k - A_0 x_1^k x_2^j x_3^i - A_0 x_1^i x_2^k x_3^j + g_3(x_2, x_3, x_1)$$
$$= f(x_1, x_2, x_3) = A_0 x_1^i x_2^j x_3^k - A_0 x_1^j x_2^i x_3^k - A_0 x_1^k x_2^j x_3^i - A_0 x_1^i x_2^k x_3^j + g_3(x_1, x_2, x_3),$$

which implies the monomial term $A_0 x_2^i x_3^j x_1^k$ must be in $g_3(x_1, x_2, x_3)$. Thus

$$f(x_1, x_2, x_3) = A_0 x_1^i x_2^j x_3^k + A_0 x_1^k x_2^i x_3^j - A_0 x_1^j x_2^i x_3^k - A_0 x_1^k x_2^j x_3^i - A_0 x_1^i x_2^k x_3^j + g_4(x_1, x_2, x_3).$$

Finally, applying the permutation $\pi = (132)$, the polynomial $f(x_1, x_2, x_3)$ is of form

$$A_0 x_1^i x_2^j x_3^k + A_0 x_1^k x_2^i x_3^j + A_0 x_1^j x_2^k x_3^i - A_0 x_1^j x_2^i x_3^k - A_0 x_1^k x_2^j x_3^i - A_0 x_1^i x_2^k x_3^j + g_5(x_1, x_2, x_3)$$

$$= A_0 \begin{vmatrix} x_1^i & x_2^i & x_3^i \\ x_1^j & x_2^j & x_3^j \\ x_1^k & x_2^k & x_3^k \end{vmatrix} + g_5(x_1, x_2, x_3),$$

where $g_5(x_1, x_2, x_3)$ is a homogeneous polynomial of degree t such that the coefficients of the monomials $x_{\pi(1)}^i x_{\pi(2)}^j x_{\pi(3)}^k$ for any $\pi \in S_3$ are zero.

Thereafter, if we apply what we did for $f(x_1, x_2, x_3)$ to $g_5(x_1, x_2, x_3)$ by following the same steps for the other possible triple partition, $t = i_1 + j_1 + k_1$, we get

$$f(x_1, x_2, x_3) = A_0 \begin{vmatrix} x_1^i & x_2^i & x_3^i \\ x_1^j & x_2^j & x_3^j \\ x_1^k & x_2^k & x_3^k \end{vmatrix} + A_1 \begin{vmatrix} x_1^{i_1} & x_2^{i_1} & x_3^{i_1} \\ x_1^{j_1} & x_2^{j_1} & x_3^{j_1} \\ x_1^{k_1} & x_2^{k_1} & x_3^{k_1} \end{vmatrix} + g_6(x_1, x_2, x_3),$$

where $A_0, A_1 \in \mathbb{F}_q$ and $g_6(x_1, x_2, x_3)$ is a homogeneous polynomial of degree t.

Since the number of the triple partition of t is finite, we may continue the above procedure until all possible partitions are over. Finally, the polynomial f is of form

$$f(x_1, x_2, x_3) = A_0 \begin{vmatrix} x_1^i & x_2^i & x_3^i \\ x_1^j & x_2^j & x_3^j \\ x_1^k & x_2^k & x_3^k \end{vmatrix} + A_1 \begin{vmatrix} x_1^{i_1} & x_2^{i_1} & x_3^{i_1} \\ x_1^{j_1} & x_2^{j_1} & x_3^{j_1} \\ x_1^{k_1} & x_2^{k_1} & x_3^{k_1} \end{vmatrix} + \cdots + A_d \begin{vmatrix} x_1^{i_d} & x_2^{i_d} & x_3^{i_d} \\ x_1^{j_d} & x_2^{j_d} & x_3^{j_d} \\ x_1^{k_d} & x_2^{k_d} & x_3^{k_d} \end{vmatrix},$$

where A_i's in $\mathbb{F}_q$. Thus $f(x_1, x_2, x_3) \in E_q(3, r)$ by the definition.

In general, for any polynomial $F(x_1, x_2, x_3)$ satisfies the condition (7), we may write $F(x_1, x_2, x_3)$ as follows

$$F(x_1, x_2, x_3) = f_3(x_1, x_2, x_3) + f_4(x_1, x_2, x_3) + \cdots + f_r(x_1, x_2, x_3),$$

where f_i's are homogeneous polynomials of degree i for $i \in \{3, 4, \ldots, r\}$. From above for any $i \in \{3, 4, \ldots, r\}$, $f_i \in E_q(3, r)$. Thus $F(x_1, x_2, x_3) \in E_q(3, r)$, which completes the proof. $\square$

Let

$$K = \left\{ P * \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}, \middle| P \in \mathcal{P}_3, a, b \in \mathbb{F}_q, a \neq b, a \neq -2b \right\} \subset GL(3, q).$$

Lemma 3.6. *Let $A \in K$ and $f(x_1, x_2, x_3) \in E_q(3, r)$, the function g defined as $g(x_1, x_2, x_3) = T_f(A)$ belongs the set $E_q(3, r)$.*

Proof. Firstly, assume that $A = \begin{bmatrix} b & a & a \\ a & b & a \\ a & a & b \end{bmatrix}$. Let $f(x_1, x_2, x_3) \in E_q(3, r)$. Under the transformation $T(A)$ in (2), we have variables $x_1 \mapsto bx_1 + ax_2 + ax_3$, $x_2 \mapsto ax_1 + bx_2 + ax_3$ and $x_3 \mapsto ax_1 + ax_2 + bx_3$.

Let $g = T_f(A)$:

$$g(x_1, x_2, x_3) = T_f(A) = f(bx_1 + ax_2 + ax_3, ax_1 + bx_2 + ax_3, ax_1 + ax_2 + bx_3)$$

Let $\pi = (12) \in S_3$.

$$\begin{aligned} g(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) = g(x_2, x_1, x_3) &= f(bx_2 + ax_1 + ax_3, ax_2 + bx_1 + ax_3, ax_2 + ax_1 + bx_3) \\ &= -f(bx_1 + ax_2 + ax_3, ax_1 + bx_2 + ax_3, ax_1 + ax_2 + bx_3) \\ &= -g(x_1, x_2, x_3), \end{aligned}$$

second line comes from the fact $f(x_1, x_2, x_3) \in E_q(3, r)$. Similarly, when $\pi = (13)$ or $\pi = (23)$, we get $g(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) = -g(x_1, x_2, x_3)$.

On the other hand, when $\pi = (123) \in S_3$, we have

$$\begin{aligned} g(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) = g(x_2, x_3, x_1) &= f(bx_2 + ax_3 + ax_1, ax_2 + bx_3 + ax_1, ax_2 + ax_3 + bx_1) \\ &= -f(ax_2 + bx_3 + ax_1, bx_2 + ax_3 + ax_1, ax_2 + ax_3 + bx_1) \\ &= f(bx_1 + ax_2 + ax_3, ax_1 + bx_2 + ax_3, ax_1 + ax_2 + bx_3) \\ &= g(x_1, x_2, x_3). \end{aligned}$$

Similarly, when $\pi = (132)$ we obtain $g(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) = g(x_1, x_2, x_3)$.

Finally from above, we can characterize the multivariate polynomial $g(x_1, x_2, x_3)$ as follows for any $\pi \in S_3$:

$$g(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)}) = \begin{cases} -g(x_1, x_2, x_3), & \pi \text{ is an odd permutation,} \\ g(x_1, x_2, x_3), & \pi \text{ is an even permutation.} \end{cases}$$

Thus by Lemma 3.5, $g = T_f(A) \in E_q(3, r)$. In the same manner, this approach can easily be applied to the other elements of the set K. So the proof is completed. $\square$

Theorem 3.7. *The automorphism group of the $SRM_q[3, r]$, where $q \geqslant r > 3$, code family contains a subgroup isomorphic to K, i.e. $SRM_q[3, r]$ is invariant under the transformations come from K.*

Proof. By the definition of $E_q(3, r)$, the members of degrees 3 and 4 in $E_q(3, r)$ are the sets

$$P = \left\{ a_{012} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix} \middle| a_{012} \in \mathbb{F}_q \right\}$$

and

$$Q = \left\{ a_{013} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^3 & x_2^3 & x_3^3 \end{vmatrix} \middle| a_{013} \in \mathbb{F}_q \right\},$$

respectively. Let $B \in \mathbb{F}_q^{3 \times 3}$ be an invertible matrix such that $B = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. It is clear that

when $f \in P$ and $g \in Q$, the following conditions must hold

$$T_f(B) \in P \qquad \text{and} \qquad T_g(B) \in Q.$$

Thus, by the properties of the sets P and Q, the coefficients of the terms

$$x_1^3, \ x_2^3, \ x_3^3, \ x_1 x_2 x_3, \ x_1^4, \ x_2^4, \ x_3^4, \ x_1 x_2 x_3^2, \ x_1 x_2^2 x_3, \ x_1^2 x_2 x_3, \ x_1^2 x_2^2, \ x_1^2 x_3^2, \ x_2^2 x_3^2$$

in the outputs, after applying the corresponding transformations $T_f(B)$ and $T_g(B)$ must be zero. Furthermore, in the same outputs, the sum of the coefficients of the terms in the pairs

$$(x_1 x_2^2, x_1^2 x_2), \ (x_1 x_3^2, x_1^2 x_3), \ (x_2 x_3^2, x_2^2 x_3), \ (x_1 x_3^3, x_1^3 x_3), \ (x_1 x_2^3, x_1^3 x_2), \ (x_2 x_3^3, x_2^3 x_3)$$

must individually be zero. So we have 19 different equations over $\mathbb{F}_q$. When we solve these equations for the unknowns $a, b, c, d, e, f, g, h, i$ with the help of the computer algebra system SageMath, we get the solution set S as

$$\left\{ \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}, \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}, \begin{bmatrix} a & b & b \\ b & b & a \\ b & a & b \end{bmatrix}, \begin{bmatrix} b & a & b \\ a & b & b \\ b & b & a \end{bmatrix}, \begin{bmatrix} b & a & b \\ b & b & a \\ a & b & b \end{bmatrix}, \begin{bmatrix} b & b & a \\ b & a & b \\ a & b & b \end{bmatrix}, \begin{bmatrix} b & b & a \\ a & b & b \\ b & b & a \end{bmatrix} \middle| a, b \in \mathbb{F}_q \right\}.$$

For the invertibility for each element of the above set, $a, b \in \mathbb{F}_q$ satisfies the conditions $a \neq b, -2b$. Thus the solution set will be the set $K \subset GL(3, q)$. By combining this with Lemma 3.6, the set K is the maximal set in $GL(3, q)$ such that $E_q(3, r)$ is invariant under the transformations come from K. $\qquad \square$

We give examples of $SRM_q[2, r]$ and $SRM_q[3, r]$ for some q, r values, respectively.

Example 3.8. *Let* $q = 5$, $n = 2$, $r = 4$ *and* $(i_1, i_2) \in \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3),$ $(1, 4), (2, 3), (2, 4), (3, 4)\}$. *For a matrix* $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \right.$ $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 4 & 3 \end{bmatrix}, \left. \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix} \right\}$

and $\alpha \in \mathbb{F}_5$, *the following equation holds:*

$$\alpha \left[(x_1^{i_1} x_2^{i_2}) - (x_1^{i_2} x_2^{i_1}) \right] = (ax_1 + bx_2)^{i_1} (cx_1 + dx_2)^{i_2} - (ax_1 + bx_2)^{i_2} (cx_1 + dx_2)^{i_1}$$

Example 3.9. *Let $q = 7$, $n = 3$ and $r = 5$. Then $(i_1, i_2, i_3) \in \{(0, 1, 2), (0, 1, 3), (0, 1, 4), (0, 2, 3)\}$.
Let*

$$g_1(x_1, x_2, x_3) = x_1 x_2^2 + x_2 x_3^2 + x_1^2 x_3 - x_1^2 x_2 - x_2^2 x_3 - x_1 x_3^2,$$

$$g_2(x_1, x_2, x_3) = x_1 x_2^3 + x_2 x_3^3 + x_1^3 x_3 - x_1^3 x_2 - x_2^3 x_3 - x_1 x_3^3,$$

$$g_3(x_1, x_2, x_3) = x_1 x_2^4 + x_2 x_3^4 + x_1^4 x_3 - x_1^4 x_2 - x_2^4 x_3 - x_1 x_3^4,$$

$$g_4(x_1, x_2, x_3) = x_1^2 x_2^3 + x_2^2 x_3^3 + x_1^3 x_3^2 - x_1^3 x_2^2 - x_2^3 x_3^2 - x_1^2 x_3^3,$$

and

$$\mathcal{A} = \left\{ P \begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} \middle| P \in \mathcal{P}_3, a, b \in \mathbb{F}_7, a \neq b, a \neq -2b \right\}.$$

Then

$$E_7(3, 5) = \{a_1 g_1 + a_2 g_2 + a_3 g_3 + a_4 g_4 \mid a_1, a_2, a_3, a_4 \in \mathbb{F}_7\}.$$

*For the matrix $K \in \mathcal{A}$ and $g \in E_7(3, 5)$, the polynomial $T_g(K) \in E_7(3, 5)$ by Theorem
3.7. Since the $SRM_7[3, 5]$ code is a type of polynomial evaluation codes, as in Example
1.4, $SRM_7[3, 5]$ is invariant under the corresponding transformations come from the $\mathcal{A}$.*

Note that the determinants of matrices in solution set must be non-zero. For example, the
solution set on $\mathbb{F}_5$ for $n = 2$ does not include the element $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$ because its determinant is
zero. Additionally, as the parameter n increases, the values of q and r should be adjusted
accordingly.

3.3. **The general case.** Determining all transformations under $GL(n, q)$ that leaves the
SRM code invariant for a general n is quite challenging. For this, there needs to be a general
method to identify such transformation. Nevertheless, we can predict the solution set that
leaves the SRM code invariant for a general n and is a subgroup of the affine linear group.
However, we have not established that this set may include all possible linear transformations
under which SRM codes are invariant. We leave the task of finding a generalized method
for this problem for future work.

4. CONCLUSION

Our work aims at determining the set of affine-invariant transformations. The linear
automorphism groups of SRM for $n = 2$ and $n = 3$ over the field $\mathbb{F}_p$, where p is any prime
number is proven in this paper. For $n = 2$ and $n = 3$, we find that the exact set generated
by transformations remaining SRM codes invariant is a subgroup of the affine linear group.
For different values of n, different techniques were used to determine this set. Therefore,
we could not give a general proof for an arbitrary n, and leave it an open problem of the
complete determination of the automorphism group $Aut(SRM)$ for any $n > 3$. We state our
conjecture below.

Conjecture. *Let J_n be the $n \times n$ all one matrix, I_n be the $n \times n$ identity matrix and $\mathcal{P}_n$
be the set of permutations of order n. Let M be a subset of $GL(n, q)$ defined as $M = \{P((b - a)I_n + aJ_n) \mid P \in \mathcal{P}_n, a, b \in \mathbb{F}_q, a \neq b, a \neq (1 - n)b\} \subset GL(n, q)$. Then, the automorphism group of the $SRM_q[n, r]$ for $q > r > \frac{n(n-1)}{2}$ contains a subgroup isomorphic to M, i.e.,
$SRM_q[n, r]$ is invariant under the transformations in M.*

The proof of the invariance of $SRM_q[n, r]$ codes under the transformations, which come from the set M, may be similarly done to the proof of Lemma 3.6. Notwithstanding, in order to show that the set M is the complete set in this manner is quite challenging to follow the same techniques.

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