DEVELOPMENT OF AN INSTRUCTIONAL UNIT FOR DEVELOPING KINDERGARTEN STUDENTS’ ALGEBRAIC REASONING PRIOR TO FORMAL ARITHMETIC EDUCATION

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ABSTRACT

DEVELOPMENT OF AN INSTRUCTIONAL UNIT FOR DEVELOPING KINDERGARTEN STUDENTS' ALGEBRAIC REASONING PRIOR TO FORMAL ARITHMETIC EDUCATION

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The aim of this study is to develop an instructional sequence that focuses on algebraic reasoning before arithmetic education. The study is based on Davydov's approach to early algebra education, which promises a better understanding of variability and functions in higher grades. The instructional design includes creating effective learning trajectories and materials for use in kindergarten mathematics courses. To achieve this goal, the study investigated the following research question: How can kindergarten students' algebraic reasoning be supported by a proposed instructional sequence? A hypothetical learning trajectory was developed and adapted for the kindergarten level based on the Davydov & Elkonian curriculum of first-grade mathematics. Based on the Design-Based Research perspective, classroom activities aligned with this hypothetical learning trajectory were further refined during the implementation of instruction based on student learning outcomes. Following the implementation and testing of the trajectory through 20 in-class and online sessions with 10 students, an effective learning sequence and practical instructional materials were developed. Students' progression through the trajectory, related to design principles, contributed to the theory of how students develop algebraic learning in early grades. Piaget's conservation principles were effective in students' perception of structures in the equations. The data showed that students could develop an algebraic understanding of equality and operations at the kindergarten level by

adapting the Davydov-Elkonian curriculum with modifications on materials and symbolization in activities.

Keywords: Early Algebra, Davydov, Kindergarten Students, Mathematics Education, Design-Based Research

ÖZ

ANAOKULU ÖĞRENCİLERİNİN ARİTMETİK EĞİTİMİ ÖNCESİNDE CEBİRSEL MUHAKEMELERİNİN GELİŞTİRİLMESİ İÇİN EĞİTİM İÇERİĞİ GELİŞTİRME

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Bu çalışmanın amacı, aritmetik eğitiminden önce cebirsel akıl yürütmeye odaklanan bir öğretim dizisi geliştirmektir. Çalışma, daha yüksek sınıflarda değişkenlik ve fonksiyonların daha iyi anlaşılmasını vaat eden Davydov'un erken cebir eğitimi yaklaşımına dayanmaktadır. Öğretim tasarımı, anaokulu matematik derslerinde kullanılmak üzere verimli öğrenme yolları ve materyaller oluşturmayı içermektedir. Bu amacı gerçekleştirmek için çalışmada şu araştırma sorusu incelenmiştir: Önerilen bir öğretim dizisi ile anaokulu öğrencilerinin cebirsel akıl yürütmesi nasıl desteklenebilir? Birinci sınıf matematiği Davydov & Elkonian müfredatına dayalı olarak, anaokulu düzeyine uyarlanmış ve geliştirilen bir varsayımsal öğrenme yolu oluşturulmuştur. Tasarım Tabanlı Araştırma perspektifine dayalı olarak, öğrenci öğrenme çıktıları doğrultusunda sınıf aktiviteleri bu varsayımsal öğrenme yolu ile hizalanmış ve öğretimin uygulanması sırasında daha da rafine edilmiştir. 10 öğrenci ile 20 sınıf içi ve çevrimiçi oturum boyunca öğrenme yolunun uygulanması ve test edilmesinin ardından, etkili bir öğrenme dizisi ve pratik öğretim materyalleri geliştirilmiştir. Öğrencilerin tasarım ilkelerine dayalı olarak öğrenme yolu boyunca ilerlemesi, öğrencilerin erken sınıflarda cebirsel öğrenmeyi nasıl geliştirdiği teorisine katkıda bulunmuştur. Piaget'in miktarın korunumu ilkesinin belirlediği

sınırlar öğrencilerin denklemlerdeki yapıları algılamasında etkili olmuştur.Veriler, öğrencilerin anaokulu düzeyinde Davydov-Elkonian müfredatını materyaller ve aktivitelerdeki sembolleştirme düzenlemeleriyle uyarlayarak eşitlik ve işlemler konusunda cebirsel bir anlayış geliştirebildiklerini göstermiştir.

Anahtar Kelimeler: Erken Cebir Eğitimi, Davydov, Anaokulu Öğrencileri, Matematik Eğitimi, Tasarım Tabanlı Araştırma

To my son and my daughter

To all children in the world

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CHAPTER 1

INTRODUCTION

Algebra is generally associated with relations, functions, variation, and modeling languages through symbolization, generalization, and acting on generalized structures by the language of symbols (Kaput, 2008) and “well-defined methods of manipulation” (Katz & Barton, 2007, p. 185). Defining algebra and distinguishing algebra from arithmetic is difficult (Cai & Knuth, 2011; Kaput, 2008), particularly in the early grades of teaching algebra (Kaput, 2008). Through systematic strategies for computations, relational properties, operations, and dynamic problem-solving, algebra is fundamental to most mathematical domains (Katz & Barton, 2007), including arithmetic, where rules and generalizations occur in computations (Cai & Knuth, 2011). In this context, algebra is sometimes referred to as generalized arithmetic (Kaput, 2008; Katz & Barton, 2007).

Algebra has been seen as generalizations and anything that consists of operations, systems, and computations underlies an algebraic point of view. From a historical perspective, algebra started by explaining general strategies to solve numerical problems and evolved to studying operational systems. It gradually evolved from explanations through words to the total use of symbolization (Katz & Barton, 2007). Even with the help of notational representations it created, algebra formed more of a language (Stacey & Chick, 2004) to communicate and also a base for studying mathematics.

Central to most mathematical areas and foundational to everyday mathematics and every branch of mathematics, the importance of algebra education is unquestionable. Although intertwined, arithmetic and algebra are taught at different educational levels: arithmetic in elementary school and algebra in middle and high school (Cai

& Knuth, 2011). This separation may be the cause of problems in learning algebra (Kieran, 2007). Learning algebra is found to be difficult and seen as problematic in education (Cai & Knuth, 2011; Kaput, 2008; Stacey & Chick, 2004). Seen as a “gatekeeper,” mathematics education studies focus on difficulties in understanding algebraic topics and developing algebraic curricula, as these challenges significantly impact mathematics learning (Cai & Knuth, 2005, p. 1).

These major problems involve difficulties or misconceptions of algebraic thinking and the initial introduction of algebra after arithmetic education, where new concepts challenge students' arithmetical thinking. Consequently, studies focus on introducing algebraic thinking in earlier contexts, creating a smooth transition from arithmetic to algebra, or teaching arithmetic from an algebraic perspective without conflicting with algebra.

Transitioning from the arithmetic of four operations with numbers to the algebra of generalizations, variables, functions, and operational properties introduces many misconceptions and difficulties (Kieran, 2007; Ndemo & Ndemo, 2018), which may persist and affect future mathematics learning. While learning arithmetic, certain limitations or misunderstandings may hinder the future learning of algebra (Ketterlin-Geller, et al., 2007). For example, due to the emphasis on "finding the answer" in arithmetic, understanding the equal sign as an operator rather than as a symbol of relational equivalence, is common (Byrd, et al., 2015; Kieran, 1981; Saenz-Ludlow & Walgamuth,1998). This misconception not only affects young students but also has long-lasting effects on college mathematics (Fyfe, et al., 2020). Particular tasks are recommended at “the very beginning of early algebra education” to support students’ understanding of the structural meaning of equation signs (Stephens, et al., 2013, p. 173). Therefore, why not enforce tasks that support structural understanding when students first encounter equal signs?

Similarly, focusing on finding unknown values hinders the understanding of variables (Rosnick, 1981). Functions and relations between quantities become the most difficult subjects as a result of a limited understanding of equality and variability. Moreover, from a Piagetian perspective, it is believed that algebra can only be understood by secondary-grade students who have reached the level of abstract thinking.

To overcome problems related to algebraic understanding, researchers have begun to focus on early algebra education. This involves introducing algebraic ideas, such as the use of letter notation, in a less formal and earlier way, or defining pre-algebra stages to facilitate a smoother transition from arithmetic to algebra (Herscovics & Linchevski, 1994; Linchevski & Herscovics, 1996; MacGregor & Stacey, 1998; Pillay, et al., 1998; Van Amerom, 2003) This transition is supported by using familiar contexts through word problems and visual demonstrations.

The dominant approach in early algebra education is to support future algebraic learning by focusing on functions through the generalizations of solutions to problems or pattern detection among number sets or geometric figures in the early grades (Blanton, 2010). Another common approach is not bringing traditional algebraic curriculum earlier but rather incorporating concepts such as unknowns through familiar contexts (Carraher, et al., 2008). This approach also involves teaching arithmetic in a way that supports algebraic reasoning (Cai & Knuth, 2011), such as learning equal signs as symbols of balance relations and focusing on operational properties from the very beginning of arithmetic education. This helps to prevent contradictions and misconceptions during the transition from arithmetic to algebra (Ramirez Uclés, et al., 2022; Warren, 2003).

In addition to these approaches, Davydov offers a radical solution for algebra education. Based on Vygotskian perspectives, he argues that there is no age limit for abstract thinking, and algebraic discussions can be advanced in the proximal zone of development, where students can develop their understanding with the guidance of a knowledgeable other. Following Hegel's perspective, Schmittau and Morris (2004) state that Davydov introduces theoretical algebraic discussions as early as the 1st grade of elementary school, even before arithmetic education. This approach involves first teaching operations algebraically, then arithmetically, designing instruction with a deductive hierarchy where arithmetic is viewed as a numerical subset of algebra. Consequently, this method aims to prevent inconsistencies, cognitive conflicts, and the hindrance of arithmetical thinking while transitioning to algebra education, and ultimately fostering algebraic reasoning from the start.

From a Piagetian perspective, students can not be taught anything at any age. Opposing Piaget, Bruner's work challenged age boundaries for learning and identified stages to go through for learning a new concept (Bruner & Kenney, 1965; Conway, 2007). These stages are; enactive, iconic, and symbolic stages, which are compatible with Davydov's curriculum, where concepts are handled starting with real-life illustrations and iconic representations, then connected with formal algebraic expressions. Davydov (1988) stated the importance of "concrete activity" in investigations for the abstraction of learning concepts (p. 195). Starting actions with concrete materials in mathematical investigations is also suggested by APOS (Action-Process-Object-Schema) Theory at early grades (Arnon et al., 2014).

Davydov's curriculum has been shown to be effective in developing strong algebraic understanding at early ages, as supported by replicative studies (Dougherty, 2008; Schmittau & Morris, 2004;). His work has inspired numerous studies aimed at establishing better connections between algebra and arithmetic and providing strong bases for algebraic understanding across different age groups (Dougherty & Venenciano, 2007; Eriksson & Jansson, 2017; Okazaki et al., 2006; Tortora & Mellone, 2017). Regarding the relationship between arithmetic operations and algebraic reasoning, integrating algebraic concepts into arithmetic has been proposed as a remedy for difficulties in arithmetic, demonstrating that algebra is not inherently more difficult than arithmetic (Gerhard, 2009).

Early algebra studies have influenced the revision of curricula significantly. Traditionally, algebraic education around the world and in the US began at the secondary level, aligning with Piagetian stages of abstract thinking. However, affected by recent research on early algebra education, the US curriculum now includes patterns and generalization of numbers as early as elementary grades

(Schimittau & Morris, 2004), aiming to foster relational thinking and understanding of functions.

In these early stages, relationships often begin with considering the relations among concurrent numbers in a series defined by an operation. This approach helps in understanding the trend of change in a function. However, it sometimes neglects emphasis on the independent variable. The focus on the patterning of numbers tends to be discrete and can pose challenges when transitioning to understanding continuous functions.

Schoenfeld and Arcavi (1988) stated that variables define the transition from arithmetic to algebra, and generalizations of patterns should be mastered before using variables. This suggests a transition from arithmetic to algebra from a relation of numbers to a variable perspective. Davydov’s perspective indeed takes a reverse approach compared to traditional methods. His curriculum begins by introducing relations involving variables first. This means considering non-discrete, unknown, and varying quantities and exploring their relationships through equations and inequalities. According to Davydov, algebraic theoretical thinking about operations starts with an understanding of the dynamicity of the equations. He advocates that this approach to algebraic thinking can begin even before the introduction of numbers and arithmetic. Davydov's philosophy states that starting mathematical reasoning empirically prevents thinking theoretically (Schimittau & Morris, 2004).

Davydov's curriculum introduces equations and multiple types of solution sets early on, aiming to enhance students' understanding of equal signs and variables rather than just the concept of the unknown. Theoretical thinking on operations and their properties begins immediately with instruction on equalities. Students act on equalities and inequalities with addition and subtraction operations and discuss adding or subtracting equivalent quantities to reform equalities. They model the $A \pm B = C \pm D$ form of equations with real-life situations. In Davydov's approach, after discussing equations and operations theoretically through unknown quantities, number sense is developed based on the concept of equality relations, where one quantity relates to another as a multiple. Unitization is introduced not through discrete counting of some units, but by understanding measurement through multiplicative relations between quantities in equations in the form A=kB, where k is a counting number. This multiplicative relationship is then integrated with addition, laying the groundwork for problem-solving involving multiple unknowns, which is the final topic in the Grade 1 Mathematics Book of Davydov's curriculum (Davydov et al., 1995).

Davydov's curriculum shows promise for developing strong algebraic reasoning skills, but it may potentially delay instruction in arithmetic (Schmittau & Morris, 2004). Implementing this curriculum also presents challenges related to cultural adaptation, teacher education, and acceptance (Mellone et al., 2021; Schmittau & Morris, 2004; Sidneva, 2020). These factors highlight the need for careful consideration and adaptation when introducing Davydov's approach into different educational contexts.

In the Turkish curriculum, there is a focus on developing a relational understanding of equality from an early age. The recent Kindergarten curriculum emphasizes comparing objects based on different attributes, like color, shape, and length, as well as ordering or classifying objects based on these attributes (Ministry of National Education [MoNE], 2013). While the curriculum does not explicitly mention the use of > and < signs, teachers often incorporate activities involving these signs in their classrooms. In 1st Grade, equality is exemplified by real-life objects. Then, objects are compared and ordered based on length, volume, and weight with the help of units. Addition takes place with traditional one-sided equations in Grade 1. Relational understanding of equality is stressed in the 2nd Grade by introducing two-sided operations by the objective “He/She recognizes the meaning of 'equality' between mathematical expressions represented by the equal sign. It is emphasized that the equal sign does not always imply the result of an operation but also indicates the balance (equality) between the mathematical expressions on both sides. For example, 5+6=10+1; 15-3= 18-6; 8+7 = 20-5; 18= 16+2” ([MoNE], 2018, p. 33). After getting reluctant to traditional equations with operations on the left side, students might develop an operational meaning that hinders the relational meaning of the equal sign (Byrd et al., 2015; Sfard & Linchevski, 1994) or they might develop a tendency to focus on solving the operation even/if they develop a relational understanding of the sign (Lee & Pang, 2023). Moreover, the mentioned objective stresses that the equal sign does not always have an operational meaning. We propose that equal sign should always be acquired to have relational meaning, which will not contradict the traditional equations presented in the 1st Grade. It takes a long time to alter the operational understanding students might bring from kindergarten and develop a relational understanding even in Grade 1 (Falkner et al., 1999). Hence, starting teaching the equal sign with a relational meaning at an early age is essential.

Starting with discussion on equality rather than operations fosters a relational understanding of the equal sign. Davydov's approach is not only promising for a meaningful understanding of equality but also for understanding variables. Adapting Davydov's curriculum to the kindergarten curriculum offers an opportunity to develop a robust algebraic foundation before formal arithmetic education in Grade 1, particularly focusing on equations involving addition and subtraction operations.

Arithmetic education traditionally begins in kindergarten with counting, and many students are familiar with solving simple arithmetic problems even before starting school (Kilpatrick et al., 2001). Drawing from Davydov’s deductive perspective, advocating for starting algebra education before arithmetic can help prevent reluctance toward arithmetic reasoning. Therefore, introducing algebraic concepts aligned with Davydov's approach in kindergarten can adequately prepare students without delaying arithmetic education or compromising traditional elementary education goals (Stephens et al., 2021).

By adapting Davydov’s deductive perspective at the kindergarten level, the goal of this study is to mitigate difficulties in learning algebra and facilitate the transition from arithmetic to algebra by emphasizing variables over unknowns, continuous variables over discrete variables, and relational over operational reasoning by addressing inconsistencies from the outset.

1.1 Purpose of the Study

This study aims to develop an instructional sequence for the kindergarten level that adapts Davydov's approach (algebra before arithmetic) and focuses on the algebraic understanding of equations in the form of A±B=C±D. In this study, unitization in the multiplicative form of A=kB expression is not included. Introduction of numbers and operation with numbers is left to formal arithmetic education in Grade 1. Bruner's modes of algebraic representations (Bruner & Kenney, 1965) will be enhanced in structuring activities, and APOS Theory (Dubinsky & McDonald, 2001) will be used to ensure and assess the development of students' algebraic understanding throughout the activities. The following research questions will guide a Design-Based Research study to develop and improve the intended instructional sequences.

1.2 Research Questions

The research questions outlined below guide a Design-Based Research study aimed at developing, revising, and refining an instructional sequence.

General Research question: Based on Davydov's approach, how can kindergarten students' algebraic understanding of equations be effectively supported before arithmetic education?

1. What is an adapted learning trajectory for supporting kindergarten students' algebraic understanding of equations from Davydov's non-numerical perspective?
    a) To what extent do kindergarten students learn equations with addition and subtraction with an adaptation of Davydov's curriculum for first graders?
    b) What are kindergarten students' strengths and difficulties in understanding the equations in the adapted trajectory?

2. What are the effective and practical activities for supporting kindergarten students' algebraic understanding of equations from Davydov’s non-numerical perspective?
    a) Which characteristics of the activities help kindergarten students understand and resolve their difficulties in comprehending equations?

1.3 Significance of the Study

This study has three main significances: 1) contributing to the literature on teaching equality and quantity in earlier grades by filling the gap with an algebraic perspective, and 2) providing an evidenced-based trajectory for teaching algebra at the kindergarten level. 3) generating effective and practical activities to support the trajectory.

First, there is limited research on teaching algebra in the early grades. Studies of early algebra focused on one of the major problems of understanding of equal sign. The problem with equal signs is that students have operational understanding rather than relational understanding. Several studies have explored understanding and teaching equality in a structural way (Matthews, et al., 2012; Stephens et al., 2021), suggesting the use of various equation structures. Matthews et al. (2012) developed a test for relational understanding of equal signs and assessed among 2nd to 6th graders. Stephens et al. (2021) used this framework to observe students' progress in understanding the equal sign during an early algebra intervention from kindergarten to Grade 1. These studies emphasize that understanding equality is fundamentally relational. Based on these studies we can draw conclusions about how students develop a relational understanding of the equal sign in arithmetic. However, there is a significant gap in understanding how students develop a relational understanding of equality and equations with unknowns.

There are some studies about students' understanding of indeterminate quantities in the 1st grade through numerical generalization for interpretation of variables. (Brizuela, et al., 2015) These studies have shown that first graders can use symbols to interpret variables and manipulate a mathematical expression with variables. The findings are promising for understanding variables at early ages, leading to a proposed trajectory for developing representations of quantity to variables in Grade 1 (Blanton, et al., 2017). Building on these findings, Ventura et al. (2021) revised the trajectory to include kindergarten and Grade 1. Their results were consistent with those of Blanton et al. (2017) showing that Grade 1 students can interpret variables and work with them in meaningful algebraic expressions. While kindergarten students could produce some symbolic notation for indeterminate quantities, they struggled to perform operations or use these symbols in algebraic expressions. These studies showed that with the right opportunities, students can reason about variables and construct algebraic expressions using unknowns, providing hope for early algebra education.

However, the variables used in these studies were discrete numerical quantities, and generalization was achieved by analyzing patterns in number tables. Adopting Davydov's perspective necessitates studying the teaching of continuous variables such as volume, weight, and length.

These studies on teaching equality and variables primarily address discrete numerical quantities, whether indeterminate (variables) or determinate numbers. We aim to extend this research by incorporating non-numeric continuous quantities. Adopting Davydov's perspective, this approach focuses on developing a relational understanding of equality and enabling kindergarten students to work with continuous unknowns in equations.

There is no study applying Davydov's trajectory perspective at the kindergarten level, but some studies have demonstrated its effectiveness starting from Grade 1 (Dougherty, 2008; Schmittau & Morris, 2004), which gives hope for teaching equality with continuous variables in early grades. In the Measure-Up Project

(Dougherty, 2008), students compared continuous quantities and used additive and multiplicative reasoning to interpret algebraic equations at the Grade 1 level, as in Davydov’s trajectory.

To avoid delaying formal arithmetic education in Grade 1, there is a need to adapt Davydov’s trajectory to the kindergarten level. This adaptation would support the understanding of the relational meaning of equal sign and addition/subtraction operations involving unknowns. Introducing these concepts before formal arithmetic education can build a strong foundation for relational reasoning, thus preparing students for more advanced mathematical concepts.

Hence, this study's results will illuminate students' understanding of equality and continuous quantities as early as kindergarten. The results will clarify students' difficulties and how they overcome them while developing their understanding at the kindergarten level. This study will contribute to the literature on early algebra education by focusing on equality and quantity, filling the gap by exploring the understanding of continuous non-numeric quantities and algebraic equality at the kindergarten level.

The second significance of this study is its outcome as a learning trajectory. Not only will students' difficulties and strengths in understanding equality and quantities be illuminated, but also, through advancing design-based perspectives, a trajectory for students' improvement on quantity, equality, and equations with addition and subtraction involving unknowns will be developed empirically. This trajectory will provide a connected explanation of students' learning and progression through these concepts. Students’ learning progression will be assessed by APOS Theory. However, there is little research using APOS Theory to develop or assess students' algebraic learning in early grades (Arnon et al., 2001). Hence, this study results will also contribute to APOS Theory, at early ages.

Moreover, the trajectory will serve as a guide for further studies aimed at developing comprehensible algebraic reasoning at the kindergarten level. Curriculum development that supports algebra from very early ages is possible through this trajectory, which will be validated based on design-based perspectives. Studying further revisions and improvements on the learning trajectory and instructional sequence are also possible based on the implementation conditions. Thus, this study's trajectory will be a valuable tool for educators and researchers, offering a proven framework for teaching algebraic concepts to young learners and paving the way for continuous enhancement of early algebra education. The design principles, as a practical outcome of this study, explain its third significance. Practical activities and materials will be developed to support the learning trajectory. These ready-made activities and materials, proven to be practical and supportive of the learning trajectory based on a design-based perspective, will facilitate the adoption of kindergarten curriculums and simplify implementation for teachers. By providing evidence-based, user-friendly resources, this study ensures that educators can readily integrate these activities into their teaching practices. The design principles will guide the creation of effective instructional materials that align with the learning goals, making early algebra education accessible and manageable for both teachers and students.

Briefly, this study and its theoretical outcomes will enable us to explain students' development of algebraic understanding through an adapted Davydov curriculum at the kindergarten level. Moreover, the adapted trajectory and supportive activities developed through this study will benefit future curricula and instructional practices.

1.4 Limitations of the Study

The study's results will yield a learning trajectory along with supporting practical activities. While the resultant learning trajectory may not be optimal, it can be refined through analysis and implementation of micro-design cycles over a semester. Furthermore, the findings are constrained by the characteristics of the classroom environment. Testing the efficacy of the trajectory in different classroom settings and exploring alternative learning trajectories can be achieved through macro design cycles.

1.5 Definition of Terms

Algebraic reasoning

From the point of view of symbolization and beyond, algebra involves patterns, generalizations, rules, and actions taken on those generalizations. Algebraic reasoning can be defined by how individuals perceive and act upon these rules, highlighting differences in their approaches (Kaput, 2008). From this perspective, algebra education centers on patterns and generalizations, primarily within number sequences. In this study, our perspective focuses on the rules and actions involving operations with continuous quantities, aligning closely with Davydov's approach.

Learning trajectory

A learning trajectory is the learning path that is partially or wholly planned before implementation and is open to adjustments based on the conditions during the trip (implementation) (Simon, 1995, p. 136).

Hypothetical learning trajectory

A hypothetical learning trajectory is designed before classroom intervention and includes objectives, defined activities, and a predictive learning process. The hypothetical learning trajectory is tested and refined in the procedure, and it is hypothetical in the sense that it is based on the "prediction of the path which learning might proceed" until a resultant path is accomplished (Simon, 1995, p. 135).

Quantity

"Quantities are attributes of objects or phenomena that are measurable; it is our capacity to measure them—whether we have carried out those measurements or not that makes them quantities" (Smith & Thompson, 2007, p. 101).

Variables

Variables can be defined through symbolization or a placeholder to represent indeterminate quantities that are assumed to be or can be varying in mathematical sentences. In algebra education, variables are mostly used as tools for representing generalizations, underlying dynamicity in quantities, or deriving solutions for unknowns in equations (Schoenfeld & Arcavi, 1988).

Genetic decomposition

"From a cognitive perspective, a particular mathematical concept is framed in terms of its genetic decomposition, a description of how the concept may be constructed in an individual's mind. This differs from a mathematical formulation of the concept, which deals with how the concept is situated in the mathematical landscape—its role as a mathematical idea" (Arnon et al., 2014, p. 17).

Mental Structures

Individuals make sense of mathematical concepts by building and using specific mental structures (or constructions), which are considered in APOS Theory to be stages in the learning of mathematical concepts (Arnon et al., 2014, p. 17).

CHAPTER 2

LITERATURE REVIEW

With secondary school education becoming compulsory worldwide, algebra has become a fundamental part of the curriculum and the primary difficulty in mathematics education (Kaput, 2008). Despite extensive research and numerous interventions, algebra remains one of the major challenges in learning and teaching mathematics (Warren et al., 2016). The difficulty arises from the cognitive gap between the arithmetic calculations students are accustomed to and the algebraic systems they are newly encountering (Herscovics & Linchevski, 1994). Never before have students focused on rules, relations, generalization, symbols, operation by unknowns, and variables, all of which require a cognitive shift in problem-solving. In algebra, perception and the context of the problem determine how to interpret expressions (Sfard & Linchevski, 1994). This presents another challenging step for students. Equality acquires a new relational meaning, and numbers are replaced with unknowns and variables in calculations. Operating with these unknowns and variables necessitates understanding operational properties (Sfard & Linchevski, 1994).

Not only do quantities pose a challenge, but the abstraction of operations in calculations also presents difficulties. Warren (2003) indicated that the difficulty in readiness for algebra stems from a lack of understanding of "arithmetic operations as general processes" (p. 133) and pointed out the importance of recognizing patterns and generalizations in operations for an abstract understanding of arithmetic. Beyond calculations, the abstraction of operations needs to be more prominent and should be connected to real-life language. These changes require students to develop new cognitive abilities. Additionally, their reluctance toward arithmetical processes hinders their ability to focus on algebraic structures. These factors contribute to the difficulty in transitioning from arithmetic to algebra.

In the following sections, we will address these difficulties and possible solutions in algebra learning in the literature that guided this study.

2.1 Transition from Arithmetic to Algebra

In the transition from arithmetic to algebra, students encounter the challenge of conducting operations on unknowns, a new concept for them. The didactic cut emerges with the introduction of unknowns and operations on them. Strategies that students apply to arithmetical problems and equations are not effective for solving algebraic problems and equations, which require the use of an operation on unknowns, causing the didactic cut (Filloy & Rojano, 1989). Thinking about this problem from Davydov's perspective, if students were to learn to develop algebraic strategies and operations on unknowns first, these strategies would then apply to arithmetic problems and equations, thereby eliminating the didactic cut. Students' tendency to assign numbers when solving algebraic equations shows the cognitive gap between numerical operations and the ability to operate on unknowns (Herscovics & Linchevski, 1996).

In addition to operations with unknowns, Herscovics and Linchevski (1994) emphasize the importance of understanding the equal sign for solving algebraic equations. Arithmetic and algebraic problems/equations may differ in the positioning of operations, which can lead to misunderstandings and misreadings by students because of their arithmetical-solving practices. Their intention was to improve arithmetic education designed to support algebraic reasoning, and algebraic arithmetic, drawing attention to the understanding of the meaning of the equal sign.

According to Warren (2003), not only abstraction of operations but also generalizations of their properties, their use in real-life language, and their symbolic interpretations are essential for the transition from arithmetic to algebra. Therefore, there is also a need for developing activities that support everyday language to express generalizations of algebraic properties.

One way to bridge arithmetic to algebra is to challenge students to discover informal strategies based on their prior knowledge of arithmetic to solve algebraic problems at the pre-algebra stage. However, reasoning algebraically in those types of activities does not guarantee to be successful in interpreting algebraically (Van Amerom, 2003). Algebraic problem solving differentiates from arithmetical problem solving in that, to solve algebraic problems you must first symbolize the problem situation to operate on it. This difference in solving problems or equations introduces a cognitive difficulty in transitioning from arithmetic to algebra, described as a "didactic cut" by Filloy and Rojano (1989) and as a "cognitive gap" by Herscovics and Linchevski (1994). To overcome this difficulty, a gradual transformation to letter notation is suggested (Carraher et al., 2017, Mason, 1996). Another "learning leap" appears in the transition from discrete to continuous variables (Boote & Boote, 2017, p. 456). Davydov's perspective has the potential to eliminate this difficulty by starting with continuous variables such as volume, length, and height, and then continuing with discrete/numerical calculations.

In the following sections studies on equal sign and quantity will be presented to explain major difficulties and cognitive demands in algebra education. Then early algebra studies will be discussed as a solution to these challenges. Among these, the studies inspired by Davydov's approach will be detailed to explain the focus of this study.

2.2 Understanding of Equal Sign

The most problematic and misleading meaning of the equal sign is as a command to interpret the resultant, a perception rooted in arithmetical learning that generally does not pose an issue until algebraic reasoning is required. Perceiving the equal sign as a "do something signal" persists through elementary school and is even observed in early high school students, often leading to misunderstandings and errors in solving equations (Kieran, 1981, p. 317). Falkner et al. (1999) found more than 90% of students in primary school showed misconceptions in solving the equation: 8+4=? +5, and the percentage does not lower with aging. Misconceptions were specifically “finding a result for the left side of the equal sign”, or “adding up all numbers in the equation”. In another study (Stephens et al. 2013), higher percentages of correct responses to solving the equation “7+3=? +4” were observed in higher grades of primary school, however, the percentage was lower than 50%. Gürel and Okur (2018) reported that misconceptions of adding numbers on the left side or adding all numbers in the equation: 83+14=? +16 can be persistent in 7^{th} Graders with a percentage of 25%, while these misconceptions were not observed among 8^{th} graders, where algebra courses take place.

This misconception stems from students' conceptualization of the equal sign are rooted in students' conceptualization of equal signs as an indication of a process rather than a static relation (Sfard & Linchevski, 1994). Students often think that the left side of the equal sign represents a process to be carried out, while the right side merely the result, due to their arithmetical background where the equal sign signals the completion of calculations (Sfard & Linchevski, 1994). This interpretation is reinforced by the typical structure of arithmetical computations, where the equal sign is seen as the final step in a series of operations rather than a symbol of equivalence. As a result, students may overlook the importance of the equivalence of quantities on both sides of the equal sign, focusing instead on executing operations and obtaining results (Sfard & Linchevski, 1994).

Byrd et al. (2015) defined students' explanations, such as “something is equivalent to something else” or “balanced on both sides,” as the relational definition of the equal sign. Explanations such as “end of question” and “a symbol to let you know the answer is next” are not relational meanings of equal sign. Other specific non-relational meanings of the equal sign are arithmetic-specific, such as “it means when you add something, get the total” or “the number you add, subtract, divide, and multiply.” Arithmetic-specific interpretations are found to be more hindering than non-arithmetic-specific interpretations among non-relational understandings. This underscores how initiating the learning of equal signs in a solely arithmetic context can be detrimental to future algebraic learning.

Knuth et al. (2006) explained how understanding the equal sign is related to solving equations. They conducted a study with 6^{th}, 7^{th}, and 8^{th} graders and reported that all grades had a low relational understanding of the equal sign. They categorized students' understanding of the equal sign as either relational or non-relational. They found that students with a relational understanding of the equal sign performed significantly better in solving equations.

Recent studies also show that college students might have a non-relational understanding of equal signs which affects their algebraic abilities. Fyfe et al. (2020) categorized the understanding of the equal signs as relational, operational, and other non-relational types. Shockingly, they found that 1 out of 6 college students, whom they refer to as adults, held only the operational meaning of the equal sign and performed significantly lower than others in solving algebraic equations.

Mathews et al. (2012) categorized students' understanding of equal sign into four levels: rigid operational (operations on the left), flexible operational (operations on the right), basic relational (operations on both sides), and comparative relational (transformations applied, without solving). They found that students from Grades 2 to 6, with higher levels of understanding of the equal sign were more capable of solving simple algebraic questions with letters as variables and performed better on questions demanding advanced relational thinking about transformations and the preservation of equality.

Their categorization of understanding of the equal sign is compatible with the categories in another study. Stephens et al. (2013) used three categories of understanding equal sign in an assessment of 3^{rd}, 4^{th}, and 5^{th} graders' understanding, which are "operational", "relational-computational", and relational-structural (p. 174).

Operational understanding of equal signs can be associated with the "rigid operational" level in Mathews et al.'s (2012) studies, where students understand the equal sign as calculating the left side. Relational computational understanding can be associated with the "basic relational" level, where students perform calculations on both sides and indicate equality between them. Their reasoning is limited to operating and comparing, and they cannot focus on structural properties.

In the last type of structural understanding, students focus on structures and properties to evaluate equations and do not need to carry out operations (Stephens et al., 2013). This is similar to the "comparative relational" level that students can do transformations without actually solving operations to determine the correctness of equations such as "67 + 86 = 68 + 85" (p. 320).

Consistent with the literature, Stephens et al. (2013) also found that the operational understanding of equal sign is common among students, who mostly have difficulty seeing structures in arithmetical operations. However, they remarked that some tasks, such as "5 + 3 =__ + 3" evoke students to see structures in the operations. Hence, they concluded that making students investigate tasks that underline operational properties and structures can be beneficial prior to algebra education.

Using non-traditional formats in arithmetical problems was tested through an experimental study for their effects on developing students' understanding of equivalence at the age of 8 (McNeil et al., 2011). The study showed that using non-traditional formats improved students' understanding of the equal sign, equation encoding, and equation solving more effectively than traditional formats or no intervention.

In an earlier intervention study (Falkner et al., 1999), it took one and a half years for 1st and 2nd graders to develop a relational understanding of the equal sign through contextual problems and discussions on the correctness of non-traditional expressions of equations. The challenge was to shift students' interpretation from the operational meaning of the equal sign, which they might have carried over kindergarten. The authors observed an operational meaning in kindergarteners, even though they had no prior formal education on the equal sign. This highlights the importance of constructing a correct understanding of the equal sign as early as possible, as changing this conception becomes more difficult over time.

Lee and Pang (2023) outlined the difficulty of developing a relational understanding. They emphasized that even after students develop a relational understanding, they might depend on their operational understanding of unfamiliar questions. They defined this type of understanding as “simultaneous operational and relational” (SOR) understanding (p. 561). Discussions on non-traditional tasks did not help much; instead, this issue was more effectively addressed by using a pan balance to simulate equations.

A remarkable study on building a relational understanding of equal signs is conducted by Stephan et al. (2021). They started the intervention in kindergarten and continued into the first grade. Besides non-traditional tasks, they used balance scales in kindergarten to compare and illustrate equations with addition on both sides. Based on the studies of Falkner et al. (1999) and Lee and Pang (2023), starting intervention from kindergarten and using pan balances are effective strategies for developing a relational understanding of the equal sign. However, the illustration they used in the first grade requires knowledge of physics because balance is not solely associated with the weights or amounts of numbers but also with their distance from the center. They begin comparing unknown weights on balance in kindergarten, though the unknowns they compare are discrete and numerical. Hattikudur and Alibali (2010) also pointed out the importance of comparison tasks in understanding equality. In their experimental study, they showed that using greater and less than signs together with the equal sign helped better than using only equal signs in these tasks for a relational understanding.

Our aim extends this approach by having students compare unknowns in a weight balance context, where unknowns are continuous variables and non-numerical, as outlined in Davydov’s trajectory. We will then proceed with operations on unknowns, inclusive of arithmetic calculations, based on Davydov's deductive perspective.

2.3 Understanding of Quantity

Usually, quantity or quantitative reasoning is misleadingly associated solely with numbers and numerical calculations. "Quantities are attributes of objects or phenomena that are measurable; it is our capacity to measure them—whether we have carried out those measurements or not that makes them quantities" (Smith & Thompson, 2007, p. 101). Discrete quantities are represented through numbers, whereas continuous quantities such as area, volume, and weight can be represented through non-numerical symbols. (Ellis, 2011; Stavy & Babai, 2016).

The development of algebraic reasoning involves two distinct stages: firstly, progressing from numerical calculations to operations involving fixed unknowns, marking a shift from operational to structural algebraic thinking. Secondly, advancing from operations with unknowns to the use of variables represents the transition from structural to functional algebraic thinking (Sfard & Linchevski, 1994). By adopting Davydov's perspective we aim to make students operate by unknowns and focus on structures. This approach excludes numerical (operational) computations. In this study, the teaching of variables is not objected to but discussed through varieties of solutions.

In the Early Algebra Learning Progression (EALP) for Grades 3 to 7, researchers attempted to develop instructional strategies to foster functional thinking in early grades (Blanton et al., 2015). Building on insights from the GEAARR project, they observed and reported that students could grasp functional thinking earlier than anticipated, illustrated by tasks like relating dogs to the number of eyes and tails. From kindergarten to 5th grade, students demonstrated the ability to identify and interpret patterns in numerical data across different contexts, including pattern detection in number sequences, an area typically challenging for elementary students (Blanton & Kaput, 2004).

Within the EALP Project, starting in 3rd grade, students begin by interpreting functional relations between two numerical datasets derived from contextual problems through a learning trajectory. This trajectory starts with the relational understanding of equality, progresses through additive operational properties, and culminates in modeling and solving linear equations involving unknowns and variables. The curriculum further integrates multiplication in equation-solving, facilitating a developmental progression from operational to structural understanding, emphasizing the use of unknowns. Ultimately, the focus shifts towards fostering a functional understanding through variability and variable notation, underpinned by a generalized arithmetic perspective for interpreting variables (Blanton et al., 2015).

Blanton et al. (2015) empirically developed a learning trajectory using a design research perspective to cultivate functional thinking among 1st-grade students by generalizing relationships. Through contextual problems, students engaged in exploring additive function types, such as $y = x$, $y = x + x$, $y = x + x + x + x$, $y = x + 1$, $y = x + 2$, $y = x + 3$, $y = x + x + 1$. To elucidate how Grade 1 students generalize functional relationships between two quantities, researchers did not rely on existing frameworks designed for older students, aiming to avoid constraining their perceptions of data. They observed that students demonstrated the capability to develop either recursive or functional thinking when generalizing patterns. This ability to shift between a specific or more generalized view of relationships stems from their proficiency in using notational interpretations (Blanton et al., 2015, p. 542).

Brizuela et al. (2015) reported on how first-grade students can use variable notation to interpret relations by focusing on interviews with four students. They found that even young students can use letter notations to explain relationships between quantities and act on variable notations. During their progression, students displayed some difficulties with notational interpretations. They might use letters to represent objects or indeterminate quantities, consider the letters' ordinality, or avoid using letters with numbers in expressions. The authors argued that using letter notation in Grade 1 might not make the transition smoother, but it is promising in discussing variability and understanding functions. In this study, we decided to introduce letter notation by making it easier through pre-given letters on objects, using letters familiar to the students.

The trajectory of learning variables in the first grade is explained in relation to the use of letter notation to interpret the unknown (Blanton et al., 2017). Evolving from representing an object to representing a variable, students might go through 6 stages:

- Level 1: Pre-variable / pre-symbolic
- Level 2: Pre-variable / Letters as labels or as representing objects
- Level 3: Letters representing variables with fixed, deterministic values
- Level 4: Letters representing variables as arbitrarily chosen values
- Level 5: Letters representing variables that are varying unknowns
- Level 6: Letters representing variables as mathematical objects

At the level of 6, students not only represent variables using letter notation, but they also carry out operations on the constructed algebraic expressions. This ability allows them to treat variables as algebraic objects that can be manipulated and acted upon (Blanton et al., 2017).

This trajectory was revised by Ventura et al. (2021) at the kindergarten level, and in their study, they analyzed data from 8 kindergarteners and 8 Grade 1 students, reporting their learning levels on variables. Levels 2 and 3 are not observed among kindergarten students in the study. Kindergarteners showed an understanding of variables up to Level 5, where they could use letters as indeterminate quantities and quantities can vary. However, they could not operate on the constructed notations or create a meaningful expression representing relationships. Therefore, these kindergarteners often struggled to define new letter notations or connect different letter notations in a single. Moreover, they often struggled to define new letter notations or connect different letter notations in a single expression. Level 6 understanding was also rare among Grade 1 students; only 2 out of 8 students demonstrated the ability to operate with letter notations, and one of these students was unable to maintain this understanding in the post-interview.

Following Davydov's trajectory, operating with variables is beyond our scope; instead, we focus on interpreting determinate fixed quantities with symbols. These trajectories emphasize pattern detection among number sets to determine relations between quantities, conducted through arithmetical calculations. In each subject, quantities are numerical, arithmetic is known, and expressions are built on them. Their perspective is generalized arithmetic for algebra, where students use letter notations to generalize arithmetical relations.

In this study, we aim to teach algebraic operations to which arithmetic will apply, adopting a deductive perspective following Davydov. We will use continuous types of variables for quantities rather than discrete numerals for more inclusive learning about quantity.

Two studies have explored the use of continuous variables with young children. One study aimed to enhance sophistication in mass measurement (Cheeseman et al., 2014). This study involved Grade 1 and Grade 2 students and developed a learning trajectory. Another study focused on length measurement with Kindergarteners, Grade 1 and Grade 2 students (Sarama et al., 2021). Both studies began with non-discrete comparisons of given objects based on weight or length. The goal in both was to teach unitization for measuring continuous variables, specifically length, and weight. Initially, students constructed non-standardized units and subsequently learned to use standard units in measurement. In both studies, they were not expected to perform operations with continuous variables.

Another study inspired by Davydov is the Measure-Up Project (Dougherty, 2017), which focused on comparing continuous variables such as area, volume, and length. This project examined how to make unequal objects equal and determine the amount that can be added or subtracted to achieve equality. The concept of difference was central, and there were no operations with different quantities independent of this concept. The discussion of “how to make equal” immediately led to the need for unitization. Operations were conducted with these units and their relation to a whole, represented in equations such as “E + E + E + E = W, W = 3E + E, W – 4E = 0” to conclude multiplicative relations such as 4E=W (Dougherty & Venenciano, 2007, p. 454). Questions of the form A±B=C±D, as seen in Davydov’s trajectory during discussions of “equal, not equal, equal again” before unitization, were not explicitly included in the Measure-Up Project. The project’s focus appeared to be on measuring continuous quantities and creating unitization. This study will focus on teaching operations on definite and continuous quantities.

2.4 Early Algebra Education

Early algebra education has been proposed as a solution to the challenges faced in traditional algebra education. However, early algebra does not teach algebraic concepts earlier (Carraher et al., 2008). Problems that arise in higher grades will not be resolved by introducing them earlier. They argued that early algebra should not overload existing curricula but it should connect algebraic topics to existing ones through contextual problems and gradual symbolization instead. They criticized the idea of limiting instruction based on developmental readiness and maturation. To improve students' functional thinking in earlier grades, specifically in Grade 3, they used contextual problems where students construct numerical data sets and investigate relations between them. They found that there is a cognitive leap that hinders students from thinking functionally, as they tend to focus on specific numbers instead. Their study suggested that it is essential to investigate the appropriate conditions for effective early algebra education.

A common approach in teaching algebra at earlier grades is through patterns. As emphasized in Carraher et al.'s (2008), functional thinking requires the detection of patterns among number sets. Consequently, patterning has gained attention in early algebra studies, and activities focusing on constructing patterns are included in curricula as early as possible. Even for students as young as 3-4-year-old, activities are designed to help them iterate patterns using geometric shapes (Lee et al., 2016; Rittle-Johnson et al., 2015). Lee et al. (2016) suggest several pre-algebra activities to support future algebraic thinking. These activities include matching and sorting, identifying, following, and creating patterns, comparison of concrete materials based on qualitative attributes, and Venn diagrams to classify objects.

Moyer et al. (2004) provided empirical evidence that the U.S. elementary school curriculum supports functional thinking through patterns and relations through K-5 grade levels. In their study, they emphasized that at the kindergarten level, it is crucial to include not only repeating but also growing patterns to ensure a comprehensive understanding of functions in the future. Hence, the researchers recommended constructing new knowledge on students' existing understanding and cautioned against rushing into symbolization in early algebra education.

In addition to patterns and generalizations for fostering functional thinking, early algebra studies also emphasize teaching arithmetic from an algebraic perspective to ease the transition from arithmetic to algebra. This perspective views algebra as "generalized arithmetic," helping students bridge the gap between these two domains smoothly (Lee & Wheeler, 1989, p. 41). Understanding the laws of arithmetic supports early algebra by focusing on the generalizations and properties of operations (Schifter et al., 2008). Slavit (1998) pointed out the importance of operational sense in algebraic thinking, defining it as not only the ability to perform operations but also understanding their underlying structure, use, relationships with other operations and structures, and potential for generalizations.

Addition and subtraction can be taught through "the set model, the number line model, and the function machine model" (LeBlanc et al., 1976, p. 3). A functional understanding of operations is essential in understanding arithmetic from an algebraic perspective. Particularly, an algebraic understanding of addition requires a relational understanding between input and output sets rather than the joining of amounts (Carraher et al., 2000).

Four activities that are common in both arithmetic and algebra, aiming to connect these two areas were defined by Russell et at (2011):

- understanding the behavior of the operations,
- generalizing and justifying,
- extending the number system, and
- using notation with meaning. (p. 44)

For example, to understand the behavior of operations they had students compare different operations to each other. Students focused on the similarities and differences in behaviors of addition and subtraction (Russell et al., 2011). They argued that including these four factors in the curriculum not only supports early algebra but it is also essential for arithmetic education.

Linchevski (1995) argues that pre-algebra activities can take place even in early arithmetic and defines a pre-algebra course as “algebra with numbers and arithmetic with letters” looking from a generalized perspective of arithmetic calculations, and using letter notations in expressions for generalizations (p. 113).

2.5 Studies Utilizing Davydov's Perspective in Early Algebra Education

Among early algebra education studies, Davydov’s curriculum offers a unique perspective, proposing the introduction of algebra even earlier than arithmetic (Schmittau, 2005). Developed between the 1950s and 1960s for grades 1-3, Davydov’s curriculum is based on Vygotskian perspectives (Eriksson & Jansson, 2017). It focuses on structured scientific development rather than building on students’ prior knowledge, as in the constructivist perspective (Schmittau, 2004). Davydov (1982) emphasized understanding principles over solving problems in mathematical thinking, and he opposed teaching numbers and operations by simply focusing on their procedural aspects. Instead, he stressed the importance of understanding the “origins of numbers and arithmetical operations” from a theoretical perspective (p. 225). Davydov's algebraic instruction begins in 1st grade with comparing quantities and representing relations with letter notations. It then continues with addition and subtraction operations until it defines the number system from a multiplicative perspective of unit counting. Davydov’s instruction begins with exploring relationships between continuous quantities such as length, area, volume, and weight, rather than relationships between numerical data sets. Operations are conducted on these continuous quantities algebraically, and numbers are subsequently constructed based on the multiplicative relationships between quantities.

Davydov’s curriculum was tested in a US context in a three-year-long study (Schmittau, 2004). Schmittau and Morris (2004) explained that while children in the US had pre-algebraic experiences that were numerical, Russian children studying Davydov’s curriculum had pre-numerical experiences that were algebraic. They argued that in Davydov’s perspective, arithmetical/numerical operations were concrete applications of algebraic operations. However, this approach might be criticized for its view that numbers in Davydov’s approach appear primarily as symbolic representations of ratios between definite quantities. The study found that Davydov's approach is powerful for developing students' algebraic reasoning even in elementary school, though it may delay the learning of arithmetic.

Another study implementing the Davydov curriculum is the Measure-Up project (Dougherty & Venenciano, 2007; Okazaki et al., 2006). For Grade 1, the instruction centers on constructing quantities and numbers through measurement and unitization. Students use letter notation for quantities in comparisons. They interpret the relation between quantities with >, <, = and use properties such as transitivity to deduce new relations in equality and inequality concepts. Then, they discuss how to equalize quantities by focusing on the difference in addition and subtraction. They subsequently construct numbers through unitization in comparisons and express equations with additive and multiplicative relations between two quantities. Around measuring concepts, it is shown that students in Grade 1 can compare quantities, reason by transitivity, create equalities with addition and subtraction, and interpret multiplicative relations between quantities by repetitive addition following Davydov’s trajectory.

Other studies have also been inspired by Davydov’s approach. Eriksson and Jansson (2017) reported three key algebraic activities designed in a pilot project for 7-year-old students. In the first activity, students learn =, >, < signs by comparing dots on two dice. This activity focuses on numerical and non-algebraic comparisons but only underlies the relation between two sides. The second task focuses on unitization in the comparison of volumes and includes discrete counting in relations. The third task involves expressing equations with one-side addition using wooden rods. This activity is similar to the paper strips activity in Davydov’s trajectory, where students are asked to cut or paste strips to make them equal. However, in Davydov’s activity quantity is continuously manipulable, and the student determines how much to cut or paste. In Eriksson and Jansson’s (2017) activity, adding quantities means combining them rather than increasing the quantity.

Schmittau and Morris (2004) reported that differences in learning theories and cultural contexts present challenges in adapting to Davydov’s perspective. Sidneva (2020) examined the implementation of Davydov’s curriculum and assumed that teacher experience and school readiness would impact its adaptation. However, the researcher found no differences in student success based on motivation or teacher backgrounds. Besides, Mellone et al. (2021) also pointed out the importance of teachers’ perspectives and flexibility in adapting the Davydov-Elkonian curriculum, noting that a rigid curriculum could hinder successful adaptation.

Gerhard (2009) addressed the difficulty of curriculum change in 1st year as it will get an adverse reaction from teachers and parents, and proposed using Measure-Up project activities in grades 1 to 5 to meet the needs of low-achieving students and improve their arithmetic abilities, even if they had prior arithmetic instruction (the reverse of Davydov’s approach). The study showed that some low-achieving students, after participating in Measure-Up activities, began to use algebraic solutions to arithmetic problems. This led to the conclusion that algebra is not inherently more difficult than arithmetic and can help students approach arithmetic problems with algebraic reasoning. It also supports the idea that algebraic reasoning underlies arithmetic operations. Another issue in adapting Davydov's curriculum at the primary level is the use of letter notation. This is also pointed out by the researcher that letter representation can be problematic, as it may be unclear whether a letter represents a quantity or an object.

To sum up, all studies inspired by Davydov's perspective focus on Grade 1, involving students aged 6 and 7. No study specifically addresses the adaptation of the perspective of algebra-before-arithmetic perspective for kindergarten-level education. As mentioned earlier, even kindergarten students may have arithmetic-specific reasoning before formal education, which can hinder algebraic thinking (Falkner et al., 1999). There is a need to adapt Davydov's curriculum at the kindergarten level to introduce algebraic thinking about quantities and equalities from the very beginning, potentially alleviating the issue of delaying arithmetic education in Grade 1.

By adopting a design research perspective, we can adjust learning trajectories based on students' needs and develop ready-made activities to facilitate implementation for teachers. Following Davydov's perspective integrating action-based activities into the kindergarten curriculum is also more feasible, as it offers flexibility and provides ample time and space for using mathematical toys with informal symbolization.

Studies utilizing Davydov's approach are promising for an adaptation at the kindergarten level. However, maturation may create problems in understanding certain topics such as transitivity which has very low development among students below age 8 (Smedslund, 1963), and is highly associated with understanding the conservation of amount based on Piaget's Theory (Owens & Steffe, 1972). A design-based research perspective in the adaptation procedure will facilitate the refinement and optimization of strategies for addressing specific problems.

2.6 Studies on Teaching and Learning Mathematics in Preschool and Kindergarten

Preschool and kindergarten mathematics education has gained importance due to its potent effects on future mathematical learning (Claessens & Engel, 2013; Jordan et al., 2012). Duncan et al. (2007) reported meta-analysis results of six longitudinal studies, finding early mathematical skills as the most predictive effect of school readiness for later achievement. Nguyen et al. (2016) reported that advanced counting skills in pre-K skills are most potent in 5th-grade mathematics achievement. Basic counting, patterning, geometry, measurement, and data skills are not significantly related to fifth-grade mathematics achievement. Patterning abilities might be correlated with higher grades where algebraic learning takes place. Similarly, Watts et al. (2018) investigated the effects of preschool math achievement on a late elementary school. They concluded that mathematical readiness is effective in fifth grade but not as much in fourth-grade achievement.

Kindergarten mathematics is dense and has gained importance as first-grade mathematics (Bassok et al., 2014). Intervening with mathematics instruction at the preschool level can significantly enhance students' learning progress throughout their elementary school years (Dumas et al., 2019). Hence, there have been many interventional studies on teaching mathematics at the preschool or kindergarten level in recent years. In their meta-analysis, Wang et al. (2016) concluded that interventional studies can be highly effective if they focus on a limited subject with an extended amount of time, suitable in a grade-level environment.

Recent interventional studies on kindergarten and preschool have been mainly interested solely in numbers and operations (Arnold et al., 2002; Curtis et al., 2009; Jordan et al., 2012; Kidd et al., 2008; Monahan, 2007; Ramani & Siegler, 2008; Sood, 2009; Tarim, 2009; Young-Loveridge, 2004), whereas some studies also included algebra (Chard et al., 2008; Clements et al., 2011; Klein et al., 2008; Papic et al., 2011; Pagani et al., 2006; Starkey et al., 2004), geometry (Aunio, 2005; Clements et al., 2011; Chard et al., 2008; Clarke et al., 2011; Pagani et al., 2006;

Sophian, 2004; Starkey et al., 2004), and measurement (Aunio et al., 2005; Char et al., 2008; Clarke et al., 2011; Clements & Sarama, 2008; Fuchs et al., 2001; Pagani et al., 2006; Sophian, 2004; Starkey et al., 2004) tasks besides numbers and operations (Wang et al., 2016). Casey et al. (2008) focused on only spatial abilities and geometry in their intervention. Studies about teaching algebra at this grade level mainly focus on patterns (Chard et al., 2008; Clements et al., 2011; Klein et al., 2008; Papic et al., 2011; Rittle-Johnson et al., 2015; Starkey et al., 2004).

There are also studies related to algebraic tasks other than shape patterning. Pagani et al. (2016) studied the grouping of objects, fractions, and number lines in addition to counting. Besides, Pasnac (2006) studied grouping, the extension of series, and oddity in addition to numeracy at preschool age. Rather than shape patterning or repetition and extension of geometric patterns, Ventura et al. (2021) studied the generalization of numerical patterns for variable notation at the kindergarten level. In this study, kindergarten students could notate unknown quantities with letters and recognize patterns in number sequences but could not operate on unknowns. Khosroshahi and Asghari (2013) showed that kindergarteners can reason algebraically in similar numerical tasks, including unknowns and operations on them, without using algebraic symbols. They discussed the necessity of formal notations at this early age to reason algebraically.

The analysis of studies on teaching and learning mathematics in preschool and kindergarten shows that none comprise a relational understanding of non-numerical quantities and operations on unknowns with algebraic properties, as in Davydov's approach.

2.7 Theoretical Background: APOS Theory

APOS is a constructivist theory developed by the Research in Undergraduate Mathematics Education Community (RUMEC) (Arnon et al., 2014). They aimed to improve Piaget's work on reflective abstraction to explain students' learning in postsecondary mathematics (Dubinsky & McDonald, 2001). They explained changes in students' mental constructions while solving problems with the assimilation and accommodation of schemas (Asiala et al., 1996).

They associated actions with Piaget's active schemas, processes to operations, and defined objects as the encapsulated processes on which further actions and processes can be taken. Schema consists of all of these procedures, which is related to Piaget's schemata or Tall and Vinner's concept images (Asiala et al, 1996). Assimilation and accommodation regulations are not explained through these stages in the figure, but they are related to the idea of generalization in APOS Theory (Arnon et al., 2014).

> "Sometimes new Actions, Processes, or Objects can be assimilated to a previously constructed Schema by establishing new relations among the components of the Schema. In other situations, a Schema may be related to one or more different Schemas that lead to the construction of a new, more extensive Schema." (Arnon et al., 2014, p111).

Actions are taken on previous mathematical concepts, then they are interiorized into processes. Processes are encapsulated into mathematical objects, which can be de-encapsulated into processes (see Figure 2.1). Schemas are the organization of mental constructions; actions, processes, and objects (Asiala et al. 1996). Mental constructions have a "circular feedback system" (Dubinsky, 1991, p. 106). However, there is no strict linear improvement through stages from action to process to objects in the construction of schemas (Arnon et al., 2014).

The action stage is the first step and it plays an essential role in the development of the other stages (Arnon et al., 2014). It is the stage where new mathematical knowledge is retrieved from external stimulus, and performed by the student by the guidance of the external instructions. In this stage, students can perform operations step-by-step, by reminding his/her self or by external guidance (Dubinsky & McDonald, 2001).

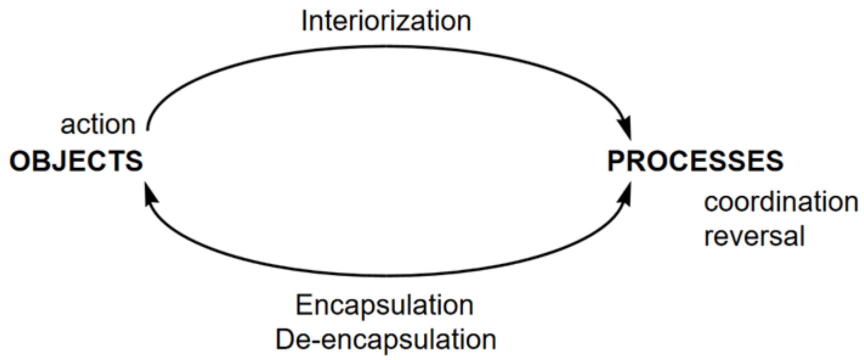

Figure 2.1. Constructions for Mathematical Knowledge Through APOS Theory (Asiala et al., 1996, p. 6)

By the repetition of the actions or reflecting on them (Asiala et al., 1996), they are interiorized or coordinated into processes (Arnon et al., 2014). In the process stage, students perform the same actions (Asiala et al., 1986). As Arnon et al. (2014) stated, "In particular, Processes are interiorized Actions" (p.20). However, by becoming fluent in those actions, they do not need to perform them by reminding themselves of all the steps internally (Arnon et al., 2014), nor do they depend on external instructions anymore (Asiala et al., 1986). It is the same action, but students do not perform the action in the same way and can think about the action. Hence, he/she can think about the reverse of the procedure and compose it with other internalized processes (Dubinsky & McDonald, 2001).

When the student encapsulates the process, that means seeing the process as a totality, that process becomes an object (Dubinsky & McDonald, 2001). By reflecting on the processes, the student recognizes operations can act on the processes, and perform those operations. This reflection empowers the abstraction of the subject (Dubinsky, 1991). This means the student uses the encapsulated totality as an object on which he/she can act (Asiala et al., 1986). Arnon et al. explain the role of actions in the encapsulation of objects as; "In particular, Processes are interiorized Actions, and mental Objects arise because of the application of Actions. New Actions lead to the development of higher-order structures. For instance, in the case of functions, performing operations on them spurs their encapsulation as objects." (Arnon et al., 2014, p.20). Dynamic structures of processes become static structures in objects (Arnon et al., 2014), and those structures can be de-encapsulated into the processes in manipulations (Asiala et al., 1986). Encapsulation of processes into objects is not an easy procedure (Asiala et al, 1986; Arnon et al., 2014).

Schema is the composition of processes and objects with their connections to each other. (Asiala et al., 1986). Concept images are related to mathematical structures (Tall & Vinner, 1981) while schemas describe mental structures in a concept (Arnon et al., 2014). Schemas act as static objects in their connections to higher-level schemas (Asiala et al., 1986). Comparison between schemas of students may help explain how students develop certain mental construction and students' achievement on a topic can be tested through these schemas. Schemas help to develop genetic decompositions to describe how certain mathematical learnings are acquired in detail (Dubinsky & McDonald, 2001).

Instructions can be constructed based on preliminary hypothetical genetic decompositions, which are detailed descriptions of learning through schemas of mental constructions (Arnon et al., 2014; Dubinsky & McDonald, 2001). In this study, we use Davydov's trajectory as a hypothetical trajectory and combine the objectives of this trajectory with hypothesized APOS levels to determine hypothesized genetic decomposition on learning equations.

Even though APOS Theory was developed to understand how college-level mathematics is learned, it has also been applied in studies at the elementary level. Arnon et al. (2001) investigated the learning of equivalence sets in the concept of fractions with 5th graders using APOS Theory. They highlighted the cognitive difference between Piaget's formal and concrete operational stages and developed their instruction to start with actions involving concrete objects (Arnon et al., 2014). They illustrated the difference between the stages of APOS at post-secondary and elementary levels with the following figures, showing that while the actions are taken on concrete objects at the elementary level, the resultant objects are abstract in both cases.

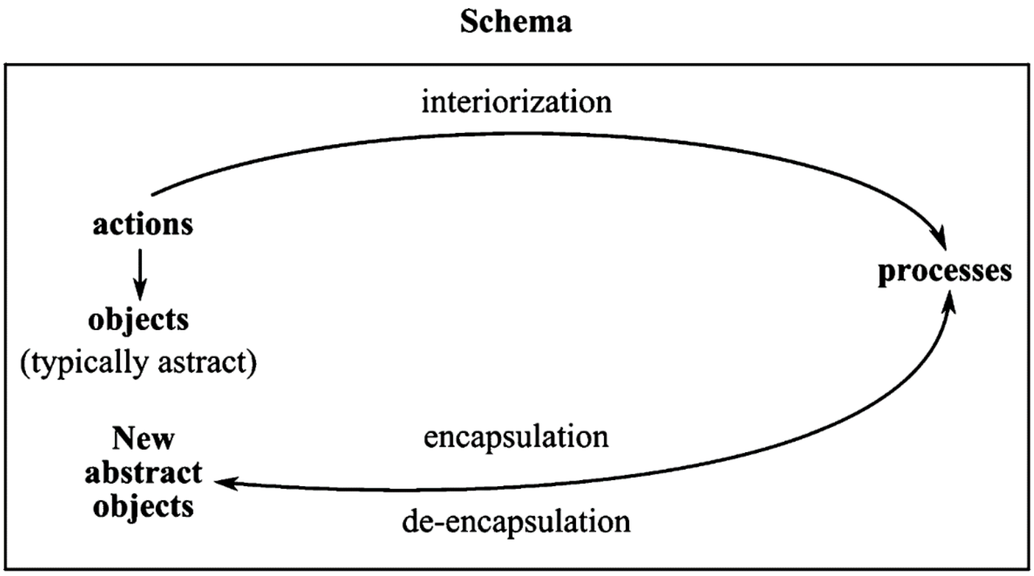

Figure 2.2. APOS for Postsecondary Students (Arnon et al., 2014, p. 153)

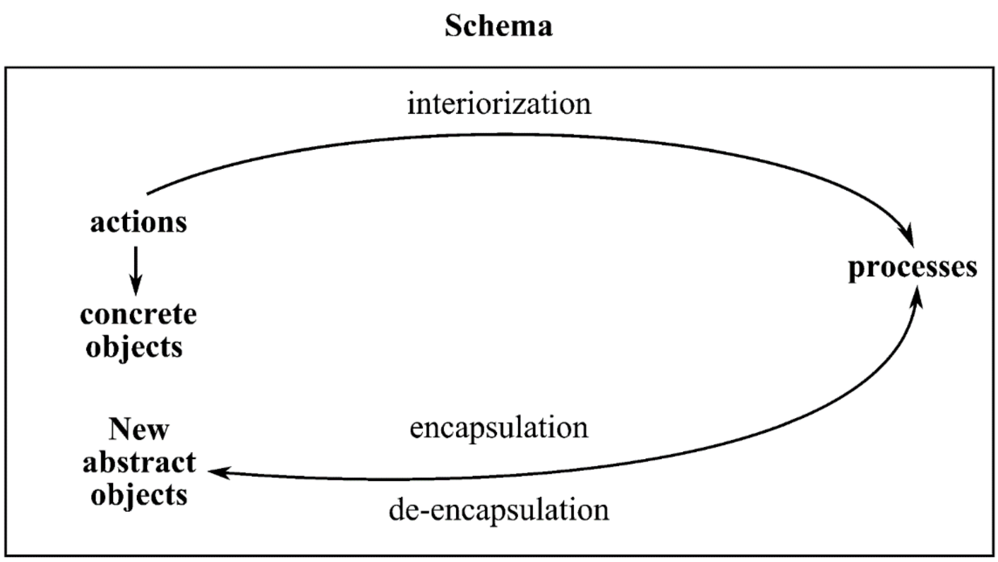

Figure 2.3. APOS for Elementary School Students (Arnon et al., 2014, p. 154)

Arnon et al. (2014) concluded that APOS Theory enhances the development of a meaningful path for learning when adapting higher-level mathematics subjects to the elementary school level. In this study, we will also advance APOS stages to ensure students' learning of algebraic topics through investigations with concrete materials at the kindergarten level. Even though Arnon et al.' study enlightens a starting point for initiating actions at the elementary level, APOS Theory is usually used in investigations of algebra learning at secondary and graduate levels. (Şefik et al., 2021). The implication of APOS Theory at this very early age (kindergarten) will contribute to the APOS Theory of learning algebra.

CHAPTER 3

METHODOLOGY

The purpose of this study is to adapt the Davydov-Elkonian Mathematics Grade-1 curriculum to the kindergarten level. For this adaptation, a design-based study is conducted to develop an instructional sequence and observe its effects on kindergarten students' algebraic learning based on APOS Theory.

In this chapter, the design of this study, phases of conducted design-based research, context, implementation, and analysis procedures are explained in detail. Trustworthiness is discussed at the end of the chapter.

3.1 Design of the Study: Design-Based Research

Design-based research or design research was initiated by Brown in 1992 with the term design experiment, to develop complex classroom interventions to test and refine developed designs by formative assessment procedures (Brown, 1992; Collins et al., 2004). Adopting design perspectives in educational studies, prototypical instructional designs are systematically tested and refined iteratively, and theoretical explanations for learning are signified. The outcome as developed design and explanation of "why the design works" outlines practical and theoretical aspects in design-based research (Cobb et al., 2003, p. 9). Gravemeijer and Cobb (2006) explain that "the purpose of design experiments is to develop theories about both the process of learning and the means designed to support that learning." (p. 18). This explanation highlights the contribution of design-based research to theory in two ways: by defining processes of learning and design principles that support learning, and by developing a practical instructional sequence. Design research is; interventionist in a real context, iterative in cyclic procedures of design, evaluation, and revision, process-oriented in interpretations, practicality-oriented in designs, theory-oriented in the construction of designs, and in contribution to theory by results (Van den Akker, et al., 2006).

These distinctive characteristics of design-based research make it appropriate for designing this study. Cognitive abilities may play a big role at the kindergarten level and adaptation procedures must be constructed and assessed carefully. One-shot design and testing may not result in success as we expected. However, our aim is to develop effective and practical instruction. To satisfy and observe student learning, each step taken must be assessed and further steps must be built upon it. Design-based research will help construct a working trajectory and practical instruction due to its cycles of assessment and refinement procedures. The effectiveness of the instruction will be tested in a natural classroom environment not in laboratory settings, based on Desing-based study principles, which will confirm practicality (Brown, 1992; Cobb et al, 2003; Gravemeijer & Cobb, 2006). Most importantly, guided by theory, students' algebraic learning will be monitored and explained in each step through design-based research procedures.

Advancing the design-based research, the study will produce both theoretical and practical outcomes (Gravemeijer & Cobb, 2006). These include an adapted trajectory for kindergarten-level algebra from Davydov’s perspective and effective activities that support this trajectory. The first outcome contributes to early algebra education theory at the kindergarten level, while the second provides practical benefits for curricular improvements. These outcomes are directly answering our main research questions:

- What is an adapted learning trajectory for supporting kindergarten students' algebraic understanding of equations?
- What are the effective and practical activities for supporting kindergarten students' algebraic understanding of equations?

Gravemeijer and Cobb (2006) define design research by discussing it in three phases: (1) preparing for the experiment, (2) experimenting in the classroom, and (3) conducting retrospective analyses (p.19). A local instructional theory is constructed in the first phase, which will be refined and developed further in the second phase through "cycles of design and analysis" (Gravemeijer & Cobb, p.24). In the last phase, after development and revision through experimentation, the whole data is analyzed to contribute to local instructional theory. One of the most common models of design-based research is Bannan's model which determines phases of design-based research as Informed Exploration, Enactment, Evaluation: Local Impact, and Evaluation: Broader Impact (Bannan, 2009). Reeves (2006) used a four-staged design model to construct an educational technology; analysis of practical problems, development of solutions, iterative cycles of testing and refinement, and reflection to produce "design principles" and enhance solutions. In all models, a preparation stage exists where problems and possible solutions are investigated, which will be tested and refined in implementation cycles and evaluated at the last stage to reflect on the theory (local or broader).

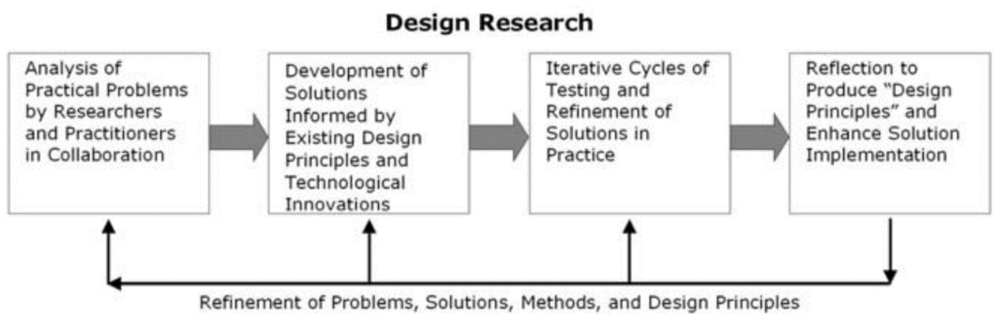

Figure 3.1. Design-Based Research Model for Educational Technology Research (Reeves, 2006, p. 60)

We adopted Reeves's (2006) four-stage model for the design of the study. Analysis of the problems step is adopted as pre-investigations. For the development of the solutions phase, we constructed a hypothetical learning trajectory (HLT) in the study.

The “iterative cycles” stage of Reeve’s model corresponds to the refinement of the HLT by implementation and testing. Finally, reflections on the construction of “design principles” are represented in the further investigation step where the student’s learning progress (theoretical findings) and criteria to support learning progress (design principles) are investigated in depth. This final step is also called retrospective analysis by Gravemeijer and Cobb (2006) for the contributions to local instructional theory, which also supports our aim for developing an algebraic learning trajectory at the kindergarten level. A hypothetical learning trajectory is a construct composed of objectives and defined activities for supporting these objectives, which are tested and refined in the procedure. HLT is hypothetical in the sense it is based on the “prediction of the path which learning might proceed” (Simon, 1995, p. 135).

The following figure illustrates the design of this study in four phases: pre-investigation, construction of hypothetical learning trajectory (HLT), implementation, assessment, refining of HLT, and further theoretical analysis. Arrows in the figure indicate the sequence in the model. Cycles of piloting, implementation, and revision of the HLT are shown with rounded arrows. These phases will be explained in detail in further sections.

Pre-investigation	Construction of first HLT	Implementation and revision of HLT	Theoretical analysis
Literature Review Davydov's trajectory by pilot Piaget's conservarion Test Pre-interview	Assessment of Davydov's trajectory based on APOS Theory (Adopting Davydov's) Determining objectives based on APOS levels (genetic decomposition) Designing activities based on Bruner's modes of representation	Pilot Lecture 1,2...,20 Analysis Revise HLT Repeat ...	Theoretical explanation of students' learning progress based on APOS Design Principles of activities supporting

Figure 3.2. Stages in the Design Model of the Study

3.2 Pre-investigations

Pre-investigation includes literature review, exploration of Grade-1 Mathematics Book with a 5-year-old kindergarten student (Ecem, a pseudonym), investigation of kindergarten students' readiness through Piaget's conservation test, and pre-interview with kindergarten students in real-life knowledge of volume, weight, and length contexts. All of these pre-investigations helped to construct the first HLT.

Firstly, the first 72 pages (up to the section on numbers) of the Grade-1 Mathematics Book (Davydov et al., 1995), based on the Davydov-Elkonion Curriculum, were reviewed with kindergarten student Ecem through one-to-one interventions to observe a kindergartener's perspective and abilities in this context. She had remarkable success in completing tasks in the book, except she had some difficulty in grouping and area problems. Her struggles and attitudes provided initial insights. Literature plays an important role in the explanation of her struggles. At this age, cognitive abilities may be important for capabilities. According to Piaget's conservation Theory, before age 7, most children have no thought of the conservation of amounts when they are partitioned or dispositioned. Davydov's book was full of part-whole problems, volumes of cups problems, and most challenging area problems. Even though Ecem had conservation of amount, she had some difficulty in area tasks. Hence, Piaget's conservation test was decided to be conducted on kindergarten students we targeted in classroom implementation before we constructed trajectory and activities.

While the literature on Davydov-inspired studies, non-numerical mathematical studies at the kindergarten level, and algebra and arithmetics education in earlier grades helped structure our instructional design, Davydov's trajectory and context of implementation and students' prior knowledge dominated structure. The following sections will detail how pre-tests, Piaget's conservation Test, and pre-interviews as pre-investigations contributed to decisions taken to construct the first HLT. Then construction of HLT and adoption from Davydov's trajectory will be given in the second phase of research design under the heading of construction of the first HLT.

3.2.1 Piaget’s Conservation Test

Our intention was not to improve or force students’ cognition on conservation with our activities. We decided to assess their readiness for these activities by typical Piaget’s conservation test questions; the same number of coins on two different length rows, sharing a chocolate bar, and pouring liquid from one cup to another. These tasks are open to discussion; however, they were internally consistent. A student gives the correct answer to all or gives an incorrect answer to all.

The task of pouring liquid from one cup to another was particularly important to us and was directly incorporated into our activities focused on the concept of equality. To effectively teach the principle of balance, it was essential to include not only weight but also the variable of volume. This ensures a comprehensive understanding of equality in different contexts. Additionally, area-related questions were closely connected to the concept of dividing chocolate bars, which helped illustrate practical applications of area measurement in sharing. On the other hand, counting coins was not relevant to our activities because it involves discrete measurement rather than continuous variables, which are central to our focus on balance and equality.

We assigned these three tasks to our participants, comprising 10 students in a kindergarten classroom in a public school. Only two of ten demonstrated that they had reached the level of Piaget’s conservation of amount. The remaining eight clearly showed that they do not understand the amount is preserved when it is partitioned or displaced.

Based on the results of the pre-investigation on students' readiness to understand the conservation of amount, we modified all activities. We eliminated the activities on the concept of area. It was also difficult to interpret equality in certain areas. For the volume of cups, we decided to use an instrument to see equivalence. For weight, we used a balance scale as an instrument, and for volume, we used identical cylindrical transparent cups. It worked like balance scales as a measurement instrument for observing equality and interpretation of equality can be just placing equal sign in between. As a result, the measurement of the volume of cups turned out to be the measurement of height on identical cylinders.

Through our work with Ecem and two additional pilot students, as well as pre-interview observations and feedback from kindergarten teachers, we found that discussing variables such as volume, area, size, height, length, and width would be challenging and require additional effort. Furthermore, focusing on how to compare these variables correctly or learning to discriminate and name them could divert the students' attention away from the core discussions on equality. This study aims to focus on one concept at a time, following expert opinion, with equality as a major theme. Pre-interview results showed that students can interpret comparisons of objects based on multiple attributes. Therefore, we could implement our activities relying on students’ ability to interpret different variables/attributes. Building on their previous knowledge, we can focus on multiple types of variables/attributes and their equality.

3.2.2 Pre-interview

A pre-interview was conducted before implementation to assess students’ familiarity with algebraic signs, different attributes, equality, and operations in the context of weight. The questions were contextual and may have had an instructional role in addition to assessing their prior knowledge. Some students gained new insights during the implementation of instruction or the post-interview. The last question about balance scales was particularly enlightening for instructional purposes. Questioning their thoughts served as an inquiry method, helping them realize or learn new concepts. These context-based questions were kept brief, lasting 5-10 minutes, with no guidance toward correct answers.

The results of the pre-interview indicated that they were familiar with algebraic signs, some were aware of the equality sign (n=4), and very few knew the plus sign (n=2). One guessed an unequal sign, related to an equal sign. They all can considered different attributes of objects for comparison without or with little guidance.

From the results of the pre-interview, we concluded that students were capable of interpreting different attributes/variables depending on the activities. Balance scales were relevant for them from playgrounds, but they need more theoretical discussion on equality and operations, as well as practical observations using the scales. While addition correctly influences balance movement, subtraction may cause inconsistencies, as seen in Ecem’s case. Not only paperwork, as in Davydov’s book (Davydov et al., 1995), but also extensive observation and action are necessary.

The problem with subtraction in the context of equality is not just about predicting movement but also about preserving equality when subtracting or identifying the larger side after subtraction, which proved challenging for most students. The algebraic interpretation of equality and operations was unfamiliar to all students.

In addition to contributing to instruction, the results from the pre-interview, post-interview, and in-class implementation can be used to assess students’ individual progress or the overall success of the classroom in the program. See the Findings Chapter for detailed results of the pre-interview.

3.3 Construction of First HLT

The first HLT was adopted based on Davdov’s trajectory. In constructing the HLT, we followed these steps: First, Davydov’s trajectory was outlined. Second, the objectives for this study were aligned with Davydov’s trajectory. Third, the activities based on these objectives were constructed.

In this dissertation, APOS Theory is used as the algebraic learning framework, to guide the construction and revision of the trajectory, as well as to observe students’ algebraic understanding during implementation. Therefore, Davydov’s trajectory and the construction of the first HLT will be explained based on this theory.

3.3.1 Davydov's Trajectory on Teaching Equations Through Quantity, Equality, and Operations

The aim of this study is to teach kindergarten students algebraic equations adopting Davydov's trajectory. There are three main components of equations to focus on: variables, equality, and operations. These three components act on each other and compose equations. It means their action and dynamicity, and properties of these actions form the knowledge of equations. Representation of the algebraic knowledge of equations appears as knowledge of notation. Notation is not just representation but also a shortcut of communication of mathematical actions, making it a significant concern of algebra.

In the following table, you can see components/domains of learning on equations and the major topics under these components. These major topics are derived from Davydov's Grade 1 Book, and directly included in the study, with a difference in algebraic notational interpretations. Our pre-investigations and teacher opinion suggested not to include full letter notation, but to use iconic pictures for representing objects/quantities. (A "try-out" activity for letter notation using beans was planned but could not be implemented.)

Table 3.1 Teaching Domains in Equations with Addition and Subtraction in the Form: A±B=C±D

Variables	**Equalities**	**Operations**
Continuous not discrete Multiple attributes: Height, weight, volume (as water height). Area is excluded Variable rather than unknown - Construction for the missing. - Multiple answers (infinitely many)	Equality/inequality A=B A≠B A>B B<A Relational properties Symmetry Transitivity - Ordering 3 obj - Constructing scale	Increase/decrease action with (+, -) signs Increase/decrease amount (how to make it equal) One side operation A+B=C, A-B=C, A=B+C, A=B-C How to make equal A>B => A=B+C or A-D=B Recognize C=D
Algebraic Interpretations	**Equations**	
Davydov full letter notation This study: No letter notation Use of pictures to represent quantities of objects. Less writing of signs, with more activities involving choosing or placing.	Double side single type operations in the form; A+B=C+D & A-B = C-B Operational properties 1.A=B => A+C=B+C, A-C=B-C 2.A=B & C>D => A+C>B+D & A-C<B-C 3.A-B=C => A=B+C 4. Symmetry for addition integrated into discussions Modeling equations	

The teaching domains in equations with addition and subtraction listed in the above table are taught in the following trajectory/sequence in Davydov’s Grade 1 Mathematics Book. The following summary highlights Davydov's trajectory's keystones and main learning themes.

Table 3.2 Keystones in Davydov’s Trajectory (Davydov et al., 1995)

Concepts	**Trajectory keystones**
Equality	Different attributes of objects
	=, ≠ signs in comparisons
	Part-whole equality
	How to make equal: verbal
	>, < signs in comparisons
	Part-whole grouping
	Determine attributes based on relations.
Transitivity	Ordering four objects
	Transitivity enactive
	Use and create an intermediary
Operations	Increase/decrease to make equal
	Use of +/- signs to make equal
	Increase/decrease amount
Equations	Operations on both side
	Modeling real life with equations
	Introduction of units and numbers
	…

Davydov’s trajectory of Grade 1 (Davydov et al., 1995) before the introduction of units and numbers can be outlined in terms of learning objectives as follows:

1. How to compare objects properly based on their properties.
2. Interpret equal and unequal quantities with equal or unequal lines.
3. Interpret equality with equal and unequal sign.
4. Discuss equality in part-whole situations.
5. Discuss how to make daily life objects equal in quantity.
6. Use <, > signs to interpret the relation between quantities.

7. Use <, > signs in-part whole situations.
8. Imagine and construct quantities based on given relations.
9. Construct quantities of an unknown third quantity based on relation to other quantities: transitivity with enactive representations.
10. Match given relations with given situations.
11. Determine types of attributes of quantities based on the given relation.
12. Order 3 or 4 objects using <, > signs.
13. Use transitivity to deduce the third relation from the given two symbolic relations.
14. Construct an intermediary or equivalent scale to compare distant objects: use of transitivity.
15. Explain how to achieve equality by determining which side to manipulate, including the meaning of increase and decrease.
16. Describe where an increase or decrease occurs in a verbal interpretation based on a given situation that is initially unequal and then made equal (one-sided).
17. Determine the appropriate sign choice for increases and decreases based on a given situation that is initially unequal and then made equal (for both sides), starting with real-life examples and progressing to full algebraic interpretations.
18. Interpret increases and decreases to achieve equality with addition or subtraction on one side, and assign new letter notations such as c or d to expressions like a+b or a-b.
19. Define the difference, added, or subtracted part to achieve equality in the context of water height.

20. Pay attention to the amount in the concept of difference: The importance of determining the exact amount needed to increase or decrease to achieve equality.

21. Discuss how to make inequality by changing equality situations through increasing and decreasing.

22. Equal, not equal, equal again: properties of operations, equal amount added, subtracted to preserve equality.

23. Modelling and matching real-life situations to algebraic expressions of equations or inequalities of one or two-sided addition and subtraction.

The objectives in Davydov's trajectory are included and detailed in the construction of the first HLT. Adaptations are explained in the next section, which covers the construction of instructional design objectives.

3.3.2 Construction of Instructional Design Objectives

To facilitate algebraic learning, we utilized APOS Theory to construct the HLT. APOS Theory, which encompasses actions, processes, objects, and schemas, was also employed to observe and assess students' understanding. To develop objectives based on the APOS Theory, we first assessed Davydov's trajectory through the APOS levels. Then, the objectives in the first HLT were designed to align with these levels in Davydov's trajectory. We adhered to the APOS definition provided by Dubinsky and McDonald (2001) throughout all stages of this study: analyzing Davydov's trajectory in terms of algebraic understanding, designing our trajectory and activities, and evaluating learning stages at the conclusion.

> Action: the transformation of objects perceived by the individual as essentially external and as requiring either explicitly or from memory, step-by-step instructions on how to perform the operation.
>
> Process: When an action is repeated, and the individual reflects upon it, he or she can make an internal mental construction called a process in which the individual can think of performing the same kind of action but no longer with

> the need for external stimuli. An individual can think of performing a process without actually doing it and, therefore, can think about reversing it and composing with other processes.
>
> Object: constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it. (Objects can be used in other processes)
>
> Schema: individuals' collection of actions, processes, objects, and other schemas which are linked by same general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving the concept.
>
> (Dubinsky & Mc Donald, 2001, p.275)

Firstly, we analyzed Davydov's trajectory in terms of the APOS steps. The trajectory showed alignment with the APOS learning steps that will ensure effective learning of equations with addition and subtraction. In the following, the main trajectory is outlined, with its corresponding APOS steps for the major learning components aligned horizontally.

As seen in the figure, one step may include the object level of learning while it is put in action of some other learning. It is important to recognize that each object level of learning occurs at the process level of the next learning, as the definition indicates. Object level is gained either before or at the time when it is used in some other processes. Although it may have been acquired earlier, the primary observation of the object level occurs when students can apply it in new processes. However, the need to use new procedures might force students to objectify it. Details for each learning stage will be illuminated by the research results.

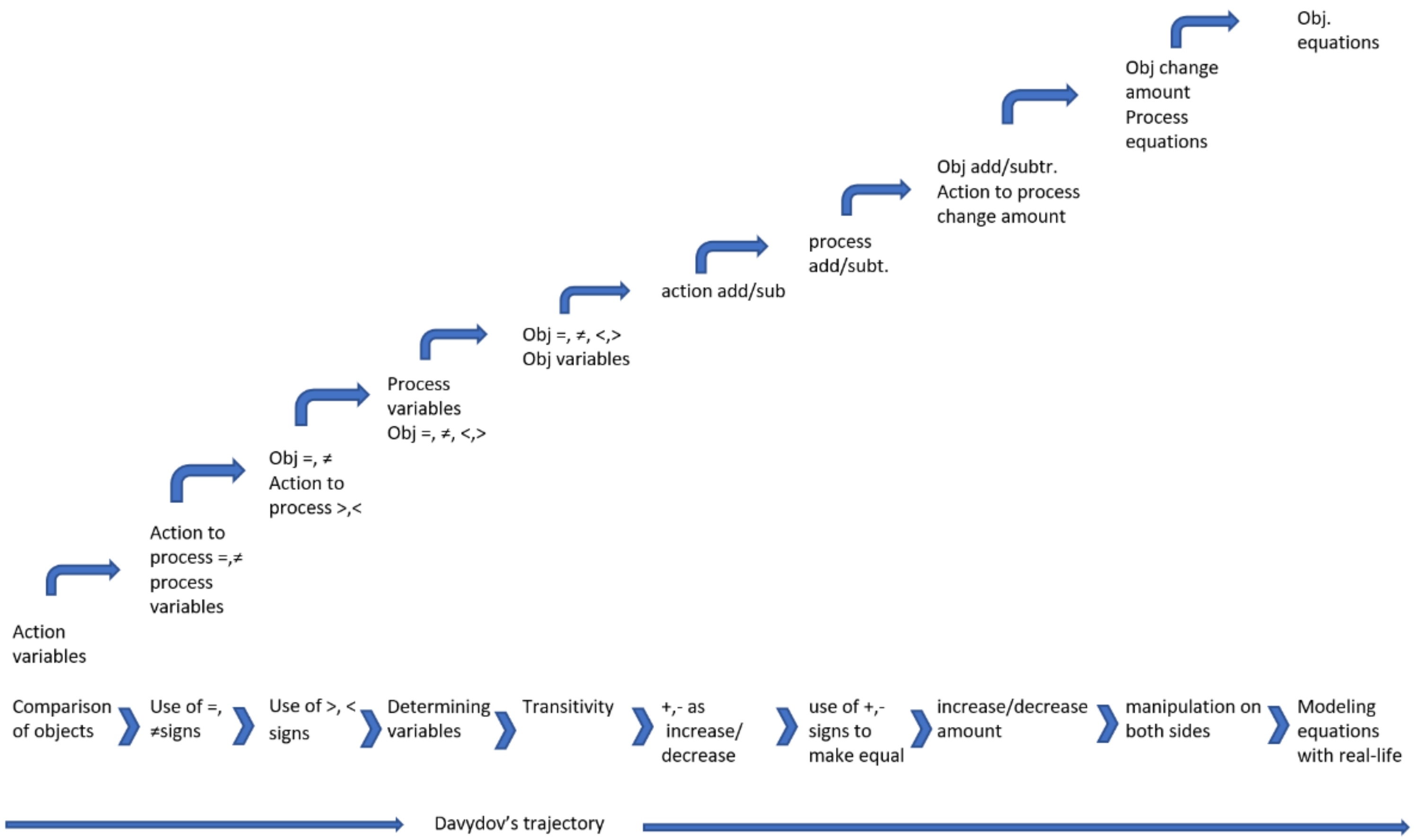

Figure 3.3. Davydov's Trajectory Based on APOS Levels

Since this figure shows the steps in the sequence of the trajectory, it may be difficult to see the object-process dominance of specific learning components. Therefore, the APOS steps for the major components/domains of algebraic learning variables, equality, operations, and equations are resented separately below for clarity.

Variables

Object: Construct variables using transitivity

Process: Determine variables based on relations

Action: (How to) compare variables

Equality

O: Use of =, ≠, >, < signs and concepts to solve transitivity problems

A, P (=): Use of >, < to interpret unequal comparisons

A: Use of =, ≠ to interpret comparisons

Operations

O: operational properties by manipulation
on both sides

P to O: interpretation of increase decrease amount

A to P: choose of sign +, - to represent increase or decrease to make equal

A: increase or decrease to make equal verbal

Equations

O: modeling real life by equations

P: operational properties by manipulation on both sides

A: how to make equal

As seen, Davydov's trajectory is aligned with APOS steps, which will ensure learning. Adopting Davydov's trajectory, we prepared activities taking similar actions. From a design research perspective, we tried to satisfy learning based on APOS levels in each learning outcome (to satisfy, it is open to evolving in the procedure). Details can be seen from the table of our first HLT, explaining how/why each activity will lead to the referred APOS levels for the intended objectives. This research results will show if assumed/hypothesized mental actions (genetic decomposition) taken in those steps will result in those APOS levels or will result in different paths of mental actions. While we try to compose a working, purposeful instructional design for a younger age throughout this hypothetical learning trajectory, students' level of understanding based on APOS Theory will be assessed through students' actions. In other words, instructions will be ensured to satisfy the hypothetical learning trajectory through intended mental actions (genetic decomposition), and the resultant learning trajectory will be the mental actions observed in the students' learning procedures.

Following this consistent trajectory, we based the trajectory and objectives for each classroom lecture on APOS Theory. Each learning experience begins with actions on known objects using provided algorithms. Through repeated actions, we aim to develop an understanding at the process level. Then, the developed processes are used as objects in new actions and processes after encapsulation.

There are 32 lectures designed for the first HLT, to be conducted in 16 days in 8 weeks. Each week, there will be two lecture days. Each day, two complementing lectures were planned to be implemented consecutively, where each would take about 30 minutes. The hypothesized APOS levels (genetic decomposition) and objectives as student behaviors for each lecture in the first HLT are listed in Table 3.3. Hypothesized learning trajectory will evolve into the resultant learning trajectory after implementation procedures, revealing a resultant genetic decomposition based on students learning through the activities.

Table 3.3 Objectives and Respective APOS Levels in the First HLT

L	APOS	First HLT objectives
1	Action =, ≠	1. The student interprets equal and not equal signs. 2. The student compares objects and uses equal and not equal signs to interpret relations based on size. 3. The student uses balance scales to compare the weight of objects and interprets the relation using equal and not equal signs. 4. The student differentiates height, length, volume, and weight as different variables.
2	Process =, ≠	1. The student uses equal and not equal signs to interpret a relation in a part-whole context.
3	Process =, ≠	1. The student uses equal and not equal signs to compare volumes of cups.
4	Process =, ≠,	1. The student reports the comparison of volumes of objects symbolically on the paper with =, ≠signs 2. The student reads the symbolic interpretation of equality and inequality and checks it with concrete objects.
5	Action >, < Object =, ≠	1. The student interprets inequalities with greater or smaller relation. 2. The student uses >, < signs to interpret relations 3. The student interprets (verbally) how to make equality from greater or less than relations
6	Algebraic notation	1. The student uses the first letter of his/her name as notation. (planting bean)
7	Process >, < Object =, ≠	1. The student uses >, <, = signs to interpret (without reminding the algorithm) the comparison of weights. 2. The student manipulates both sides /increases or decreases play dough to make equal-weighted pieces.
8	Process >, < Object >, < Transitivity	1. The student uses >, <, = signs to interpret the comparison of volumes (as a new continuous variable) of cups. 2. The student uses two relational interpretations of three cups to guess the third relation (transitivity property).
9	Object =, ≠, >, <	1. The student finds suitable objects for a predetermined relation, finds an equal and unequal object, and interprets the relation between them by using =, ≠, >, < signs.

Table 3.3 (continued)

10 Object =, ≠, >, <	1. Given the relation between two objects, the student determines the attribute for the comparison
11 Object =, ≠, >, <	1. The student completes the unknown/variable in the given relational interpretation (equality, inequality, >, <) by drawing 2. The student discusses the variability of the drawing
12 Mid-assessment/ Repetition	1. The student uses signs on worksheets 2. Given two different-sized paper strips, the student cuts a long paper strip to make it equal to a shorter one 3. Given two different-sized paper strips, the student glues an extra paper strip to make it equal to the longer one 4. Given two different-sized paper strips, the student interprets/shows how much paper to cut or add to make paper strips of equal length
13 Action sequence	1. The student orders 3-4 objects and puts relevant signs between them based on their relation: with toys
14 Process sequence	1. The student orders 3-4 pictures and puts relevant signs between them based on their relation: with cards
15 Action, transitivity	1. Given two relations among two of three objects, the student determines the relation of the third comparison.
16 Process transitivity construction	1. Given two objects and their relation to a third unknown object, the student draws/constructs an unknown object.
17 Object transitivity Action intermediary	1. The student uses his height or a rope as a scale to compare two stable and distant objects by concluding from their relation to both.
18 Process intermediary Action notation	1. The student constructs scales to compare distant objects. 2. The student uses the same notation to indicate same-size objects. Squares activity
19 Reverse process intermediary	1. Given two objects and their relation to a third one (intermediary), the student constructs, and draws the third object
20 Object/process intermediary	1. The student uses equal-sized scales to represent measurement. Report/graph plant height
21 Object =, ≠ Action +, -	1. The student verbally interprets on which side to increase or decrease to make/satisfy equality (play dough)

Table 3.3 (continued)

22 Object =, ≠ Action +, -	1. The student discusses increase/decrease in volume context to make equality
23 Object =, ≠ process +, -	1. The student chooses the correct sign +/- to interpret increase or decrease on sides to satisfy equality. (weight & volume context)
24 Object =, ≠ Process +, - Action equation with one side +/-	1. Given symbolic interpretations (worksheets) the student chooses the correct sign +/- to interpret the increase or decrease on sides to satisfy equality.
25 Object =, ≠ Process +, -	1. The student uses + and – signs to construct equations with one-side addition/subtraction in part-whole context.
26 Process +, - Action increase amount Action equation with one side addition	1. The student determines addition amount to make equality 2. The student interprets a quantity as the addition of one to another Animal height game: one-side addition
27 Process +, - action increase amount	1. The student uses +/- signs to interpret operations to make equal-length 2. The student enactively investigates increase and decrease amount (difference amount) to create equal length (paper strips)
28 Process +, - Process increase amount	1. The student interprets the increase amount iconically 2. The student compares increase amount of different situations increase amount: plant height
29 Object +, - Object >, < Object increase amount	1. The student discusses how to make equality, unequal, and equal again by addition and subtraction 2. The student interprets effects of addition or subtraction of the same amount on both sides on equality (in volume and weight)
30 Action equations With two-side addition	1. The student models equalities with two-side addition 2. The student uses algebraic notation to interpret equalities with addition on two sides (in height context)
31 Reverse-process equations	1. The student models symbolic equations with one-sided addition or subtraction in the enactive mode of representation: paper strips

Table 3.3 (continued)

32 Modeling equations	1. The student reads equalities and inequalities based on real-life models 2. The student uses algebraic equalities and inequalities for real-life designs.

The following table shows the alignment of the first trajectory (first HLT) and Davydov's trajectory based on APOS levels for learning subjects; equality, quantity, transitivity, operations, and equations. Planned activities to support learning of the subjects are included in the table.

Table 3.4 Summary/Keystones of Davydov's Trajectory and Its Adaptation as the First HLT with APOS Levels and Designed Activities

	APOS	**Davydov**	**First HLT**	**Activities**
Equality & Quantity	Equality action-process	Equality-inequality	Equality-inequality	Compare objects
	Inequality action-process	Greater-less than	Greater-less than	Compare objects
	Pre-action increase/decrease	How to make equal: iconic	How to make equal: enactive & verbal	Balloons, play dough
	Action to process Quantity	Determine variable	Determine variable	Paperwork, card play
Transitivity	Action ordering	Ordering	Ordering	Order 3-4 objects
	Action transitivity	Construct based on relations Guess the third relation	Transitivity Construct based on transitivity	Guess the third relation
	Object quantity Action intermediary Pre-action equal scale	Create intermediary	Create intermediary	Compare objects in the classroom
	Object equal scale Process to object Quantity	---------	Squares: fixed quantity notation	Color notation squares
	Object transitivity	Transitivity symbolic	Construct based on transitivity	Draw intermediary

Table 3.4 (continued)

	Pre-action +/-	Verbal Inc/dec to make equal	Enactive increase/decrease to make equal	Playdough and water cups
	Action +/-	+/- signs to make equal: iconic steady: first this, then this	+/- signs to make equal: continuous manipulation	Enactive and on paper: to make equal
	Action increase/decrease amount	Increase/decrease amount to make an equal, continuous quantity	Increase/decrease amount to make equal	Part-whole
	Action Equation One-side operation	One-side add/subtract to make equal: iconic continuous	One side addition to make equal: Find unknown: height: fixed quantities	Animal height game
	Action difference amount	Exact amount	One side adds/subtracts to make equal	Paper strips
Operations	Object increase amount		Compare increase amount	Compare plants
	Two-side op. Equations action	Two-side addition and subtraction: equal not equal, equal again	Two-side addition and subtraction: equal not equal, equal again	Discuss how to make equality again
	Equations process		Two side addition: find unknowns	Animal height game
	Modeling two-side equations	Matching real-life examples with equations	Create a model of equations	Paper strips
Equations	Modeling equation with one side addition And inequality		Use expressions of equality and equations to model	Rainbow

3.3.3 Designing Activities

Activities are designed based on Bruner’s Theory of representation; enactive-iconic-symbolic stages (Bruner, 1966). In the enactive stage, the learner interacts with the physical world and discovers new learning with concrete materials. In the iconic stage, the learner gains insight into the new learning and creates visual images in mind or uses visual imitations of the concrete world to create meaning. At last, symbolic stage, he/she can use symbolic figures and notations and communicate through them (Conway, 2007). In the development of learning from the enactive to the symbolic stage, Bruner (1965) pointed out the importance of concrete investigation with the words;

> “Note that constructions can be "unconstructed and reconstructed even when the child does not yet have a ready symbol system for doing so abstractly. In short, construction, unconstruction, and reconstruction provide reversibility in overt operations until the child, in Piaget’s sense, internalizes such operations in the symbolized world.” (p.52).

Boundaries to learning a concept are not limited by maturation but by mastering through these three stages (Conway, 2007).

In this study, extensive time is devoted to investigations with concrete materials and providing a connective path to symbolic algebraic representations as we advance these stages. For each learning objective, activities are designed through all of these three stages, starting with concrete manipulatives, material, toys, etc., developing algebraic discussions by iconic representations, and going forward to symbolism with algebraic notations as much as possible at this age. Adhering to this order of representation is consistent with the APOS levels and ensures that alignment. The enactive stage is aligned with the Action level. Repetition of actions with toys (enactive) or iconic representations satisfies understanding of those actions as processes. Algebraic symbols are first introduced on concrete material, and students use these symbols in iconic modes in the processes. Advancing symbolic representations, students can act on it or use it in new algebraic actions, which supports object-level understanding.

See Appendix A for details of how each activity is chosen based on Bruner’s mode of representation and APOS Theory. In this appendix table, the activities designated for the first HLT, and explanations of how each action contributes to the development at each APOS level can be observed. However, research results will either confirm or refute the hypothesized relation between these actions and APOS levels, leading to a better understanding of how these levels develop through the actions.

The following figure presents samples from enactive, iconic, and symbolic representations used in this study in weight context. Students enactively represented on concrete manipulatives (measurement tools). In iconic mode representation, measurement is not explicit. Objects are icons of the implicit attributes (weight, volume) in the expression. Pictures of objects are used as symbols of the attribute in algebraic expressions.

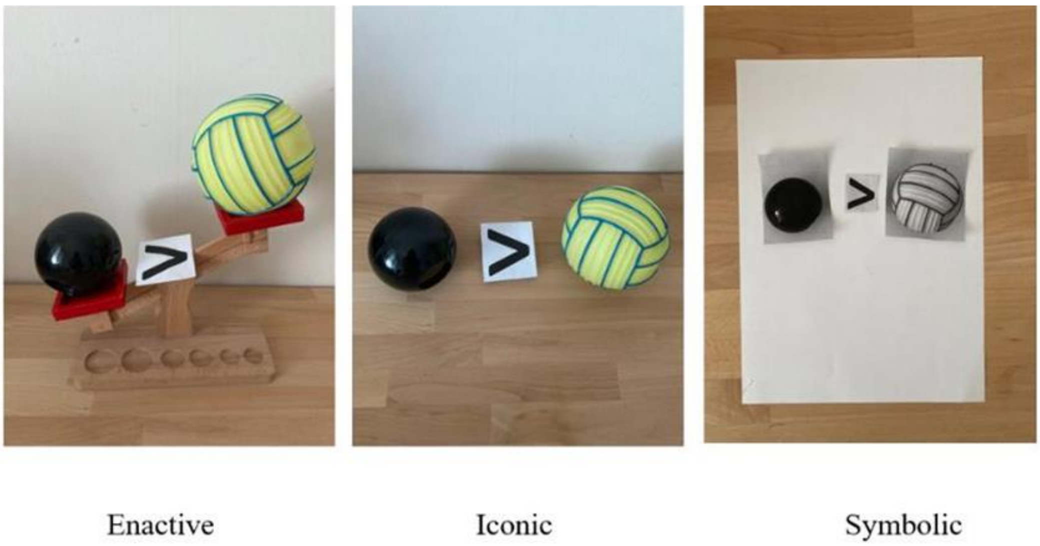

Figure 3.4. Samples From This Study Based on Bruner’s Modes of Representation

3.4 Implementation

Activities in the first HLT were initially planned to be implemented over 8 weeks, comprising 32 lectures during 2nd semester. However, due to pandemic regulations, 5 in-class lectures are followed by 12 online lectures. The final 3 lectures were conducted in the classroom, making a total of 20 lectures completed.

After designing the objectives and sequence for the first HLT, activities were pre-designed for each objective. Each activity was initially tested separately with 2 pilot students. The results from these pilot tests, which served as the laboratory phase for activity design, helped to revise and refine the activities.

Subsequently, each activity was implemented in a classroom environment. In-class results were evaluated through mini-interviews with students during lectures and after-class notes taken by the researcher and the teacher. The achievement of the objectives, observed difficulties, and student motivations informed revisions to the next activity and the overall HLT. The revised activity was then pre-tested with pilot students before being implemented in the class. The results of this second activity informed the design of subsequent activities, with pilot testing continuing to guide further development. (see Figure 3.5) These cycles continued throughout the entire semester. Sometimes, pilot testing could last 2-3 weeks before implementation to ensure a smooth flow or to allow additional time for corrections and re-piloting of activities and alternatives.

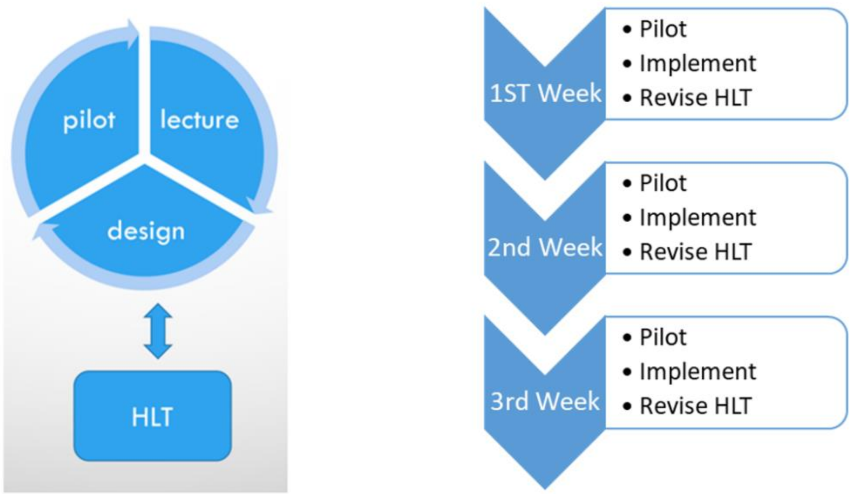

Figure 3.5. Development Cycles of HLT and Implementation Flow

Briefly, designing the next class was based on:

- In-class assessments through individual mini-interviews

- After-class notes
- Pilots
- Hypothetical learning trajectory

The implementation in the class also serves as a revision of the activity itself. Laboratory and real environment situations may differ. The focus of Design-Based Research is not on implementing all activities and then testing to see if they work but on the evolution of the activity for optimal efficiency. If we were to first design HLT, implement it entirely with pilots, and then use it in a classroom to see results, the error estimate would be larger. Pilot and in-class implementations differ significantly. Design should be developed through real classroom experience and assessment.

If no pilots were included in the study, we would have very little information about what we would face during implementation. For example, if we encounter difficulties during classroom implementation and the activity is not working, we design alternative/additional activities to overcome these difficulties and fill learning gaps.

We have the opportunity to immediately test these with pilot students and then bring them to class because we continue working with pilots. Pilot and in-class lectures worked in cycles, sufficiently feeding into and aligning with each other, resulting in very similar HLTs.

Assessment of the lecturing in these cycles led to major and minor changes in the HLT. For example, as a major change, letter notation was simplified to photos, and the focus on variables shifted to the practical use of variables. While classroom objectives helped to ensure satisfaction with each lecture and development of activities, further detailed analysis based on APOS theory was needed for theoretical contributions. This analysis would also provide empirical findings to define principles for designing effective activities.

3.4.1 Participants

There are two students in individual piloting and ten students (five girls and five boys) involved in classroom implementation. All students are in kindergarten, at a public elementary school, with a mean age of 61 months, ranging from 50 to 66 months old. They had no formal education of addition and subtraction prior to this study. Few (3) of them can add and subtract verbally with numbers less than 10. One of them showed an interest in addition and subtraction in written form. They did not even know the terms for addition and subtraction or the symbols representing them. Based on the results of Piaget's conservation test, they lacked an understanding of the conservation of amount, with the exception of one student.

3.4.2 Setting

The study was conducted in a public school during the second semester of the 2020-2021 academic year. The school is on the property of a factory; the parents are primarily from the working class. The school has two kindergarten classes. Implementation took place in only one of them. During the implementation, COVID-19 pandemic regulations were in effect. As a result, the students attended school two days a week for 3 hours each day. Implementation took two lecture hours for two days each week in the classroom. After three weeks of classroom implementation (six lectures), 12 lectures are implemented online. The remaining two lectures are implemented in the classroom, completing 20 lecture hours of implementation.

Ten students were taught by one teacher, with an intern/trainee assisting for half of the classroom implementation time. The researcher was present in the class, guiding students. In the online lectures, mothers assisted students with using manipulatives and materials.

Through warm-up activities in the previous semester, students became familiar with the researcher, videos, and similar classroom activities. There were three warm-up activities. In the first one, students used different-sized circular stamps to cover equal areas. In the second one, they discovered different surfaces of square prisms, cylinders, cubes, and triangular prisms by painting and stamping wooden toys. In the third activity, students used two dice with animal pictures and numbers on the other to count and perform animal steps in a competition context.

Not all students attended every class. Out of 10 students, one student did not attend pre-interview and post-interview. However, this student made remarkable contributions during classroom implementations.

3.4.3 Data Collection

Video and audio recordings are used to capture students' behaviors and verbal interpretations during implementation and interviews. Students' written works, kindergarten teacher's reflection notes, and field notes are other kinds of collected data.

3.4.3.1 Observations

To observe the behaviors of 10 students during in-class implementations, four cameras, and two audio recorders were used. Students worked at 2 round tables, with their seating arrangements changing for each activity. For each table, 2 video cameras recorded the students from different angles. For online lectures, audio of each student and video of online meetings are recorded. Transcriptions of the students' verbal interpretations and detailed explanations of their behaviors were used to analyze their learning. Both the researcher and the classroom teacher were present to guide the students. The researcher conducted inquiries and mini-interviews to scaffold students' learning and encouraged think-aloud sessions to reveal their understanding and thinking processes.

3.4.3.2 Interviews

The data related to students' preconceptions on the topic, their real-life experiences, and their knowledge of signs were collected through pre-interviews. The post-interviews were conducted to lay out their final conceptions and schema on the algebraic concepts. Both interviews included the same items, with additional items included in the post-interviews as described in the following Table of Specifications. We adjusted the extension of inquiry during interviews based on each student's knowledge of the topic to expose the boundaries of their understanding. These semi-structured interviews allowed for questions to be directed informally or algebraically. See Appendix B for the semi-structured interview items.

Table 3.5 Table of Specifications for Interviews

APOS Level	**Objectives**	**Item #**
Process signs =,>, <	The student interprets name of signs =, >, <	1 signs & objects
Process equality	The student uses =, >, < signs to interpret equality and inequality in several contexts: height, width, weight, volume.	1 signs & objects
Process quantity	The student interprets equality for several attributes for comparison of quantities.	1 signs & objects
Transitivity	The student predicts the third relation based on two weight comparisons, deduces the third relation by transitivity.	2 Animals
Action Increase/decrease	The student verbalizes increase/decrease to make it equal.	3 Wood/plant
Process +/- signs	The student interprets increase/decrease with signs +/-	3 Wood/plant
Process Increase/decrease amount	The student interprets increase/decrease amount to make equal.	3 Wood/plant

Table 3.5 (continued)

Process equations with one-side addition/subtraction Object difference amount	The student models equations with one-side addition/subtraction. The student uses difference amount to model addition from given subtraction.	4 Apples
Equations with two-side addition process	The student models equations with two-side addition weight context	5 Children balance
Process increase/decrease amount Process equations with two-side addition	The student finds unknown in one-side and two-side addition The student models equations with two-side addition height context	6 Animal height game
Properties of operations process Object addition/ subtraction amount	The student guesses comparison of results for adding or subtracting same or different amounts based on initial situations in volume context.	7 Properties
Reverse process equality/inequality Multi-solution action	The student constructs quantities based on a given relation to a quantity. The student interprets multi or single solution to quantity based on given relations when asked.	8 Drawing circles

Note: Items #2, 6, 7, and 8 belong to only post-interview. Others are included in both pre- and post-interviews

The interview items are semi-structured in that the researcher proceeded based on the students' responses. Especially in the pre-interviews, students' knowledge of signs or language capacity to interpret attributes affected the follow-up of the interview items. The researcher used algebraic and formal expressions or informal daily life language based on the student's ability to answer the first items.

The researcher interviewed students individually in their classrooms, separated from their peers. Each interview took about 15 minutes and was videotaped. In the interview questions, no enactive manipulatives were used. Students are provided with pictorial illustrations of physical worlds, =, <, > signs, and pencils for written items (see Appendix B).

3.4.3.3 Student's Written Work

As students investigated algebraic topics enactively, there was no written work in some of the lectures. When there was written work, they were collected and further analyzed through content analysis, searching for patterns or differences in their works to gain insight into their understanding. Written works and photos taken by mothers of students' enactive works were essential for online lectures. Written works did not only consist of drawing or writing on paper items; it also included cut or paste manipulations.

3.4.3.4 Debriefing with Design Team Members

The researcher had meetings with the design team members to design the activities. Member opinions were taken verbally and audio recorded and through feedback on instructional documents. These meetings took place at the beginning of the instruction and in the middle of the implementation procedure to decide revisions in the trajectory.

Debriefing between the researcher and the kindergarten teacher occurred before, during, and after implementing the activities. Debriefings were noted by the researcher. These debriefings helped plan the lecture, handle problems during implementation, and reflect on the strengths and difficulties in the implemented lecture. Before the lecture, the kindergarten teacher evaluated the activity in terms of difficulty based on students' levels and suggested implementation planning. During the implementation, debriefing helped to decide how to proceed lecture, support students with additional activities, respond to particular difficulties, or report remarkable learning of students to each other. Awareness of other students' learning procedures connected and empowered discussions and inquiries throughout the class. After the implementation of each lecture, kindergarten teachers' reflections are gathered through the following questions:

- Was the lecture successful in creating intended learning, or was it difficult?
- Was the lecture engaging and fun for students?
- Is there anything to change or improve in the lecture?

3.4.3.5 Field Notes

The researcher took field notes in written and audio-recorded form during and right after the implementation. The notes included conclusions on students' learning or struggles and observations on teaching strategies by kindergarten teachers or researchers that emerged out of the plan. These field notes worked as the first analysis and implications of data; some were used in revising further lectures, and some helped understand why specific learning or difficulties occurred during implementation.

3.5 Analysis

Due to Design-Based Research perspectives, analysis is an ongoing process. The following table (Table 3.6) lists analysis tools and their contributions to each phase of design-based research. In this table, analysis procedures are associated with their purpose and data collection tools in each research phase; construction of the first HLT, implementation and revision of HLT, and retrospective analysis. In the first phase, the first analysis is conducted on students' understanding of conservation by Piaget's conservation test, with three questions on conservation of volume, area, and number quantity. Then, a qualitative content analysis was conducted on the pre-interview data to understand students' informal knowledge of the topic and their achievement of the pre-determined objectives described in the table of specifications for interview items (see Table 3.5).

Table 3.6 Design-Based Stages, Data Collection Tools, and Data Analysis Procedures

	Construction of first HLT	Implementation and Revision of HLT	Retrospective Analysis
Aim of Analysis	- Determine readiness - Starting point	- Effectiveness of activities - Revision for HLT	- Theoretical analysis on APOS level progress - Design principles
Data Collection	-Piaget's conservation test -Pre-interviews	- Field notes - Mini-interviews - Peer debriefing	- Video analysis - Post-interviews
Data Analysis	- Content analysis	- Ongoing assessment	- Constant comparative -Thematic analysis

During the implementation process, analysis cycles occurred at every stage to assess the effectiveness of activities. The analysis was based on objectives described for each activity regarding the intended APOS Levels (see Appendix A). Effectiveness was measured by the success of the majority of students in the classroom. Any difficulty in understanding was addressed through further instructions which included adding extra activities for support or revising the subject's difficulty. These revisions could occur during the lecture, with changes in implementation made immediately.

Classroom field notes, after-class assessments by researchers on achievement and difficulties through immediate reflection or video checking, peer debriefing, and reflections by kindergarten teachers were all parts of the analysis process. Individual student difficulties were addressed immediately. If a pattern of difficulty was observed among students, new activities or revisions were investigated, piloted, and added to the trajectory to meet the objectives or revised objectives as needed. This cyclic process of implementation, analysis, and HLT revision forms the core of the study, ensuring effective activities throughout the trajectory.

Further analysis of student progression on APOS Levels through activities and the effectiveness of these activities was conducted after all implementation was concluded. Videos were transcribed, capturing students' verbal responses and behaviors with thick descriptions (Gravemeijer & Cobb, 2006), and analyzed using qualitative data analysis methods. Four themes guided open coding for research questions and APOS levels for each activity: student response, lecture flow, APOS levels, and design principles. APOS Level code is included to answer the theoretical research question of how students' learning progressed through implementation. Open coding on students' responses provides information for the APOS theme. The design principles theme addresses the research question of why these learning progressions occurred. Lecture flow and inquiry are expected codes to contribute to the explanation of design principles. In other words, these four preliminary themes, which guided open coding, come from the design of the study and research questions. Under each theme, codes are given to specify the focus on open coding. By constant comparative methods, resultant codes and themes will emerge from the data in retrospective analysis.

Table 3.7 Preliminary Codes and Themes Guiding Open Coding

Themes	**Codes**	**Themes**	**Codes**
Student Response	Aha moments Student difficulties algebraic intuition motivation	Apos Levels	action stage process stage object stage anchoring points
Lecture Flow	Introduction of each step Order of instruction Enactive-iconic-symbolic	Design Principles	Materials Inquiry Teacher needs

These categories evolved as the analysis progressed. Students' responses and their understanding at various APOS levels highlighted the trajectory, while the lecture flow identified effective activities that supported this trajectory. Design principles were articulated in relation to material characteristics and inquiry methods, illustrating why these activities were effective in supporting specific APOS levels through student interaction with manipulatives and the teacher. Resultant themes and categories are given in the table below:

Table 3.8 Resultant Themes and Categories Used in the Retrospective Analysis

Themes	**Categories**	**Codes**
Algebraic concepts/topics	Equality	Equality/inequality
		Relational properties
		Symmetry
		Transitivity
		Ordering three obj
		Intermediary
	Operations	Increase/decrease action (+,-)
		Increase/decrease amount
		One side operation A+B=C, A-B=C, A=B+C, A=B-C
		how to make equal A>B => A=B+C or A-D=B Recognize C=D
	Equations	Double side single type (+/-) operation A+B=C+D, A-B = C-B
		Operational properties 1.A=B => A+C=B+C, A-C=B-C 2.A=B & C>D => A+C>B+D & A-C<B-C 3.A-B=C => A=B+C 4. Symmetry for addition integrated into discussions
		Modeling equations
	Variables/Quantity	Continuous vs discrete
		Multiple attributes
		Variable vs unknown
		Construction for the missing.
		Iterating answers (infinity)
		Addition of two quantities to construct another

Table 3.8 (continued)

Notation	Algebraic Interpretation	Verbal interpretation
		Enactive interpretation
		Iconic interpretation
		Symbolic interpretation
Student Response		Student Difficulties
		APOS evident responses
APOS Levels	Action	Understand algorithm sign
		Choose appropriate sign
		Placement of sign correctly (notation)
		Carry out algorithm with self-talk
		Follow algorithm instructions by reminding
	Process	Carry out algorithm without reminding / fluent algorithm
		Carry out the algorithm in a new context
		Self-consider reverse process
		Enforced reverse thinking
		Composition of processes
	Object	Use in the new actions and process
Interaction	Teacher interaction	Enhancing - Language - Inquiry
		Hindering Teacher's difficulties
	Interaction with manipulatives	Enhancing: Choice of material
		Hindering: - Students' lack of physical world experience/knowledge - Manipulative limitations
Lecture Flow		Daily-life example
		Previous class reminding
		Pre-algorithm
		Verbal algorithm
		Action algorithm
		Repetition of algorithm
		Dictation of algorithm
		Reminding algorithm in a new context
		Reverse process inquiry
		Use of prior concepts in new actions
Underlying Algebraic Intuition		- Symmetry - Transitivity

The data analysis focused exclusively on individual student progress. Students' advancement through APOS levels served as empirical evidence of the activities' effectiveness on these levels. Specific student actions acted as indicators of APOS stages, enriching theoretical insights into their progression. The design principles were determined by the characteristics of the activities, shaped by students' interactions with manipulatives and inquiry. These elements, along with the lecture flow, were instrumental in defining effective activities as tangible outcomes of the study.

3.5.1 Definition of APOS Levels for Analysis

Students' APOS levels were deduced using the following table which defines indicative behaviors used for coding each APOS level. These definitions were applied to both interviews and classroom implementation data. Additionally, APOS levels and expected behaviors as objectives were noted for each activity (see Appendix A). As previously mentioned, Dubinsky & McDonald's (2001) definition of APOS level was considered for a holistic perspective on assessments and for making additional inferences. In this definition, the actions, processes, and objects mentioned refer specifically to algebraic actions, processes, or objects. For example, a student might use an algebraic object in an action that takes place in a real-life context, but this does not necessarily indicate an algebraic action. Hence, we cannot conclude student has an object level of understanding.

Another clarification is needed regarding the algebraic process-object definition. It is distinct from the procedural-conceptual understanding dichotomy. Students who apply processes in new contexts may demonstrate evidence of conceptual understanding, but this does not equate to an object-level understanding. New contexts help us observe a student's ability to perform algebraic actions without relying on a memorized algorithm, indicating a process-level understanding. APOS Levels specified in the context of learning equations at the kindergarten level are given in the following table.

Table 3.9 Definition of Codes for APOS Levels Used in Analysis

APOS level	**Student behaviour**
Action level equality	The student uses =, <, > signs in limited contexts when reminded.
Process level equality	The student uses =, <, > signs fluently in several contexts without reminding. Student finds objects based on given 0, <,> relation. (reverse process)
Object level equality	The student uses equality in actions of operations or equations.
Action level quantity	The student compares objects based on a type of quantity when reminded.
Process level quantity	The student compares quantities in several contexts fluently. The student decides which quantity type is used in given comparisons. (reverse process)
Object level quantity	The student uses quantities in actions of operations or equations.
Action level operations	The student uses +/- signs when reminded to interpret increase decrease.
Process level operations	The student uses +/- signs for increase/decrease fluently.
Object level operations	The student uses +/- in the action of increase/decrease by an amount, equations, or properties of operations.
Action level increase/decrease amount	The student interprets how much increased/decreased to make equal when asked.
Process level increase/decrease amount	The student finds unknown amounts in equations.
Process level increase/decrease amount	The student finds unknown amounts in equations.
Object level increase/decrease amount	The student reasons by +/- amount in properties of operations.

Table 3.9 (continued)

Action level equations	The student compares and interprets equality/inequality of one/two-sided operational interpretations when reminded of algorithms.
Process level equations	The student interprets/models equations with operations on one or two sides.
Object level equations	The student discusses the properties of operations in equations.

3.5.2 Definition of Design Principles

Design principles are not something that is coded or discovered in the data through analysis. They do not emerge directly as results of analysis but rather as conclusions drawn from those results. Determining design principles involves inferring from the findings, and they are presented as suggestions, recommendations, or applications of the research outcomes. While coding, they do not manifest as codes for specific student or teacher behaviors; instead, they arise from the interactions among other codes and categories. To understand design principles, the following literature discusses what design principles entail and how to formulate them. Subsequently, we can explore the relationship between design principles and other emerging themes.

According to Bakker (2018) “principle” has different meanings;

- Value, ethical norm
- Criterion
- Guideline, heuristic, advice
- Prediction

Design principles may be written in the form of the last three meanings. Design principles explain the criteria of the activities to reach a certain aim/learning. They may be in the form of prediction; “if you proceed in this…., you probably achieve this result”. The form of a design principle may include theoretical (from literature), empirical (research findings), and advisory base (observations, experience). Design principles are not necessarily specific, and they are open to (or maybe as advice to) revision or future testing “hypothetical nature of principle” (Bakker, 2018, p. 51).

Van den Akker (2013) has a formulation for writing a design principle;

> “• If you want to design intervention X [for purpose/function Y in context Z]
>
> • then you are best advised to give that intervention the characteristics C1, C2, . . . , Cm [substantive emphasis]
>
> • and to do that via procedures P1, P2, . . . , Pn [methodological emphasis]
>
> • because of theoretical arguments T1, T2, . . . , Tp
>
> • and empirical arguments E1, E2, . . . , Eq”
>
> (p. 67)

Van den Akker’s (2013) template guided our reporting design principles. In conclusion, general design principles for teaching algebra at the kindergarten level will be discussed theoretically. Findings for each specific learning activity will reveal design principles based on students’ interaction with activities, instructors, or manipulatives and the stages they undergo.

Characteristics of the activity and manipulatives, guided inquiry methodologies, lecture flow procedures, theoretical insights detailing stages for achieving each APOS level, and empirical evidence from student responses and their levels of understanding will contribute to formulating design principles. These principles are derived from interactions at the micro-level for each activity and learning step.

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3.5.3 Interaction of Themes

Lecture flow, as an instructional activity, and its impact on students' learning (including their responses and evaluation on APOS levels), form the foundation for design principles in this study. Lecture flow serves as a teacher intervention, eliciting responses from students that correspond to specific APOS stages. At times, students' responses, such as encountering difficulties, prompt further teacher intervention (for correction or support). The effectiveness of these interventions in relation to students' development provides insight into how learning unfolds.

Beyond interventions, the nature of the activity and the use of manipulatives are crucial criteria for learning. Design principles aim to elucidate the criteria for effective activities and interventions, addressing the "what" questions of our research. Meanwhile, analysis of APOS levels contributes to understanding how students engage in algebraic reasoning, addressing the "how" questions of research from a theoretical perspective.

3.5.4 An Example of an Analysis Procedure

Theoretical analysis and inferences from the analysis of the first lecture are summarized below by examples from the analysis of the first lecture:

1. Detailed transcripts were openly coded under four themes and predefined categories. At the right column (see Table 3.10) detailed transcript of data is placed. Students' responses regarding indicator behavior at APOS levels and underlying algebraic structures in their responses are determined. Lecture flow and design principles are coded and associated with the students' responses.

Table 3.10 Example From Open Coding for Lecture 1

Design principles	**Student response and APOS levels**	**Algebraic concepts/ topics and intuitions**	**Lecture flow and teacher/ manipulative interactions**	**Transcript**
Let enactive interpretation be on the action with toys on the scale or between water tubes; iconic interpretation be between objects but not with the action manipulatives; symbolic interpretation be with the photos of the object placed on the paper	Understanding the action of determining, choosing, placing equal or not equal sign is not a problem anymore for the student. They are at the **process stage.** They can follow the procedure freely in different situations and contexts	Notation Equality Intuition a=a	First presentation of iconic interpretation How to form **iconic** interpretation based on the situation is shown at this stage but not expected from the students yet.	Trials go on. The researcher encourages Ekim to think aloud. Taking two identical wooden toys, Ekim finds them equal. R: Which sign do you choose? Ekim finds the correct sign and puts it on the table The researcher removes wooden toys from balance and puts them on two sides of equal sign.

2. Students' APOS levels and individual behaviors were documented to observe their progress. Criteria for developing indicator behaviors are deduced from lecture flow, instructional inquiry, and interaction with manipulatives.

Table 3.11 Coding APOS Levels for Individual Behaviors in Lecture 1

Student	**APOS**	**Behavior**	**Criteria**
Eylem	Process =, ≠	Explains process procedure Reverse process Discuss the inequality of identical toys' weights Diff: balance for weight, higher one is heavier, inexperience 300-500	Explaining own process Available identical toys Inquiry into which one is heavier: Student own strategy: quantitative reasoning

Table 3.11 (continued)

Didem	Absent Action =, ≠	At home, equal or not equal exercise. We assume action level.	
Ekim	Process =, ≠	Process freely without reminding Process in a new context without reminding	Repetition of algorithm New contexts
Aylin	Process =, ≠	No reminding Discovering equal and unequal toys (reverse process)	Reportage for equal and unequal objects.
Medine	Process =,	Action repeated by different material toys of the same size	Investigator. Repetition of algorithm Lots of experience with various toys
Ufuk	Process =, ≠	Automatic in new context weight Tendency to find equality Investigation of addition A+B=C+D	Lots of experience with various toys Investigator
Bekir	Process =, ≠	Discuss unequal weights of identical toys Tendency to discover equality Addition to lighter size to make equal A+B=C	Discussion on equality Lots of experience with various toys Investigator
Hasan	Process =, ≠	Different faces for comparison of obj Investigation of addition A+B=C	Affected by a warm-up activity Repetition of algorithm
Yaman	Process =, ≠	Difficulty: Focus on number of objects for equality p.8 Tendency to find equality Multiple toys No reminding algorithm	He likes numeric calculations Repetition of algorithm (Investigator)
Ali	Process =, ≠	No reminding algorithm Tendency to find equality	Boys seem to be affected by each other for the investigation of equalities with multiple toys.

3. Patterns were detected through the constant-comparison method (Bakker, 2018) in students' behaviors, identifying stages and indicators of APOS levels for each

algebraic learning domain. This stage provides information for common indicator behaviors for certain APOS levels (theoretical) and the reasons for their development (design principles). (See Table 3.12)

Table 3.12 Determining Patterns of Students' Behaviors and Reasons Behind Them in Lecture 1

Students	**Student behavior evidence for process level**	**How/why**
Medine, Ekim, Aylin, Ali	Follows the algorithm fluently without reminding	Repetition of actions
Ekim,	Carries out the algorithm in a new context without reminding	New context, weight
Medine, Eylem,	Explains the algorithm/process in his/her own words	Make std explain actions
Hasan, Eylem, Bekir	Discusses equality for different attributes of objects	Use of different materials for same-size toys
Bekir, Eylem, Ali	Tends to find equal objects; testing identical toys	Exposure to identical toys
Hasan, Yaman, Ali Bekir	Tends to discover equalities, multiple toys	-Lots of toys -Inquiry into how to make equal
Aylin, Eylem, Ekim	Finds objects based on a given sign (reverse process)	-Inquiry -Tendency to match two signs

4. For each lecture, APOS levels on algebraic learning domains were determined by the level of the majority of students. All learning levels observed in the lecture, along with specific difficulties, were reported for each domain. Progress through each learning domain was noted as steps towards the intended learning level, contributing to theoretical outcomes. Each progress step was explained by design principles that supported the learning stages. (See Table 3.13) Additionally, students’ algebraic intuitions, materials used in the activity, and teacher needs were documented alongside the design principles (see Table 3.14). This stage organized the development of stages and related design principles specific to the lecture.

Table 3.13 Connection of Design Principles to Learning Outcomes in Lecture 1

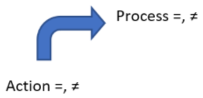
APOS Level: 		**Variables** **Action** **limited to size, height, weight** **discrete materials**	**Notation** **Enactive, Action**
Action steps . Verbal interpretation of sign names =, ≠ . Verbal interpretation of equality for the comparison of 2 objects . Inquiry for choosing of right sign for interpretation of equality (algorithm) . If not successful in choosing of sign, dictation of the algorithm . Repetition of algorithm . Students' explanation of the algorithm.	Process steps . Following the algorithm fluently . Carry out the algorithm in a new context without reminding . Explains the algorithm in his/her own words . Discuss equality for different attributes of objects . Tendency to discover equalities and find equal objects . Finds objects based on a given sign (reverse process)	. Pre-action level for variables: exposure to different attributes of objects for all students . Action level for variables: interpretation(verbal) of how they are equal for all students . Process for variables: Discuss equality for different attributes of objects (Hasan, Eylem, Bekir) activity . Comparison of objects . New contexts for comparison . Recognition of multiple attributes of objects	. All action to process level for enactive representation of equal and unequal sign except Medine is at action level and has difficulties. . Sign meaning . choose of sign rather than writing . Placement of sign correctly . Reporting weight comparisons on the table is an iconic way of algebraic interpretation.
Design principles: . definition of signs direction as the action . connect the sign to comparison through matching/choosing inquiry . Dictate the algorithm if needed. . make students verbalize the algorithm	-identical toys: reverse process -different material toys - lots of experience, -different attributes exploration -students verbalizing their actions.	. use of how equal questions . multiple types of materials and toys . discuss unexpected (inequality) results of weight comparisons for identical toys	. Interpretation of sign direction as action . make students report their comparison of weight on the table . Direct students to find equal and unequal weights for reporting

Table 3.14 Algebraic Intuitions, Material Properties, and Teacher Needs in Lecture1

Students' algebraic intuitions:	**Materials:**	**Teacher needs:**
- a=b => b= a not seen and given - used intermediary (fingers) - a=a - a+b=c+d - a+b = e - a+b=a+b quantitative reasoning Eylem 300-500	Balance scales: inappropriately working, the student thinks about equality free from measurement problems Identical toys: equality discussions Check what if we change order symmetry discovery on balance	. Material experience . Develop language to address attributes.

5. For each lecture implementation, findings were reported theoretically by documenting progression through APOS levels and indicator behaviors specific to each level within the algebraic domains (see Figure 3.6).

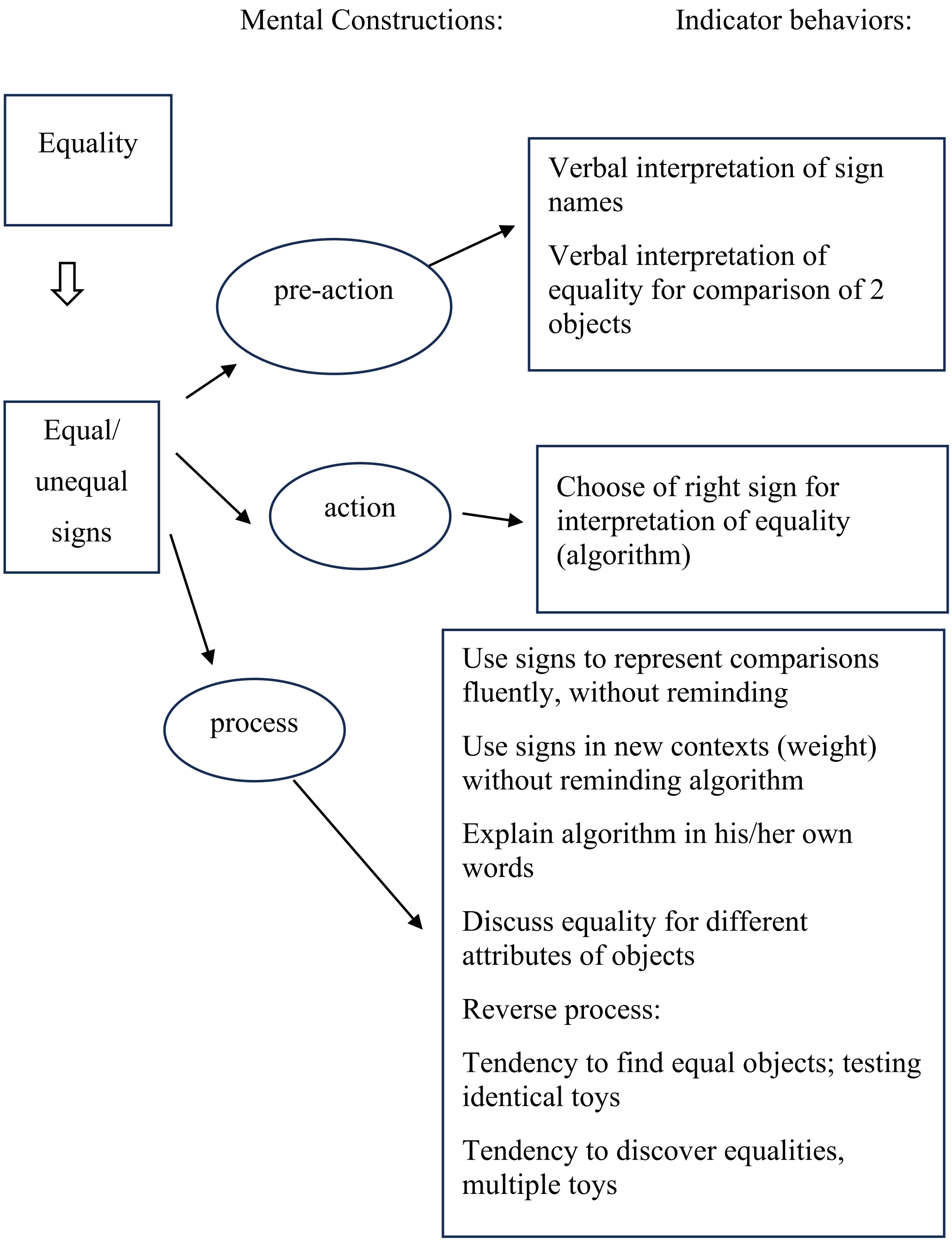


Figure 3.6. Mental Constructions and Indicator Behaviors in Lecture 1

6. For each lecture, practical outcomes were reported as design principles to support each algebraic learning domain (see Figure 3.7):

Design principles for interpreting equality with equal and unequal signs for process level:

- Lots of experience and experience in different contexts help becoming fluent in the algorithm.
- Different material same size toys provide anchor for discussion on weight comparison
- Identical toys trigger to test equality in weight context; which is a typical thinking type of reverse-process: finding objects based on a given relation.
- Encourage students to find equal (weighted) toys which forces reverse-process, while also being motivational.

Figure 3.7. Design Principles for Supporting the Learning of Equality in Lecture 1

3.6 Trustworthiness

Design-based research is a type of qualitative naturalistic research, where we talk about trustworthiness instead of validity and reliability. Validation of constructs and assessments for correct inferences, and transferability of inferences are important for the trustworthiness of the study results (Gravemeijer & Cobb, 2006). Credibility, transferability, dependability, and confirmability are considered under trustworthiness (Lincoln & Guba, 1985).

Activities based on HLT of Davydov involve one-to-one projection of objectives for validation. In design-based research, the design team plays a crucial role in this validation process. The team comprises the researcher, a PhD student specializing in mathematics education who has experience in designing grade 1 activities for teaching addition and subtraction operations, and a kindergarten teacher with a master's degree in education management.

Piloting the activities with two students serves to triangulate and validate each learning step. This piloting step creates a laboratory-controlled environment to construct and test activities for intended learning outcomes. The PhD student’s insights and reflections, particularly on teaching operations, guided the initial construction and subsequent revisions of the activities as needed.

The kindergarten teacher contributed not only to designing activities and adjusting difficulty levels but also enhanced instructional techniques to ensure actions aligned with student understanding during implementation. This interaction influences HLT and informs further task designs. Peer debriefing with the kindergarten teacher ensures the credibility of inferences drawn from the study. Additionally, the researcher's dual role as an instructor provides continuous observation throughout the study. Prolonged engagement and persistent observation further enhance the credibility of the study's findings.

For interviews, expert opinions were sought. Initially, a detailed first interview was not necessary at the beginning of the design-based research, given that it involved entirely new knowledge that had not been implemented before. In design-based research, assessment primarily focuses on monitoring the progress of the processes. Initial assessments are used to gauge students' readiness. However, to triangulate data collection on student progress, pre-interviews were matched with post-interviews. Pre-interviews were structured with a level of formality by the researcher to elicit students' informal knowledge on the topic. In addition to Table of Specifications and expert opinions, interviews were piloted with two kindergarten students to assess the effectiveness of the interview items. The credibility of inferences was enhanced through prolonged engagement, persistent observations, peer debriefing, and triangulation of data.

The effectiveness of the activities and students' achievement of the intended objectives were assessed through classroom observations, as well as in-class and after-class reflections by both the researcher and the kindergarten teacher. Both researchers and kindergarten teachers took on the role of instructors during lectures, guiding students through inquiry and devising strategies for supporting their understanding. Following each lecture, the researcher designed subsequent lessons and revised the HLT based on their interpretations, as well as input from the kindergarten teacher's reflections and ideas at each step. The implemented lecture and plans for the next classroom lesson were discussed between these two team members, with consultation from the third member of the design team when necessary.

To assess students' APOS levels, 5% of classroom implementation videos and interviews were jointly analyzed with a second coder who holds a PhD in mathematics education. There was only one instance of mismatching in the coding, which was resolved by clarifying the definition of the relevant code, achieving 100% agreement.

Thick descriptions were provided in terms of applications and environment to ensure replicability. Barab and Squire (2016) emphasize that replicability also depends on the role of the researcher.

> "It is also the responsibility of the design-based researcher to remember that claims are based on researcher-influenced contexts and, as such, may not be generalizable to other contexts of implementation where the researcher does not so directly influence the context." (Barab & Squire 2016, p. 10)

Hence, the researcher's role not only influences confirmability but also transferability in this study.

3.7 Researcher and Kindergarten Teacher Roles

The researcher actively participated in the implementation as an instructor, bringing a different background compared to a kindergarten teacher. While a kindergarten teacher also served as an instructor, she was guided by the researcher throughout all processes. The kindergarten teacher and the researcher belong to the design team, working together to improve HLT. They are also observers in the implementation to determine students' strengths and difficulties in the lectures. They informed each other, and brought solutions to problems of design by consensus during implementation. The researcher has a role in ensuring mathematical appropriateness in scaffolding, while kindergarten teachers develop pedagogical strategies in the implementation. Briefly, the researcher and the kindergarten teacher have participant-observer roles in the study (Creswell, 2002). Both design team members being present in the classroom environment ensured consensus on the interpretation of data (Gravemeijer & Cobb, 2006) and enabled development for the instructional design during implementation (Barab & Squire, 2004).

3.8 Ethical Issues

This study is found to be appropriate regarding ethical issues by the Middle East Technical University Human Subjects Ethics Committee (see Appendix C) based on its structure, used materials, and instructions. Necessary permission for implementation in a public school is taken from the Ministry of National Education (see Appendix D).

Parents are informed about the implementation, data collection procedures, and privacy in the use of data. Parents' consents are taken by permission forms (see Appendix E). Students' confidentiality in video recording and voluntarism is ensured. Students' names are kept pseudonyms in the reporting and their data is not shared with others.

The participant teacher is informed about the study, and her consent is taken (see Appendix F) for the use of video and audio recordings of implementation and debriefing sessions.

CHAPTER 4

FINDINGS

The resultant learning trajectory is completed in 20 total in-class and online lectures, approximately 30 min on average. In this chapter, the results of 20 lectures will be given separately, in three parts: lecture plan, APOS stages, and design principles, by answering research questions:

- What is an adapted learning trajectory for supporting kindergarten students' algebraic understanding of equations from Davydov's non-numerical perspective?
- What are the effective and practical activities for supporting kindergarten students' algebraic understanding of equations from Davydov's non-numerical perspective?

Presenting the results of each lecture will begin with the plan of the activity, outlining learning objectives, lecture flow, and revision on HLT. Next, the learning progression in the sense of APOS Theory achieved for each algebraic domain in the lecture will be documented to address the theoretical research question on developing a learning trajectory. Finally, the characteristics of the instruction to meet the criteria for an efficient learning trajectory will be presented under the title "Design Principles".

Activities are initially planned based on HLT, piloted, and then implemented in the classroom environment. Revisions continue during implementation, allowing the researcher to adapt objectives and modify instruction based on students' progress, thereby enhancing the design of effective activities. Not only the current lecture but also future HLT are revised based on classroom implementation. HLT evolves into a resultant learning trajectory, throughout effective activities which are ensured by pilot (laboratory) and classroom (natural) testing. The learning trajectory is called a hypothetical learning trajectory until all implementations and revisions are completed. After that, it is called the resultant learning trajectory.

In the findings of each lecture, under the title “Plan of Lecture”, we present the generated activities as implemented lectures, along with objectives belonging to the resultant trajectory. These resultant objectives are explained with reference to their origins in the initial HLT, as well as the revisions made during piloting and classroom testing. For a summary of changes for each lecture, including the first HLT and resultant learning objectives, see Appendix G. This section also explains mini-cycle procedures, the implementation process, and the results from daily and weekly analysis.

The second subtitle is “Theoretical Findings” focusing on activities found to be supportive of students' advancement. APOS levels are reported by indicator student behaviors for related APOS levels of understanding in each algebraic learning domain: equality, variable, operations, equations, and notation. How students progressed through these APOS levels and what specific difficulties they encountered are explained under this title. This section is concluded with a theoretical (retrospective) analysis through “how” questions.

The “why” (and “in what circumstances”) questions clarify the last title “Design Principles”. These principles guide how to characterize activities from theoretical and empirical evidence to support intended learning outcomes. Hence, under this title, design principles are specified for learning progression in the sense of APOS Theory. They are not intended to restrict activities but to regulate actions needed for each learning step. The resultant trajectory will be presented after documenting the results of all 20 lectures.

4.1 Results of Lecture 1

This is the introductory and most important lecture. Equality and quantity learning is introduced and gates to other topics open in this first lecture as all learning topics improve on meaningful learning on equality and quantity. Sufficient time and attention are dedicated to discussion and free experimentation. It took approximately 40 minutes on the first day and was followed by 2nd lecture on the same day. Nine students (out of 10) attended class. The absent student was supported by homework.

4.1.1 Plan of Lecture 1

The activity aims to teach equal and unequal signs and interpret equalities or inequalities with these signs. Interpretation of equality includes the following algebraic domains regarding APOS levels:

- equality conception: use of equal and unequal signs at action APOS level,
- variables: consider different attributes for representing quantities at the action APOS level,
- notation: choose the correct sign in the enactive stages

To satisfy targeted APOS levels at the mentioned domains, students are expected to fulfill the following objectives:

1. The student interprets equal and not equal signs verbally.
2. The student compares objects and uses equal and not equal signs to interpret relations based on size.
3. The student uses balance scales to compare the weight of objects and uses equal and not equal signs to interpret the relation.
4. The student uses different variables/attributes (which she already knows) to interpret equality.

The fourth objective in the first hypothetical learning trajectory expects the student to differentiate height, length, volume, and weight as distinct variables. Due to pilot applications and pre-investigations in class, this initial plan includes some modifications that differentiate it from Davydov's curriculum. Davydov's approach begins with teaching students to compare attributes such as length, height, width, weight, volume, and area. Pilot results showed that verbalizing these attributes, treating the area as quantity, and making accurate comparisons are confusing for students at this age. Moreover, comparing volume (and area) may be difficult due to the lack of maturation based on Piaget's conservation theory. Pre-investigation into Piaget's conservation levels showed that about 80 percent of students do not hold the concept of preservation of quantity. Hence, attributes compared are limited to what students already know and express in their own words. The focus is on interpreting multiple attributes to address multiple contexts for equality, and students are encouraged to explain how and why they are equal to clarify any confusion about equality. In future lectures, the context of volume will be turned into height comparisons with the help of appropriate manipulatives. Following the general trajectory from Davydov, the scope of the subject domains and teaching strategies may be adjusted. Our goal is to adapt Davydov's curriculum at the preschool level by modifying their explanations to better fit the needs of young learners.

Students know what equal or not equal means in their daily lives. In this activity, the use of equal and not equal signs for the comparison of two objects is a new algorithm for them. Experience with lots of toys enables practice for remembering and applying algorithms themselves, which evolves into the process stage. Using concrete objects provides an enactive representation of equality.

The lecture is implemented through the following steps;

- presentation of signs =, ≠
- verbal interpretation of equality for comparison of 2 objects/toys
- inquiry for choosing of right sign for interpretation of equality (algorithm)
- repetition and dictation of algorithm with new toys
- make students interpret different attributes for comparison and interpretation of equality, including height as an attribute
- introduce weight as a different attribute for comparison
- introduce how to compare weight with balance scales.
- repetition of weight comparisons and enactive interpretation of equality in terms of weight

Following these steps, all students managed to show success in achieving the mentioned objectives throughout the lecture period. The progression of the students on these objectives will be explained based on APOS Theory in the following section.

4.1.2 Theoretical Findings of Lecture 1

Equality

Students are expected to use the equal and unequal signs at the action level; which means they are expected to choose the correct sign for a comparison when asked, with the help of guidance on the algorithm. All of the students showed evidence that throughout the activity they progressed from action level to process level in the interpretation of equalities. The following schema shows indicator behaviors for the mentioned APOS levels. These indicator behaviors reveal how students progressed through APOS stages and can also be used to determine their APOS levels for interpreting equality.

Pre-action levels are not mentioned in APOS Theory, which eventuated from our data seems to be important at this grade level. The pre-action stage may refer to pre-knowledge required for learning new knowledge or can be understood as a level of understanding on the topic but not at the action level; actually, carrying out the algorithm. Commonalities of pre-action stages will be discussed in the discussion chapter. Now, it has to be understood as a prior level. In this lecture, it is observed that prior to presenting the algorithm, the presentation of sign names is essential and may take time for remembrance. Hence, each future lecture is adopted to create strategies to learn new signs and remind previous ones. Teaching of equal sign needed to use actions of hand gestures (teacher strategy). (Even sign learning seems to start as action evolved to static objects then later.)

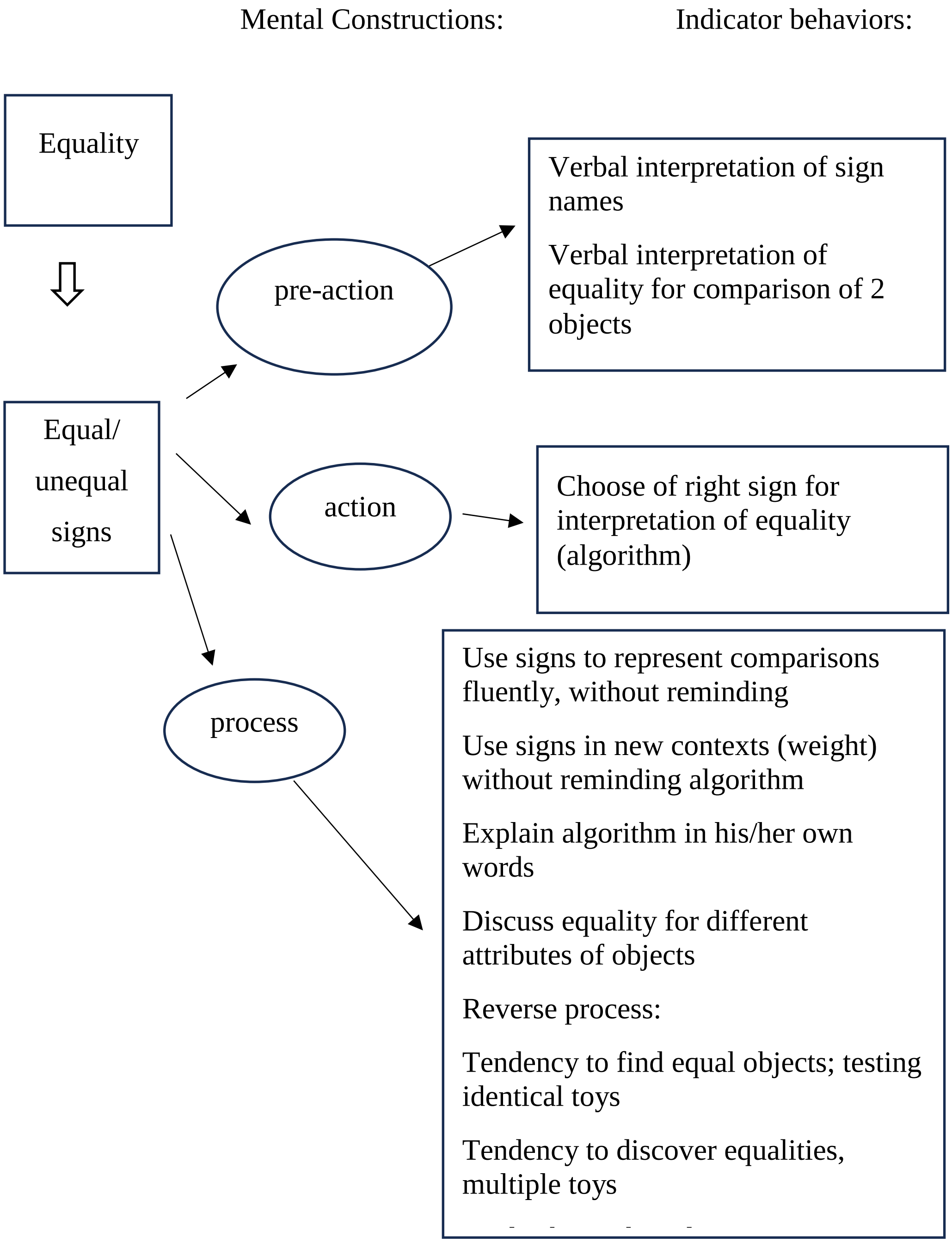


Figure 4.1. Schema for Learning Equality in Lecture 1

Verbal interpretation usually acts as prior knowledge and comes prior to the algebraic interpretation of new algorithms. In this lecture, verbally interpreting equality is essential and appears as a pre-action. Matching with the newly learned signs, this stage turns into an action level of using equal and unequal signs for algebraic interpretations. Making students verbalize equality not only outlies their knowledge of equality in real life but also bridges/anchors it to its algebraic interpretation.

After the presentation of equal and unequal signs, the investigation starts with the equality of simple objects. Students tend to interpret identical toys as being the same or equal, but they can interpret equality based on size and height when asked to do so. Some students focused on the surfaces of wooden blocks being equal because they had a lecture on investigating surfaces. Students have a tendency/motivation to find equal objects. Height is a common attribute they use when they are guided to find non-identical toys that are equal in size. Inquiry is deepened into questioning "How objects are equal based on one attribute while they are not equal based on another attribute?". At first height and width are used in inquiry as different attributes. Weight context helped to clarify this discrimination. Before the presentation of weight comparison by balance scales, they recognized that same-size objects may have different weights when they weighed objects by hand.

Repetition of comparisons, and introduction of new contexts; particularly height and weight, helped students' progress to the process-level understanding of equal and unequal signs. Their tendency to find equal objects increased motivation in the weight context, as it was challenging. Finding equal objects proves they can reverse-process the interpretation of equality with algebraic signs by starting from the sign (equal) and finding objects based on it. Because of the presence of identical toys and the challenge of finding equal-weighted toys, this lecture evokes the motivation of finding equals and intuitively initiates the reverse process. Some students (e.g., Aylin) tend to stick to the procedure (algorithm). Guiding these students to find objects based on both signs and report their findings (ironically, on the table) enables reverse-process thinking. Moreover, this guidance encourages notational interpretation and refocuses students on inequality and unequal signs, which they may lose attention to while investigating equal objects.

Even without being required, most of the students showed evidence they could reverse-process, all of them became fluent in the algorithm of using signs, and some were capable of transferring the algorithm to weight context automatically. At the end of the lecture, these capabilities were common and fluent among all students, showing they were at the process level.

Variables

In this lecture, for a variable domain, students are expected to use different attributes for interpreting the relation between quantities at the action APOS level, which means they are expected to explain "how objects are equal/ or not" based on quantitative attributes, size, height, width, and weight when asked. Recognition of multiple attributes serves as a pre-action level. Students expose this recognition by enactive investigations of comparing objects based on different attributes they know and verbally interpreting "how" or "based on what" they are equal. Through inquiry, attributes they know are oriented to quantitative attributes.

At the end of the lecture, all the students were able to refer to these quantitative attributes in equality of objects which shows they are at action level for quantities. Some students were even able to discuss equality for multiple attributes of the same chosen pair of objects. Procedures they have undergone, and how they accomplished this level will be explained in this chapter.

If not guided on height or width comparisons, students may focus on the appearance of the compared objects for equality. Some students interpreted equality based on the similarity of the surfaces of toys, which was motivated by a previous activity where they reported/found different shapes on the surfaces of wooden blocks. Students could discuss equality based on different attributes, but putting equal or unequal signs based on one non-trivial attribute, such as width, was a little bit confusing for them. Students were free to explain equality based on whatever they wanted. Guiding students' comparisons based on pre-determined attributes; height and width gain attention but are not sufficient to evoke quantity and create a meaning of equality differentiated from being the same. This guidance only helped the recognition of objects' being equal based on one attribute but not based on another attribute. Interpretation of equality in multiple attributes starts after this recognition. This whole stage worked as a pre-action to the understanding of quantity.

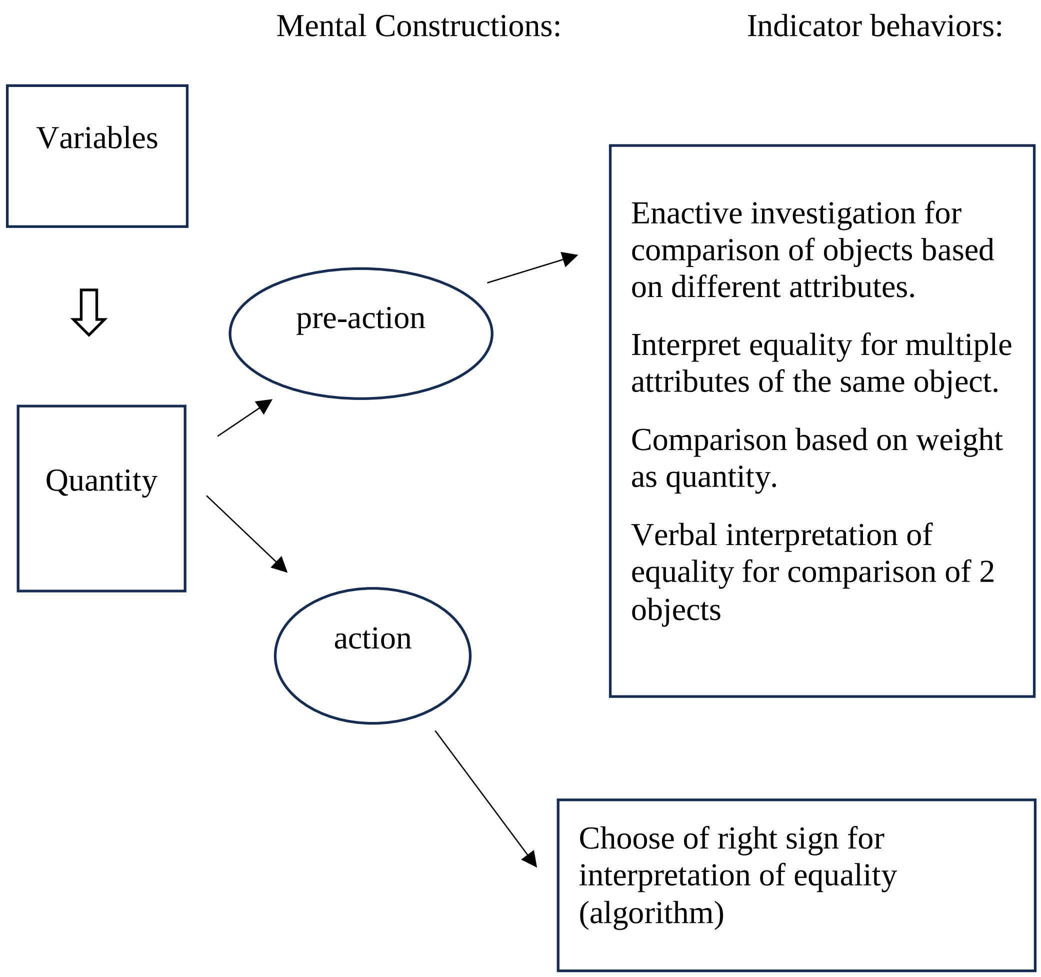


Figure 4.2. Schema for Learning Quantity in Lecture 1

By the use of weight comparisons; most importantly, students understand comparison is made not between objects, equality is not mentioned as being the same; but comparison is on quantity. Moreover, quantity is obtained from multiple attributes of objects if multiple (size, weight, height, and width) contexts are discussed together in the same objects being compared. Quantity exists independent of the object and the context; which makes it more close to objectifying. (Quantity is a matter of measure, and equality is a representation of the relation between quantities, not objects compared.). In addition, weight context made it explicit which side is bigger and promoted quantitative reasoning. quantity becomes a matter of comparison rather than similarity when they see the comparison as “one is more and the other is less”.

Eylem had quantitative reasoning associated with numbers 300-500 in weight context Being one bigger makes students focus on quantity inequality, which evoked Eylem to think about big and small numbers, associating the heavier side with bigger numbers. She had some difficulty like other students in the weight context. She can state which side is heavier, and which number is bigger, knows the heavier side has to take a bigger number, but stated sides incorrectly at first. After an inquiry into questioning, the mentioned knowledge she has, she corrected herself immediately. Students seem to be confused in weight context, because they sometimes associate the higher side with being bigger, especially when subtracting reduce take away something to make it equal (not only in kids, I had the struggle in my first experience with the chicken balance game). When asked to interpret which side is bigger or heavier, they had no problem. Hence, experiencing a lot in the weight context was essential for beginning activity. Other students also had similar problems even though they had non-numerical reasoning. This activity is also essential because they got experience on weight comparison for the first time; which side is heavier, and how they become equal.

> Eylem: 300-500
>
> Researcher: Which one is 300, and which one is 500?
>
> Eylem: This one is 300, and this one is 500 (referring to the heavier one as 300, because it is down. She interprets the upper side to be bigger and assigns 500)
>
> Researcher: Which one is heavier?
>
> Eylem: This one is 500 (pointing to the one above) (finding: even though they understand inequality, they might have trouble identifying the heavier one. Pay attention to separate sub-learnings).

Researcher: Is the 500 one heavier, does the heavier one stay up or go down?

Someone: Down.

Eylem: (Immediately says) down.

Researcher: Yes.

Eylem: Then this is 300, this is 500 (correctly shows).

Researcher: High five.

In piloting, no identical toys were provided to students, following Davydov, assuming that students would focus on quantity not being identical when provided different toys, but it was hard to find equally weighted toys. The use of identical toys not only initiated the reverse process of equality but also created a learning opportunity on quantity. Some students faced a cognitive conflict after they compared identical toys on balance scales and found they had different weights. This cognitive conflict is resolved through discussions on why they are differently weighted. Especially wooden blocks caused this problem. Errors in factoring are shown as one cause for the difference. Even though toys have the same shape, same size, and are produced from the same material, students could recognize that they may have different weights. This made them abstract and discuss the weight of the objects they compared.

Weight was a suitable measure for them to comprehend comparison quantitatively. Weight context also enabled to investigate following algebraic expressions intuitively:

- a=a
- a+b=c+d
- a+b = e
- a+b=a+b

The first and last one is triggered by the presence of identical toys and if they have equal weights. All investigations are carried out through comparisons of non-manipulatable discrete objects (the next lecture will include continuously manipulable objects: play doughs compared in weight context). Finding equality is again motivation. No investigation of addition is guided but permitted. Hence this lecture was pre-action level for addition operation for some students whereas they are motivated to make equality by adding. The next lecture will focus on how to make equality, intentionally creating a pre-action stage for operations.

Yaman is an outlier student in the classroom. (He has an analytical and mindful understanding of mathematics throughout the semester.) He is interested and experienced in arithmetic with addition and subtraction. His knowledge of arithmetic may hinder some focus on the algebraic knowledge learned, giving chance to observe the difference in algebra education before and after arithmetic. In this lecture, he focused on the number of toys and had some difficulty focusing on quantity in the beginning. When asked whether two objects are equal or not, he focused on a number of the objects and concluded equal reasoning by 1=1.

> Researcher: Are yours equal, Yaman?
>
> Yaman: Uh-huh (meaning yes).
>
> Researcher: What is equal about yours, Yaman?
>
> Yaman: Two of them.
>
> Researcher: Because there are two of them? Is this one equal to this one, Yaman?
>
> Yaman: No.
>
> Researcher: Then you will choose this sign. These two are not equal to each other.

Notation

As planned, all students could use equal and unequal signs in the enactive mode of notation by going through the following procedure:

- learn the name of the sign
- choose the correct sign for comparison
- place the sign on the enactive mode of representation

We began with directly giving equal and unequal sign names, contradictory to Davydov. Davydov's interpretation with long and short lines for inequality seems more appropriate to focus on quantity but our piloting showed that children are confused about learning this interpretation and associating it with the real sign interpretations. Moreover, most of the students knew the equal sign earlier, and in our pre-interviews, we showed all signs, including the unequal sign. Students have intuitions that a line crossing over something indicates canceling of the sign, which will ease the learning of "not equal". We focused on learning signs and matching them with verbal interpretations of equality and inequality.

Learning the name of the sign was also a challenge at this age. The kindergarten teacher's intentions to teach signs as the action of two moving arms in orientation helped to teach equal signs. Even sign learning seems to start as action evolved to static objects later. The unequal sign is thought to refer to the canceling of equality. After learning the sign, matching the sign with the situation becomes problematic. Even if verbalizing signs and situations correctly, the student may not match these situations correctly as in Medine's case, which needs a dictation of the algorithm. Matching the correct sign, the student is expected to use the sign in enactive mode. Placement of the sign in the correct place (between objects) and the correct orientation is challenging at this stage, which is solved through guidance and lots of experience. Medine used $A=B\neq$ incorrect representation when asked to use both signs. Other students had no problem matching the sign with the compared objects and representing equality, but only some orientation of the sign was difficult for some students.

In the weight context, students were expected to use signs on the balance scales as enactive representations. It is found to be difficult to move onto the iconic mode of representation in this lecture for some students (Ekim). However, some students (Aylin) also could interpret iconically based on weight through guidance on the representing comparison on the table rather than on the balance scale, or through ordering reportage of one equal and one not equal situation based on weight. Motivation to find equal loose attention on using signs and interpretation. Reporting by two signs encourages interpretation.

Figure 4.3. Eylem's Enactive Representation of Identical Toys with Different Weights

4.1.3 Design Principles for Lecture 1

Design principles for interpreting equality with equal and unequal signs for action and process levels

Action:

- To alter students' difficulties in matching signs to comparison results; be precise in expressing alternatives (signs: =/≠) rather than asking what to do. The inquiry should follow as

"What is the name of this sign"

"Are these toys equal or not?"

"Which sign should you choose then?"

- If a student knows the names of the signs and interprets equality relations verbally but cannot match the correct sign, the dictation of the algorithm may be a solution (Medine's case). Connecting the two, sign and relation, may be difficult.

- Making students verbalize the algorithm while they use interpretation enhances self-guidance, which is a part of learning at the action level.
- Do not reject interpreted equality; ask “how” or “based on what” they are equal to encourage them. The “why” questions make them step back.

Design principles for interpreting equality with equal and unequal signs for process level:

- Extensive experience and exposure to different contexts help students become fluent in the algorithm.
- Different material, same size toys provide an anchor for discussion on weight comparison
- Identical toys trigger to test equality in weight context, which is a typical thinking type of reverse process: finding objects based on a given relation.
- Encourage students to find equal (weighted) toys that force reverse-process while being motivational.

Design principles for understanding quantity at (pre-)action level

- Design materials for exposure to multiple types of materials and toys and build inquiry into how they are equal to make students enactively investigate and compare objects based on different attributes.
- Support recognition of different attributes through inquiry for staging a base for action level of comparison on quantities.
- Have students verbalize different attributes that indicate quantities, such as height and weight, to help them decontextualize quantity as a measure.

These were what we expected before, and so how we designed the activity. What experimentation evidenced is that;

- Provide identical toys for students’ investigation, because the presence of identical toys not only enforces the reverse process for equality but also creates a cognitive conflict when they have different weights. This cognitive conflict is resolved by a discussion on why identical toys are not equal in weight. Discussions promoted a focus on weight as a comparison of quantity. Abstraction of weight as a quantity eventuated independent of objects that are identical in appearance.

Design principles for enactive mode notation of equality with equal and unequal signs

- Give sufficient time and attention to students' learning of sign names. Presentation of sign direction as action helps students differentiate signs.
- Make students report their comparison of weight on the table to encourage iconic representation when students seem to be ready, showing evidence she/he can interpret equality verbally in terms of quantity.
- Direct students to find objects with equal and unequal weight for their reports to encourage them to use and learn both signs and their correct placement and orientation. Reporting their interpretation of equality provides notational learning and algebraic communication.

4.2 Results of Lecture 2

This lecture is given after Lecture 1 on the same day. It has two activities in it; play dough weight and part-whole equality. These two activities seem very different, but both have common aims; firstly, to provide new contexts, continuous and part-whole contexts, supporting process level of equality, and secondly to enable change on the sides of the inequalities to make equality creating a pre-action level for operations. Play dough activity took about 20 minutes, while the part-whole activity took 30 minutes. These times may sound too long for activities. These are not the amount of time to complete tasks for students. Mini-interviews to ensure each student's learning and to observe their thinking took the majority of time. When they complete the task, which they are guided through, they are freely exploring and playing with the materials provided which also supports their experience and learning. Those times are also recorded, observed, and analyzed to conclude their learning. One student is missing in class. Make-up is conducted by the researcher, which is added to the data. The analysis of her (Didem) data indicates her individual learning..

4.2.1 Plan of Lecture 2

In the first HLT, playdough activity was not in this lecture, but it was after greater or less than the subject, in "how to make equal" activities, to anchor increase and decrease. Pilots and in-classroom implementation showed that students have intuitions and tendencies to make equal and part-whole activity also supports this intuition. Dividing playdough into two activities is decided to be moved earlier to Lecture-2 by the researcher and kindergarten teacher based on the following reasons:

1. In the initial plan, there was no discussion on increase or decrease, or how to make things equal at this lecture. However, discussing equality in Lecture-1, along with piloting and classroom implementation, showed that students' tendencies and early intentions led to discussions on how to achieve equality earlier than we expected. These discussions could not be ignored. Although this activity was originally planned for later, it was moved here to address students' intuitive understanding of achieving equality. Our approach is to present concepts in three stages: first intuitively (through actions: enactive investigations/pre-action), then verbally (through inquiry: verbal interpretation/pre-action), and finally algebraically (with signs: algebraic algorithm/action). This process requires time and concentration in different contexts. Hence, discussions can be introduced earlier whenever necessary to align with students' intuitive grasp of the concept.
2. Investigations in the first lecture were limited to discrete comparisons. There was a need to include continuous manipulatives in the weight context. Discussions on variables and equality must include all continuous and discrete variables and manipulations to prevent inconsistencies and overgeneralizations Relying solely on greater/less than comparisons would make the discussion of equality in weight incomplete without continuous

materials. Also, students might think that certain manipulations or variables are associated with certain signs or subjects. Hence, to address this, we aimed to include continuous variable manipulation in the concept of equality using equal and not equal signs. In this class, manipulations involving increases and decreases with continuous weight activities, such as using play dough, were well-suited to this approach.

3. Physical world experience with weight context, particularly using balance scales, was insufficient and required more attention for understanding equality in weight context. In some cases, a higher scale was incorrectly interpreted as representing a larger quantity, especially when integrating this understanding with other issues, such as numbers or achieving equality. The emergence of the need for more experience in weight equality was identified as a priority to address misconceptions. Decreasing weight proved to be confusing for making equal comparisons, even for older students. Using play dough, which offers flexibility in manipulations, was found to be helpful in addressing this issue, as discrete objects were challenging for observing weight equality.
4. The entire activity includes an intuitive understanding of equality, so it is better to address this issue earlier. We did not want to start with increase and decrease activities using discrete variables, which appeared as discrete parting in part-whole activity. Beginning with objects that allow for continuous dynamicity and partitioning aligned well with the deductive approach of Davydov for algebraic generalizations. Before going on with discrete part-whole activity, weight comparisons should include continuous variables to compare first. Without this, the trajectory might lack coherence until volume comparisons are introduced, potentially leading students to perceive weight as discrete. Therefore, all comparisons involving weight, volume, and height should include discrete and continuous elements.
5. Students get quickly tired and unmotivated to see the same toys again (pilot result). Completing the investigation in a single session, whenever possible,

appears to be more effective for maintaining motivation and fostering in-depth discussion.

Hence objectives of the Lecture 2 are revised from:

"The student uses an equal and not equal sign to interpret a relation in part-whole context."

To:

1. The student uses balance scales to partition play dough into two equal masses by increase/decrease actions.
2. The student uses an equal and not equal sign to interpret a relation in a part-whole context.
3. The student manipulates (increase/decrease) one side for the satisfaction of equality in part-whole activities

Play dough activity is the continuum of the weight comparison activity in Lecture-1; students are given a piece of play dough and asked to separate it into two equally weighted parts. It has two particular aims:

1. Equality subject in continuous variable context at the process level of the equality sign.

2. "How to make equal" subject in continuous variable context at the pre-action level of increase /decrease.

As observed in pilot studies and Lecture 1 results, students initially tend to focus on increasing quantities rather than decreasing them to achieve equality. To address this, the amount of play dough is fixed to prevent free experimentation, thereby guiding students to decrease the heavier side. Additionally, determining which side to decrease has been identified as another challenge (based on pre-investigation and pilot results), which requires further experimentation. This lecture provides a solution to address that issue.

In the second part of the Lecture, a part-whole activity is conducted, which also requires manipulation to achieve equality, but in a part-whole context. The alignment of objectives between this activity and the play dough activity can be seen in the following list:

1. The concept of equality in discrete part-whole context at the process level of the equality sign
2. The concept of "how to make equal" in a discrete part-whole context at the pre-action level of addition/subtraction (adding or taking apart).

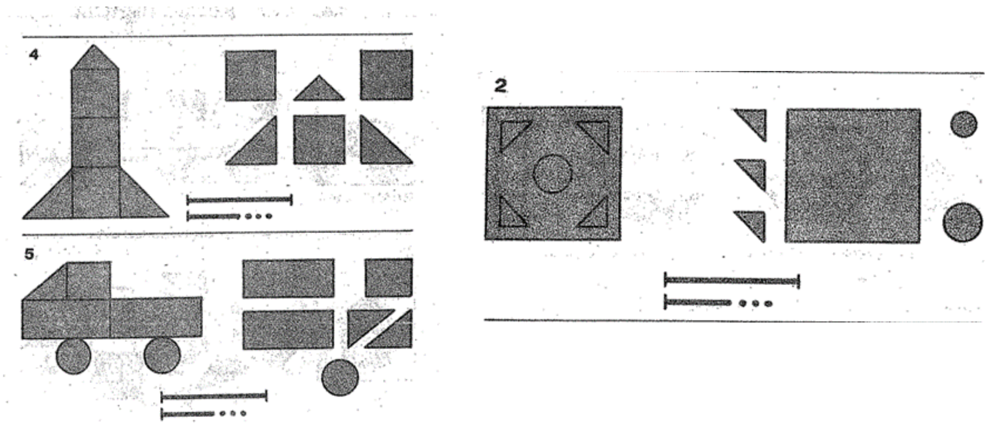


Figure 4.4. Part-whole Equality (Davydov et al., 1995, pp. 18-19)

Figures for part-whole items are taken from Davydov's book (Davydov et al., 1995). Working on these figures, it is expected to serve as an iconic mode for equality, whereas the previous class was actually on enactive investigations. The figures include three items; one representing an equality situation (a rocket), and two others representing inequalities that require manipulation of adding and taking some parts (a square and a truck).

In the figures, Davydov's initial interpretation of the equal or unequal sign is represented. When two given figures are equal, lines are drawn to show equality; when they are unequal, one side is left smaller. When you take apart and add some parts to make equality; you cannot talk about one side being bigger than the other.

Thus, Davydov's first interpretation of the unequal sign does not indicate that one side is larger, as inequality typically suggests.

Moreover, the figures do not correspond to the interpretation of one side being larger. Including this interpretation, without anchoring it to one side being larger, only adds another step to remember before actually using equal and unequal signs (pre-investigation). We removed this step, focusing instead on the actual equal and unequal signs to be chosen based on the figures and placed between them in this lecture.

Lecture Flow:

Based on the objectives and designed activities explained above, Lecture 2 is implemented in the following order:

- Introduction to making things equal (setting focus); show unequal objects and continuously manipulable objects such as balloons; inquire into both increase and decrease to achieve equality, focusing on manipulating one side at a time to reach equality.
- Given a fixed amount of play dough (continuously manipulable objects), students are expected to experience themselves partitioning it into equal weights by decreasing one side while increasing the other to achieve equality.
- Inquiry into which side is heavier to ensure physical world understanding and connect it to how to increase and decrease to achieve equality.
- Part-whole equality with one equal case and two unequal cases with iconic figures are presented one by one, and students are expected to choose the correct sign for each.
- Discussion of equality in the context of discrete parts relative to the whole.
- Inquiry into “how to make equal” unequal situations: encourage manipulation of one side by taking away and adding parts to achieve equality with the whole.

Following these steps, students could use equality and inequality signs in weight and part-whole contexts. The mentioned objectives are satisfied by the majority of students. However, for the first objective, instead of decreasing/increasing, some students (3 out of 10) had different strategies to create equal-weighted play dough pieces. Difficulties and progression in students' learning will be explained further based on APOS Theory.

Further implications to trajectory and further recommendations:

After implementation, the play dough activity seemed to be difficult for some of the students, because simultaneous manipulation on both sides was needed to achieve equality when the play dough amount was fixed. Hence, further activities and inquiries were oriented toward including only one side manipulation. Moreover, it is recommended to revise this activity to include one-side manipulation: for example, fix an amount and create equal-weighted chunks to enforce decreasing or guide intentional increases or decreases on one side with additional inquiries in a freely manipulable environment using continuously manipulable objects.

Secondly, we decided to add additional activities in the trajectory, for part-whole equality, because it was difficult for some students. For additional activity, Lego toys were planned to be used to provide a more intuitive context, helping students remember how wholes are composed of parts. Moreover, it was decided to present the activity in enactive mode to provide a more hands-on approach. The iconic mode, as outlined in Davydov's book (Davydov et al., 1995), did not succeed as expected. Thus, the enactive stage was incorporated into the trajectory and is recommended for future implementation. However, due to pandemic regulations, the Lego activity had to be conducted in iconic mode once again.

4.2.2 Theoretical Findings of Lecture 2

The aim of the second lecture on the first day is to continue processes of equality with new contexts, continuous and manipulative variables/quantities (play doughs), and part-whole equality. In the previous lecture, quantities that were compared were stable (discrete), but now, they can be manipulated (continuous). The "How to make equal" inquiry starts in this lecture, which will evolve into operations later. Results won't be given based on the activities but based on the APOS Theory they support; firstly, process level for equality, and then pre-action level for operations.

Equality

The main APOS level intended in this lecture is the process level of equality, which is fluent in using equal and unequal signs for compared objects. The previous lecture includes a comparison of discrete objects. This lecture includes two different contexts: continuous variable, and part-whole equality for progression at the process level.

All students show the level of process; fluent in using signs (process level), using equality to manipulate quantities (reverse process), and interpreting equality in a new context. They had no difficulty using equality in the continuous context of weight, but they had difficulty transferring their knowledge to part-whole equality. The researcher and teacher needed to remind the algorithm (to 4 students out of 10) to use equal or unequal signs to interpret equality between parts and whole. Some other students (2 out of 10) had difficulty interpreting part-whole equality and inequality with equal and unequal signs. These students (6 out of 10) seem to be at the action level for interpreting equality in a part-whole context. However, the problem is not about remembering algorithms or not knowing how to use signs. The problem is about understanding how parts are equal to the whole.

There are some reasons observed causing this problem. Mainly, the parts do not directly make the whole; some pictures are confusing (square), which means they overlap or need some construction procedure to make the whole. In this situation, students have explained what part-whole equality means as "if we can produce this whole from these parts". Secondly, the activity is in iconic mode. Guided one-by-one matching of parts helped students to determine equality. In addition, cutting pictures of parts and trying to compose the whole is used as a strategy to investigate equality in an enactive mode.

The addition of Lego activity to the trajectory is decided to clarify part-whole equality in enactive mode. The results of this activity revealed that part-whole context is perceived as a new context for students and cannot be used as an iconic stage of equality, which was our first intention. Part-whole equality should also start with enactive investigations.

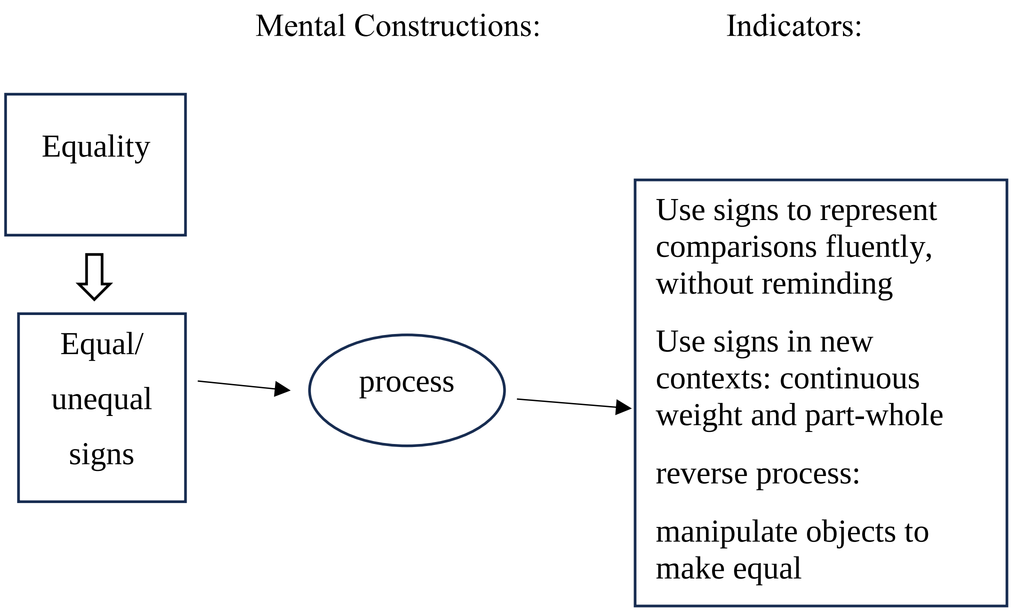


Figure 4.5. Schema for Learning Equality in Lecture 2

Operations

The operations action starts with answering the “how to make equal” question through verbal interpretation of increase and decrease with the pre-algorithm action stage. In this lecture, no verbal interpretation of actual increase or decrease is expected. However, students use increase or decrease actions to make equality in both play dough and part-whole activities, which is even before the verbal pre-action level for increase and decrease (operations). Focus seems to make an increase/decrease (play dough), or to take away/add (part-whole) in activities. They are implicit aims under the questioning/inquiry of how to make equality out of inequality. Implicitly we aim to appeal to students' intuitions to make those actions enactively, but it is early mention about increase/decrease verbally at this stage. This stage is brought back earlier, to meet students' intuitions on making equal. Pre-action stages play an important role in making students ready for action. While verbal pre-actions are the anchor to algorithms of algebraic actions, enactive pre-actions usually take place even before that, as investigations or acts. Hence, we tried to embed those stages in the trajectory whenever possible and appropriate. We took the range of the distribution of a subject in the trajectory as wide and as early as we could to support long-term remembering and learning.

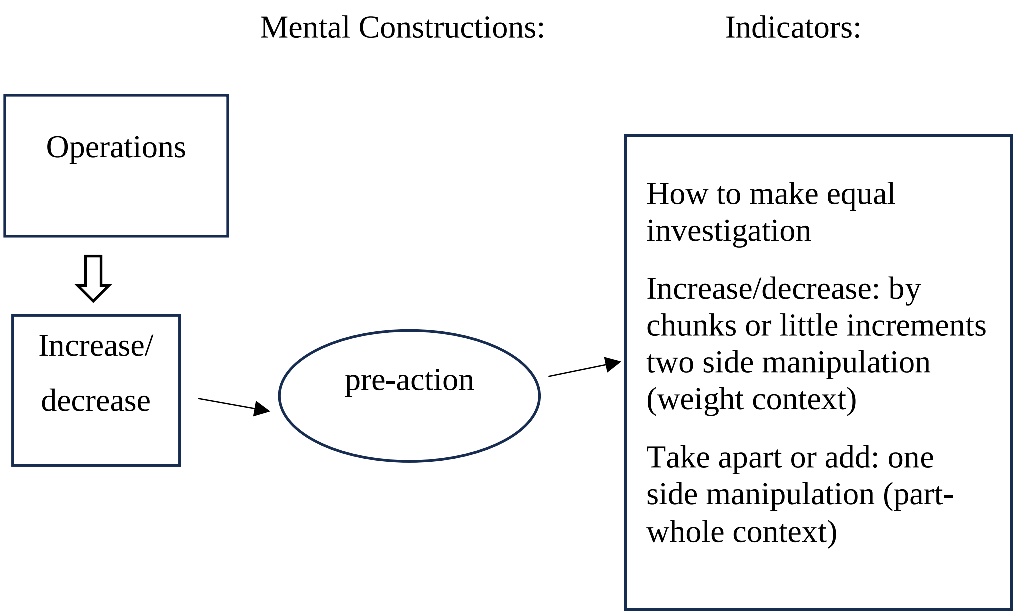


Figure 4.6. Schema for Learning Operations in Lecture 2

If we could define this stage as an action level for increase/decrease, we could say that students use equality as an object in this action. However, this is a prior stage, but not an actual algebraic action of increase/decrease. Moreover, students focus on creating equality, seeing equality as a process (reverse process), not a static state yet.

The student tries to compose an equality situation. In play dough and part-whole activities, the focus is on creating equality action, rather than increase decrease.

Playdough activity:

Students' capacity promised their early intuitions to make equal to be at a higher level to make equal, and we expected them to decrease heavier ones and add it to the lighter side intuitively. However, with some students, it did not work very well for two reasons; firstly, limited play doughs require additional understanding steps for manipulations, not singly but on both sides. Thinking about increase/decrease amount and partitioning was difficult (Ekim, Medine. 2 out of 10). Secondly, which side to decrease was difficult to determine due to a lack of experience in weight comparisons physically trying to reduce the upper scale (Ekim, Medine). Given unlimited play doughs would also make it easy to experience physically which side to increase/decrease when one manipulation at a time is adequate. Pre-investigations showed that even for adults, intuitively decreased higher scale is emergent when quantities are not comparable and structured. This problem does not count to be very important, there is no misunderstanding in terms of algebra. The student wants to decrease bigger size to make it equal. However, this misleading physical inexperience creates an opportunity to discuss equality, the bigger side in weight and reducing the heavier side. Reflecting upon these concepts will improve understanding of quantity.

Students (3 out of 10) who have difficulty modifying sides of balances by increasing/decreasing actions in the play dough context, tried the strategy of taking equal amounts/sizes of play doughs and comparing them by balance scales. 3 other students had another strategy of changing play dough on the scales in small increments. All others (4 out of 10) could successfully decrease the heavier side and increase the lighter side intentionally. For the majority of the class (7 out of 10), playdough activity seems to evoke increase and decrease actions.

Part-whole activity:

Students were expected to take away excessive parts by putting a cross on them and add missing parts by drawing. Adding or taking away was not difficult for students when they were directed to manipulate parts. Some students (4 out of 10) had difficulty interpreting equality after manipulations on the paper. They are guided by instructors. However, cutting excessive parts would be more beneficial to see equality. In addition, students who had difficulty modifying play doughs on balance scales also had difficulty in manipulating parts to make them equal to whole (2 out of 10). They needed close guidance to manipulate and interpret equality after manipulations. They both seem to interpret equality/inequality but are not ready to act on it.

Briefly, these two activities succeeded for the pre-action level for operation where the play dough activity served pre-action for increase/decrease, and the part-whole activity completed as increase/decrease amount.

Variables

Play dough activity serves a continuous context, which is a new context for quantity, supporting process-level understanding. Moreover, they engage change in quantity in an inequality relation, and change is continuous. This must be promising for a better understanding of the quantity. However, some students were not capable of focusing on one side being bigger, and manipulations were through increase and decrease. Part-whole activities also are not a superiority of one side being bigger, the focus is lost on quantity totally, to similarity. We could say we change quantity in play doughs, but not actually changing quantity algebraically, not operating on quantity algebraically, but a step ahead. Quantity might be still in progress for the process level until the bigger/smaller concept is. For some students change is on the object rather than quantity, even in playdough activity.

While play dough activity serves as a continuous context for quantity, there is no explicit quantity focus in part-whole activity, it is more on the similarity. However, part-whole activity underlies the message that quantity consists of parts. (It is recommended further as a Lego activity, which was not unearthed until retrospective analysis, uses two whole figures made out of equal or unequal parts for comparison. Students can separate to see if they are equal in quantity, to focus on the quantity to compose rather than the similarity of the resultant figures. Composition action will be executed from the procedure, which makes students hesitate to decide on equality. In this way the discussion will be on the equality of inclusion, rather than part to whole (which makes a difference.))

Focus on creating equal parts may lead to a loss of focus on the overall quantity. Part-whole equality shifts attention to similarity instead of quantity. In our study, it worked effectively at the pre-action stage for addition and subtraction, a stage not present in Davydov’s trajectory.Davydov only used this stage as being equal or not in a part-whole context, not discussing how to achieve equality. We expected students to make things equal by adding and subtracting parts from the same side, which again misplaced the focus on quantity. The preservation of quantity in subtracted and added parts was not discussed. In the previous class, students discussed the equality of objects in terms of the quantity they represent as an attribute. This might be why they paused in this context because they did not know which attribute to use to compare parts to the whole. The change was also based on the shape rather than the quantity.

The playdough activity focused more on changing the quantity to achieve equality. Most of the students seemed to manipulate quantity to make it equal (7 out of 10). Three of them (Didem, Ali, and Hasan) changed quantity by small increments while others increased or decreased to make it equal successfully, taking away from the heavier side and adding to the lighter side. Three students (Bekir Ekim, and Medine) took equal amounts and then tested on balance scales. This might have originated from not knowing how to increase or decrease in the weight context (Ekim, Medine), or not concentrating on quantity, in the weight context, yet. Bekir was shocked when identical toys had different weights in the previous lecture. In this lecture, he took equal-shaped playdough pieces by using a mold and then placed them on the balance scale to test their equality. Hence, we cannot conclude that all students perceived weight as a quantity measure.

Small increments of change in quantity seem to be the result of being careful to balance weight, as balance scales are sensitive to little differences. It makes a good starting point to understand continuity in change of quantity.

Notation

Playdough activity is enactive in representation, which students ignored and focused on creating equality. Not to create confusion, an iconic representation of play dough equality is not expected. Interpreting equality on the tables rather than the comparison manipulative represents comparison based on size, rather than other quantity types for students. Hence, this lecture does not aim at this stage yet. There appeared some intentions for iconic representation but it seemed confusing and remained as teachers showed off.

Part-whole activity is planned to be in iconic representation mode; being an on-paper work. One-to-one matching may be difficult, or how parts compose the whole may need explanation through cutting and paste. This activity turns into an enactive mode in terms of the necessity to address these problems. More essentially, they struggle to interpret equality for then and now situations algebraically. The inclusion and exclusion of parts on the paper are completed by students. However, they may not see equality, because resultant figures are not similar. To make it look more similar, especially cutting off excluded parts rather than crossing over, might help those students.

Orientation of signs is still a problem for some students even if they choose the correct ones. They may place sign notations in the wrong way. (Ekim, Medine), or they may create a sequence of a=b≠ to use all signs (Hasan).

4.2.3 Design Principles for Lecture 2

Process Equality

- The play dough activity serves as a new context for equality using continuously manipulable materials. The part-whole activity not only acts as a new context but redefines equality in its context. However, discrete material alternates with the play dough activity's nature. For further implications and explorations, the revision of the part-whole activity to a Lego activity that compares two holes to each other in terms of their inclusive parts is suggested. At least this part-whole activity can be carried out in enactive mode by using scissors to cut parts and bring them together to compose wholes.
- Be careful about the language and provided materials. When pictures of parts and wholes are provided in separate papers, students may think of equality of the size of papers provided to them. The precision of inquiry and what they understand from words affects a lot at this age.

Pre-action operations

- Play dough activity enactively, part-whole activity iconically enables investigations for increase and decrease. While one supports continuous change and sets out pre-action for increase/decrease action, the other uses discrete manipulation and bases pre-action for change amount (increase/decrease by an amount).
- "How to make equal" inquiry and ease of change in both activities encourage doing operations on equality. However, equality is not perceived as an object yet, because the focus is on making equality, displaying the reverse-process level for equality.
- In the play dough activity, the focus is on increase and decrease actions, which students can control in small increments change. However, the change amount is not as visible as it is in part-whole activity. Part-whole activity inspires taking away and adding actions naturally because what is missing or extra is clear in terms of parts.

- Drawing missing parts creates an analogy for addition, while the exclusion of parts is simulating subtraction by crossing over. Interpretation of equality before and after manipulations of parts can be facilitated by an enactive mode of representation.
- In this study, addition and subtraction are included together in the items of part-whole activity. “One manipulation at a time” case could be included for clarity.
- In play-dough activities, also increase and decrease actions take place together in manipulations, but on different sides. Play dough activity can be revised to include an unrestricted amount of play dough to experience increase and decrease controllably. Then, students should be supported by guidance for also using decreased actions to make equality. Instead of directly guiding through decrease, a reference weight can be used to create equal-weighted chunks of play dough to make equal to. However, in this way, the flexibility to manipulate both sides will be lost. Providing a free amount of play dough and no ordering for division into two, but creating an equal amount of play dough pieces is another suggestion.
- If practicing on balance scales does not alleviate the problem of deciding which side to decrease due to physical world inexperience, inquiry by “which one is heavier” question creates the needed anchor. The student verbally interprets which side is heavier and decides which side to decrease instead of automatically decreasing the higher side.

Quantity

- Using continuously manipulable material (play dough) helps students develop an understanding of variables.
- Manipulations for achieving equality underpin the understanding of continuous change in quantities.
- Allowing two-sided manipulation for change is crucial to prevent misconceptions and preserve consistency in understanding of variability.

- Continuously manipulable material and the capability to create equality in weight context prevents expectancy vs measurement conflict.
- Part-whole context makes it easy to change and take away, grounding an analogy for addition and subtraction.

Notation

- Start the activity part-whole with the enactive stage, or turn it into an enactive mode through cut-and-paste actions in need.
- Make students interpret inequality and equality at each stage, before and after making equal to keep them focused on the interpretation. In part-whole activity, interpretations may be assisted by enactive modes.

4.3 Results of Lecture 3

3rd lecture took place in 2nd day of the first week which took 45 minutes. The equality concept is explored in a volume context, which demands plenty of time being new and complex. Accurate measurement and connection of measures to the capacity of cups are additional learning challenges. 2 students' data are missing because they were absent. No make-up is provided and volume context is delayed until greater/smaller context for these two students (Eylem, Didem). They were both successful in previous contexts. The findings chapter does not aim to reveal students' individual learning progress throughout the activities. Data relies on features of designed lectures and how attendants' learning is affected. Moreover, in natural classroom settings, where design-based research takes place, there is always a similar concern of attending. If it is an important keystone, it is made up for the student. If future lectures have the potential to close up, students are assisted in opportunity.

4.3.1 Plan of Lecture 3

In this lecture, students compare cups based on their capacity/volume. To compare cups, students fill two different cups with salt and then pour each cup into identical transparent cylindrical measuring cups. This turns volume comparison into height comparison, to make it comprehensible for students who do not hold Piaget's conservation of amounts. Then, they use equal and unequal signs to interpret relation results from the comparison. This activity aims to teach equality in a new context volume for process level of equality and to teach iconic representation of equality in a volume context. The following objectives address these aims;

1. The student uses equal and not equal signs to compare volumes of cups.
2. The student interprets equality of volumes of cups iconically.

2nd objective is added later to the hypothetical learning trajectory. In the first plan interpretation of equality of cups seemed to be trivial. However, it creates an additional challenge for students to interpret iconically. Iconic representation is where it appeared first. In weight context, some students could report iconically based on weight when asked. Yet, iconic representation did not appear as objective in the weight context as it was difficult for the majority. Volume activity is convenient for iconic interpretations because measurement tool distinguishes measurement from cups. On weight comparison when students represent enactively on balance scales, objects being compared are present on the sides of the equality sign being on the balance pans. In volume context, when the sign is placed on the measurement tool, what you see is salt, but not cups. The need to represent equality with cups emerges to interpret which cup is equal/or not equal to which. By the nature of this activity iconic representation gains importance and becomes intrinsic whereas it was confusing in weight context because it reminds size in iconic mode. However, it is not straightforward and needs stimulation. It took 2 lecture-hour investigations to ensure all students made accurate comparisons and interpreted their comparisons iconically. The investigation was held through the following steps.

Lecture flow:

- Reminding the previous lecture on equality in weight context.
- Introduction of comparison of volume, a tool for mapping volume to height
- Introduction of iconic representation for volume comparison
- Lots of experience with comparison and placement of a sign
- Encourage to find equal cups that are not identical, and report equality between these cups.

Encouraging students to find equal cups has two purposes. Firstly, previous lectures showed finding equality creates motivation. Secondly, non-equal cups are obvious in shape and capacity, but students need to consider height, and size issues to choose cups in close capacity and then experience their being equal or not by measuring. Volume comparison gains meaning and importance in representing equality.

The next class would be on the same day on reporting equality between cups in the symbolic mode of representation. Then, students would be reading each other's reports. Based on the equality relation they read, they would conduct a comparison experiment to check if the peer's report is correct. This would facilitate the action of creating symbolic representations, and reverse process for this action by reading and making sense of the created algebraic expressions. This lecture on symbolic reportage of volume comparisons is held up until teaching of greater and less than signs due to two challenges. Firstly, students are not ready for symbolic representation as they met iconic representation recently. Half of the students had difficulty representing iconically on their own in a volume context. The second challenge comes from the measurement error in the volume context. Comparisons of cups mostly result in inequality; which would cause mediocrity/commonness in reportage and checking reports. Using ">, <" signs besides equality will solve this problem.

4.3.2 Theoretical Findings of Lecture 3

Equality

This lecture aimed to process level for understanding equality preserving volume as a new context. As the majority of students (7 out of 8 attendants) show evidence at the process level for using equal and unequal signs in volume comparison, the lecture is successful at its purpose. The one remaining student was able to state equality in volume context verbally but had some difficulty choosing the correct signs (Hasan). Focusing on finding equality, they all can be said to be at the process level. (Reverse process).

The comparison tool was helpful and no students had doubt or confusion using it to compare cups and interpret equality based on the comparison. The inquiry included the following questions in order: "Did you compare? Did it become equal or not?", "Which two cups do you compare?" "Then, are these cups equal or not?". These questions firstly pay attention to measurement tools to make students interpret equality, then change focus from measurement tools to measured cups to match comparison results to the relation between cups.

From the very beginning of the lecture, some students (3 out of 8: Aylin, Medine, and Yaman) showed their capability to transfer their knowledge of equality into this new context of the volume. All got fluent in interpreting equality in this lecture, after going through the following steps;

- compare volumes by tool (cylindrical cups)
- verbal expression of equality
- reporting by placing sign between cups. (Algebraic iconic representation)
- focusing on finding equality (Reverse process)
- showing equal cups by iconic representation

These stages represent what students could do gradually, also pointing out the flow of the lecture. Guidance took place to make students encounter all of the steps. Going through these steps, they succeed in the mentioned indicator behaviors addressing the process-level understanding of equality in a volume context. These behaviors evident from the data are summarized in the below figure.

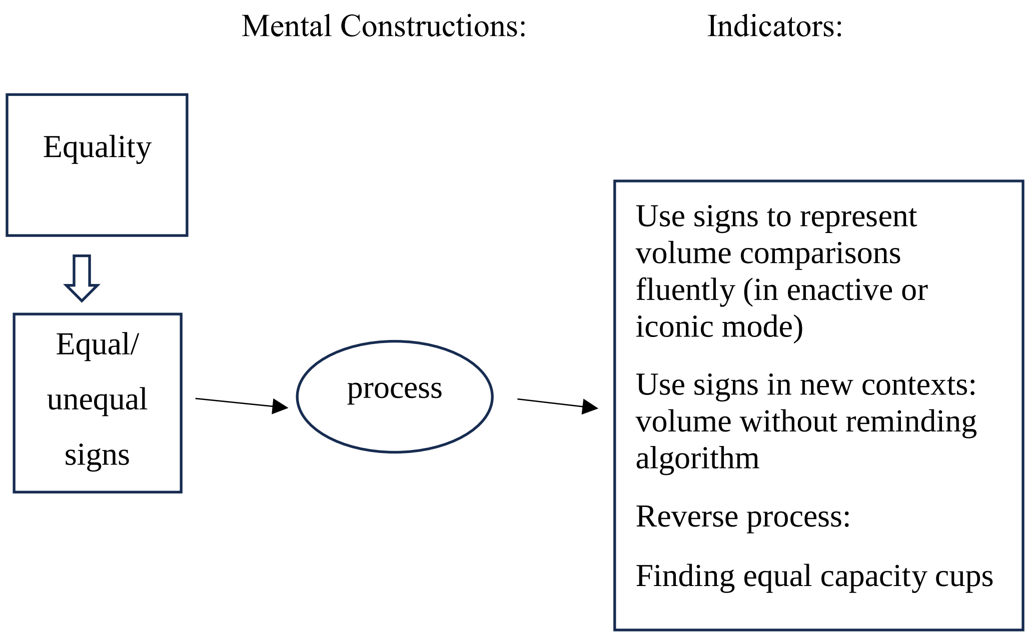


Figure 4.7. Schema for Learning Equality in Lecture 3

Variables

The volume serves as a new context for understanding quantity after weight context, which is expected to lead to process-level understanding. However, in the weight context, students confused weight and size for iconic representations, which shows they have difficulty conceptualizing quantity from the object itself. Hence, in weight context, quantity preserved action level progress. In volume context, no student had difficulty relating measurement results to objects being compared. Moreover, by interpreting results iconically, they were not misled by the size of the cups. Directing students to discover equality between different cups may be one of the reasons, for eliminating cups being the same or not in the first place of choosing cups to compare. After comparison, relating measurement to cups and interpreting based on the measurement was not difficult for them (except for one student, Hasan) after guidance. Most of them (5 of 8) became fluent in these interpretations, while the others correctly interpreted by guidance, showing they had no problem understanding what they were comparing and interpreting. The volume context seems to address the concept of quantity more effectively.

When envisioning volume as heights in identical cups, it becomes clearer that one side is bigger compared to the weight context, where some students mistakenly assume that the higher side is bigger. Another reason for conceptualizing volume as quantity stems from the use of measurement tools. These tools do not directly compare objects but rather their capacity, such as the amount of salt they hold. Thus, capacity is distanced from the object itself.

Besides all this, this activity indicates an action level for quantity, as it is still in progress (until we observe reverse process thinking on quantity through "based on what?" questions for equality). Only one student (Ufuk) expressed "capacity" for interpreting the equality of cups. This lecture, while introducing a new context, does not guarantee advancement in the quantity concept but still contributes to constructing an understanding of quantity as a measure of comparison.

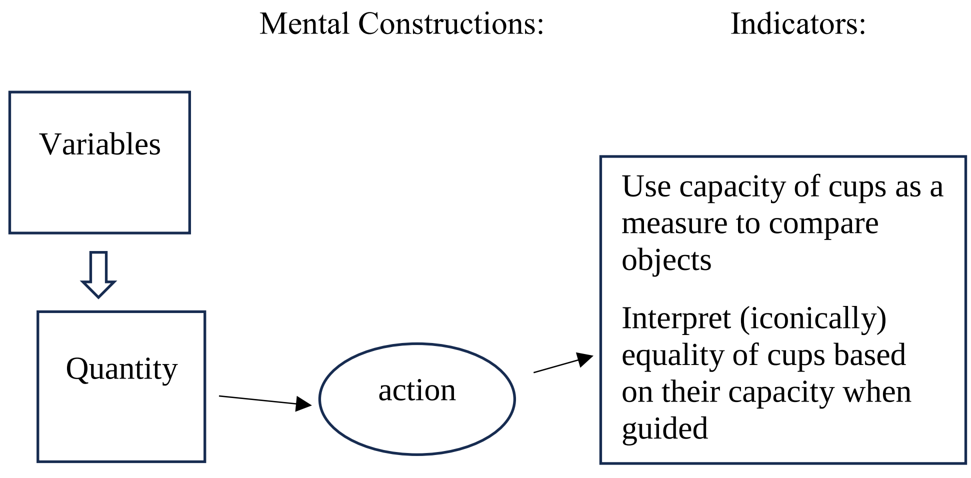


Figure 4.8. Schema for Learning Quantity in Lecture 3

Volume context needed more investigation and discussion. Motivation for finding equal cups prevents students from trying different shapes. They choose similar size cups, mostly based on height or precisely small cups to small cups. Hence, they miss opportunities for finding equalities for different shapes. To prevent this problem, activities can be more structured to satisfy certain comparisons. The investigation was not structured, but through guidance, students were directed to compare the capacity of wider to taller cups. The results of the comparisons could be discussed deeper, based on size, height, and volume. This would distinguish volume as a quantity measure from size.

The material used (salt) is continuously manipulable, but the comparison is between fixed quantities (capacity of cups). Even if quantity is nonmanipulable and fixed, creating that amount needs continuous manipulation. Filling of the salt and pouring reveal continuous change physically, which will add up to students' experience and envision.

Notation

Volume context empowers iconic representation as the measurement procedure separates the measurement aspect from the measured objects. We measure the volume/capacity of cups by the salt they contain, then interpret iconically using the objects themselves: cups. When interpreting comparisons by icons, what is referred to is their attribute of them. Moreover, the measurement tool lies in differences in quantity. In weight context one side is heavier, but not apparent as in volume context.

Even the majority of the students (7 out of 8) could use signs perfectly on enactive mode without reminding (process level on enactive mode), 4 of them carried it to the process level of iconic representation. In contrast, four others needed guidance or reminding for iconic representation till the end of the lecture. Hence, this activity is assigned to be action level for interpreting iconically in a volume context. Even though some (3 out of 7) students could transfer their knowledge on equality in volume context in enactive mode, only one of them was automatically representing iconically (own, without guidance) from the beginning (She was also the only one who could iconically represent in weight context by reporting comparison results of one equality and one inequality on the table without hesitation.). Others needed repetition and encouragement for representation in iconic mode to actualize it or become fluent in it. Directing to find equal cups was one of encouragement for iconic representation (Medine, Aylin, Ekim). Inquiry for anchoring measurement to the compared objects is another method for stepwise teaching of the iconic representation. Stepwise inquiry consisted of the following questions: “Which cups did you compare?”, “Were they equal or not?” “Which sign did you choose to put between the cups.” These questions aim to connect cups to measurement, then measurement results to the relation between cups. At last, the chosen sign is placed between cups, working as an iconic representation of capacity and volume. An additional template would be helpful for iconic representation where students can place their compared objects and sign the resultant comparison. In further lectures, it is planned to place templates as much as possible.

Guidance for the iconic representation relied on starting with enactive representation. The researcher assumed that iconically interpreting the relation between cups (using the cups) is trivial. However, the struggle of some students or their tendency to put signs on the measurement tool proved it to be non-trivial. (In this lecture, the cognitive connection of measurement results to the objects was sufficient. It became more complicated in >, < signs in volume context in symbolic mode, where the kindergarten teacher’s step-wise inquiry helped more.)

The following lecture would be on reporting, reading, and checking each other’s reports in symbolic mode. It moved forward for three reasons. Firstly, they learned about iconic representation. Iconic representation was not as straightforward as we thought. Hence, symbolic representation requires extra time at a later time after students strengthen their understanding of iconic representation (at least process level). Secondly, using only unequal signs has a higher probability in a volume context, as experienced in the first half of the lecture. This can make all reporting monotonous, and checking the reports will consistently result in correct inequality findings. The inclusion of >, < signs would be more meaningful and interesting.

Lastly, reporting and checking peers' reports activity would be revisited after >, < signs are taught. Repeating the same activity decreases their enthusiasm. It is better to delay, as the former activity would also comprise this one.

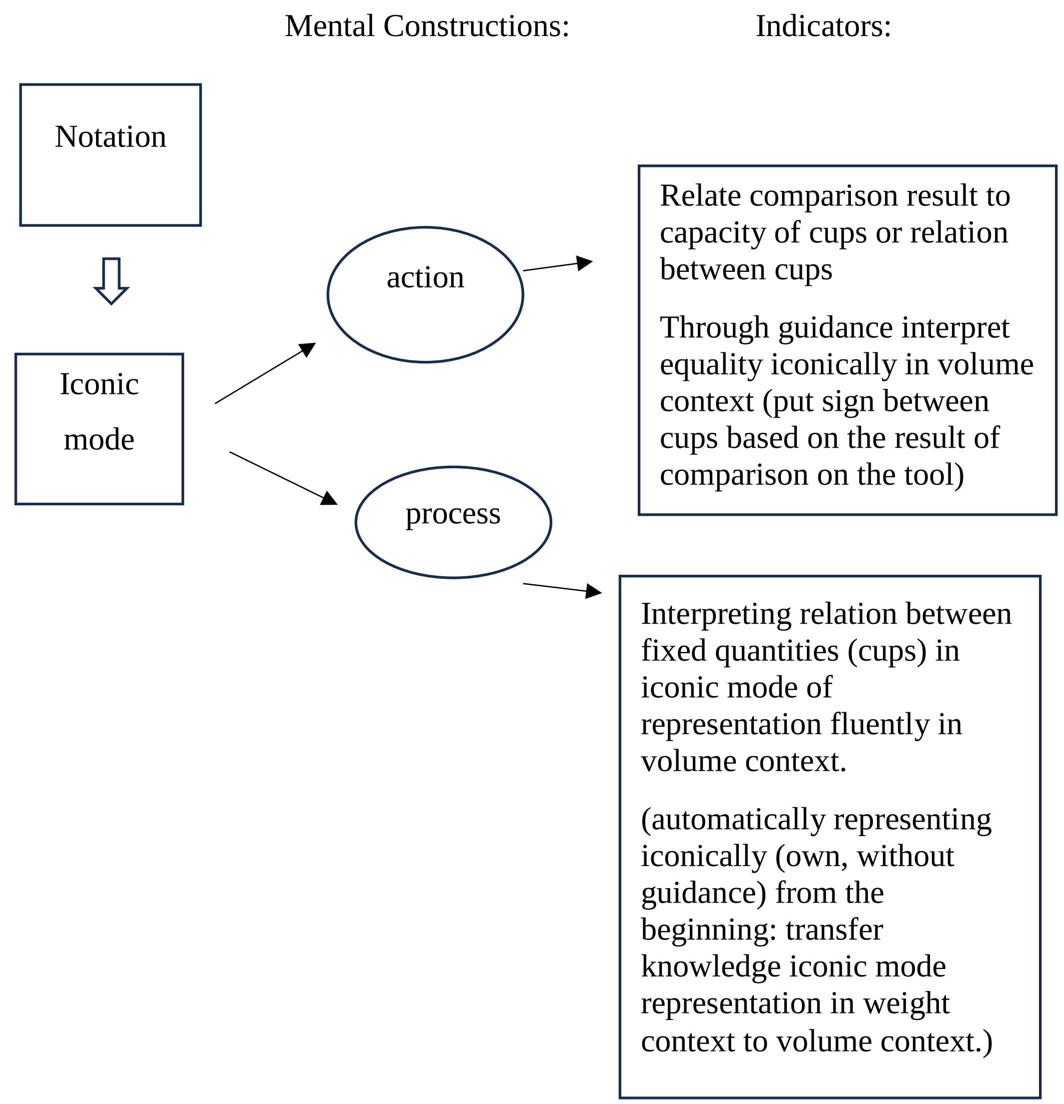


Figure 4.9. Schema for Learning Notation in Lecture 3

Interpretation of equality in iconic mode requires "based on what" thinking, as putting signs between cups is based on their capacity, not based on size or height. However, until we observe reverse process thinking on quantity through direct "based on what" questions for equality, we cannot deduce students have a proper understanding of quantity in their interpretations. They may be one-to-one projecting their representations. Half of them were fluent and showed no hesitation. They seem to be interpreting consciously based on the capacity of objects. Iconic representation contributes to the understanding of quantity.

4.3.3 Design Principles for Lecture 3

Equality

- The teacher can guide students to compare certain kinds of cups so as not to miss opportunities to find equalities for different shapes. However, first classes are recommended to be less structured, so by getting more exposure to manipulatives, students may connect the physical world to mathematical deductions.
- Finding equality should be a part of the activity to support reverse process thinking and also improve motivation for investigations.
- By being restricted to choosing different cups that seem equivalent to see if they are equal, students are oriented away from choosing the same cup and seeing that it is unequal due to measurement errors.

Variables

- Using a tool to measure volume changes a 3D variable (volume) to a 2D variable (height) for comparison. (regarding the readiness test on Piaget's conservation theory). However, interpretation should be based on the capacity of the cups.
- Focusing on finding equal cups, students tend to compare similar-sized cups. Structured inquiry can improve investigations to compare certain cups with different shapes, such as those that are wider or taller.
- The volume context elevates understanding of quantity. For students, volume is a new attribute that is unpredictable from appearance. Comparing one

volume to another reveals its superiority, which distinguishes it from merely being an attribute to representing a magnitude.

Notation

- Stepwise inquiry should connect comparison results to the relation between compared objects through questions such as “Which cups did you compare?” “Were they equal or not?” and “Which sign did you choose to put between the cups?” Interpretation of the relation between objects based on volume forms iconic representation.
- To support iconic representations we can form templates, where they can place their compared objects and sign the resultant comparison. It is recommended with the teaching of equal and >, < signs together. It will work as a reminder algorithm tool till it gets automatic.
- Finding equality not only encouraged investigations but also created a natural motivation to use iconic representations to show their findings to teachers.
- Being inexplicit based on shape, volume context necessitates iconic interpretation.

Materials

- Having different shapes but equal volume cups helped with experimenting but did not guarantee equality due to measurement error.
- Having equal cups with different colors helped in discovering equalities.
- Comparison tools can be non-sectional to prevent oversensitivity for comparisons to reduce measurement error.

4.4 Results of Lecture 4

This lecture took place in the second week. It is the introduction lecture for >, < signs. After using>, < signs for simple object comparison, the lecture continues with weight context. Investigation and discussions took about 50 minutes long. nine students out of 10 attended class. (Aylin did not attend).

4.4.1 Plan of Lecture 4

To teach how to use >, < signs to interpret inequalities, this lecture is planned around the following objectives in the first HLT.

Planned objectives:

1. The student interprets inequalities with greater or smaller relation.
2. The student uses >, < signs to interpret relations.
3. The student interprets (verbally) how to make equality from greater or less than relations.

The first two objectives aim for action level for >, < sign use for interpreting inequalities, while the third objective is for the pre-action stage for addition and subtraction operations through verbal interpretation as increase/decrease. Each context is centered around equality and how to make things equal from Davydov's perspective. If not equal, discussion on how to make it equal prepares students for further operations topics. (While teaching operations, discussion on how to make equal facilitates properties of operations.) This lecture is planned to discuss how to achieve equality, through verbally interpreting as increase/decrease, by examples of balloons and apples. However, before verbal pre-action on the increase/decrease, we studied it enactively, with a balloon example in the second lecture. Enactive pre-actions appear essential and less demanding/challenging than verbal pre-actions. The balloon example was successful as being an analogy for increase/decrease. Students have control over balloons to make them equal. In the apple example, there are two apples then one is bitten, and then the other one should be bitten, too, to make it equal to the first one. It seems a good analogy to enforce decrease/subtraction with an equal amount. However, students at this age have difficulty accepting the equality of two objects (pilot and first lecture results). Hence, the apple example is removed. Through the balloon example, 3rd objective has partially been accomplished in 2nd lecture. Lecture 2 objective #3 is: "manipulate (increase/decrease) one side for the satisfaction of equality in part-whole activities," and Lecture 2 included demonstrations with two balloons at the beginning. Some students already could interpret increase/decrease verbally in the balloon example, and play dough activity in 2nd lecture. Verbal interpretations would require extra challenge for students in this new topic. Hence, 3rd objective is omitted for this lecture, delayed for introduction to operations. This lecture focuses on interpreting bigger or smaller objects verbally and with algebraic signs. The following objectives define the revised aims of Lecture 4:

Revised objectives:

1. The student interprets inequalities with greater or smaller relation.
2. The student uses >, < signs to interpret relations.

Interpreting inequality with new signs, the lecture starts connecting new signs to unequal sign. For this reason, students are expected to represent simple object comparisons with equal and unequal signs, which they knew before, and then presenting new signs, students are asked to replace unequal signs with these new signs. Actually >, < signs are provided to students as single signs, which are twisted to place in the proper orientation. At this age, students do not know how to read words or expressions from left to right. We did not teach equal signs with orientation as well. We did not say, “This is equal to this,” but “These are equal (to each other),” emphasizing balance.

The lecture follows a structured path for replacing unequal signs, connecting to previous learning, ensuring the use of inequality, and aligning them all in the discussion of the replacement of signs. Finally, the use of signs is continued with a weight context, as in every topic we learn. Moreover, the weight context promotes quantitative comparison for inequality situations.

Lecture flow:

1. Presentation of new sign >, < for indication of bigger and smaller.
2. Student finds objects based on given sign =, ≠, based on size.
3. Inquiry into which signs we can use for equality and inequality situations.

4. Students use >, < sign instead of ≠ sign to interpret comparisons of discrete objects based on size.
5. Students interpret weight comparisons by >, < signs.

This last step also helps to understand which side is heavier. The previous lecture addressed the lack of experience in the physical world of comparing weight. This step takes students one step back for condensation on weight comparison and interprets which side is bigger. It also teaches how to interpret being bigger based on weight.

While teaching >, < signs at the action level, this lecture uses =, ≠ signs. Replacement of unequal signs needs reflection upon it. However, only one student could achieve this, which will be described in the theoretical findings. Hence use of =, ≠ signs does not necessarily indicate an object-level understanding yet. Lecture flow stimulates following APOS level learnings;

Action >, <, Process =, ≠

1. reverse process =, ≠, (find objects)
2. action <, >: interpretation of inequality with <, > signs
3. action to process <, >: interpretation of inequality with <, > signs, in new context: weight

4.4.2 Theoretical Findings of Lecture 4

Action level inequality:

This lecture successfully teaches >, < signs at the action level. Only two students had difficulty carrying out the algorithm (Ekim, Medine). One of them (Ekim) had poor attention and was ignored due to her unwillingness. However, her loss of motivation seemed to originate from her poor understanding of quantity. She does not interpret equality relation between two objects based on any attribute related to volume, height, or weight, but limits herself to comparing the size of some parts.

R: What is bigger, Ekim? (according to what)

E: The circle is bigger. (There are two wooden blocks: a cylinder and a square prism. She points out that the circle's surface is bigger than the square's, but there is no significant difference in size between them.)

R: Have you looked at their heights?

Ekim shakes her head, meaning no.

R: Look, their heights are the same.

To overcome these problems, structured comparisons can be created by limiting toys. For example, toys of the same kind or similar shapes but different heights would initiate comparisons based on height.

Another student's (Medine) struggle was with the orientation of the sign. Previous sign interpretations depended on only choosing the correct sign, but now she needs to interpret relation by the orientation. Moreover, she had difficulty seeing the sign as a static object. Teaching which side is bigger and which side is smaller by actions did not work for her, while others had no trouble. She moved the sign like a paper plane to interpret relations; the sign worked like an arrow indicating an action.

Seven students could carry out the algorithm of using >, < signs to interpret relations based on size or weight. Three of them (Ufuk, Yaman, Ali) became fluent, and two (Yaman, Ali) of which were preferring unequal signs sometimes. We can conclude that this lecture not only served for action level but also supported process level understanding for some of the students. The lecture level is determined as action due to the majority's understanding, which needs support and reminding in the algorithm. Two students (Eylem and Hasan) had difficulty transferring the knowledge to weight context. The difficulty was not because of understanding the interpretation of the relation with the signs but determining the heavier, bigger side in weight context. Asking, "Which side is heavier?" helped them correct immediately.

Inquiry for teaching algorithm of using <, > signs is as follows:

- Find objects based on previously learned signs (=, ≠)
- Which signs are appropriate for the situation? Discuss inequalities.

- Interpret bigger and smaller sizes for inequalities.
- Replace ≠, with >, < signs

However, when it came to replacing the unequal sign through discussion, only Yaman was successful. Yaman is interested in numbers and arithmetic. From the very beginning, his arithmetical abilities hinder equality with non-numerical quantities. For being equal or not, or being smaller or bigger, he has numerical or quantitative reasoning in mind rather than the shape or properties of the objects compared. The sentence is mostly correct, but it could be improved slightly for clarity: The others might not develop the concept of equality solely based on quantity, or they might think only one answer can apply to a situation. However, it seems they just acquired it as a new concept independent from the previous ones.

Despite his successful understanding of the signs conceptually, Yaman preferred using unequal signs. Restrictions for signs might regulate acceptance of the new signs. But this stage needs clarification of the replacement conceptually. Whereas most of the students could not understand replacement, rather than restricting signs, students were guided to verbally interpret which object was bigger when they used unequal signs. Then, they are directed to use newly learned signs to indicate the bigger side. Interpreting bigger or smaller object based on an attribute, not only indicates their understanding of quantity but forms a pre-action level being a prerequisite for the interpretation. Together with the notational orientation, an algorithm for the interpretation of the new signs is accomplished. Orientation of the sign demands reminding its algorithm, standing as a distinct ability/knowledge.

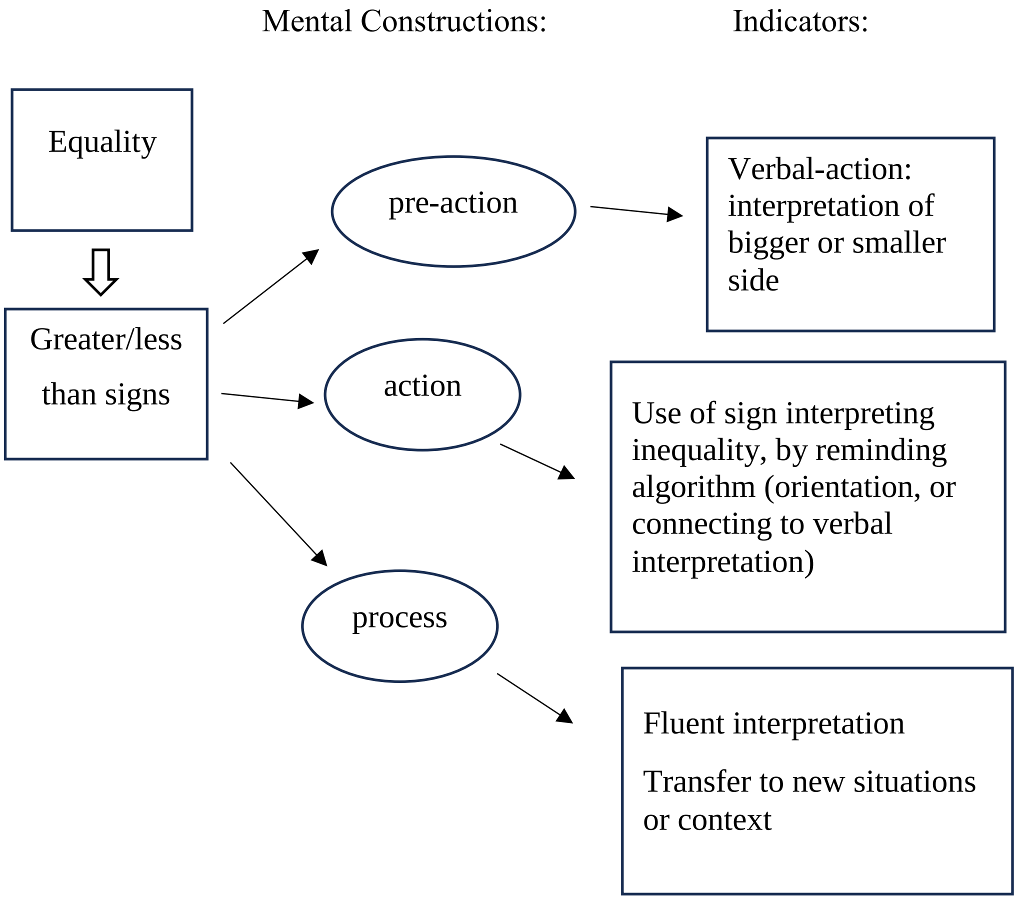


Figure 4.10. Schema for Learning Equality in Lecture 4

Weight context did not guarantee process level being new context, but supports action level as it indicates one side being bigger. Additional restricted contexts for example height could be helpful. Moreover, it is a difficult context to interpret the bigger side. Not guarantee the process but it satisfies the action level enforcing algorithm on quantities.

All of the students were fluently using =, ≠ signs interpreted equality for different attributes of objects, could do reverse-process, and they could relate inequality with new signs >, < which shows they can compose it with other processes (Dubinsky & Mc Donald, 2001). Only one student (Yaman) could reflect on the use of >, < signs instead of inequality. That might be an indicator of his object-level understanding of equality relations. Other student's use of inequality sign in unequal situations hinders using equal/unequal signs.

> R: Very good, well done. Now listen to me. You found that these two shapes are not equal, Medine. Yaman, listen. Did you find them to be equal? No, you found they are not equal. Well done, Yaman. Now, can we put an equal sign between these two shapes?
> Students: No.
> R: Why not?
> Some students: Because their heights are not equal.
> R: Yes, exactly. It means they are not equal, so can we put this sign? (unequal sign)
> Students (all): Yes.
> R: Can we put this sign? (greater/less than sign)
> Students (all): No.
> Yaman: yes
> R: Yes? Yaman, why yes?
> Yaman: Because one side shows small, and the other side shows big.
> R: Yes, indeed. (all the students are shocked, especially Eylem) We have a big side and a small side, don't we? But not everyone may have it. Medine, do you know which one is big and which one is small?
> Medine: This one is big.
> R: That's right. So, can we put this sign between them?
> Medine: No.
> R: Why not?
> Medine: Because they are not equal.
> R: They are not equal, that's correct. But one is big, and one is small. Which one can we use to show which one is big? Let's everyone take this sign, and hold it in your hand. Show the big side to the big side and the small side to the small side. Put it between the two. Let's see, put it between their pictures.
> Some students: I did it. (Medine, Eylem, and Ali put the sign unattentively, and Yaman put it correctly)

Yaman was a little bit cautious about using new signs after this conversation. His level of consciousness in his reply was obvious from his gestures and consistent throughout the lecture. He is one of the students who considers quantity in comparisons rather than shape and always uses the sign in the correct orientation. Using unequal signs, and greater-less than sign alternatively was not challenging for him. For some students, it was difficult to use signs to indicate the relation between sides, but they were matching situations with signs. For these students, guidance for the orientation of the signs takes extra effort, specifying an extra level of understanding.

In addition, the lecture allowed some specific mathematical deductions while students were freely experiencing. Eylem compared herself to friends based on height and put signs between them. She said, “I am bigger than you and bigger than you. So, you two are equal, but I am bigger than both of you”. It was a wrong deduction, but it showed she had conservation on quantities and could relate three quantities in a row using the newly learned signs. She was using newly learned correctly in her own created situations. However, the weight context challenged her to transfer new knowledge. (Eylem’s incorrect algebraic intuition: $a>b$, and $c>b \Rightarrow a=c>b$)

Another mathematical intuition aroused from Bekir’s experimentation with weight balance on which he likes to experience additive relations. After composing a relation of $a+b=a+b$ on the balance scale using two types of identical toys, he added another toy to one of the balance pans and immediately concluded that $a+b<a+b+c$. He knew how to change equality and how the result will be affected. He did not forget to interpret the result with new learned sign $<$. (Bekir’s correct algebraic intuition: $a+b=a+b \Rightarrow a+b<a+b+c$)

Allowing various materials and giving sufficient time throughout different contexts provided students with experience and discussed equalities and inequalities in multiple aspects associated with quantity.

Variables

In the first part of the lecture, students are free to choose attributes of objects for comparison. Still, choosing an attribute defining a quantity was problematic for some students (Ekim). Size and height were discussed in the comparison of simple objects.

In the second part, comparing based on a pre-determined attribute weight is more appropriate for focusing on the quantity. However, the weight context has its difficulties. For Hasan, defining heavier as bigger was complicated. When placing

>, < signs, he considered size rather than weight. Some other orientation problems (Ufuk) might address the same difficulty. Another problem with weight context is that it might be complicated for some students to determine the bigger side without guidance, as in Eylem's case. When thinking about the bigger side, she automatically considered a higher balance pan. Weight context ensured quantity comparison. Due to difficulties and students' inexperience in weight context, starting teaching of >, < signs structured around height context would be more apparent beforehand.

In both parts, comparison is between discrete and non-manipulable objects. The first part was confusing in quantity. However, in the second part, not only weight context aided focusing on quantity, but in contrast >, < signs helped to clarify the quantity concept by focusing on one side more. In other words, it is a double side benefit, based on the student's difficulty. Weight context provided natural context for these signs, while signs helped students who had difficulty understanding quantity in weight context.

In previous lectures, students could see quantity differences as images in a volume comparison context. Hence, we assigned the previous lecture for action level of quantity (comparison). In this lecture, it is not only observed visuals but also students interpreted quantity comparison verbally and algebraically by the use of signs, indicating in "one is bigger" relation. For the comparisons, not being the same or not, but being bigger or smaller became the aspect, which will be turned into more or less relation in the volume context in the following lecture. Not only through the attribute contexts but also in the equality relations, quantity knowledge improves and evolves.

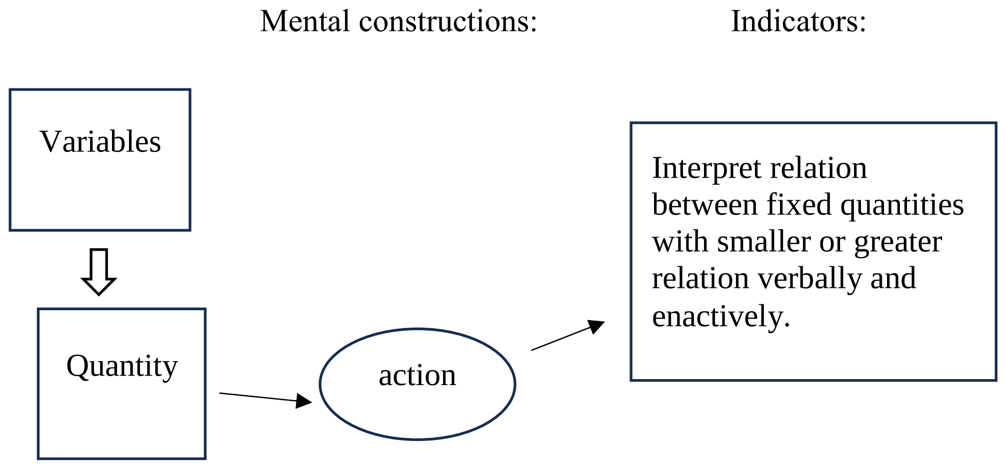


Figure 4.11. Schema for Learning Quantity in Lecture 4

Notation

This lecture recommends only using enactive mode in notation; students put signs between objects or on a balance scale in weight comparison. They were all enactively notating the relation between quantities with equal and unequal signs. Sometimes, they needed to be reminded to use new signs in their comparisons (Ali, Yaman, Bekir). Some (Hasan, Didem) needed a reminding algorithm for the orientation, which side shows bigger, which side shows smaller. Some of the students became fluent in the use of new signs (Bekir, Ufuk, Yaman, Ali).

Big and small are not new concepts for students. However, the interpretation of signs is complicated, as observed. Hence, it is proved to be a good decision that this whole lecture is dedicated to new sign and their connection to inequality. If it also included how to make equal progress as it was in the first plan, it would be difficult and would not fit in a day lesson. (Also, it is due to the difficulty of comparing weight.)

Orientation of the sign demands reminding its algorithm, standing as a distinct ability/knowledge (Ali, Ufuk, Medine). These students having difficulty in orientation seem to choose signs based on the situation, not to relate or connect the sides. Thus, they may even place a sign pointing upwards.

Explanation of the sign played an important role in how they used it. There are no two signs: >, <. There is only one, and it is dynamic. First, it was planned to include two static signs, and students would show off with two banners. At the beginning of the lecture, when signs are introduced, students would choose the sign and hold up the banner based on the situation given. However, the interpretation is confusing by matching. The single sign was adequate and comprehensible.

The researcher tried teaching signs as actions based on previous class findings, which showed that teaching equal signs as actions worked.

> R: What am I doing to show this? I am holding the big side to the big, wide-open side. It is showing the small side; it is getting smaller and smaller, showing the small one. Here is the small side. Then it gets wider and wider, showing the big one. Did this confuse you a bit?
>
> Ekim: Yes.
>
> Aylin: No.
>
> Ekim: My head is spinning.

Especially this type of kinetic explanation confused Medine;

> R: How will we show with the big-small sign? Show it to the big one.
>
> Medine: This one is getting smaller and smaller. It becomes small. Then it goes like this: bigger and bigger.
>
> R: Medine, it does not move like this; it stays like this. The big side shows the wide one. This side has shrunk and become small. This shows the small side. Our sign will stay like this, okay? Now, let us see, Ufuk ….

Following a static explanation of the sign, which matches objects to the sides (point on corresponding sides, not decrease), worked better;

> R: They are not equal, but one is big, and one is small. We can use this big-small sign to show which one is big. Let everyone take this sign, hold the big side to the big side and the small side to the small side, and put it between the two. Let us see, put it between their pictures.

4.4.3 Design Principles for Lecture 4

Inequality

- Provide all signs learned and do not restrict choice. Previous knowledge of ≠ signs may prevent/hinder the use of >, < signs. However, it will strengthen the relationship between the two, when included and discussed together. Appreciate the use of unequal signs when appropriate. Then, by inquiry, make the student verbalize which side is bigger, ask whether >, < signs are appropriate, and encourage to interpret with >, < signs.
- Replacement of unequal sign with >, < signs is complicated because inequality did not solely develop on the quantitative reasoning up to now, or students may believe only one answer applies to a situation.

Quantity

- Even discussed in previous lessons, quantity could be underestimated or not attended by students for comparisons. Inquire into their thought when expressing inequalities and how they connect them to the newly learned >, < signs. Based on the algorithm, they will compare objects and put signs in between based on the quantity. However, they will need to consider quantity to compare objects (to set up a comparison) to use these signs. It seems that learning this new knowledge builds on the previous knowledge of quantity. Indeed, it provokes quantification in comparisons, proving the importance of the subject in the trajectory.
- Learning of <, > signs and how to use them to interpret the relation between quantities is challenging enough that a lecture should be dedicated to it. No additional inquiry on "how to make equal" is applicable unless students become fluent in the use of these signs.
- Weight context creates extra difficulty in determining the bigger side. Do not question the bigger side or heavier side, but the question "Which has bigger weight" in inquiry to connect the weight to being bigger. Besides difficulties,

weight context generates satisfactory circumstances to discuss quantitative relations in comparisons.

Notation

- Teaching signs as static objects, which have corresponding indicator sides for being smaller or bigger, deserves dedication of sufficient time and effort. Static explanation of matching sides of the sign to compare objects ("wider side shows bigger one, narrow side shows smaller one") is mathematically compatible and creates long-term knowledge.
- Systematically remind notation for comparisons, even for reverse-process activities. Verbal interpretations for their interpretations should be requested to understand and underly their thoughts and develop mathematical language for equality and quantity concepts.
- To focus on one learning at a time, iconic representation for weight context delayed for its complexity. Enactive notation keeps it simple and focused.

4.5 Results of Lecture 5

The previous lecture was introductory to interpreting inequalities with >, < signs with simple object comparisons, and additionally in weight context. This lecture continues the use of >, < sign in volume context. It has two main parts: interpretation in symbolic mode, reading peer's reports, and checking by enactive investigations. Reporting took about 35 minutes reporting, while checking reports took 15 minutes.

All ten students attended class. Aylin did not attend the previous class, the introduction on >, < signs with simple object comparison and weight context. She learned these signs in this lecture in volume context for the first time with a little individual introduction. She was a quick learner and did not face any difficulty.

4.5.1 Plan of Lecture 5

This lecture aiming process level in the use of >, <, = signs and continues with volume context. We started symbolic representation with equal unequal signs in Lecture 3, which stayed at the action level. We continue with additional >, < signs for symbolic representation and aim to carry it to the process level, including the reverse process of symbolic representation immediately after. Hence, it includes two stages of activities: symbolic representation of three comparisons in volume context and reading and checking peers' symbolic representations.

Reporting and checking back was planned earlier with using only equal and unequal signs for the fourth lecture:

Objectives for symbolic representation in the first HLT:

1. The student reports comparison of volumes of objects symbolically on the paper with =, ≠signs.
2. The student reads symbolic interpretation of equality and inequality and checks it with concrete objects.

We thought reading and checking back symbolic representations would be more meaningful, including >, < signs. Reporting volume with >, < signs was planned to be in transitivity concept in the first HLT.

Objectives for using >, <, = signs in volume context in the first HLT

1. The student uses >, <, = signs to interpret the comparison of volumes of cups.
2. The students use two relational interpretations of three cups to guess the third relation (transitivity property).

Symbolic representation and transitivity concepts are both problematic concepts that deserve the dedication of separate and sufficient time. Mainly, the transitivity concept was found to be challenging in pilot studies. Hence, transitivity will be handled longer and in a more structured environment. Measurement errors in volume context would create divergencies and problems for stability in class for correct answers and deductions for transitivity. Hence, it should be structured around obvious experimentations. Additionally, symbolic representations should be understood for deductions of transitivity, as it appeared in Davydov's book (Davydov et al., 1995). Supporting symbolic representation before the transitivity concept is more meaningful. Moreover, the transitivity concept is decided to be placed after the ordering concept in the learning trajectory, as a pilot study shows that ordering inquiry is the only working strategy we discovered. In this lecture, symbolic representation plays a central role, together with its reverse process: reading symbolic representation. Reading algebraic representation means deducing meaning from the written expression. Checking its correctness works as the reverse process of the algorithm of comparing objects and interpreting equality between them. Volume context creates the perfect environment for symbolic expressions. However, it requires considerable time, as students cannot get fluent even in iconic representation the first time they encounter volume context (based on Lecture 3 results). Hence, we decided to handle symbolic representation with volume context and transitivity in a separate time. Briefly, supporting process level understanding in using >, <, = signs in volume context for developing symbolic interpretation and promoting reverse process in symbolic representation; Lecture 5 is constructed around the following objectives:

Revised objectives for Lecture 5:

1. Report: The student interprets the comparison of volumes of cups by >, <, = signs symbolically on paper by using pictures of compared cups as letter notation. (symbolic representation)
2. Read report and check: The student reads/uses a symbolic representation of a peer's comparison and checks with manipulatives if the comparison is valid. (reverse process for algebraic notation in symbolic representation)

To acquire these objectives, the lecture proceeds in the order below:

1. choose two cups and compare volumes with the help of identical cylinders.

2. use pictures of cups to represent comparison symbolically on paper.
    a. place cups near corresponding cylinders. (If necessary for the student, step back for enactive representation; place sign between identical cylinders.)
    b. place sign between cups (iconic representation)
    c. place pictures of cups near corresponding cups.
    d. carry sign between pictures
    e. stick pictures and signs on paper for reporting
3. repeat comparison and reporting three times
4. change reports with peers
5. peers check comparisons by cups based on peers' reports

2nd step is extended after classroom implementation. We thought symbolic representation would be straightforward because representative cup photos seemed clear. However, representing the relation between these photos was challenging for students. The kindergarten teacher developed major steps from "a" to "e" to help students relate compared amounts to the pictures of cups. The reportage was not tricky once this connection was acknowledged.

Trajectory:

- discrete/fixed quantity comparison of a continuous variable (volume)
- iconic representation of >, <, =
  (process >, <, =, new context: new variable volume)
- symbolic representation of >,<, =
- read the given symbolic representation >, <, =
  (reverse process, algebraic/symbolic notation)
- model the given symbolic representation with a real-life experiment
  (reverse process, algebraic notation/ equality

4.5.2 Theoretical Findings of Lecture 5

Equality

The majority (8 out of 10) of students are capable of fluently interpreting inequality relations with >, < signs, demonstrating that this lecture supports process-level learning of using >, < signs in representing inequalities with new context volume. Four of these students were immediate in representing without help, showing they were already at the process level and successfully transferring previous knowledge to the new context.

The remaining (2 out of 10, Medine & Ekim) had difficulty using these signs correctly showing they are still at the action level of using them. Both struggled to remember to use sign independently and needed guidance and reminding. Unequal signs might still be hindering them (Medine), which also evidences their level. Reminding the "wider side to bigger object algorithm" or inquiry into which side is bigger, pointing to the sides of the signs, helped them remember. Step-wise inquiry connecting bigger objects to bigger/wider sides and smaller objects to narrow sides took attention to see the signs as static objects having particular sides.

Not only using >, <, = signs in new context volume, the process level is supported through the transformation of representation modes because the process of using signs is composed with the process of symbolic notation. In the reporting procedure, when enactive representations are transferred into symbolic representations, students need to use these signs. In reading reports for checking, they reverse processes using signs.

The following schema gives mental constructions for greater/less than signs for Lecture 5 (see Figure 4.12).

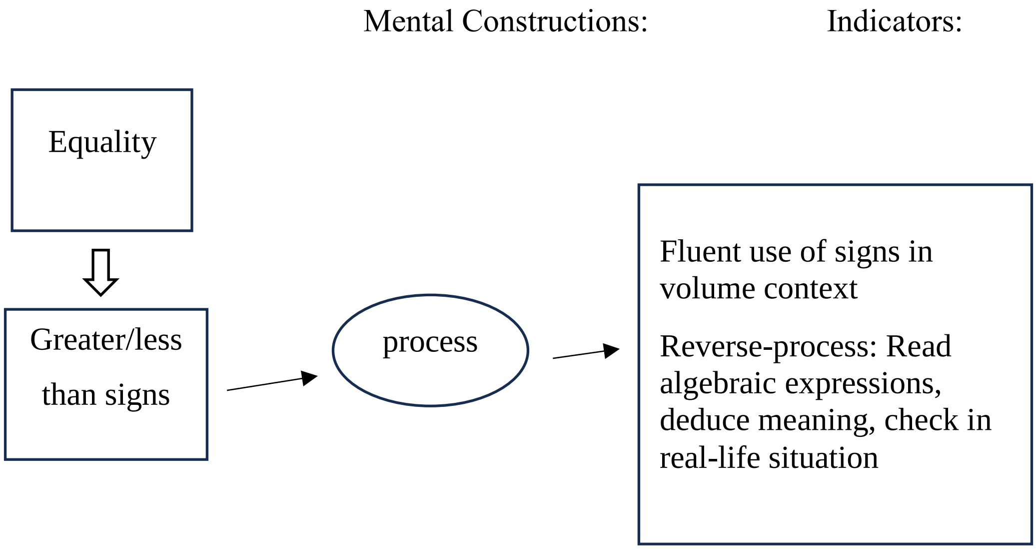


Figure 4.12. Schema of Learning Equality in Lecture 5

Variables

Volume is a continuous quantity, but again, we work with fixed quantities. The challenge about quantities in this lecture is that it is conserved through modes of representation. Representing interpretation based on the quantity type can be problematic for some students. Quantities are transferred through representation modes through step-wise one-to-one projection guidance. In reading reports, students need to interpret the relation between quantities verbally. Didem forgot all the processes we go through for volume comparison and assessed the peer's report based on size.

Verbal interpretations are essential at this stage to observe students' deductions based on the reports. If a student reads symbolic relation as "this one takes more" verbally, we can conclude that he/she is considering the volume/capacity of the cup in the picture. "This one is more" response is also acceptable. However, if he/she says, "This is bigger," he/she might be considering the size. Students should be questioned and guided if the deduction is valid based on the size but not volume (as in Didem's case). These interpretations not only give clues when checking reports but also in the interpretation of relations in different modes. These types of verbal interpretations give clues about their transfer of quantity. Students tend to depend on size rather than comparison in iconic modes (previous findings from Lecture 3), possibly in symbolic modes.

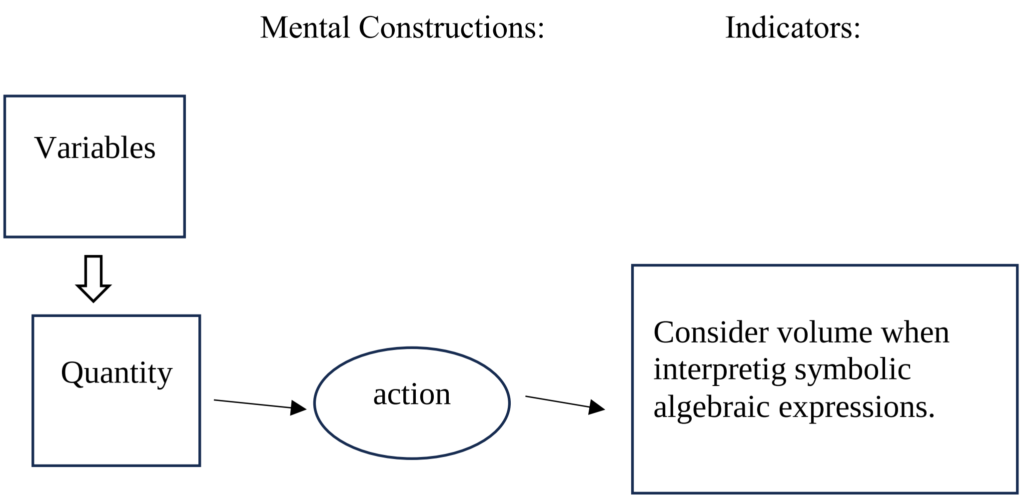


Figure 4.13. Schema for Learning Quantity in Lecture 5

Notation

This lecture is assigned to be teaching action level for symbolic representation of >, <, = signs in volume context, as it is planned to be. 4 of the students (Ekim, Medine, Yaman, Ali) constantly needed step-wise inquiry to construct symbolic representation, proving that they are at action level. 2 of them (Ekim & Medine) had difficulty in determining >, < signs also. Five of the ten students (Eylem, Didem, Ufuk, Bekir, Hasan) improved the process level for symbolic representation after guidance for only reporting the first comparison.

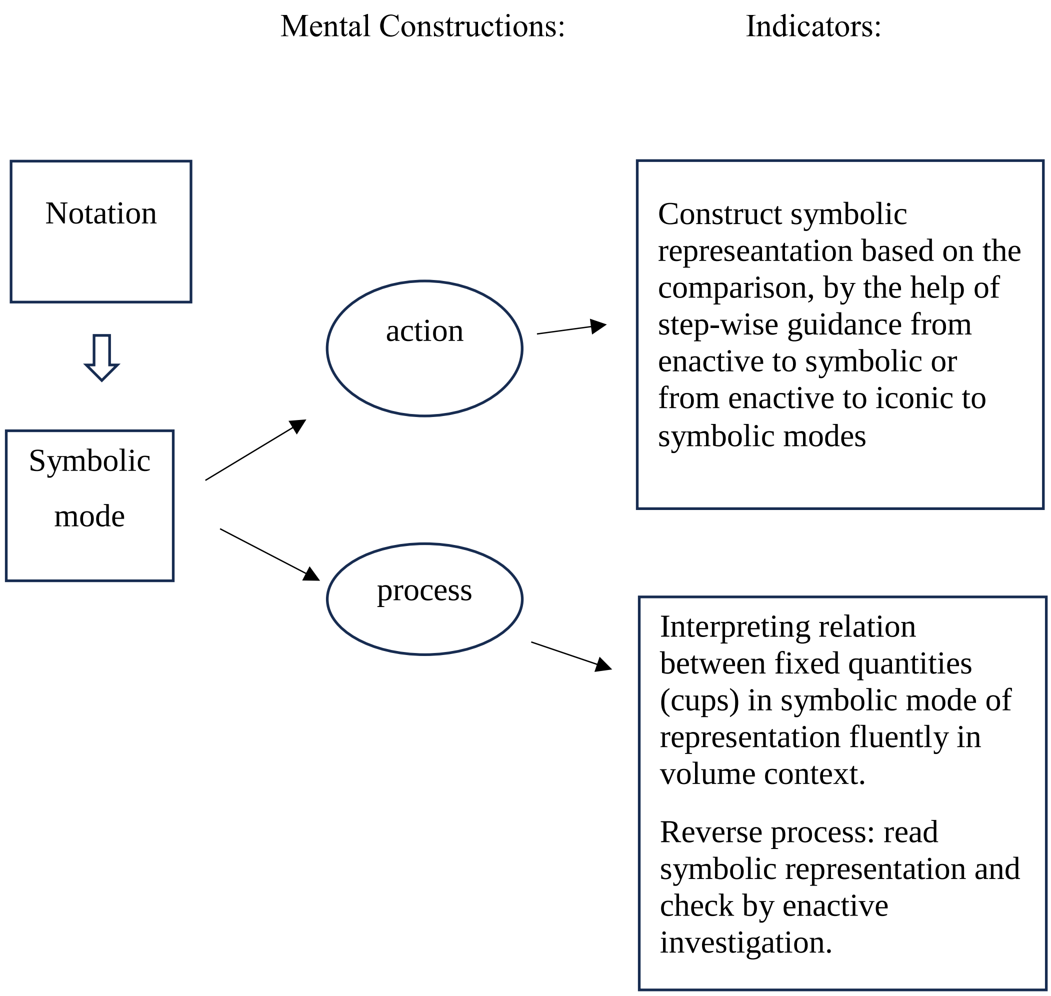


Figure 4.14. Schema for Learning Notation in Lecture 5

The researcher tried to connect enactive investigation to symbolic representation by bringing representative pictures and asking relation between them based on the comparison result. This guidance does not involve iconic representation. The kindergarten teacher's strategy of stepwise inquiry to go from enactive to iconic, and iconic to symbolic representation worked better. The stages for this step-wise inquiry are summarized below:

Enactive to symbolic representation stages:

a) place cups near corresponding cylinders. (enactive)
b) place sign between cups (iconic)

c) place pictures of cups near corresponding cups. (match picture (algebraic) notation with iconic representation)
d) carry sign between pictures (reflect iconic to symbolic)
e) stick pictures and signs on paper for reporting (full symbolic notation)

One student (Aylin) needed no guidance, and she was immediate and auto for symbolic interpretation after photos of the cups were provided to them. She had conservation of amount based on Piaget's testing, which might be the reason that she easily connected pictures to amounts in comparison without receiving a step-wise algorithm to connect these two.

For the second part of the lecture, checking reports back: two students (Aylin and Bekir) were immediately reading and checking reports without guidance, and one student (Hasan) became fluent in checking reports. Two out of 10 students (Medine and Ekim) had difficulty reading symbolic representations. Medine could read interpretations by guidance through row by row. The reverse process of symbolic representation flowed from the symbolic to the enactive mode in the following order (reversing the kindergarten teacher's strategy);

symbolic to enactive:

a) read symbolic representation
b) put cups on symbolic representation (iconic)
c) test by comparison (enactive)

4.5.3 Design Principles for Lecture 5

- Guide students through the connection between representation modes, as it is not obvious for them to represent quantities with symbols, even if they are pictures of the objects they represent. A row that guides from enactive to iconic and then iconic to symbolic is suggested, which they could easily see without skipping any stage or representation modes. Even when reading reports and checking for correctness, include all stages in reverse: symbolic to iconic (simply by putting cups on the representation), then iconic to enactive.

- Reporting and checking back reports can be conducted one by one to prevent confusion in reading symbolic interpretation. Focus on one algebraic expression at a time. Then, repetition of these procedures helps students become fluent.
- Encourage students to use signs in each procedure; even when they assess a peer’s report, ask for correct notation.
- Encouraging verbal interpretations is also essential to make students connect equality to algebraic representations and focus on the quantities in their comparisons or interpretations. Encourage verbal interpretations of back and forward processes. In their symbolic interpretations or when they assess peers’ reports, be cautious about students’ use of quantities, which can be detected through their verbal interpretations.
- Arrange materials carefully. Include a variety of equal and unequal cups, which are not apparent in size. Appreciate equal results by students’ interpretation and welcome measurement errors. Focus on the interpretation, not the precision of comparisons, unless there is a logical misunderstanding, such as not filling measured cups.
- For this activity, we can restrict students to using $>$, $<$, $=$ signs, as $\neq$ signs still appear to be hindering some students. Make sure they understand the replacement relation between $\neq$ and $>$, $<$ signs.
- Value the use of all signs. If they stick to one type of representation, peer-check activity will allow variety.
- Representative photos of cups are better in color to match.
- For checking the correctness of reports, the “try it” comment is more understandable and encouraging than the “test it” comment.

4.6 Results of Lecture 6

Due to pandemic regulations, this lecture was conducted online. The online lectures continued for 12 sessions. Students were video recorded and individually interviewed during these lectures to observe their improvement.

This first online lecture is a mid-assessment for previous learning, while it assures clarification on some complex topics, such as part-whole equality and symbolic representation based on weight. All ten students attended the lecture; one student, Bekir, lacked material until the last item. It lasted about 30 minutes.

4.6.1 Plan of Lecture 6

This lecture is a paper-work activity, which assesses their previous learning on signs, among the contexts that students had difficulty and which we wanted to revisit. There are additional or revisited topics in the items we wanted to close up. Naturally, items of the paperwork have the potential to teach those topics, while we restrict guidance while they learn to assess their understanding of previous topics. In this way, we assessed their pre-knowledge of the new contexts or topics, observing their using or transferring their existing knowledge. Using the same context, or items as in the previous lectures would be boring and meaningless as we already observed their knowledge before in that way. New context and topics added to the assessment while assessing their knowledge did not break the chain we constructed between lectures. Each lecture teaches new concepts while requiring the use of previous knowledge on the topic, which allows us to observe student's level of knowledge. As well, this lecture functions as an activity and assessment at the same time. The difference is that contexts and topics are chosen among those we wanted to close up, which will be explained in this section through item objectives listed below:

1. The student constructs an unknown quantity based on a given algebraic relation to another quantity by >, <, = signs

2. The student uses =, ≠ signs to interpret part-whole equality given by symbolic figures (Lego photos).
3. The student uses >, <, = signs to interpret relations symbolically based on given representations of weight comparisons.

The first objective is to assess students' use of signs and reverse-process level indicators by their ability to construct quantities based on the signs and quantities pre-given to them. There are three parts in the item assessing this objective. In each part, there is blank space for their drawings of an object, a sign, and a reference drawing. The first part is a tree; they are expected to draw a smaller one. In the second, they are provided an ice cream with a cone, which they would draw a bigger one. In the last one, they will draw a pencil equal to the given one. The construction of variables in the equation works as an assessment for sign knowledge. It is a reverse process such as finding objects, but at a higher-level perspective. Now, they draw an unknown object by given relation: information with comparison to another object and sign for the result of comparison. (As explained before, the assessment was not a copy of previous lectures. Because learning is assessed at the process level, performing in a new context is essential. Hence, drawing objects acted as a new context (new era) but not a new algebraic concept.)

Construction or imagining for the unknown object is often addressed in Davydov’s book (Davydov et al., 1995) as it is not only a higher level of reverse-process aim but also allows students to think about multiple solutions to an unknown. We make students feel flexible but do not expect multiple drawings/solutions. The variety of their solutions will be discussed in a later class (see Lecture 11) for a more vital awareness of multiple solutions.

In the second objective, a problematic topic, part-whole equality, is revisited, also students’ remembrance of =, ≠ signs is assessed. As decided before, based on the results of Lecture 2, Lego toys are used to accept parts composing a whole easily. It was decided to be enactive, using Lego toys, but we could use/deliver only drawings of the toys. This item also has another implicit purpose: reminding the use of unequal signs because it is overshadowed by >, < signs. In this item, students choose the correct sign among =, ≠ signs for given part-whole relations.

(The first construction item could also be extended, using unequal sign. There was no item like that in Davydov. However, it can be evaluated for future implementations. However, it should be studied carefully because unequal may represent many things that students may consider and ruin being a smaller, bigger idea. Ordering of those signs is another study topic.)

3rd objective is assessment on the use of >, <, = signs in symbolic mode of representation. Weight context is problematic for deciding which side is bigger; understanding that the heavier side is lower needs some experience, and more importantly, reporting the comparison with iconic representation may be hindered by the size of the compared/weighted objects, as we observed during lectures. In previous lectures, we used iconic representation for weight comparisons and symbolic representation for volume comparisons. However, for the first time, they are expected to represent weight comparison symbolically in this lecture. Another new concept is that student interprets a symmetric version of the algebraic relation of weight comparisons. This appears as an experience rather than an inquiry into symmetry. Hence, in the last item students are provided three different situations of weight comparison in picture mode (which was difficult to interpret based on pilot results.). Then, below are representative comparisons where the student chooses the sign among >, <, = alternatives. Relations are also asked in reverse order to experience symmetric property. (Verbalizing or discussing this property conceptually is difficult based on pilot results, so we aimed solely at experience, as we did in most activities.)

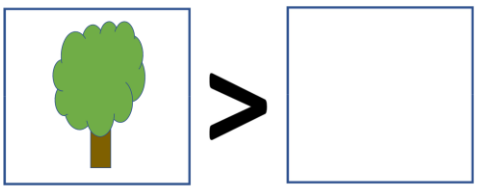
The lecture flow is summarized by the item description and trajectory in the following. Students were not left alone with the items; the researcher explained one part of each item without intervening or guiding their response, and they accomplished each step at the same time.

Items:

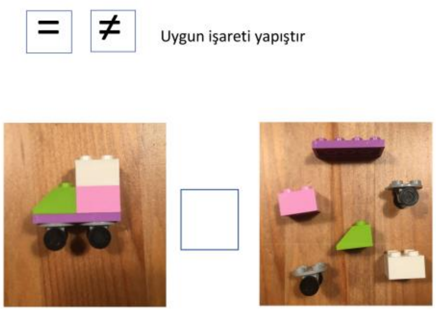
1. Construction: a. smaller, b. bigger, c. equal
2. Part-whole equality: a. equal b. not equal
3. Symbolic representation: weight context a. smaller, b: bigger c: equal

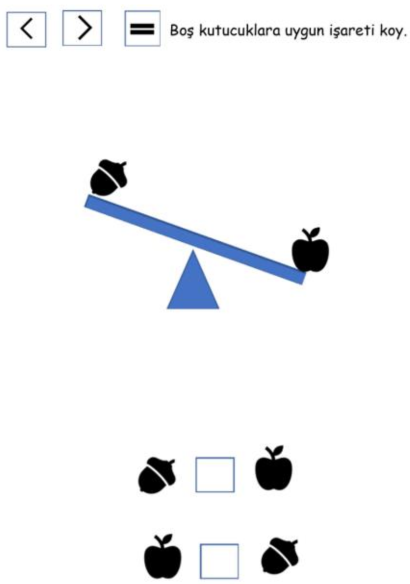
Trajectory

- Process =, >, <: reverse process, continuous variable: size. Construction of unknown
- Process part whole equality
- Process symbolic representation



Item 1, part 1





Item 2, part 1

Item 3, part 1

Figure 4.15. Sample Items in Mid-assessment in Lecture 6

4.6.2 Theoretical Findings of Lecture 6

Equality and inequality

Responding to the first item, 8 out of 9 students proved to be at the process level (reverse-process evidence) using >, <, = signs by drawing objects based on pre-given objects and their relation to them. No problem occurred in remembering or understanding the use of signs, but constructing the quantity based on a relation was a new topic for them. They were all handled successfully. Only Medine had some difficulty with signs consistently through all items, showing. She appears to be at action level for the use of signs. Aylin hesitated for the last part (drawing equal-length pencils); she wanted to write "1" as a number of pencils because her mother guided her to do so. (The researcher is careful about mother distractions.) They drew a smaller tree in the first part without much difference. In the second part, when they were expected to draw a bigger ice cream on a cone, their drawings varied; some preferred little change, some drew bigger scoops, and some drew a greater number of scoops, all consistent with the relation. Varieties were sufficient for discussion on multi-solution later. Findings on their level of understanding confirm results from previous lectures, with a difference. Ekim improved from the action to the process level in using >, < signs. Bekir did not have the material for the first two items, however, his response on the last item shows he can use all of the signs fluently. We can conclude that nine of the ten students are at the process level of using signs based on the results of all lectures/mid-assessment.

In the second item, they need to match =, ≠ signs with part-whole equality situations. Only Medine's result was unclear. On the other hand, Ekim had difficulty seeing some parts of the whole (Lego construction). Others (7 students) completed the item easily.

Medine also had difficulty in the last item, interpreting weight comparisons symbolically; she tended to use unequal signs instead of using >, <. Her use of signs in previous lectures was poor and she needed guidance based on quantity consideration or reminding sign orientation algorithm most of the time.

Variables

Quantities that are expected to be constructed in the first item are variables. Construction of them is continuous. However, we do not expect students to create multiple solutions, while they have experience in this continuous environment. Nonetheless, the construction of those quantities is a reverse process for quantity comparison, which we expect at this level. Eight of them completed all parts of this item correctly. In contrast, Medine was not clear in drawing the tree in the first item, and needed inquiry into which one is smaller. As a result, we observed, that students were capable of constructing quantities, as well as they were choosing quantities based on relations. Reverse-process quantity as the amount is comprehended.

Some students had difficulty in the third item, reporting weight comparison. However, the difficulty did not originate from their misunderstanding or misinterpreting weight as quantity.

Notation

7 of 10 students did not face any difficulty interpreting weight comparisons symbolically. Most of them were immediate in completing tasks. They successfully apply their knowledge on weight context, even when they see it for the first time, proving they are at the process level for symbolic interpretations. Three of these seven students were at action level in symbolic representation in volume context, showing evidence they improved to the process level in this lecture. Four of them were already showing signs that they were at the process level in the volume context in the previous lecture, and this lecture proved their capability to transfer their knowledge in a new context.

Three other students (Ufuk, Hasan, and Medine) struggled with the symbolic representation of weight comparison results. In contrast, in the previous lecture, all three students were observed to improve from action to process level in symbolic representation in a volume context. This might be from their inability to transfer knowledge on symbolic representation, or their poor understanding of it. The previous lecture was on the symbolic representation of volume. However, symbolic representation is not discussed through variables. Step-wise inquiry for symbolic representation of volume had to be on matching cups to the representative picture. One of these three students (Ufuk) successfully interprets volume as quantity (Lecture 3 results). His problem with symbolic representation of weight occurs as misleading figures in the item and his inability to connect symbolic representation to a given comparison situation. He preferred equal signs to interpret their relation because acorns and apples are given in similar sizes.

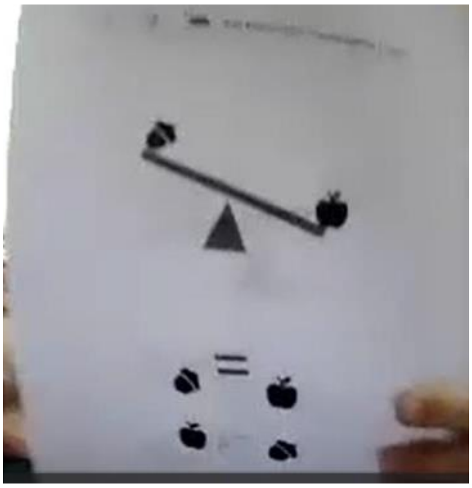


Figure 4.16. Ufuk’s Misinterpretation of Equality Based on Weight Symbolically

The researcher guided him to focus on weight comparison for interpretation.

> R: What symbol did you put, Ufuk?
>
> Ufuk: Equal.
>
> R: So, they seem equal in size, right? Then, when you put them on the scale, which one weighs more?
>
> Ufuk: Apple.

R: The apple is heavier. The weight of the apple is greater, okay? So, you should put the sign accordingly.

After this conversation, Ufuk still has not changed the sign. Some time passed

R: What did you do, Ufuk? You made it equal. They seemed equal in size, but I am asking about their weights, Ufuk. Put the sign according to which one is heavier, okay? Not according to their sizes but according to their weights.

He completed other parts of the item correctly after guidance. Hasan is another student who had difficulty interpreting symbolically. He had a good understanding of signs. However, he depends solely on his imagination of the compared objects and rejects guidance on the interpretation based on weight comparisons. Even though he interprets referring to the weight of the objects, not size, he does not conclude it from the illustration given. He ignores all the guidance to connect illustration to interpretation. He even does not show action level at symbolic representation in weight context. Inquiry is restricted to the following, not to interfere with assessment results.

R: What did you do, Hasan?

Hasan: Tweet, tweet. (duck compared to the clock)

R: Hasan did something different. Why did you do that? Tell me. Which one is larger?

Hasan: Duck.

Hasan: The clock is light; the duck is heavy.

R: Is that so? The clock is light, the duck is heavy, right? Do you have a clock at home or a duck?

Hasan: No.

R: But you imagined it that way, didn't you? All right, there is a scale there. What does it say about their weights?

Hasan: It says “equal.”

R: It says “equal” about the weights, huh? Hmm, I see.

Hasan nods yes

R: But do you think the duck is larger?

Hasan thought a while and said “yes” only. (not confidently)

R: If we need to put a sign according to their weights, what would be the sign between them?

Hasan: Equal (pause) unequal.

R: Unequal, right? Are they not equal, Hasan? (actually, they are equal in illustration)

Hasan: Yes.

R: Okay, I understand, my dear. I think everyone did it. We can say goodbye now.

Possibly, the inquiry is poor in connecting illustration to symbolic representation because the researcher does not ask the students to interpret based on their weights, as illustrated in the picture. Instead, she solely asks, “If we need to put a sign according to their weights, what would be the sign between them?” The inquiry is distinct in the interpretation of illustrated comparison and on their weight, which might push the students to think about their actual weight.

The last student who had difficulty in symbolic representation was Medine. Her difficulty arises from her weak understanding of sign use. She is at action level for >, < signs. Hence, she responded to the first two parts of the item wrongly. She could interpret symbolically with equal sign in the third part, with an orientation problem, she also had in previous lectures. She correctly chooses and matches signs correctly, but orientation is problematic. Especially in >, < signs she needs guidance. Little guidance on paying attention to the wider, narrower side helps her.

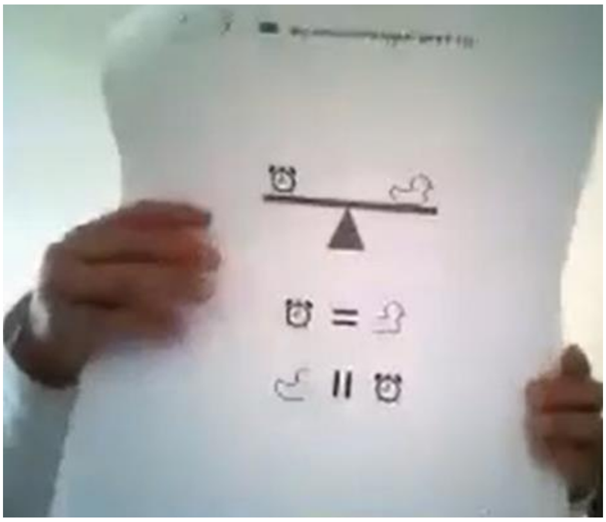


Figure 4.17. Medine's Use of Equal Sign Vertically

This assessed students' transfer of learning on symbolic representation on volume comparison to new context weight. Previous lectures and this lecture showed that connecting symbolic representation to comparison situations is difficult. Enactive representation on a given illustration of weight comparison could be asked, before moving on to symbolic representation.

Another remarkable finding on sign notation is that structured paperwork assessment helped one student recognize two modes of greater/less than sign on the paper, which they learned as a single sign changing directions in need.

> Eylem: There are two signs, one small and one big.
>
> R: Yes, but their directions are different, right? They are looking in different directions.

They saw two signs from the instructions above and also when writing them down below. They then symbolically interpret the relation between quantities with their symmetry. We did not aim to teach symmetric property, so we did not ask them about it. No students revealed intention or awareness of the symmetric property.

4.6.3 Design Principles for Lecture 6

- Ask which one is bigger based on weight comparison and guide to focus on weight comparison for interpretation created the connection between comparison and the representation.
- Support students who had difficulty in symbolic interpretation with enactive representation.
- Working ideas on paper is a new activity for them, and assessing their knowledge on paper is difficult. Guide them through how activities work and what the instructions are. Communicate a lot to guide them through items or understand their thinking.
- New context volume, paper-work, reverse-processes quantities through construction, symbolic representations, altogether supported process level of understanding signs.
- Be careful in structuring items to prevent misinterpretations. The real-life context of compared objects or pictures of them may be hindering. Try to choose similar-sized and weighted objects in real life and also in illustrations, which may differ a little bit: like apple and pear. In this way, the item becomes more effective in teaching but may become weaker in the assessment of symbolic interpretation. At least try to prevent systematic errors in the items. Namely, do not provide situations where a bigger one is heavier, which allows students to choose a bigger object to be the heavier one in interpretations. Do not also provide situations that give clues about their weight in a real-life context.
- Be sure objects in figures touch the pans of balance scales.

4.7 Results of Lecture 7

Lecture 7 is the second online lecture which is about a new topic; ordering, built upon using >, < signs from previous learnings. Lecture 7 lasted about 30 minutes. 8 out of 10 students attended. (Hasan and Ali did not attend.)

Lecture 7, being the second online lecture, is revised to fit online circumstances, even the objectives did not change too much. Online lectures are kept neat, focused, and short. Structure is constructed around instructions to be easily followed by all students at the same time. Then, all students are individually interviewed for presenting their work and questioned to observe their understanding. Materials are chosen from home environment, or delivered to them in need.

4.7.1 Plan of Lecture 7

This lecture aims teaching ordering; creating a sequence of objects using >, < signs. By piloting sequence of Davydov; ordering objects is decided to be taken earlier than transitivity. In Davydov's Book (Davydov et al., 1995), by the comment "deduce", transitivity is used first time implicitly for deduction of unknown relations between objects that are not compared directly to each other, but their relation to another object is known. Then trajectory continues with ordering of 3-4 objects. After ordering, transitivity is handled by deducing third relation out of two relations.

The first hypothetical trajectory, following Davydov's, included deduction using transitivity implicitly in a volume context. Pilots and classroom implementations showed that students have trouble with that kind of deduction due to a lack of transitivity intuitions, which Davydov's trajectory depends on, to deduce third relations symbolically. Pilots showed that transitivity had a chance to be thought through ordering strategy (1 student out of 2 students). Hence, the deduction on volume context is canceled. Transitivity is decided to be thought after the ordering concept because it is constructed on ordering strategy. A summary of trajectory adaptation can be seen in the following table.

Table 4.1 Trajectory Change in Transitivity

Davydov	**First HLT**	**Last Trajectory**
- using transitivity for deduction of the third relation intuitively	- using transitivity for deduction of the third relation in volume context intuitively	- reporting dual comparisons of three objects in a volume context
- order four objects	- order 3-4 objects	- order multiple objects and extend the sequence
- transitivity in symbolic representation	- transitivity in symbolic representation	- transitivity by ordering strategy

At first, we aimed to teach ordering at the action level while they use >, < as objects. Children would order stones found in the school garden based on their weight. They would enlarge the sequence by comparing new stones to the stones on the row, using the relational information they got. Weight context is chosen because it is not apparent and needs an iconic representation of relational information. This relational information would be the algebraic objects they use in ordering algorithms. However, weight comparison seemed difficult for symbolic interpretations and deductions for ordering. Hence, it is simplified for comparison based on the size of the stones. In this way, using >, < signs stays as a process composed with the ordering process.

Davydov points out the recursive relation in ordering four objects of similar shape but different sizes. We aimed to at least show the extension of the sequence to students by ordering stones. Due to the impossibilities of online lecturing, discussions and ordering needed to be simplified and restricted to include fewer objects. However, the aim of enlarging the sequence is still presented in the aims.

In the first HLT, ordering is planned for two lectures; the first will use simple objects, while the second will use pictures and illustrations to order as a game.

First HLT objectives:

1. The student orders 3-4 objects and puts relevant signs between them based on their relation: with toys
2. The student orders 3-4 pictures and puts relevant signs between them based on their relation: with pictures

Revised HLT:

1. The student orders at least 4 objects based on their size and uses the> sign to interpret the sequence.
2. The student extends the sequence of ordered objects based on size

As in Davydov's Book (Davydov et al., 1995), we started by using similar object comparisons. They prepared four toys belonging to the same class, such as balls, cars, or dolls, to order during the lecture. They also drew and cut out four members from their family to compare and sequence based on height. At last, they brought some stones for extending the sequence activity. The lecture flow is summarized in the following, which will be explained further.

Lecture flow:

- Ordering algorithm: presentation by balloons
- Ordering four toys (enactive representation)
- Ordering drawings of 4 family members (symbolic representation)
- Interpret two relations out of a sequence of family members (reverse process)
- Compare two stones and place a sign between
- Extend sequence with 3rd and 4th stones
- Imagine extending further.

Algorithms for ordering in the lecture flow:

- Ordering four toys; algorithm: find the biggest and second biggest. Put a sign between them. Continue with the third one and then the last.

- Ordering drawings of four family members: Order four family member drawings from biggest to smallest. Take two of them, compare them, and put a sign between them. Take another two, compare, and put a sign between them.
- Ordering four stones: Take two stones, compare them, and put a sign between them. Take the third stone, place it in the proper place, and put a sign. Take the fourth stone randomly, place it in the sequence with the proper sign, and continue the sequence as much as you can.

There seems to be too much restriction for order, from biggest to smallest; however, continuing the sequence is possible by doing it. Allowing to order from smallest to biggest is also possible. However, they might not have a sense of order from left to right at this age as they do not know writing. It gives a starting point for ordering (Whether they use right or left, they start with biggest). They might need specific directives. If they start by putting a relevant sign between two objects and then add a third one related to one of them, the sequence might be disordered; signs would be disoriented in the sequence.

After creating a sequence of family members, we wanted students to look for a relationship between two members. We aim to reverse the process by this, making students recognize dual relations composing the sequences. If it was not online but an in-class lecture, the relation of members in the sequences deserved more time and step-by-step visualization to look for recursion in the relations. As we will see later, ordering three objects is also essential and will be used for transitivity. Looking for relations between two items out of a sequence will also create a base for transitivity activity, where we will try to infer relations between 3 objects.

Trajectory:

Action sequence: algorithm of ordering (4 objects; enactive)

- state biggest and the next, represent relation with > sign
- put the third biggest in the row with > sign

- complete the sequence with the smallest

process:

- repeat the algorithm with new objects (iconic representations of family members)

reverse process:

- interpret the relation between two objects out of the sequence

extending sequence:

- adding items in both directions (enactive)

Extending the sequence gives a chance to mention infinity. Implementation was not complete in the discussion of infinity; however, it provided a good starting point. A more focused in-class activity of ordering stones would better support the discussion. It enables adding to the row as much as possible, not only going through the imagination of extending the sequence. Extensions not only on the ends but also in the rows have the potential to fruitful discussions on infinity

4.7.2 Theoretical Findings of Lecture 7

Equality (inequalities)

All of the eight students were capable of using>, < signs fluently in comparisons and ordering. Aiming action level at ordering four objects, this activity helped students order at least four objects from bigger to smaller with the correct use of >, < signs. 6 out of 8 students became fluent in ordering and needed no help, while two students (Ekim, Medine) needed guidance or help with ordering representations with signs.

Use of >, < signs in process level indicators in ordering activity appears as:

- Fluent use for interpreting dual relations (already from previous lectures)

- Use >, < signs for ordering four objects, with step-wise directions on extending a relation between 2 objects

Action ordering indicator:

- order four objects from bigger to smaller, using signs >, <

Process level indicator behaviors for using>, < signs are also indicators of action level in the ordering algorithm if supported by guidance. If no guidance is needed, and students can order four objects fluently using these signs, they are at the process level for ordering.

The use of signs fluently is also indicated in previous lectures. The ordering algorithm builds upon the use of these signs. Students compare two objects and interpret the relation between them using these signs, then extend relational interpretation using third and fourth objects through relative comparisons. The use of these signs does not significantly indicate students' object-level understanding of relations. They may know the object level, but this ordering activity requires procedural action of comparing and representing the objects. Hence, this lecture did not provide evidence of students' use of greater/less than relations as objects in the ordering procedure. If the lecture were built upon weight comparisons as planned in the first place, they would have to depend on the iconic interpretations of weight comparisons for ordering. In this way, the iconic relational interpretation of object1>object2 would be the algebraic objects students will use in the ordering procedure. However, in this lecture, students order based on not pre-given or pre-obtained relations. Dual interpretations and comparisons appear to be procedurally composed in the ordering process.

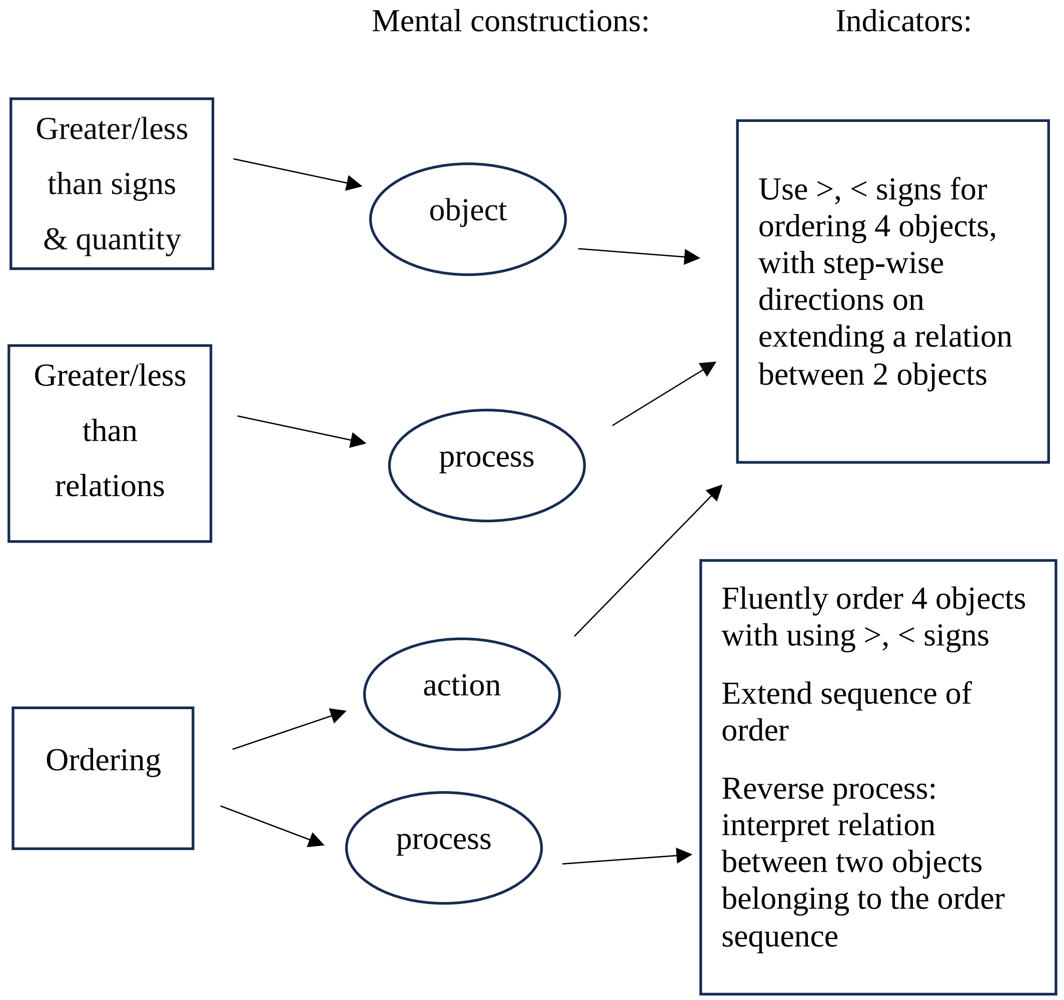


Figure 4.18. Schema of Learning Inequality, Quantity, and Ordering in Lecture 7

When they become fluent in ordering four objects, they do not refer to or start with dual comparisons anymore because they compare them in their minds priorly. It means they do not follow the first thought algorithm but just use >, < signs to construct the sequence. When they extend the sequence, the dual comparison procedure is again composed into ordering process. It is not easy to separate ordering from comparison relation as a new algebraic procedure. It is the continuum of interpretation of equality/inequality. However, using the signs in the comparison procedures becomes an object for those interpretations as we see. Students no longer refer to wider and narrower side analogies and use these signs without hesitation to interpret comparison results or construct ordering sequences. From this perspective, >, < signs became objects, while a>b did not become yet.

We expect the type of algebraic representations to become objects in the increase/decrease topic where students meet “how to make equality out of inequalities,” acting on inequality situations by increase/decrease actions to create equality situations. Ordering and transitivity concepts come just before the increase/decrease topic. The following figure shows the alignment of ordering and transitivity in the trajectory, which seems disconnected from the trajectory of teaching equations: There is nothing about equations for ordering and transitivity concepts. Our sequencing of lectures and Davydov’s instruction do not directly connect anything about transitivity to the subsequent topic increase/decrease or the further topics. Connection is in the development of mental constructions. In ordering subject, the student finds an area to use >, < signs as an object component. Then, ordering helps transitivity, which uses two relational interpretations as object components to deduce the third one. Ordering or transitivity processes are not used further in any way. Connection is on the mental constructs: as object >, < signs to object relations to use, which will be acted upon to make equal in the following topic increase/decrease. (Transitivity could not be used as objects again. No deduction on interpretation is easy for our students. However, it supported process level for inequality.)

Despite its unnecessity in terms of components of equations, ordering and transitivity create extra support for the trajectory in terms of mental constructs. It creates a soft transition from process to object mental construct for greater/less than relations. Moreover, ordering sequences and transitivity opens horizons to many important algebraic topics; with thinking quantities and relations in the system (which students struggled with due to lack of conservation). In this lecture, students could relate and interpret more than one quantity and then, with reverse processes, focus on dual relations within the sequences of relations properly.

Quantities: How to compare a, b action			
Equality of quantities =, ≠ action a, b object a=b action			
Inequality of quantities >, <, a>b action			
Find and create unknown quantities from relations a>b, a=b reverse process	Ordering a>b>c>d >, < object a>b process in composition	Transitivity a>b & b>c => a>c a>b object	Creating scale: Unknown construction using/composing transitivity
Increase/decrease: how to make inequalities equal by operations +, - action a>b, a=b objects			
Increase/decrease by an amount: Difference amount: find unknown quantity in equations with operations +/- a action b=c+? / b-? = c action			
Properties of operations in equations +/-a objects a±b=c±d action			

Figure 4.19. Transitivity in the Whole Trajectory

Variables

Comparisons are between fixed and non-manipulable variables. However, the ordering stones activity has the potential to discuss extending sequences on the ends and in the middle of the sequence. It would be better to actively investigate the extension of the sequence of stones as much as possible in the school garden. Students would gather stones, which would extend the sequence to any place. Online lecturing limited the number of stones they experienced. Sequence extension is presented by the researcher and discussed further for infinity.

Students have a poor imagination of infinity; usually matching with the biggest number they know (Bekir). One of the students stated infinity as “every day” (Didem), and one stated as “many days” after (Medine). Eylem and Aylin stated as “never-ending.” Ekim and Yaman had no idea about infinity. After Bekir tried to state the biggest number he knew, the discussion evolved to “Is there a bigger number than this one?”. In each number stated, students could find a bigger number by doubling it: “Two hundred thousand is bigger than hundred thousand” (Medine); by adding one to it, “thousand-thousand is bigger than 999 thousand” (Eylem).

Discussion on infinity through recursion is possible in this activity, which may support number sense and sets. The density of sequences may also be sensed through extension in the middle. This activity is out of Davydov’s trajectory, limited to ordering four objects. However, Davydov also allows infinitely many solutions in constructing quantity activities. Ordering stones activity can also be structured to support multiple solutions by asking students to create a proper stone for a particular place, extending the sequence as we first planned. However, discussions were poor and led to the imagination of infinity only due to online lecturing conditions.

Notation

In this lecture, students no longer had trouble determining the orientation of the>, < sign. They started to see the sign as a static object (even Medine) and easily determined the orientation without reminding the wider or narrower side by the teacher or themselves, they could use it fluently in their representations of ordering.

Interpreting ordering is objected to in an enactive mode of representation. Ordering toys or stones is in enactive mode; sequencing consists of objects themselves. Ordering family members is done through students' symbolic drawings of the family members. They represented the heights of the family members correspondingly. While ordering pictures, they use symbolic pictures to represent the heights, but they might solely depend on the pictorial height in order. Whether they refer to family members' height in their verbal interpretations is not apparent. In discussions, the researcher pointed out the real-life reflection of what they interpret, which is trivial primarily in real life and in their drawings, which made no clear differentiation. The activity of ordering family members did not support symbolic representation, whereas it created a successful context for ordering and reverse-ordering processes.

4.7.3 Design Principles for Lecture 7

- Beginning ordering by the biggest (or smallest) creates a connection between what they already know (ordering objects) and what they will learn (using > sign notation in a sequence).
- Starting by comparing two objects and then adding the third and fourth objects to the row supports the idea of extending sequences.
- Once students get fluent in the ordering, they may forget this algorithm and order objects directly based on size. This is not necessary, but the algorithm may be reminded in the stone activity to initiate and structure the extension of the sequence.
- Extend sequences purposefully, in a single direction (bigger/smaller side always) to support the infinity idea, or keep end points fixed, placing mid-items to support the idea of density (such as infinity in real numbers between two items). The inquiry should be very structured for those purposes.

- Remember the one learning at a time principle, and it is kept simple, focusing on order. Based on weight or volume is difficult at this stage.
- Ordering and the idea of infinity can be supported by quantitative reasoning, but be careful that students might have incorrect knowledge of numbers.
- Use similar types of objects to make it easy to compare based on size. Stones may also have a decision problem regarding size. Weight context would solve the problem, including the = sign in the sequencing.

4.8 Results of Lecture 8

Lecture 8 is on the transitivity property, where two dual relations between three objects are appropriate to conclude the third relation. Transitivity is thought in a volume context. Students determine the relation between the capacity of two cups based on their relation to another one. It was the third online lecture, which took 35 minutes, for two investigation activities of transitivity. The first activity took 22 minutes, while the second took 13 minutes, and both had the same algorithm. All ten students attended the class.

4.8.1 Plan of Lecture 8

Because it is a complex topic for students (pilot results), the lecture is kept neat and focused on two structured investigations on only two transitivity situations. It is built around symbolic interpretation and volume context because transitivity needs deductions based on pre-determined or given relations of non-obvious situations. Materials are chosen carefully so as not to give clues about their volume. Students experiment with relations between fixed quantities in this lecture. However, experimentation is step-wise structured. The researcher provides three cups to compare, and in each step, they compare two of them, which the researcher told them to do. After all students do one comparison, a second comparison is done simultaneously. Comparisons are chosen in the order which allows the deduction of the third comparison out of the first two comparisons.

The objective in the first HLT:

1. Given two relations among two of three objects, the student determines the relation of the third comparison.

Revised objective:

1. Given three objects, the student experiences and reports two comparisons (in an order) and guesses the third relation.

As seen in the objectives, the purpose is directed from determination to guessing based on relations because it was challenging to pilot. Moreover, given relations are changed to enactively experienced relations based on piloting results. One of the pilot students (Kerem H) could not consider anything about transitivity or any idea that would lead to transitivity. However, for the other piloting student (Kerem A), ordering objects in the given relations worked as a strategy to guess the third relation, which encouraged us to try it with the students in the classroom. Ordering quantities is an implicit objective used as a strategy for transitivity deductions. Piloting showed that iconic or symbolic representations make it more difficult to understand variables in a system of relations. Hence, we stepped back and started with enactive comparisons. Additionally, based on the piloting results (Kerem A), volume context and inclusion of equality relation made deductions easier. (Kerem H struggled in transitivity activities. He just depended on size and height for comparisons.)

The lecture Flow is planned as in the following;

- Pre-chosen 3 cups were given to students. 2 of them were in equal volume, and one was different than the others.
- Two comparisons are conducted and reported, one of which results in an equality relation
- Ask students which cups are left for comparison.
- Make students guess the result of the last comparison.

- Make them order three cups in terms of volume if needed.
- Let them check their guess through measurement.
- Continue with the new 3 cups; comparing, reporting, ordering, and guessing for the last relation.

In the implementation, making students guess for the last relation immediately directed them to check by comparison. Hence, the ordering did not work as a strategy for guessing. As a result, we could not accomplish the objective following this lecture flow. Thus, we decided to revisit transitivity further. There are two different lecture flows suggested for further implications (which will be used in our second attempt at teaching transitivity.):

1.
- Report two relations based on comparisons symbolically
- Order three objects based on these two comparisons (through which one is the biggest, which one is the smallest inquiry) (iconically)
- Guess, then interpret the result for the third comparison out of your ordering

2.
- Report one relation after one comparison
- Compare the third unknown to one of the unknowns in the represented (related) relation and report
- Extend the first reported relation as a sequence with the new knowledge (relation) obtained (ordering step: extending sequence) (symbolic or iconic)
- Report the third relation that is asked for.

There are also other reasons which made this lecture unsuccessful in the objective. Volume comparisons are vulnerable to measurement errors. Equality relations made measurement errors more effective. Students' investigations differ a lot due to these reasons, making deductions impossible. Make-up activity would eliminate these by controlling investigations through researcher conduct and show activity. (It would also make it similar to Davydov's pre-given comparisons. In Davydov's book (Davydov et al., 1995), transitivity is expected based on ready-given comparisons and structured on symbolic representations. Hence, deducing the third relation by transitivity is definite. Because we enactively investigated relations, measurement errors affected structures. They also have difficulties interpreting symbolic representations. Hence, a make-up lecture on transitivity will be better based not on symbolic representations but on visual experimentations. However, these experimentations will be shown by the researcher, for guaranteed results.)

Intermediary construction is another learning objective from Davydov also included in the first HLT after transitivity. Creating an intermediary seems as a result of reasoning by transitivity. However, it seems more intuitive and can be considered as a base or a support for transitivity. Hence planned make-up activity for transitivity will be placed after creating scale lecturing. Make-up instruction will be considered and structured carefully. Fulfilling needs purposefully, implementation will be delayed even further. Lecture 16 in the first HLT also aimed to teach transitivity at the process level by constructing quantities. The objective was "Given two objects and their relation to a third unknown object, the student draws/constructs the unknown object.". due to the difficulty of the transitivity topic, this lecture and objective are canceled.

In addition to the suggested lecture flows mentioned before, the following revisions in this activity are considered to achieve the intended objective:

- Eliminate measurement errors: Dictate specific results in comparisons or do not use equality relations. Use non-obvious cups.
- Do not try to depend on intuition but depend on more structured symbolic representation and ordering procedures
- Dedicate more time to ordering objects in different contexts of variables/ attributes before this lecture.
- Strengthen symbolic representation in ordering and extending sequences to use ordering as a base inquiry to converge transitivity idea.

4.8.2 Theoretical Findings of Lecture 8

Pre-Action transitivity

There are two transitivity activities through the investigation of relations between three cups. Mainly, out of 3 cups, two of them are compared, and the relation between them is reported. Then, another two are chosen, and their relation is also reported. This lecture is constructed around one equality relation. If the first two relations include equality relation, then deduction of the third relation is possible. Cups and their relation are provided as follows:

1st activity cups: coffee mug = tall cup > short cup

2nd activity cups: plastic cup = bowl < bottle

The first comparison was between a coffee mug and a short cup in the first activity. All of the students found coffee mugs> short cups. The second comparison was between a coffee mug and a tall cup. While we expected an equality relation, only three students found equality, one student found a tall cup > coffee mug, and most of the students (6 out of 10) found a coffee mug > tall mug. These incorrect responses (and divergence in their findings) result from measurement error. In volume context, especially, finding equal cups is difficult. Unfortunately, designing transitivity activity with cups with different volumes may give clues based on their size. (The second activity is more appropriate, not giving clues). Out of these two relations, we cannot deduce the relation between tall cups and short cups mathematically. However, students tend to see taller cups to be bigger. Hence, they could deduce the correct relation between tall cup > and short cup, but their reasoning is unclear.

To understand their reasoning, students are asked to show the biggest, middle, and smallest cups and order them. Three students (UE, Hasan, Ali) found the relation coffee mug = tall cup. These three students hesitated to choose the biggest cup, consistent with their findings. One student found the relation to be taller cup > coffee mug. His ordering is also consistent with his findings. Independent from their comparison results, some students (4 out of 10: Ekim, Medine, Didem, Bekir) had difficulty ordering or verbally stating the biggest cup. They depend on the height of the cups for order. The researcher asked one of the students (Ekim): “Is there a cup bigger than the one you are holding (which she shows as the biggest)? Is there any cup which takes more water than this cup?”. Then, three students (Ekim, Bekir, and Didem) corrected their ordering.

One of the students (Medine) struggled to correct her ordering. The researcher asked her, “When we compared this and this cup, which was bigger, Medine?”. She showed the correct cup based on her volume comparison but did not correct her ordering. After a while, the researcher asked which cup was the biggest. She showed tallest. The researcher reminded volume: “We compared these two. Which one was bigger?” She showed again based on the volume comparison. She insisted on the same reasoning at two different times. She knows verbal interpreting based on volume comparisons but interprets bigger based on height. More discussions are needed on quantity, what does means to be big in volume, and why we depend on volume for cups. Teachers should be aware of this issue. Height is also a quantity and she interprets being big based on height correctly.

In the second experiment, we started by comparing equal cups, bowls, and plastic cups. Three students (Ekim, Didem, and Bekir) found equality between them. One student (Ufuk) found a “bowl>plastic cup.” Others (Ali, Yaman, Medine, Aylin, Hasan, Eylem) found “plastic-cup>bowl.” In the second comparison, a bottle is compared to a plastic cup. All found the bottle to be bigger, but Medine may still have a problem with the use of >, < signs correctly. She signed “plastic cup>bottle” while showing the bottle to be the bigger one. She corrected her sign after reminding the sign algorithm to a wider side to a bigger object. For the last comparison between bottle and bowl, Eylem said they might be equal. It is not the result of her interpretations, as she found “plastic cup > bowl” and” bottle>plastic cup.” Then she immediately changed her mind and showed the bottle to be the biggest. Five students out of 10 (Ali, Didem, Eylem, Bekir, Aylin) correctly guessed bottle>bowl. When asked why, they did not have a reply.

Classroom results were aligned with pilot results, as students understood relations separately, but it may not be easy to understand relations within a system. When ordering objects, some students do not consider the relations they noted in the system unless guided. They might base their ordering on size (height in this case). Even if they correctly interpret ordering based on volume, they might order based on height (Medine). The previous lecturing was on order but depended on the size as a variable/attribute. This lecture results show that ordering objects was learned contextually, not transferred to new contexts automatically. Hence, we cannot say they are at the process level. This result calls for more time spent ordering concepts with different contexts. At ages 5-6, we see that no concept is transferred automatically to other contexts, even though we have discussed them before. We can say learning occurs context-based even though the majority would consider volume in their verbal, symbolic interpretation and ordering. However, we cannot say they conclude based on them, using transitivity. Ordering based on volume context should be dedicated to lecture time before this transitivity activity in volume context.

The findings in this lecture did not address the action level for transitivity. Even though many students could deduce the third relation correctly., we do not know their reasoning behind it. In addition, the lecture flow does not necessarily support a systematic procedure for transitivity (alternatives/revisions are suggested). However, through enactive investigations, we settled a pre-action stage before a planned additional lecture on transitivity. Some students could even be guided to use their reports to order cups based on their volume. This shows a significant foundation for using relations for deduction and staging for transitivity.

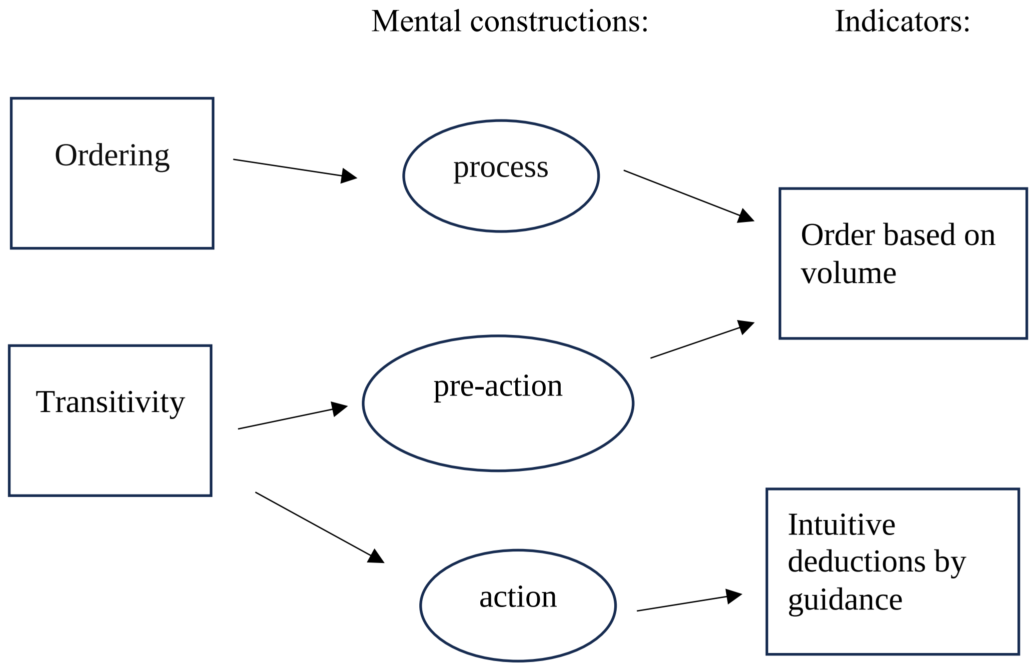


Figure 4.20. Schema of Learning Transitivity in Lecture 8

Process =, <, > relations

Our aim for this activity was for students to become at object level understanding for =, <,> relations because they use these relations as algebraic objects in transitivity processes. However, students' reasoning by transitivity (intuitively) depending on reported relations could not be observed clearly. Their reports of relations are constructed through comparisons procedurally and composed into ordering processes (as in the previous lecture), not becoming objects for deductions. Some students could refer to their reports in the ordering process. In addition to previous lecture results, some students could refer to relations reports for ordering when guided. Referring reports may be seen as using these relations or reverse-processing to deduce real-life meaning. They showed no evidence of acting on relations or using these relations to extend sequences for ordering. The use of relations through ordering and transitivity is complicated to explain. However, the use of these relations in the trajectory of equations is apparent in how they become objects acted upon by operations.

Process Ordering three objects

Some students were confused when asked to order cups because they tend to size for ordering, even though they could symbolically report relations based on volume. The previous lecture was on ordering solely based on size or height. Volume acts as a new context for order, and some students are still at the action level of ordering based on volume. Half of the students (5 out of 10) had difficulty ordering based on volume. Some reason may be the difficulty of representing based on volume (and weight) as it appeared in previous lectures. Some reason may be the difficulty of ordering not from direct measurement but using symbolic representations. Students' ordering strategies were not on the extension but on determining the biggest, middle, and smallest quantities. Ordering based on volume and weight is strictly suggested to be studied before this lecture on transitivity.

Variables

This became an influential activity for volume discussion for determining what is big for cups. In dual interpretations, students used volume without any problem because they had prior experience, but in ordering some of them used height. In addition to being a new context for ordering, ordering based on the volume brought the discussion of "what does it mean to be big for cups." Teachers should be aware of what students consider in their ordering. Ordering based on height can be appreciated, but students also should be oriented to consider the capacity of cups in ordering.

Notation

Representations of relations in a volume context are in symbolic mode. No student had difficulty symbolically reporting dual relations. Ordering based on height iconically was straightforward for some students. They could read and depend on their reasoning on the reports without doubt when guided. Ordering based on volume improved these students' ability to represent iconically based on volume. They can represent symbolically and read algebraic/symbolic representations successfully. All students at least showed process and reverse process behaviors in symbolic representations. Only Medine had difficulties in using signs.

4.8.3 Design Principles for Lecture 8

- Teaching of ordering three or more objects should be revisited in all contexts of variables you want to teach. Dedicate more time to ordering objects in different contexts of variables/ attributes, and strengthen symbolic representation in ordering and extending sequences to use ordering as a base inquiry to converge transitivity idea.
- Guide students to order unknowns, which are noted by relations in a system, as pre-action for transitivity
    - Guide them to represent their ordering iconically or symbolically
    - Guide them to depend on representations
    - Guide them through the ordering algorithm by extending sequences
- Create a need for guessing the third relation out of the system;
    - That need is to be consistent with the context (variable)
    - Try to reduce the bias of size
        - Choice of the suitable material
        - Lots of experience on the variable context, experience, and discussion on what is bigger (which variable is reasonable to consider) in terms of related context
- Using equality relations may be avoided if it would be heavily on the student's own experience and one-to-one discussions are limited. Measurement error ruins the structure of transitivity and guesses for the third relation in classroom discussions.
- Ordering activities before transitivity, are suggested to be supported also by equality relations.
- Be aware that ordering based on volume is not straightforward. Appreciate order based on height, but direct students for capacity.

4.9 Results of Lecture 9

The previous Lecture was on transitivity. This lecture applies the transitivity property. Students use transitivity to compare two distant objects by constructing a movable object. The lecture is adapted to online situations. Nine students attended the online lecture, while Medine was absent. The lecture took 30 minutes.

4.9.1 Plan of Lecture 9

In Davydov's Book (Davydov et al., 1995), transitivity takes a prominent place, from drawing unknowns or ordering objects based on given relations to creating a scale to compare distant objects. These activities focus on concluding a third relation with the help of two pre-determined relations, which ensures the use of the transitivity property.

In this lecture, students are first introduced to creating a scale for comparing two distant objects. They construct a scale and compare it to distant objects to look for their relation to the scale. Then, they develop a relation between these distant objects from their relation to the scale. Constructing the scale equal to one of the distant objects is the easy and handful strategy used in Davydov's Book (Davydov et al., 1995).

In the first HLT, creating scales will be taught through guidance at the action level, whereas students will need to use transitivity property to construct scales. Hence, students' mental constructions on transitivity were expected to be at the object level.

Aiming these mental constructions, the objectives of the first HLT were designed as follows;

1. The student uses their height or a rope as a scale to compare two stable and distant objects by deducing their relation to both.

Revised objectives of the implemented Lecture 9 are:

1. The student creates an equivalent scale for an object to compare it to another distant object.
2. The student interprets the result of the comparison in terms of the distant objects, not in terms of the scale he/she used.

Based on the previous classroom implementation, transitivity is found to be difficult. Hence, the deduction is removed from the main objectives. The change in the objectives also underlies the use of equal scales only to make comparisons easier.

Lecture 9 starts with an introduction on how to compare distant or unmovable objects with the help of another object. Then, students complete two activities to compare distant objects. In the first one, students compare the heights of the table, kitchen counter, and bathroom sink. Students are asked, “Which one is higher, your table or kitchen counter?”. First, they measure the height of the table by marking the same level on their bodies with their hands. This is constructing an equal scale to the table. Then, they go to the kitchen to compare this measure to the height of the counter. They come back and interpret which is higher: the table or the kitchen counter.

This was an obvious and easy way of using transitivity. To make up for the previous class's failure on transitivity, the researcher increased the level of transitivity with another object: the bathroom sink, which is added to be compared to the table and kitchen counter. Students are asked to order these objects by guessing. Students are expected to converge to intuition on transitivity without notations or actual use of the property.

In the second activity, students cut a rope representing their height and used it to compare their height to unmovable objects around them. This activity repeated the algorithm, where they constructed an equal scale to their height.

Lecture Flow:

- (Pre-algorithm) Exemplifying how to use a movable object to compare two distant/unmovable objects

- Compare movable objects to unmovable objects one by one and come up with a result based on their equality or not.

- (Algorithm) Creating equal scales for comparing unmovable objects:
  - Create a measure/scale of height (with the help of their hands and bodies) equal to the height of the first unmovable object
  - Carry the scale and compare it to the second unmovable object.
  - Conclude a relation between two unmovable objects by the relation between the scale and the second unmovable object.
- (Repeat algorithm) Use a created scale to compare with another (third) unmovable object.
- Interpret comparison as the relation between measures of two distant objects.
- (Order three unmovable objects) Interpret the highest and lowest objects based on comparisons or representative measures. (Anchoring, ordering)
- (Repeat algorithm) Use the rope as a scale to compare two unmovable objects (rugs)
- (Repeat algorithm/process) Use rope as an equal scale for student's height to compare with unmovable objects, and interpret relations between student's height and the objects, restricted to the relation between two objects, not a sequence (or ordering for 3-4 objects)

Trajectory

- Continuous quantity manipulation to make equal: use equality (object =)
- Substitution of equal objects in relations (use transitivity with equality: object transitivity with equality)
- Interpretation of the relation of objects using their relation to another constructed object

Further suggestions on learning trajectory:

The trajectory is complicated. Creating and using the scale as a moving version of the object seems to be underlying transitivity. There appears to be a very naïve version of it. To see if there is an underlying understanding of transitivity, using discrete objects (nonmanipulative objects) as scale and lots of practice observed and guided by the instructor is suggested for further studies. We had no chance during online classes. Manipulating a continuous quantity and creating an equal scale is easier than using discrete equal objects. It was the reason why we preferred equal scales in online lectures. Using equal scales may hinder the use of discrete objects as scales. They may even mark discrete objects to create equal lengths, making discrete objects turn into continuously manipulable tools. Discrete object use is essential for understanding transitivity. Creating equal scale does not act as a distinct object of transitivity. When a student uses continuously manipulable material, he/she may not see it as a distinct object but as a measure using a ruler (as it will be seen in the next activity better). Using movable but discrete objects for comparing distant objects (classroom situations will give chance a lot with lots of toys), students may see the movable object as another object. Moreover, they will have a chance to use non-equal objects for comparison, leading to transitivity with inequalities. Our first plan existed to create an intermediary, which also took someplace in Davydov's Book (Davydov et al., 1995). Students' poor understanding of transitivity in previous classes and lack of conditions due to online lecturing reduces transitivity. Transitivity first began with volume construction, which created the struggle. We thought the size would be trivial; however, distant objects would make it nontrivial and more meaningful. As a result, for future studies, we recommend trying trajectory, starting with using equal objects or intermediaries to compare distant objects and then transitivity.

4.9.2 Theoretical Findings of Lecture 9

Create a scale for comparison; Action to process

Interpretation of comparisons between two distant objects without referring to scale was expected from students. Lecture 9 achieved its objective for all students at an action level mental construction on creating a scale for comparison of distant objects. 6 out of 9 (Aylin, Ekim, Didem, Eylem, Bekir, Hasan) students show clear evidence that they developed their level to process as they could use scale fluently and interpret comparison results referring to unmovable objects (free of scale). Action to process mental construction using the scale in comparisons is empowered through the following algorithm:

- Manipulation of a continuous variable to create an equal quantity scale to an object (reverse-process equality)
- Substitution of the equal amount scale instead of the object in comparison to other objects (includes use of transitivity intuitively)
- Interpretation of the comparison results referring to a substituted object
- Creating and using a scale to compare two distant objects and fluently state the relation between them without any guidance or recommendation.

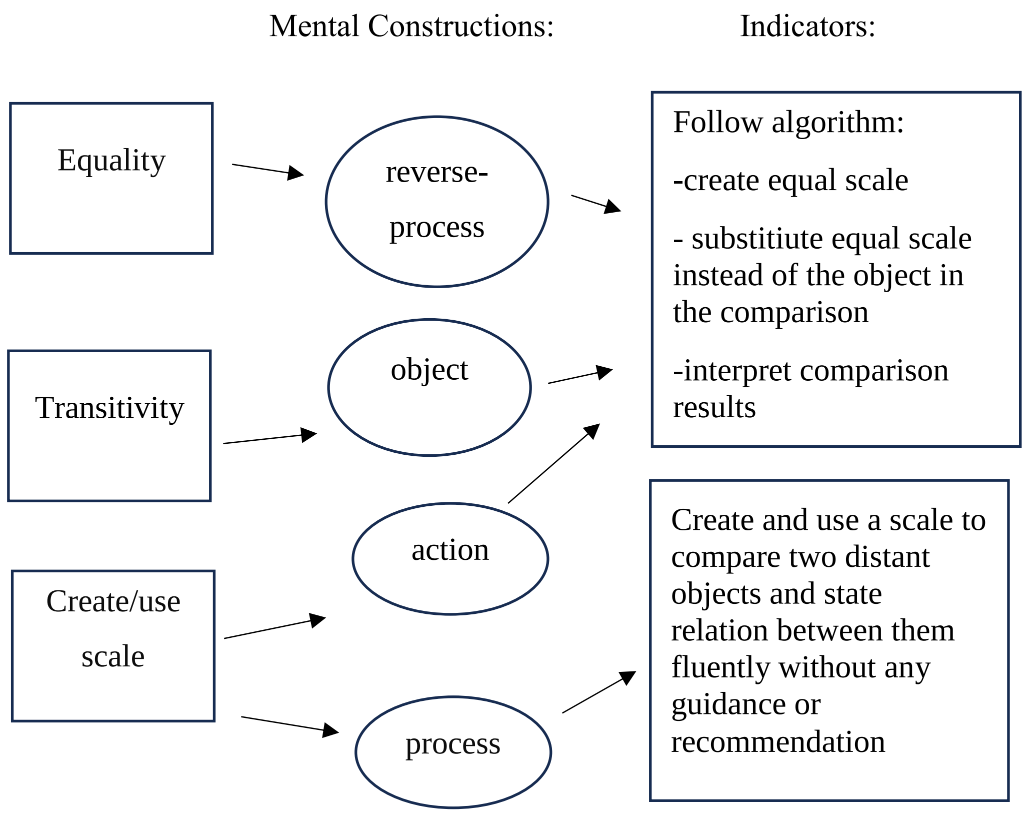


Figure 4.21. Schema of Transitivity, Ordering, and Using the Scale in Lecture 9

Transitivity

Substitution of the scale in comparisons underlies the understanding of transitivity relation with an equality relation in the systems (1) and (2).

(1) If Obj1 = scale & scale > Obj 2, then Obj1 > Obj 2
(2) If Obj1 = scale & scale < Obj2, then Obj1 < Obj2

In addition to these, in activity one, holding heights corresponding to each compared object with both hands, Eylem had another algebraic intuition (3);

(3) If Obj1=scale1 & Obj2=scale2 & scale1>scale2, then Obj1>Obj2

These types of transitivity properties, including relations, are students' mental constructions, which make it possible to accept the idea of creating a substitute for an object and conclude if this substitute is smaller/bigger than the reference object is smaller/bigger than the compared object.

Deduction is guided through the following steps;

- Obj1 is compared to Obj2 with the help of a scale.
- Scale is constructed equal to Obj1.
- Scale is compared to Obj2.
- Relation between Obj1 and Obj2 is concluded from the previous relations.

The second activity fulfilled this deductive algorithm more effectively. In the first activity, scales did not occur as different objects but acted as measurements, which caused an understanding of transitivity to remain implicit throughout the actions. Students used their bodies to imprint the heights of the compared objects. More than the objects themselves, all of their equivalent scales appeared to be compared, making it easier for students to conclude. However, the second activity was more meaningful because the equivalent scale of students' heights enabled comparing their selves to objects. This chance created motivation to make several comparisons. They were voluntary for stating comparison results based on their heights and the objects compared. (If it were a classroom lecture, we would use their heights directly as inequivalent scales, probably causing difficulties for deductions.)

As a result, transitivity was not solely based on deductions from symbolic representations, which students struggle with. However, it could be used intuitively to construct equivalent scales and to deduce relations between distant objects based on their relation to another object (scale). Transitivity joins the action of creating scale as an object while creating equal scale is observed as a reverse process for equality composed within the new process.

Variables

Distant objects are fixed quantities, while the created scale is continuously manipulable. Created quantity is singular, fixed to one of the objects. However, the continuous manipulability of the length is a new context for them.

This lecture is limited to height and length attributes. Ordering was not challenging because comparison is based on height. Moreover, inquiry by making students state the highest, middle, and lowest objects helped them order immediately.

Notation

This activity does not include notation, not even enactively. Only verbal interpretations are used throughout the investigations.

4.9.3 Design Principles for Lecture 9

- The motivation of comparing their height to objects around them creates a motivation for interpretation free of the scale, even without asking. If not motivating, creating a scale should bring meaning to comparing distant objects.
- Make students interpret "what is compared to what" if their interpretation of the result does not include them. They might only state as equal, big, or small. Encourage verbal interpretations of the relations.

- After students interpret their deductions on the comparison of unmovable objects with the help of the created scale, you may simply ask "why" to awaken intuitions on the transitivity. Even if they do not have a direct response, inquiry into why takes attention to the equality of the scale.
- Support students in ordering by asking for highest, lowest, and medium in height context.

4.10 Results of Lecture 10

Lecture 10 is the fifth online lecture. It is about creating scales to measure distant objects, a continuum from the previous lecture. The main difference is that it is a paper-work activity that has two parts: using scales and ordering. Grouping same-size objects and representing them symbolically in the ordering activity facilitates new learning on quantities. All ten students attended the online class, which took about 40 minutes.

4.10.1 Plan of Lecture 10

This lecture aims to process the level of creating and using equivalent scales, strengthening the previous lecture's learning through many comparisons on paper-work activity.

In the first HLT, aims are represented in objectives as in the following:

1. The student constructs scales to compare distant objects.

2. The student uses the same notation to indicate same-size objects

Based on these objectives, the activity is designed to include a comparison of squares drawn on paper with the help of a string. Students are expected to use string to construct an equivalent scale to one of the squares. Then, they use this scale to find other equivalent squares and paint all of them the same color to indicate the same-size squares. In this way, students would use transitivity through all equal relations to find same-size squares.

Create scale:	sq1=string
	&
Use scale in comparisons:	string = sq2 & string = sq3 & string=sq4 …
	⇩
Color equal squares:	sq1=sq2=sq3=s4

An additional activity is planned to strengthen previous learning on ordering and transitivity. Students are asked to compare and order squares using color as their interpretation. The objectives are also revised.

Revised Objectives

1. The student constructs scales to compare distant squares.

2. The student uses the same color notation to indicate same-size squares

3. The students use colors as a notational representation to order squares based on size.

This revision in the activity and objectives extended learning in three main topics: Firstly, it would include a comparison of unequal squares, which needed transitivity with one equal and two unequal relations. It will support previous learning on transitivity. Secondly, students will order these squares and their knowledge of order will be strengthened with new context. This structure will resemble the previous Lecture 9 for creating scales to compare and order distant objects. What is different is that, in this lecture, we do not directly compare objects but a set of objects with the same size. This fact explains our intention of learning on the third topic: Quantity. Color notation of the same size is symbolic notation for a fixed quantity. Objects (same-size squares) are defined to be in the same set (color) by their quantity being the same. The coloring of the squares would probably be sufficient to indicate the set. However, we wanted students to use this color as a symbolic representation of quantities in comparison and ordering. This level of understanding in quantity does not exist in Davydov's trajectory. Davydov uses letter notations to denote a quantity belonging to an object. Same-sized different objects are not labeled with the same letter, but equality between them is represented by the relation between different letters. Based on the revised objectives below lecture flow is implemented in Lecture 10.

Lecture flow:

1. Given the different sizes of squares and a string, the instructor shows how to use string to construct a scale equal to one of the squares and compare it to other squares to find all equivalent squares. (Reminding algorithm for creating equivalent scale) (Transitivity with all equals) Find all equal squares. (Reverse algorithm for using scales. The relation is given, and students find objects based on the relation)

2. The instructor recommends that students color all equal squares in the same color. (Preparation for color notation by assigning colors directly on the objects: enactive representation)

3. The instructor wants students to choose two colors and makes students compare corresponding squares by using string. (creating an algorithm for comparing distant objects) (no reminding algorithm)

4. The instructor recommends that students interpret relations based on the comparison using colors. (color notation in the interpretation of relations process: symbolic representation)

5. At last, students are expected to order the size of the squares by their colors. (color notation in the ordering process: symbolic representation) (3 of the students' works represent iconic rather than symbolic; the size of the colors in algebraic notation differs.)

Trajectory:

- Reminding algorithm to create equal scales (action creating and using scales)
- Grouping all equals (action quantity sets)
- Creating notation for equal quantities (pre-action symbolic notation by color coding)
- Using scales for comparison of distant objects. (process creating and using scales)
- Interpretating comparison results by using colors (action symbolic notation by color coding)
- Ordering quantities of square sizes by using colors (action to process symbolic notation by color coding) (repetition of the algorithm in a new context)

When we compare the size of the squares, we mean comparing the length of one size. This is why we choose squares to make comparisons easier by just comparing one side using a linear string. In addition, creating an equivalent scale is easier in length compared to weight and volume contexts.

4.10.2 Theoretical Findings of Lecture 10

Create/use equivalent scale and transitivity

In finding equal squares activity, students are asked to use strings to find equal squares. 4 out of 10 (Medine, Ali, Ekim, Bekir) students were at action level in using scales, needing some help for comparisons by string scales. From the beginning of the lecture, three students (Eylem, Aylin, and Ufuk) had mental constructions at the process level. They need no help or reminding to use scales in comparison, showing they transferred their prior knowledge to a new context. Three other students (Yaman, Hasan, Didem) improved their knowledge from action to process level after reminding in the first comparisons.

Students went through the following stages/algorithms for using the equivalent scale

- use string to create an equivalent scale to one of the squares
- compare string/equivalent scale to other squares to find equal ones
- determine the equality relation between the squares with the help of the scale

There may be small measurement errors (Aylin), or students may be incapable of creating a correct scale. Regardless of these difficulties with measuring, we noted how they used scales to interpret equality relations verbally or through colors. Some specific problems appeared, such as finding only pairs of equals (Ekim, Medine) or irrelevant coloring (Ali).

Four students preserved action mental constructions during the lecture, all of whom had different difficulties. Bekir is a perfectionist at measurement. His mom helped during the process of creating and using scales. Medine did not attend the previous lecture on using scales. Thus, she needed guidance for these procedures. Ekim and Ali had no guidance, working on their own. Ekim was able to find pairs/equals for squares. However, she could not reflect her ability to use scales in reporting relations or ordering. In prior lectures, she had difficulties ordering constantly. It is not apparent if she struggles only with ordering and comparing. She could use scales in previous lectures and interpret relations verbally. Her difficulty seems to result from her inability to represent symbolically, which obscured our observations about her understanding in this lecture. Ali had the correct ordering. However, his coloring was irrelevant. His only verbal interpretation (explaining his use of scale) is, "There are two unequal ones." He interpreted unequal squares he discovered. The researcher replied, "You will color unequal ones in isolated colors." The researcher thought, he finished all comparisons and found only two squares that were not equal to any of the squares. Probably, he was behind the class, and he was at first comparisons. He successfully stated the result of using scales, that he found the squares unequal. Then, he colored squares indistinctly based on the researcher's command. Maybe he successfully used scales independently but could not understand how coloring works. Whenever he found them unequal, he chose a distinct color. Skipping the step for finding equal ones, his image looked irrelevant.

Figure 4.22. Ali’s Paperwork in Squares Activity

In the first part of the lecture, students needed to use transitivity between all equal relations to find equal squares by creating an equal scale. In the second part, we asked them to report dual relations between different colored squares. We expected them to use transitivity with one equality and two inequality relations by creating an equivalent scale again. Mostly, they did not need to use a scale to make comparisons, and so did not need to use transitivity with inequalities. In some cases, when squares are similar in size or the same size with different colors due to measurement errors (Aylin), students need to use the scale again to compare different size squares. In those cases, they referred to using scales again, but with one equality and two inequality relations.

Aylin reported red = blue, which shows she used scales for comparison again. In the first part, her scale was not always correct, painting two equals with different colors. In the second part, based on her coloring, the difference in size was not apparent. Hence, she needed to use scales to compare again. She was surprised to find them equal because she was supposed to color the same if they were equal. Even the researcher told her to leave equality that way because it was correct; she preferred to correct her coloring.

Briefly, the lecture was successfully supported using scales for all students with many comparisons between squares. Coloring equal squares created a purposeful and motivational context for the subject.

Relations and ordering

All students successfully interpreted relations, and the majority were fluent and auto in ordering (some (Ufuk, Hasan) could not reach till the end.). The lecture revealed students' mental constructions on relations and ordering at the process level. Even Ali, who had difficulties using scales, showed he could interpret and order. His ordering and relations were relevant to the coloring of his work. Ekim's ordering was irrelevant to the situation; her coloring of squares. However, she showed she knows how to order well. She used colors distinctly, and in interpretation, she drew bigger colors with a bigger size. (Her first interpretation of the relation purple > blue is the researcher's first example. She copied directly. Others are irrelevant.)

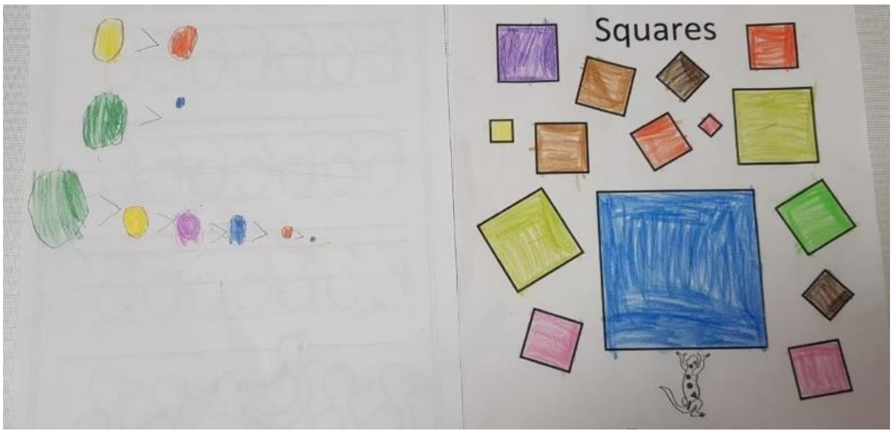


Figure 4.23. Ekim's Paperwork in Squares Activity

The ordering process in this lecture developed through the following steps:

- comparison of two random colors and interpretation of the relation between them
- comparison in pairs as much as possible in a limited time. Not all combinations are expected.
- ordering all colors, from the biggest to the smallest

Comparison and ordering may be trivial. Students may not need to use any scale because, after coloring squares, dissimilarity becomes apparent.

All students could use > and < signs as objects in the new ordering process. Only Ekim used the signs non-interpretive in real-life situations, but she could also use > sign correctly to interpret dual relations and ordering. Ordering is still a process for them because we did not have an activity to use order as an object. This activity is just a new context and new level of symbolic representation for ordering to encourage that type of notation. Hence it was effective in supporting the process level for relations and ordering.

Quantity-Variables

In squares activity, fixed quantities are compared to each other by manipulating a continuous scale. Manipulating it to a fixed quantity underlies an implicit understanding of a continuous quantity, which any quantity can be marked on / can be constructed from.

Different from all other lectures and Davydov, this lecture has another focus on quantity. Quantity does not belong to a single object but represents a set of objects in interpreting their relations or ordering. Once discussed as different attributes, quantity evolved into comparable amounts belonging to objects. Now, quantity is independent of the compared object and represents a fixed amount set in the relation by advancing color notational coding.

Although the activity seems suitable for thinking about the quantity; the language was not supported enough. There was not enough time to interview them individually. No evidence is observed of students thinking about the quantity as a set of squares from their verbal interpretations. However, they had no confusion in dual comparisons or ordering. For example, students might think of just two squares in their comparisons, but nobody asked which blue square to which red square. It may be evidence that they know all the same color squares have the same quantity in the comparison.

Ali's case is exceptional. Ali compared brown to brown, which he colored the same because they are the same size. Then he interpreted them as being equal: "brown=brown." It is correct but a sign that he does not see the color code as a quantity; instead, he considers squares themselves in comparison. He compared two distinct squares, not two distinct colors (not quantities represented by them). His coloring of other squares also contributes to this evidence. No squares are colored based on their equality. No color code, but random coloring can be observed in his work. Squares of the same color with different sizes occur, and primarily, different colors are used. Only two brown squares are equal in size, which he interpreted later as brown=brown. His color notation is unsuccessful. However, his interpretation of relations between squares by their colors is correct. This indicates another result. Accurate relational interpretation proves their quantitative understanding of color notation, and accurate coloring of equals is necessary.

Algorithms need to be clear, and one-to-one communication is essential at their age. Ali misunderstood the guidance on the coloring, which resulted in this confusion. Online lecturing limited observations on students' work, which could be controlled during the procedure. Other students' works showed that this lecture added a new dimension to the understanding of quantity with the help of color notational coding.

Notation

In the previous section, we explained how color notational coding enhanced our understanding of quantity. The color notation also improved symbolic representation. Until this lecture, students used pictures of objects as symbols in their interpretation. The color of the objects, in this case, the color of a set representing a quantity, is a higher level of symbolism.

All students are at the process level for the symbolic representation of signs and ordering. However, for color coding, they are in action to process the level of symbolization. 6 out of 10 students showed evidence that they could use the color as a symbol in interpreting relations and orders. Two students (Medine and Aylin) might be adjusting the size of the colors in order, but not apparently. Their use of colors in interpreting dual relations is symbolic. Ekim represented bigger colors with bigger sizes. Moreover, coloring had no connection to real-life situations. Her disjoint notation does not represent/reflect any other object or quantity, typifying an enactive representation. Briefly, Lecture 10 supports symbolic representation through color-coded notation at the action level.

Figure 4.24. Correct Symbolic Interpretation with Adjusting Size of Symbols

The algorithm of color code notation is given the first time when they are interpreting dual relations; then, they are requested to order. Children may transfer/integrate the knowledge of color coding and use the algorithm automatically in new context ordering, or some students may hesitate to color define a quantity (as in Medine and Aylin’s cases). In that case, they need to show the size difference in the order sequence, as it was when they first learned ordering in the enactive mode of representation. Mental constructions for symbolic notation with color coding of fixed quantities are defined below (due to student observations in this lecture):

Action:

- use color notation through guidance
- resizing when passing from relations to ordering.

Process:

- use of colors automatically when asked to order, without resizing.
- Color represents code for fixed quantity, symbolic.

The reverse process for color notation will be held in rainbow activity. In the rainbow activity, the object's color will be used as notation in algebraic representations. Students will read the algebraic representation and create real-life situations out of it. Due to potential difficulties based on the results of this lecture, in rainbow activity, quantity color codes will be provided in sizes compatible with the relations.

In Lecture 10, color code symbolic notation, using scales, interpreting relations, and ordering are orchestrated successfully, advancing each other and elevating a new understanding of quantity.

4.10.3 Design Principles for Lecture 10

- Measurement errors may exist in this activity as the squares are small and look similar. Unless the student's technique in creating a scale is wrong, errors may be ignored. Finding all equals and notating all equals with the same color is more important.
- Be precise in commands and check if they follow the algorithm step by step giving sufficient time in each step of measure, compare all, and color all. Ordering with color notation will build on these steps if concluded correctly and consciously, where students will need to see equal-sized squares in a set and quantity as an amount represented by a notation.
- The inclusion of order is essential in this activity so that we do not miss the opportunity to see the quantity concept from an object's point of view (quantification).
- Encourage students to improve their language on quantity through discussion. At least the instructor should use it each time to ensure thinking about the quantity: use plural words for squares and address quantity

comparison: “Which one is bigger: the size of blue squares or the size of green squares?”

- Appreciate equality relations in the interpretation of dual comparisons and ordering quantities. Guide them to reflect on their color-coding procedure. Coloring solely in pairs is common which can be prevented from the beginning.
- Transitivity of equal relations is not the only mathematical intuition students encounter in this lecture. Squares become members of a set by their relevance on quantities. Constructing sets can be prompted after this lecture.

Materials:

- Provide squares of different sizes but close to each other to create a purpose for using scales.
- Include various numbers of equal squares, but not mention their numbers. Avoid bigger and smaller squares (not to confuse number vs size).
- Pre-given stickers could be beneficial to enforce color-coding.

4.11 Results of Lecture 11

Lecture 11 Online is the sixth online lecture consisting of three parts: constructing signs with wooden sticks, discussion on multi-solution on the prior mid-assessment results, and graphing plants height. All of the activities are constructed during a lecture time of 40 minutes. Only Ali was not an attendee.

4.11.1 Plan of Lecture 11

The first activity in the lecture was constructing signs they learned $=$, $\neq$, $<$, $>$ with wooden sticks. This lecture brought some fun while reminding signs. The construction of signs focuses their attention on the parts of the signs that will strengthen their remembrance. They did not construct or write in the previous lecture but used signs provided to them.

The second and third activities are the main activities built around the objectives of different topics. The second activity, discussion on multiple solutions, was not in the first designed HLT, whereas the third activity, graphing plant heights, was part of the first HLT.

The first HLT objective was:

1. The student uses equal-sized scales to represent measurement. (plants)

By the addition of activity on multiple solutions, objectives are revised objectives:

1. The student recognizes multiple solutions to construct objects based on >, < relations.
2. The student uses equal-sized scales to represent measurement. (plants)
3. The student verbally interprets the change in height.

In the second activity, students' solutions on the mid-assessment for constructing quantity items are shown to students. Students reflect on the solutions for discussion on the appropriateness of multiple results.

Lecture Flow: Recognize multi-solutions

- See two different solutions to a “>, <” construction problem
- Reflect on the difference between the two solutions to the problem.
- See all other solutions (10) together
- Discuss if all can be correct
- Explain why all are correct.
- Discuss extreme solutions to the question
- Repeat stages with another “>, <” problem
- See a solution to create an equal object to a given problem.
- See all other solutions
- Discuss why they all look similar.

In the third activity, they will measure their planted beans for the third time using a string. They stick the string on the paper to form a graph of height over time and observe changes in the height of the plant through their graph.

Lecture Flow: Change in plant height

- Use string to measure plant: represent quantity by an equivalent scale
- Stick string on the paper: graph plant height (for the third time):
- Interpret change in plants' height: interpret change using/reading the graph

4.11.2 Theoretical Findings of Lecture 11

Wooden stick signs

The researcher showed how to make a ">" sign from sticks. The researcher asked students to create an equal sign. Students got confused because there were no joints. Then researcher showed how to do it. The researcher asked students which sign was left. Most of the students replied with an unequal sign. Finally, students constructed unequal signs. They all could remember signs and names and constructed signs correctly. Remembering of unequal sign was remarkable, as we had not used it for a while. All their knowledge of signs seems permanent.

Variables

There were three items for constructing quantities based on pre-given relations: drawing a smaller tree, bigger ice cream, and equal pencil. Firstly, the researcher shows two different solutions to drawing a tree smaller than a given one, explaining varieties to students. The researcher then presents all solutions to the students and asks students if all of the solutions are true. One of the students (Eylem) replied, "No," while four students (Ekim, Bekir, Hasan, and Didem) replied, "Yes" confidently. Bekir could explain why they can all be correct in his own words.

R: Why all answers are correct?

Bekir: Because they are small. If they show like, then they are correct.

The researcher continued the discussion by asking what if a tree was drawn extremely small. Didem and Bekir replied, “It is OK”. The researcher asked the reverse to make them recognize the limits and ensure they were not auto-replying. The researcher asked if the tree was drawn. Very big four students (Didem, Bekir, Aylin, Hasan) replied, “No,” all attending discussions from the beginning. Only one student (Eylem) hesitated in discussing multiple solutions in item 1. Her hesitation was not due to size but due to image differences. She wanted to see all the trees looking like the reference one. The researcher explained why all these varieties are correct. Then researcher continued with the second item: drawings for ice cream bigger than a given one.

The researcher asked if two different drawings of a bigger ice cream were correct. Eylem replied, “No,” while Didem replied, “Yes” again. The researcher shows all the different drawings and asks if they are all correct. Five students replied “yes” (Bekir, Medine, Aylin, Didem, Hasan), while Eylem insisted on replying “no.” (She might be considering other attributes, wideness or height, or size to assess results. She admitted that all pencils are drawn correctly in item 3 when we discussed height as the measure, we consider inequality. Expressing based on what we are comparing is essential.)

Inquiry to infinity

Discussion on the multiple solutions in the second item of bigger ice cream continued with the inquiry on infinity. Infinity was discussed in a previous lecture, in ordering before. The researcher questioned how big we could draw the ice cream bigger than the given one: as big as a tree, as big as the biggest, bigger to infinity:

> Aylin: If the paper was very big, it could be possible.
>
> Eylem: As big as the earth
>
> Hasan: This much
>
> Yaman showed above his head
>
> Student: As big as the sun

Didem: As big as the sky

Medine: Up to infinity

Researcher: How big is infinite?

Bekir and Eylem: It means it never ends.

Eight students out of 9 attendees had recognized and accepted multiple solutions and considered extending solutions as much as possible until now. (1 student Ufuk no response) Infinity inquiry was on how big a solution can be. Discussions on the number of solutions have not been conducted yet. Infinity is a new topic for them. Moreover, we never focused on representing discrete numbers of objects before, we kept focus on continuous quantities. In this activity, "multiple solutions are correct" is discussed but not reflected on the circumstances they exist. They only experienced infinite and single-solution cases.

The last item was confusing for students because it was a struggle to draw equal-height pencils in mid-assessment also. The researcher had to prove equality in height by the help of using a scale to Yaman. He was cautious in his drawing, matching the endpoints of equal pencils with lines. Then researcher appreciated their work; all students' drawings were equally successful, and she told how they looked similar to each other's, all having the same height. Eylem and Didem approved the researcher. Then the researcher asked; "Why they are all equal?" referring to students' pencil drawings. Medine replied: "Because all are the same thing." Their ability to explain and use words is limited. However, they know in terms of how they are equal. The researcher explained further: "Yes, all are the same thing because we drew them looking at the same pencil, and equal to it. Thus, all are equal to this pencil, all look like this, all look like each other.". multiple and single solution discussion ends here.

With the help of this lecture, we could make students recognize the possibility of multiple solutions. Built on the prior environment, where they had the flexibility to have multiple solutions with the help of continuously manipulable variables, they had no difficulty accepting multiple solutions. In the trajectory to find and reflect on multiple solutions as objects, we can say they are at the pre-action level. At the end of the whole semester, we expect them to be able to find multiple solutions as process-level mental construction. We can list the whole trajectory for multi-solution as follows;

- Pre-action: Flexibility to have multiple solutions in discrete and continuous tasks
- Pre-action: Recognize multiple solutions can be possible/correct
- Action: Interpret multiple/other solutions when recommended
- Process: Find multiple solutions fluently
- Object: Discuss properties of solution sets when multiple solutions are possible and reflect on solutions or situations.

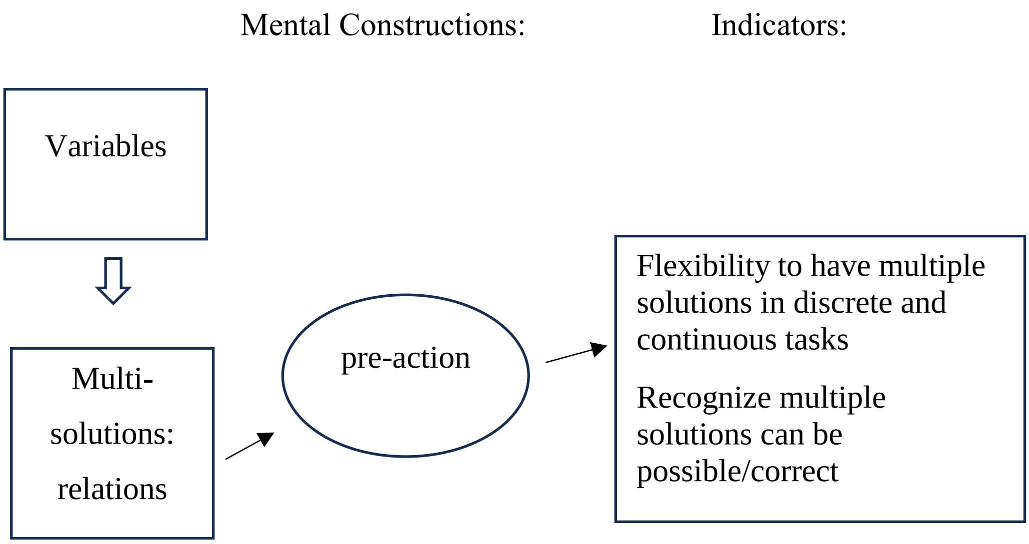


Figure 4.25. Schema for Multi-Solutions in Lecture 11

Variables

The lecture goes on with the graphing height of plants students grow. In this lecture, they made the third measurement. The researcher starts the activity by asking about graphs:

R: Let us see if our plants got higher or shorter

Eylem: Mine got higher too much (showing plant) (interpreting difference on the plant)

Didem: Mine is this much (interpreting height at present)

Bekir: It got bigger than mine; how it that happen that much? (comparing his plant to another)

R: All yours become beautiful. Now we will measure them by strings and stick them on the paper as we did before.

Eylem: Teacher, mine was very little, now it is very big

R: Good job. Now stick your strings on the papers…… (Eylem and others stuck strings and showed them to the researcher.) Good job, Eylem. Eylem, did it grow? Is it the same as last week, or did it grow higher?

Eylem: It grew/elongated

R: Yes, it did.

Eylem: They are not the same; they are different (stating the relation between last week's string and today's)

After this conversation, the researcher checked the students' work and asked them if their plants changed. Eight of them stated that their plants grew higher. (Ekim's plant had died.). They could use strings to create equivalent pieces and use them to construct graphs. They could interpret change in the plant based on the graph verbally. This lecture is a preparation for a further lecture, where they will be questioned on the amount of change. Verbally interpreting change creates verbal pre-action for increase/decrease actions by +/- signs. Then increase/decrease by an amount will be questioned on these graphs.

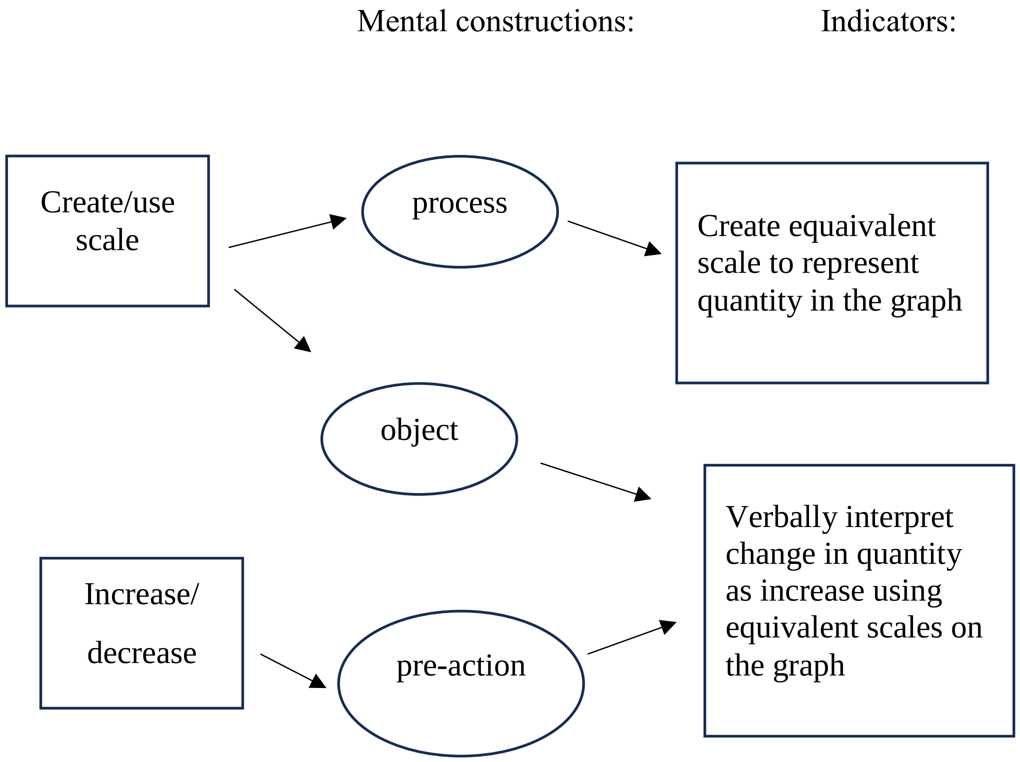


Figure 4.26. Schema for Learning Quantity and Increase/Decrease Actions in Lecture 11

Plants activity: Trajectory on quantity:

- Represent quantity by an equivalent scale (use of scale for graphing, while graphing is the process of creating scale)
- Interpret change in quantity (while interpreting change, students use those scales to represent quantity in time and then interpret change referring to them. These scales became objects in the observation of change)

4.11.3 Design Principles for Lecture 11

Multi-solution

- Build multi-solution discussion on examples where students had experience with flexibility on multiple-solutions, do not bring outer examples. Even if students could not bring multi-solutions to the tasks, flexibility brings diversity to their answers, providing examples needed.
- Construction of quantities creates a flexible environment for multi-solutions, creating pre-action mental constructions for multiple solutions.
- Accepting others' solutions is a way to recognize multi-solutions as a pre-action level. Another way to recognize or find multi-solutions can be directly asking the student if she could draw another one or find another solution to a given situation. (This comes later in our trajectory.)
- It also starts the multi-solution idea with a reverse process view. (not creating multi-solutions but assessing given solutions as multi-solution)
- Discuss extreme solutions to evoke the boundaries and infinity of solutions.
- To clarify diversity and equality in multiple solutions, always inform students about in terms of which attribute (size, height, etc.) quantities are constructed when constructing and assessing.
- Be aware of students' imaginations and hesitations, which may hinder their focus on discussing solutions. Perfectionism or drawing preferences may also affect their arguments.

Change in height: Plant activity

- The activity naturally evokes students' interest in the change of height. Comparison and interpretation of change (among time and plants) appear naturally but can also be supported through questions. Emphasize comparison between heights.
- Change amount will be discussed through these graphs after increase/decrease and addition/subtraction subjects are learned. In this lecture, stating change lays the foundation for change amount. It forms pre-action mental construction for +/- as an increase/decrease topic through

verbal interpretations of increase/decrease. Encourage students to use increase/decrease in their verbal interpretations of height changes.

4.12 Results of Lecture 12

Lecture 12 is the seventh online lecture. It is an introductory course to operations, starting with the names and meanings of plus and minus signs. It took 40 minutes, and only Hasan did not attend the class.

4.12.1 Plan of Lecture 12

Following Davydov’s trajectory, operations are first introduced through a discussion of how to make equality. Inequalities are played as algebraic objects; students are expected to act on them with operations to create equality situations. Now, students do not see equalities and equalities as a process; they are static objects, where operations become procedures.

In our first HLT, operations are designed to be introduced in three separate lectures; objectives and activities are listed below:

HLT Lecture 21

Objective: The student verbally interprets on which side to increase or decrease to make/satisfy equality.

Activity: Students interpret which side to increase or decrease and actively investigate increase and decrease in a weight context using play dough.

HLT Lecture 22

Objective: The students discuss the increase or decrease in volume context to make equality

Activity: Students continuously manipulate salts in cylinders to investigate increase/decrease in volume context.

HLT Lecture 23:

Objective: The student chooses the correct sign +/—to interpret the increase or decrease on both sides to satisfy equality.

Activity: In volume context, students use signs enactively.

HLT Lecture 24:

Objective: Given iconic interpretations (worksheets), the student chooses the correct sign +/- to interpret the increase or decrease on sides to satisfy equality.

Activity: The student chooses signs to make equal from given unequal situations on paper items.

In the first week, students enactively investigated increase and decrease to make equal in a continuous context weight with play doughs. Moreover, they verbally interpreted the increase in Lecture 11. Hence, we eliminated HLT Lecture 21 and HLT Lecture 22. Starting with volume context would make it more difficult. We started with iconic representations of simple height and length comparisons and how to make an equal inquiry, as in Davydov's Book (Davydov et al., 1995). Our revised objective for Lecture 12 is:

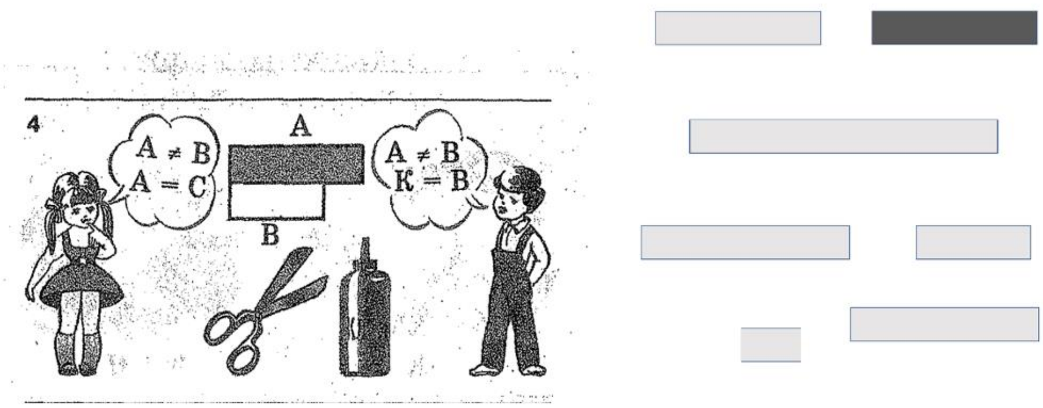
Revised objective: The student chooses the correct sign (+/-) to interpret the increase or decrease on sides to satisfy equality.

There are two activities to investigate +/- signs enactively. Both activities demand action by using signs through the "how to make equal" inquiry. Given the inequality situation; students determine the correct sign for a side of inequality to make it equal to the other side. In the first activity, they are free to choose a side and discuss making equality by manipulating both sides, which means using both signs. In the second activity, students are expected to manipulate only one side. However, several situations enable them to experience both signs.

1st Activity: Paper Children Height: There are two paper children simulations in which students can increase and decrease their heights. Students choose the correct sign for one child to make it equal to the other and execute their choice as increase or decrease based on the sign to make equal.

2nd Activity: Paper strips: There are several paper rectangular strips, one black while the others white. Students are expected to make all strips equal to the black paper strip. Firstly, they choose signs for increase/decrease, then activate increase/decrease through cut or paste to make equal strips. One of the strips is equal to the reference black strip, which needs no manipulation.

2nd activity is adapted from Davydov's Book (Davydov et al., 1995), where two unequal strips exist, and cutting or adding strips appear as two different options to make them equal to each other without using +/- signs. (This activity was before using +/- signs, in Davydov's trajectory. We adapted changing both sides to make equal out of equality subject in weight context with play doughs). In this lecture strips activity, we restricted students to manipulate a single side and learn signs connected to the actions while we made them experience and repeat several situations.



(a) (b)

Figure 4.27. "How to Make Equal Strips" Activity Versions in (a) Davydov's Book (Davydov et al., 1995, p. 27) and (b) This Study

Lecture Flow:

- Introduction of sign names as plus and minus; explaining plus for increase and minus for decrease.
- Showing how signs act on inequality to make equality: heights of researcher and daughter example. (From the first week, students are reluctant to this example for increase and decrease to make equal. Now they learned signs additional to this. One learning step at a time they had no difficulty understanding.)
- Individual enactive investigations to make paper children's heights' equal by choosing right sign. Students are individually interviewed to observe if they learned signs and manipulation correctly.

Trajectory:

- Manipulations of inequality to make equal (object inequality, action increase/decrease)
- Match operations with increase/decrease actions (algorithm using +/- signs)
- Experience operations on both sides to make equal
- Experience operations forced on one side to make equal
- Actualize operations on one side by changing adding/subtracting quantities (pre-action increase/decrease amount)
- Repetition of the algorithm on one side manipulation to make equal

4.12.2 Theoretical Findings of Lecture 12

Operations

The lecture starts by asking the names of the “+” and “-“ signs. All attendees (9 out of 9) knew the name of a plus sign, while only one student (Eylem) interpreted the name of the minus sign as “subtraction”. The researcher introduced the names of the signs and explained that they are used for increase and decrease. The researcher demonstrated how to use these signs to achieve equality. In the demonstration, the researcher compared her height to her daughter’s height. She questioned, “What to do to make heights equal”. 5 (Ali, Yaman, Bekir, Eylem, Ekim) students immediately suggested an increase in the daughter’s height to make it equal, while 3 (Bekir, Eylem, Ekim) of them could also suggest a decrease in the researcher’s height. These correct and immediate responses originate from their reluctance to increase and decrease to make equal from the first week. This verbal interpretation of increase/decrease acts as a verbal pre-action mental construction for operations. It was built onto the pre-action mental stage of enactive investigations of increase/decrease actions to make equal in weight context in the first week (play dough activity). The researcher explained how to use +/- signs for those increase and decrease actions. This explanation formed an algorithm for increase/decrease actions using +/- signs.

After introducing signs and their use, students took their paper children to enactively investigate increase and decrease height individually. All students are interviewed through their use of signs correctly to make students' height equal. Only one student (Ali) had assigned the wrong signs to children to make them equal. His elder sister suggested he use the + sign for higher children and the – sign for lower children. Ali’s sister is biased that more quantity deserves a plus sign, and less quantity deserves a minus sign. Other students had no difficulty choosing which sign to make equality. This shows lecture is successful in teaching +/- signs as increase and decrease. Given an inequality situation, they learned they needed to increase the smaller side, with addition operation, and the opposite. After we ensure their use of both signs for increase and decrease action, a third activity is conducted, in which they would assign signs more systematically.

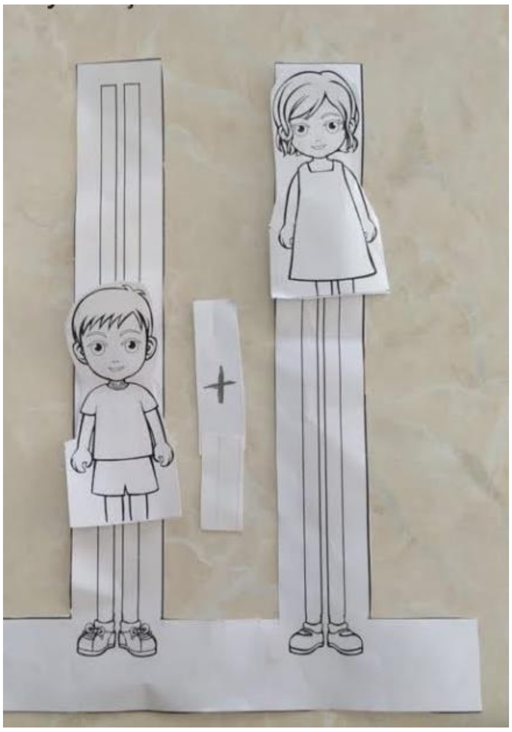


Figure 4.28. Medine: Children Height; Assign + Sign to the Short Child.

In the third activity, they were provided paper strips, one of which was black. They again enactively investigated how to make equality by increasing/decreasing actions and using/assigning signs for their actions. They had several strips for repeating the algorithm. Different from the previous activity, one-sided manipulation is forced. The black strip was not manipulated, while students cut or pasted others to make them equal to the black one.

Steps for making strips equal are described by the researcher as follows:

- Compare one of the strips to the black strip.
- Decide what to do to the white strip: increase or decrease to make it equal to the black one.
- Write a plus or minus sign on the white paper, which is reminding the action to take

- Then cut the excessive part, if you use a minus sign, or add an extra part if you use the plus sign

The majority of the students (7 out of 9) could perform both actions by assigning correct signs, including no manipulation for the equal-sized paper. Only two students (Ali and Medine) needed step-wise guidance until the end of the lecture. Medine usually has difficulty remembering signs, so she needed guidance in each activity. Ali was confused about assigning signs again in the strip activity as in the children's height activity.

R: Which sign did you write there?

Ali: Plus

R: Is that long or short?

Ali: Long

R: What happens if you put plus on it if it is long

Ali: It gets longer

R: Yes, it gets longer. Then, do they become equal?

Ali: No

R: Which sign then

Ali: Minus

R: Good job, minus sign

Ali knows what to do to achieve equality, in terms of increase and decrease actions. He needed reminding of the algorithm for choosing the correct sign for those actions. A similar conversation with him appeared in the children's height activity. Due to his constant need for step-wise guidance, he is assigned to be at action action-mental stage for using +/- signs. Ali performed all others correctly after this conversation. Yaman had no final report of the "equal strips" activity, but he had verbal interpretation for increase and decrease actions and he exposed choosing of correct signs in mini-interviews correctly. After all students performed operations to make equal, the researcher asked for generalization through questions: "What did you do to the longer strips." "What to do, if the strip is shorter than the colored one?"

Answering these questions, students also developed formal language for operations; replying "I reduced/subtracted" and "I added on" (Bekir). The researcher appreciated their response (Yaman, Eylem) of suggesting change on the black strip, but directed students to change white strips for now.

All these activities for action level on operations are built around "how to make equal" inquiry. "How to make equal" inquiry starts with an inequality relation, and acts on it by operations to create an equality situation. Here inequality relations in the form of a>b become algebraic objects acted on by increase/decrease actions. Briefly, mental constructions belonging to Lecture 12 are summarized in the figure below:

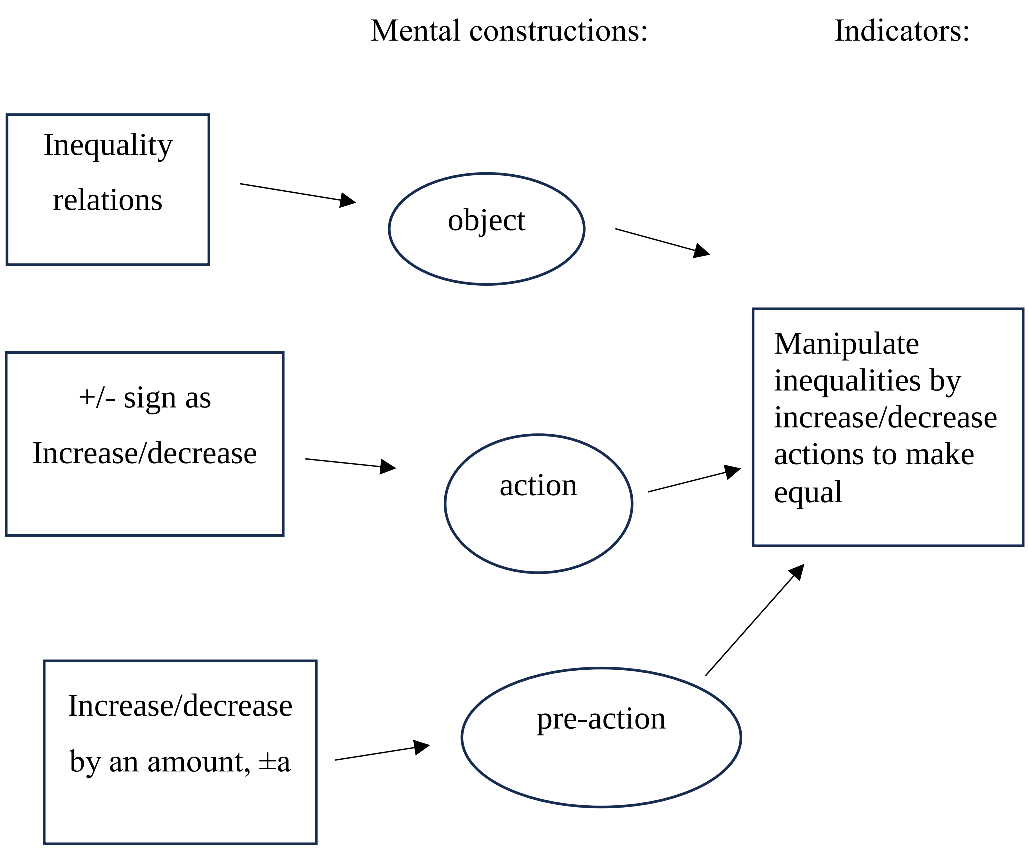


Figure 4.29. Schema of Learning Equality and Operations in Lecture 12

As described before pre-action stages of enactive investigations and verbal interpretation were conducted before this lecture. Lecture 12 supports increase/decrease action levels for addition and subtraction operations. Students will become fluent in those actions, improving to a process stage on performing operations. In the activities of Lecture 12, they performed also increase/decrease amounts while they were manipulating to make equalities. However, we did not want students to express verbally how much to increase or decrease to achieve equality. Hence, for learning increase/decrease (addition/subtraction) by an amount this lecture found an enactive pre-action stage. In further lectures, they will be supposed to express increase/decrease amount to make equal verbally, for verbal pre-action level. (The increase/decrease process and quantity process are composed of addition/subtraction by an amount process together). Later, performing addition/subtraction by an amount will be in our trajectory. When performing an increase/decrease by an amount becomes a process of operations, finding unknowns defines its reverse process. At last, we want them to reflect on the effects of certain addition/subtraction amounts to the equations, where operations become algebraic objects reflected upon. Reminding Lecture 12 preserves an action level, the whole trajectory of operations can be summarized as in the following figure:

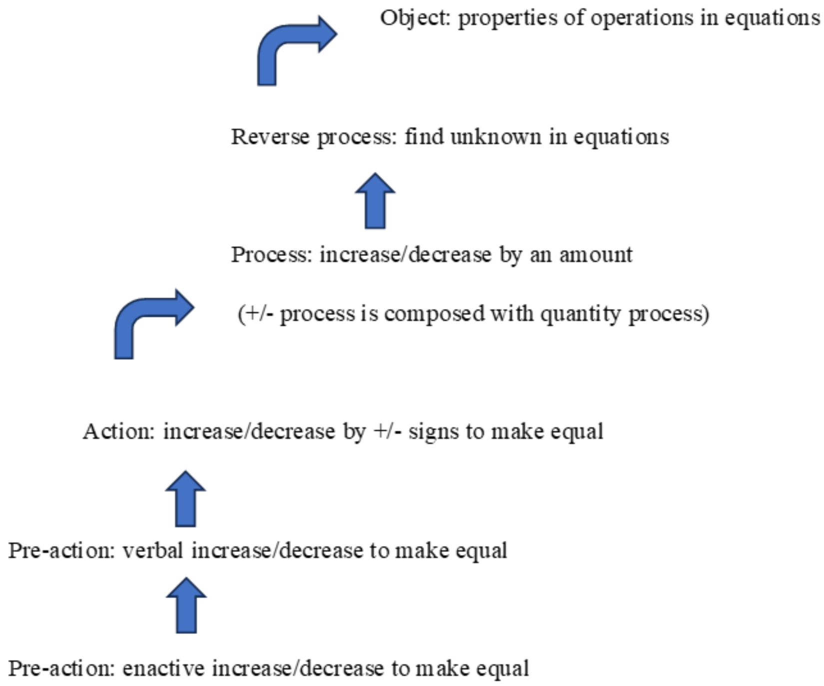


Figure 4.30. Trajectory of Operations

Variables

In this lecture, continuous variables height and length are used. These variables are chosen for their ease of visualizing increase and decrease. Students acquired quantity and comparison between quantities before. Now, they manipulate quantity by increasing and decreasing actions to equal it to another quantity. In the children's height activity, manipulation is continuous, while in the “equal strips” activity, manipulation is piecewise. “Equal strips” activity requires determining the difference amount between quantities. Cutting and pasting based on different amounts in one piece is a common solution. If the addition amount is not obtained, students use discrete amounts added onto each other until equality is satisfied (Eylem, Aylin). Aylin’s addition of extra pieces until it reaches an equivalence can be seen in her work. Eylem stated her addition of discrete pieces as: “I joined five paper pieces.”.

Figure 4.31. Aylin’s Solution to Making Equal Strips Activity

The addition and subtraction of discrete pieces set the pre-action stage as an enactive investigation for the difference amount. Students manipulate sides of the inequalities by adding or subtracting some amount of quantity. (Underlying algebraic intuition is a difference amount. Equivalence of increase and decrease amount is not discussed.) Here, addition or subtraction amount is a new form of process, but not in algebraic notational form yet. Students’ mental stages are at the pre-action level in the form of enactive investigations for difference amounts to make equality.

The action is on inequalities. Hence, in this lecture inequality relations become algebraic objects. The researcher emphasized these relations before manipulations to make equalities by questioning: “If the strip is shorter than the colored one (inequality relation), what do we do?” Didem immediately replied, “We should get paper bigger”.

Didem’s reply shows that action manipulation is not only objecting to change on the inequality, but change is on the quantity. Relation has two components: inequality/equality, and quantities. To change the inequality relation to an equality relation, students act on quantities by addition/subtraction. Didem’s reply also underlies a continuous change in the quantity, pointing out that it gets bigger. Her interpretation aligns with our aim of teaching operations as actions of increase/decrease. Eylem's reply, "I joined five pieces to make equal," might seem like an addition from a "coming together" perspective. However, she also increased the quantity by adding pieces until it equated. Other students' interpretations also aligned with the idea of change in quantity, and they all enactively investigated the change amount throughout the activities.

Notation

Paper children's heights and paper strips are hands-on materials that students worked on both enactively. They assigned signs and performed increase/decrease actions directly on the manipulatives.

4.12.3 Design Principles for Lecture 12

- Before this lecture, discussing how to make equalities and verbally interpreting increase/decrease actions are essential, as they play an important role in pre-actions to operational actions.
- Start inquiry by inequality relation to build operational actions on it. Emphasizing quantities helps students to determine increase/decrease actions. Once increase/decrease actions are determined for particular side of the inequality, the representative sign can be assigned. In the last step, students should be asked to manipulate the sides / and perform the actions based on the signs they assigned. This last step constructs the action mental stage for addition/subtraction. They need this step, to perform addition/subtraction actions and see the result in terms of the equality relation.
- Emphasize both equality and quantity in interpreting relations, to act manipulations and observe changes in them structurally.

- Design activities to include both signs and both increase/decrease actions. If students are confusion, make sure they use both signs and both increase/decrease actions in their way of understanding.

4.13 Results of Lecture 13

This is the eighth online lecture to teach signs as positive/negative directions in a "distance" context. It took about 20 minutes. Only one student, Aylin, was absent.

4.13.1 Plan of Lecture 13

In the concept, we aim to teach the signs associated with positivity as an increase and negativity as a decrease. In the Previous class, the increase decreased the meaning of the signs, but it was only discussed how to make them equal.

Objective: Student performs actions of increase/decrease by an amount by moving forwards and backward with fixed lengths.

Activity: Students are provided two dice. One dice includes animals, while the other includes signs. Students roll two dice at the same time. Based on the result student will move forward or backward with imitating animal steps.

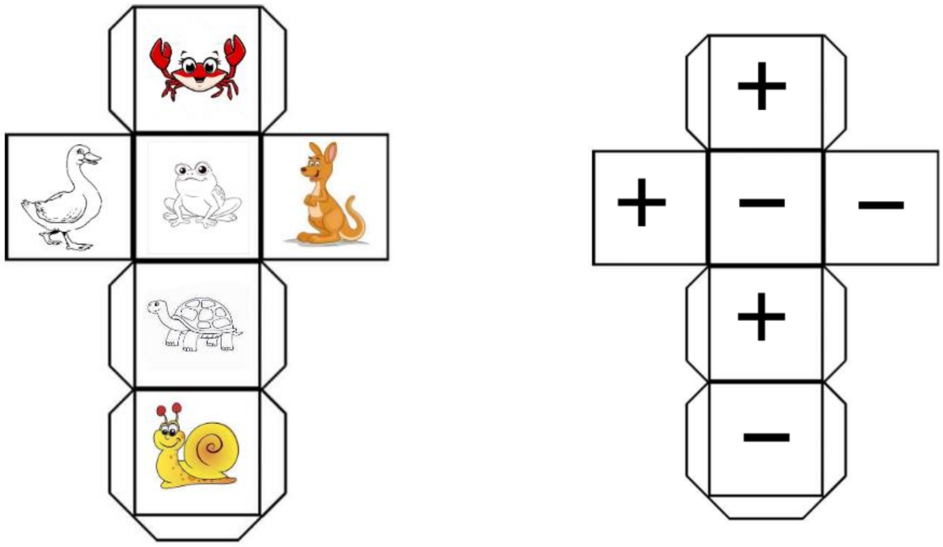


Figure 4.32. Animal and Operation Dice Used for Animal Steps Activity

This lecturing is out of Davydov’s trajectory. Davydov always includes equality relations in operations, and operations are built on “how to make equal” inquiry. Based on piloting and lecture results, we decided to strengthen the positive and negative meanings of the signs. Thinking operations increased and decreased with the equality situation was difficult sometimes. This lecture is designed to teach action of increase or decrease by an amount free of thinking about equality.

In the previous class, manipulatives were continuously dynamic and increased/decreased with a fixed difference based on pre-given or pre-constructed situations. In this lecture, students learn signs as positive and negative movements with multiple fixed/discrete amounts/quantities. As they roll two dice, one defines orientation by signs; the other defines jump size by animals for fixed quantity. They will simply perform situations where quantity varies based on the animal dice. (In warm-up activities (to meet and get used to students), there was a similar game: animal jumps and numbers dice. In that activity students were having difficulties even in which way they moved forward.)

The positive and negative meanings of signs will also create bases for future learning on number lines. We do not mean positivity and negativity as related to signs with their dictionary meanings, nor do we directly interpret the actions verbally as positive and negative. We wanted to emphasize the increase and decrease in the meaning of the signs. Moreover, when piloting in operation activities, we also faced the difficulty of falling behind the zero line to investigate subtraction in quantities. Our activities are based on free investigations where students increase and decrease quantities of their choice or by chance. In Davydov's activities, there are iconic pictures of situations, and no negativity below zero exists. Our piloting with subtraction in volume context showed a need to make this connection. Increase or decrease by an amount will be investigated freely in this activity in Lecture 13 before it is given in volume context. Piloting this lecture was also successful and motivational for students because they were active in the motions.

Recommendations for future interest: We had no dice with numbers but it can be added as a third dice representing constants in the equation.

Lecture flow:

- Show animal dice and teach animal jumps/steps to students.
- Remind +/- sign meaning as increase decrease
- Tell students when plus comes to move forward, but when minus comes to move backward because it means decrease. The aim is to reach the camera, which defines forward orientation.
- Perform +/- actions by rolling two dice simultaneously. Repeat the algorithm several times.
- Play a game as a race between two people.

4.13.2 Theoretical Findings of Lecture 13

Operations

All students enjoyed and learned +,- signs as positive and negative directions in their movement. The lack of equality made this lecture just focus on the sign direction. Movement requires magnitude in its nature. The animal dice determine the quantity of the movement.

Students performed where dice were components of this algebraic process. “a” varies by the animal dice. Repeating action by throwing dice several times, strengthens the perception of “+” and “-” as a process together with the quantity “a” process, composing the process of increase/decrease by an amount “±a”. Composition is not only performed but also visible through the result of dice. Two dice come together, composing the image as “± animal.” The previous lecture discussed increase/decrease to make equality, where the focus was on the operation, and the amount was not interpreted explicitly but investigated enactively. In this lecture, the pre-action of investigating an increase/decrease amount evolves into the action of increasing/decrease by an amount, where the amount is determined and performed by animal steps.

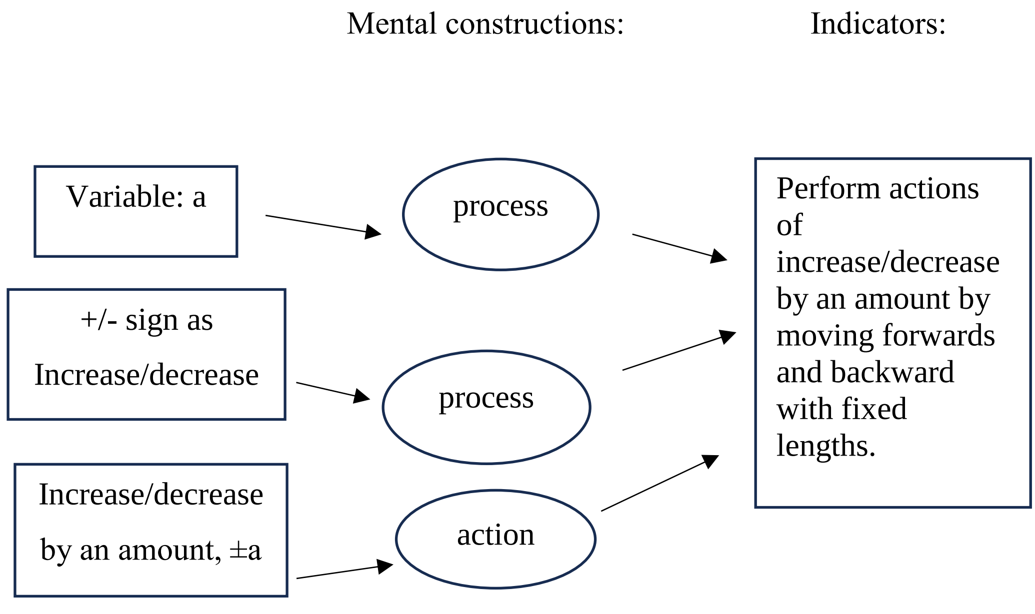


Figure 4.33. Schema for Learning Quantity and Operations in Lecture 13

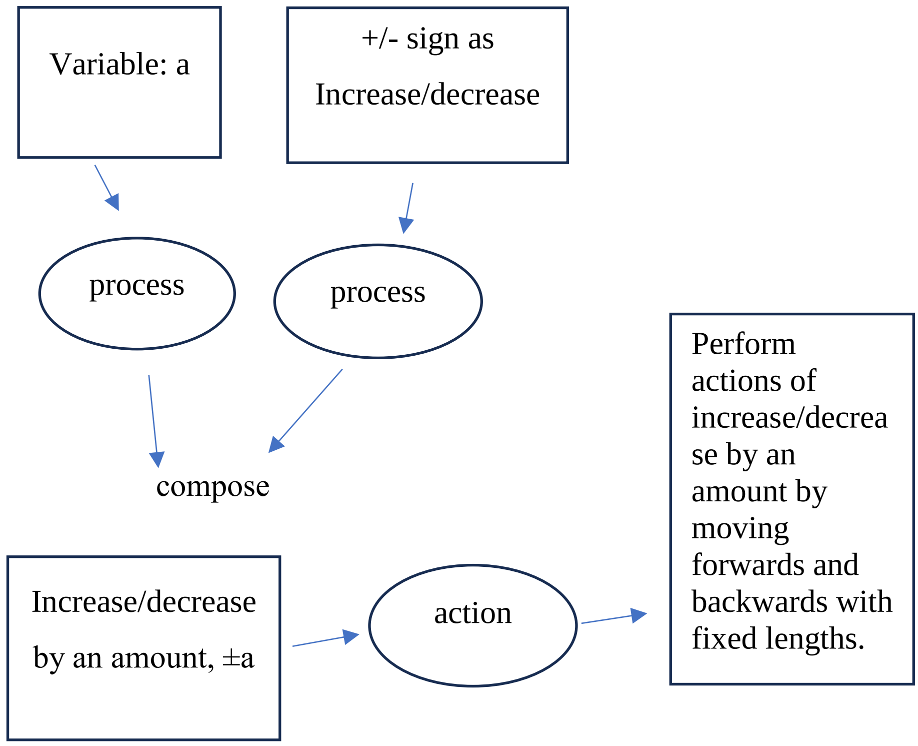


Figure 4.34. Composition of Quantity and Increase/Decrease Processes into Increase/Decrease by an Amount Process in Lecture 13

Variables

Quantities are fixed relative to each other, based on students' perceptions. In previous lectures, students also used different fixed quantities for comparisons. With dice, randomness enables variety in a limited number of fixed quantities. Length context improves understanding of continuous quantities. Students experience different lengths by fixed quantities of animal steps. Continuity in the movement underlies the idea of continuity in the distance context, which can be associated with the number of lines on the horizon.

Notation

Enactive animal steps students perform are interpreted through symbolic representative pictures. The action is from symbolic representation to an enactive investigation.

4.13.3 Design Principles for Lecture 13

- Include pictures of animals whose steps vary clearly.
- Students start games/actions from the middle of the room. The chance of coming + can be increased by putting more plus on dice to eliminate problems about not being able to perform negative movements when space is not enough. Reaching the target is not easy with equal chance.
- Do not let students turn back when moving backward. This will also prevent confusion on negative and positive movement on a number line.
- Racing helps to limit and structure/organize actions. Racing to a peer evokes questioning of the effect of the +/-, as well as the size of the quantity.

4.14 Results of Lecture 14

Lecture 14 is the ninth online lecture. It is a continuum of Lecture 13, performing increase/decrease by an amount in a new context volume. Two students, Hasan and Ali, are absent. Eight students attended the online class, which took about 30 minutes of investigations and individual interviews.

4.14.1 Plan of Lecture 14

Lecture 14 aims to teach increase/decrease by an amount, like Lecture 13. Lecture 13 has an easier context to start, which helped students understand the new topic. Volume and weight contexts can be confusing, especially in new topics. In this lecture, an increase/decrease by an amount action is expected to improve the process level of students' mental constructions by working several times and with the help of the new context volume.

Similar to Lecture 13, Lecture 14 is an outlier to Davydov's trajectory, as it does not construct the action of increase/decrease on the "how to make equal" inquiry.

However, we wanted to strengthen our understanding of algebraic objects independent of equality. For this reason, we designed the objective and the activity as described in the following.

Objective: The students increase/decrease a quantity by a fixed amount in a volume context. (perform ±a)

Activity: Students are provided two dice; one has + and – signs on, while the other has photos of cups with different volumes. Starting with half-full identical cylinders, students roll both dice and perform increase/decrease by cups in turns with their partners (mothers or siblings). Who fills the cup first wins the race. There is a blank surface in cup dice to represent zero quantity.

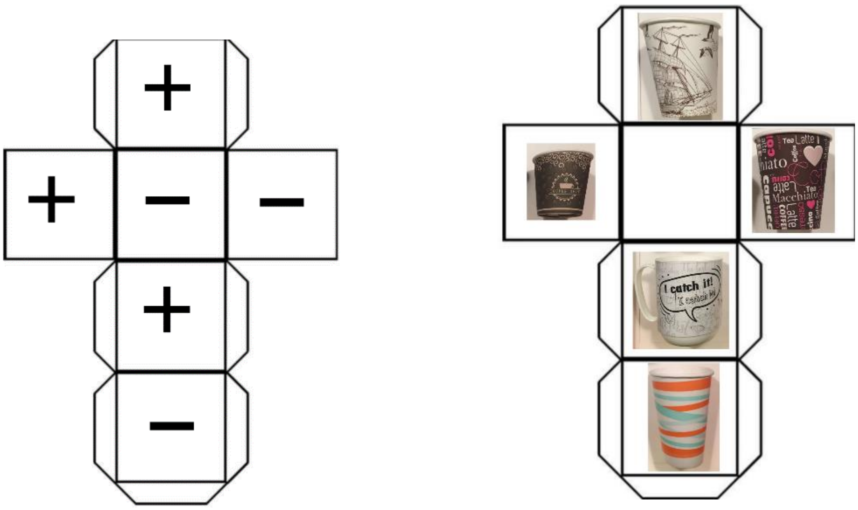


Figure 4.35. Operation and Cups Dice Used in Lecture 14 for Increase/Decrease by an Amount in Volume Context

Lecture flow:

- Make students fill the identical cylinders half.
- Explain the algorithm of increase/decrease by an amount:
    - Rolling two dice at the same time.

    - Based on the result, take the represented cup (amount), pour it in the cylinder, or take it away from the cylinder to increase/decrease the amount of water in the cylinders. (algorithm)
    - Explain how to perform zero quantity in dice.
- Guide students through dice rolling and performing actions till one of the cylinders is full.
- Interview each student, on their perceptions of operational actions and quantities.

Trajectory:

- Determine signs and cups by rolling dice, constructing a symbolic representation for +/-a.
- Associate signs to increase/decrease action and pictures to the capacity of cups (reading symbolic interpretation).
- Perform increase/decrease actions in volume context.

In this lecture, pre-given algebraic representation is read and increase/decrease actions are determined by the student based on this interpretation. Then, they perform actions in real-life situations. The trajectory was in reverse order in the first lecture on +/- signs, which was constructed around the “how to make equal” inquiry (Lecture 12). In Lecture 12, based on the real-life situation, students determined increase/decrease to make equal, and then they assigned +/- signs for increase/decrease. We then wanted them to perform actions of increase/decrease similar to this lecture. Davydov’s trajectory does not include these forward actions; instead, “first this (inequality), then this (inequality)” situations are given to students. Action required between these “first this, then this” situations is expected to be imagined by the students through questioning where it increased or decreased. We included performing these actions to create the connection between Lecture 12 and

Lecture 14 and promote action-level mental construction for increase/decrease by an amount concept.

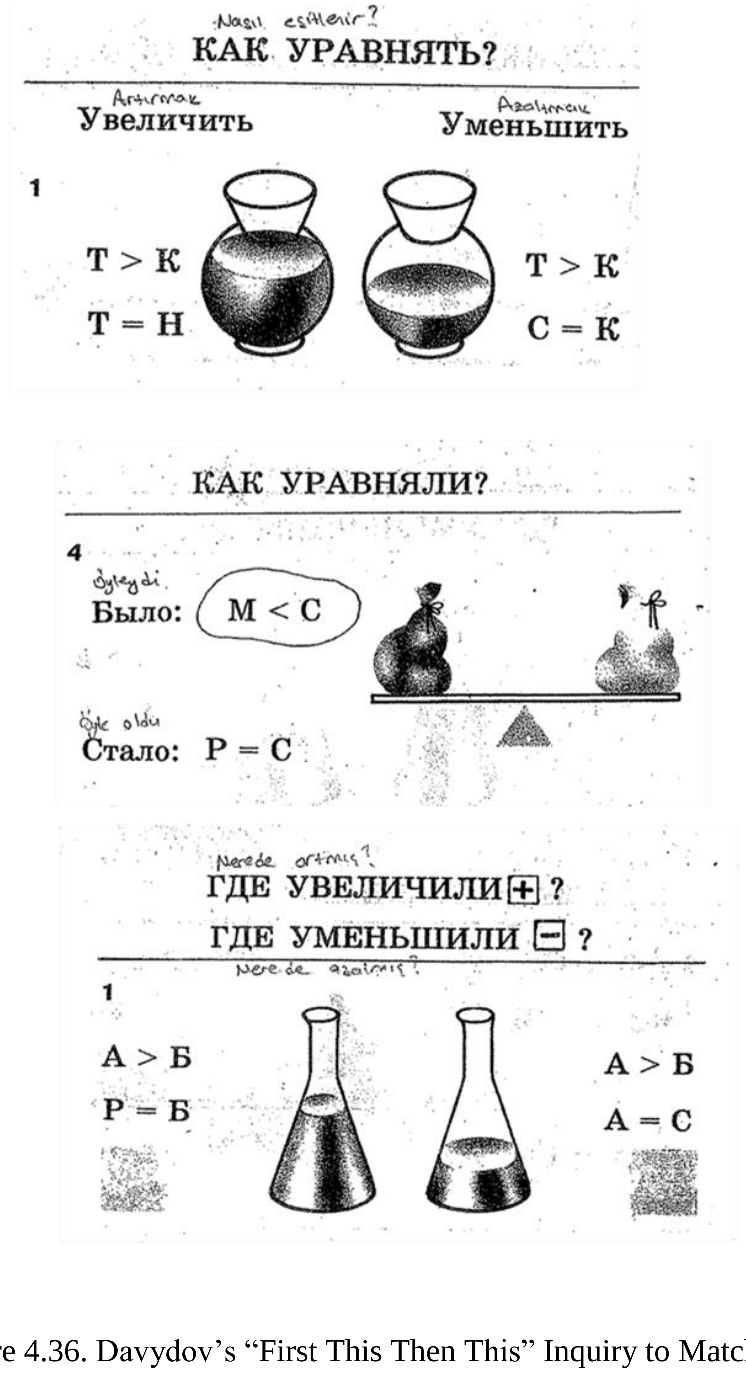


Figure 4.36. Davydov's "First This Then This" Inquiry to Match +/- Signs to Increase and Decrease Actions (Davydov et al., 1995, pp. 48-50)

4.14.2 Theoretical Findings of Lecture 14

Activity is a race between two people to fill their cylinders. Firstly, how to take individual actions of increase/decrease amount is explained, then race starts. The researcher recommended that students fill two identical cylinders almost to half. They have to be equal, but not certainly half. One cylinder is for the students themselves, while the other is for partners. Students rolled two dice at the same time. Took the cup represented in the dice. The algorithm was explained as decreasing or increasing by the amount(cup) in your cylinder. The researcher added that there will be no action if the blank side comes in cup dice because it represents no amount/quantity.

After the algorithm was given, each student was directed individually through their actions. Then, it is passed to the other competitor (mother or sibling). Students expressed who won after one turn, one roll for each competitor. The activity could also be designed that way, with one roll for each side, and observe results. We left this investigation by observing different amounts and operations on the sides of equality for the last lectures. In this lecture, we wanted to focus only on actions of increase/decrease by an amount in volume context. Getting fluent in these actions will make comprehending further lectures on the properties of operations easier. To reflect on operations as objects, they should be perceived as processes first. The researcher guided the competition until one of the competitors reached the top. Through race, we wanted students to get motivated and experience actions as much as possible. Researcher reminded + means increasing, - means decreasing when needed. At the end of the race, the researcher interviewed students individually to see how they interpreted their actions.

Two students (Bekir and Yaman) understood the game wrong; they thought that when minus came in the dice, they filled the partner’s cylinder. The guidance of the researcher corrected them. Then, they completed the actions correctly afterward. Explanation of games step-by-step is essential. Individual mini-interviews are also conducted to observe and ensure their level of understanding through the following questions;

- Which cylinder is yours to determine what to increase/decrease? What is the quantity
- What you do when each sign comes, increase/decrease action
- How much to increase: relate with the dice.

As a result, 6 out of 8 students showed evidence that they fluently increased and decreased by fixed amounts in volume context without guidance. All of them performed correctly through guidance. Hence, this lecture can be assigned to support action to process level for increase/decrease by an amount in volume context.

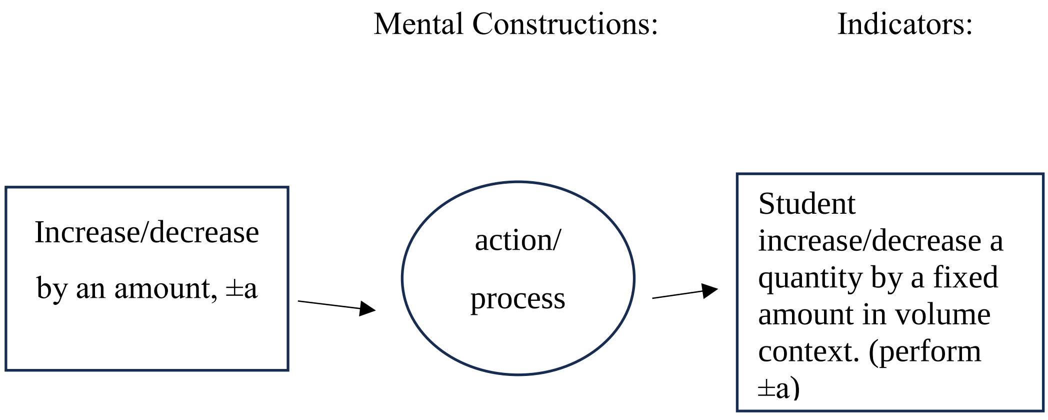


Figure 4.37. Schema for learning Increase/Decrease by an Amount in Lecture 14

Variables

Students performed "±a" using dice, where "a" has five positive possibilities and zero amount. Students may have difficulty performing where to increase and decrease; pouring water from where to where may be challenging to understand (Medine). Expressing the cylinder amount is the consideration of the quantity, and we increase it from outside with an amount. Hence, throughout mini-interviews, students are questioned about where they increased/decreased and how much they increased/decreased. They could all perfectly pour and take away the required amount. 2 students were able to express how much to increase/decrease verbally. They usually interpreted the increase/decrease amount; by showing cups and saying, "This much." One of these students, Aylin, interpreted the increase/decrease amount by generalization, even without the "How much?" question directed to her. When the researcher asked, "What did you do when plus comes?" she interpreted the action with the amount, even without expecting: "I filled my cylinder with water as much as the cup."

Students could also perform zero quantity by doing nothing. Students interpreted blank cups as changing turns (Ufuk, Didem) or doing nothing (Aylin, Ekim, Eylem).

4.14.3 Design Principles for Lecture 14

- Understanding the game rules can be more complicated than the algebraic actions. Step-wise guidance and simultaneous application help to follow and guide students.
- Through individual interviews questions: what to increase/decrease, and how much to increase/decrease to awaken students on quantities.
- Plus and minus signs have equal chances, making it difficult to win and fill the cups. As suggested in the previous lecture, plus signs may be placed more on dice. Moreover, students' joy of having a positive and bigger quantity in their turn reveals their understanding.

4.15 Results of Lecture 15

Lecture 15 is the 10th online lecture, which has two parts: addition/subtraction operations in part-whole context and transitivity in weight context. Only one student (Ali) did not attend the class, but he completed items correctly later, by guidance. The lecture took 40 minutes.

4.15.1 Plan of Lecture 15

Lecture 15 has two parts, objecting to two different topics. In the first part, HLT is followed, while the second part aims to make up the transitivity topic, which students needed help with in previous lectures. In the first HLT, the objective of the 25th lecture was;

1. The student uses + and—signs to construct equalities with one-side addition/subtraction in a part-whole context.

Without revision, this objective is applied to Lecture 15. Part-whole context was also used in equality with iconic pictures and Lego toys. Lego activity for part-whole equality was found to be challenging to study in pictorial mode; some students (who do not have Lego) had difficulty recognizing some parts in the constructed Lego model (Results of Lecture 6). Hence, part-whole items are built to include clear and distinct visions. Wooden blocks are chosen for two items as they do not lose any parts in vision (Legos are intertwined). For one item, animal figures are used. Wholes are represented with organized items, while parts are placed randomly to indicate which side to manipulate/change by increase/decrease actions to construct the whole.

Lecture flow for the first part on addition subtraction in part-whole context:

- Show parts and whole, ask if they are equal or not
- Ask what to do to parts make equal/ to make the whole: increase or decrease parts
- Place related sign + or –
- Then ask which part to decrease or increase and place the picture of parts under the sign (showing the picture of the piece helps to differentiate and place)

In the second part of the lecture, the transitivity subject is revisited in a weight context. Previous lectures were not as successful as expected due to measurement errors. In this lecture, comparisons are structured and stabilized by the activity where the researcher is experimenting, and students are observing and reporting the results of the comparisons. It is expected to create a synchronized environment for transitivity inquiry. For transitivity in weight context, additional objectives to the trajectory in Lecture 15 are designed as follows;

2. The student reports two dual comparisons of 3 objects (in weight context)
3. The student concludes the third relation based on two relations between 3 objects (in weight context)

The second objective fulfills the purpose of symbolic representation in a weight context, which some students had difficulty with before. Moreover, it will stage needed relations to deduce the third relation by transitivity. The third objective addresses the use of transitivity property to deduce the third relation. The transitivity property used in this lecture will be based on three inequality relations, different from previous lectures on transitivity using at least one equality relation. In this lecture, the newly designed trajectory for transitivity will be tested. Firstly, students will be expected to order objects, and then they will deduce the third relation. This trajectory/ordering is hypothesized based on previous lecture results on transitivity.

Lecture Flow for the second part on transitivity:

- Present three different weighted but similar-sized objects.
- Compare two of them: biggest and middle weighted ones (Follow smallest-to-biggest, or biggest-to-smallest order, for ease of ordering and deduction. We followed the biggest-to-smallest order in dual comparisons)
- Students report 1st comparison by pictures symbolizing the weights of the objects.
- Show second comparison; middle to the smallest weighted objects.
- Students report 2nd comparison by pictures symbolizing the weights of the objects.
- Ask students to order these three objects.
- Ask students results of the third comparison based on their order and report comparison.

- Check the result of the third comparison through an enactive investigation.

4.15.2 Theoretical Findings of Lecture 15

Operations

In the first part, three different part-whole items are provided to students. Items include part-whole situations where they have equal signs between. Students are expected to operate parts to make them equal to the whole. First, two items include part-whole situations with one side manipulation to make it equal to the whole. Students add and subtract items to satisfy the equality relation, following the "how to make equal" inquiry. Construction of operation with amount appears first time here with the help of easy manipulation of simple objects as parts. They do not take out or add parts, as it was in the first week of part-whole how to make an equal inquiry. They had enactively investigated change without referring to any operation signs, but in this lecture wrote symbolic representations of addition and subtraction together with the addition or subtraction amount. Templates and guidance helped all students to complete the first two items correctly. As in the first representation of +/- signs, the inquiry started with the "how to make equal" question. Then, the choice of sign is associated with the determined action. At last, students placed what to add or subtract in the equation. In this lecture, students constructed equations with one-side manipulation through actions of increase/decrease by an amount with the help of "how to make equal" inquiry. With the help of templates, the construction of the equations could be established iconically.

In the third item, we wanted to make students operate on two parts sets and use two different operations to satisfy equality to make them equal to one whole. This is the first step to approach equations with operations on two sides. They operate on sides distinctly but make them all equal to the whole, which is placed in the middle of them. Prior items were completed with the guidance of the instructor. Students' performance on this item revealed their level of learning on constructing equations with one side operation.

Figure 4.38. Ali’s Work on Equations With One-Side Operation in Part-Whole Context and Transitivity Activity.

Four students out of 9 attendees immediately completed operations on two sides in the third item, immediately when they saw the item. One student (Yaman) completed one side operation (addition) immediately, then concluded the other side during a given time. One other student (Hasan) completed the third item himself in the given time. 3 (Didem, Medine, Ekim) students needed guidance on “how to make equal,” “which sign to choose,” and “what to add or remove to make equal” for the third item. One student who did not attend class completed items correctly after class. It can be concluded that 6 out of 9 students can perform operations and construct equations with one side operation without direct guidance, 5 of which are observed to be fluent in the process. This lecture is an introduction to equations with one-side manipulation. The lecture accomplished teaching equations with one-side addition/subtraction at the action level. Moreover, it supports process level for operations by the ease of manipulatives in part-whole context. The following figure summarizes the mental constructions observed in this lecture;

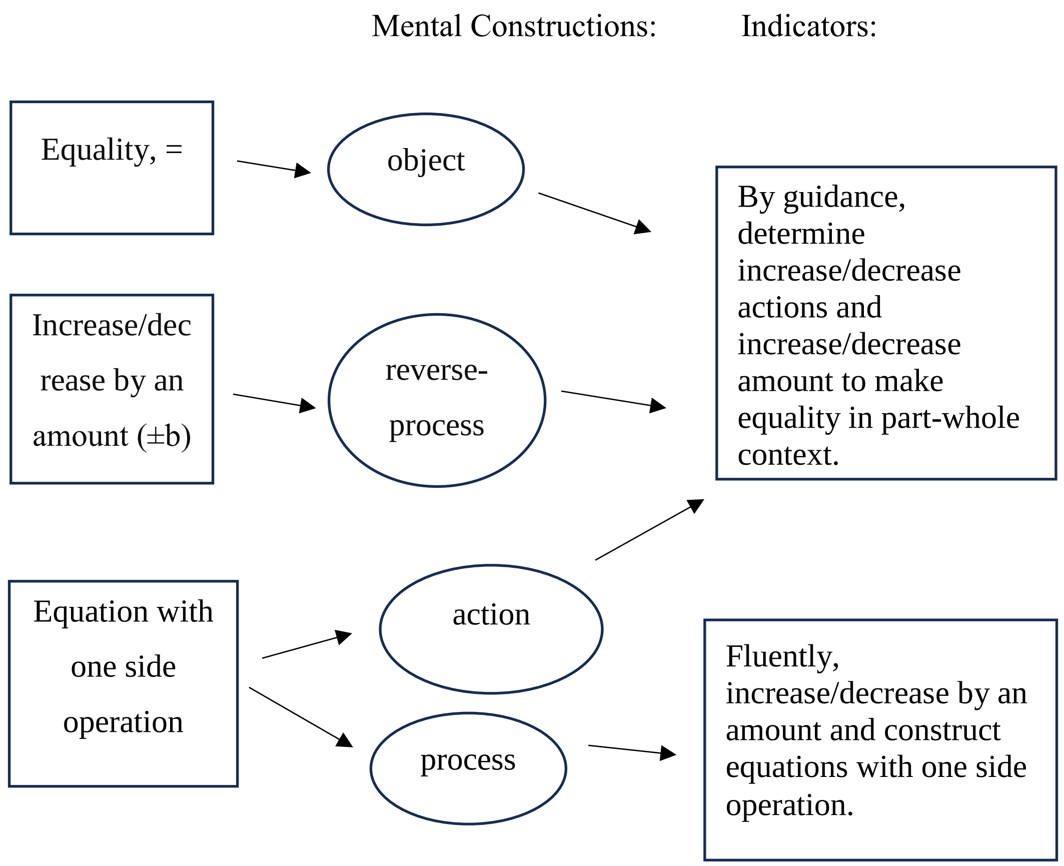


Figure 4.39. Schema for Equality, Operations, and Equations in Lecture 15

As described in Figure 4.39, equality is an algebraic object in creating equations with one side operation. Equality between parts and whole is pre-given to students. Through how to make equality inquiry, students complete equations determining increase/decrease by an amount. Hence, increase/decrease by an amount is composed of reverse processes in the action of creating an equation with one side addition/subtraction operation. The guidance needed in actions of constructing equations with one-side operation is in the following order;

- What is missing or an extra part to make the whole?
- How to make equal: add or take away, increase or decrease?
- Which sign to choose for increase/decrease?
- What to add or subtract from here?

This guidance procedure helped Didem and Ekim to construct equations. Medine needed extra help because she had difficulty focusing on which side to manipulate. Even though she knows signs and what increase/decrease means, she struggled with interpretations. She also needed help in previous activities in part-whole contexts. She may need to investigate add or take away enactively. Surprisingly, she could interpret subtraction but could not interpret addition. She usually has the problem of transferring her knowledge in new situations. She learns context-based and requires extra guidance relating enactive real-life situations to algebraic interpretations. Algebraic objects are not static for her, and she cannot focus on them. Even though she struggles with interpretations or iconic pictures, she is interested in the topics, understands them conceptually, and her verbal interpretations are accurate.

Through individual mini-interviews, the researcher ensured all students understood and completed the operations on part-whole tasks correctly. In the second part of the lecture, transitivity in weight context is accomplished.

Transitivity

Transitivity with one equality relation was visited earlier, in volume and height context. In this lecture, transitivity is discussed by comparison of three objects, all of which have different weights. Researcher shows enactive comparisons of three toys, and students report results of dual comparisons. Three objects are a wooden cube, a toy car, and a dog. Firs two comparisons are cube > car & car> dog. Comparison result between cube & car is expected from students after they order these three objects based on their weight.

Eight of nine students were capable of guessing and interpreting the third relation immediately, while one of them (Hasan) only ordered three objects but did not interpret the third relation symbolically. (Materials were not sufficient to express all relations and ordering. Students needed to interpret ordering with symbolic pictures. Provide enough pictures next time; consider ordering also.)

One of the major difficulties in weight transitivity is that students may have an idea about the weight of actual objects, which handicaps thinking by the measurements. In the pilot study, there were three animals compared based on weight. Pilot student A thought the elephant was bigger in real life, so he did not relate the iconic weight comparison, where the giraffe is heavier, to his interpretation of the relation between the elephant and the giraffe symbolically. This was the reason why reasoning by transitivity was complex for him. Symbolic representation may occur distinct from the comparison situation. In representations, real-life knowledge hinders their mathematical deductions. Students may depend on size for representing by >, < signs. However, this was not the problem in this activity; all students could represent comparisons symbolically based on weight when the researcher showed the first two comparisons to them. The problem occurs when they are asked to deduce the third result or order objects. They may forget about previous relations and depend on their real-life knowledge. In this activity, some students think the wooden toy is heavier, for example, and ignore comparison. Medine was one of the students who correctly concluded the third relation, but her explanation needed to be more sufficient. When she is asked for dual comparisons again, she can interpret all of them one by one, through guidance. However, she could not order them. She is confused about ordering objects based on the previous relations she interpreted; she insisted on considering real-life sizes for objects: “Dog is small, the car is very big.”

Like Medine, other students may also need help explaining their reasoning by transitivity, based on their capacity in language. Ekim explained her reasoning for “wooden cube>car” as “because wood is very big.” Aylin’s explanation included all objects and their relation to each other; “because the wood is heavier than the car, and the dog is even lighter than the car.” Aylin’s explanation proves her dependence on transitivity in her reasoning. Others' fluency in guessing the third relation also might indicate the success of this lecture on carrying out mental construction on transitivity from action level to process level of understanding. Step-wise interpretations in weight topic and ordering strategy contributed to the success. Their immediate response to the third relation, even before ordering, addresses the fact that the sequence of the relations (cube > car & car> dog => cube>dog) helped them deduce the third relation easily.

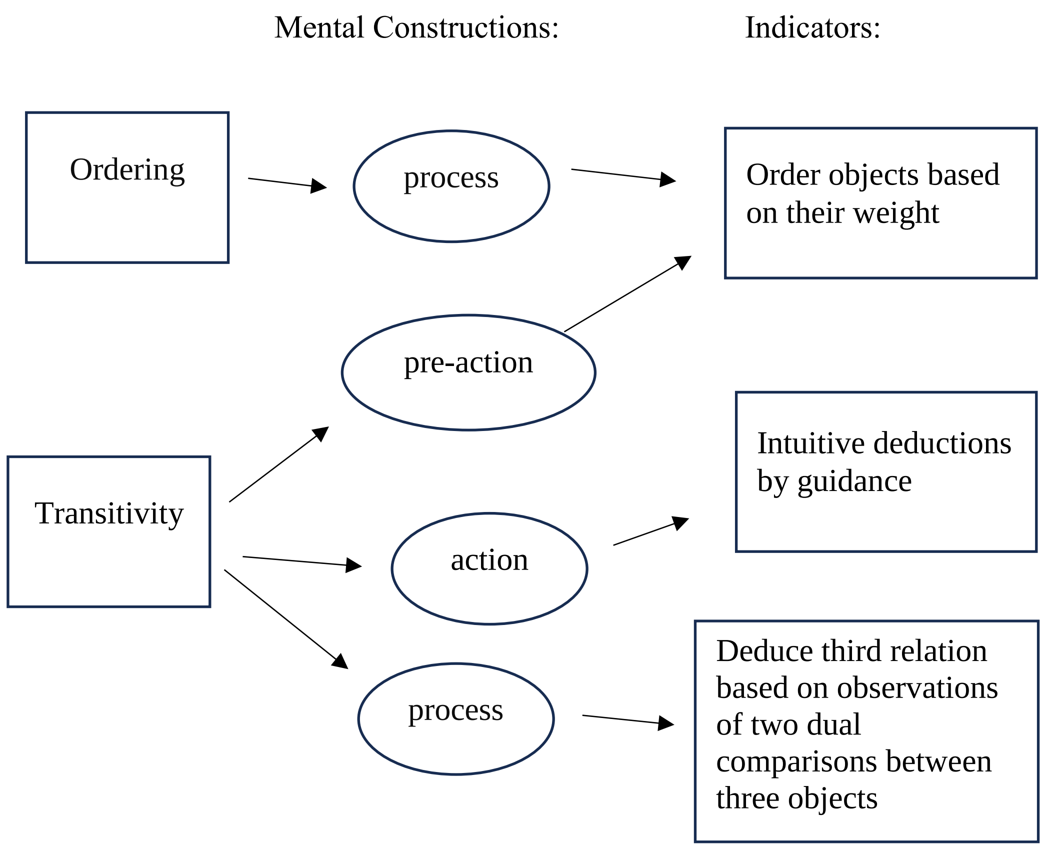


Figure 4.40. Schema for Ordering and Transitivity in Lecture 15

Transitivity is a difficult topic, and determining bigger based on weight is also difficult for students. After several attempts and systematic organization, reasoning by transitivity is achieved. However, this transitivity is based on their enactive observations. We don't know if they can reason by pre-given symbolic interpretations of relations to deduce another relation.

In the last minutes, the researcher illustrated a third relation, to make students check their deductions. Then researcher showed all combinations of comparisons and their reverse again. Even though we had never discussed directly what happens when we compared objects in reverse; they all responded correctly and immediately. In mid-assessment, they were not asked conceptually but were expected to interpret reversed versions of comparisons, which they were successful at. The way we taught >, < signs also allows thinking in symmetry because they have no direction/orientation fixed for interpreting and reading interpretations throughout the activities.

Notation

In the first activity; operations in part-whole context, notation is in the iconic mode of representation. They iconically investigate equality situations and construct equations with one-side operations, using signs and pictures of toys representing added/subtracted items. They were all capable of constructing equations with one side operation symbolically and got fluent in the third item in their representations. Only Medine needed constant guidance for the symbolic representation of operational actions during activity.

Part-whole context is observed to make an easy start to determine and interpret addition/subtraction amount before volume and weight context. In previous lectures, they actualized addition/and subtraction by an amount in a volume context. Still, they did not determine how much to increase/decrease or what is addition/subtraction amount (reverse-process addition/subtraction amount). This is a prior step for finding unknowns in equations. The part-whole context created easy determination and symbolization for unknowns in the equations. It needed to place the sign before it supported forward processes on addition/subtraction amounts. It staged pre-actions for finding unknowns with the help of iconic representations. Finding the unknown is easily recognized as missing or extra parts in part-whole context.

In transitivity activity in weight contexts, interpretation was expected in symbolic mode, based on enactive observations, with the help of representative pictures of objects. All students were successful and fluent in symbolically representing relations based on weight.

4.15.3 Design Principles for Lecture 15

- Starting equations with one-side addition/subtraction in a part-whole context is practical and makes it easy to focus on unknowns.
- If it was not an online lecture, first, enactive investigation is suggested to construct wholes from parts instead of pictures. We suggest using pictures on paper templated with blank squares (to place +/- signs and parts) to enforce operation on one side and empower symbolic representations for addition and subtraction by an amount.
- Remind use of signs connected to the actions of increase/decrease. Add or subtract parts can move attention on operations away from increase/decrease actions. Follow the inquiry as follows:
    - Are they equal?
    - What to do part to make equal to whole; increase or decrease?
    - Which sign to choose for the determined action?
    - By which piece to increase/decrease, what to add or subtract?
- Using colored pictures helped students to recognize parts in the part-whole activity.
- Rather than repeating the algorithm in transitivity topic, dedicate more time to a more structured activity, which strictly follows the inquiry;
    - starting with symbolic interpretations of relations,
    - then symbolic interpretation of the order of objects,
    - then deducing the third relation based on the order.

Ordering of objects occurs sometimes, implicitly, before symbolically interpreting it. In that case, the student can immediately deduce the third relation.

- In the transitivity activity, asking comparisons in the order of hierarchy (such as a>b & b>c) helps in deducing the third relation. Comparisons from smallest to biggest or biggest to smallest order are suggested. Alternatives can be tested later.
- Provide extra representative pictures of compared objects for interpretation of order.
- Use similar-sized objects in weight comparison to prevent misleading conclusions.
- Refer to the attribute when you ask for interpreting comparisons or ordering each time. Students' consideration of attributes may not be stable among contexts. They tend to think based on real-life properties of objects, independent from previous findings on classroom investigations.

4.16 Results of Lecture 16

Lecture 16 is the 11th online lecture. It aims to teach finding unknowns in equations with one-side addition. Eight students attended class, and one student (Hasan) was lectured briefly after class. One student, Medine, was absent and did not take any make-up. The lecture took about 40 minutes.

4.16.1 Plan of Lecture 16

Davydov's trajectory focuses on one-side addition in equations, then continues with two-side addition. Lecture 16 focuses on equations with one-side side addition in height context. It uses a game model, "Math Forest," where animals are represented by numbers and respective heights (see Figure 4.41). You can put animals on top of each other and create equality between heights of groups. Each animal has different heights, representing different numbers from 2-9, while only two are equal to each other, representing number 1. We covered all numerals on the animals and measurement tools to construct unknowns and create algebraic equations with quantities represented by animals (see Figure 4.42). Moreover, we added an extra wooden line so students can compare animals and observe equalities nearby, not based on number line (which we also covered). We will use this game in teaching equations with two-side addition in Lecture 17.

Figure 4.41. Math Forest Game

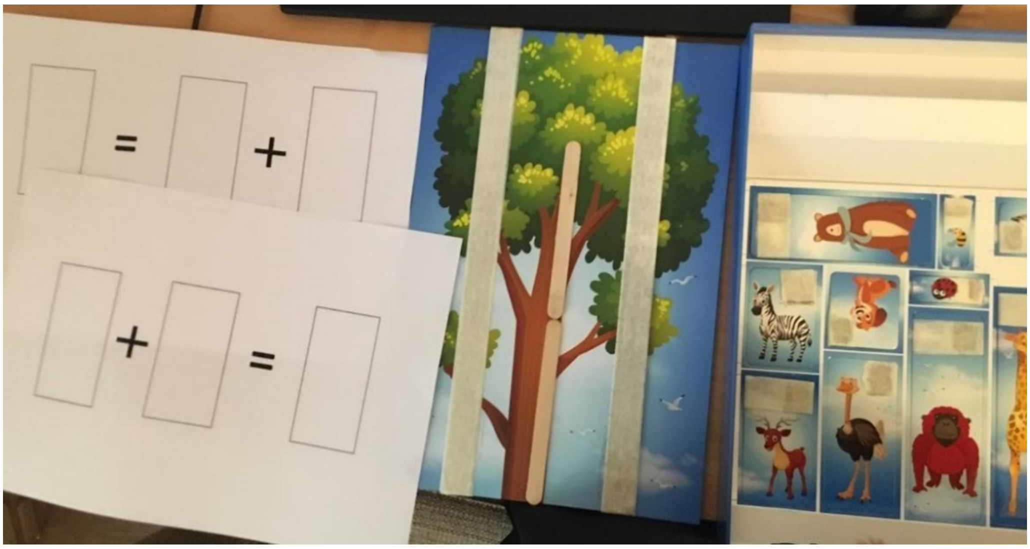


Figure 4.42. Using Equation Templates and Covering Numbers in Math Forest Game

Students are directed to 8 questions and guided to find unknowns in the equation. They are provided extra templates, including three blank boxes, equality, and + signs, in which they can place animals and construct equations. They use a measurement tool (which has a tree on it) to compare sets of animals, try and find unknowns, and then place the animals on the template of equations. The template can be used in any orientation. Questions are not asked to direct an orientation but to indicate equality/balance.

Based on the first HLT, the objectives of the 26th Lecture were:

1. The student determines the addition amount to make equality.

2. The student interprets a quantity as the addition of one to another.

Revised objectives are:

1. The student determines the addition amount to make equality.

2. The student finds unknown in an equality with one side addition.

3. The student recognizes multiple solutions to equations with two unknowns (a =? + ?)

Revision of the objectives does not indicate a change in the focus of the lecture; Lecture 16 aims to teach equations with one-side addition and finding unknowns in one-side addition as planned. The second objective in HLT is canceled because verbal interpretations are complicated for students. Hence, we only guide them through increasing heights by adding other heights in the animal height game. 2nd and 3rd revised objectives discriminate items used in Lecture 16. We decided to add an item with two unknowns, empowering the investigation and interpretation of addition on equations and developing an understanding of multiple answers. Students are expected to discover equal sums and use them instead of each other by the 3rd objective.

There are eight tasks students are guided through to find unknowns in equations with one-side addition.

1. Addition to one side to make equal: a=b+? ("What do you need to add onto the ostrich to make it equal to the giraffe?" type of questioning.)
2. Addition to one side to make equal (repeat algorithm): a=b+?
3. Equality situation: a=b+? when a=b (This item asks students to recognize the equality of two animals: Ladybug=bee).
4. Addition to one side to make equal: a=b+? Two solutions, ladybug and bee: use the same quantity of objects instead of each other
5. Addition to one side to make equal (repeat algorithm): a=b+?
6. Addition to one side to make equal (repeat algorithm): a=b+?
7. When unknown is the sum: ?=a+b
8. Equation with two unknowns in the form: a=?+?, multi-solutions recognition and finding

As seen from the items, we do not start by adding two quantities and asking sum of them as a result; we leave it to the end. Every question is focusing on finding the unknown in an equation. This strengthens the idea of equality of two sides, rather than add and find solution command. Starting with “how much to increase to make equal” questions, we follow Davydov’s trajectory of how to make equal by increase/decrease actions. Hence, items start with comparing two animals and continue questioning where to increase and then what to increase. Some questions are verbal, then some are directly asked on a template based on students’ understanding.

4.16.2 Theoretical Findings of Lecture 16

The lecture started with the presentation of manipulatives and how to use them, by item 1. The researcher showed two animals and how to compare them on the manipulative. The researcher asked how to make them equal by adding another animal and then showed how to test their equality. At last, the researcher explained how to place the animals on the measurement to the template. Students imitated the solution to the first item. 6 (Aylin, Yaman, Bekir, Eylem, Didem, Ufuk) students could immediately understand and use the template correctly. Eylem suggested the addition of two animals instead of one answer. One student (Ekim) needed guidance for item 1. Another student, Ali, had difficulty understanding because he was absent from the last two lectures, where addition and subtraction by an amount were discussed. Ali’s struggle continued through the whole lecture, while others got fluent after the second item. The researcher reminded the use of the plus sign for increase actions in the second item to help Ali and make other students focus on the increase.

In the third item, two animals were equal in height, and students could interpret equality immediately. Aylin interpreted their equality by writing an equal sign between them. The researcher put them on the template and explained nothing to do. The researcher changed the places of equal animals on the template to show it does not matter where they are; they are equal, and we add nothing. The equality of two animals is used in the next item as an addendum.

(a) (b)

Figure 4.43. Aylin’s Interpretation of (a) “Bee=Ladybug” and (b) Aylin’s Use of a Template

In the fourth question, the researcher asked, “By which animal should we increase gorilla to make it equal to ostrich?”. Stress is on the amount and the increase meaning of addition connected to previous classes. The difference between ostrich and gorilla is 1 unit in the game, which is equal to bee or ladybug. Some students replied bee; some students replied ladybug for a solution. The researcher pointed out these two different solutions: “Some say ladybug, some say bee!” Ufuk replied, “Because they are equal.” Ekim said, “Both works.” The Researcher explained, “We can put this or this because they are the same” by showing bee and ladybug onto each other to show their equality. The researcher said, “I can put this or this instead of it.”. Not on the template, she used them interchangeably on compared sets of animals to show students that equality is preserved. The researcher sometimes did not use measurement tools. Students placed animals next to each other to see equality.

The fifth question repeats the algorithm, starting with comparing two animals. Six students (Aylin, Eylem, Bekir, Ekim, Ufuk, Didem) out of 8 attendees were fluent in their responses, and four of them were fluent in using templates (Aylin, Eylem, Bekir, Ekim) to construct equations by one-sided addition in this item.

In the 6th item, which is also for repletion of the algorithm of finding unknowns in equations of the form a=?+b, the researcher emphasized the meaning of the plus sign as the addition of two quantities rather than increasing one. Moreover, the researcher associated verbal questions with the template, addressing the placement of unknowns in the equation:

"What will become a bear if I add it with a deer?" (addition of two quantities meaning)

"What comes to this blank box?" (template-based questioning)

The researcher directed the 7th item directly on the template. 7th item asks for the sum: ?=a+b. It is to make an equal process. They (Yaman, Bekir, Ufuk, Aylin, Eylem, Didem, and Ekim) immediately replied item correctly. They had no confusion. Yaman also improved using the template to construct equations by this item. Ali and Hasan had difficulty finding the answer and using the template, even with guidance.

8th item is on addition of two unknowns in the form: a=?+?. The researcher directed questions and waited for responses. When there was a different response, she pointed out. Students were not shocked or confused to see other solutions. They accepted the solution of their friends. Moreover, they composed an alternative solution (Bekir) or used an equivalent to modify their solution (Eylem). The researcher illustrated all solutions by putting on and nearby, and asking students to do the same. They all imitated/showed multi-solutions successfully.

(a) (b)

Figure 4.44. Multiple Solutions of (a) Ufuk and (b) Didem

In finding unknown procedures, the majority of the students successfully determined the increase amount, created equations on templates, and tested their equality. They successfully recognized multi-solutions and used them interchangeably. This lecture accomplished all the intended objectives except for two students, Hasan and Ali. One attended late, and the other missed the last classes on operations.

In previous lectures, students learned operations as actions, then operations by an amount as actions. They determined the operation and amount on both sides of inequalities to make it equal. Also, this lecture starts with the inequality situation to build equality. Differently, the operation is given, and the construction of equality depends on finding missing quantities. Moreover, the unknown does not only appear with the operational action but is also represented by the result/addend. (Previous lectures focused on the construction of the operations by the amount.) This means that Lecture 16 focuses on constructing equations, with addition on one side, or given an equation (with the defined operation), students find the unknown quantity. These processes refer to the action level of equations. Addition operation, quantity, and equality are objects of equations constructed in Lecture 16. Students developed action or process-level mental constructions during the lecture on the constructions of equations and finding unknowns satisfying the equations. Addition by an amount process is composed in the processes of finding unknowns and constructing equations. See Figure 4.45.

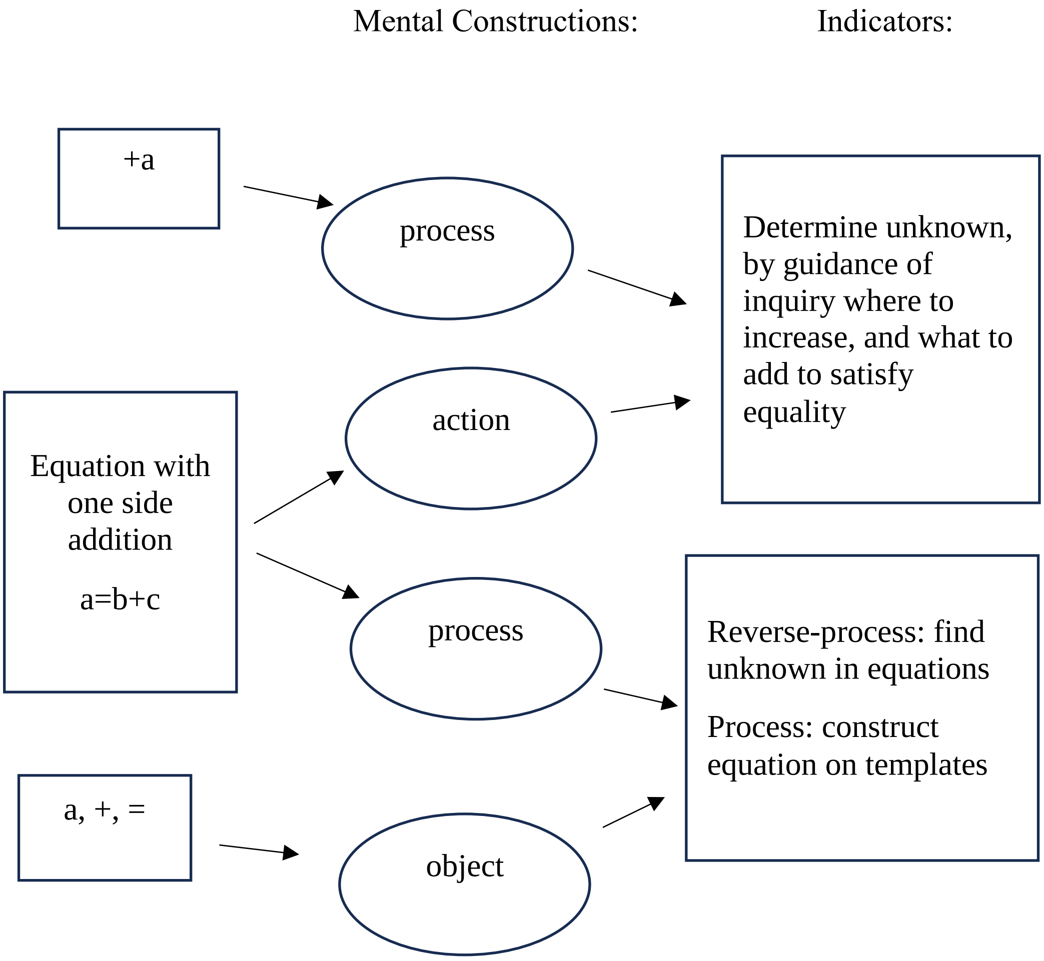


Figure 4.45. Schema for Learning Equations With One-Side Addition in Lecture 16

Quantity

Items have the potential to teach different aspects of quantities as unknowns. Height is a continuous type of quantity, where fixed quantities are used in this lecture. The increase of the quantities is also by fixed quantities. There exists an equality between two objects in height in item-2. Two animals are the same in quantity and can be used instead of each other. Hence, quantity gets abstracted one more step away from being particular to an object. (Students develop quantity from object to quantity to variable.)

Moreover, items support multi-solutions. Not only are equivalent objects used instead of each other, but equivalent sums are also discussed as solutions to an item (last item-8). (Variability in the equations is sensed through multiple solutions, but dependency between quantities is not discussed in any of our activities.)

Notation

Templates helped transfer enactive representations to iconic representations. They worked as algebraic structures of equations, with the placement of unknowns. Except for Hasan and Ali, students had no difficulty using templates and relating them to enactive representations.

At first, we wanted the template to be different: the addition sign should be on top rather than by the side. It would be similar to part-whole activity and imitate height addition directly. Then, this type of displacement side by side would also be used in further classes. We did not have extra time to study all kinds of templates. The researcher asked the kindergarten teacher's advice; she said they could understand side by side. Then, we implemented it and were not disappointed. Placing a plus sign on top could be tried to be more intuitive in height topic.

4.16.3 Design Principles for Lecture 16

- Start inquiry on comparing quantities and actions to make them equal. Which side to increase and how much to increase are the main questions that should be asked to build on their prior knowledge and construct operations as an increase/decrease in the quantity.
- Questioning which signs to use for increase/decrease actions can also be revisited in need.
- Repeat finding unknown algorithms and using templates for constructing equations to make students fluent.
- Ask questions to find unknowns verbally and on the templates for connecting each other. Then, ask, referring to templates only to get used to equation

expressions. Templates help students interpret and create algebraic meaning in equations.

- Include different types of equations where the unknown is placed in different places.
- Ask your questions focused on the equality of two sides rather than the resultant of addition to support the meaning of equal sign as balance.
- Use equivalent height objects to support quantitative understanding, creating a set of objects belonging to a fixed quantity. Then, operations are conducted on quantities rather than on objects themselves.
- Create items that enable multi-solutions, where quantities or their sum can be used in terms of each other.
- Measurement tools and games are not necessary for this activity. Using different height/length objects and a template is sufficient.

4.17 Results of Lecture 17

Lecture 17 is the last and 12th online lecture. It is a continuum of Lecture 16 on equations with addition on one side. Lecture 17 uses the same game Math Forest (animal height) to investigate equations with addition on both sides. Eight students attended online classes, and two of them (Medine and Didem) were absent. Medine was absent in Lecture 16, the previous class on equations with addition. The lecture took 40 minutes.

4.17.1 Plan of Lecture 17

Using the same tools as Lecture 16, this lecture aims to teach equations with addition on both sides. It is aligned with Davydov’s trajectory, focusing solely on addition, first on one side, then including addition on both sides. Addition on one-side topic starts with “how to make equal” inquiry in Davydov’s and our trajectory. Addition on two sides begins with the “equal, not equal, equal again” inquiry in Davydov’s trajectory, which means equality is turned into inequality by the addition of a quantity on one side, and then it is made equal again by adding the same amount on the other side. This approach preserves making an equal inquiry. Lecture 17 also includes an "equal, not equal, equal again" inquiry, but in the second part of the lecture. In the first part, finding unknowns and modeling equations with two side additions centralize the activity.

In the first HLT, additions on two sides of the equations were planned after discussion and modeling of increase/decrease amount (and difference amount as increase or decrease amount to make equal). We decided to change the trajectory; and placed equations with two-side addition immediately after equations with one-side addition. Hence, modeling two-sided addition and discussion of "equal, not equal, equal again" on equations is placed earlier in height context. Discussion and modeling of increase/decrease amount is embedded in difference amount subject in Lecture 18. Discussion of "equal, not equal, equal again" in volume context is embedded into Lecture 20, with addition and subtraction operations on both sides. See the change in the trajectory in the table below:

Table 4.2 Change in the Trajectory: Equations and Difference Amount

HLT Lecture #	**Fist HLT**	**Implemented Lecture #**	**Last trajectory**
26	one-side addition: animal height	16	one-side addition: animal height
27	Increase amount: paper strips enactive, use signs +/- Difference amount		See Lecture 18
28	Compare the increase amount: plant growth.		See Lecture 18

Table 4.2 (continued)

29	"Equal, not equal, equal again" volume and weight context Properties of operations +/-	17	Model two side addition: animal height "equal, not equal, equal again" in height context
30	model two sides addition: animal height		
	See HLT Lectures 27&28	18	Difference amount: plant growth
31	Model one-side equations in real-life context: by paper strips		
32	Rainbow Activity	19	Rainbow Activity
		20	Properties of ±a in volume context

Reasons for the change in trajectory:

1. We decided to continue modeling equations with one-side addition, with two-side addition in height context to prevent students' generalization of operations being one-sided (and conserving the meaning of equality as solving for it). To make it equal, they changed one side only in Lecture 16. However, they are reluctant to add on both sides from the weight context, where they add multiple objects on balance scales due to free experience. Using a manipulative "Math Forest" toy, investigating and modeling two-sided addition would be easy and meaningful.
2. Discussing addition or decrease amounts is a difficult topic and requires one-to-one face-to-face guidance. This manipulative "Math Forest" toy is easier to adapt to online lecturing with the additional use of a template representing equations. Hence, continuing with the manipulative and delaying the

discussion of increase/decrease amount to the in-class lecture was meaningful.

3. In the 29th lecture of the first HLT, addition/subtraction on both sides would be discussed through an “equal, not equal, equal again” inquiry. Adding or subtracting the same amount is trivial for students. Revisiting volume and weight context would be unattractive for them. We decided to build this lecture on their knowledge of how addition/subtraction of equal amounts on both sides affect equations. Added amounts are different objects but have to be equal in quantity to preserve equality (which also aligns with Davydov’s trajectory in weight context (bunny and squirrel example; see last-interview item?)). This approach is the reverse process for investigating the property of the addition of equal quantities on both sides of the equation.

For these reasons, the objectives of HLT-29 and HLT-30 are revised and integrated into the objectives of Lecture 17

Objectives of HLT-29:

1. The student discusses how to make equality, unequal, and equal again by addition and subtraction (in volume and weight context)
2. The student interprets the effects of addition or subtraction of the same amount on both sides of equality (in volume and weight context)

The objective of HLT-30:

1. The student models equations with two-side addition (in height context)
2. The student uses algebraic notation to interpret equations with addition on two sides (in height context)

Revised objectives for Lecture 17:

1. The student finds the unknown in the equation with two side additions in height context. (includes HLT-30 as iconic representation objectives)
2. The student finds multiple solutions to equations with two unknowns.

3. The student adds equal amounts to both sides to preserve equality. (reverse-process for HLT-29 objectives)

Lecture 17 is developed on two activities to accomplish defined objectives: finding unknown in equations with addition on two sides and addition of equivalent sums to preserve equality. Multiple solutions result from equivalent sums, which students are reluctant to from the previous lecture. The lecture starts with finding unknowns in pre-given equations defined on templates. Then, an extension of equations with equivalent sums is conducted through an "equal, not equal, equal again" inquiry. The template representing equations used in this lecture differs from the template used in Lecture 16; this template includes additions on both sides. By using this template, the following questions are asked in order;

1. Find the unknown in the equation: a+?=c+d
2. Find the unknown in the equation a+?=c+d (repetition of the algorithm)
3. "Equal, not equal, equal again" inquiry: (a+b=c+d) (a+b+e≠c+d) (a+b+e=c+d+?)
4. "Equal, not equal, equal again" inquiry: a+b=c+d => a+b+f=c+d+? (repetition of the algorithm)
5. "Equal, not equal, equal again"; attention on multi-solutions for addend, how to replace with equivalents.

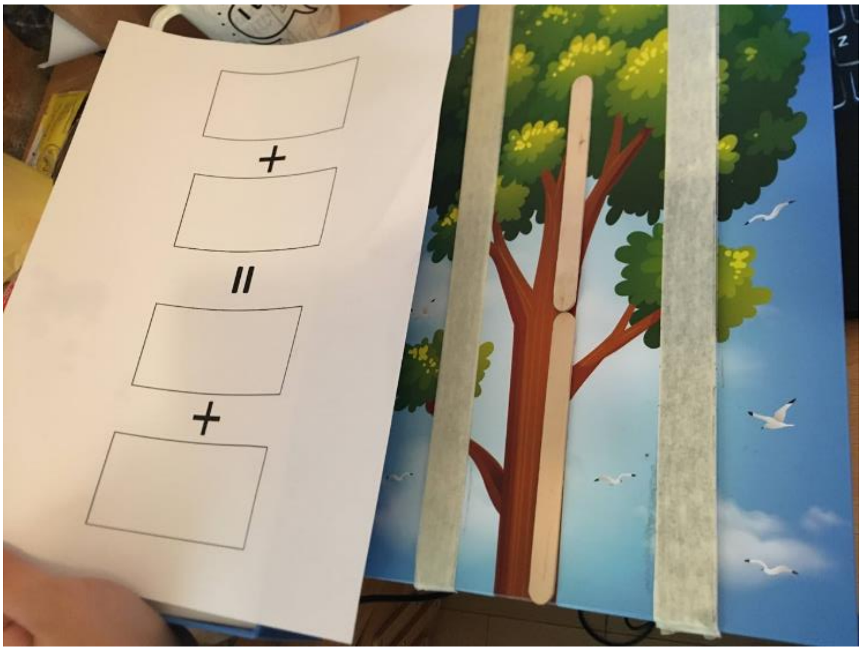


Figure 4.46. Template Used in Learning Equations With Two-Side Addition in Math Forest Game

4.17.2 Theoretical Findings of Lecture 17

The researcher directed questions on the template, asking students what comes in blank space. Students found and illustrated solutions with the help of measurement tools. Most of the students (6 out of 8, Eylem, Ekim, Aylin, Ufuk, Bekir, Yaman) represented their solutions on templates even without being asked to do so in the first two questions of finding unknown in equations with addition on two sides. The researcher explained how illustrations on measurement tools and representation on templates are related to each other. However, two students, Hasan and Ali, had difficulty interpreting on template, even though they found correct answers for unknowns using measurement tools.

The third question is on the “equal, not equal, equal again” inquiry to teach equivalent addends on both sides of the equation. The third question starts by finding the unknown in the equation with addition on both sides. After students found the answer unknown, the researcher put the animals on the measurement tool, visualizing the equality. Then, she created inequality by adding an animal on one side. The added animal is “ladybug,” which has an equivalent “bee.” She asked students, “Is equality broken?” Students replied, “Yes” (Aylin, Ekim, and others). Before they were asked “how to make equal again,” some students shouted, “Bee” (Aylin, Ekim). After the researcher directed the question, “How do we make equal again?” most of them (Aylin, Ekim, Hasan, and others) quickly replied, “Bee” to be added on the other side. It shows they know how to make it equal again by offering equal quantity. By offering equal sums, their performance on different items also proves that.

The researcher then added the bee for equalizing. Then she asked, "If we remove the ladybug, what do we do to make it equal again?" (expecting the response, "Remove the bee"). Aylin had a good reply: "Put the ladybug." It was a normal reply because they had only seen adding in this context. The researcher said, "Yes, we can put the ladybug again or remove the bee. Let's remove the bee. We removed the ladybug and bee. Now I put a duck here and broke the equality." The researcher continued with a new question, breaking equality by adding a duck on one side. Some students immediately replied, "Bee and ladybug" (Ekim, Aylin, Ufuk) because the sum of ladybug and bee equals a duck. This sum is the only solution. Then researcher questioned what if equality is broken with the addition of the giraffe on one side. Students had different answers as sums. The researcher verbalized all students’ answers. Hasan asked, “Which one is correct?” Aylin replied before the researcher: “We can put both of them.” The researcher explained how each answer is correct and can be used instead of each other to achieve equality. (Hasan was not attentive in previous classes by total concentration due to the internet; the multi-solution task was unclear to him.) It was decided to discuss multi-solutions in the 5th item, but the nature of the tasks allowed students to recognize and discuss multi-solutions earlier.

In the 4th item, the researcher asked the question without showing it on the measurement tool, focusing on the addend to break the equality. The addend part is placed somewhere else, and equivalents are placed nearby (See Figure 4.47).

Students could generate multiple solutions. Ekim, Aylin, and Eylem interpreted the use of equal quantities (bee and ladybug) instead of each other.

Figure 4.47. Adding Equivalent Sums to Break and Reform Equality

In the fifth item, the discussion of multi-solution deepened more, from recognition to generation of multiple solutions. Firstly, seven of eight students (except Hasan) replied “How to make equal again” correctly, coming up with three different solutions. Some students (Aylin and Yaman) generated multiple solutions even before they were expected to do so. Then, all students generated solutions as much as possible, showing all equal sums nearby.

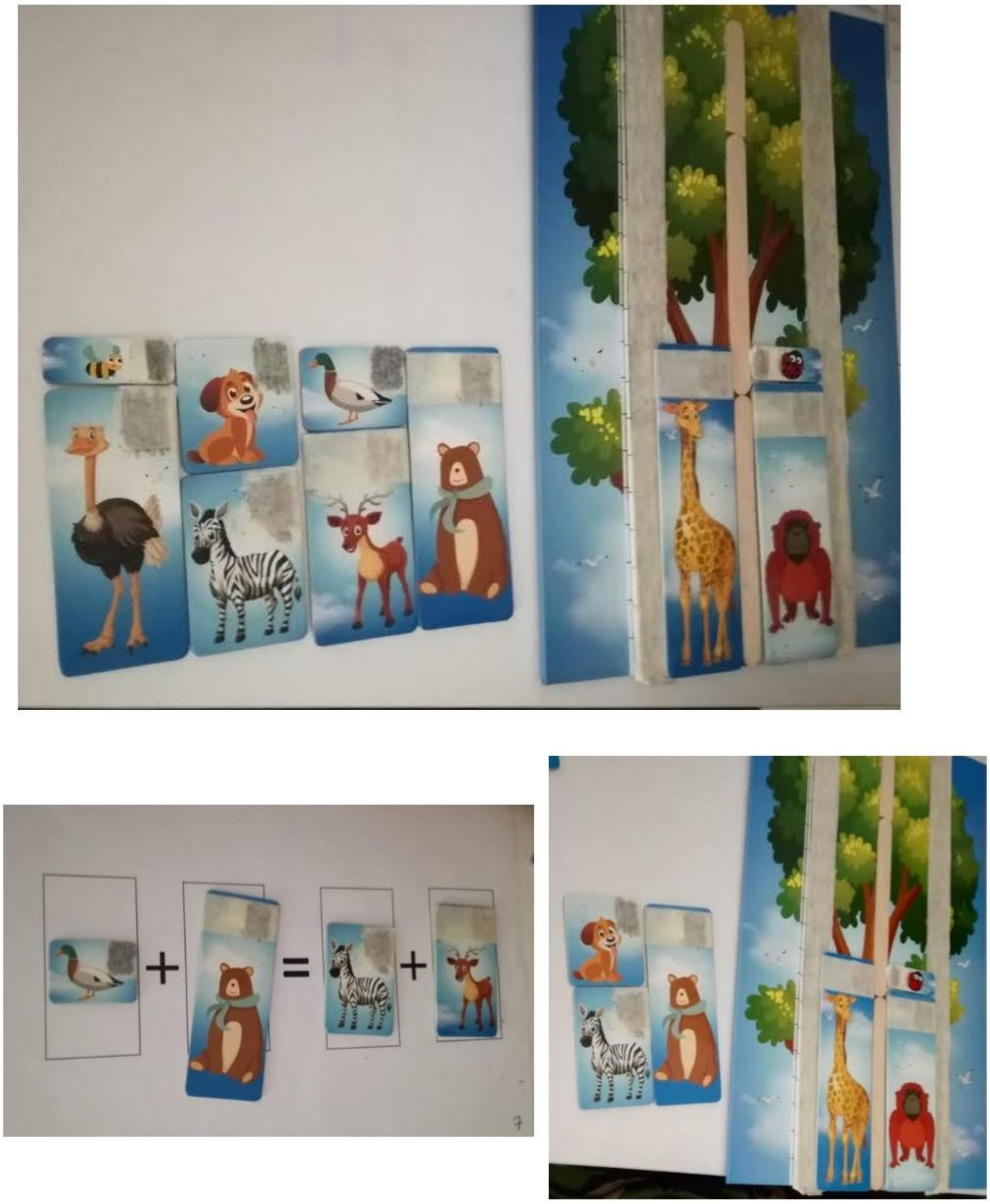


Figure 4.48. Yaman’s Interpretation of Equivalent Sums and Use of a Template

The majority of students (6 out of 8) were automatically using templates and enactively constructing on measurement tools, also finding unknowns in the equations represented by templates. Some students (Ali, and Hasan) needed guidance through the whole lecture. Hence this lecture addresses action to process level for construction of equations with addition on both sides. Students were successful in the first attempt of constructing equations with addition on sides, because they were reluctant to two-side manipulation from the beginning of equality, and they learned to construct equations with one-side addition with templates in the previous lecture.

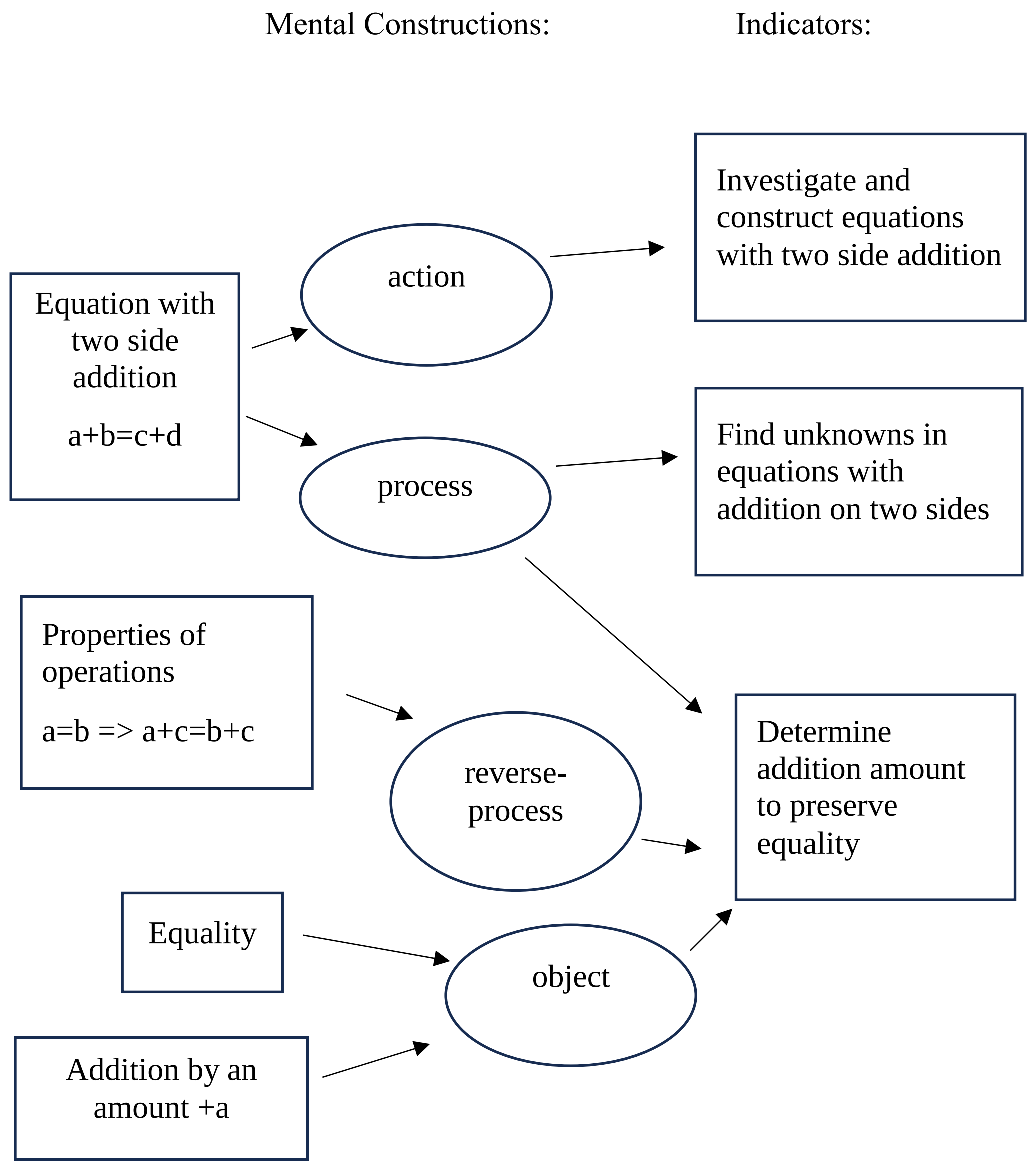


Figure 4.49. Schema of Operations and Equations in Lecture 17

Students had no difficulty determining addition amounts to preserve equality in equations broken by addition on one side. They used the property of addition of equal amounts on sides in the reverse process to determine the addition amount needed. Moreover, they used equivalent quantities and tried to make equivalent sums instead because they had many experiences with equivalent sums in the previous lecture. Equality and addition of equal amounts are objects acquired to process property of operation by equivalent amounts on equations, where they operate on equality thinking about adding equal quantities as a construct on both sides to preserve equality.

Variables

The idea to start with equal-height objects bee and ladybug, strengthens the concept of quantity. First, it made it easier to understand that objects that are the same/equivalent in quantity can be used instead of each other. Then, the "equal, not equal, equal again" inquiry is started with the addition of one of these objects which has an equivalent. They worked as equal quantities added on sides, satisfying the property. Bee and ladybug, which are different objects but equal in quantity, helped students to abstract quantity concepts from the object. (Use of similar objects, for example, same-colored wooden blocks equal in length, might not have the same effect because students would think they are the same object.) In the first lectures, the same objects resulted in different quantities in weight comparisons. In this lecture, different objects resulted in equal quantity. In both ways, quantification is supported, by thinking independent of the objects they belong to.

Ladybug and bee worked in the equations instead of each other or in the same way for addition amount. Students had no difficulty acknowledging, nor did they need to experience, the result of the manipulatives. They reasoned by their equality to construct the equations. It proves they acknowledged quantity. There were no other equivalent objects; hence, students used sums of quantities to create equivalent quantities. Then, they used these summative quantities successfully in the equations. This also shows that they acquire sums as a single quantity preserving the property of addition in the equation.

Quantity sums provided multi-solutions earlier than we expected. Students initiated discussion on multi-solutions. They all recognized multiple solutions and experienced alternatives through enactive investigations.

Notation

Students experienced equations enactively using measurement tools or placing animals nearby for comparison. Templates helped them construct equations iconically. Finding unknown questions is also directed solely at templates or placing animals nearby for comparison. Templates helped them construct equations iconically. Finding unknowns questions are also directed solely on templates. 6 of 8 students had no problem and were fluent in using templates and associating equations on templates with the enactive mode of representations.

4.17.3 Design Principles for Lecture 17

- Designing manipulatives including equivalent quantities would be easier and more symbolic to experience structures of equations and the property of adding the same quantity on both sides. However, with the help of the Math Forest game as manipulative, the mentioned property occurred in the level of reverse-process to construct equivalent quantities because rather than investigating the addition of the same quantity on sides of the equation, they had no choice but create an equivalent sum to the added amount. This shows they knew the property, so they wanted to create an equivalent.
- Manipulative supports reverse-process mental construction for the property of addition of equal amount. However, this property is initiated by equivalent objects: ladybug and bee. Presentation or experimenting with their equality and inclusion of them in the first question about the property is essential.
- This manipulation with objects having different quantities empowered the investigation of multiple solutions by constructing sums belonging to equivalence sets.

- Template use is effective in constructing equations and finding unknown questions.
- For investigation of the property in the “equal, not equal, equal again” inquiry, using the prior equation as the first equality situation is confusing. Try having additional equivalent quantities for creating the first equality situation. This would construct a more appropriate construct for the property. In the way we conducted an investigation, the focus was on added parts rather than equations.
- Rather than having many experiences, give sufficient time for each step and focus on discussions.

4.18 Results of Lecture 18

Lecture 18 is in the class and is conducted through individual interviews. The subjects of this lecture are increase/decrease amount to make equal, equality of increase and decrease amount as difference amount. Students are individually lectured by using their graphs of plant heights. Each interview/ mini-lecture took about 10-15 minutes. Nine students were interviewed/lectured. Yaman was absent.

4.18.1 Plan of Lecture 18

This lecture is on interpreting the increase/decrease amount to make it equal. Students are expected to recognize, increase, or decrease amount to make two objects equal to each other. It is a difference amount between two objects; however, we do not aim for interpretation as a difference amount. The lecture aims for enactive investigation by using +/- signs only.

Investigations of increase/decrease amount are represented in objectives of the 27th and 28th lectures in the first HLT. In HLT-27 students are expected to make two paper strips equal to each other focusing on the increase/decrease amount as being equal. In HLT-28, they were expected to interpret changes in graphs and compare the increase amount among peers’ graphs. These lectures are delayed for reasons mentioned in Lecture 17. Objectives of these two lectures are embedded into the activity in Lecture 18, which includes the interpretation of individual graphs. Objectives of HLT-27, HLT-28, and revised objectives for Lecture 18 are listed below.

HLT-27 objectives

1. The student uses +/- signs to interpret operation to make equal length (paper strips)
2. The student enactively investigates increase and decrease amount (difference amount) to create equal length (paper strips)

HLT-28 objectives

1. The student interprets the increase amount iconically in height context (plant height)
2. The student compares an increase amount of different situations in height context (plant height)

Revised objectives for Lecture 18

1. The student interprets the increase amount iconically in height context (plant height) (HLT-28)
2. The student determines addition and subtraction amounts to make them equal.
3. The student experiments and recognizes equality of addition and subtraction amount (difference amount)

The first objective of HLT-28 is reflected in the first revised objectives. The second objective of HLT-28 was eliminated for three reasons: they had no chance to observe other plants' growth, the lecture was decided to be through individual interviews, focus was concentrated on the difference amount. The student interprets change in their graphs; no comparison between graphs exists. Other objectives are revised versions of objectives in HLT-27 updated. The “equal paper strips” activity in

Lecture 12 objected to the use of signs in making equal strips. Objectives in Lecture 18 focus on the increase/decrease amount using relative signs.

Lecture 18 is conducted through individual interviews on analysis of students' own graphs representing the height of plants. Their graph has three measurements of height, and they used strings to interpret height for three consecutive weeks. The lecture follows the inquiry below:

- Interpret change: At first, students are expected to interpret the growth of their plants as an increase in height. Then, they are asked how much it increased/decreased (usually increased). One student reported a decrease in height.
- How to make equal: After interpreting change, interviews continue with the "How to make equal" inquiry. Students focused on the first two measurements and asked how to make them equal by adding and subtracting. They are guided to show how much increase and decrease is needed to achieve equality.
- Represent equation with addition and subtraction iconically: They cut strings to represent the addition amount and perform addition by that amount by taping them on top of the shorter string to visualize equality. Then, taking the tape out, the students' graph is turned to the initial situation. Then, students are asked to use strings to show how much to cut the longer string to make equality. They show and cut out the string representing decrease amount.
- Experience difference amount: Students are asked which string is longer or equal, referring to increase and decrease amounts. Students compare strings and see the results. If they cannot observe equality students are guided for difference amount by comparing the height of people.

Trajectory: investigate difference amount in continuously manipulable height context:

- Interpret change in quantity as an increase
- Interpret the increase amount iconically

- Determine the increase amount and decrease amount to make equal (using scale)
- Interpret equations with addition and subtraction iconically
- Compare the increase amount to the decrease amount to make equal
- Interpret equality between increase and decrease amounts to make it equal.

4.18.2 Theoretical Findings of Lecture 18

All students could verbally interpret change in plant height correctly, as “increase” or “decrease.” Two trends appeared among students when asked to show how much increase/decrease. Most students (4 out of 9) had a poor interpretation of increase/decrease amount to make equal. They focused on ending points of compared heights rather than focusing on the amount. It could be observed through their use of hands. These students tried to join endpoints (Eylem) or showed end levels/points and cutting points to make them equal (Hasan, Bekir). Sometimes, their verbal interpretations reflected their level-vise equalization as “up to this point” (Didem). These students could choose the right signs for increase/decrease actions to make equal and perform cut and paste using end levels.

Three students (Aylin, Medine, and Ufuk) interpreted the increase/decrease amount as a length. Ufuk showed difference amount by hand pointing end and increase level simultaneously for the increase amount. Medine traced the whole length to show an increase amount. Aylin interpreted the increase amount as “one finger length”. These three students were also the only ones, who immediately interpreted the equality between increase and decrease amount to make equal. Aylin even interpreted decrease amount by referring to increase amount string and saying “This much”. Medine and Ufuk couldn’t explain their reasoning but were confident in their response. Medine put an equal sign immediately between strings to interpret equality without comparison. Ufuk proved his conclusion of equality by comparing two strings representing increase and decrease amounts.

One student, Ali, needs clarification in interpretations and the inquiry was not completed for a difference amount, because he was unwilling to do so. Hasan was not questioned on the difference amount, because he was confused with a decrease in his data. Three students (Eylem, Bekir, Didem) could not interpret equality between increase and decrease amount. They relied on comparison; measurement error misled them to interpret inequality. Ekim's interpretation of the increase/decrease amount was unclear. She showed where to cut for a decrease amount. She also relied on the comparison of strings to interpret equality. After a quick comparison, she found them to be equal in joy.

Students enactively investigated increase/decrease amount or found unknowns as fixed quantities for increase amount in prior lectures. This is the first place where they verbally interpret the increase/decrease amount or create equivalent scales for interpreting the increase/decrease amount to make it equal. Some students focused on the added/subtracted amount as length, while others focused on the endpoints of action to determine up to where they would perform increase/decrease and make them seem equal.

Students investigated difference amount, enactively to interpret equality of increase and decrease amounts to make equal. In this investigation, some students saw increase/decrease amount as constructs/objects that are equal to each other. They could interpret equality between increase and decrease amount without comparing them because they acquired them as equivalent constructs/objects. They use these constructs to achieve equality rather than increase/decrease actions until they reach equality. Students who use increase/decrease processes until reaching an equal level are at process level for increase/decrease amount because they focus on construct addition/subtraction to satisfy the equality. Students are expected to construct equations and find unknown addition/subtraction amount in this activity. In constructing the equation, they might have a process-level understanding of the increase/decrease amount and procedurally focus on constructing the addition/subtraction amount. Then they need to compare these amounts to conclude an equality relation between them. However, students with an object-level understanding of increase/decrease amount could interpret increase/decrease amount as length and did not need to compare to test their equality. They already knew the difference amount and constructed the increase/decrease amount based on their knowledge of the difference between the two quantities. The relation between mentioned mental constructs is summarized in the Figure below:

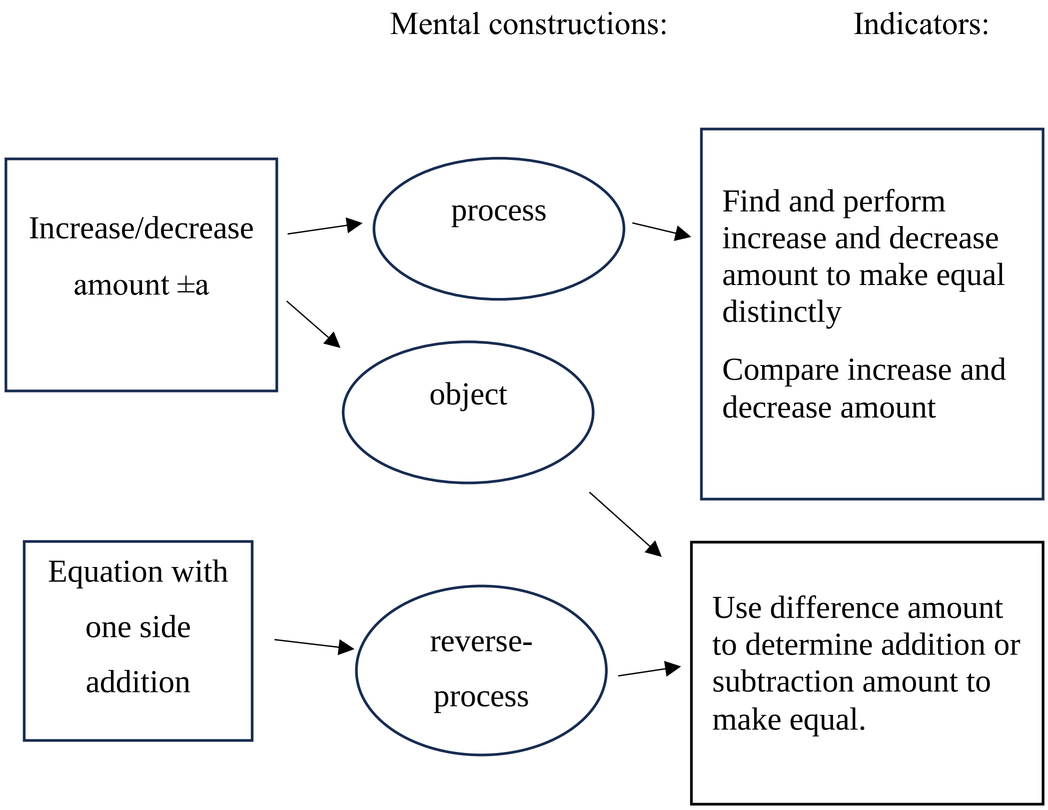


Figure 4.50. Schema of Using Equation With One-Side Addition and Increase/Decrease Amount in the Concept of Difference Amount in Lecture 18

For those students who could not associate increase/decrease amount in plant height context, or who could not explain the reason for equality, the researcher showed dotted lines to show difference amount (See Figure 4.51). Even this expression did not work for Didem and Eylem. The researcher then guided them by showing the difference between the heights of the students and herself, which might work as a real-life example. How much to grow for the student, or how much height to be reduced in the researcher’s height, is shown by the difference amount between height. The researcher and student were standing side by side, which made visualizing the difference amount easier. This might be the reason why this inquiry was successful for Didem and Eylem.

(a) (b)

Figure 4.51. The researcher Explains the Difference Amount: (a) Medine and (b) Ufuk

Quantity

In the enactive mode of investigation, plants' height is manipulated through string lengths. Context turns into length from height after plant heights are reported by graphs using equivalent scales. Continuity in quantity is sensed through manipulation in constructing scales for graphing and in addition/subtraction to make equalities. Unknown as addition/subtraction amount is associated with the difference of pre-constructed quantities. All students successfully interpreted the height of plants by the length of strings. They all manipulated quantities correctly to find addition and subtraction amounts. Some could (3 out of 10) reason by the difference of quantities in determining unknown.

Notation

Students interpret equations with one-side addition and investigate difference amount in the enactive mode of representation. They are iconic lengths representing plant height, but they manipulated strings as they are the objects of enactive investigation. The choice of +/- or equal signs was correct in all student's work. However, for iconic representation as addition in constructing equations (as equality between longer string and shorter string plus addition amount), students are guided by the researcher. This guidance was stepwise; first, choose of sign for increase, place the increase amount on top, and then represent equality.

(a) (b)

Figure 4.52. Interpretation of Addition: (a) Eylem and (b) Bekir

All students could be guided through the iconic mode of interpretation of addition step-wisely. Only one student, Aylin, could interpret addition and subtraction symbolically. She used strings as quantities in the equation, written in linear form. She substituted the increase amount for the decrease amount in the equation (See Figure 4.53.).

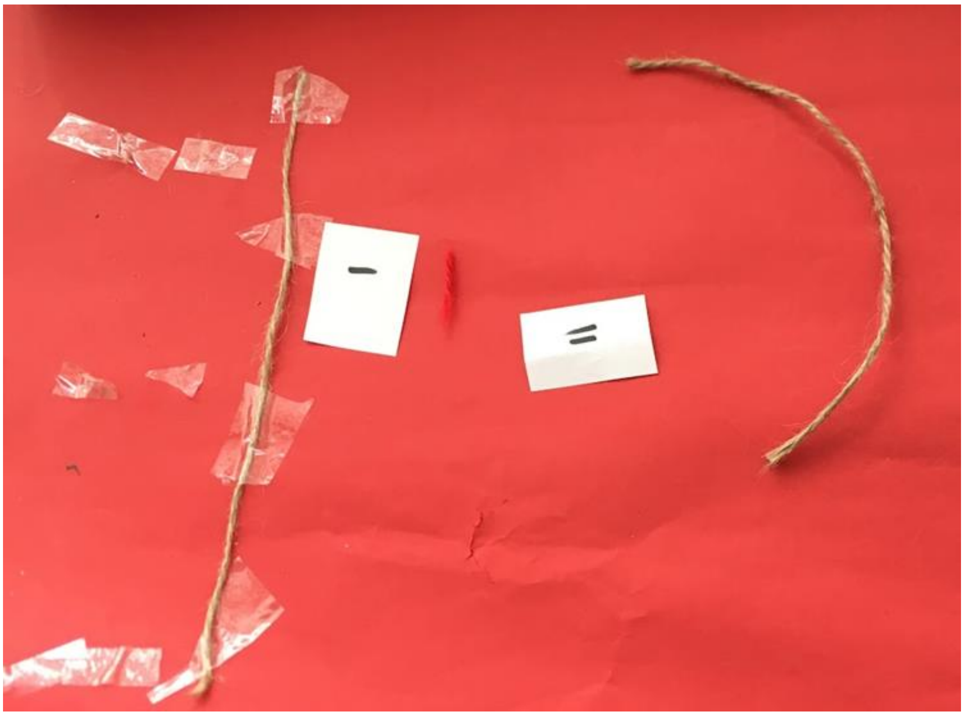


Figure 4.53. Aylin’s Interpretation of Equation with Subtraction; Red String: Increase Amount Substituted as Decrease Amount in the Equation

4.18.3 Design Principles for Lecture 18

- Step-wise guidance is essential for constructing equations, including operations, at least for addition. This inquiry follows:
    - choose where to increase, choose the correct sign for increase,
    - placement of signs,
    - determination of addition amount,
    - placement of addition amount,
    - representation of equality by equal sign.
- Bring strings together or compare people's heights to explain the difference amount. Explanation by dotted lines doesn’t work for children who do not reason by equivalence of increase and decrease amount.

- Activity is sensitive to measurement error. Some students relied on the comparison to investigate the difference amount. Measurement error affected their conclusions.

4.19 Results of Lecture 19

Lecture 19 was an in-class implementation about reading symbolic interpretations of equations with one-side addition. Students obtained secondary colors by using equations and inequalities defined by primary colors as recipes to construct a rainbow by play dough. The lecture took 45 minutes. Before the lecture, students observed the rainbow and colors of the rainbow outside the classroom for about 15 minutes of experimentation. 8 students attended class. Ekim and Yaman was absent.

4.19.1 Plan of Lecture 19

This lecture is auxiliary to Davydov's trajectory. Modeling equations are part of Davydov's trajectory, but there is no meaning of addition as two parts coming together to construct a quantity. The meaning of addition focuses on the increase in quantity. In this lecture, students join two colors. We wanted to develop a fun lecture, adopted on secondary colors subject, including a read and use of equations and equality/inequality relations. This lecture also aims to connect their knowledge of addition as an increase to the meaning of addition as two quantities coming together, which they will encounter as a functional definition for addition later. In addition to Davydov's trajectory, modeling with equations is given in systems, including an equality interpretation.

Placement in HLT and objectives did not change much. Objectives are elaborated to focus on quantity. Preserving quantity in the system among expressions/equalities is a difficult additional object for students.

Objectives in HLT-32

1. The student reads equalities and inequalities based on real-life models
2. The student uses algebraic equalities and inequalities for real-life designs

Revised Objectives for Lecture 19

1. The student determines quantities of objects based on equality and inequality relations in weight context.
2. The student reads and uses algebraic interpretations of equality/inequality relations and equations with one-sided addition in a real-life weight context.
3. The student preserves the quantity represented in the interpretation to use it in the addition equation (relations and equations are connected in a system).

In this lecture, students construct rainbow colors by using recipes to determine quantities of a mixture based on recipes given in systems of equations with addition and equalities/inequalities. For example, the orange color is obtained by a mix of yellow and red in equal quantity. Quantity is compared to weight attribute. Equality is represented by colored circles in the interpretation. The addition of red and yellow in the equation becoming equal to orange represents a mixture. We call it a system because to make the mixture using the equation, students must preserve the quantity defined by the equality situation.

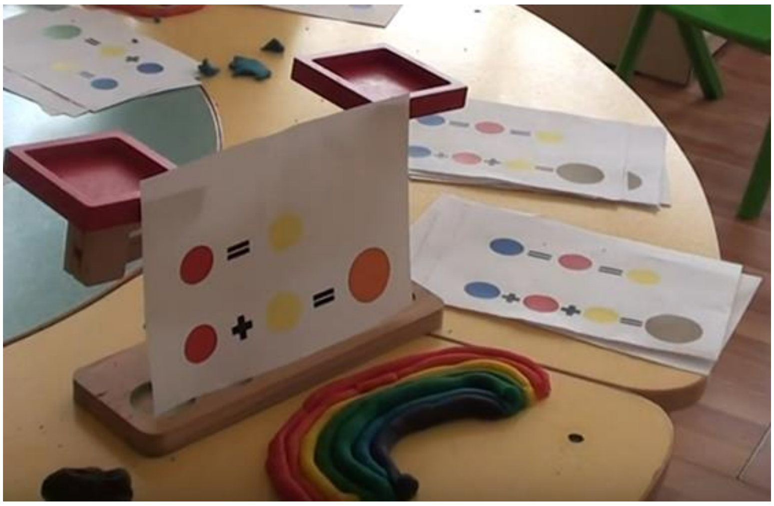


Figure 4.54. Example From Classroom Implementation: Eylem

In addition to 3 primary colors, blue, red, and yellow, there are four secondary colors, orange, purple, green, and turquoise, interpreted by four systems below:

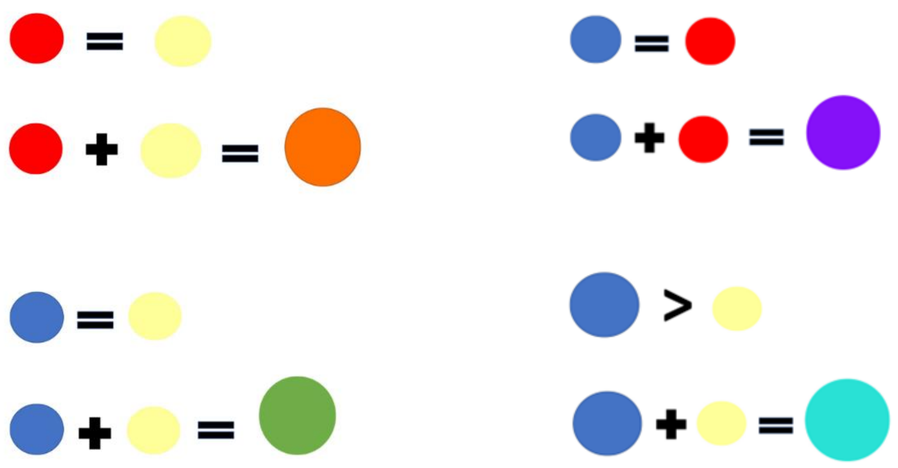


Figure 4.55. Recipes for Secondary Colors Interpreted with Systems of Inequalities and Equations.

Additionally, after they complete the rainbow, they are asked to obtain brown by using the system:

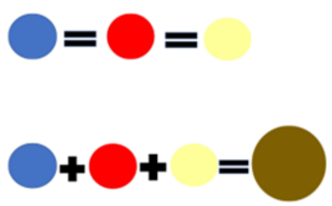


Figure 4.56. Recipe for Brown Color with Equality and Addition of Three Colors

This is not the first time students use color notation in algebraic representations. In the prior Squares Activity, they used colors to represent equal-sized squares in inequality relations. In this lecture colors represent relative play dough quantity in the system. In the Squares Activity, in symbolic representation, the size of the colors was not correlated with their actual size. For this lecture, piloting also included manipulatives representing all circles of the same size in the symbolic interpretation of systems. One of the pilot students did not attend symbolic representations to construct rainbow colors. The kindergarten teacher and researcher (as design team members) decided to represent a larger quantity by a bigger circle in symbolic representations to make it easier for students and not to contradict their perception because some of them had difficulty in Squares Activity to use color as notation independent from size.

Lecture Flow:

- Follow rainbow colors, starting with red, then present the recipe for orange in order.
- Ask students to read and explain expressions. Explain what the recipe tells.
- Ask how to make equal amounts of playdough direct using pan balances.
- Direct and control each student to read and follow instructions for comparison and mixture of playdough.
- Go on with the algorithm of Turquois, then purple recipe red=blue
- Discuss the reason behind getting different shades of turquoise (if any) (multiple-solution)
- At last, present the recipe for brown color. Guide through how to make three equivalent pieces through dual comparisons.

4.19.2 Theoretical Findings of Lecture 19

Students were presented with seven colors of rainbows and provided with three colors of play dough and balance scales. To make secondary colors, they received recipes in the order of the rainbow, from red to purple. In the first recipe for orange, they all ignored using balance scales to create equal pieces of red and yellow; only

Bekir used balance scales to construct equal-weighted pieces. Medine follows him. Other students are reminded to use balance scales.

Students were individually questioned and guided to observe how they read and use recipes in the form of systems, including an equality relation and equation with addition. Students ignored reading the addition equation. They created equal pieces by looking at equality. In some cases, just looking at the recipe they thought they mix red and yellow to make red. They mix a little from one and a little from the other. When asked how much to take from each, one student (Medine) replied, “There are two pieces of yellow (in the recipe) and one red.” She ignored all algebraic expressions. The researcher explained recipe says red is equal to yellow, so take equal pieces. Eylem told herself she had taken them equally already. The researcher asked, “How did you take equal?” No one referred to comparison with balance scales. After asking this question, Eylem said they were equal, so she took equal.

The first inequality/equality relation of comparison worked for students, but the second representation of addition didn’t mean much. They did not attend to the expression of addition but just mixed colors. No reading was observed. The researcher took students’ attention to the expression by saying, “The equation says: we add yellow and red.” This activity refers to addition as “coming together,” which confused students while reading the recipes.

Explanation of recipes and recommendations for using balance scales helped them focus on the equality of the mixed colors. They all used balance scales correctly and fluently to make equal pieces, by increasing and decreasing actions (Students preferred adding doughs to make equal, researcher reminded also taking some as decreasing to make equal.). When they needed extra playdough, they created more by taking equal pieces using pan balances (Ufuk, Eylem). Bekir checked and ensured equality between the resultant colors, which he used in his rainbow (He is always a perfectionist).

Some students started loudly reading recipes provided to them (Ufuk, Bekir), mostly focused on the equality relation. Some students (Ali and Hasan) ignored the reading recipe for turquoise color and took equal amounts of yellow and blue play dough. Bekir explained the recipe loudly, which made them correct themselves. The presentation of turquoise color improved their attention on the algebraic expressions in the recipes.

Bekir explained the recipe of turquoise as: “Blue is big, yellow is small” How they read the expression shows how they think about the sign. They don’t have a sentence to read from left to right. However, dependency on quantities is not reflected in his interpretation.

Hasan and Medine had difficulty with the recipe for brown. They placed small pieces of all three colors on the same pan (See figure). They also had difficulty interpreting relations on the recipe related to real life. For example, when asked Ali what the recipe says for brown, he said, “We make equal.” he meant that all colors will be equal. However, Hasan only read the names of signs “equal, equal” while pointing at them.

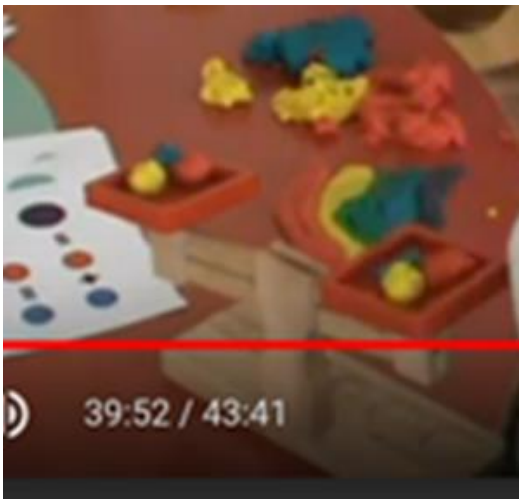


Figure 4.57. Hasan Trying to Create Equal Pieces of Three Colors to Make Brown

Some students could take similar small pieces and obtain all equal to each other by chance (Aylin, Eylem, Didem) and then check for the equality of all combinations between three colors (Aylin). Some students were reluctant to work systematically to create equal pieces more than two (Bekir). Some students (Ufuk) get confused in the process while making the third color equal. The struggle for equalizing the third color results from the difficulty of holding one color constant. For brown, Ufuk added and subtracted by increments to satisfy equality, saying, “I am adding and adding, but it does not make equal.” He equalized yellow and blue first, then put blue aside, saying, “Let it be sat here” when yellow stayed on the pan. He began to put red on the other side. He increased the red by increments, but it became heavier than the yellow. He took some from red but couldn’t equalize again. He suggested to add to yellow. The researcher intervened at that point to guide for holding a constant.

R: If we add to yellow, will it remain equal to blue or not?

Ufuk: No

R: Then we should change to red. Do we need to increase or decrease red?

Ufuk: Decrease

Then, he decreased red by increments and satisfied equality. Then, he mixed all the colors.

Briefly aiming for students to read and realize symbolic equality and equations in real-life situations, we observed that they heavily depend on teachers' verbal interpretations. Guidance, making them read aloud, and creating diversities in the algebraic interpretations help them reverse-process in modeling, especially for equality relations. Before this lecture, they constructed quantities based on an equality relation (=, < signs) to a fixed quantity. In this lecture, they are provided the relation, and quantities are not fixed, continuously manipulable (and dependent on each other by the relation). They construct both quantities based on the relation determined. It means they reverse-process creating equality relations based on a real-life comparison. The majority of them were (6 out of 8) successful in interpreting dual relations and using them in constructing quantities in weight context on their own. However, equality between the three quantities was confusing for some students (2 out of 8).

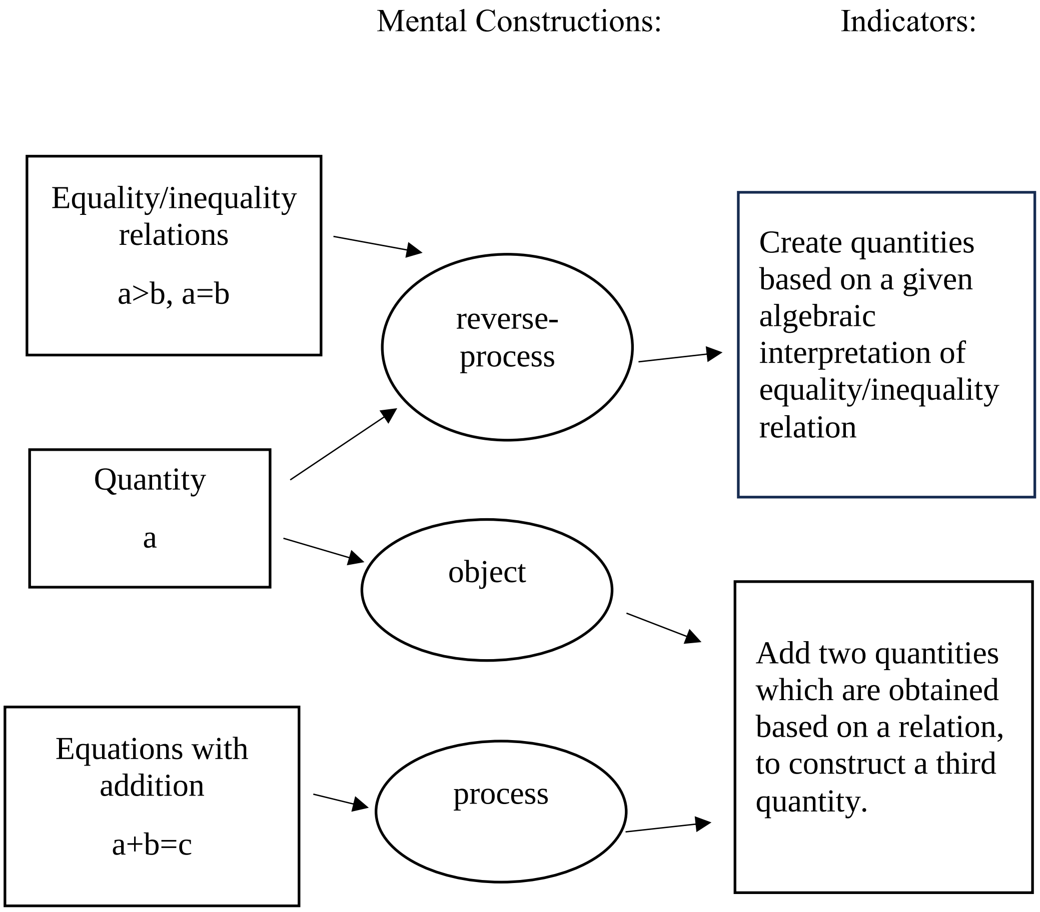


Figure 4.58. Schema for Equations and Relations in a System in Lecture 18

Quantities obtained by the reverse process of equality relations pre-given to students are preserved in the process of addition to create new quantities based on an equation. We could say that quantity obtained is used in the addition operation process, if the resultant would be compared to something, or the resultant will be determined in quantity. However, in this activity, students used these models (equations) to create the third quantity out of it, where the equation and equality in the system define the relation between all three quantities. We have no evidence from students' verbal interpretations for the use of equation and preservation of quantity obtained by equality relation, but mental constructions are defined on their enactive investigations.

Quantity

Quantities are continuously manipulable. They are variables related to each other, with no fixed amount. Students freely manipulated and constructed quantities. We aimed to make them use these quantities preserved in the system. However, students did not attend equations in the system.

Notation

Symbolic notation for equations with addition, reverse-process not construction but reading and realizing equations and inequalities.

4.19.3 Design Principles for Lecture 19

- Students ignored the second algebraic representation. If they focus, they just state the first expression. Only the first algebraic relation might be enough to complete the activity. This calls for a revision in the activity.
- This lecture is incompatible with increasing quantity, meaning for addition. We wanted to emphasize coming together meaning for addition. But it is out of trajectory. Students had difficulty understanding addition expressions because of this reason. Hence, use language to connect both meanings. The words should be chosen to express adding meaning of addition as they are used to do. Not “We will add red and yellow,” but “We add red onto yellow.”
- Students still prefer increase action to make equal in weight context. Remind decrease action, significantly to ease holding one constant in making brown play dough.
- Individually interview and guide students on their interpretation of algebraic representations. First, ask them to read the expression, then ask, “What might that mean?” in the real-life context.
- Students ignore algebraic representations when they understand the assignment. Create the need for reading recipes: algebraic representation, by diversities.

- The quantities in the equation are given in different sizes according to the relation. Try using same-size notation to improve symbolic representation abilities.

4.20 Results of Lecture 20

Lecture 20 is an additional activity to the first HLT. In previous lectures, Students constructed equations with addition on two sides and used the addition of equal amounts as an operational property in a height context. This lecture investigates how addition/subtraction by an amount affects equality and inequality situations in a volume context. It took 45 minutes to examine in class. It is the last lecture on implementation. Nine students attended class; only Yaman was absent.

4.20.1 Plan of Lecture 20

In previous lectures, students constructed equations with addition on two sides. To investigate the property of operations, they used the addition of equal amounts to preserve equality and investigated the equality of increase/decrease amount to make equality. In volume context, they performed addition and subtraction by an amount. We wanted to revisit addition and subtraction operations by amount to observe the effects of adding/subtracting equal amounts in a more structured construct in a volume context.

Objectives defined for Lecture 20 are;

1. The student operates addition and subtraction on equalities based on given expressions such as +/-a in volume context.
2. The student experiences operational properties on equalities (the starting point is equality).
3. The student realizes and compares algebraic expressions such as a+/-b in a volume context.

4. The student experiences operational properties on equations (starting point changes).

Students will investigate $\pm a$ properties and compare $a \pm b$ situations in a volume context. There are two levels of activities in this lecture to investigate operational properties. Firstly, students will start with equality and use dice to perform $\pm a$ with their partners. Before performing, students are expected to guess who will win. In this level, they are expected to acquire positive signs resulting in a higher level than negative signs in comparison cylinders; the addition of a bigger amount will result in a higher than the addition of a smaller amount, and the subtraction of a bigger amount will result in a lower than the subtraction of a smaller amount:

$b>c \Rightarrow A+b > A+c$, $A-b < A-c$, $A+c > A-b$, $A+b=A+b$, $A-b=A-b$

Students are reluctant to operations in the first phase, from Lecture 14, where they performed increase/decrease by an amount in volume context (perform $\pm a$). In the second level, the initial amount is not equal but is determined by another dice with bigger cups to satisfy positivity in subtraction cases. In this level, starting amount plays and addition/subtraction by an amount play a role in determining the result. Students will try to guess addition/subtraction by equal amounts in different quantities. They are expected to guess or investigate the following situations:

(1) $A>B>c>d \Rightarrow A\pm c>B\pm c$

(2) $A+c>B-d$

Briefly, they will reflect on addition/subtraction by equal amounts and different amounts on equality and inequality relations. For the investigation lecture is conducted through the lecture flow:

- Remind students what +, - means in the context of the action of filling and pouring away.
- Each student has one cylinder cup. They work in pairs. Competition based. They first fill their cup nearly half, but equal to each other.
- Take two dice, one representing signs and one representing cups.

- Pairs roll two dice simultaneously and place them on the table for representation. Students place corresponding cups near the dice. They guess for the result who wins. Then, perform increase/decrease by amount. Teacher guidance for taking each guess and then performing for results is important.
- To prepare for the second try, students are commanded to equalize their cups again.
- After several tries, add a cup of dice and cups to the performance. Make sure the cylinders are empty at the beginning. A bigger dice distinguishes which one to roll first and fill the cup. Bigger dice include pictures of bigger cups (one cup is identical to one from the little dice). ±
- First, bigger dice are rolled, and the corresponding amount is poured into the empty cylinders. Then sign dice are rolled with the small dice, printed on little cup photos, to perform addition or subtraction with an amount. This stage is different from the previous one, by determining starting point by a dice.
- Three dices form an algebraic expression of the unknowns and one addition or subtraction sign between ($A \pm c$). Students guess and perform results.
- During this game, the researcher guides students in assessing the expressions and guessing the result. Without rolling dice randomly, the researcher places them on the table, forming particular expressions of $A \pm c$ vs. $B \pm d$. The researcher asks "what if" questions for certain combinations in a structured inquiry to help students see the properties of operations and quantities affecting both sides.
- After guessing the result, students perform the expression of three dice. Start by initial amount, they perform addition/subtraction by an amount and interpret the equality/inequality relation between them. Who guessed correctly wins; both students can win.

4.20.2 Theoretical Findings of Lecture 20

Starting with equality, all students could perform the first actions of increase/decrease by an amount correctly. Usually, they forget to guess before comparisons and create an equality situation before performing actions defined by dice. When guided and interviewed individually, they could compare addition and subtraction by an amount before executing operations. All of them (9 out of 9) could correctly guess the result for addition and subtraction by the same amount and addition by the different amount on equality. However, some students (Ali, Hasan, Medine, Ekim) struggled to guess subtraction results by different amounts on sides of equality.

Ali and Hasan had difficulty performing negative actions, subtraction by an amount. The researcher guided them through actions and made them see if their prediction was correct through an enactive investigation. While pouring out by amounts, Ali recognized that Hasan would win because his cup was tiny. Hasan admitted the result by saying, "If he had a smaller cup, he would win." Guidance with Ekim also relied on a similar development in her understanding through investigation. She investigated subtraction by different amounts with Eylem. Eylem predicted correctly, while Ekim thought hers would be much because, in her dice, the cup is bigger.

R: Which one is bigger?

E: Mine is bigger (correct)

R: What will be a lot then here

E: Mine…

R: Why?

E: Because my cup is very big.

R: You will pour away very big. Let's do it. See, Eylem's remained bigger. Why did this happen, Ekim?

E: Because I poured some more.

R: Because your cup is bigger. When you poured you took much salt from here. Eylem's cup is small, she decreased a little. It is ok if you are confused. We check by doing to see.

E: Teacher, I understood; because Eylem's cup is little, a little is taken off. Because it decreased little, here remained a lot. (own explanation)

Ekim could explain the reason by the actions of subtraction and how much is subtracted procedurally. In subtraction, pouring away took a little longer, so she associated it with the procedure. Her mental constructions of decrease by an amount points out process level. Guidance helped Ekim, not because she saw the result, but experienced the procedure. It was similar in Ali's case, where he corrected his reply while pouring away before seeing the result.

Medine's development has a similar procedure. She predicted wrong when she first met subtraction with different amounts. She thought a bigger cup would result in higher. After enactively investigating, she observed result is equality, so cups must be equal to each other. The guidance did not improve her understanding, but her experience with subtraction and reflecting on dice improved her understanding of subtraction by an amount. She could reason by subtraction amount seeing it as a stable construct, not referring to the process of long pouring as Ekim did, but referring to how much decrease on the manipulative. The researcher asked, -b vs -c where b>c. She replied correctly. The researcher asked, "Why?". She held the smaller cup on the comparison manipulative (one of the identical cylinders), saying, "Because this is little, this will come out of it." (See Figure 4.59.) She did not even need to demonstrate actions.

Figure 4.59. Medine Explains How Much to Decrease by the Cup on the Cylinder

Briefly, predicting, checking for results, and performing subtraction by amount enactively helped students discover operational properties. One of the properties is investigated also through reverse processes. Predicting the results of addition or subtraction by equal amounts on both sides was trivial for all students. The reverse process for this property is observed in investigations of Medine. In the investigation of subtraction by two different cups, she found the result to be equal (measurement error):

> R: Why equal? Is this cup, or is this one bigger?
>
> M: They are both the same (showing the cups, like the same level) because they are equal (referring to the cylinders, talking about equality). (She explained without recommendation)

When students are expected to predict the results of comparisons, they have a common misconception of ignoring signs at the beginning of the lecture. When both are plus signs, no difficulty occurs. However, if both signs are minus (Ekim, Medine, or even one of them is minus while the other is plus (Ali), they ignore the sign and think a bigger cup will win. Enactive experiences helped them overcome this difficulty, at least in situations started by equality.

The second phase of the activity investigates different starting points to investigate operational properties. They investigate not the effect of operations but algebraic expressions in the form of a±b, represented by three dice rolls. They reflect on by comparison and operate these constructs enactively. Similar problems occurred in this phase. Students might ignore the whole expression and focus on a particular part. In the investigation of A-b vs A-c when b≠c when Ekim focused on starting points.

> Ekim: "This might be equal because these are equal" (for starting cups)
>
> Researcher: are these equal (subtracted ones)
>
> E: no
>
> R: Which one is bigger?
>
> E: This (own)
>
> R: And this one is small (Eylem's). First, imagine pouring these (initial big cups), then reduce by this from Eylem's and by this from yours.
>
> Ekim sighs "Ha" (understanding sign)
>
> R: Which will be bigger, who wins?
>
> E: Eylem (she knows)
>
> R: Yes, you know. Guessing is essential, not winning.

It might be easy to compare A+b and A+c situations for Didem, but it is difficult to subtract equal amounts from different starting points. Didem had "A-c," and Aylin had "B-c," where A>B. Aylin guessed the result correctly. Didem did not.

> R: Why do you think so
>
> D: Because a bigger one came to me (she thought she lost because A is subtracted from c, and the question is in the form of subtraction of different amounts from the same amount)
>
> R: No, this is the one we put first. We put the big one first then we decrease from it.

Didem read the algebraic expression in the wrong direction. This problem occurred because of her missing understanding of order in the expressions. She read the expressions from right to left. This wrong displacement also occurred in Eylem's interpretations, which were corrected by the researcher. Before this lecture, we had no activity on writing equations with subtraction. In this lecture, we try to teach order by taking action with the bigger dice/ bigger cups first in addition and subtraction procedures. However, it is difficult to structure items based on their age, mainly when they operate subtraction/addition actions without writing or expressing a starting point before. What they did was to operate action on sides until now.

The problem of structures of equations was also problematic for Bekir. He could comment on the A-b vs. A-c situation but could not reflect on the A-b vs. B-c situation. Starting with a different amount or reading the expression was difficult for him.

Aylin is the only student who could always comment on A±b objects correctly. Her explanations are also remarkable, with his language abilities and correct connections on mathematical topics. When A-b is compared to A-c, she reasoned her response by saying: “because it will be decreased less” pointing out the bigger side by explaining the decrease amounts relative to each other and expressing the operational action simultaneously in her sentence.

In this lecture also Eylem could understand A±b expressions as algebraic construct. The difficulty of the second phase for some students reminded the researcher about students’ levels on Piaget’s conservation test. The researcher conducted a quick test on the volume item of Piaget’s conservation of amount test. Ekim and Medine could not interpret equality between two cups by pouring one from the other, as they could not at the beginning of the whole implementation. However, Aylin and Eylem confidently replied to this item. This might be the reason why some students have difficulty seeing A±b as constructs and reflecting on or comparing them. They just focus on particular aspects, operations, and equal or different amounts they see in the expressions. Creating meaning out of the whole is difficult for them. When they pour the first quantity (determined by the first dice) into the cylinders, it loses connection with the expression. They comment on the operational property, add subtracted amount, observe results at most, and lose attention to the structural property of the expression.

Quantity

Students compare addition/subtraction of fixed quantities in this lecture. However, starting investigations with equality, students used three different types of manipulation: continuous manipulation, incremental change, and begin by fixed equal quantities. Didem filled cups equally, pouring on one by continuous manipulation until it became equal to the other. Similarly, Aylin filled cups equally in half by choosing a point. She stopped in command of researchers when they became equal. Aylin compared and equalized, by small increments, almost continuous manipulation of adding. Medine also manipulated with increments to make it equal. Ali suggested that Hasan use the same cups to fill, and then they continued manipulation with increments to satisfy equality. Similarly, Eylem suggested Ekim fill with identical cups to make an equal start. Then, they filled and did not even check for equality.

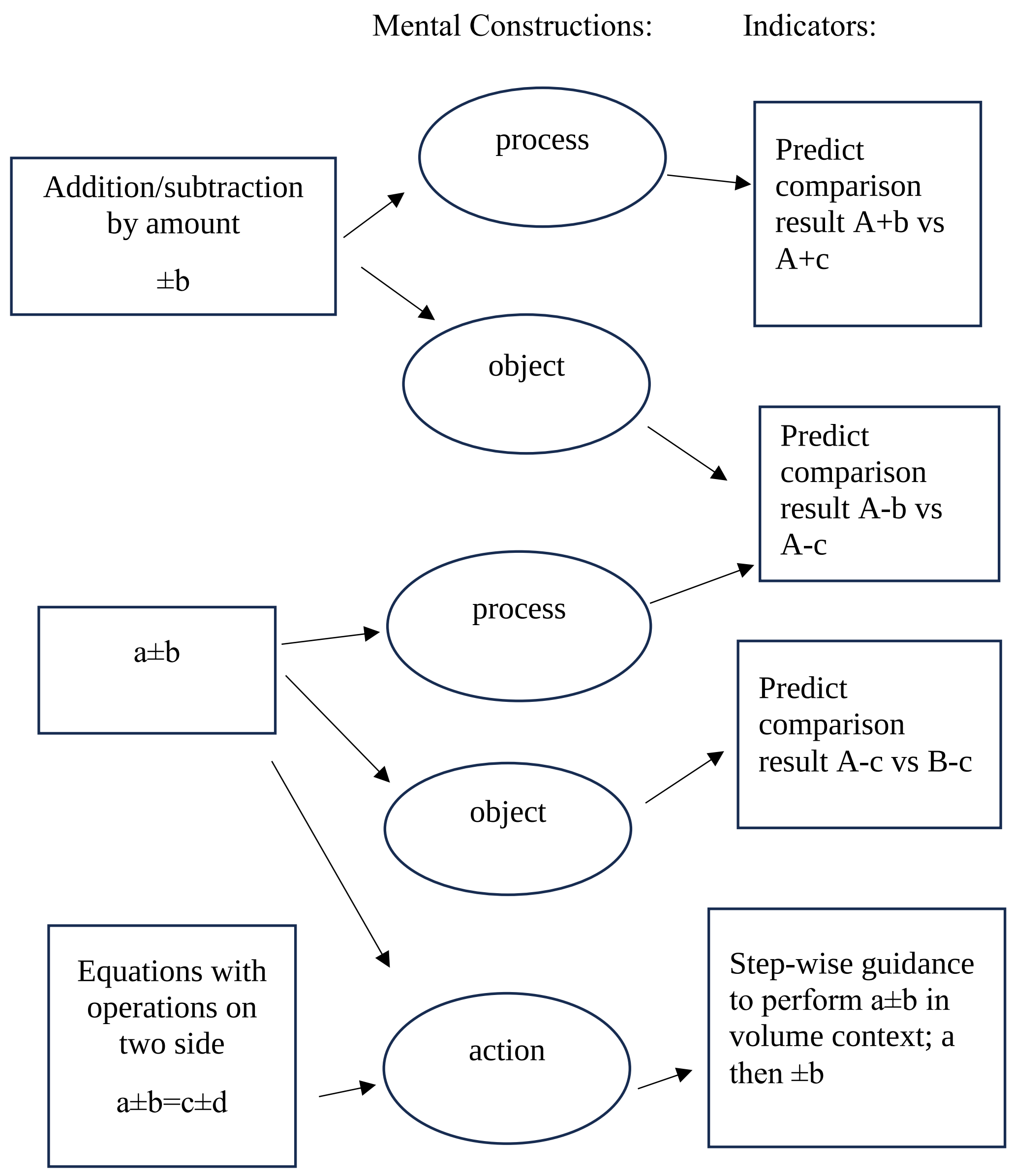


Figure 4.60. Schema of Equations in Lecture 20

Notation

We used templates to create equations with addition on two sides in height context, in Lecture 17. In Lecture 17, the template placement of the quantities did not matter because of the commutativity of addition. This lecture also includes subtraction. In prior lectures, they acquired subtraction as a decrease, and the starting point was not a problem in their actions of decrease. In the first phase of Lecture 20, they investigated the decrease by quantity effect on sides of equality. The second phase starting point is added to the algorithm to obtain a±b constructions. First, “a” is added to cylinders, then ±b is performed on. When dice were rolled and three dice came together to construct A-b, students could not reflect on it, even read the expression from right to left, and commented by referring to “-A”. In prior lectures, they learned equality as balance and operations acting on the sides. No ordering was mentioned, nor was the reading of expressions based on a starting point before.

In this lecture, we thought dice would work as the template we needed to construct subtraction. We used two different dice; one is bigger than the other. The bigger one contained photos of bigger-sized cups on it. Students were given an algorithm to throw these bigger dice at first and perform it. (Piloting of this lecture concluded to add this algorithm of performing bigger dice “A” at first; throwing three dice and performing all at once was difficult.) The student threw a second dice with the dice indicating +/- operations. They form ±b and perform it on the sides. Later, they are directed to throw three dice at the same time. Placement of three dice in the wrong order (Eylem) and reading from right to left (Didem) problems occurred with subtraction cases. Students are guided to place bigger cups/dice first and operate subtraction from them. To resolve this problem, constructing addition and subtraction expressions not from left to right but from top to bottom can be tested for future implications.

This lecture required more than one hour of lecture time. Investigations of operations on equality as the first phase would have been sufficient. The second phase with a±b construction needed additional lecture time for active investigations and algebraic interpretations.

4.20.3 Design Principles for Lecture 20

- The activity supports many free investigations that are semi-structured by using dice, with a limited number of cups to ease comparison. However, inquiry through certain properties of operations and structures of equations is important if it does not occur naturally.
- Time is insufficient to discuss all properties individually in a classroom setting. Design activities more structured to lead to investigation of several of them for all students. Limiting the number of cups represented in dice was helpful.
- We used equal-volume cups in two different dice to obtain zero quantity after subtraction. However, no observation occurred. Differentiating the size of cups is helpful for predictions.
- Measurement error occurs in the volume context, which may lead to difficulties in the results of investigations but may also lead to opportunities, as in Medine’s case.
- Avoid racing, focus on guessing. Racing decreases motivation with peers for some students. Try to connect positive feelings in mathematical investigations. Understanding a game is complex, and it is hindered by previous games on volume. Students try to fill up their cups. Revision of the game structure can be discussed.
- Investigations through structured discussion worked for students to understand properties and their own experience on subtraction by amount.
- Additional activities or templates are needed to construct and understand a±b structures.
- Consider templates for a±b constructs, from top to bottom rather than from left to right, to make them easy to construct, read, and perform.

4.21 Resultant Learning Trajectory

The resultant learning trajectory is given in comparison to the first HLT and Davydov's trajectory in the following table. See Appendix G for a detailed comparison and alignment of objectives in the first HLT with 32 lectures to objectives in the last/revised trajectory with 20 lectures. Reasons and outcomes of trajectory changes are reported under the results of each lecture in the section of the lecture plan.

Table 4.3 Summary of HLT Adaptation and Change

	APOS	Davydov	First HLT	Last trajectory
Equality &quantity	Equality action-process	Equality-inequality	Equality-inequality	Equality-inequality
	Inequality action-process	Greater-less than	Greater-less than	Greater-less than
	Pre-action increase/decrease	How to make equal: iconic	How to make equal: enactive & verbal	How to make equal: enactive
	Action to process quantity	Determine variable	Determine variable	Multiple types of quantities
Transitivity	Action order	Ordering	Ordering	Ordering
	Transitivity action	Construct based on relations Guess the third relation	Transitivity Construct based on transitivity	Transitivity: Guess the third relation
	Object quantity Action intermediary Pre-action equal scale	Create intermediary	Create intermediary	Create equal-scale
	Object equal scale Process to object quantity		Squares: fixed quantity notation	Squares: fixed quantity notation
	Object transitivity	Transitivity symbolic	Construct based on transitivity	Order for transitivity
Operations	Pre-action +/-	Verbal increase/decrease to make equal	Enactive increase/decrease to make equal	
	Action +/-	+/- signs to make equal: iconic steady: first this, then this	+/- signs to make equal: continuous manipulation	+/- signs to make equal: continuous manipulation

Table 4.2 (continued)

Operations	Action increase/decrease amount	Increase/decrease amount to make an equal, continuous quantity	Increase/decrease amount to make equal	Increase/decrease by an amount and then compare fixed quantities
	Action one-side operation equation	One-side add/subtract to make equal: iconic continuous	One side addition to make equal: Find unknown: height: fixed quantities	One side addition Find unknown: height: fixed quantities animal height
	Action difference amount	Exact amount	One side addition or subtraction to make equal	See further
	Object increase amount		Compare increase amount	
Equations	Two-side operation properties Action equations	Two-side addition and subtraction: equal not equal, equal again	Two-side addition and subtraction: equal not equal, equal again	Two-side addition: equal not equal, equal again: multi-solution Animal height
	Process equations process		Two side addition: find unknowns Animal height	
	Modeling two-side equations	Matching real-life examples with equations	Create a model of equations Paper strips	Difference amount Action recognition plants
	Modeling equation with one side addition And inequality		Use expressions of equality and equations to model Rainbow	Use expressions of equality and equations to model Rainbow
	Properties of operations			Experience properties of addition subtraction on two side

The table below is the resultant trajectory defined by objectives and APOS levels for each designed activity. The designed activities evidentially support the resultant trajectory through the lectures from 1 to 20.

Table 4.3: Resultant Trajectory

	Objectives:	APOS level of concept
Lecture 1	1. The students interpret equal and not equal sign 2. The student compares objects and uses equal and not equal signs to interpret relation (action to process level) 3. The student uses balance scales to compare the weight of objects and uses equal and not equal sign 4. The student uses different variables/attributes (which she already knows) to interpret equality	Action =, ≠ Action variables
Lecture 2	1. The student uses balance scales to partition play dough into two equal masses by increasing/decreasing actions verbally. 2. The student uses equal and not equal signs to interpret a relation in a part-whole context. 3. The student manipulates (increase/decrease) one side for the satisfaction of equality in part-whole activities	Process =, ≠ Action increase/ decrease
Lecture 3	1. The student uses equal and not equal signs to compare volumes of cups 2. The student interprets the equality of volumes of cups iconically (notation)	Process =, ≠ Action iconic notation
Lecture 4	1. The student interprets inequalities with greater or smaller relation. 2. The student uses >, < signs to interpret relations	Action >, < Process =, ≠
Lecture 5	1. Report: The student interprets comparison of volumes by >,<, = signs symbolically on paper by using pictures of compared cups as letter notation. 2. Read report and check: The student reads/uses a symbolic representation of a peer's comparison and checks with manipulatives if the comparison is accurate.	Process >,<, = Action symbolic representati on
Lecture 6	1. The student draws an unknown figure based on a given algebraic relation to another figure with >,<,= signs. 2. The student uses =, ≠ signs to interpret part-whole equality given by symbolic figures (Lego photos). 3. The student uses >,<,= signs to interpret relations symbolically based on given representations of weight comparisons.	Process >, <, = Process symbolic representati on
Lecture 7	1. The student orders four objects based on their size and uses> sign to interpret the sequence 2. The student extends the sequence of ordered objects based on size.	Action >,< sequences
Lecture 8	1. Given three objects, the student experiences and reports two comparisons (in order) and guesses the third relation.	Action, transitivity

Table 4.3 (continued)

Lecture 9	1. The student creates an equivalent scale for an object to compare it to another distant object. 2. The student interprets the comparison result in terms of the distant objects, not the scale they used.	Action creating scale Object transitivity
Lecture 10	1. The student constructs scales to compare distant squares 2. The student uses the same color notation to indicate same-size squares 3. The student uses colors as a notational representation to order squares based on their size.	Process creating scale Action notation
Lecture 11	1. The student recognizes multiple solutions to construct objects based on >, < relations. 2. The student uses equal-sized scales to represent measurement. (plant height)	Pre-action multiple solution Object equal scale
Lecture 12	1. The student chooses the correct sign +/- to interpret the increase or decrease on sides to satisfy equality.	Object =, ≠ Action +/ -
Lecture 13	1. The student dramatizes, +/- size as action of moving forwards and backwards (understanding of +/- signs as positive and negative directions)	Action +/- Pre-action +/- with a fixed amount
Lecture 14	1. The student increases/decreases a quantity by a fixed amount in a volume context. (perform ±a)	Action +/- with a fixed amount
Lecture 15	1. The student uses +/- to make equal in part-whole context 2. The student uses an ordering strategy to deduce the third relation for a transitivity sequence (make-up for transitivity)	Process ±a Action transitivity Object ordering
Lecture 16	1. The student determines the addition amount to make equality. 2. The student finds unknown in an equality with one side addition. 3. The student recognizes multiple solutions to equations with two unknowns (a=?+?)	Action Equation with one-side addition
Lecture 17	1. The student finds unknown inequality with two side addition in height context. 2. The student finds multiple solutions to equations with two unknowns. 3. The student adds equal amounts to both sides to preserve equality.	Action multiple solutions

Table 4.3 (continued)

Lecture 18	1. The student determines addition and subtraction amounts to make them equal (in a continuous context). 2. The student uses both addition and subtraction to manipulate both sides of an inequality to make it equal. 2. The student experiments and recognizes the equality of addition and subtraction amounts.	Process addition/ subtraction amount Object Addition/ subtraction
Lecture 19	1. The student determines quantities of objects based on equality and inequality relations in a weight context. 2. The student reads and uses algebraic interpretations of equality/inequality relations and equations with one-sided addition in real-life weight context. 3. The student preserves the quantity represented in interpretation to use it in addition equations (relations and equations are connected in a system).	Read the model and realize equality and one-side addition in a system.
Lecture 20	1. The student operates addition and subtraction on equalities based on given expressions such as ±a in volume context 2. The student experiences operational properties on equalities (the starting point is equality). 3. The student realizes and compares algebraic expressions such as a±b in a volume context 4. The student experiences operational properties on equations (starting point changes).	Process operations Object operations Action properties of operations

Mental constructions of learning equations based on the resultant trajectory can be summarized in the following schema (Figure 4.61). In this schema, learning concepts, which are keystones for learning equations, are interpreted in rectangles. Arrows indicate the use of the learning concept in the new mathematical concept. Mental constructions noted on the arrows represent the type of mental construction that takes place in the relation between two learning topics. For example, one learning topic may be used as a process in a new mathematical action and as an object in another one. Some processes are composed in new actions. From this schema, it can be hierarchically deduced, which topics take place with what kind of mental constructions in a new learning action.

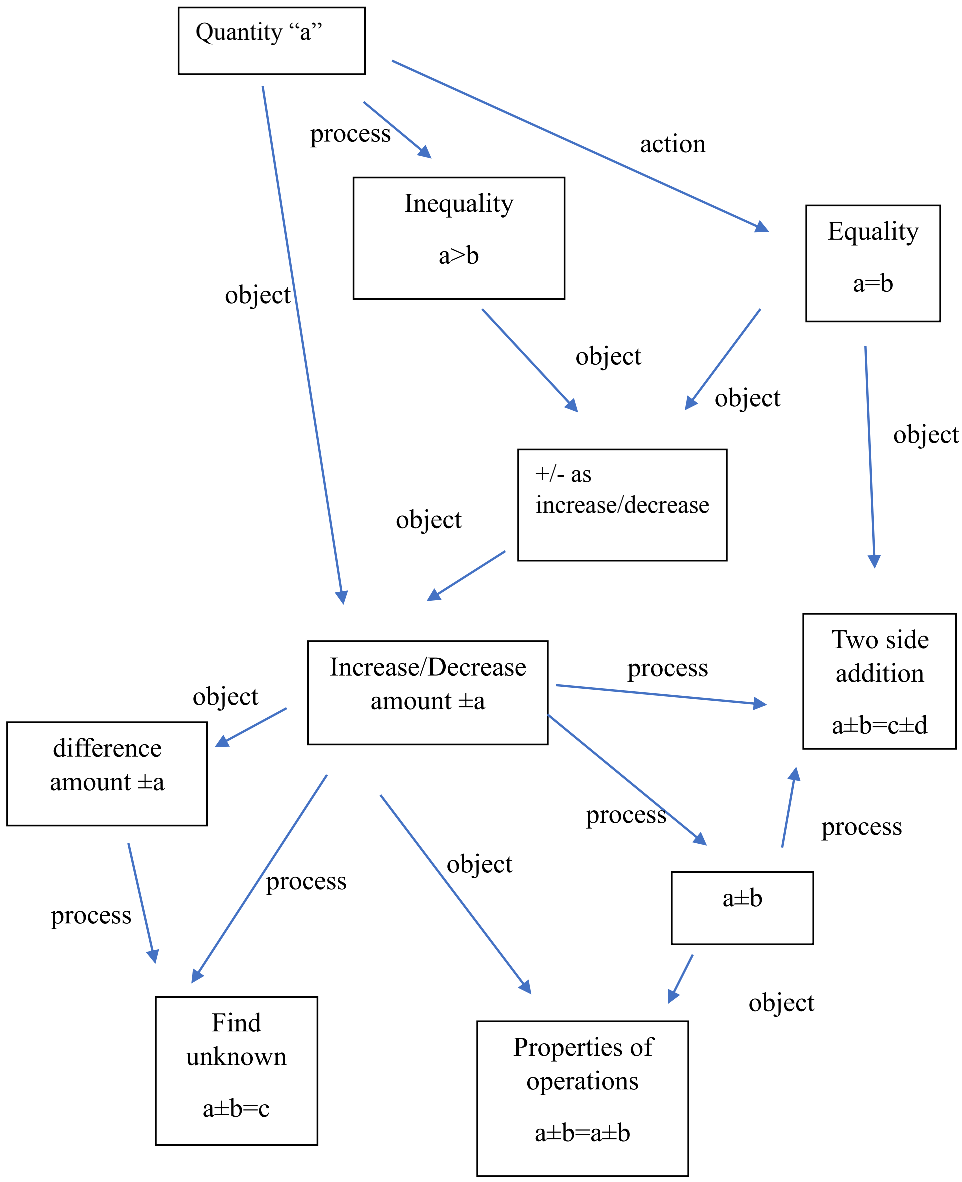


Figure 4.61. Schema of Mental Constructions for Equations Based on the Resultant Trajectory

4.22 Interview Results

In this section, pre-interview and post-interview results will be documented and discussed together with the implementation findings. Firstly, pre-interview and post-interview have common and different items as mentioned in the methods chapter. As trajectory developed additional items were added to observe especially learning in the last lecture. Findings are presented in terms of students' APOS levels on particular algebraic topics. These interviews are conducted for triangulation of data. Both interviews and implementation data are analyzed qualitatively, and consistency between them will be evaluated and presented qualitatively.

The results of the pre-interview for each student in each item are summarized in the following table. Their prior knowledge of the concepts: of quantity, equality, comparisons, weight, width, height, increase, decrease, addition, and subtraction was exposed and used for the design and revision of the activities. Pre-interview was conducted for three main purposes: to explore students' progress after implementations compared to post-interview results individually, to bring out what prior knowledge they bring to classroom implementations and what implementation put on it, to design classroom instruction based on their prior knowledge or real-life experience about the concepts.

9 students out of 10 attended pre-interviews. Students had little or no knowledge of signs. Some know the names of +, and - signs (3 out of 9). Only one student knew the name of the equal sign, two others knew its use as a "write result of operation" comment. They knew size comparison, using equal, same, and bigger words. They know height, weight, and width; while some may have difficulty with the word "width". Pre-investigations on Piagetian conservation showed that they had no understanding of the conservation of amount. Hence, volume was not questioned in the pre-interview.

They had a good understanding of increase and decrease, which we could base our instruction on. Some of the students (1 out of 4) who knew the names of +/- signs, had difficulty matching these signs with increase and decrease. Some (2 out of 9) students had difficulty with the words increase and decrease, they instead used “get more” or “become less”.

Some students (2 out of 9) had a good understanding of difference amount and could interpret it in subtraction and addition concepts. Most of them (5 out of 9) have a good understanding of the comparison of weight context in the “children's balance” item. Some of the students (2 out of 5) had some problems with subtraction to make a balance in weight even though they had no difficulty in addition.

Considerations on designing instruction based on pre-interview results are presented in the Methodology Chapter. In this chapter, we present the pre-interview results to explain learning progress. Each student's progress matters individually, which is a part of the analysis.

Briefly, pre-interview results interpret students’ real-life knowledge of the topics. First of all, pre-interview results are used to build algebraic learning on their prior real-life knowledge in the design of activities. Items in pre-interview are not directed algebraically, not in written or verbal form. The researcher used daily-life terms to observe their experience in real life about quantity and balance. Post-interview items are directed in algebraic forms. Pre-interview shows that students do not know about signs, and their names, depend on height or size to determine bigger objects, and know balance in weight context but have difficulties maintaining balance. They have no experience with addition and subtraction not in written form.

Table 4.4 Pre-interview Results

	Name of =,≠,>,<,+,-	**=, ≠, >, +,-**	**“a”**	**+/-, +/-a**	**a=b±c Difference amount**	**a+b=c+d weight**
Eylem	+,- signs	For only incr&dec	width	Verbal and sign	Draw 5 guesses, total is more, add – (based on count)	Heavier is down. compare and balance, add and subtract to make a balanced, no algebraic expression
Didem	Only + sign	-	Knows width	verbal	Draw after counting, total is more, add +	Learned which is heavier balance by adding and subtracting
Ekim	-	-	No width	Verbal by help	Draw, total is more, add -	Confused about real-life case heavier is down
Aylin	+,-, (=: no name) signs	For only incr&dec	Width, consider volume for deciding on a bigger	Verbal and sign	Draw 9 after counting, if the total is more, add +	Balance and compare balance by adding and subtracting.
Medine	-	-	Knows width	verbal	Draw 1, total is more, add -	Heavier is down. balance by adding, wrong for positioning
Ufuk	+, -, =	For inc/dec	No width	Verbal and signs	-no sense Arithmetic, equality no balance but execution, inquiry with some help	Balance and compare balance by addition. No algebraic expression

Table 4.4 (continued)

Bekir	+,-, (=: no name) signs	+	width	Verbal but no sign even knows the names of the signs	Number guess draw, total is more, add -	How to balance one by one, and by adding
Hasan	-	-	Width, Takes more (volume) confused	Confused verbal	Draw by guess, total is more, add + but because both are much.	Confused about weight, knows heavier but position wrongly
Yaman absent						
Ali	-	-	Width	verbal	Draws by guess, total is more, add +	Balance and compare balance by adding and subtracting

When compared to post-interview results, they upgraded and associated their real-life knowledge of balance, size, and increase/decrease actions to equality, quantity, and operations with their algebraic representations. Multi-solutional cases, finding unknowns in equations, and discussion of properties of operations on equality are other particular concepts they improved. Post-interview also reveals some major difficulties; symbolic representations and transfer of knowledge in the new context. When they were asked about construction in weight context in symbolic representation, the majority had difficulty reading and realizing algebraic expressions in real-life situations. However, they performed better in height context, which they are reluctant to from lectures on this topic, and successfully interpreted the addition of quantities becoming equal to other quantities in verbal algebraic representations. Their success outlined strengths of instructional sequence, while their difficulties revealed areas to support through revision and call for further investigation of teaching mentioned topics at this grade.

Table 4.5 Post-interview Results

	Name of: ≠,>,<,+,-	P: =, ≠, >, +,-	P: “a”	P: transitivity	P: +/-, +/-a	P: a+b=c+d weight and height	P: properties of +a	P: a=b±c O: Difference	P =,< A: multi-solution	O: ±a P: Properties of ±a O: a±b
Eylem	+	+	+	+	+	+	+	+	+	+,+,+,+,+,+
Didem	+/-	+	+	-	+	focus 3 quantity verbal	+	-	+	+,+,+,-
Ekim	+	+	+	-	+	-	-	-	+	+,+,+,+
Aylin	+	+	+	+	+	+ and reverse +	+	+	+	+,+,+,+,+,+
Medine	+	+	+	+	+	-	+	-	+	+,+,+,+
Ufuk	+	+	+	+	+	height + weight - verbal	-	+	+	+,+,+,+,+
Bekir	+	+	+	+	+	verbal	+	-	+	+,+,+,+
Hasan	-	+	+	+	+	-/-	+	-	+	+,+,+,-
Yaman absent										
Ali	+/- No +/-	+/- No +/-	+/- Weight -	-	verbal	-/-	-	-	+	+,+,+,+

These findings are consistent with the implementation findings. Students also had difficulty in using templates, constructing and reading expressions, and transferring knowledge in new contexts in lectures. Particular reasons and solutions are explained in each lecture in detail. Findings on post-interview are compared to students’ individual development through topics for triangulation to validate our inferences out of implementation by students' understanding levels on post-interview.

Table 4.6 Students' Individual Development in Implementation Through Lectures

Lecture #	**Ali**	**Yaman**	**Hasan**	**Bekir**	**Ufuk**	**Medine**	**Aylin**	**Ekim**	**Didem**	**Eylem**	**APOS**
1	+	+	+	+	+	+	+	+	+	+	A: =, ≠ A: "a"
2	+	+	+	+	+	+	+	+	+	+	P: =, ≠ P: +/-
3	+	+	+	+	+	+	+	+			P: =, ≠, P: "a" A: iconic notation
4	+	+	+	+	+	-		-	+	+	A: >, < P: =, ≠
5	+	+	+	+	+	- +	+	- +	+	+	P: >, <, = A: symbolic notation
6	+	+	+ -	+	+ -	-	+	+	+	+	P: >, <, = P: symbolic notation
7		+		+	+	+	+	+	+	+	A: ordering
8	-	+	+	+	+	-	+	-	-	-	A: transitivity
9	+	+	+	+	+		+	+ -	+	+	A: mid-value O: transitivity
10	- +	+ +	+ -	- +	+ +	- +	+ +	- -	+ +	+ +	P: mid-value A: notation
11		+	+	+	-	+	+	+	-	-	A: multi-solution
12	+	-		+	+	+	+	+	+	+	O: =, ≠ A: +, -
13	+	+	+	+	+	+		+	+	+	A: +, - P-A: +/-a
14		+		+	+	+	+	+	+	+	A: +/- a
15	+	+	+	+	+	-	+	- +	+	+	P: +/- a A: transitivity O: ordering

16	+ -	+	+	+	+		+	+	+	+	A: a=b+c P: multi-solution

Table 4.6 (continued)

17	+ -	+	+ -	+	+	+	+	+		+	A: a+b=c+d A: multi-solution
18	+		+	+ -	+ +	+ +	+ +	+ -	+ -	+ -	P: +/-a O: +/-a difference amount
19	+		-	+	+	-	+		+	+	O: a>b O: a+b=c
20	+	+	+	+	+	+	+	+	+	+	O: +/-a A: properties of +/-a

Post-test results (see Table 4.5) reveal students' mental construction schemas on equations. Students' individual developments through classroom implementation (see Table 4.6) are compared with these schemas. Further, development on each topic is evaluated with the results of related lectures. Students' individual APOS levels in each lecture are given in the table relative to the APOS level assigned for the lecture. "+" means the student achieved at least a determined level, while "-"means the student had difficulty in that topic and his/her APOS level is below the majority in class. Blank spaces indicate the student's absence in that lecture. Students' individual development, and classroom implementation results are consistent with post-interview results, except for some differences. For example, Hasan's schema based on post-interview results coincides with his learning trajectory through lectures (see Figure 4.62). He had difficulty in two-sided addition, difference amount, and subtraction of different amounts in properties both in the post-interview and during classroom implementations.

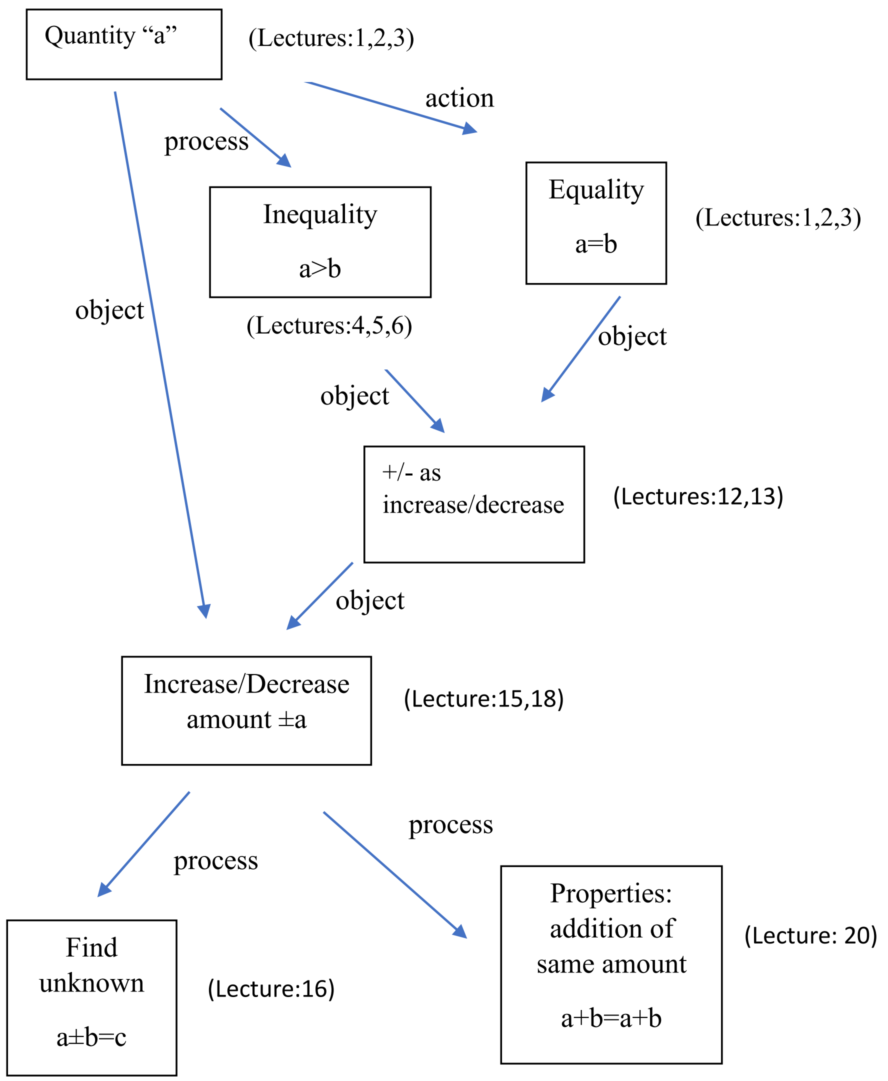


Figure 4.62. Hasan's Schema on Equations Based on Post-interview Results Matched with His Learning Trajectory Through Lectures

Some inconsistencies appeared between the schemas based on post-interview results and individual progression in lectures. One reason for the inconsistencies might be that post-interview is limited to time and items based on students' attention. Some items had implicit learning objectives, creating inconsistency. Coding as "+" or "-" in items of post-interview may not correlate with individual progress in the classroom. Hence, results are compared qualitatively and not restricted to the result tables. Inconsistencies are investigated for the possible reasons. For example, for the difference amount concept, only Aylin, Ufuk, and Medine successfully reasoned by difference amount to construct equations of addition and subtraction in Lecture 18. However, post-interview results showed that Eylem, Aylin, and Ufuk successfully constructed equations with addition and subtraction symbolically reasoning by difference amount. Eylem seemed to understand guidance about the difference amount in the lecture and showed her understanding in her response for the post-interview item. Medine's difficulty was because of her not attending lectures about the construction of equations with one-side and two-side addition. Her inability in symbolic representations continued in further lectures and other post-interview items. She could not command addition on two sides for weight and height context. Except for Medine's difficulty, the results of this item correlate with the related lecture. This explains why this difference does not indicate an inconsistency but Medine's difficulty in symbolic representations.

Another problem in post-interview results in the item about the construction of equality in weight and height context. Students had difficulty in weight context. Hence, the researcher directed the item in the height context as it appeared in Lecture 17. Students could verbally interpret the addition of items and their equality, but some (Ali, Hasan, Ufuk, Bekir, Didem) had difficulty reading and depending on symbolic representations. Symbolic representation was overshadowed in this lecture while investigating unknowns. Ufuk and Bekir had difficulty in symbolic representation but could verbally interpret equality with four unknowns. (Inconsistency between their post-interview and classroom implementation results does not occur among their schema of understanding equations but originates from their inability to interpret algebraically in different modes.) Especially Ali and Hasan could not catch up with the classroom in Lecture 17, and they both could not reply to this item. Didem could reply item using three unknowns representing the equation as one side addition. She was absent in Lecture 17 but attended Lecture 16 on equations with one side addition in height context.

A remarkable consistency is between the results of the last post-interview item and Lecture 20. Students (Eylem, Aylin) who had an object-level understanding of constructs in Lecture 20 could correctly answer all questions in the last post-interview item. Other students could interpret properties based on the addition/subtraction amount. However, they could not reason by the starting amount in the last post-interview item, consistent with their Lecture 20 results.

Briefly, students' individual development is consistent with their success in post-interviews, which proves our inferences from both data validates each other.

CHAPTER 5

CONCLUSION AND DISCUSSION

The main purpose of this study was to develop a learning trajectory for teaching equations from an algebraic perspective before arithmetic education through cyclic implementation, analysis, and revisions of designed activities. Furthermore, the purpose of the study included a detailed analysis to explain how students' understanding of algebraic concepts developed, which was interpreted using APOS Theory. Additionally, the study aimed to assess why certain activities supported the resultant trajectory, leading to the generation of design principles.

This chapter discusses the results of the study through the relevant body of literature under four major topics. In the first part, the research questions will be addressed by summarizing the developed trajectory and common design principles, reflecting on both the theoretical and practical outcomes of the study. Secondly, the theoretical contributions to early algebra education and APOS Theory will be discussed. In the third section, the practical implications of the study will be provided. Finally, suggestions for further revisions and studies will be given.

5.1 Outcomes of the Study: Discussion of Findings

Advancing a design-based research perspective, this study yielded two key outcomes: a theoretically grounded learning trajectory and practical design principles for activities supporting this trajectory. This section will address the research questions by presenting these outcomes and reflecting on their implications for the literature. Research questions guiding this study were:

General Research question: Based on Davydov's approach, how can kindergarten students' algebraic understanding of equations be effectively supported before they receive arithmetic education?

1. What is an adapted learning trajectory supporting kindergarten students' algebraic understanding of equations from Davydov’s non-numerical perspective?
    a) To what extent do kindergarten students learn equations with addition and subtraction with an adaptation of Davydov’s curriculum for first graders?
    b) What are kindergarten students' strengths and difficulties in understanding the equations in the adapted trajectory?
2. What are the effective and practical activities for supporting kindergarten students' algebraic understanding of equations from Davydov’s non-numerical perspective?
    a) Which characteristics of the activities help kindergarten students understand and resolve their difficulties in comprehending equations?

The results of this study revealed that by adapting symbolization and contexts in Davydov’s learning trajectory, kindergarten students' algebraic understanding can be supported through investigations with concrete manipulatives in balance contexts (General research question). By simplifying formal symbolization from letter notation to pictorial and color-coded notation, and modifying the investigations related to the concepts of area and volume, Davydov’s trajectory has been successfully adapted to the kindergarten level (Research question #1).

With this adaptation, kindergarten students can develop an algebraic understanding of equality, quantities, and addition/subtraction operations on both sides of the equalities (Research question #1a). While students faced difficulties in reasoning by transitivity, they developed a strong understanding of equality and operations acting on equalities with non-numerical quantities (Research question #1b).

Enactive investigations of equalities and operations with balance manipulatives in weight, height, length, and volume context helped students understand algebraic concepts of equality, quantity, and addition/subtraction. Reporting enactive representations first iconically, then algebraically, helped them connect and model investigations with concrete manipulatives to algebraic expressions (Research question #2). Free explorations with manipulatives, the use of various objects in investigations, and a focus on different attributes helped students quantify and understand equality and operations algebraically across several contexts (Research question #2a).

These conclusions of the study will be detailed and discussed within the related literature in the following sections. Section 5.1.1 will address the first research question by presenting the resultant trajectory in each domain—equality, quantity, operations, and notation—to explain the extent to which students learned algebraic concepts. Section 5.1.2 will address the second research question by outlining the design principles and characteristics of the activities that supported the algebraic understanding of kindergarten students.

5.1.1 Theoretical Outcome: Resultant Trajectory

As one of the outcomes, this study generated an evidence-based, effective trajectory for teaching equations with addition and subtraction at the kindergarten level from an algebra-before arithmetic perspective. We followed Davydov's trajectory to design the first hypothetical learning trajectory in teaching equations to kindergarten students from an algebraic perspective. Adaptation continued through revisions of the hypothetical learning trajectory until it evolved into a resultant learning trajectory. In the adaptation procedure, the trajectory did not change a lot for teaching quantity, equality, and operations. See Table 4.2 for a summary of the adaptations and changes made to HLT, comparing Davydov's original trajectory with the resultant trajectory developed in the study. The only change in the order involved moving the instruction on equations with two-side addition to precede the teaching of difference-amount in equations with one-side addition. In the “using scale” subject, a square activity was added to Davydov’s trajectory to aid in the notational interpretation of equivalent quantities. Revision from Davydov’s to the first HLT, and subsequently to the resultant trajectory, primarily involved simplifications rather than exclusions or reordering of the topics. These simplifications were both contextual and notational. Contextual simplifications were made based on students’ developmental levels concerning Piaget’s conservation of quantity. The area context was removed and the volume context was turned into height. Letter notations were reduced to pictorial representations of quantities due to students’ illiteracy and their difficulty in representing objects with outer icons.

Adapting Davydov’s trajectory without sacrificing the teaching of equality, quantity, and addition and subtraction operations, the resultant trajectory was successfully adopted to the kindergarten level. The effectiveness of this adaptation was proven through the evaluation of APOS levels pre-determined for both Davydov’s trajectory and the resultant trajectory.

Not only activities and dedicated time for the subjects but also objectives were revised based on students’ learning. Hence, the resultant trajectory was evidenced based on students’ indicator behaviors through each learning step, as documented in the findings chapter. In other words, students’ learning progression created the trajectory, allowing us to interpret this trajectory as students’ achievement throughout the topics. In this section, using the resultant trajectory, we will reflect on the students’ achievements in the learning domains of quantity, equality, operations, and notation with related literature.

5.1.1.1 Resultant Trajectory on Quantity

One of the components in the equations is variable/quantity. Quantities are defined by Smith and Thompson (2007) as “attributes of objects or phenomena that are measurable; it is our capacity to measure them—whether we have carried out those measurements or not that makes them quantities" (p. 101). In Davydov's perspective, non-numerical quantities are used in equations. Numbers are constructed with ratios of quantities. Until numbers are presented, quantities are compared, composing an equality/inequality relation. Numbers are represented as multiplicative relations between quantities, which is out of the scope of this study.

Teaching variables in equations begins with teaching how to compare different attributes correctly in Davydov's trajectory. Those attributes include area, weight, volume, and length, all of which can be measured in quantities (Ellis, 2011). Comparison of objects based on area and volume would cause problems because students don't have conservation of quantity. We started by interpreting equality between objects based on different attributes students already knew. The area attribute was omitted, while volume was shifted into height with the help of manipulatives. Objects have different attributes, and quantities of objects can be compared based on these attributes. Comparing attributes and interpreting equality between objects based on several attributes did not guarantee understanding of quantities. Some students reasoned by quantities to interpret equalities, while for some of them, attributes did not define quantity in comparisons, but remained as different aspects of objects. They simply used measurement tools based on the attribute to decide on equality without mentioning which quantity was greater. They continued interpreting equality between objects regarding non-quantitative aspects of them (i.e. similar shape or parts). Quantity was acquired for all students when >, < signs were used to interpret inequalities, leading them to understand that objects conserve a quantity in the comparisons based on the attributes of consideration. The use of >, < signs underlined one quantity is greater than the other, enforcing the idea that equality is associated with quantities being in equal amounts.

This finding showed that teaching >, < signs was essential in the early stages of teaching quantity from a non-numerical perspective. After this stage, comparison shifted from being between objects to between the quantities associated with those objects. This idea was strengthened by comparing the same objects based on different attributes. Different attributes created different contexts for quantities and helped quantity become an object independent from being specific to objects. Abstraction of quantity away from objects was an important and non-trivial stage. Using different attributes as various contexts, and applying greater/less than relations helped the quantification of compared objects. As stated in design principles, presenting different objects with equal quantities in a specific attribute, similar-looking objects with different quantities, or discussing equality based on different attributes of compared objects all helped the quantification process. Iconic representation of comparison results, based on implicit attributes such as volume and weight, revealed students' understanding of the quantity abstracted from the objects. This was evidenced by their use of icons to represent quantity.

After quantity was acquired, students used this knowledge for comparisons and interpreted equality/inequality based on different attributes. The second stage involved constructing quantities based on given relations, essentially reversing the process used to determine equality between two quantities. In this stage, the quantity was treated as unknown in the given relation, and students either found or constructed a suitable object to meet the requirements of the relation.

Finding unknown quantities continued in the equations with addition or subtraction operations. Before that, students increased or decreased quantities by addition or subtraction to achieve equality. This process of adjusting quantities turned into finding the added or subtracted amounts as unknowns in equations. These adjustments were applied to continuous variables such as length, height, volume, and weight (Stavy & Babai, 2016). Some activities required continuous manipulation, while others involved working with fixed quantities to find unknowns in equations. Using continuous variables in comparisons and operations was promising for enhancing students' future understanding of continuous variables as opposed to discrete ones. Therefore, Davydov's perspective not only reorders arithmetic and algebra but also shifts from discrete to continuous variables before discrete numerical or countable comparisons. Boote and Boote (2017) discuss the difficulty of transitioning from discrete to continuous variables as a "learning leap" (p. 456) where students need to understand both the similarities and differences between these two types of variables. They call for a better-organized transition starting earlier. Davydov's inclusive and deductive perspective has the potential to solve this transition difficulty, as it shifts the transition from arithmetic to algebra.

Creating quantities based on given relations naturally resulted in an infinite number of possible answers in both Davydov's and our trajectories. In addition to Davydov's trajectory, we stressed the concept of multiple solutions. We developed a trajectory that gradually introduced students to recognizing and generating multiple answers for finding unknowns in equations. Firstly, students learned to extend sequences of inequalities by systematically adding bigger objects to conceptualize infinity. Secondly, they experienced creating quantities based on given inequalities and equalities, which led them to observe and discuss multiple and single solutions, as well as boundaries in these solutions. Finally, in equations involving two-sided addition, students recognized and generated multiple solutions for finding unknowns by manipulating fixed quantities. The use of combinations of fixed amounts enabled them to come up with several answers. Moreover, they developed an understanding of quantities as sums of two other quantities, leading to the formulation of equivalence sets composed of equivalent sums.

The following is the resultant trajectory of quantity in this study:

- Comparison between objects for equality as similarity
- Comparison of different attributes of objects for equality
- Quantity as a comparison measure/consideration for inequalities (by use of >, < signs)
- Conservation of amount/quantity in different modes of representation (represent quantity by an icon or symbol)
- Determine the attribute based on the given relation
- Construct quantity based on a given relation. (reverse-process) quantity as unknown

- Recognize multiple solutions to construct unknowns in inequality relations
- Interpret equivalent quantities with the same notation.
- Find the unknown quantity as an increase/decrease amount
- Find multiple solutions for equations with addition on two sides: quantities can be added to construct another quantity (quantity operated on), and equivalent sums: equivalent sets of sums

The last step in developing a learning trajectory for the concept of quantity involved extending Davydov's trajectory. Davydov discusses "equal, unequal, equal again" by adding a quantity on one side of equality to break equality and then adding another equal quantity on the other side to make it equal again. In this study, students did not add an equal amount but used an equivalent sum to make it equal again, which allowed them to determine all equivalent sums and use them as substitutions of each other.

Blanton et al. (2017) studied the trajectory of variables for 1st graders, and Ventura et al. (2021) revised this trajectory to include both first graders and kindergarteners. When comparing these trajectories to our learning trajectory for quantity, students began at Level 2 where letters are used as labels or to represent objects that correspond to the first stage in our trajectory They then progressed to Level 3, where letters are understood as representing variables with fixed deterministic values. The difference is while they used letter notation, we used pictures as symbols. By color coding, quantity is interpreted with color notation and abstracted as being representative of an equal quantity set. Based on Ventura's study, students may be willing to change or not the letter notation of a quantity when they are operated on, or represent different quantities with the same notation at Level 5. In Davydov's perspective, every object has a different letter notation even if they are equal in quantity, which is also represented by using different symbolic pictures for each object. However, by using color coding, we abstracted quantity and defined a representation/notation independent from the object itself. At the last stage, students in this study could operate with unknowns which potentially contributed to Ventura's 6th level, where no kindergarteners achieved this stage in their study.

These comparisons are based on the use of quantity and representation of quantity. However, Blanton et al.'s (2017) and Ventura et al.'s (2021) studies focused on teaching variables, with all assessment questions including varying unknowns. Our objectives and achievements concerning variability were limited to recognizing, finding, and iterating multiple solutions in activities involving finding or creating unknown activities. The selected contexts were continuous, which helped in understanding variability. In addition, in studies of Blanton et al. (2017) and Ventura et al. (2021), stages of knowledge of quantity were defined through the use of letter notation, while we disintegrated notation to explain the understanding of quantity more comprehensively. We evaluated verbal, enactive, and iconic representations in addition to symbolic representations as algebraic expressions of quantity. Furthermore, in this study, students investigated continuous non-numerical quantities, whereas in Blanton et al.'s (2017) and Ventura et al.'s (2021) studies, unknowns were discrete and countable objects. This difference makes the comparison of trajectories even more difficult in terms of evaluating students' learning and development.

5.1.1.2 Resultant Trajectory on Equality

For students, equality initially meant being the same. They focused on finding identical toys or interpreting geometric similarities between some parts of the objects they compared. The equal sign was presented with its reverse, the unequal sign as in Davydov's trajectory. Students engaged with different contexts, such as volume weight, and part-whole relationships. However, for some students, equality continued to mean being the same or similar until greater/less than relationship was introduced. They did not consider the quantity being compared but used the measurement tools to decide whether it was equal or not. After being taught the >, < signs, these students still tended to use unequal signs instead. This tendency might be due to the difficulty of contexts such as weight and volume, where interpreting which one is bigger posed a challenge. The “>” and “<” signs enforced reasoning by quantity by clearly determining the bigger side. This finding is consistent with the experimental study by Hattikudur and Alibali (2010), which indicated that including inequality signs in the instructional sequence helped students understand equality in a relational structure more effectively than an instructional when compared to the instructional sequence including only equal signs. Deciding which side is bigger likely helped students see the sides of the relationship as algebraic constructs, thereby creating a balance relation between them.

In this study, students began learning about equality through balance relations in each context. As a result, they developed a relational understanding of the equal signs. Some students progressed to a “relational-computational” understanding, while others developed a “relational-structural” reasoning (Stephens et al., 2013, p. 174). Asking about the properties of adding and subtracting equal and different amounts to equalities helped students reflect on the structures of addition and subtraction by amount. This reflection facilitated a deeper understanding of the algebraic structures underlying these operations.

For inequality situations, only students who had a mature understanding of the conservation of amount could reflect on the structures such as a±b and compare them algebraically without needing to test operationally. Stephans et al. (2013) suggested tasks supporting relational, structural understanding of equations in arithmetic for early algebra education. Operational understanding of equal sign can persist and cause problems when solving algebraic problems, even in college grades (Fyfe et al., 2020). Non-relational (operational) understanding of equal sign can be classified as arithmetic-specific or non-arithmetic-specific, with arithmetic-specific interpretations posing a greater obstacle to early algebraic learning (Byrd et al., 2015)

This study eliminated the hindrances of arithmetic from the beginning and taught equality in a relational way. Direction was never presented, and it did not matter in equations for students, who could operate on both sides of equations algebraically.

Misconceptions of solving from left to right or viewing equality as a "solve it" response did not occur in our students' behaviors. This outcome resulted naturally from using manipulatives that emphasized balance and allowed manipulations on both sides. Students verbally and algebraically interpreted equality between them.

This finding conforms to a recent study by Lee and Pang (2023), who pointed out the difficulty of building a relational understanding of equality and how pan balances helped in achieving this understanding. Lee and Pang (2023) defined simultaneous operational and relational (SOR) conceptions of equal sign. They found that even when students possess a relational understanding of the equal sign, they often revert to an operational conception when faced with an unfamiliar equation. Improving students' conception of the equal sign was difficult until they engaged with a pan balance. This tool helped students understand the quantitative balance between sides, leading to a more stable relational understanding. The "$a \pm b = c$" form of equations hinders relational understanding in the "$a \pm b = c \pm d = e$" type of equations (Lee & Pang, 2023). To address this, we started with pan balance and focused on operations that change the balance or imbalance situations. Students consistently operated on equality or performed operations to construct "$a \pm b = c \pm d$" types of equations and worked with inequalities to construct "$a \pm b = c$" types. Consequently, students developed a relational understanding of equal sign in equations.

Moreover, equality or inequality relations became algebraic objects acted/operated on by unknowns, which is a major difficulty even for high school students, causing a cognitive gap between arithmetic and algebra (Linchevski & Herscovics, 1996). In this study, it was not difficult for students to identify equal signs as algebraic objects. They constructed equality relations by combining equality and quantity processes and then operated and reflected on these relations to grasp how to change and preserve equality.

Furthermore, students operated by equivalences of quantities on equations. Understanding equality and equivalent algebraic expressions as substitutes is important for solving algebraic problems (Nicaud et al., 2004). Students might have difficulties in differentiating equality and equivalence in algebraic expressions, often reasoning by particular numerical equality rather than by equivalence of algebraic expressions (Saldanha & Kieran, 2005). By using a non-numerical approach in this study, the problem of numerical reasoning can be prevented. The students could reason by the equivalence of sums of unknown quantities to maintain equality. They can use these sums as substitutes for each other in equations. In other words, if $e = a + b = c + d$, then they can conclude $f + e = f + a + b = f + c + d$ (See "equal, unequal, equal again" activity in Lecture-17).

The trajectory for equality in this study can be summarized as:

- Equal sign and unequal sign for the determination of similarity
- Equal sign and unequal sign for the determination of equality based on different attributes
- Greater/less than signs related to an unequal sign for the determination of the quantitative relation between sides in inequalities.
- Operating on inequalities by addition and subtraction to achieve equality: one-side operational equation
- Operating on equalities to make them unequal, and then equal again by addition: additional properties on equations
- Operating on equations by equivalence sets/relations to maintain equality.
- Operating by addition and subtraction on two sides of equalities and inequalities: properties of operations

Briefly, the equality concept evolved by associating inequality with greater/less than relations, which evoked a quantitative understanding. Then, equality and quantity concepts became objects of equality relations ($a \leq b$). These relations are acted/operated on by addition and subtraction. See Figure 4.61 for the resultant schema on equations.

This trajectory supporting the relational understanding is found to be effective in preventing misconceptions about the meaning of the equal sign. Students developed a relational understanding of equal signs, and they had no misconception of perceiving the equal sign as “solve for”. Research studies have shown that misconceptions regarding the equal sign are prevalent even at higher grade levels. (Falkner et al., 1999; Gürel & Okur, 2018; Kieran, 1981; Stephens et al. 2013)

Studies suggest that introducing non-traditional formation of tasks, such as operations on the right side of the equation before formal algebra education, can help address these misconceptions (McNeil et al., 2011; Stephens et al., 2013). Even kindergarten students may have an operational view of the equal sign and carry forward misconceptions from their early exposure to equality in solving simple addition problems (Falkner et al., 1999). McNeil et al. (2011) suggest that exposing students to the equal sign outside of the arithmetical context can help counteract their resistant, unidirectional operational meaning of the equal sign. Hence, this study’s intervention successfully prevents such misconceptions by grounding the concept of equality in a non-arithmetical context and by employing non-traditional operations on both sides of the equation from the very beginning.

5.1.1.3 Resultant Trajectory on Operations

The trajectory for operations starts by acting on equality. As in Davydov’s trajectory, increasing and decreasing actions start with “how to make equal” discussions. These discussions do not start with the introduction of +/- signs, but much earlier, almost as early as the equality concept. In these discussions, students engage in actions of increasing and decreasing amounts without yet interpreting these actions explicitly. Students learn they can use both types of actions to achieve equality. They learn to use both types of actions to achieve equality, understanding them as tools for balancing sides rather than as formal arithmetic operations. The first introduction of +/- signs occurs in the activities where students assign the correct signs to sides of inequalities to make them equal to the other side. Therefore, plus and minus signs are presented simultaneously for the given situation, associated with the given increase/decrease actions both of which can be applied. Davydov presents this association with before and after situations and asks assignment of signs based on the actions taken to make sides equal. In this study, with the help of continuously manipulable manipulatives, students enactively investigated increase/decrease actions (as in other activities, enactive investigations dominate). After determining these actions to make sides equal, and assigning signs associated with actions, students enacted the increase/decrease actions. At this stage, how much to increase/decrease is enacted but not interpreted explicitly yet.

Due to measurement errors, the concept of finding the unknown was almost impossible to discover/investigate in weight and volume contexts. Prepared quantities would not result in expected equations on students' perception. Moreover, learning signs and associating them with actions is somewhat complicated for the kindergarten level. Therefore, we added an extra step to strengthen positive and negative actions associated with the plus and minus signs. This involved an activity where students moved backward and forward by different animal steps, initiating an increase/decrease action by a predetermined amount independent of equality. These increase/decrease actions by an amount are revisited in volume context, stressing the quantity in the expression $\pm a$. While $\pm a$ is always discussed in the equation "$b \pm a = c$" in Davydov's trajectory, we wanted to objectify the $\pm a$ statement on its own. We also wanted to make students understand these actions by amount properly, because they had to think about two things simultaneously: equality and the actions in the "how to make equal" discussion. In this way, $\pm a$ is investigated in the forward process rather than being reverse-processed in finding $\pm a$ as unknowns in equations, in making equal processes. Racing activities in volume and length contexts using $\pm a$ constructs helped students compare two constructs, and reflect on which is greater, or advantageous in an informal but effective way. This enactive/informal investigation was more structured in the final activity in the volume context, where students observed the effects on equality situations.

After acting by $\pm a$ constructs, finding unknowns in equations with one-side addition and two-side addition is studied in height context (animal height game). Students found addition amount to make two unequal quantities equal to each other. Hence, addition and addition amount "a" are handled in equation "$b = a + c$" as unknown. The form of the equation does not matter, facilitated by the height context and templates used. In equations with two-sided addition, students used the property of adding two equivalent quantities on two sides to make them unequal and equal again.

After finding unknowns in the equations with addition, the concept of difference amount was individually taught to students by constructing equations with one-side addition and subtraction in a length context. Firstly, students graphed the plant heights by strings at three different times. They observed and interpreted changes and increase amounts on the graphs. The change in a quantity (height of plant) over time was observed, and the amount of increase was verbally and enactively interpreted. Then, they constructed equations by interpreting the increase and also decrease amounts to make two strings (representing plant height) equal to each other. They were asked about the equality of increase and decrease amounts to understand the difference in amount.

In finding the increase or decrease amount to achieve equality, students engaged in actions that treat $\pm a$ as a process. Rather than simply performing $\pm a$, they reverse-processed it; in other words, they identified the $\pm a$ when processed forward, would achieve equality. In the last activity, adding or subtracting equal amounts on the sides of the equalities was discussed. In that activity, before enactively testing and seeing the results students reflected on addition and subtraction of fixed amounts by comparing $\pm a$ to $\pm b$ constructs, where the relation between a and b is known. Objectifying $\pm a$ on equalities was not difficult for all students in this activity. However, comparing and reflecting on $a \pm b$ constructs was difficult for those students who had no conservation of amount based on Piaget's test. This lecture on the properties of operations was added later in the trajectory. We thought it would be difficult for this grade level, but we wanted to try it with some simplifications. This lecture helped us to conclude about mental constructions related to students' limitations.

There was another activity introduced before the discussion of properties of operations, which does not align with this trajectory or Davydov's trajectory. We wanted to teach up to equations with addition on two sides and complete trajectory with a modeling activity: the Rainbow Activity. In this activity, secondary colors were given in an equation as the sum of two primary colors. However, this activity does not align with the trajectory; because it represents addition as the joining of two colors (or two pieces of play dough) to compose another color. As a result, students had difficulty to make sense of the equations in this context. It introduces the concept of joining, which does not align with our trajectory's emphasis on addition as an increase in quantity. Traditional teaching often defines addition through this joining model, viewing the binary operation on particular sets (Carraher et al. 2000) and as a set model for addition (LeBlanc, 1976). This model allows the exploration of commutativity and associativity properties in addition, which is reflected naturally in the balance contexts used in this study.

Properties of operations were investigated through their effects on the equality/inequality in our trajectory. There was also a number line model in this study (Animal Steps Game) for addition and subtraction operations, which we wanted to reflect in our activity using +/- signs for initiating forward and backward movement. Davydov's trajectory and our trajectory predominantly support the functional model for addition, because $\pm a$ actions are treated as operations on any initial quantity. Investigating change in a quantity over time and associating these changes with the operations further supports relational functional thinking. The Rainbow Activity and Animal Steps Activity are embedded in our instructional sequence to connect students' operational understanding with the set model and number line model that they will encounter in primary school.

The trajectory for operations in this study can be summarized as:

- Increase or decrease actions to achieve equality

- Determine actions on both sides to achieve equality

- Associate +/- signs with the increase/decrease actions to achieve equality
- Increase and decrease as positive negative direction (number line model: Animal Steps Activity)
- Increase or decrease by an amount
- Find unknowns in equations with one-side addition
- Find unknowns in equations with two-side addition: addition of equal amounts property in reverse-process
- Interpret change in quantity
- Interpret the increase amount
- Discover difference amount through equality of increase/decrease amount
- Use the addition model in real life (set model: Rainbow Activity)
- Properties of operations

Through this trajectory, both continuously manipulable quantities and fixed quantities are operated.

5.1.1.4 Resultant Trajectory on Notation

Activities for the first hypothetical trajectory were designed based on Bruner's modes of algebraic representations. Each concept would be revisited firstly in enactive modes through physical investigations, secondly in iconic modes through paperwork activities, and finally in letter notational symbolic modes of algebraic representations. In this study, we could not achieve in letter notational symbolic mode in any concept, and used photos of objects to symbolize objects for three reasons:

- Students did not know the names of letters and had difficulty distinguishing them

- They had difficulties with iconic representations and pictorial symbolic representations. Through modes of representation, they lost focus on the quantity of the objects based on the attributes but reason by the size of the pictures themselves. Their difficulty in transferring between modes may be a result of their lack of conservation of amount.
- Pictures were also representative of quantities, which may be treated as symbols for quantities. Activities were designed first to be represented through comparison tools as an enactive mode, then as an iconic mode based on an implicit attribute, and finally as a symbolic mode through pictures.

Students' transfer between modes of representation was not as straightforward as we expected. They had difficulty associating pictures with the enactive investigation. We had to use a strategy to associate enactive to iconic, then iconic to symbolic mode through one-to-one correspondence in the following way;

- Compare toys on the measurement tool
- Put a sign on the measurement tool to represent enactive measurement.
- Put the compared toys near the measurement tool
- Move the sign on the measurement tool between toys
- Place pictures near toys
- Move the sign between the toys to between representative pictures

In the symbolic mode of representations in relations and equations, unknowns are represented by blank boxes. These unknowns may be a fixed quantity or a varying quantity. Davydov used illustrations of curtains hiding objects to represent unknowns iconically or letters to represent unknowns symbolically. In Davydov's notation system, every object is assigned a different letter notation, even if they have the same quantity. Moreover, when an object's quantity is increased or decreased through operations, its letter notation is changed to reflect its transformation into a new object. In Davydov's system, the letter notation primarily represents the object itself. In this study, pictures of objects were used to notate objects or fixed quantities related to different attributes of objects. In addition to pictures, color code notation was used in the Squares Activity, where color codes represented quantities through sets of objects rather than the objects themselves.

Notating quantities by letters is challenging for younger students. Brizuela et al. (2015), in an investigation with kindergarten to 2nd-grade students, reported: "We provided evidence that first-grade children, of approximately six years of age, can develop a variety of understandings about variable notation." (p.57) even if they do not claim these students perfectly understood what symbols mean as quantities or variables. Students have difficulties choosing letter notation for specific quantities and interpreting problem situations with these letter notations in algebraic expressions. Ventura et al. (2021) exemplified this difficulty in a first-grade student's work on the Candy-Box Task. A student notated the number of candies in a box with the letter "a". Then, adding two more candies, he notated the new situation as $a + 2 = a$. As the authors denoted, he struggled to assign new letters for quantities in revised situations. However, he might also notate the number of total candies a child might have at two different times with the same letter notation. The authors also noted that some kindergarten students had this emergent use of letter notations for quantities or variables; no one could use these notations correctly/purposefully in algebraic expressions (Ventura et al., 2021).

Khosroshahi and Asghari (2013) showed how a 6-year-old student could succeed in algebraic tasks without using letter notations. Van Amerom (2002) also pointed out the importance of reasoning by algebra over formal symbolizing for bridging the gap between algebra and arithmetic because "algebraic notation is not necessary for algebraic reasoning" as a result of their study with 6 and 7 graders (p. 54). Van Amerom suggested informal symbolization and discovering algebraic strategies for pre-algebra classes. In this study, students showed they could use pictures as informal notations for quantities and successfully used them in algebraic expressions and operations. In addition to Davydov's trajectory, in this study, color-code notation was used to upgrade the symbolization of quantity in Squares Activity (see Findings Chapter, Lecture-10).

5.1.1.5 Resultant Trajectory on Transitivity

By transitivity property, we mean transitivity between dual equality/inequality relations between three objects ($a < b$ & $b < c$ => $a < c$). Trajectory on transitivity started with ordering three or more objects. Then, based on two relations/comparisons between three objects, students were expected to deduce the third relation in a volume context. It was not easy for students to imagine the third relation, moreover, volume context brought measurement errors causing disorders in the volumes of compared cups. Then students learned how to create an equivalent scale to an object and use this scale to compare the object to a distant one. Creating an equivalent scale and using it in the comparison of two distant objects require the use of transitivity property, including one equality relation. After learning how to create and use scales, the students used this understanding to find all members of equivalent sets in Squares Activity. The transitivity property was then revisited in the weight context with more structured enactive investigations. Students struggled to reason using the transitivity property in a volume context. Thus, in the final activity, before students were asked to express the third relations, they were guided to order the three objects based on the dual comparisons. They then deduced the third relation based on the order. However, while this strategy worked for the majority of the students, some of them struggled with ordering and deducing the third relation.

The summary of the trajectory is as follows:

- Ordering objects
- Transitivity in volume context
- Creating and using an equivalent scale to compare distant objects
- Using an equivalent scale to determine elements of equivalence sets
- Transitivity in weight context using ordering strategy.

In Davydov's trajectory, the transitivity property is assumed to be intuitional, and tasks are designed to use symbolic notations for deductions or to create quantities based on the relations and using the transitivity property. However, it required extra effort to guide our students and to make them follow the ordering strategy in the transitivity trajectory. Interpreting “being bigger” based on volume and weight was always difficult for some students. After several attempts and systematic organization, reasoning by transitivity is achieved. However, students’ reasoning by transitivity is based on their enactive observations. We do not think they could reason using pre-given symbolic interpretations of relations to deduce another relation. Based on the results of the project “Measure-Up” which aimed to develop an Early Algebra curriculum following Davydov’s perspective for Grades 1 to 5, starting with measuring continuous quantities and constructing number sense on multiplicative measurements, Dougherty (2008) reported that Grade 1 students can reason by transitivity by the following dialog:

> “Imagine the following dialogue in first grade as Caylie and Wendy compare three volumes, D, K, and P:
>
> “I think that volume D is greater than volume K,” said Caylie.
>
> “How do you know that, Caylie? We didn’t directly compare those two volumes,” said Mrs. M.
>
> “Well,” said Caylie, “we found out that volume D is equal to volume P and volume P is greater than volume K, so volume D must be greater than volume K.”
>
> “I agree with Caylie,” said Wendy. “Because volume D and volume P are really the same amount so if volume P is greater than volume K, then volume D also has to be greater than volume K.”
>
> (Dougherty, 2008, p.389),

However, it was not easy and clear for kindergarten students in this study. The results of our study are more compatible with the results of another study about kindergarteners' performance on transitivity (Owens & Steffe, 1972). In their study, students were required to use transitivity between two dual relations of discrete quantities, between three sets of objects; stars, squares, and circles, to conclude the third relation. They were also asked questions to determine if they conserved the number of objects when they were rearranged in the illustration. There were 42 students ranging from 65 months to 75 months, which is similar to the age of this study's participants. The study concluded that students' conservation performance on the "as many as" relation, which corresponds to Piaget's conservation test we used, is correlated with their achievement on the transitivity test. Conservation on "more than" or "less than" relations are found to be insignificantly correlated to transitivity. This result is consistent with our results, where students who did not have an understanding of the conservation of amount (majority of our students) could not reason by transitivity property of relations.

Only Aylin and Ekim reasoned by transitivity immediately, as they were the only ones who demonstrated conservation of amount. This suggests that transitivity is a complex concept for kindergarten school because the majority of the students had no conservation of amount for "as many as" relations.

For being unrelated to the trajectory of equations, and difficulty due to conservation of amount, transitivity can be excluded from the trajectory for further studies. However, creating equivalent scales had some benefits more than supporting or using transitivity:

- It prepares students for measurements by introducing the concept of comparing quantities through a consistent reference, which later connects to numerical measurements as multiples of determined scales, as seen in Davydov's trajectory.
- It plays an important role in graphing changes in quantities (e.g., the plant height activity).
- It underlies a continuous view of quantity. It is a scale, like a ruler, without discrete numerals placed on it. Continuous manipulation of a scale and the idea of continuous change of quantities may have been gained before having measurements by classical rulers indicating cm/inches.

5.1.1.6 Interplay of Equality, Quantity, Operations, and Notations in Learning Early Algebra

Equality and quantity concepts can be thought to be developed together. However, they are both elevated after the presentation of >, < signs. They became objects of equality/inequality relations. Acting on inequality relations, one-sided equations are composed to make equality. Operation appeared as increase and decrease actions by fixed amounts. Then, acting on equalities, the properties of operations were discussed through comparison. These properties were used to break and reconstruct equalities, building knowledge of two-sided operations on equations. Notations helped mathematical communication and enabled using prior concepts as objects in new algebraic processes. The gradual development of notations helped students abstract algebraic concepts out of the objects they are defined on. (see Findings Chapter, Figure 4.61for the schema for equations)

5.1.2 Practical Outcome: Design Principles

The design principles for each lecture were given in the findings chapter in detail. In this section, we will point out common design principles throughout the activities. We followed Davydov's trajectory, and designed activities supporting it. Adaptation at the kindergarten level was achieved with the help of these design principles. These design principles explain how/why activities support learning trajectories.

5.1.2.1 Free Experimentation

The concepts of equality, quantity, and operations are enactively experimented with physical manipulatives. Free experimentation allows students to choose the materials and the way they experience the physical world of related mathematical contexts. This enables observation of students' underlying mathematical reasoning in the investigations. For example, they choose identical toys to compare or add on the balances to investigate $a = a$ or $a + b = a + b$ situations in the weight context. Students initiate free investigations based on their mathematical intuitions or assumptions.

Kindergarten students have little experience with volume and weight context, especially in subtraction. Free experimentation with sufficient time, improves their physics experience in these attributes. Students not only build mathematical knowledge on their physical world experience, but they also learn how to interpret physical conclusions mathematically. Moreover, they developed their strategies leading to important algebraic learnings; such as incremental change of quantities to make equality. Activities, not only support the comparison of continuous variables but also enable enactment of continuous change in quantity.

As a result, we suggest giving sufficient time for free experimentation in activities. If students have not already discovered by themselves through free investigations, they can be guided through discussions, or directed to guided investigations.

Besides its advantages, free experimentation also brings its difficulties. Measurement errors and the limitations of manipulatives may distract from the investigation procedures. Measurement tools, measured objects, and students' measuring may not be as precise as we expected. Turning the disadvantages of physical experience into opportunities for learning can be achieved through discussions of possible reasons for the unexpected situation. These discussions can facilitate reflection on the subject and help the abstraction of it (Dubinsky, 1991).

Free experimentation in the activities can be supported through the following stages in order:

- Free experiment
- Sufficient time
- Guide for specific learning
- Welcome measurement errors and the imperfect environment of comparisons
- Discuss and reflect on possible reasons for unexpected results

5.1.2.2 Quantification

Abstraction of quantity from objects is an important and non-trivial stage. Students might refer to objects rather than their quantity in algebraic expressions. In this study, quantification is supported through five key strategies. Firstly, studying quantity as different attributes (different types of continuous variables) supported abstraction by improving context independence. Through several contexts, students compared objects and interpreted equality based on the quantity defined by the attribute. Different contexts are visited in different lectures. Secondly, for the comparison of two objects, students were expected to interpret equality for different attributes of objects. This also helped think away from the objects themselves and referred to different attributes of objects in comparison results. These are what we expected and designed our activities based on before implementation. These two strategies supported taking a step in reasoning by quantity. However, the equality may be interpreted between objects based on some properties by using certain comparison manipulatives, but it does not essentially indicate a quantity in the algebraic statement for some students. Including >, < signs to interpret comparisons emerged from our data as a third strategy of supporting quantification. Thinking by greater/less than relations signified quantity in comparisons.

Another strategy supporting quantification was observed during the investigation of identical objects resulting in different quantities in comparisons. Students expected identical toys would have equivalent weight. Discussion led them to think about the actual weight of the objects, rather than being the same. Finally, investigating different objects and finding equality between them helped students quantify comparisons. Briefly include all varieties of toys (similar and distinct, continuously manipulable or fixed quantity) in investigations with physical experiments to improve students' reasoning by quantity. Davydov did not include identical objects in comparisons. We included and took advantage of them in the discussion on quantities. However, we limited students to comparing different objects in some activities to make them discover equal quantities for different objects.

Briefly, quantification can be enhanced by;

1. Using different contexts
2. Interpreting equality based on different attributes of compared objects
3. Using >, < signs
4. Including identical objects, with different quantity
5. Including different objects of the same quantity.

5.1.2.3 Notation

Bruner's mode of representation worked, with some modifications on the level of symbolism. The same color or pictorial symbols are appropriate for this grade. However, the step-wise connection between enactive, iconic, and symbolic modes of representation is essential, as it is not straightforward for this grade. Templates can be developed, and lectures should be designed to address all the modes of representation in every context. Be precise in expressing which type of attribute/variable you refer to in discussions so as not to confuse students. They tend to reason by the size of objects rather than based on the attribute, even transferring between notations.

5.1.2.4 Motivation

Motivation is essential for engagement in topics. The only and strongest motivation is having fun for the majority of the students. Building conceptual discoveries around games is important. Finding equality, and winning through determination of bigger quantities creates motivation and meaning for algebraic concepts. Free experience or gaming is more fun than structured investigations. Build lectures around free experiences or games that lead to investigation/observation of certain algebraic expressions. Using dice helps to structure the investigation. Then, create a motive to continue the investigation, like winning, filling the cup, finding equality, etc.

5.1.2.5 Sequencing Activities

Sequencing of subjects emerged in the following order through activities: seeding, teaching of signs, investigating, reporting, reading, and modeling. The seeding stage involves mentioning the algebraic topic in informal ways much before it is taught, whenever possible. It completes the relativeness, creates connections, and gives time for acknowledging as being prior steps to algebraic actions. Stages of algebraic concepts coincide; i.e., some learning in the modeling stage includes the seeding of a further learning concept. We value the seeding stage at the kindergarten level and suggest including it as much as possible associated with future learning even leading to outer horizons. Individual student progress may not follow a linear path through APOS levels. Therefore, it is important to design activities that revisit stages of learning in different contexts and at different times.

5.1.2.6 Online Lecturing

If students have access to manipulatives and are given time for individual interviews during online education, the experience can be similar to in-class instruction. The primary difference is the observation of students' work, which can be more challenging online. Activities can be adapted for online conditions. If comparison manipulatives and paper templates can be provided to students, other manipulatives can be substituted with items they have available at home.

5.2 Theoretical Contributions

By adapting Davydov's approach at the kindergarten level, the findings of this study contribute to the theory of early algebra education. Analyzing these findings in relation to the use of APOS Theory for designing and assessing the trajectory at younger ages may further enrich APOS theory.

5.2.1 Contributions to Early Algebra Education

Davydov's perspective differentiates from others in early algebra education by teaching algebra before arithmetic in the 1st Grade. Adopting Davydov's trajectory in kindergarten makes three significant contributions to early algebra education theory: kindergarten algebra trajectory, boundaries of early algebra, and assessment of Davydov's trajectory based on APOS Theory.

5.2.1.1 Adopted Algebra Trajectory at Kindergarten Level

This study demonstrates that Davydov's perspective can be successfully adapted to the kindergarten level by simplification of notation and restriction on contexts. Hence, quantity based on continuous variables, equality/inequality relations, and addition/subtraction operations can be learned at the kindergarten level. Kindergarten-level early algebra studies mostly focus on starting algebra by patterns to improve relational thinking (Wang et al., 2016). Some kindergarten studies developed learning trajectories on generalizations and operations by indeterminate variables, whereas quantities were discrete numerals (Ventura et al., 2021). In this study, as in Davydov's trajectory, quantities are continuous variables that are compared, manipulated, and operated on. The most related study is the Measure-Up project following Davydov's trajectory for 1-5th graders (Dougherty, 2008). This study aimed and succeeded in initiating algebraic by Davydov's approach in kindergarten, before formal arithmetic education. Following Davydov, with some regulations on notations and contexts, students as young as kindergarten could compare and operate on unknowns with a non-numerical perspective and continuous variables rather than discrete numerals. Letter notation is simplified to photos of objects in algebraic expression and could be improved to color notation to represent equivalent quantities in expressions. Contexts for enactive investigations are restricted to height, length, weight, and volume as height, to meet students' level of understanding on conservation of amount.

With those regulations, the resultant trajectory explains how kindergarteners learn algebraic equations theoretically, and design principles explain how it can be adapted to teaching algebra at early grades. These are what implementing Davydov's trajectory at the kindergarten level contributes to the theory of early algebra. However, there are also alterations we made in adaptation. Major changes are explained as simplification of notation and variables. There are also additions to Davydov's trajectory in the implementation. The most remarkable is the discussion on multiple solutions in equations and inequalities. Extending sequences, representing the equivalent set of quantities with color notation, and constructing equivalent sums are other additional concepts taught. Depending on Davydov's trajectory and empowered by enactive investigations, our implementation supported understanding equality, quantity, unknowns, solution sets, continuous variables, relations, operations as change, and sequences at the kindergarten level.

5.2.1.2 Defining Boundaries of Early Algebra

Regulations made for the adaptation at the kindergarten level give clues about boundaries, limitations of algebra at early grades, and how we can alter them. We found that Piaget's conservation of amount is a boundary in learning some concepts, so we made alterations to handle students' limitations. Pushing the limits, we hit on some learning boundaries. Piaget's conservation of amount is essential in defining boundaries of what students at this age can learn. Due to a lack of conservation, students struggle in particular contexts such as area and volume, the symbolism of quantities, and the preservation of quantity in expressions. We reduced area, and modified volume investigations into height comparisons. For symbolism, we thought not knowing letters would cause problems. However, the association of the quantity to the symbol that represents it is not straightforward at this age. Switching between modes shows that students cannot conserve knowledge in the prior mode to represent it in the new mode. This was common among students who had no conservation of amount. These students could not construct or reflect on algebraic objects such as

“$a \pm b$". Lack of conservation of amount also makes it impossible to reason by transitivity, as discussed earlier. These problems show how Piaget’s conservation of amount is related to learning specific algebraic topics in Davydov’s trajectory. Design principles and regulations, such as simplifying symbolism and reducing variable types, provide strategies for altering limitations. These adaptations explain what we can teach as algebra at this grade level.

5.2.1.3 Evaluation of Davydov’s Trajectory Based on APOS Levels

In this study, Davydov’s trajectory, and students learning in our trajectory are investigated by APOS Theory. We implemented a constructivist assessment method, APOS Theory, to analyze a Vygotskian trajectory: Davydov (Schmittau, 2011, p.71). Firstly, Davydov’s trajectory is decomposed based on the APOS levels. Each subject in Davydov starts with actions of components of the taught topic. Actions are developed through new context and reverse processes, then used in new actions being objects. We showed that Davydov’s trajectory is consistent with APOS stages, which proves it ensures algebraic learning and develops gradually in stages. The adaptation procedure protects matching the steps taken based on the APOS Levels. Then, students’ learning is assessed based on APOS Levels. Hence, algebra learning at early/younger ages is explained by APOS Theory for the first time. APOS levels enabled an explanation of each learning stage for any topic in detail. Assessment of Davydov’s trajectory matched with the observed students’ progression through concepts based on APOS Theory. Hence, Davydov’s trajectory is also empirically proven to be consistent with the APOS Theory.

5.2.2 Contributions to APOS Theory

We explained students' progression on algebraic concepts through the APOS Theory at the kindergarten level. There is limited research applying APOS Theory to elementary education (Arnon et al., 2001), and none specifically at the kindergarten or first-grade level 1. Most studies on algebra education using APOS Theory focus on secondary and graduate levels (Şefik et al., 2021). Arnon et al. (2014) explained the differences between APOS for elementary school (4^{th} and 5^{th} grades) and postsecondary (after K-12) school students, noting that while postsecondary students initiate actions on abstract objects, elementary students develop actions on concrete objects. They suggest that working with concrete objects helps make abstract mathematical concepts more accessible to young learners, with actions on concrete objects facilitating the understanding of “abstract mathematical objects in a child’s mind” (Arnon et al., 2014, p.153). Students internalize actions on concrete objects by imagining these actions in their minds, which leads to the development of a process level (Arnon et al., 2001).

Similarly, in this study, it was observed that students performed actions in their minds before actions on concrete objects and initiated their investigation/actions based on those abstract algebraic actions. Like reasoning algebraically in solving arithmetical problems, students reasoned algebraically to investigate concrete objects. We called them as pre-actions. Before using algebraic expressions with concrete objects in the action stage, they showed they had abstract algebraic reasoning in their minds, either through modifying enactive investigations based on it or explaining it verbally. Hence, we categorized them as enactive pre-action and verbal pre-action. At the kindergarten level of algebra, we found that these stages are essential in students learning.

We build algorithms for new algebraic actions on students’ verbal or enactive pre-actions, which can be seen as prior informal knowledge related to the newly learned algebraic topic. These pre-actions sometimes function as prerequisites or prior knowledge but are not necessarily algebraic in nature. They are not merely pre-existing knowledge that students bring to the classroom; rather, they are intentionally developed through the lectures

As we stated, pre-action stages appear in two forms: enactive or verbal, where there is no essential order. In general, enactive pre-actions occur when students' algebraic reasoning is reflected in enactive investigations but cannot be interpreted verbally. We guide students to interpret concepts verbally if they do not do so on their own, even though they are not yet able to use algebraic expressions at this level. We then make connections between algebraic signs and these verbal interpretations.

Pre-action stages play an important role in making students ready for algebraic actions. While verbal pre-actions are the anchor to algebraic actions, enactive pre-actions take place even before that, as investigations or acts. As a result, we tried to embed those stages in the trajectory whenever possible and appropriate

Recognition is another pre-action we observed and embedded as a strategy to teach specific topics, if not all. It is about recognizing others' actions and enactive investigations, or it helps to scaffold the discovery of some algebraic learning. We used this stage in multi-solutions to make students recognize other solutions, and build an algorithm for constructing new solutions on this recognition. Another recognition phase is discovered in the difference-amount concept. When students could not reason by difference amount, we made them recognize that increase and decrease amounts are equal to each other. Although it was beyond the scope of this study, interpreting and using the concept of difference amounts can be developed based on this recognition.

Algebraic actions are given through an algorithm connected to pre-actions. Understanding and following these algorithms are also found to be difficult for kindergarten students. Learning algebraic signs and expressions requires extra effort, even after enactive and verbal pre-actions. After students become fluent in algorithms, the process stage is supported through different contexts and reverse processes. Then in new algebraic actions, processed algebraic objects are observed to be used as algebraic objects. The order of development between process and object levels is not clear. Whether algebraic processes become objects and then they are used in new actions, or they became objects when they are forced to be acted/reflected on new actions is not clarified through our data. Difference amount activity is a good example of this causal relation. Two students reasoned by difference amount in the new action, which shows they had an object level of knowledge on difference amount already, while another student developed knowledge of difference amount when she had to reflect on in the new action. Arnon et al. (2014) describe the progression between APOS Levels as not being always linear. In our study, we observed that moving back and forth between action and process levels often occurred across different contexts in our study. However, we also observed when a student reaches the object level of understanding, he/she might use it procedurally, reflecting the duality of object and process levels as described by Sfard & Linchevski (1994). Despite this procedural use, the student typically retained their object-level understanding when encountering new concepts.

5.3 Practical Contributions and Implications

In addition to theoretical contributions, this study has provided a practical instructional sequence and activities. Advancing design-based research, this study developed activities within natural classroom environments, ensuring both the applicability of theory in real-world settings and the practicality of implication. Theoretical outcomes indicate that students at the kindergarten level can learn equality, quantities, and addition/subtraction operations algebraically. These activities were not only aligned with the learning trajectory but also proved practical for implementation in a public-school setting with a group of ten students.

As previously mentioned, challenges in implementing Davydov's trajectory include the postponement of arithmetic education and acceptance by authorities, teachers, and parents. Introducing algebra in kindergarten addresses the issue of delaying formal arithmetic education until 1st grade. Moreover, initiating algebra education from Davydov's perspective is crucial, as early intervention is essential—students often face arithmetic problems even before starting school (Falkner et al., 1999). Curricular revisions are possible based on the trajectory proposed in this study. The activities developed, along with accompanying design principles, provide teachers with practical tools that facilitate easier implementation.

In the Turkish Elementary Mathematic Curriculum (Ministry of National Education [MoNE], 2018), algebra starts in 6th grade and is heavily studied in 8^{th} grade. In the 1st grade, addition and subtraction are given in one-sided equations where the resultant is always on the right side, supporting a “solve for” meaning for equal signs. The 1st-grade curriculum includes the balance concept to compare objects in a weight context to interpret which one is heavier and not associated with arithmetic. In the 2nd grade, students see equation structures with operations on the left side or both sides. In the 3^{rd} grade, >, < signs are learned to compare numerical quantities. There is an attempt to support algebraic reasoning in early grades, but concepts are presented in a fragmented manner. In our trajectory, we introduce operations on both sides of an equation, emphasizing their use in balance and imbalance situations from an algebraic perspective. Moreover, our study demonstrated that students could solve for unknowns in two-sided equations within an arithmetic context before reaching 2nd grade, where such equations typically appear. Students were able to transfer their understanding seamlessly to arithmetic problems, proving that our trajectory aligns with and even enhances the elementary curriculum. Rather than obstructing arithmetic education, our trajectory strengthens it by laying a solid algebraic foundation.

The results of this study may be used to develop more structured manipulatives and templates to investigate algebraic structures. These manipulatives can be either hands-on manipulatives/toys or technological tools/games, which will enable individual or parent-guided learning, resulting in a more manageable and worldwide algebra education.

Adopting Davydov’s deductive perspective, this study not only introduces algebra before arithmetic but also emphasizes continuity over discrete variables and relational thinking over operational thinking. This approach has the potential to inform the development of spiral curricula that introduce mathematical concepts such as continuity, sequences, variables, and relations through informal investigations at younger ages.

5.4 Suggestions for Further Studies

Advancing a design-based research perspective, we ended up with an adapted learning trajectory. However, we do not claim it is the best learning trajectory. Suggested revisions on the trajectory for a 2nd cycle and additional study concerns will be presented in this section.

Unequal sign has been found to be unnecessary and potentially hindering when teaching the concepts of < and > in the trajectory. Moreover, since the unequal sign is not included in the kindergarten curriculum, omitting it may make the implementation more acceptable to authorities. While the unequal sign can be seen as a complement to the equal sign, serving as its reverse, its use may cause students to focus more on the concept of equality rather than on understanding quantities. Furthermore, the concept of 'being unequal' is not always related to quantities, especially in the part-whole examples.

While addressing the three cases of equal, greater, and less than may seem complicated, focusing on these through solution cases that address quantity could be more effective. Presenting the equal, greater than, and less than signs simultaneously can evoke quantification in comparisons. Dynamic models, where the equal sign (=) transitions into greater than (>) or less than (<) signs by changing angles, could help students better understand which side is bigger and which side is smaller.

It is easier to use =, ≠ signs in weight and volume contexts, as interpreting greater or less than with > and < signs can be more challenging in these scenarios. In future studies, spending more time in these contexts to learn signs is recommended. For this grade level, using =, >, < signs while excluding ≠ seems to be a practical solution, but it is mathematically incomplete and may affect future understanding of solution sets for systems. Evaluating and testing both trajectories based on their advantages in further cycles would be beneficial.

Similarly, transitivity stands as an outlier in the trajectory of learning equations. Transitivity is mostly related to inequality relations, and it was difficult for kindergarten students because they had no conservation of amount. Exclusion of transitivity is another revision suggested to be tested in further trajectories.

Letter notation has been simplified to pictures of objects. It is suggested to further investigate new trials for the gradual development of letter notation, or possibly color code notation. Additionally, exploring the relationship between symbolic or letter notation and Piaget's Theory on conservation of amount would be valuable.

Variables naturally emerged as flexibility in constructing quantities based on relations and as multiplicity in solutions for equations. The trajectory can be enriched to better ground the concept of variability. Although symbolizing variables may be difficult at the kindergarten level, there is a need to connect the notation of non-numerical quantities in Davydov's trajectory to the notation of variables as a generalization of numbers, as discussed in the literature (e.g., Ventura et al., 2021). Dynamic environments in enactive investigations offer the potential to investigate variability by adjusting quantities and corresponding changes on the other side based on defined algebraic expressions. While we have followed Davydov's trajectory closely to adopt a proven trajectory to an earlier grade level, any improvements that could enhance algebraic learning deserve further studies through new cycles.

This study also indicated that student exhibited algebraic reasoning underlying their enactive pre-actions, indicating that they can think about algebraic structures even before formal investigation. Thinking about arithmetic from an algebraic perspective is crucial, yet recognizing structures in arithmetic equations can be challenging for some students. Davydov's approach helped prevent arithmetic from hindering algebraic thinking. One reason arithmetic may obscure the recognition of structures in equations is that students often approach equations like a+b=c by reading and solving them from left to right, focusing on procedural methods. Another reason might be an over-reliance on arithmetic operational procedures. The relationship between the ability to identify algebraic structures in arithmetic problems and the algebraic intuitions discovered in this study deserves to be studied further to explain how arithmetic education hinders algebraic intuitions.

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APPENDICES

A. Objectives, Activities, and Theory Behind Choosing Activities in the First Hypothetical Learning Trajectory

Lecture #	Apos level of concept	Objectives	Activity	Theory behind
Lecture 1	Action =, ≠	1. The student interprets equal and not equal sign 2. The student compares objects and uses equal and not equal sign to interpret relations based on size (action to process level) 3. The student uses balance scales to compare the weight of objects and uses equal and not equal signs to interpret the relation 4. The student differentiates height, length, volume, and weight as different variables	Comparison of simple objects based on size and weight comparison: use toys to compare objects interpret different variables, and ask for the weight of the same height objects. (pilot result) use different objects, balloons, water cups, books, and human height to discuss equality on different examples (to refer to variables) use a balance scale to compare the weight of objects	Students know what equal or not equal means in their daily lives. Using equal and not equal signs to compare two objects is a new algorithm for them. Experience with lots of toys will be good practice for remembering and applying the algorithm themselves, which will evolve into the process stage. Using concrete objects will provide an enactive representation of equality. Balance scales and inquiry into variables construct the first step on variable/ quantity
Lecture 2	Process =, ≠	1. The student uses equal and not equal signs to interpret a relation in a part-whole context.	Constructing whole from its parts on paper. Davydov's part-whole activities for being equal or not	Not entirely, but more symbolically, for iconic representation, working on paper activities are included. However, if students have difficulty, step back to the enactive stage; make students cut pieces with scissors so they use pieces as concrete objects to try to construct whole from parts.
Lecture 3	Process =, ≠	1. The student uses equal and not equal signs to compare volumes of cups	Guessing and comparing volumes of cups using identical cylinders.	New context volume and Continuous variable

Lecture 4	Process =, ≠, double-sided	1. The student reports the comparison of volumes of objects symbolically on the paper 2. The student reads the symbolic interpretation of equality and inequality and checks it with concrete objects.	Reporting (letter notation) comparison relations between volumes of cups and assessing peers' reports by comparisons.	Double-sided process Symbolic self-interpretation Read interpretation symbolic algebraic, assess through concrete objects relation.
Lecture 5	Action >, < Object =, ≠	1. The student interprets inequalities with greater or smaller relation. 2. The student uses >, < signs to interpret relations 3. The student interprets (verbally) how to make equality from greater or less than relations	Comparison of simple objects and discussion on how to make them equal: apple and balloon examples Verbal interpretation to make equal	Inequality turns out to be greater/smaller than relation Inequality and equality are used (as objects) to interpret newly learned greater or smaller relations.
Lecture 6	Algebraic notation	1. The student uses the first letter of his/her name as notation.	Planting beans and giving letter notations. (stem activity, discussion on how plants grow, parts of plants can be embedded as science. Labeling plants with the first letter of names, reporting growth, and comparing growth will be done through lettering, which will complete symbolic(algebraic) notation.	Letter notation is essential for algebraic notations. First impressions can be given at this stage for the need to discriminate among plants. Reporting the height and comparison among different plants will add to the use/need of this letter notation.
Lecture 7	Process >, < Object =, ≠	1. The student uses >, < , = signs to interpret (without reminding algorithm) comparison of weights(as continuous variable) of play doughs 2. The student manipulates (concrete/enactively) both sides /increases or decreases playdoughs to make equal-weighted pieces	Comparing play doughs in weight context and creating equal weighted pieces. Hands-on manipulation to make equal	New context for greater smaller, continuous variable, manipulation of play doughs on both sides. To make equal. Use equality (as object) base target to manipulate greater less than

Lecture 8	Process >, < Object >, < Transitivity property	1. The student uses >, <, = signs to interpret the comparison of volumes (as a new continuous variable) of cups. 2. The student uses two relational interpretations of three cups to guess the third relation (transitivity property)	Comparison of volumes of cups. Transitivity of relations. Given 3 cups, investigate two dual relations and predict the last relation for the following situations a=b ve b>c ise a>c a>b ve b>c is a>c	The new context for continuous variables, volume Use interpretation of relation with signs >, < to investigate transitivity property, which will carry >, < to object level. Predict and investigate enactively.
Lecture 9	Object =, ≠ Object >, <	1. The student finds suitable objects for a predetermined relation, finds equal and unequal objects, and interprets the relation between them	Based on the given relation, discover toys in the classroom. Using pictures as iconic representations of objects, and choose of cards based on given relations.	The reverse process is essential for encapsulating into object.
Lecture 10	Object =, ≠ Object >, <	1. Given relation between two objects, the student determines the attribute(variable type) for the comparison	Based on given objects and the relation between them, determine attributes in comparison. Determine attributes using identical cylinders and balances. Given cards and relations between them, discuss attributes.	The reverse process extended to variables. We call attributes variables because activities that include continuously changeable attributes (e.g., weights of playdough, water height) act like variables (not just unknowns) in algebraic interpretations.
Lecture 11	Object =, ≠ Object >, < construction	1. The student completes the unknown/variable in the given relational interpretation (equality, inequality, >,<) by drawing 2. The student discusses the variability of the drawing	Based on a given relation and attribute, completing a given picture of an object	Manipulation on the variable (size): Variability, Multiple, infinite solutions, Reading algebraic interpretation, Self-construction, Work-sheet, more symbolic Fill in the gaps, determination of x, not unknown but variable situations given first.

Lecture 12	Mid-assessment and repetition	1. The student uses signs on worksheets 2. Given two different-sized paper strips, the student cuts a long paper strip to make it equal to a shorter one 3. Given two different-sized paper strips, the student glues the Ekstra paper strip to make it equal to the longer one 4. Given two different-sized paper strips, the student interprets/shows how much paper to cut or add to make paper strips of equal length	Assessment and repetition of worksheets Part-whole Paper strips enactive investigation to make equal.	Assessment It also improves symbolic more algebraic understanding, as it is on paper. The new context of length, continuous, and self-control of unknown The first step was discussing how to make it equal by increasing/decreasing cut and pasting the amount of increase/decrease)
Lecture 13	Action =, ≠, >, < sequences	1. The student orders 3-4 objects and puts relevant signs between them based on their relation	Comparison and ordering of 3-4 objects using toys.	Using relations in sequence context.
Lecture 14	Proces s =, ≠, >, < sequences	1. The student orders 3-4 pictures and puts relevant signs between them based on their relation	Symbolic representation of comparisons by using cards and then on-paper activities	more symbolic already known comparison between 2 obj process is used. New context, new situation. Maybe for ordering, this process will turn into object parts of the ordering process. Ordering three or more objects can help think about transitional properties between 3 objects. Hence, this activity is planned to be earlier.

Lecture 15	Action, transitivity	1. Given two relations among two of three objects, the student determines the relation of the third comparison.	Given two dual relations between 3 objects, guess the third relation.	Non-investigated or observed but based on pre-given relations. Not apparent in size as in Davydov's book (Davydov et al., 1995). Just relying on symbolic interpretation, the student should determine the third relation. Guessing is not based on observation but on algebraic knowledge. Second and higher step for transitivity. The activity includes worksheets with pictures of objects and their relations, as in Davydov's book (Davydov et al., 1995).
Lecture 16	Process transitivity construction	1. Given two objects and their relation to a third unknown object, the student draws/constructs an unknown object.	Drawing the third object based on its relations to the other two objects.	Unknown construction.
Lecture 17	Object transitivity Action intermediary	1. The student uses their height or a rope as an intermediary to compare two stable and distant objects by concluding from their relation to both.	I am using strings as an intermediary to compare distant objects in the classroom.	Using transitivity to compare distant objects, transitivity becomes an algebraic object.
Lecture 18	Process intermediary Action notation	1. The student constructs an intermediary to compare distant objects. 2. The student uses the same notation to indicate same-size objects	Comparing distant squares by using strings. Using notation to interpret squares.	Using an intermediary to determine the class of equal-sized objects. Letter notation to a set of fixed quantities makes quantity belong to an equivalent set rather than specific to an object.
Lecture 19	Reverse process intermediary	1. Given two objects and their relation to a third intermediary), the student constructs and draws the third object	Drawing intermediaries based on given relations to compare distant objects.	Construction of intermediary

Lecture 20	Object or process intermediary	1. The student uses an equal-sized intermediary to represent measurement.	Observing plant heights, and making the first measurement and graph using string.	The first step is for measurement and graphing. Students learn that an equal-sized intermediary represents the one it is compared to. Graphing occurs in a continuous variable, not with numbers as usual.
Lecture 21	Object =, ≠ Action +, -	1. The student verbally interprets on which side to increase or decrease to make/satisfy equality	Discussion of increase and decrease in weight context to make equal.	Enactive investigations for increase and decrease actions to make equal.
Lecture 22	Object =, ≠ Action +, -	1. The student verbally interprets on which side to increase or decrease to make/satisfy equality	Discussing increase and decrease to make equal in volume context	New context for increase/decrease actions
Lecture 23	Object =, ≠ Action +, -	1. The student chooses the correct sign +/—to interpret an increase or decrease on both sides to satisfy equality.	Assigning +/- signs to increase/decrease actions to make them equal in volume context.	Use of algebraic expressions for verbal interpretations of increase/decrease actions
Lecture 24	Object =, ≠ process +, -	1. Given more symbolic interpretations (worksheets), the student chooses the correct sign +/- to interpret the increase or decrease on sides to satisfy equality.	Worksheet activities to use +/- signs to make equal.	Symbolic representation mode to use +/- signs in increase/decrease actions.
Lecture 25	Process +, - Action equality with one side addition/subtraction	1. In part-whole examples, The student uses + and – signs to construct equalities with one-side addition/subtraction.	Use of +/- signs in part-whole contexts to make equal.	Anchoring for increase/decrease amount for cont. Variables. It includes known parts for addition or subtraction For 1st-grade students, it may be extended to equations with double-side addition/subtraction

Lecture 26	Object =, ≠ Process +, -	1. The student determines an addition amount to make equality 2. The student interprets a quantity as the addition of one to another	Animal Height Game. Addition on one side to make equal to the other side.	Still, addition refers to action, but the addition/increase amount is not interpreted. It is an object of the addition process. The addition means an increase of one by a certain amount.
Lecture 27	Object =, ≠ Process +, -		Use the plus and minus signs to increase and decrease the lengths of paper strips to make them equal.	New context+ more symbolic+ testing at the same time
Lecture 28	Process +, - Action increase amount	1. The student interprets the increase amount iconically 2. The student compares an increase amount of different situations	Second measurement for plant height. Interpreting increase in height. Comparison of increase amounts.	The change concept is central to algebra. Comparing change amounts is complicated. This activity can be a complete research title. We aim to focus on the interpretation of the increase/change amount. Together with letter notation, comparison of increase final heights of plants may strengthen algebraic interpretation skills.
Lecture 29	Process +, - Action to process increase amount	1. The student discusses how to make equality, unequal, and equal again by addition and subtraction 2. The student interprets the effects of addition or subtraction of the same amount on both sides in an equality.	Using identical cylinders to discuss how to make equal, unequal, and equal again in a volume context.	Not only making equality but also destroying equality. It focuses on how actions (operations: addition and subtraction) affect equality. The question of how to make them equal again forces students to think about the relation between actions taken on both sides. Probably, this step will carry the action level to at least the process level as it makes students not only learn how to increase or decrease but also has the potential to make them think about the increase decrease amount. (especially to make equal again)

Lecture 30	Process +, - process increase amount	1. The student models equalities with two-side addition 2. The student uses algebraic notation to interpret equalities with addition on two sides	Animal Height Game Interpretation of equations with addition on two side	
Lecture 31	Reverse process +, -	1. The student models equalities with one-sided addition or subtraction	Using paper strips to create real-life models of given equations with addition or subtraction on one side.	The reverse process is an essential and handy step for objectifying. Student turns, reads algebraic interpretation, and visualizes it in real life.
Lecture 32	Object +, - Object >, <	1. The student reads about equalities and inequalities based on real-life models 2. The student uses algebraic equalities and inequalities for real-life designs	Observing colors of the rainbow. Constructing rainbow out of play dough by obtaining secondary colors from primary colors by reading expressions of equations with one-side addition, and equality/inequality relations.	The rainbow activity serves as a new interpretation and context and shows useful, fun parts of algebraic expressions. We do not know where actually +,- becomes an object. It may be even earlier than we expected. Implementation will enlighten the process of becoming an object.

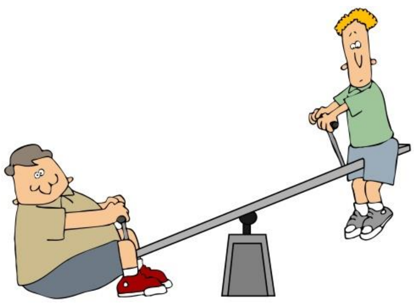
B. Semi-structured Pre- and Post-Interview Questions

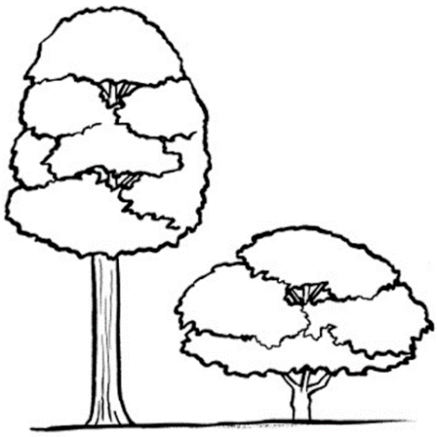
Soru 1:

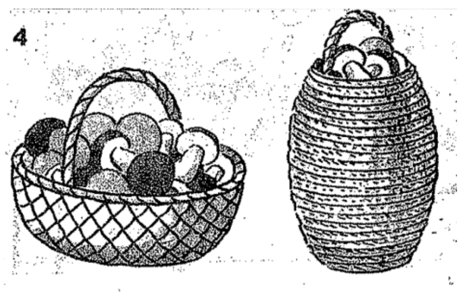
a) Bu işaretleri tanıyor musun?

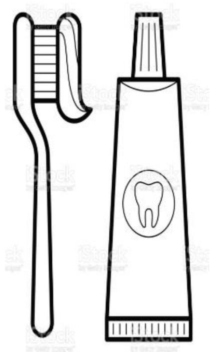
$=$ $\neq$ $>$ $<$ (işaretler kartlarla verilir, kağıt üzerinde değil)

b) Bu işaretlerle bu resimleri eşleştirebilir misin?









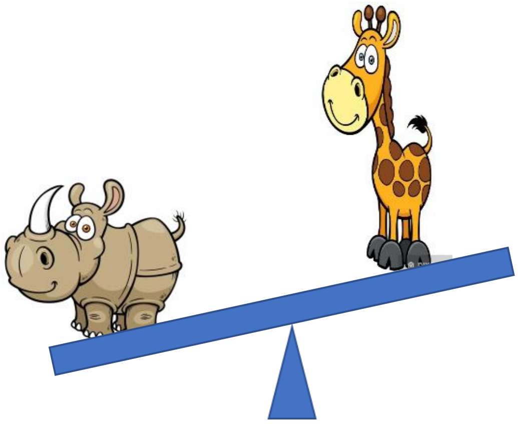
c) Neye göre eşleştirdin? Sorusunu sor.

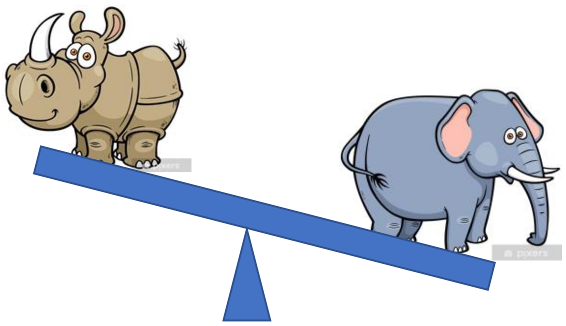
İşaretleri bilmiyorsa her bir resmi anlatması beklenir? Sence bunlar eşit mi, yoksa biri daha mı büyük? Sorusu yöneltilir.

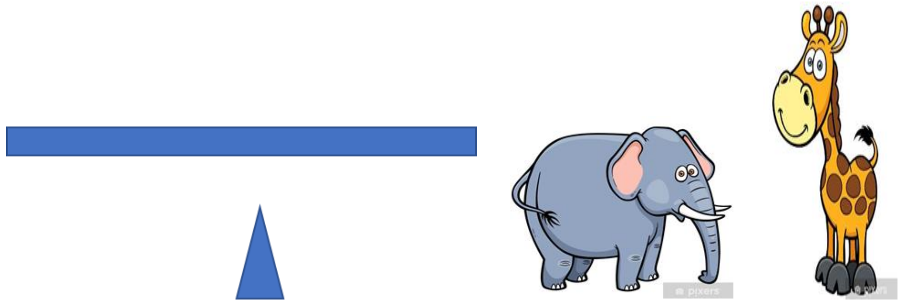
sence eşit olanlar hangisi? Hangisi büyük küçük nesneleri gösteriyor? Sence bu resimde eşit mi değil mi?)

Neye göre büyük, neye göre küçük? Sorularıyla büyük küçük eşit eşit değil kavramları ve değişkenler üzerine konuşulur.

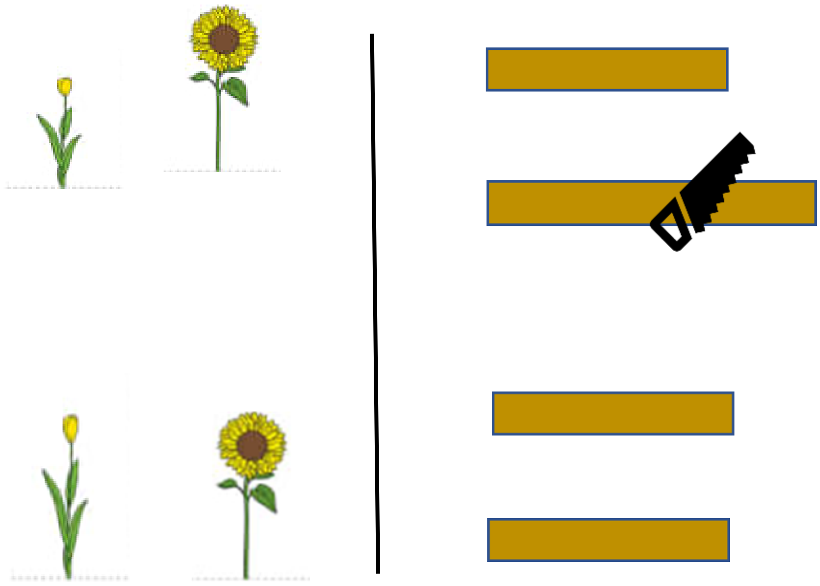
Soru 2: Hayvanlar arasında verilen ikili ağırlık ilişkilerine göre 3. Ölçümün nasıl sonuçlanacağını tahmin etmesi beklenir. Hayvan resimleri kesilerek verilir. Terazi için kesilen dikdörtgen üçgen ile iğne yardımıyla birleştirilir. Öğrenci hareketli terazi kolunu fil ile zürafa arasındaki ağırlık ilişkisini ifade etmek için kullanır.







Soru 3:

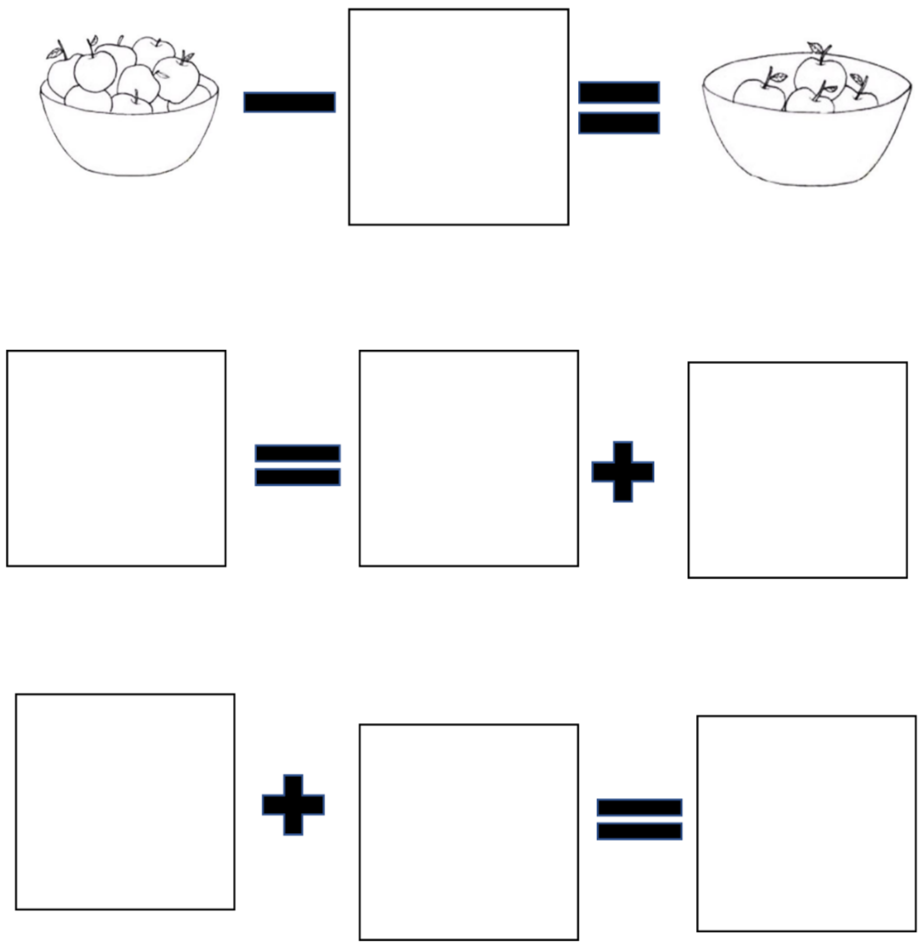


Bu resimlerde ne olduğu sorulur, anlattırılır. “önce böyleymiş, sonra böyle olmuş” şeklinde açıklanır. “Bu resimleri anlatır mısın? Hangi çiçek uzun? Sonra ne olmuş?” soruları yöneltilie.

İkinci adımda artı-eksi işaretlerinden hangisi ile değişen nesneyi işaretleyeceği sorulur.

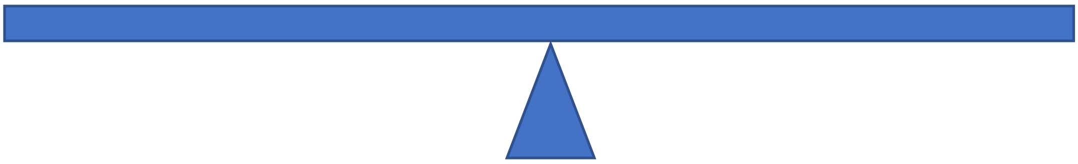
Artma-azalma miktarı: Bu soru üzerinden artma azalma miktarı ne sorusu sorulur. “Ne kadar artmış?” “Ne kadar azalmış?”

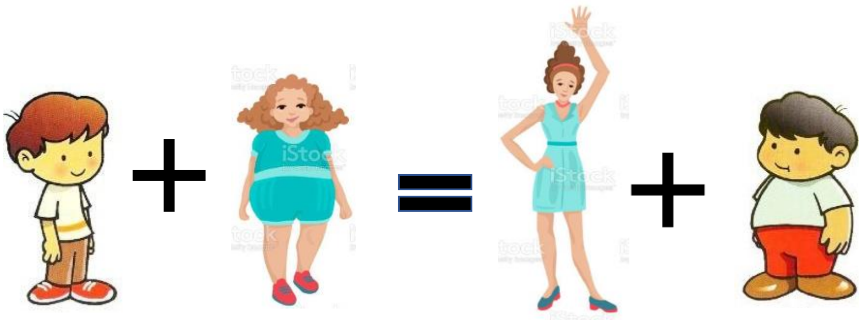
Soru 4: “Sence bu ifade ne anlatılmak istenmiş?” “Bu boş kareye ne çizmemiz gerekiyor?” Sorularıyla denklem ifadesi yorumlatılır. Bunlar kağıt üzerinde değil kartlar ile sunulur. Ve sonrasında kartların aşağıdaki toplama ifadelerine yerleştirmeleri istenir.



Soru 5: İki tarafta toplama bulunan denklemin günlük hayat ile modellemesi

Hareketli terazi modeli oluşturulur. Kağıt üzerinde üçgene iğne ile tutturulmuş ince uzun dikdörtgen hareketli kalası temsil eder.





Yukarıda çocuk resimleriyle ifade edilen denklemin terazi üzerinde modellenmesi istenir. Dengeyi

Soru 6: Hayvan boyları oyunu üzerlerindeki rakamlar gizlenerek oynanır.

Öncesinde oyunun tanıtımı yapılır ve boy kavramına odaklanıldığı belirtilir.

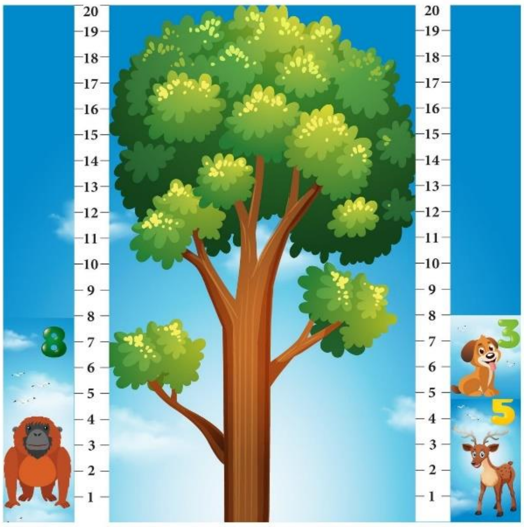
a) A=?+B tarzı sorular cebirsel ifadeleriyle verilerek sorulur. Cebirsel ifade doğru algılanamazca öğrenciye soru sözel, o da algılanamazsa şekil üzerinden sorulur. (sadece post-teste)

  - Eşitlemek için ne tarafa eklenmeli?
  - Ne kadar eklenmeli?
  - Bunu işaretlerle nasıl ifade edersin?

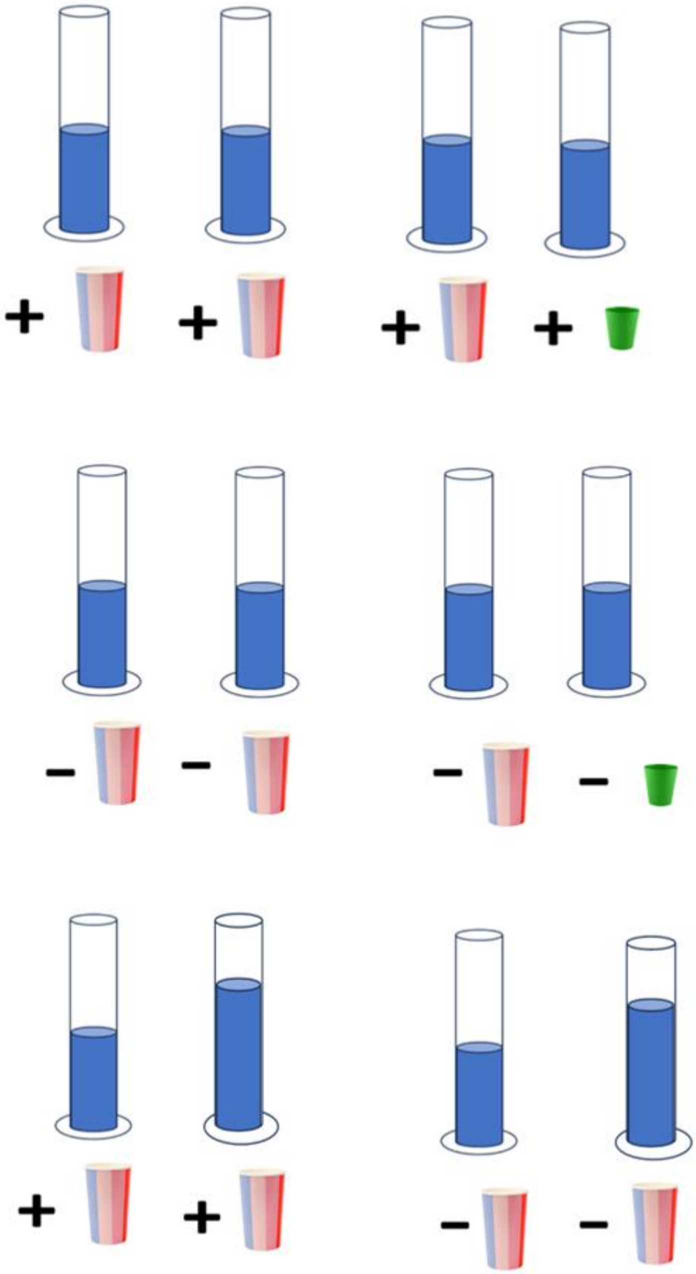
- Buradaki ifade de boşluğa ne gelmeli? Soruları yöneltilir.

b) A+B=C+D tarzı ifadeler verilerek bunların şekil üzerinde modellenmesi istenir. Şekil üzerinde verilen toplamların da cebirsel ifade edilmesi istenir.

c) A+?=C+D tarzındaki tek bilinmeyenli ifadeleri çözmesi istenir.



Soru 7: Aşağıdaki şekillerle ve işlem ifadeleriyle verilen durumların sonucunun karşlaştırılması istenir.



C. METU Human Subjects Ethics Committee Report

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

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17 ŞUBAT 2021

Konu : Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi : İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Doç. Dr. Bülent ÇETİNKAYA

Danışmanlığını yaptığınız Sevgi SOFUOĞLU'nun "Okulöncesi Öğrencilerin Aritmetik Eğitimi Öncesinde Cebirsel Muhakemelerinin Geliştirilmesi İçin Eğitim İçeriği Geliştirme" başlıklı araştırmanız İnsan Araştırmaları Etik Kurulu tarafından uygun görülmüş ve **052-ODTU-2021** protokol numarası ile onaylanmıştır.

Saygılarımızla bilgilerinize sunarız.

Prof. Dr. Mine MISIRLISOY
İAEK Başkanı

D. Permission From the Ministry of National Education

T.C.
ANKARA VALİLİĞİ
Milli Eğitim Müdürlüğü

Sayı : E-14588481-605.99-28705584 29.07.2021
Konu : Araştırma İzni

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
(Rektörlük)

İlgi : a)MEB Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 2020/2 nolu Genelgesi.
b)07.07.2021 tarihli ve 221 sayılı yazınız.

Enstitünüz Matematik ve Fen Bilimleri Eğitimi EABD doktora programı öğrencisi Sevgi SOFUOĞLU'nun **"Okul Öncesi Öğrencilerin Aritmetik Eğitimi Öncesinde Cebirsel Muhakemelerin Geliştirilmesi için Eğitim İçeriği Geliştirme"** konulu çalışması kapsamında İlimizde yer alan Devlet okulunda öğrenim gören anasınıfı öğrencilerine, uygulama talebi ilgi (a) Genelge çerçevesinde incelenmiştir.

Yapılan inceleme sonucunda, söz konusu araştırmanın Müdürlüğümüzde muhafaza edilen ölçme araçlarının; Türkiye Cumhuriyeti Anayasası, Milli Eğitim Temel Kanunu ile Türk Milli Eğitiminin genel amaçlarına uygun olarak, ilgili yasal düzenlemelerde belirtilen ilke, esas ve amaçlara aykırılık teşkil etmeyecek, eğitim-öğretim faaliyetlerini aksatmayacak şekilde okul ve kurum yöneticilerinin sorumluluğunda gönüllülük esasına göre uygulanması Müdürlüğümüzce uygun görülmüştür.

Bilgilerinizi ve gereğini rica ederim.

Turan AKPINAR
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Gereği:
Orta Doğu Teknik Üniversitesi

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E. Parrent Permission Form

Veli Onay Formu

Sevgili Anne/Baba,

Bu çalışma Orta Doğu Teknik Üniversitesi doktora öğrencisi Sevgi Sofuoğlu tarafından yürütülmektedir.

Bu çalışmanın amacı nedir? Çalışmanın amacı, anaokulu seviyesindeki öğrencilerin, eşitlik, toplama, çıkarma ve denklem kavramlarına, terazi modeli çerçevesinde, giriş yapabilmeleri ve ileride cebirsel alanlarda sağlam matematiksel temeller oluşturabilmeleri için sınıf içi etkinlikler tasarlamaktır.

Çocuğunuzun katılımcı olarak ne yapmasını istiyoruz?: Bu amaç doğrultusunda, 2020-2021 Öğretim Yılı 2. Dönemi boyunca, haftada üç gün birer ders saati olmak üzere, çocuğunuzdan kefeli terazi ile ağırlık karşılaştırma, su kapları ile doldur boşalt yaparak hacim karşılaştırma, kağıt kesme yapıştırma ile uzunluk karşılaştırma etkinliklerine katılmasını isteyeceğiz ve cevaplarını/davranışlarını ses kaydı ve görüntü kaydı biçiminde toplayacağız. Sizden çocuğunuzun katılımcı olmasıyla ilgili izin istediğimiz gibi, çalışmaya başlamadan çocuğunuzdan da sözlü olarak katılımıyla ilgili rızası mutlaka alınacak.

Çocuğunuzdan alınan bilgiler ne amaçla ve nasıl kullanılacak?: Çocuğunuzdan alacağımız cevaplar tamamen gizli tutulacak ve sadece araştırmacılar tarafından değerlendirilecektir. Elde edilecek bilgiler sadece bilimsel amaçla (yayın, konferans sunumu, vb.) kullanılacak, çocuğunuzun ya da sizin ismi ve kimlik bilgileriniz, hiçbir şekilde kimseyle paylaşılmayacaktır.

Çocuğunuz ya da siz çalışmayı yarıda kesmek isterseniz ne yapmalısınız?: Katılım sırasında sorulan sorulardan ya da herhangi bir uygulama ile ilgili başka bir nedenden ötürü çocuğunuz kendisini rahatsız hissettiğini belirtirse, ya da kendi belirtmese de araştırmacı çocuğun rahatsız olduğunu öngörürse, çalışmaya sorular tamamlanmadan ve derhal son verilecektir.

Bu çalışmayla ilgili daha fazla bilgi almak isterseniz: Çalışmaya katılımınızın sonrasında, bu çalışmayla ilgili sorularınız yazılı biçimde cevaplandırılacaktır. Çalışma hakkında daha fazla bilgi almak için Sevgi Sofuoğlu ile (e-posta: e142596@metu.edu.tr) ile iletişim kurabilirsiniz. Bu çalışmaya katılımınız için şimdiden teşekkür ederiz.

Yukarıdaki bilgileri okudum ve çocuğumun bu çalışmada yer almasını onaylıyorum (Lütfen alttaki iki seçenekten birini işaretleyiniz.

Evet onaylıyorum___ ***Hayır, onaylamıyorum___***

Annenin adı-soyadı: ______________ Bugünün Tarihi:________________

Çocuğun adı soyadı ve doğum tarihi:________________

F. Informed Consent Form for Participant Teacher

ARAŞTIRMAYA GÖNÜLLÜ KATILIM FORMU

Bu çalışma ODTÜ Ortaöğretim Matematik Eğitimi Bölümü doktora öğrencisi Sevgi Sofuoğlu tarafından yürütülmektedir. Bu form sizi araştırma koşulları hakkında bilgilendirmek için hazırlanmıştır.

Çalışmanın Amacı Nedir? Bu çalışmanın amacı okulöncesi seviyede cebirsel altyapıya uygun matematik eğitimi verebilmek için, okulöncesi öğretmenlerinin sahip olması gereken matematik alan bilgisini belirleyebilmektir.

Bize Nasıl Yardımcı Olmanızı İsteyeceğiz? Çalışma süresince, 2020-2021 2. Dönemi'nde,sınıfınızdaki okulöncesi öğrencilere cebir etkinlikleri uygulamada araştırmacının eşliğinde öğretmen rolü ile ve her etkinliğin sonrasında karşılaşılaştığınız zorluklara yönelik kısa mülakatlara katılımcı olarak bu çalışmaya katkı sunmanızı bekliyoruz.

Katılımınızla ilgili bilmeniz gerekenler: Bu çalışmaya katılmak tamamen gönüllülük esasına dayalıdır. Herhangi bir yaptırıma veya cezaya maruz kalmadan çalışmaya katılmayı reddedebilir veya çalışmayı bırakabilirsiniz. Araştırma esnasında cevap vermek istemediğiniz sorular olursa boş bırakabilirsiniz.

Sizden toplanan veriler tamamen gizli tutulacak, verilere sadece araştırmacılar ulaşabilecektir. Bu araştırmanın sonuçları bilimsel ve profesyonel yayınlarda veya eğitim amaçlı kullanılabilir, fakat katılımcıların kimliği gizli tutulacaktır.

Araştırmayla ilgili daha fazla bilgi almak isterseniz: Çalışmayla ilgili soru ve yorumlarınızı araştırmacıya e142596@metu.edu.tr adresinden iletebilirsiniz.

Yukarıdaki bilgileri okudum ve bu çalışmaya tamamen gönüllü olarak katılıyorum.

İsim Soyad	Tarih	İmza
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G. Comparison of First HLT and Resultant Trajectory

L#	First HLT objectives	L#	Resultant Trajectory objectives
1	1. The student interprets equal and not equal sign 2. The student compares objects and uses equal and not equal sign to interpret relations based on size (action to process level) 3. The student uses balance scales to compare the weight of objects and uses equal and not equal signs to interpret the relation 4. The student differentiates height, length, volume, and weight as different variables	1	1. The student interprets equal and not equal sign 2. The student compares objects and uses equal and not equal sign to interpret relation (action to process level) 3. The student uses balance scales to compare the weight of objects and uses an equal and not equal sign 4. The student uses different variables/attributes (which she already knows) to interpret equality
2	1. The student uses equal and not equal signs to interpret a relation in a part-whole context.	2	1. The student uses balance scales to partition play dough into two equal masses by increase/decrease actions verbally. 2. The student uses equal and not equal signs to interpret a relation in a part-whole context. 3. The student manipulates (increase/decrease) one side to make equality in part-whole activities
3	1. The student uses equal and not equal signs to compare volumes of cups	3	1. The student uses equal and not equal signs to compare volumes of cups 2. The student interprets the equality of volumes of cups iconically (notation)
4	1. The student reports a comparison of volumes of objects symbolically on the paper with =, ≠signs 2. The student reads the symbolic interpretation of equality and inequality and checks it with concrete objects.		Included in Lecture 5
5	1. The student interprets inequalities with greater or smaller relation. 2. The student uses >, < signs to interpret relations	4	1. The student interprets inequalities with greater or smaller relation. 2. The student uses >, < signs to interpret relations

	3. The student interprets (verbally) how to make equality from greater or less than relations		
6	1. The student uses the first letter of his/her name as notation. (The first step is to use letter notation to interpret and compare the heights of their plants.) planting bean		Beans are planted before all lectures to save time. Letter notation is given up. Lecture 6: They did the first measurement with rope and glued it on the paper.
7	1. The student uses >, <, = signs to interpret (without reminding the algorithm) the comparison of weights (as a continuous variable) of play doughs 2. The student manipulates (concrete/enactively) both sides /increases or decreases play dough to make equal weighted pieces		It was implemented in Lecture 2 without >, < signs.
8	1. The student uses >, <, = signs to interpret the comparison of volumes (as a new continuous variable) of cups. 2. The student uses two relational interpretations of three cups to guess the third relation (transitivity property). Delayed, see Lecture 8	5	1. Report: The student interprets the comparison of volumes by >, <, = signs symbolically on paper by using pictures of compared cups as letter notation. 2. Read report and check: The student reads/uses a symbolic representation of a peer's comparison and checks with manipulatives if the comparison is true. >, < relation; volume, discrete comparison
9	1. The student finds suitable objects for a predetermined relation, finds equal and unequal objects, and interprets the relation between them by using =, ≠, >, < signs.		Embedded in previous Lectures. Beginning activity for Lecture 4
10	1. Given relation between two objects, the student determines the attribute(variable) for the comparison		Embedded through discussions in previous lessons as interpretation based on various types of variables difficult to construct: Davydov's images not clear (pilot result) emergently and naturally discussed in lectures.
11	1. The student completes the unknown/variable in the given relational interpretation (equality, inequality, >, <) by drawing		This lecture is embedded into the mid-assessment. . They know how to choose . They will construct a new context assessment.

	2. The student discusses the variability of the drawing		
12	1. The student uses signs on worksheets 2. Given two different-sized paper strips, the student cuts a long paper strip to make it equal to a shorter one 3. Given two different-sized paper strips, the student glues an extra paper strip to make it equal to the longer one 4. Given two different-sized paper strips, the student interprets/shows how much paper to cut or add to make paper strips of equal length		Cutting strips activity is related to the increase/decrease amount concept. Hence, it will be used in addition/subtraction concepts in a structured way.
		6	Mid-assessment 1. The student constructs an unknown quantity based on a given algebraic relation to another quantity by >,<,= signs. 2. The student uses =, ≠ signs to interpret part-whole equality given by symbolic figures (Lego photos). 3. The student uses >,<,= signs to interpret relations symbolically based on given representations of weight comparisons.
13	1. The student orders 3-4 objects and puts relevant signs between them based on their relation: with toys	7	1. The student orders 4 objects based on their size and uses the> sign to interpret the sequence 2. The student extends the sequence of ordered objects based on size.
14	1. The student orders 3-4 pictures and puts relevant signs between them based on their relation with cards		
15	1. Given two relations among two of three objects, the student determines the relation of the third comparison.	8	1. Given 3 objects, the student experiences and reports two comparisons (in an order), and guesses the third relation.
16	1. Given two objects and their relation to a third unknown object, the student draws/constructs an unknown object.		
17	1. The student uses their height or a rope as a scale to compare two stable and distant objects by concluding from their relation to both.	9	1. The student creates an equivalent scale for an object to compare it to another distant object. 2. The student interprets the result of the comparison in terms of the distant

			objects, not in terms of the scale he/she used.
18	1. The student constructs scales to compare distant objects. 2. The student uses the same notation to indicate same-size objects Squares activity	10	1. The student constructs scales to compare distant squares 2. The student uses the same color notation to indicate same-size squares 3. The student uses colors as a notational representation to order squares based on their size.
19	Reverse process Creating scale 1. Given two objects and their relation to a third one(intermediary), the student constructs draws the third object		Canceled due to difficulty of transitivity
20	1. The student uses equal-sized scales to represent measurement. Report/graph plant height	11.	1. The student recognizes multiple solutions to construct objects based on >, < relations. 2. The student uses equal-sized scales to represent measurement. (plants) 3. Student verbally interprets change
21	1. The student verbally interprets on which side to increase or decrease to make/satisfy equality (play dough)		First week
22	1. The student discusses increase or decrease in volume context to make equality		
23	1. The student chooses the correct sign +/- to interpret the increase or decrease on sides to satisfy equality. (weight & volume context)	12.	1. The student chooses the correct sign +/- to interpret the increase or decrease on sides to satisfy equality. Height context Strips cut-paste: length context
24	1. Given symbolic interpretations (worksheets), the student chooses the correct sign +/- to interpret the increase or decrease on sides to satisfy equality.		
		13.	The student dramatizes +, - size as the action of moving forwards and backward with a variety of lengths.
		14.	The student increases/decreases a quantity by a fixed amount in a volume context. (perform +/-a)
25	1. The student uses + and – signs to construct equalities with one-side addition/subtraction in a part-whole context.	15.	1. The student uses + and – signs to construct equalities with one-side addition/subtraction in a part-whole context. 2. The student reports two dual comparisons of 3 objects (in weight context)

			3. The student concludes the third relation based on two relations between 3 objects (in weight context)
26	1. The student determines an addition amount to make equality 2. The student interprets a quantity as the addition of one to another Animal height game: one-side addition	16.	1. The student determines an addition amount to make equality. 2. The student finds unknown in an equality with one side addition. 3. The student recognizes multiple solutions to equations with two unknowns (a =? + ?) Animal height game: one-side addition
27	1. The student uses +/- signs to interpret operation to make equal-length 2. The student enactively investigates increase and decrease amount (difference amount) to create equal length (paper strips: cut and paste)		See Lecture 18
28	1. The student interprets the increase amount iconically 2. The student compares increase amount of different situations increase amount: plant height		
29	1. The student discusses how to make equality, unequal, and equal again by addition and subtraction (in volume and weight context) 2. The student interprets the effects of addition or subtraction of the same amount on both sides of inequality (in volume and weight context)	17	1. The student finds unknown in an equality with two side addition in height context. 2. The student finds multiple solutions to equations with two unknowns. 3. The student adds equal amounts to both sides to preserve equality. Animal-height game two sides height context
30	1. The student models equalities with two-side addition (in height context) 2. The student uses algebraic notation to interpret equalities with addition on two sides (in height context) Animal height with two sides addition		
	See HLT Lectures 27&28	18	1. The student interprets the increase amount iconically in height context (plant height) 2. The student determines addition and subtraction amounts to make them equal.

			3. The student experiments and recognizes equality of addition and subtraction amount (difference amount) Difference amount plant height
31	1. The student models symbolic equations with one-sided addition or subtraction in the enactive mode of representation by using paper strips		Letter notation is taken out of trajectory.
32	1. The student reads equalities and inequalities based on real-life models 2. The student uses algebraic equalities and inequalities for real-life designs Rainbow Activity	19	1. The student determines quantities of objects based on equality and inequality relations in a weight context. 2. The student reads and uses algebraic interpretations of equality/inequality relations and equations with one-sided addition in a real-life weight context. 3. The student preserves the quantity represented in interpretation to use it in addition equations (relations and equations are connected in a system). Rainbow Activity
		20	1. The student operates addition and subtraction on equalities based on given expressions, such as +/-a, in a volume context 2. The student experiences operational properties on equalities (the starting point is equality). 3. The student realizes and compares algebraic expressions such as a+/-b in volume context 4. The student experiences operational properties on equations (starting point changes)

CURRICULUM VITAE

Surname, Name: Sofuoğlu, Sevgi

EDUCATION

Degree	Institution	Year of Graduation
MS	METU Secondary Science and Mathematics Education	2015
BS	METU Mathematics Department	2009

FOREIGN LANGUAGES

Advanced English

PUBLICATIONS

Sofuoğlu, S. (2015). *Reasoning about and graphing the relationship between covarying quantities: The case of high school students and prospective mathematics teachers* (Master's thesis, Middle East Technical University).

Sofuoğlu, S., & Çetinkaya, B. (2017). Kovaryasyonel değişkenlerin grafikleştirilmesinde düzgün ve düzgün parçalı grafik çizen öğrencilerin değişim varyansı bazında düşünme süreçleri. In *3. Türk Bilgisayar ve Matematik Eğitimi Sempozyumu* (pp. 139-141), Afyonkarahisar, Türkiye, 17-19 Mayıs.

HOBBIES

Baking, Miniature, Computer Technologie