OPTIMAL PORTFOLIO ALLOCATION UNDER FRACTAL THEORY

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ABSTRACT

OPTIMAL PORTFOLIO ALLOCATION UNDER FRACTAL THEORY

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The Efficient Market Hypothesis has been dominating the literature of finance for a long time. Meanwhile, the problematic assumptions and inappropriateness of Efficient Market Hypothesis in explaining real-life financial markets have dictated the significance of developing new theories and approaches. Up to now, among the literature on finance, the Fractal Market Hypothesis has developed as an alternative to the Efficient Market Hypothesis. The Fractal Market Hypothesis is developed based on fractal geometry and fractal Brownian motions, and their applications on financial markets. In essence, the hypothesis postulates that financial markets are structured as fractals, they exhibit statistical self-similarity, and long-term memory in their time series. On the other hand, the various portfolio applications to this hypothesis are quite limited in the literature. The present portfolio studies have a number of basic problems, including the definition of covariance, inability to propose a portfolio for more than two assets, and detrending. In this thesis, a portfolio optimization approach based on Fractal Market Hypothesis is developed which takes into account these problems of the existing models in the literature. This thesis proposes a portfolio optimization method, the Mean-MFTWXDFA (Mean-Multifractal Detrended Temporally Weighted Detrended Cross-Correlation Analysis) which is based on multifractal temporally weighted cross-correlation analysis with detrending approach by geographically weighted regression. The suggested method is also compared with those of classical portfolio applications such as the Mean-Variance, Mean-Value at Risk, and Mean-Conditional Value at Risk methods. The portfolio analysis include cryptocurrency market and three diversifying assets: oil, clean energy and equity. Applications of the fractal-based portfolio do reasonably well into out-of-sample analyses and outperform conventional ones.

Keywords: Fractal Theory, Fractal Market Hypothesis, Multifractals, Portfolio Optimization, Cryptocurrencies

ÖZ

FRAKTAL TEORİ ÇERÇEVESİNDE OPTİMAL PORTFÖY SEÇİMİ

AÇIKGÖZ, TÜRKER
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Etkin piyasa hipotezi uzun zamandır finans literatürüne hakim olmuştur. Öte yandan hipotezin sorunlu varsayımlarını ve uygulama tarafında gerçek dünyadaki finans piyasalarını açıklamada yetersiz kalışı, yeni teorilerin ve yaklaşımların geliştirilmesini zorunlu kılmıştır. Bu çerçevede Etkin Piyasa hipotezine eleştiri olarak geliştirilen Fraktal Piyasa Hipotezi, finans literatüründe kabul görmüştür. Fraktal piyasa hipotezi, finansal piyasaların fraktal yapıda olduğu, istatistiksel kendine-benzerlik özellikleri taşıdığı ve finansal zaman serilerinin uzun dönem hafızaya sahip olduğu temeline dayanmaktadır. Öte yandan, fraktal geometri ve fraktal brownian hareketlere dayalı olarak geliştirilen Fraktal Piyasa Hipotezine dayalı portföy uygulamaları literatürde oldukça kısıtlıdır. Halihazırda var olan portföy çalışmalarında ise bazı temel sorunlar; örneğin kovaryans tanımlaması, ikiden fazla varlık için portföy önerememe, trendden arındırma gibi sorunlar yer almaktadır. Bu tez çalışmasında fraktal piyasa hipotezine dayalı olarak ve literatürdeki mevcut modellerin bu sorunlarını çözen bir portföy optimizasyonu yaklaşımı önerilmiştir. Bu tez çalışmasında coğrafi ağırlıklı regresyon yöntemine dayalı trendden arındırmayı temel alan Multifraktal Geçici Ağırlıklı Trendden Arındırılmış Çapraz Korelasyon Analizi (MF-TWXDFA) baz alınarak varlıklar arasındaki doğrusal olmayan kovaryans ilişkisine dayalı bir portföy optimizasyonu yöntemi olan Ortalama-MFTWXDFA yöntemi sunulmuştur. Önerilen yöntem klasik portföy uygulamalarından olan Ortalama-Varyans, Ortalama-Riske Maruz Değer, Ortalama-Koşullu Riske Maruz Değer yöntemleri ile karşılaştırılmıştır. Kripto para piyasası ve üç adet çeşitlendirme amaçlı varlık (petrol, temiz enerji, hisse senedi) portföyü üzerine yapılan analizler sonucunda, fraktal temelli portföy uygulamalarının klasik yöntemlere kıyasla, örneklem dışı analizler neticesinde, üstün performans gösterdiği bulgusuna ulaşılmıştır.

Anahtar Kelimeler: Fraktal Teori, Fraktal Piyasa Hipotezi, Multifraktallar, Portföy Optimizasyonu, Kripto Paralar

To my family

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I will seize this opportunity to extend my very sincere appreciation to my family for their overwhelming support and encouragement. You gave me strength through your support, and your belief in me is what made this achievement possible. A special thanks to my dear mother, Döndü Açıkgöz, and my sister, Gül Açıkgöz, for their endless support and encouragement.

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LIST OF ABBREVIATIONS

ADF	Augmented Dickey-Fuller Test
ARCH	Autoregressive Conditional Heteroscedasticity
BTC	Bitcoin
CAPM	Capital Asset Pricing Model
CVaR	Conditional Value-At-Risk
EMH	Efficient Market Hypothesis
ERC	Equal Risk Contribution
ETH	Ethereum
EW	Equally-Weighted
FMH	Fractal Market Hypothesis
fBm	Fractal (Fractional) Brownian Motion
GARCH	Generalized Autoregressive Conditional Heteroscedasticity
GWR	Geographically Weighted Regression
JA	Jensen Alpha
KPSS	Kwiatkowski–Phillips–Schmidt–Shin Test
Mean-MFTWXDFA	Mean-Multifractal Temporally Weighted Detrended Cross-Correlation Analysis
M-CVaR	Mean-Conditional Value at Risk
MFDCCA	Multifractal Detrended Cross-Correlation Analysis
MFDFA	Multifractal Detrended Fluctuation Analysis
MFDMA	Multifractal Detrended Moving Average Fluctuation Analysis
MFXDMA	Multifractal Detrended Moving Average Cross-Correlation Analysis
MFCCA	Multifractal Cross-Correlation Analysis
MFTWDFA	Multifractal Temporally Weighted Detrended Fluctuation Analysis
MFTWXDFA	Multifractal Temporally Weighted Detrended Cross-Correlation Analysis
MOOP	Multi-objective Optimization Problem
MPT	Modern Portfolio Theory
MV	Mean-Variance

M-VaR	Mean-Value at Risk
PP	Phillip-Perron Test
SR	Sharpe Ratio
sBm	Standard Brownian Motion
TR	Treynor Ratio
VaR	Value-At-Risk
XRP	Ripple

CHAPTER 1

INTRODUCTION

Portfolio allocation has been one of the main problems in finance literature. In his pioneering study, Markowitz [50] proposed the mean-variance (MV) optimization structure. He underlined the importance of diversification and offered a framework based on return and risk trade-off. The MV model addresses the selection of risky assets by their expected returns and variances. The idea of this framework started a new era in finance, so called the Modern Portfolio Theory (MPT). Today MV model is considered as the milestone for MPT and widely accepted for portfolio selection problems, diversification, and wealth allocation [18, 45, 60].

Following the establishment of MPT, various portfolio allocation techniques have developed. Markowitz [50] utilizes the covariance structure of assets as a risk measure. Further applications based on different risk measures are offered in the literature. For instance, researchers proposed utilizing value-at-risk (VaR) as the risk measure in the MV setting [5, 24]. The mean-VaR optimization focuses on minimizing portfolio's VaR, which aims to minimize potential loss in portfolio's value considering a specified confidence interval. Rockafellar and Uryasev [58] proposed using conditional value-at-risk (CvaR) as a risk measure in the portfolio optimization problem. Such a model allows one to minimize the expected loss beyond the portfolio's VaR, and by doing so, mean-CvaR minimizes tail risk accordingly.

Another important application of MV is to modify risk measurement according to the heteroscedastic structures in financial markets. Engle [17] and Bollerslev [7]'s studies on autoregressive conditional heteroscedasticity (ARCH) and generalized autoregressive conditional heteroscedasticity (GARCH) models pioneered the literature and revealed that financial time series contain heteroscedastic structures, suggesting that variance of the return series are not constant; instead, they vary over time. Since then, many multivariate GARCH models have been developed for estimating variance and covariance structures. By utilizing these models, Kroner and Ng [35] provided a framework for the portfolio optimization problem. The authors suggested that stochastic volatility estimations obtained from GARCH family analysis can be considered as risk measurement in the MV framework.

The aforementioned approaches in the literature of MPT are based on modifying risk measurement in the MV model. Along with the MV framework with different risk measures, various asset pricing models are also developed under MPT. Building on the framework of

Markowitz, the Capital Asset Pricing Model (CAPM) was developed independently by Sharpe [60], Lintner [39], and Mossin [53]. It is a one-factor model that considers the linear dependency relationship between a risky asset and the general capital market, and offers portfolio selection for investors. The main idea of CAPM is to consider the sensitivity of risky assets against market movements as a risk measure and set the portfolio allocation accordingly. Following the idea of CAPM, the literature of MPT has expanded the model and developed multi-factor models. For instance, the Arbitrage Pricing Theory of Ross [59] sets a linear relationship between asset prices and macroeconomic factors. The three-factor model of Fama and French [19] expands CAPM by including two additional risk factors: firm size effect and value premium. Carhart [11] included a momentum factor to the three-factor model of Fama and French, expanding the model to a four-factor model. Fama and French [20] built upon the previous study and proposed a five-factor model adding profitability and investment factors to the three-factor model. In general, factor models are based on the CAPM framework and relate investment decisions to the asset's co-movement with the market, firm-specific factors, and market dynamics.

As discussed above, MPT has various risk measures such as basic variance-covariance structures, VaR, CVaR, GARCH estimations, or market-based measures (i.e., CAPM-based factor models). The estimation of risk is the crucial part of portfolio optimization studies. On the other hand, models and approaches in the literature have undeniable deficiencies with real-world data. This is because MPT relies on the Efficient Market Hypothesis (EMH) on the theoretical foundations, so that risk measurement of financial return series is based on sample variance-covariances and the fundamental research methods are linear approaches [13, 66].

To understand the deficiencies of the models in MPT, it is necessary to discuss the core theoretical problems that MPT relies on. The efficient market hypothesis of Fama [18] has dominated the literature of finance for a long time. The hypothesis claims several questionable features of financial markets, among others, for instance, market efficiency and random walk assumption for asset prices. On the other hand, the literature shows that financial markets are not efficient and most assets do not exhibit random walk behavior. The empirical studies relying on EMH have also several non-realistic properties as well. The models based on the hypothesis mostly measure linear relationships, assume Gaussian distribution of (log) asset returns, consider stationarity, and no anomalies in the market. These are not satisfied in the real world mostly for some reasons: (i) financial markets are complex systems that cannot be explained fully by linear dependency approaches, (ii) log return series mostly have heavy-tail distribution, and (iii) financial assets have long-memory and exhibit long-range correlations [23, 40].

The Fractal Market Hypothesis (FMH) as an alternative to EMH has arisen considering the aforementioned problems. The literature discusses that FMH can explain market dynamics better than EMH [57, 56] because FMH evaluates financial markets as complex structures with nonlinear and chaotic dynamics and thus it fits the real world better than EMH [49]. FMH is based on the fractal theory and fractal Brownian motions of Mandelbrot [45] and Mandelbrot and Van Ness [49]. Further, Peters' studies [56, 57] applied the fractal theory of Mandelbrot to explain financial markets. He criticized the inefficiency of EMH and its assumptions of normality and random walk to explain financial markets. Peters suggested that financial markets are chaotic systems and display fractal behaviors and he formulated FMH using fractal theory. FMH assumes that financial markets do not behave as a random walk; instead, they are shaped with self-similarity and follow certain patterns (fractals) across different scales. Since then, various studies have approved FMH and have shown that financial markets do exhibit fractality and EMH is not relevant (see Jiang et al. [30] for a comprehensive literature review of over a thousand articles on the topic). The FMH has been drawing attention from academia recently. However, most studies in the topic focus on the fractality of financial markets and nonlinear relationships between financial assets. Only a few studies consider practical implications such as portfolio optimization under the fractal market hypothesis. This thesis fills the gap and contributes to the literature in this way.

The Mean-Variance model of Markowitz aims to maximize return for a given portfolio risk or minimize risk for a given expected return. The model uses an appropriate risk measure σ_{ij}. As discussed previously, various risk measures are defined in the literature of MPT. Here, estimating the risk component is a bit problematic. The fundamental model uses standard measures such as Pearson correlation coefficient (PCC) for such optimization and hence assumes stationarity and linearity (i.e., stable PDFs of data over time). However, as the FMH discusses, this is not a good fit for real-world cases, and financial return series are mostly nonstationary, meaning that return series have long-range autocorrelations (either linear or nonlinear) and fat-tail distributions. So it is necessary to define a well-fitted risk measurement considering the fractal structures in financial markets.

Various methodologies have been recently developed to overcome such issues. These methods rely on statistical mechanics to estimate nonlinear correlation between two nonstationary signals. Later on, these nonlinear correlation measures are utilized to model the risk measure in the MV model by a few studies. For instance, Zhou [73] developed the multifractal detrended cross-correlation (MFDCCA) model to measure power-law cross-correlations between two nonstationary time series. Further, Jiang and Zhou [31] offered the multifractal detrended moving average model (MFXDMA), which follows the original method of Zhou [73] but instead of using polynomial fitting, it uses the moving average (MFXDMA) method for eliminating local trends. These two methods are criticized by the literature due to their approach on estimating detrended covariances. MFDCCA and MFXDMA methods can result in covariances with complex numbers. As a solution to this, Oswiecimka et al. [54] and Kwapien et al. [36] proposed multifractal cross-correlation analysis (MF-CCA) for estimating detrended cross-covariance while overcoming the sign issue.

Some portfolio applications are offered in the literature. For instance, Sun and Liu [62] modified the MV model and used the detrended covariance derived from MFDCCA instead of classical covariance estimation and named the model as Mean-DCCA. Chun et al. [13] also conducted a similar approach but only considered bivariate portfolios and could not provide a general framework. Later, Zhang et al. [71] generalized Mean-DCCA to multiple assets. These studies present portfolio alternatives but they only consider $q = 2$ (medium fluctuation) and ignore other moments. Tang and Zhu [69] and Zhu et al. [75] modified the MV model and replaced covariance with the detrended cross-covariance function and offer the Mean-MFDCCA model. However, they also modified the original method of Zhou [73] and ignored the sign of detrended covariances by forcing them to be positive. While this approach of detrended covariance estimation is a common application in the literature for studies examining only the fractal relationship in financial markets, it poses a problematic issue for portfolio optimization. This is because it can lead to information loss and misinterpretation of real co-movements. In addition, Wang et al. [66] used MFXDMA for portfolio optimization and offered the Mean-MFXDMA model. Further, Madani and Ftiti [43] used the multifractal cross-correlation (MF-CCA) model and offered the Mean-MF-CCA model for optimal portfolio weights of two-asset portfolios. However, the Mean-MFCCA model lacks offering a portfolio model with n-risky assets where $n > 2$.

In general, the portfolio optimization applications proposed under the fractal market hypothesis consist of three main issues. The first problem is that when we consider the original estimation method of local detrended cross-covariance such as [62], [13], and [71] did, the final outcome possibly returns detrended cross-correlations with complex numbers. In such a case, it is problematic to estimate optimal portfolio weights with complex nonlinear correlation estimates because the closed-form solution to the MV problem may not yield an optimal solution since it is possible that the covariance matrix (constructed by cross-covariances) may not be positive-definite. On the other hand, taking the absolute value of local fluctuation functions similar to [69] and [75] causes information loss on nonlinear dependency. Second, the models mostly lack to offer a portfolio framework for several assets. Lastly, the current portfolio models rely on detrending algorithms with global parameters and the approaces may not detect local trends appropriately.

Considering the previous literature, we can offer a more developed and sophisticated model that overcomes the current problems discussed above. Wei et al. [67] improved MF-CCA of [54] and [36] and presented the multifractal temporally weighted detrended cross-correlation analysis (MF-TWXDFA) model, which can be considered as a robust version of MF-CCA. In the MF-TWXDFA model, detrended cross-covariance is estimated similarly to MF-CCA but the model modifies the eliminating method of local trends. Instead of fitting m-order polynomial to the series globally, the model offers using geographically weighted regression (GWR) method. GWR is a weighted-regression algorithm that fits the detrending polynomial regressions locally for each point in the series $X(u)$ and $Y(u)$ by using the geographically nearby elements. In this way, it eliminates local trends significantly better than the previous models, avoids spurious correlation relationships, and provides more robust results.

In this thesis, we use the MF-TWXDFA approach of detrended cross-correlation for risk measurement in the MV model and construct optimal portfolios. We propose the Mean-MF-TWXDFA for portfolio optimization. On the empirical side, this thesis studies cryptocurrency portfolios. These assets have garnered significant attention recently and hold an important place in financial markets for two main reasons. First, cryptocurrencies are technology-based assets that the projects behind them present noticeable technological solutions such as AI projects, DeFi, Web 3.0, data processing, and so on. Second, cryptocurrencies have their own market dynamics. Diverging from other assets, the crypto market has extreme returns and significant bull/bear market cycles. On the other side, the crypto market is defined by its high-risk profile and volatility. Thus, these assets should be diversified well.

This study constructs a portfolio of cryptocurrencies with three other diversifying assets; (i) equities (S&P 500), (ii) crude oil (WTI), and (iii) clean energy (iShares Global Clean Energy ETF). We choose equities because they are one of the most traded assets in the world and generally offer a good performance and stabilize the return in high-volatility portfolios. We also add two energy-based securities. Crude oil proxies energy prices while clean energy stocks hold for clean energy investments. Despite its popularity, cryptocurrencies consume a high level of energy and energy is the main input in the market. So we believe that energy sources can perform well in diversifying crypto market assets. In addition to crude oil, clean energy stocks are also added to measure both sides of the energy market (clean and dirty) and see how they contribute to portfolio diversification.

The studies on crypto market has shown that crypto assets have multifractal characteristics. In 2018, da Silva Filho et al. [15] applied multifractal approach to Bitcoin prices. The authors, based on data from September 2011 until November 2017, show that the price series of Bitcoin has a high degree of multifractality, driven by both long-range correlations and fat-tailed distributions. Stosic et al. [61] examines multifractality properties in daily price and volume changes using a multifractal detrended fluctuation analysis (MF-DFA) in fifty cryptocurrencies. They observed that the prices could be behaviorally more complex and multifractal behavior than the volumes series in large fluctuations. The study shows that the correlation structure of price changes is largely uncorrelated, while the volume changes exhibit long-term anti-persistent correlations. Cheng et al. [12] investigate the momentum effect in the cryptocurrency market with mono-fractal and multi-fractal detrended fluctuation analysis based on DFA and MF-DFA. In this study, four major cryptocurrencies are taken into consideration at different time scales: Bitcoin (BTC), Ethereum (ETH), Ripple (XRP), and EOS. The results indicate that for BTC and ETH, there is a strong momentum effect, especially in the long term, meaning that price trends show persistence. In contrast, XRP and EOS have a tendency for price reversion in the presence of large deviations. The evidence suggests that such cryptocurrencies behave like nonlinear complex systems for which market responses to events are very scale-dependent. It contributes to the literature about the dynamics of the cryptocurrency market, underline different fractal characteristics and efficiency observed for diverse digital assets. Mnif et al. [52] evaluated how cryptocurrency market has performed in COVID-19 pandemic. The authors used MF-DFA approach on five crypto assets; BTC, ETH, XRP, LTC, and BNB. They separated sample periods into pre- and post-COVID-19 periods. The study compares how the generalized Hurst exponents have evolved between two periods. The results showed that crypto market is not efficient. On the other hand, COVID-19 positively influenced on market efficiency. Kakinaka and Umeno [32] examine the asymmetrical multifractality and market efficiency of the main cryptocurrencies during the COVID-19 pandemic by using the Asymmetric Multifractal Detrended Fluctuation Analysis (A-MF-DFA).

The research work comes up with a differentiated impact of the pandemic on the short- and long-term investment horizons. At the short-term horizon, multifractality increased and inefficiency prevailed due to strong herding behavior. In contrast, at long-term horizons, multifractality decreased, meaning improved market efficiency. The authors have further pointed out a shift of asymmetric properties with dramatic changes in persistence and behavior of investors against large fluctuations in the long term. In view of the above findings, the current study presents evidence for intricate behaviors of cryptocurrency markets in crisis and thus places more emphasis on investigating market efficiency and investor behavior by taking into consideration multiple time scales. Aslam et al. [3] examine dynamic efficiency in major cryptocurrencies using MF-DFA as a technique to probe the efficiency of markets over time. This research concerns six major cryptocurrencies—BTC, ETH, XRP, LTC, ADA, and BNB, during the COVID-19 pandemic. The authors find significant multifractality across all cryptocurrencies, thus inefficiency in the market. The largest inefficiency was found in Bitcoin and Litecoin, while the smallest was observed in ADA and BNB. According to the analysis in a moving window, efficiency changed very fast in these markets and followed closely the turmoil or crisis events on the market; for instance, the pandemic.

This thesis contributes to the literature in several ways. It offers a portfolio allocation framework under FMH. As discussed previously, the assumptions of EMH are quite challenging and mostly lack fitting real-world cases. The models in MPT are insufficient to measure nonlinear and chaotic dynamics in financial markets. On the other hand, this thesis suggests using an appropriate measure for risk considering FMH. The model proposed significantly outperforms portfolio methods in classical finance such as MV, M-VaR, and M-CVaR. This thesis enriches the literature on fractal structures in the cryptocurrency market as well. The findings show that the cryptocurrency market does not show random walk behavior. In contrast, cryptocurrencies have scaling behavior, nonlinear dynamics, and long-range power-law autocorrelations with significant Hurst exponents. Lastly, although various portfolio allocation techniques are offered in the literature, our approach overcomes the deficiencies of the previous studies and further provides a more robust approach for detecting detrended cross-correlations.

The study encompasses five chapters. The introduction part here enlightens the motivation and contribution of the thesis. The second part of the study discusses theoretical background and review of the previous literature. It contains evaluation of FMH with critics on EMH, reviews the methods on estimating fractality and multifractal structures in financial markets, and portfolio diversification literature in the cryptocurrency market. The third chapter of this thesis contains research methodology applied in this study. The fourth part of the study provides empirical evidence and research results. The last chapter of the study is the conclusion and evaluates the research findings and discusses practical implications as well as suggestions for further studies.

CHAPTER 2

THEORETICAL BACKGROUND

This part of the study consists three parts. First, the review of theoretical background of the study is presented. In this part, fractal theory and Fractal Market Hypothesis are discussed. Second, the methods for estimating fractality in financial markets are presented. Third, the discussion on portfolio diversification applications in the literature is provided.

2.1 Fractals and Fractal Structures

Fractal geometry is a mathematical framework that has been developed for the invariable explanation of the majority of natural structures manifesting different degrees of self-similarity [1]. It is also applied in some artificial constructions. Generally, a fractal refers to any set of self-similar objects showing repetition of the same pattern at different scales [21].

Mandelbrot [48] defines "fractal" as a geometric objects (probably with non-integer dimensions) where each parts of the object are associated to share the similar characteristics of the whole. Basically, a fractal is considered a geometric shape that, at different scales, it reflects self-similar and has chaotic behavior. Mostly, fractals are found in nature and created by repetition of a simple process over and over in some iterative way. Fractals are described by their fractional dimensions, which exhibit their complexity and how they fill space in a non-integer dimension.

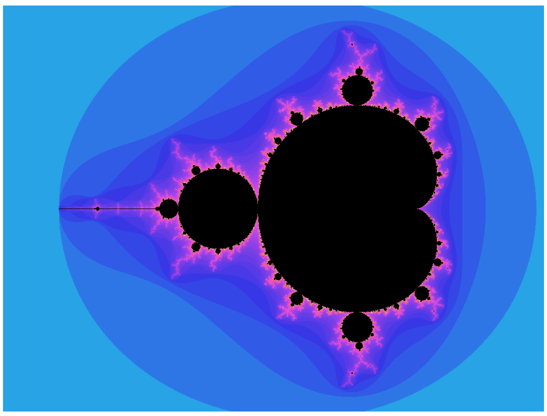

Figure 2.1: Mandelbrot Set

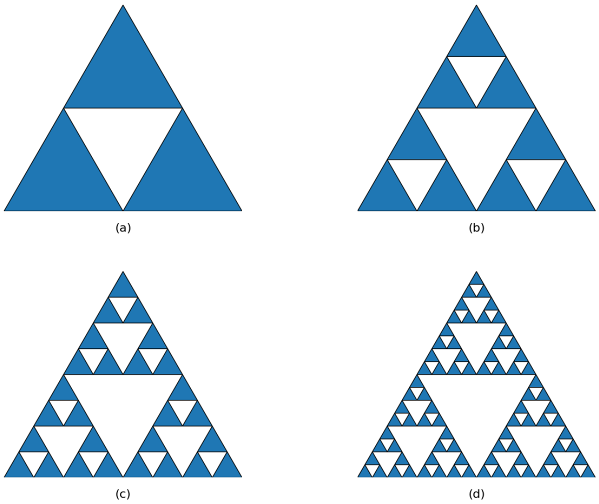

Figure 2.2: Sierpinski Set

Figure 2.1 and Figure 2.2 present two well-known examples of fractals. The figures exhibit the most common features of fractals. Fractals have self-similarity regarding their space. It reflects that smaller proportions of a fractal object share similar features of the entire structure. The object presents similar structures across various scales. No matter how zoom-in or zoom-out, fractals have similar shapes. Unlike Euclidean geometry, fractal geometrical shapes have non-integer dimensions. Thus, fractals usually have infinite complexity and complexity of the structure continues indefinitely.

Fractals can be observed both in nature and artificial structures. According to Mandelbrot [48], many patterns in nature are so complex, fragmented and irregular that they cannot fit a standard Euclidean Geometry. So, he calls these patterns or shapes as fractals. For instance, mountain ranges and rocky landscapes are fractal-like in nature.

The irregular, jagged outlines of coastlines are self-similar and scale invariant. One of the first studies on fractals was conducted by Mandelbrot [46] to understand the long of coast of Britain. As of the date, it is impossible to measure the exact long of coastlines. The scientists can only make an accurate approximation on the topic. Mandelbrot [46] discovered that coastlines have fractal shapes and statistical self-similarity. If one looks at a small part of the coastline, then basically the same type of complexity and details are found in it. Further, in his book entitled "The Fractal Geometry of Nature", Mandelbrot [48] presents a comprehensive examination on the fractality of natural objects.

Various examples of fractals are also observable in nature. Mountain ranges and rocky landscapes have fractal-like features; that is, they are self-similar in their features. They repeat the same ruggedness over and over, from peaks to valleys. Even the structure of clouds, from wispy cirrus clouds to large cumulus, forms fractals. It simply means that shapes and patterns are related to each other in a self-similar way on all scales on which they are observed. Fractals in nature include the geometry of the branching of trees and such plants as ferns, with smaller and smaller copies of the shape of the whole plant repeated in the branches and leaves. The river systems form fractal branching naturally. The tributaries branch into smaller streams in a self-similar way; thus complex networks can be formed. Geometric forms of snowflakes are also an example of symmetric, intricate fractals. Each arm of a snowflake is self-similar; it consists of smaller structures which also demonstrate the same shape as the whole. Due to the path it follows, with all its branchings and splitting to form a pattern, a lightning bolt exhibits self-similarity at different scales and hence is a fractal. The human circulatory system forms a fractal pattern with its networking of blood vessels and capillaries. It ensures efficient transportation of blood throughout the body with this branching of veins and arteries.

Though observing fractals is mostly trivial in nature, fractals are not limited to it. They are also observable in artificial structures as well. Batty and Longley [6] studied fractal geometry of urban growth and design. The authors discovered significant evidences on how human-made structures like cities have fractal patterns. Frankhauser [22] utilized fractal approaches for analyzing urban agglomerations and urban patterns and confirmed fractality in urbanization. Fractals are also visible in arts. Taylor et al. [63] showed that the artists have fractal patterns in their paintings. The historical records on criminal activities display that crime patterns have fractal dynamics and chaotic behaviors, in both individual and society levels [51, 4]. The social groups and networks of primates, both human and non-human species, perform fractal structures [68].

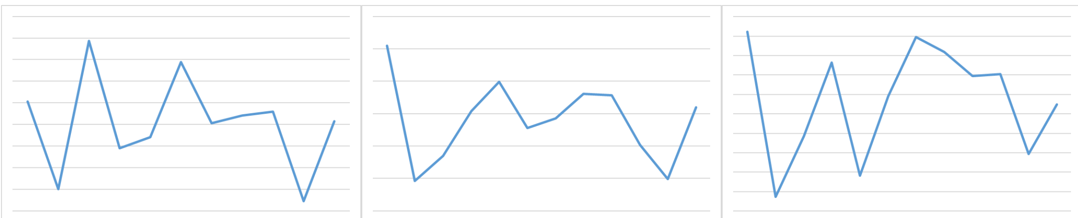

Figure 2.3: Self-Similarity in Bitcoin Returns: Daily, Weakly and Monthly Return Series

Fractals are also observed in time series and it is one of the most important study topics in fractal theory. While fractal shapes exhibit self-similarity regarding space, fractal time series show statistical self-similarity concerning time [56]. Statistical properties of fractal time series significantly differs from conventional time series. The main properties of fractal time series can be considered as fat-tail probability distribution function, gradual decline in autocorrelation function, global and local self-similarity behavior, having long-memory or short-memory in the series [37]. Figure 2.3 presents a visual example on self-similarity in Bitcoin prices. It displays twelve consecutive data from daily, weekly and monthly return series. It is quite challenging to categorize the graphs without any scales on the axes of time and return. This simple example presents self-similarity feature of fractal time series.

2.2 Fractal Market Hypothesis versus Efficient Market Hypothesis

Fractal Market Hypothesis is an alternative perspective to Efficient Market Hypothesis (EMH), claimed by Peters [56, 57] to describe capital markets more realistic than EMH. EMH is offered by Fama [18] and since then, it has been dominated finance literature.

The fundamental idea of EMH is that it assumes available information set is fully reflected to asset prices and the market is in the equilibrium state. The statement of current prices reflects all available information implies that asset price changes are independent and identically distributed. Since all past information (such as firms' fundamentals, macroeconomy, price action etc.) is already priced in the market, it implies that making a future prediction is impossible. Hence, asset prices are random walk and percentage change in price series are independently and identically distributed. The only relevant information for asset prices, let say in time $t+1$ is the information that is expected to come to the market at time $t + 1$. The market is considered to react only new upcoming information. To conclude, today's price is only related with today's news and past information has no meaning for asset returns.

EMH considers markets are consisted by large number of rational investors. The assumption leads to several consequences of the theorem. First as financial markets are of large number of investors, the prices cannot be effected by individuals. None of the actors have capability of effecting prices. Also, since the market is competitive and many market participants decide investments based on the past information and their future expectations, they assure that it is a "fair" game. Considering the rationality assumption, investors are considered to be smart. The decision-making process of individuals gives the guarantee of fair prices since EMH suggests that past information is obtained free and evaluated accurately. The only difference between investors is assumed to be their risk appetite. According to Peters [56], EMH sees the market as a community or a collective action that made up of too many people to be wrong.

Fama [18] states that EMH consists three levels of market efficiency according to relevant information subsets. The first type of market efficiency is called weak-form efficiency. In this stage of the market, current assets prices reflects only historical prices and although technical analysis cannot beat the market, with fundamental analysis or inside trading, it may be possible to earn profit. The second type of market efficiency is called semi-strong efficiency. Here, it is assumed that the market has already priced both historical price action and fundamentals. In a semi-strongly form, the only way to make profit is inside trading. The last one is called strong-form efficiency of the market. In this stage of the market, all information subsets, including historical prices, fundamentals about the economy and firm, and the special information that have not been shared with the public, are discounted by the market. So, it is certainly impossible, even for inside traders, to gain excess return in a strong-form market. However, note that it is empirically impossible to empirically test strong-form efficiency due to its reference information set [25, 34].

To summarize, the main assumptions of EMH can be grouped in two ideas: in theoretical side, we have normal distribution of asset returns, random walk behavior of asset prices (as well as asset returns) assumptions. In the empirical side, these also implies linearity of relationships in the market and stationarity (stable pdfs) of returns. Hence, the standard statistical analysis of linear models can give misleading results if the market perform nonlinear dynamical systems [56].

Now, we can turn our focus on FMH. Fractal Market Hypothesis relies on as well as inspired from two areas: (i) Hurst's study [29] and Mandelbrot's fractal geometry [45] and fractal Brownian motions (fbm) [49]. Inspired from Hurst [29]'s study, Peters [56, 57] declared that it makes no sense of that future is independent from the past. According to Peters, every action we took in the past is the reasons why we are here and will decide where will we be in the future. A human's present cannot be considered apart from his/her past. It is also observed in natural phenomenon and societies. The perspective is called "memory", which is completely refused in EMH.

Let us first start with Hurst's study on Hurst exponents and R/S analysis which holds an important place in fractal theory. Hurst [29] was a hydrologist and he worked on Nile River Dam. He measured reservoir level of Nile River over 40 years and observed that natural systems do not obey a random walk model and are not Gaussian. Instead, they follow a biased random walk model. While the next step of a random walk is completely random, biased random walk series have contains a trend with noise process. Hurst developed a robust model, R/S analysis, to test his idea and it can be applied to different time series. The model of Hurst can defined in 6 steps as follows.

Step 1: Consider a time series $\{x_t : t = 1, 2, ...P\}$. We divide the series into B number of subsequences with length G such that $B * G = P$. Also consider that $C_b(b = 1, 2, ...B)$ is the b-th subsequence and $x_{v,b}$ is the k-th observation $(k = 1, 2, ...G)$ of subsequence b. The arithmetic average of C_b is denoted by $\bar{x_b}$.

Step 2: Calculate the cumulative deviation of subsequences C_b as given below.

$$X_{v,b} = \sum_{t=1}^{v} (x_{v,b} - \bar{x_b}) \tag{2.1}$$

Step 3: The range of cumulative deviation, R_{C_b} and corresponding standard deviation S_{C_b} for each subsequence C_b is calculated as:

$$\begin{aligned} R_{C_b} &= max_{1 \leq k \leq G} X_{v,b} - min_{1 \leq k \leq G} X_{v,b} \\ S_{C_b} &= \sqrt{(\frac{1}{G}) \sum_{t=1}^{v} (x_{v,b} - \bar{x_b})^2} \end{aligned} \tag{2.2}$$

Step 4: R/S statistic for subsequent b can be obtained by $(R/S)_b = R_{C_b}/S_{C_b}$.

Step 5: The mean R/S is calculate by $(R/S) = \frac{1}{B} \sum_{b=1}^{B} (R/S)_b$.

Step 6: For each G, there is a corresponding R/S value and following the power-law, we have the equality $(R/S)_G = b * G^H$ where a is a constant. The parameter H is obtained by taking the logarithm of both sides and fit an ordinary least square (OLS) such that $\log((R/S)_G) = H\log(G) + \log(b)$.

The parameter H is called Hurst exponent. The method developed by Hurst [29] allows us to examine behavior of time series and decide whether it is a random walk or not by using Hurst exponent H. The series x_t can have three behaviors; (i) random walk, (ii) persistent, (iii) anti-persistent.

The statistical mechanics tells us that random walk series follow $H = 0.5$. In this case, the cumulative deviations increments by square root of time G. If $H \neq 0.5$, then the series x_t is not a random walk and x_t have a memory. H is bounded by $0 \leq H \leq 1$. While $0 \leq H < 0.5$, the series have short memory and anti-persistent behavior is observed. In this case x_t is mean-reverting, meaning that when x_t shows a positive increments, it is highly possible that x_{t+1} may have a negative increments and vice versa. The power of this anti-persistency is characterized by how far H is from the random walk behavior $H = 0.5$. The last behavior of a time series is long-term memory and it occurs when $0.5 < H \leq 1$. In this case, the series have long-range dependency and persistent increments. For instance, if x_t had an upward movement, it is highly expected that x_{t+1} should also have a positive increment. Since Hurst [29], H exponent has been widely used to measure long-range dependency in time series.

The second inspiration for FMH is fractal geometry and fractal Brownian motions. While empirical literature of FMH relies on Hurst's study, theoretical part is based on Mandelbrot's studies. Mandelbrot defined fractal shapes and pioneered the fractal mathematics and geometry. Fractal objects are related with shapes. On the other hand, as we discussed previously, fractals are also exist in time series as well. Fractal time series have self-similarity in statistical measures. Mandelbrot and Van Ness [49] defined fractal time series by developing theory of Fractal (or Fractional) Brownian Motion (fBm). While Fama [18] utilize random walk model, Peters [56, 57] placed fBM at the center of FMH. Note that Brownian motion is the continuous-time version of random walk model.

In 1968, Mandelbrot and Van Ness [49] developed the idea of fractal Brownian motion by generalizing Brownian motion and including fractals into Bm. Let us first consider the classical Bm so that we can understand what Mandelbrot and Van Ness contributed to the literature and how it is related with FMH. Consider standard Bm, a stochastic process $\{B(t) : t \in [0, T]\}$. It has four main characteristics. Firstly, the process has a starting point 0 such that $B(0) = 0$, and secondly it is continuous almost surely. Third, $B(t)$ has independent increments such that $B(t) - B(s)$ where $t > s$ are independent. Lastly, the increments follow normal distribution with zero mean and $t - s$ variance such that $B(t) - B(s) \sim \mathcal{N}(0, t - s)$. The fourth characteristic of standard Bm (sBm) has crucial problems in terms of financial theory. Assume that $X(t_1) = B_{t_2} - B_{t_1}$ and $X(t_2) = B_{t_3} - B_{t_2}$ are the increments of sBm. Then, if X_{t_1} and X_{t_2} have zero mean $\mathbb{E}[X_{t_1}] = 0$, $\mathbb{E}[X_{t_2}] = 0$, referring to the third feature of sBm, the claim on increments being independent equivalents to $\mathbb{E}[X_{t_1}X_{t_2}] = 0$. Applying the idea to finance as EMH does, we can consider $W(t)$ as asset prices and X_{t_k} as asset returns and we get the following implication: financial asset prices are unpredictable, random walk processes and asset returns do not have autocovariance (or autocorrelation). Therefore, return series should not have a memory, either short or long range, and it is impossible to find a pattern or repeating, self-similar structures in the series. On the other hand, Mandelbrot and Van Ness [49] offered fBm to characterize fractal time series. fBm is actually the generalized version of sBm. In their study, Mandelbrot and Van Ness criticized sBm's independent increments feature and by referring Hurst's study [29], they proposed Bm with memory or dependent increments. The main contribution of fBm, for our topic, is that it can have three types of behavior in financial time series: (i) random walk, (ii) short-term memory, or (iii) long-term memory. A fBm $\{B_H(t) : t \in [0,T], H \in (0,1)\}$ is a continuous-time Gaussian process where the increments of the process may not be independent. In fBm, autocovariance function of the process is defined as $\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$ for $t, s \in [0,T]$. In a similar fashion, the increments have the following covariance function. Consider two increments at time t_1 and t_2 as $X_H(t_1) = B_H(t_2) - B_H(t_1)$ and $X_H(t_2) = B_H(t_3) - B_H(t_2)$. Then, we have $\mathbb{E}[X_H(t_1)X_H(t_2)] = \mathbb{E}[(B_H(t_2) - B_H(t_1))(B_H(t_3) - B_H(t_2))] = \mathbb{E} = \frac{1}{2}(|t_3 - t_1|^{2H} - |t_3 - t_2|^{2H} - |t_2 - t_1|^{2H})$. As the dependence structure shows, parameter H holds an important place in fractal time series. While $H = 0.5$, the model drops to be a standard Brownian motion with no dependency in increments. While $0.5 < H < 1$, the process is said to have persistent (positively correlated) increments and long-range dependency. Otherwise, $0 < H < 0.5$ is attributed to anti-persistent behavior or short-term dependency (negatively correlated) increments. The scaling parameter H is actually the Hurst exponent previously defined by H. Edwin Hurst in 1951.

So, Hurst discovered that many natural phenomenon such as rainfalls, sunspots, weather, consist of long-range dependency and memory and he developed the Hurst exponent H to test such behaviors in time series. Later on, Mandelbrot's studies established fractal geometry, defining objects with non-integer dimensions and self-similar structures. Further, Mandelbrot proposed fractional Brownian motions and defined fractal time series with self-similar structures in statistical properties. Deriving his theory from Hurst on empirical side and Mandelbrot on theoretical side, Peters [56, 57] offered fractal market hypothesis.

Peters claims the following arguments. Normal distribution assumption of EMH is mostly irrelevant for financial return series. Instead, the series show fat-tail distribution most of the time. The market frequently generates abnormal returns (positive or negative). While EMH calls these abnormal events as anomalies in the market, FMH considers them as a part of the market dynamic. EMH's normality assumption allows and leads us to use linear statistical measures. On the other side, according to FMH, financial markets are complex systems with nonlinear interactions and therefore, linearity is not a good fit for the market and nonlinear methods are better to explain the market dynamics. Just like natural phenomenon, financial markets also have fractal structures, self-similar behavior in statistical measures. Thus, while modelling financial markets, FMH suggests to consider this fact. Financial assets do not follow random walk; instead, they follow a biased random walk with memory. So, asset returns are not independent increments, rather they exhibit long-range dependency. FMH criticizes EMH in terms of these aspects and it provides an economic and mathematical perspective to analyze the market with fractals and self-similar statistical properties.

2.3 Methods for Estimating Fractality in Financial Markets

In this part of the thesis, we propose and discuss the methods for estimating fractal structures in time series. From now on, considering the aim of the study, we solely focus on fractals in time series. As mentioned before, fractal systems are defined by their scaling exponents (Hurst exponent) H. Further, statistical physics literature have extended the relevant theory. Some series exhibit only one fractal pattern or self-similarity and these series are named as "monofractals". For instance, Hurst's study, discussed previously, is an example of monofractal methods. On the other hand, describing a fractal with only one self-similar behavior mostly inadequate to define the characteristic of the series. Multifractals are type of fractals, generalization of fractals into multiple dimensions [47]. In terms of time series, multifractal series have multiple fractal dimensions or scaling behavior at various moments q. While monofractal series are characterized by only one scaling behavior H, now with multifractals, the series have various H depends on moment q. So, multifractal time series are described by their generalized scaling exponents $H(q)$. To summarize, fractals are generalized into multifractals, allowing us to examine several self-similarity features at various moments. In addition, multifractal analysis also provides us to examine nonlinear corrrelation/covariance structures, scaling behavior, and analyzing nonstationary time series as well. By this way, multifractal methods are perfect fit for financial series since they are generally known as their nonstationarity and nonlinearity.

The following subsections present and discuss most relevant methods for estimating multifractals in the literature. Note that there are over twenty methods in the statistical physics literature and it has been growing recently. See Jiang et al. [30] for further discussion on multifractal methods. We split the following subsections into two groups: methods for univariate series and bivariate series.

2.3.1 Methods for Univariate Series

This subsection of the study discusses three multifractal methods for univariate series, namely as multifractal detrended fluctuation analysis (MF-DFA), multifractal detrended moving average (MF-DMA), and multifractal temporally-weighted fluctuation analysis (MF-TWDFA). These methods are used to describe multifractal behavior of univariate time series. For the sake of convenience, we will keep the notations as similar as possible. These three methods are differentiated in terms of their approach for detrending the series. Let us begin with defining a univariate non-stationary time series $x(k)$ with length N such that $\{x(k) : k =$

$1, 2, ...N\}$ and $\bar{x}$ is the arithmetic average of the series. Then continue with the methods below.

2.3.1.1 MF-DFA Algorithm

In 1994, Peng et al. [55] published their pioneering work. They developed Detrended Fluctuation Analysis (DFA) and examined long-range power-law correlation in nonstationary fractal time series and tested it on DNA sequences. Though it pioneered the literature, DFA was only capable of describing single fractal (monofractal) structures. Further, Kantelhardt et al. [33] generalized DFA from monofractal to multifractal and offered MF-DFA to examine the features in various moments. The model consists of six steps as follows:

Step 1: Cumulative Profile: Take the cumulative deviation $X(t)$ of the series.

$$X(t) = \sum_{k=1}^{t} (x_k - \bar{x}),\ t = 1, 2, ...N \tag{2.3}$$

Step 2: Segmentation: Divide the cumulative deviation $X(t)$ into N_s number of nonoverlapping segments with equal length s such that $N_s \equiv int(N/s)$. Notice that N may not be the multiple of s most of the time. So, in order to avoid losing the data, we repeat the procedure from starting from the end of the series. Finally, we get $2 * N_s$ number of segments with length s.

Step 3: Detrending: Fit a polynomial of degree m to each segment v to get the local trends $\tilde{X}_v(t)$.

$$\tilde{X}_v(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + ... + \beta_m t^m \tag{2.4}$$

Where t is a trend variable such that $t = 1, 2, ..s$.

Step 4: Local Variance: Local variance for each segment v is calcualted as follows. For $v = 1, 2, ...N_s$, we have:

$$F^2(v, s) = \frac{1}{s} \sum_{j=1}^{s} [(X(v-1)s + j) - \tilde{X}_v(j)]^2 \tag{2.5}$$

For $v = N_{s+1}, N_{s+2}, ...2N_s$, we have:

$$F^2(v, s) = \frac{1}{s} \sum_{j=1}^{s} [(X(N - (v - N_s)s + j) - \tilde{X}_v(j)]^2 \tag{2.6}$$

Step 5: q-th order Fluctuation: Take the average of local variances $F^2(v, s)$ over all segments v and calculate q-th order fluctuation function $F(q, s)$.

For $q \neq 0$, we have $F(q, s)$:

$$F(q, s) = \frac{1}{2N_s} \sum_{v=1}^{2N_s} [[F^2(v, s)]^{\frac{q}{2}}]^{\frac{1}{q}} \tag{2.7}$$

While $q = 0$, we get $F(0, s)$:

$$F(0, s) = \exp\left\{\frac{1}{4N_s} \sum_{v=1}^{2N_s} \ln\left[F^2(v, s)\right]\right\} \tag{2.8}$$

Where $q \in \mathbb{R}$.

Step 6: Scaling Behavior: Determine the scaling behavior and multifractal characteristic of the series. Multifractal series should exhibit scaling behavior as follows.

$$F(q, s) \propto s^{h(q)} \tag{2.9}$$

By taking the logarithm of both sides, scaling exponent $h(q)$ can be obtained via linear regression.

$$log(F(q, s)) = a + h(q)log(s) \tag{2.10}$$

Now let us discuss the inferences of MF-DFA. The model estimates detrended fluctuation function $F(q, s)$ for various time scales s and moments q. MF-DFA allows us to examine statistical self-similarity of a non-stationary series at any q value in $\mathbb{R}$. The scaling parameter $h(q)$ is the generalized version of Hurst exponent. When $h(2) = 0.5$, it indicates random walk behavior of the series. The moment parameter q has some implications. For $q > 2$, $h(q)$ exhibit behavior in large-fluctuations while $q < 2$, $h(q)$ show small behavior in small-fluctuations. The evaluation of generalized Hurst exponent $h(q)$ is similar to the standard h. A $h(q)$ value larger than 0.5 implies persistent series and $h(q) < 0.5$ indicates anti-persistent series in moment q, either in small fluctuations $q < 2$, medium fluctuations $q = 2$ or large fluctuations $q > 2$. While mono-fractal characteristic is determined by $h(2)$, to be a multifractal behavior, $x(k)$ has to show multiple self-similarity. For this aim, the generalized Hurst exponent $h(q)$ is checked. A multifractal time series is supposed to have varying scaling exponent. If $h(q)$ is constant for each q, then we conclude that $x(k)$ has only one self-similar behavior and it is monofractal since its fluctuation function does not vary over different moments q. Otherwise, $x(k)$ has several self-similarity and it is multifractal. Another clue for determining multifractality is to check the Renyi exponent $\tau(q)$. According to Zou and Zhang [76], the Renyi exponent can be utilized to reveal multifractal properties. The Renyi exponent is calculated as $\tau(q) = qh(q) - 1$ and for multifractal series, it should be a nonlinear and increasing function of q.

2.3.1.2 MF-DMA Algorithm

Multifractal Detrended Moving Average (MF-DMA) is an alternative approach to MF-DFA. Multifractal time series are challenging data to handle with, since most series, especially financial returns, have nonstationary dynamics. So, multifractal methods efficiently remove any global or local trends in the series to ensure stationarity and avoid any spurious variance or correlation. For this purpose, Kantelhardt et al. [33] used polynomial fitting to each segment v. Further, Guo and Zhou [26] offered using moving average algorithm instead of polynomials for detecting local trends. MF-DMA algorithm consists of six steps. The only difference of MF-DMA is the detrending procedure. Consider our previous example $\{x(k) : k = 1, 2, ...N\}$ again.

Step 1: Cumulative Profile: Guo and Zhou [26] defines cumulative deviance as:

$$X(t) = \sum_{k=1}^{t} x(k) \tag{2.11}$$

Step 2: Moving Average Function: Estimate the moving average function $\tilde{X}(t)$ as follows:

$$\tilde{X}(t) = \frac{1}{n} \sum_{k=-\lfloor (n-1)\theta \rfloor}^{\lceil (n-1)(1-\theta) \rceil} X(t-k) \tag{2.12}$$

Where n is the window size, $\theta \in [0, 1]$ is the position parameter, $\lfloor X \rfloor$ and $\lceil X \rceil$ are floor and ceil functions. The function calculates moving average of the series using the window $-\lfloor (n-1)\theta \rfloor$ to $\lceil (n-1)(1-\theta) \rceil$. Afterward, the moving average $\tilde{X}(t)$ is considered as the local trend and rest of the algorithm is similar to MF-DFA.

2.3.1.3 MF-TWDFA Algorithm

Multifractal Temporally Weighted Detrended Fluctuation Analysis (MF-TWDFA) is another multifractal method proposed by Zhou and Leung [74]. It can be considered as a further improvement of the previous methods. Detrending is a crucial step in multifractal methods because if we cannot get the local trends correctly, then these local trends may appear to be related in various scales and result in spurious correlation and multifractality.

In the MF-DFA algorithm of Kantelhardt et al. [33] uses a global regression for each segment v to estimate local trends. MF-DMA method suggests moving means of data for detrending. On the other hand, Zhou and Leung [74] offers using more robust approach for detrending rather than moving means or globally defined regressions. It has been suggested, in geographical studies most especially [64], that points closer together in space tend to be more closely related than those farther apart. In the same way, the principle applies to a time series dataset, where it is expected that variables closer in time will all be more related, say by having similar values, in comparison to those widely separated in time. In the context of MF-DFA, the points at beginnings or ends of a window of a time series will be more strongly correlated with their close-by points, even though those nearby ones are actually part of the neighboring window. More specifically, the farther-away within a given window can be less related compared to those from a neighboring window. It thus looks as if the points should be weighted by position in the time series.

MF-TWDFA uses Geographically Weighted Regression (GWR) method for detrending procedure. GWR is a type of varying-parameter regression model and it considers local effect in time series better than fitting global regression parameters. Suppose that the i-th point falls into the v-th local window with scale s. Then the local trend at the i-th point, denoted as $x(i)$, is decided by $x_v(i)$. But the value of $x_v(i)$ itself is decided only by points falling in its own local window, not by points that might be closer but outside the window. So in order to take into account these kinds of local effects more accurately, it's best to denote $x_v(i)$ based on nearby points. We estimate $x(i)$ by fitting a polynomial (denoted as $\tilde{x}(i)$) in each of the local windows comprising the points $\{j : |i - j| \leq s\}$. Specifically, we obtain the respective $\tilde{x}(i)$ for $x(i)$ through the moving window, MW_i, defined as $\{j : |i - j| \leq s\}$.

MF-TWDFA begins similar to the previous methods. Again, let us consider our previous example $\{x(k) : k = 1, 2, ...N\}$.

Step 1: Cumulative Deviation: We take cumulative deviation of the series.

$$X(t) = \sum_{k=1}^{t} x(k) \tag{2.13}$$

Step 2: Detrending with GWR: Obtain detrended series using GWR. In the MW_i, the polynomial fitting can be expressed as follows. The local trends $\tilde{X}(i)$ are obtained for each points i of $\{X(i) : i \in [0, T]\}$

$$\tilde{X}(i) = \theta_0(i) + \theta_1(i)i + ... + \theta_m(i)i^m + \epsilon \tag{2.14}$$

In the regression, $\theta_j(i), i = 1, 2, ...N, j = 1, 2, ...m$ stands for m order polynomial and they are fitted for each point of $X(k)$.

The parameter vector can be expressed as $\theta(m, i) = [\theta_0(i), \theta_1(i), ..., \theta_m(i)]^T$. The closed-form solution can be obtained as follows:

$$\theta(m, i) = (T^T W(i) T)^{-1} T W(i) X \tag{2.15}$$

T is a $N \times (m + 1)$ matrix where the first column is a vector of constant 1 and the rest are trend polynomials such that:

$$T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & N & \cdots & N^m \end{pmatrix}_{N \times (m+1)} \tag{2.16}$$

And $W(i)$ is a $N \times N$ weight matrix for point i.

$$W(i) = \begin{pmatrix} w_{i1} & 0 & \cdots & 0 \\ 0 & w_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{iN} \end{pmatrix}_{N \times N} \tag{2.17}$$

The diagonal elements w_{ij} of $W(i)$ represents the weight of point j to fit point i, according to their distance to point i. We have the following weighting function.

$$w_{ij} = \begin{cases} \left[1 - \left(\frac{|i-j|}{s}\right)^2\right]^2, & \text{if } |i-j| \leq s, \\ 0, & \text{otherwise.} \end{cases} \tag{2.18}$$

The rest of the algorithm is similar to MF-DFA. As underlined before, the main difference of three models is to detrend series $x(k)$ appropriately. Also notice that MF-TWDFA is the generalized version of MF-DMA. If we let $m = 0$, then the model only fits moving means for detrending and model reduces to MF-DMA.

2.3.2 Methods for Bivariate Series

In this subsection, we turn our focus on bivariate sequences. The previous section discuss multifractality in univariate series. We can obtain detrended variance from those methods. But for covariance structures of two fractal time series, one needs to apply bivariate models. The following parts present and evaluate four models, namely as Multifractal Detrended Cross-Correlation Analysis (MF-DCCA), Multifractal Detrending Moving-Average Cross-Correlation Analysis (MFXDMA), Multifractal Cross-Correlation Analysis (MF-CCA) and Multifractal Temporally Weighted Detrended Cross-Correlation Analysis (MF-TWXDFA). These methods are the generalized version of methods in univariate cases. While univariate models provide detrended variances, this chapter introduces methods for estimating detrended cross-covariances. In this way, we can obtain robust nonlinear correlations between variables. For further usage, let us define two simultaneously recorded, non-stationary time series $\{x(k) : k = 1, 2, ...N\}$ and $\{y(k) : k = 1, 2, ...N\}$ with means $\bar{x}$ and $\bar{y}$.

2.3.2.1 MF-DCCA Algorithm

In 2008, Zhou [73] developed Kantelhardt et al.'s method and introduced MF-DCCA for bivariate series. It investigates multifractal, power-law cross-correlation behaviors between two series. It consists of five steps.

Step 1: Cumulative Deviation: Determine the cumulative profile of the series $x(k)$ and $y(k)$.

$$\begin{aligned} X(t) &= \sum_{k=1}^{t}(x_k - \bar{x}), t = 1, 2, ...N \\ Y(t) &= \sum_{k=1}^{t}(y_k - \bar{y}), t = 1, 2, ...N \end{aligned} \tag{2.19}$$

Step 2: Segmentation Divide cumulative deviations $X(t)$ and $Y(t)$ into N_s nonoverlapping segments of equal length s where $N_s = int(N/s)$. Since N is not necessarily a multiple of s, this procedure may not exhaust all data points. To avoid losing data, now repeat the procedure starting from the end of the series. This gives a total of $2 \times N_s$ segments of length s.

Step 3: Detrending Estimate a polynomial of degree m for each segment v in $2N_s$. Obtain local trends $\tilde{X}_v(t)$ amd $\tilde{Y}_v(t)$.

$$\begin{aligned} \tilde{X}_v(t) &= \alpha_0 + \alpha_1 t + \alpha_2 t^2 + ... + \alpha_m t^m \\ \tilde{Y}_v(t) &= \beta_0 + \beta_1 t + \beta_2 t^2 + ... + \beta_m t^m \end{aligned} \tag{2.20}$$

Where t is a trend variable such that $t = 1, 2, ..s$.

Step 4: Local Covariance: The detrended covariance of each segment box v is calculated as follows: For $v = 1, 2, ...N_s$, $F^2(v, s)$ is calculated by:

$$F^2(v,s) = \frac{1}{s}\sum_{j=1}^{s}[X(v-1)s + j - \tilde{X}_v(j)] \times [Y(v-1)s + j - \tilde{Y}_v(j)] \tag{2.21}$$

For $v = N_s + 1, N_s + 2...2N_s$, $F^2(v, s)$ takes the form:

$$F^2(v,s) = \frac{1}{s}\sum_{j=1}^{s}[X(N - (v - N_s)s + j) - \tilde{X}_v(j)] \times [Y(N - (v - N_s)s + j) - \tilde{Y}_v(j)] \tag{2.22}$$

Step 5: Detrended Covariance: q-th order detrended covariance between $x(k)$ and $y(k)$ is calculated by averaging all local covariance functions at order q.

While $q \neq 0$:

$$F_{xy}(q,s) = \frac{1}{2N_s}\sum_{v=1}^{2N_s}[(F^2(v,s))^{\frac{q}{2}}]^{\frac{1}{q}} \tag{2.23}$$

While $q = 0$, $F_{xy}(0, s)$ is defined as:

$$F_{xy}(0, s) = \exp\left(\frac{1}{2N_s}\sum_{v=1}^{2N_s} \ln F^2(v, s)\right). \tag{2.24}$$

Zhou's method [73] allows us to determine cross-covariance of two nonstationary time series $F_{xy}(q, s)$ under various q moments. The detrended cross-covariance depends of q and s. So, MF-DCCA can generate detrended covariance relationship between $x(k)$ and $y(k)$, under different time scales s and fluctuations depending on q. For instance, in order to understand how two series behave under small fluctuations, we can check how $F_{xy}(q, s)$ behave where $q < 2$. Similarly, cross-covariance behavior under large fluctuations can be examined by letting $q > 2$.

The following relationship is expected $F_{xy}(q, s) \propto s^{h_{xy}(q)}$. The generalized Hurst exponent is obtained by taking the log of both side and fit a simple OLS such that $log(F_{xy}(q, s)) = log(C) + h_{xy}(q) \times log(s)$. The generalized scaling exponent $h_{xy}(q)$ now depends on $x(k)$ and $y(k)$ and measures dynamics between two sequences. One should exhibit that cross-correlation between two sequences is not random walk. For this aim, $h_{xy}(2)$ is evaluated. If $h_{xy}(2) \neq 0.5$, then it is concluded that cross-correlation between series is not random walk. While $h_{xy}(2) \in (0.5, 1]$, then cross-correlation is persistent and $h_{xy}(2) \in [0, 0.5]$ exhibit anti-persistent cross-correlation. Of course, behavior of cross-correlations can be examined under various q values for $q \in \mathbb{R}$. While monofractal series have constant $h_{xy}(q)$ for each q, scaling exponent varies over q for multifractal series. We can also check the Renyi exponent $\tau_{xy}(q)$ for further evidence on multifractality of cross-correlation. Also note that, letting $x(k) = y(k)$ reduces MF-DCCA to MF-DFA, the univariate version of the model. Instead of evaluating detrended cross-correlation (or cross-covariance), we get detrended variance of $x(k)$ or $y(k)$.

MF-DCCA of Zhou [73] is the fundamental model for multifractal analysis. Further researches developed their models upon Zhou's framework. Notice that MF-DCCA can yield complex values for detrended covariances $F_{xy}(q, s)$. Some studies [28, 38] in the literature modified Zhou's method to get rid of this issue. They offered taking the modulus while calculating local covariance function $F^2(v, s)$ in Step 4. However, this procedure causes information loss on the interaction between the series.

2.3.2.2 MFXDMA Algorithm

Multifractal Detrended Moving Average Cross-Correlation (MFXDMA) algorithm is offered by Jiang and Zhou [31] as an improvement on MF-DCCA and bivariate generalization of MF-DMA. The authors suggest that using moving mean of the series for detrending could give better results for estimating detrended cross-correlation between series $x(k)$ and $y(k)$. The method is similar to MF-DCCA. It only modifies detrending procedure in Step 3. The algorithm has the following steps.

Step 1: Cumulative Profile: Define cumulative summation of deviation as follows:

$$X(t) = \sum_{k=1}^{t} x(k)$$
$$Y(t) = \sum_{k=1}^{t} y(k) \tag{2.25}$$

Step 2: Moving Average Function: Calculate the moving average series $\tilde{X}(t)$ and $\tilde{Y}(t)$.

$$\tilde{X}(t) = \frac{1}{n} \sum_{k=-\lfloor (n-1)\theta \rfloor}^{\lceil (n-1)(1-\theta) \rceil} X(t-k)$$
$$\tilde{Y}(t) = \frac{1}{n} \sum_{k=-\lfloor (n-1)\theta \rfloor}^{\lceil (n-1)(1-\theta) \rceil} Y(t-k) \tag{2.26}$$

The rest of the algorithm is the same as MF-DCCA; segmentation of the series, calculating local fluctuation function, and estimating detrended cross-covariance as well as relevant statistics such as generalized Hurst exponents. Notice that the complex-valued cross-covariance problem may also occur in MFXDMA algorithm as well. Also, as it is the generalized version of MF-DMA, one can get MF-DMA algorithm for detrended variances of order q by setting $x(k) = y(k)$.

2.3.2.3 MF-CCA Algorithm

Up to this point, model improvements of multifractal methods have focused on offering various detrending procedures for better estimation of q-th order detrended cross-covariances. Oscwiecimka et al. [54] and Kwaipen et al. [36] proposed MF-CCA method, an enhancement on MF-DCCA method. They dealt with the sign issue estimating cross-covariance function $F_{xy}(q, s)$. The first four step of MF-CCA is identical to MF-DCCA. However, they defined detrended covariance function as follows:

For $q \neq 0$, we define $F_{xy}(q, s)$ as follows:

$$F_{xy}(q, s) = \frac{1}{2N_s} \sum_{v=1}^{2N_s} sgn(F^2(v, s)) |F^2(v, s)|^{\frac{q}{2}} \tag{2.27}$$

While $q = 0$, $F_{xy}(0, s)$ takes the form:

$$F_{xy}(0, s) = \frac{1}{N_s} \sum_{v=1}^{2N_s} sgn(F^2(v, s)) \ln |F^2(v, s)| \tag{2.28}$$

Where $sgn(F^2(v, s))$ is the sign function, taking value 1 if $F^2(v, s)$ is positive, and -1 if it is negative. The definition of $F_{xy}(q, s)$ in this way guarantees no imaginary part occurs in

$F_{xy}(q,s)$ and also no information is lost while taking the absolute values of local covariance functions since we preserve sign of the $F^2(v,s)$.

2.3.2.4 MF-TWXDFA Algorithm

Proposed by Wei et al. [67], MF-TWXDFA model is one of the most recent improvement on multifractal cross-correlation methods. The authors [67] generalized MF-TWDFA to bivariate series building upon the approach of MF-CCA of Oswiecimka et al. [54] and Kwaipen et al. [36]. MF-TWXDFA preserves GWR perspective of MF-TWDFA while considering the sign issue of MF-DCCA, and therefore chooses to follow MF-CCA's definition on detrended cross-covariance function $F_{xy}(q,s)$. Since MF-TWXDFA is the generalized version of MF-TWDFA, see the subsection 2.3.1.3 for using the advantages of GWR and notations.

MF-TWXDFA consists of six steps as follows:

Step 1: Cumulative Deviation: We start with taking the cumulative deviations of two non-stationary time series $x(k)$ and $y(k)$:

$$\begin{aligned} X(t) &= \sum_{k=1}^{t} x(k) \\ Y(t) &= \sum_{k=1}^{t} y(k) \end{aligned} \tag{2.29}$$

Step 2: Detrending with GWR: Obtain detrended series $\tilde{X}(t)$ and $\tilde{Y}(t)$ by applying GWR to $X(t)$ and $Y(t)$. Let $S(t)$ be either $X(t)$ and $Y(t)$ respectively and apply the following detrending procedure for both of the series. Fit m-th order polynomial to $S(t)$ by using the moving window MW_i where MW_i contains the points $\{j : |i-j| \leq \frac{s}{c}\}$ for $c \geq 1$.

$$\tilde{S}(i) = \gamma_0(i) + \gamma_1(i)i + ... + \gamma_m(i)i^m + \mu \tag{2.30}$$

Where$\tilde{S}(i)$ is the fitted value at point i and $\gamma(m,i) = [\gamma_0(i), \gamma_1(i), ..., \gamma_m(i)]^T$. The regression follows the closed-form solution given below:

$$\gamma(m,i) = (T^T W(i) T)^{-1} T W(i) S \tag{2.31}$$

The matrix T has $N \times (m+1)$ dimension. Its first column holds for model constant and the rest stands for trend polynomials such that:

$$T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 2^m \\ \vdots & \vdots & \ddots & \vdots \\ 1 & N & \cdots & N^m \end{pmatrix}_{N\times(m+1)} \tag{2.32}$$

The weight matrix $W(i)$ has $N \times N$ shape and it is estimated for each point $\{i : i = 1, 2, ...N\}$ in $S(t)$.

$$W(i) = \begin{pmatrix} w_{i1} & 0 & \cdots & 0 \\ 0 & w_{i2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{iN} \end{pmatrix}_{N\times N} \tag{2.33}$$

w_{ij} are the weights of each point $j \in MW_i$ and it weights each point j according to their distance to i with some additional penalty c.

$$w_{ij} = \begin{cases} \left[1 - \left(\frac{c(i-j)}{s}\right)^2\right]^2, & \text{if } |i-j| \leq \left\lfloor \frac{s}{c} \right\rfloor, \\ 0, & \text{otherwise.} \end{cases} \tag{2.34}$$

Apply the procedure by letting $S(t)$ equals to $X(t)$ and $Y(t)$ respectively to obtain detrended series $\tilde{X}(t)$ and $\tilde{Y}(t)$.

Step 3: Segmentation: Divide the cumulative deviations, $(X(t), Y(t))$ and detrended series $(\tilde{X}(t), \tilde{Y}(t))$, in N_s nonoverlapping segments of an equal length s such that: $N_s = int(\frac{N}{s})$. Since N may not be multiple of s, repeat the procedure starting from the end of the series so that we do not discard any data point. In total, we get $2 \times N_s$ segments of size s.

Step 4: Local Covariances: Detrended cross-covariance for each segment v is calculated as follows:

$$F^2(v,s) = \frac{1}{s}\sum_{j=1}^{s}[X(v-1)s + j - \tilde{X}_v(j)] \times [Y(v-1)s + j - \tilde{Y}_v(j)] \tag{2.35}$$

For $v = 1, 2, ...N_s$, $F^2(v, s)$ and

$$F^2(v,s) = \frac{1}{s}\sum_{j=1}^{s}[X(N - (v - N_s)s + j) - \tilde{X}_v(j)] \times [Y(N - (v - N_s)s + j) - \tilde{Y}_v(j)] \tag{2.36}$$

For $v = N_s + 1, N_s + 2...2N_s$, $F^2(v, s)$ takes the form:

Step 5: q-th Order Covariances: Obtain q-th order fluctuation function

$$F_{xy}(q,s) = \left| \frac{1}{2N_s} \sum_{v=1}^{2N_s} sgn(F^2(v,s)) |F^2(v,s)|^{q/2} \right|^{1/q}, \quad \text{for } q \neq 0 \tag{2.37}$$

$$F_{xy}(q,s) = \exp\left[\frac{1}{4N_s} \sum_{v=1}^{2N_s} sgn(F^2(v,s)) \ln |F^2(v,s)| \right], \quad \text{for } q = 0. \tag{2.38}$$

The interpretation on model results, $F_{xy}(q,s)$, $h_{xy}(q)$, $\tau_{xy}(q)$ are similar to the previous ones. The only difference is that Wei et al.[67] states the scaling exponent $h_{xy}(q)$ is valid only for $q > 0$. Negative moments are not defined in MF-TWXDFA. Also notice that letting $x(k) = y(k)$ reduces model to univariate version MF-TWDFA.

2.4 Portfolio Diversification in Cryptocurrency Market

This section of the thesis discusses the literature on portfolio diversification with cryptocurrency assets. One of the first test on portfolio diversification is conducted by Brauneis and Mestel [10]. The authors examined the risk–return profile of cryptocurrency portfolios within a traditional Markowitz mean–variance setup. In the paper, several mean–variance optimized portfolios are compared to single cryptocurrency investments and two benchmarks, the equally weighted portfolio and the CRIX index—based on daily data from 2015 to 2017 for the 500 most capitalized cryptocurrencies. The authors find that diversified cryptocurrency portfolios offer substantial risk reduction compared to individual cryptocurrencies. Quite to the point, the naively diversified 1/N portfolio had already outperformed most of the optimized portfolios in terms of Sharpe ratios and certainty equivalent returns. Their findings show that even in such a highly volatile market as cryptocurrencies, the principles of diversification can work very well at controlling risk, and simple strategies may work better than more complex optimization methods.

Liu [41] investigates the investability and diversification benefits of cryptocurrencies by analyzing the out-of-sample performance of various asset allocation models. The study used daily data from 2015 to 2018 for ten large-cap crypto currencies and then applied the six classical portfolio models: equal-weighted, minimum variance, risk parity, Markowitz, maximum Sharpe ratio, and maximum utility. The results show that diversification across cryptocurrencies significantly improve investment outcomes. It is only in the mean-utility model that the highest return and utility among others. Otherwise, no complex model outperforms the naïve 1/N portfolio in terms of the Sharpe ratio, which suggests that estimation errors to means and covariances might offset the benefits of optimal diversification.

Bouri et al. [9] examine the hedging and diversification potential of five large cryptocurrencies: BTC, ETH, XRP, LTC, and XLM, against extreme negative movements in equity markets, including indices from the USA, Europe, Asia-Pacific, and Japan. In their study, using a DCC GARCH approach, it is revealed that these cryptocurrencies would have shown different degrees of hedging and diversification capabilities. Particularly, Bitcoin acts as a hedge to USA equities, Asia-Pacific, and Japan, while other cryptocurrencies do similar duties, more so in both the Asia-Pacific and Japanese markets. Notably, the analysis indicates a time-varying nature of these abilities, thus suggesting that the effectiveness of cryptocurrency as a hedge or a diversifier may fluctuate over time. This paper concludes that, for risk management purposes, cryptocurrency assets are not homogeneous, with Bitcoin and Ethereum being the most efficient hedges in Japan and Asia-Pacific.

Demiralay and Bayraci [16] investigate the conditional diversification benefits of cryptocurrencies in stock portfolios by means of a correlation-based measure that captures the time-varying investment benefit. In this paper, data is taken on eight major cryptocurrencies and developed and emerging equity markets from August 2015 to June 2019. The authors use the DCC GARCH and DECO GARCH models to estimate conditional correlation among these assets. Results are returned that, in general, cryptocurrencies can provide diversification benefits to investors, as generally represented by low correlations with equity markets. These benefits are, however, time-varying in nature and tend to get reduced at times of turbulence, such as during the Brexit referendum and the Coincheck hack, thereby signaling an increase in the integration of markets. Offsetting the potential promise of cryptocurrencies as a source of portfolio diversification, the study concludes that their benefit is subject to exogenous shocks and changed market conditions.

Akhtaruzzaman et al. [2] examine the role that Bitcoin could play in portfolio diversification and risk management in a study of the dynamic conditional correlations of global industry portfolios and a bond index with Bitcoin. In this work, the August 2011–November 2018 period is covered with a VARMA DCC-GARCH model, generally showing that low correlations between Bitcoin with other assets make it potentially very useful for diversification. This paper identifies the optimal weights and hedge ratios of including Bitcoin in the portfolios, which show that Bitcoin can effectively hedge industry portfolios and bonds, mostly during downturns.

Ma et al. [42] examine the diversification power that cryptocurrencies can add to traditional asset portfolios. The research explores the eventual effects of adding the five most significant cryptocurrencies—BTC, ETH, XRP, BCH, and LTC—into the portfolios of stocks, technological firms' stocks, currencies, and commodities. The authors use data spanning the period from November 2015 to November 2019 to test several portfolio optimization techniques, including the Markowitz Mean-Variance analysis and Sharpe ratio optimization. The results indicate that inclusions of cryptocurrencies actually bring substantial improvements in portfolio returns and reductions in volatility across all classes, with the diversification benefits of ETH much greater than those of BTC.

Yousaf and Yarovaya [70] consider return and volatility spillovers among Non-Fungible Tokens, Decentralized Finance assets, and traditional assets consisting of oil, gold, Bitcoin, and the SP 500 by using a time-varying parameter vector autoregression model. From this study, results show that those digital assets, mainly NFTs and DeFi assets, are relatively decoupled from the traditional markets, displaying weak static spillovers. Offentlich, however, finds that during the COVID-19 pandemic and the cryptocurrency bubble in 2021, the dynamic connectedness among these asset classes increased, including potential contagion effects. The authors further demonstrate how among NFTs and DeFi assets, there exist net transmitters of return and volatility, while some are net receivers. These findings suggest that a portfolio including NFT and DeFi assets would benefit from diversification, particularly during crisis periods, due to generally low correlations with traditional assets.

CHAPTER 3

RESEARCH METHODOLOGY

This part of the thesis proposes research methodology. Dataset and research variables are explained in the first subsection, followed by descriptive statistics on data. Further, we present our model proposal Mean-MFTWXDFA portfolo optimization. We compare our model results with some well-known approaches based on EMH. Therefore, the methods from classical financial perspective; naive 1/N, Mean-Variance, Mean-VaR, Mean CVaR, are also described in this part. Our study contributes to the literature by offering a portfolio optimization approach based on FMH. The study also combines optimization techniques with two approaches: equal risk contribution (ERC) and multiobjective optimization with three objective functions; maximizing return, minimizing risk, equal distribution of portfolio risk across assets. These approaches are presented in the following subsections.

3.1 Dataset

The dataset consists of six variables, including three cryptocurrencies and three conventional assets. For crypto asset side, we choose Bitcoin (BTC), Ethereum (ETH) and Ripple (XRP) because these assets have the the largest market cap and longest available data. On the conventional side, equity market and two energy-related assets, namely crude oil and clean energy are selected. Equity class is included because the literature on cryptocurrency portfolios have exhibited several times that including equity in the portfolio enhances the performance. Cryptocurrencies consume vast amount of energy in their mining and transaction processes [14, 72]. Considering the fact that energy is the main input for cryptocurrency market, crude oil and clean energy assets are taken into account for diversification. While crude oil holds for dirty energy class, clean energy assets presents sustainable energy investments.

SP500 index data is used to represent equity market. For dirty energy asset, crude oil spot prices are obtained. For clean energy investments, iShares Global Clean Energy ETFs are used as these ETFs have the highest market cap amount other clean energy indices. Asset prices and index values are downloaded from investing.com website. The sample space spans from January 3, 2017 to December 29, 2023, including 1760 daily observations. The rest of the study uses logarithmic return series, calculated as follows: $r_{i,t} = \ln(P_{i,t} - P_{i,t-1}) \times 100$

where $r_{i,t}$ is continuously compounded return of asset i at time t and $P_{i,t}$ is the price or index data.

3.2 Descriptive Statistics

This part of the study presents and examine descriptive and visual statistics on research variables. Below, a list of descriptive statistics are presented in Table 3.1. These statistics include; basic measures on the first four moments (mean, variance, skewness, kurtosis), normality test (Jarque-Bera Test), unit root tests (Augmented Dickey-Fuller, Phillip-Perron, KPSS), heteroscedasticity test (ARCH-LM Test), and GARCH parameters for standard GARCH(1,1) model.

Table 3.1: Descriptive Statistics

	btc	eth	xrp	crude_oil	clean_energy	equity
Mean	0.2107	0.3106	0.2603	0.0174	0.0381	0.0425
Variance	21.71	40.13	61.81	9.26	3.22	1.49
Kurtosis	10.43	8.96	17.03	58.80	7.63	15.92
Skewness	-0.7898	-0.1586	1.5303	-1.9407	-0.4281	-0.8521
Jarque-Bera	8108.6	5849.3	21824.0	253037.0	4294.4	18675.0
ADF	-10.3	-9.9	-10.6	-11.9	-10.8	-11.6
PP	-1971.6	-1974.2	-1965.2	-1566.5	-1842.9	-2123.8
KPSS	0.1929	0.3614	0.2975	0.0688	0.1032	0.0359
ARCH-LM Test	10.57	17.54	40.88	214.05	102.73	415.68
GARCH α	0.1259	0.1167	0.2172	0.2411	0.0829	0.2065
GARCH β	0.8052	0.8684	0.7287	0.7554	0.9119	0.7816

According to Table 3.1, cryptocurrencies, among all assets, have highest mean and variance. Jarque-Bera test results rejects the null hypothesis of normality for all assets. Stationarity test results indicate that return series are mean-stationary. ADF and PP tests reject the null hypothesis of unit root while KPSS tests accept the null hypothesis of stationarity. ARCH-LM test is applied to test whether the series have non-constant variance over time. The results on ARCH-LM tests reject the null hypothesis of homoscedasticity and indicate heteroscedastic behavior in return series. A standard GARCH (1,1) model with ARMA(1,1) order is fitted to all series to exhibit volatility structures. BTC, ETH and clean energy volatility series display long-term persistency, while volatility of XRP, crude oil and equity are also moderately influenced by short-term shocks. The results are tested at 1% confidence level.

The sample graphs are presented in Figures 3.1 to 3.6, covering level-return-volatility series for each variable. Crypto returns display several extreme values, for instance, between 50% to 60% negative returns in one day. These assets also exhibit high volatility and several volatility clusters in the sample period. In contrast to this, conventional assets have relatively stable volatilities compared with the cryptocurrencies, with fewer extreme values. On the other hand, one can observe volatility jumps and a few extreme returns, especially during the period of the COVID-19 pandemic.

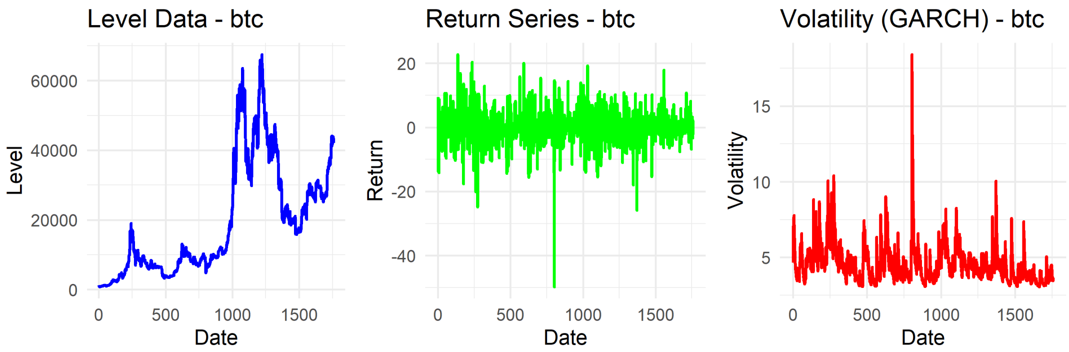

Figure 3.1: Level-Return-Volatility Series - BTC

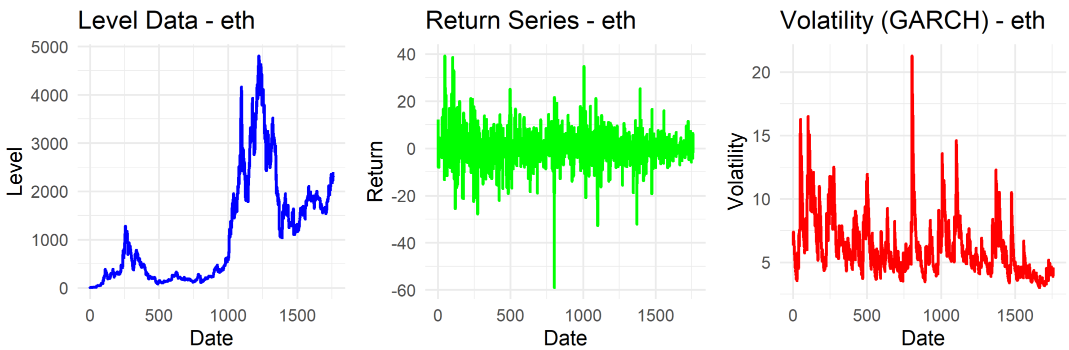

Figure 3.2: Level-Return-Volatility Series - ETH

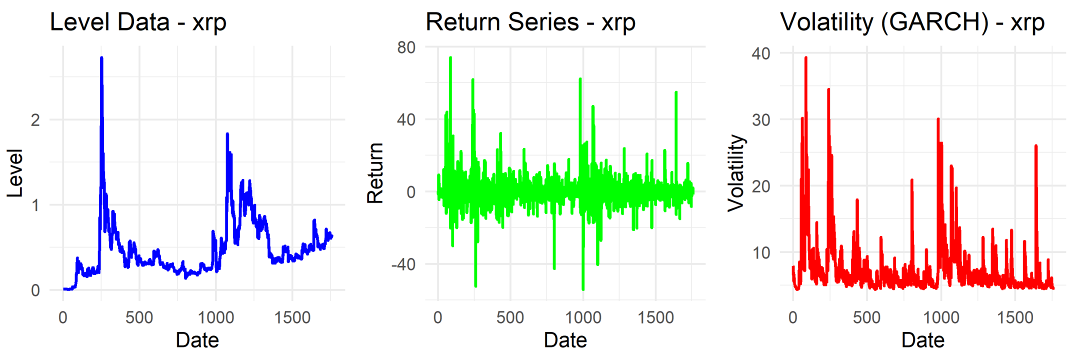

Figure 3.3: Level-Return-Volatility Series - XRP

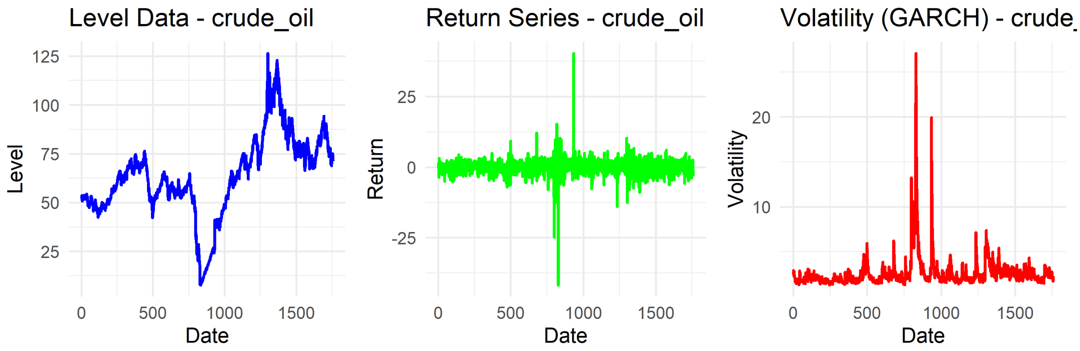

Figure 3.4: Level-Return-Volatility Series - Crude Oil

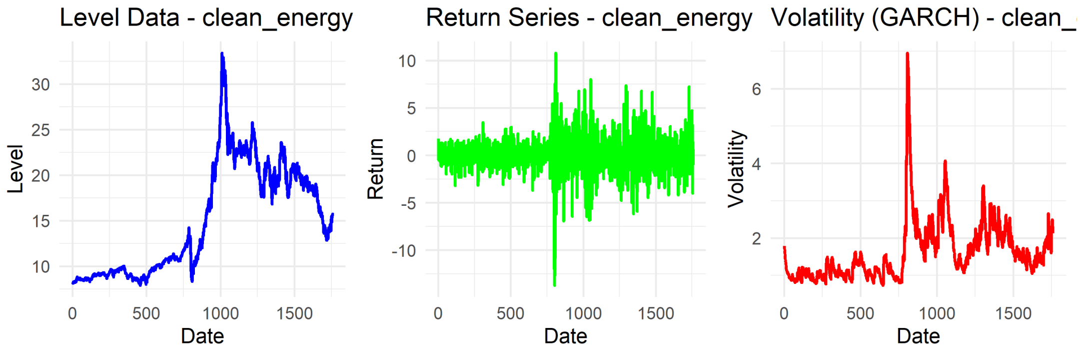

Figure 3.5: Level-Return-Volatility Series - Clean Energy

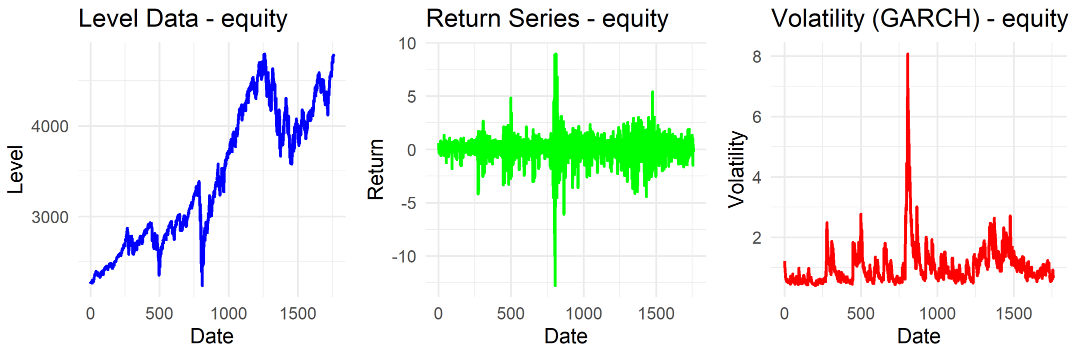

Figure 3.6: Level-Return-Volatility Series - Equity

3.3 Portfolio Construction Framework

This section propose a general framework for portfolio construction. In this study, portfolio optimizations are presented with two enhancements to classical optimization problems. First, optimization problems are solved under equal risk contribution (ERC). Developed by Maillard et al. [44], ERC is an idea for portfolio construction that equalize the risk contribution of each individual asset to portfolio's total risk. The method is applicable for various risk measures, such as variance, VaR, CVaR. Second, optimization problems are modelled with three objective functions, which are (i) minimizing risk, (ii) maximizing return, and (iii) distributing portfolio risk equally across assets. Following subsections presents the enhancements.

3.3.1 Equal Risk Contribution (ERC)

The optimal portfolios end up being very concentrated in only a small subset of the total pool of available assets or securities. Besides, mean-variance is extraordinarily sensitive to input parameters. Even slight changes in these parameters, especially in expected returns may cause large shifts in the composition of the portfolio.

In this study, we use equal risk contribution for portfolio allocation. The idea is to equalize the risk contributions of the various components of the portfolio. The contribution to risk of a component i is the proportion of the total portfolio risk that can be attributed to this component. It is calculated as a product of the allocation in subcomponent i and its marginal risk contribution, which is the change in total portfolio risk caused by a small increase in holdings of component i.

Manipulation of risk contributions has become standard practice for institutional investors, and it is often referred to as "risk budgeting". By risk budgeting, one means nothing more than looking at the portfolio from the perspective of risk contributions rather than portfolio weights. ERC replicates diversification benefits of an equally-weighted portfolio, while it considers the individual and combined risk contributions of the assets. That is to say, no asset contributes disproportionately to the total risk of the portfolio.

ERC procedure requires defining marginal and total risk contributions. Let us consider portfolio of n risky assets where w is the weight vector $w = (w_1, w_2, ...w_n)^T$. Assets have defined with variance σ_i^2 and covariance matrix Σ where components of Σ are the covariance σ_{ij} between asset i and j. Then, portfolio risk can be defined as $\sigma(w) = \sqrt{w^T \Sigma w}$. Then, according to Maillard et al. [44], marginal risk contribution of individual asset i is defined as follows:

$$\partial_{w_i}\sigma(w) = \frac{\partial \sigma(w)}{\partial w_i} = \frac{w_i\sigma_i^2 + \sum_{j\neq i} w_j\sigma_{ij}}{\sigma(w)} \tag{3.1}$$

where $\partial_{w_i}\sigma(w)$ is the marginal risk contribution of asset i. The use of the term "marginal" conveys that these values are changes in portfolio volatility for small increases in the weight of one component. Note that total risk contribution of asset i is obtained as $\sigma_i(w) = w_i \times \partial_{x_i}\sigma(w)$. Accordingly, we have the following decomposition:

$$\sigma(w) = \sum_{i=1}^{n} \sigma_i(w) \tag{3.2}$$

So, equation 3.2 provides the insight that a portfolio's total risk can be attributed to sum of individual risk components. ERC portfolio optimization has the following form:

$$\begin{aligned} w^\star &= \arg\min \sum_{i=1}^{n}\sum_{j=1}^{n} (w_i(\Sigma w)_i - w_j(\Sigma w)_j)^2 \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i = 1 \\ & 0 \le w_i \le 1 \end{aligned} \tag{3.3}$$

where $(\Sigma w)_i$ is the i^{th} row of Σw. The optimization algorithm basically aims to minimize the variance of rescaled risk contributions. ERC framework is presented for variance-covariance risk measure, however, it is also applicable for various risk measures [44]. The framework remains the same, whereas, marginal risk contributions need to be redefined for other risk measures.

Risk contribution for Value at Risk (VaR) measure is proposed by [27]. Consider portfolio with return $\tilde{r_p}$ and VaR v_p. Then, risk contribution of asset i is defined as follows.

$$\begin{aligned} \frac{\partial \sigma_p}{\partial w_i} &= \frac{\text{Cov}(\tilde{r}_i, \tilde{r}_p)}{\sigma_p} = \beta_i \cdot \sigma_p \\ \beta_i &= \frac{\text{Cov}(\tilde{r}_i, \tilde{r}_p)}{\text{Var}(\tilde{r}_p)} \end{aligned} \tag{3.4}$$

where σ_p is portfolio's risk, $\tilde{r_i}$ is the return of asset i, and β_i is the OLS coefficient between returns of asset i and portfolio. Accordingly, marginal-VaR of asset i, v_i is defined as follows:

$$v_i = \beta_i * v_p \tag{3.5}$$

Note that $\beta_p = \sum_i w_i\beta_i = 1$ and we have the relationship between v_i and v_p: $v_p = \sum_{i\in n} w_i(\beta_i \times v_p) = \sum_{i\in n} w_i \times v_i$. So, VaR component of individual assets is called $\tilde{v}_i$ where $\tilde{v}_i = w_i \times v_i$. So, marginal VaR is estimated by utilizing asset's correlation with portfolio. The contribution of each asset's VaR to portfolio's VaR is defined by $\tilde{v}_i$ and it sums up to portfolio's total VaR v_p. One can plug this value into ERC framework easily.

For CVaR portfolios, Boudt et al. [8] presents how a portfolio's CVaR can be differentiated into each component's CVaR. Consider a portfolio with return r_p, VaR v_p, and CVaR cv_p. Then, CVaR contribution of each asset $\tilde{cv}_i$ can be calculated as follows:

$$\tilde{cv}_i = w_i \times \frac{\partial cv_p}{\partial w_i} = -\mathbb{E}[w_i r_i | r_p \le -v_p] \tag{3.6}$$

One can follow ERC framework of Maillard et al. [44] after defining CVaR contributions $\tilde{cv}_i$. The idea is the same as previous approaches, that is minimizing variance of risk contributions of assets.

3.3.2 Optimization with Several Objective Functions

The basic portfolio optimization application requires one objective function to optimize asset allocation according to the objective. The fundamental model proposed by Markowitz [50] assumes that investors demand maximizing return and minimizing risk. Including ERC objective, portfolio optimization framework can be presented as below.

$$\begin{aligned}
&\min f_1(w) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
&\min f_2(w) = -\sum_{i=1}^{n} r_i w_i \\
&\min f_3(w) = \sum_{i=1}^{n}(w_i k_i - \frac{1}{n}\sum_{j=1}^{n} w_j k_j)^2 \\
&\text{s. to} \quad w_1 + w_2 + \cdots + w_n = 1 \\
&\qquad\quad w_i \geq 0, \quad i = 1, 2, \ldots, n
\end{aligned} \tag{3.7}$$

where w_i is portfolio weight, r_i is asset return, k_i is the risk contribution of asset i, σ_{ij} is the covariance between asset i and j. For VaR and CVaR risk measures, we have portfolio's VaR v_p or CVaR cv_p, rather than $\sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$.

The model in equation 3.7 is a multi-objective optimization problem (MOOP). Optimization with several objective functions can be solved using various techniques, such as graphic method, classical methods, or meta-heuristic approaches [65]. This study uses Weighted Sum Method (WSM), a classical approach with posteriori articulation of preference information method. Türkşen [65] defines WSM approach as follows: consider n objective functions $\{f_i(x) : i = 1, 2, ...n\}$ where x is a vector of decision variables and $f_i(x)$ are minimization problems for all i. The MOOP can be formulated accordingly:

$$\begin{aligned}
&\min \sum_{j=1}^{r} \alpha_j f_j(x) \\
&\mathbf{x} \in S
\end{aligned} \tag{3.8}$$

WSM reduces MOOP into optimization of one objective function. In equation 3.8, $\alpha_j \geq 0$ represents weight of objective function $f_j(x)$ and the sum of weights need to be one, for instance $\sum_{j=1}^{r} \alpha_j = 1$. Hence, MOOP in equation 3.7 can be re-defined as below.

$$
\begin{aligned}
\min \sum_{j=1}^{3} \alpha_j f_j(w) = {} & \alpha_1 \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \\
& + \alpha_2 \left(- \sum_{i=1}^{n} r_i w_i \right) \\
& + \alpha_3 \left(\sum_{i=1}^{n} \left(w_i k_i - \frac{1}{n} \sum_{j=1}^{n} w_j k_j \right)^2 \right) \\
\text{s. to} \quad & w_1 + w_2 + \cdots + w_n = 1 \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n \\
& \alpha_j \in [0, 1]
\end{aligned}
\tag{3.9}
$$

In this study, α_j is defined as $1/3$ for the following optimization problems. All the objective functions used have been given equal importance in the following optimization algorithms adopted in the study.

3.4 Portfolio Models

This section presents portfolio models used in this thesis. The main contribution of this study is to present a portfolio optimization framework under fractal theory. For this purpose, we develop Mean-MFTWXDFA portfolio optimization. To test the performance, the results of Mean-MFTWXDFA is compared with other classical models such as equally-weighted, Mean-Variance, Mean-VaR, and Mean-CVaR portfolios. The distinction between these models lies in how they assess risk in different ways. The following subsections introduces portfolio models. For simplicity, let us define notations similar to all portfolios. For the following subsections, assume a portfolio with n risky asset; where $\{w_i : i = 1, 2, ...n\}$ is portfolio weights, r_i and r_p is portfolio's expected return.

3.4.1 Equally-Weighted Portfolio Model

Also known as Naive or $1/n$, Equally-Weighted (EW) portfolio is the most basic approach for asset allocation. EW assumes that investors allocate their wealth equally to all assets. Rejecting MPT, this approach ignores variance-covariance or any risk relationship between assets. For instance, consider an investor with wealth W. Then, EW suggests portfolio allocation between n risky asset as $w_i = 1/n$ for all i.

3.4.2 Mean-Variance Portfolio Model

Mean-Variance (MV) model is proposed by Markowitz [50]. It is the first model proposed in the literature, suggesting portfolio allocation scheme according to return-risk relationship between risky assets. MV model takes the form in equation 3.7. This approach offers using variance-covariance relationship as risk measure.

3.4.3 Mean-VaR Portfolio Model

VaR is a financial measure that indicates the potential loss in value of the investment portfolio or individual security over a specified period under normal market conditions, within a set confidence level. It provides a quantifiable measure of possible downside risk in a portfolio by showing the maximum expected loss that could result. VaR is calculated as a threshold value such that the probability of loss greater than this threshold be greater than some level, represented by $(1 - c)$, where c represents the confidence level.

Among others, historical simulation, variance-covariance method, and Monte Carlo simulation are some of the different techniques for calculating VaR. Historical simulation involves simulating the possible losses from past data. Under the variance-covariance method, returns are assumed to be normally distributed; therefore, some statistical measures like mean and standard deviation are used. Monte Carlo simulation uses random sampling and statistical modeling in its estimation of potential losses. VaR is applied broadly in risk management, financial regulation, and investment strategy for the measure of potential losses that might be incurred. However, it has its limitations in assuming normality conditions in markets, probably too low for extreme risk estimation.

Mean-VaR portfolio considers VaR as the risk measure of portfolio and aims to minimize VaR. Mean-VaR optimization starts by defining portfolio's return-VaR. Recall that r_p is portfolio return and assume that it follows normal distribution with mean μ_p and varianca σ_p^2. Then, there is a $c\%$ chance that one should not experience a potential loss (return) larger than VaR v_p [27]:

$$v_p = -\mu_p + \sigma_p \times \zeta^{-1}(c) \tag{3.10}$$

where c is the confidence level and $\zeta^{-1}(c)$ is the inverse of standard normal distribution. Hence, modifying equation 3.9, Mean-VaR can be defined by the following optimization model.

$$\min \sum_{j=1}^{3} \alpha_j f_j(w) = \alpha_1 (v_p) + \alpha_2 \left(-\sum_{i=1}^{n} r_i w_i \right) + \alpha_3 \left(\sum_{i=1}^{n} (w_i \tilde{v}_i - \frac{1}{n} \sum_{j=1}^{n} w_j \tilde{v}_j)^2 \right)$$

$$\begin{aligned} \text{s. to} \quad & w_1 + w_2 + \cdots + w_n = 1 \\ & w_i \geq 0, \quad i = 1, 2, \ldots, n \\ & \alpha_j \in [0, 1] \end{aligned} \tag{3.11}$$

3.4.4 Mean-CVaR Portfolio Model

Conditional Value at Risk (CVaR), also referred to as Expected Shortfall, is a measure of risk quantifying the expected loss a portfolio or investment could realize in the worst cases beyond a given confidence level. In particular, while VaR indicates a threshold value such that the probability of losses exceeding this value lies at a given confidence level, CVaR goes one step beyond VaR by considering an average of losses beyond this VaR threshold. Therefore, CVaR is considered to be a more complete measure for tail risk because it considers extreme losses with their size. CVaR is particularly convenient for risk management and financial analysis because it makes a clear picture of potential extreme losses, hence helping investors and risk managers understand and be better prepared for the worst-case scenario. CVaR is estimated as follows according to [8]. Let $r_{p,w,t}$ be portfolio's return at time t under portfolio allocation vector w and assume that $r_{p,t,w}$ is normally distributed. Then, CVaR cv_p is calculated by the following conditional expectation, for a given VaR v_p at a pre-defined confidence interval c.

$$cv_p = -\mathbb{E}[r_{p,t,w} | r_{p,t,w} \leq -v_p] \tag{3.12}$$

cv_p is basically the expected loss beyond VaR. The Mean-CVaR portfolio targets to minimize this expected loss. Modifying equation 3.9 considering cv_p as the risk measure, we have following MOOP problem.

$$\min \sum_{j=1}^{3} \alpha_j f_j(w) = \alpha_1 (cv_p) + \alpha_2 \left(-\sum_{i=1}^{n} r_i w_i \right) + \alpha_3 \left(\sum_{i=1}^{n} (w_i \tilde{cv}_i - \frac{1}{n} \sum_{j=1}^{n} w_j \tilde{cv}_j)^2 \right)$$

$$\begin{aligned} \text{s. to} \quad & w_1 + w_2 + \cdots + w_n = 1 \\ & w_i \geq 0, \quad i = 1, 2, \ldots, n \\ & \alpha_j \in [0, 1] \end{aligned} \tag{3.13}$$

3.4.5 Mean-MFTWXDFA Portfolio Model

In this subsection, we propose our model Mean-MFTWXDFA. Recall the subsection presents MF-TWXDFA (section 2.3.2.4). We defined detrended variance and detrended cross covariance in equation 2.37 multifractal series at various q moments. Let us redefine detrended covariance $F_{xy,MF-TWXDFA}(q,s)$ for our portfolio measurement. Since we have n risky asset with returns $\{r_i : i = 1,2,...n\}$, we can define detrended cross-covariance between assets as $F_{r_i,r_j,MF-TWXDFA}(q,s)$ for $i,j = 1,2,...n$. Also note that when $i = j$, we get detrended variance of return series r_i.

We propose to use detrended cross-covariance for portfolio optimization of n risky asset. Modifying Markowwitz's classical mean variance framework, the fundamental model Mean-MFTWXDFA takes the following form.

$$\begin{aligned}
&\min \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j F_{r_i,r_j,MF-TWXDFA}(q,s) \\
&\max \sum_{i=1}^{n} r_i w_i \\
&\text{s. to} \quad w_1 + w_2 + \cdots + w_n = 1 \\
&\qquad\quad w_i \geq 0, \quad i = 1,2,\ldots,n
\end{aligned} \tag{3.14}$$

Now, let us discuss the advantages of Mean-MFTWXDFA portfolio optimization. Firstly, the proposed model is robust under non-stationarity. While classical model of Markowitz and other variants require stable pdf, risk measure under multifractality does not require such an assumption. We propose a model under FMH. It is highly recommended to utilize such a multifractal methodology when the series exhibit fractal properties. So, Mean-MFTWXDFA is applicable for fractal series and one should previously check that the Hurst exponent $h_{r_i}(2) \neq 0.5$ and the series is not a random walk and is fractal. FMH-based portfolio optimization allows us to relax the main assumptions of EMH. For instance, a multifractal risk measure such as $F_{r_i,r_j,MF-TWXDFA}(q,s)$ does not require normality assumption of asset returns and is applicable for fat-tail distributions. With Mean-MFTWXDFA, we do not only consider linear interdependence between returns, but also nonlinear intricate dynamics as well. Thus, $F_{r_i,r_j,MF-TWXDFA}(q,s)$ measures the hidden cross-covariance relationship that linear dependency models cannot catch. Compared to the preivous models, Mean-MFTWDFA is suitable for any moment $\{q : q \in \mathbb{R}\}$. While $q > 2$, $F_{r_i,r_j,MF-TWXDFA}(q,s)$ display covariance relationship under large fluctuations where the market is highly volatile. On the other hand, $q < 2$ is also preferable when the market exhibit small fluctuations. Thus, especially for dynamic portfolios, Mean-MFTWDFA allows one to adjust the risk according to the market dynamics. One of the great innovation of fractal portfolios is that they are suitable for various time scales s, which can be considered as investment periods. While the classical models consider investors with uniform investment horizons, one can easily adjust $F_{r_i,r_j,MF-TWXDFA}(q,s)$ for any s values (generally suggested $10 \leq s \leq \frac{N}{4}$).

Mean-MFTWXDFA has significant improvements on the previous fractal portfolio applications. Compared to Mean-MFDCCA and Mean-MFXDMA, our model does not ignore the sign of detrended cross-covariances and does not results in covariances with complex numbers. Relative to the Mean-MFDCCA, Mean-MFXDMA, and Mean-MFCCA, our model uses MF-TWXDFA which utilizes GWR method for detrending and therefore, it produces robust detrended series while calculating detrended cross-covariance. In this way, we eliminate spurious correlation relationships and more accurately estimate $F_{r_i,r_j}(q,s)$.

By substituting $F_{r_i,r_j,MF-TWXDFA}(q,s)$ into equation 3.9, we get the following MOOP.

$$\begin{aligned}
\min \sum_{j=1}^{3} \alpha_j f_j(w) = {} & \alpha_1 \left(\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j F_{r_i,r_j,MF-TWXDFA}(q,s) \right) \\
& + \alpha_2 \left(- \sum_{i=1}^{n} r_i w_i \right) \\
& + \alpha_3 \left(\sum_{i=1}^{n} (w_i k_i - \frac{1}{n} \sum_{j=1}^{n} w_j k_j)^2 \right) \\
\text{s. to} \qquad & w_1 + w_2 + \cdots + w_n = 1 \\
& w_i \geq 0, \quad i = 1, 2, \ldots, n \\
& \alpha_j \in [0,1]
\end{aligned} \tag{3.15}$$

CHAPTER 4

AN APPLICATION OF MEAN-MFTWXDFA PORTFOLIO ALLOCATION

This chapter of thesis covers empirical analysis and portfolio applications. In the following subsections, we first exhibit fractal behavior of six assets included in this study. Afterwards, portfolio application of Mean-MFTWXDFA is conducted and the results are compared with other classical methods; such as Equally-Weighted Portfolio, Mean-Variance, Mean-VaR, and Mean-CVaR. The analysis of this thesis are conducted using Python programming language.

4.1 Multifractal Analysis of Assets

This subsection of the study presents multifractal analysis on portfolio assets. The analysis of MF-TWXDFA is performed for individual asset to exhibit fractality of cryptos and other assets. Multifractal methods requires some pre-defined parameters. We conduct our analysis for time scales $10 \leq s \leq \frac{N}{4}$, q-power $\{q : q \in [1, 10], q \in \mathbb{N}\}$, polynomial order $m = 1$, and additional restriction on detrending c restriction for detrending $c = 1$. The results are displayed in Figures from 4.1 to 4.6. Figures express the generalized Hurst exponents $h(q)$ and Renyi exponents $\tau(q)$ for each asset. In the case of multifractal series, the Renyi exponent should increase nonlinearly with q. The variation of the generalized Hurst exponent with q moments is evidence of multifractality. Moreover, when the Hurst exponent $h(q)$ is equal to 0.5 where $q = 2$, it indicates a random walk; otherwise, it can be commented that the series is not a random walk and has memory.

Based on the findings of the multifractal analysis of the assets, the following interpretations can be made. The Hurst exponents can be observed where $q = 2$. According to the findings, $h(2) \neq 0.5$ for all assets, indicating that return series are not random walk and exhibit fractal behavior. The second evidence on multifractality is that the generalized Hurst exponent $h(q)$ is not constant and varies for all q values, suggesting that asset returns are multifractal. For further evidences, The Renyi exponent $\tau(q)$ can be assessed as well. $\tau(q)$ values display increases nonlinearly as a function of q, indicating that the series have multifractal behavior. Since fractality is confirmed for all assets, further analysis can be conducted. Based on asset-based analyses, the following findings can be reached.

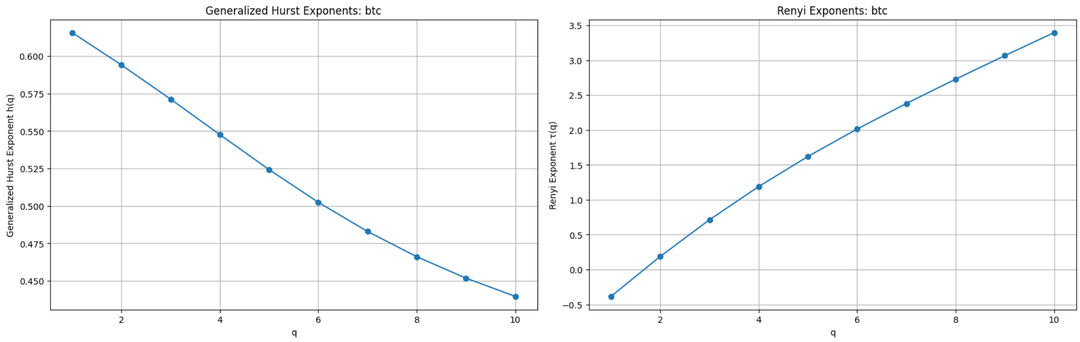

Figure 4.1: Fractal Analysis - BTC

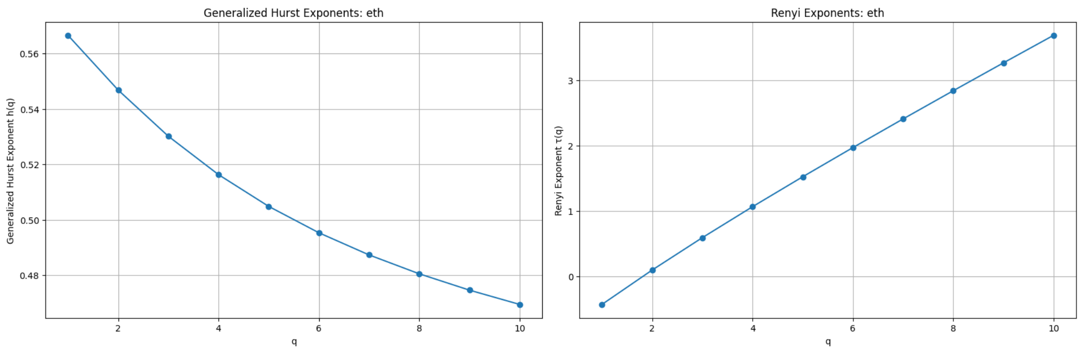

Figure 4.2: Fractal Analysis - ETH

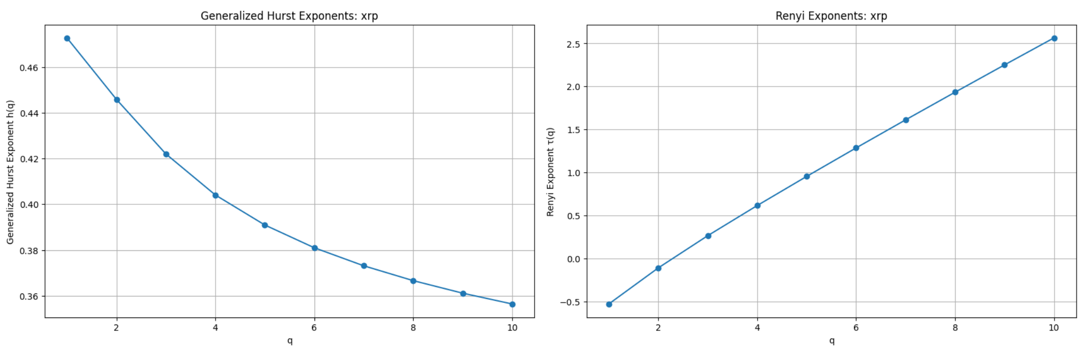

Figure 4.3: Fractal Analysis - XRP

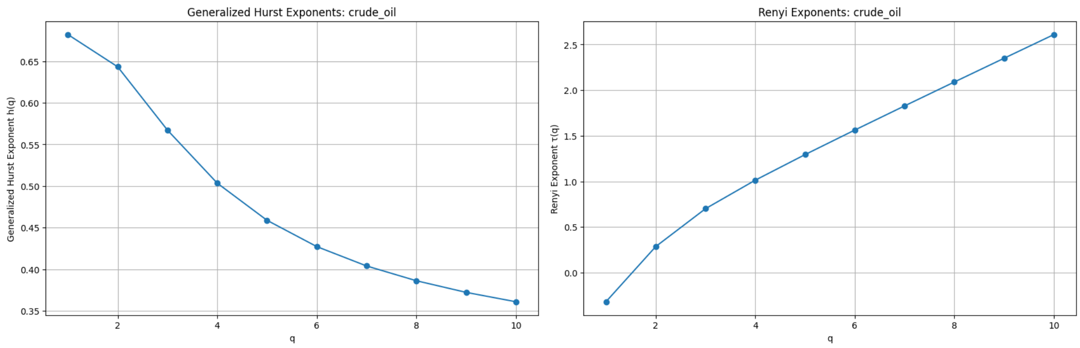

Figure 4.4: Fractal Analysis - Crude Oil

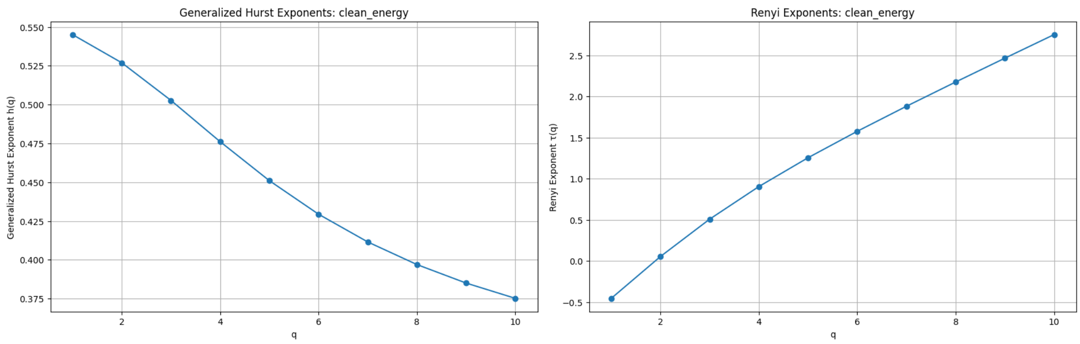

Figure 4.5: Fractal Analysis - Clean Energy

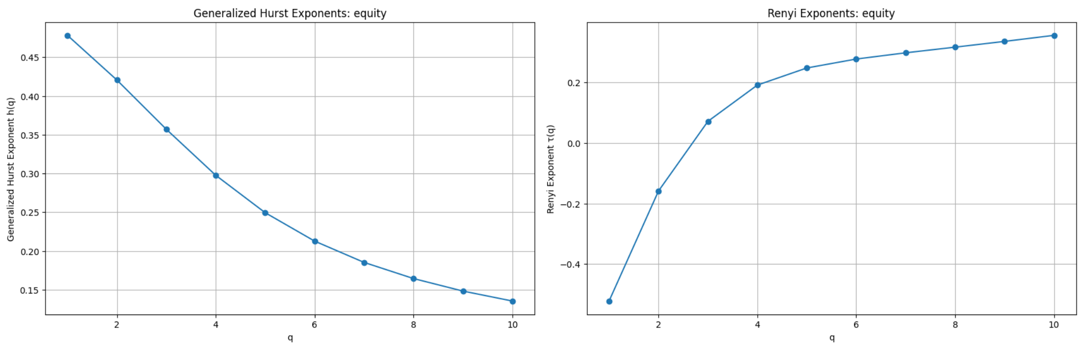

Figure 4.6: Fractal Analysis- Equity

- BTC has Hurst exponent $h(2) > 0.5$, indicating that Bitcoin returns have long-range dependency. So, positive (negative) returns are likely to be followed by positive (negative) returns. On the other hand, $q = 2$ only display return behavior in medium fluctuations. This behavior varies according to different fluctuations. For small fluctuations where $q < 2$, the long-range dependency is highest. When Bitcoin market acts relatively stable, return series have strongest long-range memory compared to other moments. Nevertheless, the market turn out to have short memory and show mean-reverting behavior in very large fluctuations. While long-range dependency of BTC returns continues for $1 \leq q \leq 6$, the market dynamics transform into mean-reverting.

- ETH performs similar behavior to BTC. In particular, the exponent $h(2) \approx 0.55$ suggests slight long-range dependence; therefore, there exists some weaker persistence in

Ethereum returns compared to Bitcoin. In the case of small fluctuations, $q < 2$, large values of $h(q)$ reflect a stronger long-range dependence. This means that during periods with relatively stable markets, ETH will hold more to the trends set in the past. By increasing the moment order to medium-sized fluctuations, $q = 2$, returns persist; though the effect is not so pronounced. For $q > 2$, this decrease in $h(q)$ can further suggest a shift in mean reverting behavior, for in such highly volatile conditions, the returns of ETH are more likely to reverse and less persistent. This pattern of varying $h(q)$ values with increasing q thus illustrates the multifractal nature of the ETH market, from persistent to more random mean-reverting behavior as volatility increases. Compared with BTC, multifractality and long-range dependence are relatively weaker in ETH.

- XRP exhibits divergent behavior from other cryptocurrencies. From the obtained values of $h(q)$, it can be deduced that there exists a anti-persistency in returns of XRP. In particular, the value of exponent $h(2)$ stays much below 0.5, which means that the returns are more likely to reverse than to persist in trends. For small fluctuations, $q < 2$, the Hurst exponent takes a value of 0.45, thus indicating anti-persistence in small fluctuation periods. The general view on XRP is that $h(q)$ decreases with increasing order q, thus exhibiting a stronger anti-persistent tendency in large fluctuations especially $q > 2$.

- Crude oil analysis results in $h(2)$ are greater than 0.5, approximately 0.65, which indicates strong long-range dependence and persistence in crude oil returns. This means that positive or negative returns are likely to be followed by similar returns, a trend-following behavior. In the case of small fluctuations, $q < 2$, it has a rather high value, above 0.65, which means very strong persistence and trend-following behavior during periods of stable markets. As one goes to larger moments q, the value of $h(q)$ decreases, indicating therefore a reduction in long-range dependence. This trend might indicate that there is a transition from persistent to less persistent or even mean-reverting behavior with increasing fluctuations. In particular, for large fluctuations, $q \geq 4$, the generalized Hurst exponent drops below 0.5, indicating a shift towards mean-reverting dynamics where the market shows a tendency to reverse trends rather than continue them.

- Clean energy data exhibits the Hurst exponent value of $h(2) = 0.53$ which is only slightly higher than 0.5, meaning that there is a long-range dependence in the returns. This implies that there is slight persistence in the trends of returns—that is, positive returns are slightly more likely to be followed by positive returns, and negative returns by negative ones. For small fluctuations $q < 2$ the Hurst exponent maximizes itself on a value around 0.55, which means that at small volatility periods the behavior of the clean energy market would have a tendency for a more trend-following type behavior. For larger moment order increases, $h(q)$ gradually decreases, which means that the correlation exhibits a progressive decrease of long-range dependence with a switch to anti-persistent (mean-reversion) behavior, characteristically evident for large fluctuations especially $q > 3$. This may imply that, though there is a weak tendency for long-range persistency in the clean energy sector, these persistences become much weaker in the face of larger market fluctuations.

- Equity returns has the Hurst exponent $h(2) = 0.42$, which is less than 0.5, meaning that there is a predominance of anti-persistent behavior in equity returns. This means that it's likely for returns to reverse in direction rather than to persist in the same direction and implies existence of short-term memory. For small fluctuations, $q < 2$, the Hurst exponent comes as 0.47, already demonstrating some weak but existing anti-persistent behavior during periods of market stability. However, when q increases, there is a major drop in $h(q)$ to values far below 0.5. This downturn indicates that with an increase in market fluctuations, the equity market is much closer to mean reversion. Specifically, for large fluctuations, $q > 2$, it keeps falling to very low values, close to 0.15 for $q = 10$, thus indicating a strong tendency for the returns to behave anti-persistent.

4.2 Portfolio Applications

This subsection presents empirical findings on portfolio analysis. We apply; (i) Equally-Weighted (EW), (ii) Mean-Variance (MV), (iii) Mean-Value-at-Risk (MVaR), (iv) Mean-Conditional Value-at-Risk (MCVaR), and (v) Mean-MFTWXDFA portfolios. Portfolio optimization algorithms are performed dynamically by using rolling window (RW) approach. RW method is a technique where dataset is updated by using a dynamic approach and parameters are re-estimated accordingly. Assume that dataset is divided into two parts, the training set from 2017 to 2022 and test set containing data of 2023. The final year of the dataset is separated as test set. By utilizing the data from beginning of 2017 to end of 2022, the optimization algorithms estimate portfolio weights on the first trading day in 2023. We can mathematically explain RW as follows. Let us define our beginning training set $\{r_t\}$ for $t = 1, 2, ...T$, where T is defined as the last day of 2022. Portfolio optimizations are conducted using $\{r_t\}$ to estimate portfolio weights for the next trading day w_{T+1}. Following this estimation, portfolio return for day $T + 1$ can be obtained. For instance, after calculating the return for the first day of 2023, portfolios are updated. We modify training set by adding the new observation r_{T+1}, while removing the oldest data point r_1. So, portfolio weights are re-estimated according to this new training set for the next trading day which is the second day of 2023. The procedure is repeated iteratively up to the last day of 2023. RW approach allows us to estimate portfolio weights dynamically and construct adaptive portfolio strategies. At the end, we come up with a total 250 daily portfolios and allocation scheme for each portfolio optimization algorithm. Mean-VaR and Mean-CVaR optimizations are performed under 95% confidence level. For Mean-MFTWXDFA, the results are presented for medium fluctuations at $q = 2$ but note that q can take any values. Mean-MFTWXDFA portfolios can be performed for various time scales, according to individuals' investment periods. For this thesis, we select $s = 20, 40, 60, 120, 240$, roughly representing one month, two months, three months, six months and twelve months investment horizons. Although s can take any values between $m + 1 \leq s \leq N$, it is better to keep $[\frac{s}{c} \geq m + 1]$ for the sake of detrending polynomials fitting.

Table 4.1: Portfolio Results Comparison

	EW	MV	VAR	CVAR	MF_s20	MF_s40	MF_s60	MF_s120	MF_s240
Return	33.7%	12.2%	13.3%	13.6%	17.1%	16.6%	16.7%	18.6%	18.5%
Risk	27.4%	15.2%	15.8%	16.5%	18.0%	18.0%	18.2%	17.3%	17.4%
Sharpe Ratio	1.038	0.459	0.509	0.505	0.660	0.633	0.628	0.775	0.762
Jensen Alpha	0.148	-0.084	-0.072	-0.071	-0.027	-0.033	-0.035	-0.012	-0.009
Treynor Ratio	0.342	0.075	0.087	0.089	0.134	0.128	0.126	0.151	0.154

Table 4.2: Portfolio Weights Summary

	MV	MVaR	MCVaR	MF_s20	MF_s40	MF_s60	MF_s120	MF_s240
w_1 mean	6.4%	7.0%	7.6%	8.3%	8.7%	9.0%	9.5%	9.4%
w_1 max	6.8%	7.3%	9.8%	11.7%	10.6%	11.2%	12.7%	11.2%
w_1 min	6.0%	6.7%	5.7%	5.2%	6.1%	6.4%	7.6%	7.0%
w_2 mean	2.6%	4.0%	4.7%	6.8%	7.2%	7.2%	6.6%	6.2%
w_2 max	2.8%	4.2%	5.3%	10.9%	10.5%	11.0%	8.5%	7.4%
w_2 min	2.4%	3.8%	0.0%	5.2%	5.1%	5.4%	5.4%	4.2%
w_3 mean	3.0%	4.0%	4.9%	8.3%	8.0%	7.9%	7.2%	8.4%
w_3 max	3.2%	4.2%	9.9%	12.4%	12.2%	11.8%	11.2%	15.1%
w_3 min	2.9%	3.9%	4.2%	3.9%	4.6%	4.9%	4.3%	3.6%
w_4 mean	15.5%	16.2%	14.4%	17.7%	16.6%	15.7%	14.7%	15.3%
w_4 max	15.7%	16.4%	17.0%	23.2%	21.7%	20.9%	18.9%	21.9%
w_4 min	15.3%	16.0%	12.4%	14.4%	13.4%	12.3%	11.6%	12.3%
w_5 mean	23.8%	24.0%	27.2%	20.7%	20.4%	20.2%	20.6%	17.1%
w_5 max	24.3%	24.5%	31.2%	28.7%	27.8%	26.0%	26.5%	24.7%
w_5 min	23.0%	23.3%	23.3%	13.5%	11.9%	13.2%	11.7%	9.3%
w_6 mean	48.6%	44.7%	41.1%	38.1%	39.1%	40.0%	41.3%	43.6%
w_6 max	48.8%	44.9%	44.6%	47.4%	53.3%	51.6%	49.8%	50.0%
w_6 min	48.4%	44.6%	37.7%	30.4%	31.3%	32.6%	33.4%	37.1%

Under the framework described above, portfolio model results are presented in Table 4.1. It presents (i) return (annualized mean return), (ii) risk (standard deviation), (iii) Sharpe Ratio, (iv) Jensen Alpha, and (v) Treynor Ratio. Figure 4.7 presents dynamic cumulative wealth graphics for portfolio alternatives, calculated based on one-dollar investment. Lastly, asset-based dynamic portfolio allocation schemes are displayed in Figure 4.8 and the summary of the portfolio weights is given in Table 4.2. The following evaluations can be made according to the results of the portfolio analysis:

- **Return:** According to the portfolio analysis, there exist wide spreads in returns among the various strategies. Firstly, equally-weighted portfolio has the highest return among all portfolios, with 33.69% annual return. Among eight optimized portfolios, Mean-Variance performs the poorest in terms of returns, providing 12.2%. Mean-VaR and Mean-CVaR portfolios have moderate returns of 13.25% and 13.56%, respectively, and promise higher return potential than the MV portfolio. Evaluating our fractal-based

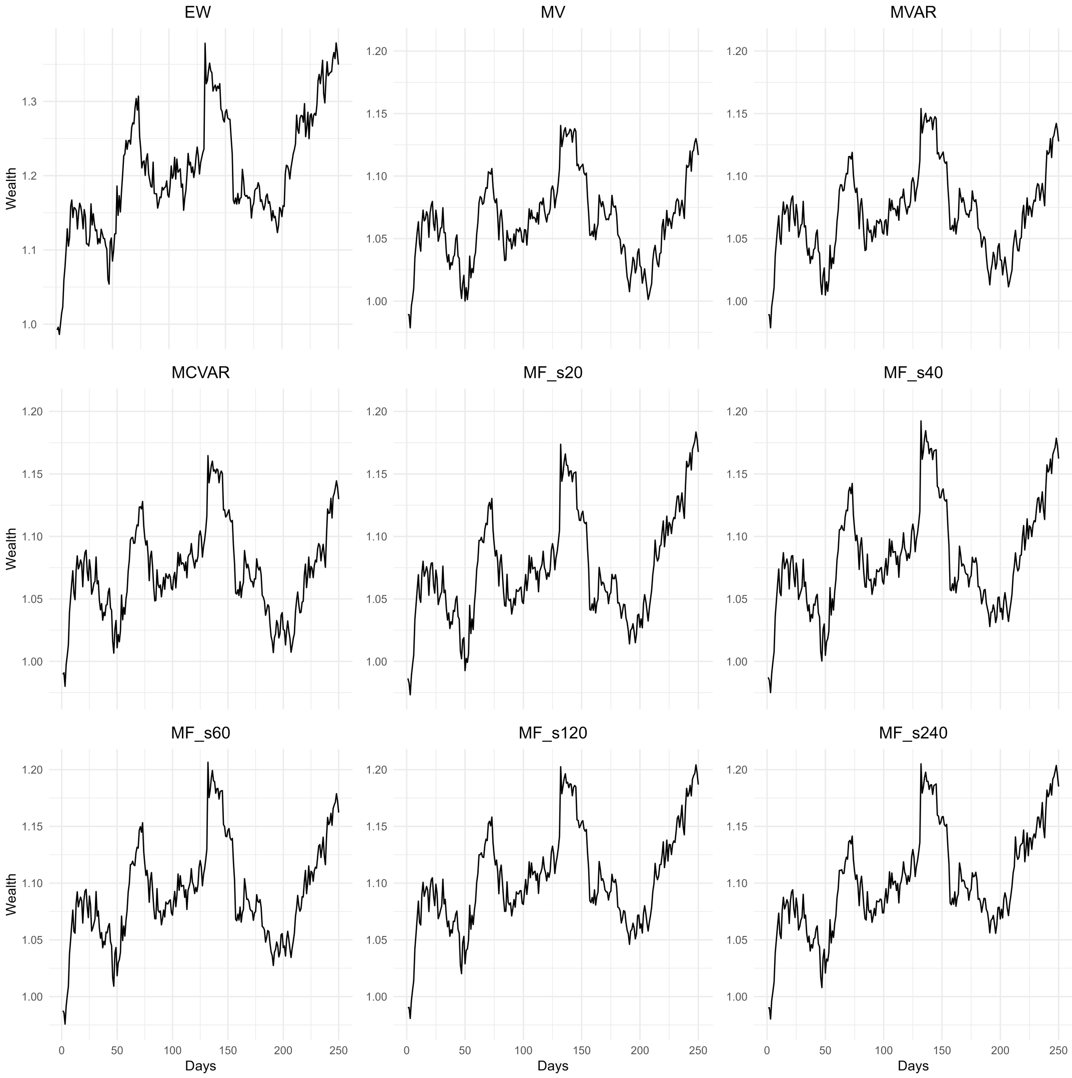

Figure 4.7: Cumulative Wealth($)

portfolios, Mean-MFTWXDFA strategies significantly outperforms other optimization algorithms. The returns of all fractal portfolios range from 16.65% to 18.62% at various time scales. The maximum return is obtained by $s = 120$, which is 18.62%. Similarly, $s = 240$ also performs the second-best portfolio return and follows this with a return of 18.49%, as the more extended time scale could capture long-term trends of the markets. Compared with the risk-based strategies, time scale $s = 20$ with a return of 17.10% is quite appealing. The returns from time scales $s = 40$ and $s = 60$ are 16.65% and 16.68%, respectively, with pretty good performance but relatively lower than the $s = 120$ and $s = 240$ strategies. In general, time scale strategies with longer-terms, especially $s = 120$ and $s = 240$, or very short investment horizons like $s = 20$ have competitive returns and outperform traditional risk-oriented strategies.

- **Risk:** The portfolio analysis results clearly display significant differences in risk levels across strategies. The basic $1/n$ portfolio has the highest risk, exhibiting 27.43% standard deviation that reflects how much this portfolio might stand to lose if the mar-

Figure 4.8: Portfolio Weights (%)

ket conditions turn against it. Despite its ease of implementation, EW model lacks of diversifying portfolios efficiently and therefore, it results in portfolios with high volatility. Mean-Variance portfolio has the least amount of risk, at 15.16%, which suggests a risk-averse attitude. The Mean-VaR and Mean-CVaR portfolios consist of 15.75% and 16.47% standard deviations respectively, thus balancing moderate risk-return balance. For fractal portfolios, investment risks increase from 17.3% to 18.2% for all the different time-scale strategies. Among the models, time scale $s = 120$ has the lowest risk at 17.3%, indicating that this approach is pretty efficient at managing the risks while capturing the market trends. The 240−time scale, with a risk value of 17.40%, stays very close, thus showing that the longer time scales handle risk quite effectively over extended periods. The scale $s = 20$ has a standard deviation of 18%. Though this is higher than the $s = 120$ and $s = 240$ strategies, it is still below the EW portfolio and it can be considered a reliable option for short-term investors. The scale $s = 40$ and $s = 60$ have risk levels of 18.02% and 18.22%, respectively, thus indicating a medium-risk profile. For the most part, these time-scale strategies, particularly $s = 120$ and $s = 240$, remain competitive in terms of risk management relative to the high-risk EW strategy, maintaining overall relatively moderate levels of risk. To conclude, among all models, Mean-Variance exhibit the lowest risk while fractal portfolios perform moderate to high level of risk.

- **Sharpe Ratio:** Sharpe Ratio (SR), calculated by dividing portfolio's excess return to standard deviation, is one of the most used portfolio metric in finance literature. SR enables us to evaluate and compare different portfolio performances based on a risk-adjusted perspective. The results pointed out remarkable differences in the SRs between variously selected strategies. According to the results, Equally-Weighted portfolio has the highest SR score 1.038, thus portraying the best performance with respect to risk-adjusted returns. On the other hand, among optimized portfolios, fractal-based models outperform the classical approaches at all time scales. From the strategies that incorporated fractal methods, time scale $s = 120$ represented the best the risk-adjusted performance with a SR of 0.775. On the other hand, a minimum variance portfolio had a remarkably low SR of 0.459, thus exhibiting less efficient risk-adjusted performance. The MVAR and MCVaR portfolios also have slightly more elevated SR, with 0.509 and 0.505, respectively, bringing out a moderate balance between risk and return. From the different time scales, the Sharpe ratio varied from 0.628 to 0.775. Further, time scale $s = 240$, with a SR equal to 0.762, is very close and simply reflects the strong risk-adjusted returns of the strategy over a longer period. Short-term investment horizon of $s = 20$ performs SR of 0.660, higher than any of the traditional risk-focused strategies (i.e. MV, M-VaR, M-CVaR), although lower than both $s = 120$ and $s = 240$. The SRs for time scales $s = 40$ and $s = 60$ strategies yield SR of 0.633 and 0.628, respectively, indicating good risk-adjusted performance but slightly inferior to $s = 120$ and $s = 240$. To sum up, portfolio strategies based on fractal-theory show competitive SR that surpass the traditional risk-focused strategies in delivering efficient risk-adjusted performances.

- **Jensen Alpha:** Jensen's Alpha (JA) is a portfolio evaluation technique that exhibits the excess return of the portfolio against the expected market return. JA for the different portfolio strategies differs significantly from one to another. EW portfolio returns a positive JA of 0.148, hence having outperformed the market and other portfolios. It means that this most simple strategy of equal weighting has been successful in generating excess returns. In contrast, Mean-Variance, Mean-VaR, and Mean-CVaR yield a negative Jensen's Alpha value of -0.084, -0.072, and -0.071, respectively. This simply means that these portfolios underperform the market. The fractal-based portfolio strategies at all time scales $s = 20$, $s = 40$, $s = 60$, $s = 120$, $s = 240$, have negative values of JA scores. Nevertheless, compared with Mean-Variance, Mean-VaR, and Mean-CVaR, the extent of underperformance is smaller in these portfolios. Among all alternatives, the $s = 120$ and $s = 240$ portfolios have the smallest negative alphas of -0.0012 and -0.009, which may imply that adjustments over longer terms might reduce underperformance and stay closer to market returns. The negative JA scores may be resulted in due to the fact that that such risk minimization strategies come at the cost of returns.

- **Treynor Ratio:** Treynor Ratio (TR), estimated by dividing portfolio's excess return to systematic risk, exhibits a portfolio's performance adjusted by its systematic risk. A higher TR portrays better performance since it measures a return per unit of market risk taken. TR score for EW portfolio is 0.342, which is the highest for all the portfolios. It means that EW portfolio has the best risk-adjusted return. Whereas the TR for the Mean-Variance portfolio at 0.075 underperforms relative to the EW portfolio. TRs for Mean-VaR and Mean-CVaR portfolios come as 0.087 and 0.089, respectively, marginally better than that of the MV, but still worse than the EW portfolio. Though these strategies are trying to minimize possible losses, the modest TRs of those indicate that the return they yield is not really compensatory of the market risk taken. The TR scores for five fractal-based portfolios significantly outperform the classical optimization approaches. Starting from a TR of 0.134 for time scale $s = 20$ portfolio, the value improves progressively through the portfolios; $s = 240$ portfolio had the highest TR score among the fractal strategies at 0.154. In general terms, the TR scores suggest that EW strategy is the best one regarding risk-adjusted return. It significantly outperforms other strategies. While the MV, MVaR, and MCVaR portfolios minimize risk efficiently, they all offer lower risk-adjusted returns captured by their TR scores. Among optimized portfolios, fractal-based approaches significantly yield improved risk-adjusted performances.

- **Wealth:** The graphs of evolving wealth make clearer how the different portfolio strategies perform against time. Equally Weighted portfolio is highest in wealth accumulation, ending up a dollar investment to $1.3495\$$, finishing at a very high level compared to the other strategies. This represents that the EW portfolio, though with high fluctuations, finally provides the best performance in terms of final wealth. On the other hand, MV portfolio display a very conservative growth pattern, with less fluctuation, and having $1.1168\$$ wealth. On the other hand, M-VaR and M-CVaR portfolios exhibit patterns similar to MV. They provide more volatile but also more aggressive wealth growth, concluding at levels slightly higher compared to MV model. The MF strategies, which come in different time scales such as $s = 20, 40, 60, 120, 240$ are biased towards a more aggressive growth profile but with larger peaks and volatility. In particular, the strategies $s = 120$ and $s = 240$ have an ending at the highest levels among all optimized portfolios. To sum up, EW portfolio is the leader in terms of wealth accumulation, while the MF strategies, especially $s = 120$ and $s = 240$ for high investment horizons and $s = 20$ for short-time scales, also portray very strong performances. The MV, MVaR, and MCVaR strategies show steady growth for conservative investment, hence moderating wealth accumulation.

- **Portfolio Allocation:** Portfolio allocation graphs provide us a perspective of investment strategies employed across different portfolio models. Mean-Variance and Mean-VaR portfolios exhibit relatively stable weights of assets. On the other hand, Mean-CVaR portfolio is also quite stable in a similar sense, but it is more adaptive compared to Mean-Variance and Mean-VaR. Mean-CVaR quickly responses to changes in the market dynamics and takes the necessary adjustments for portfolio allocation. However, multifractal strategies do produce frequent and sharp changes in the asset weights. These strategies are adaptive for fluctuating markets. The very dynamic nature of these strategies in place stands for capturing the maximum market opportunities to optimize performance rather than avoiding any possible risk sources, which makes them more responsive and aggressive in approach in opposition to the risk-focused and static strategies. Broadly speaking, allocation patterns underline how different portfolio strategies show varying degrees of intensity in risk management and market responsiveness. According to the results, multifractal approaches generate robust portfolio allocation and capture market dynamics more successfully than classical methods. Table 4.2 presents the summary of portfolio weights among the six assets; BTC (w_1), ETH (w_2), XRP (w_3), crude oil (w_4), clean energy (w_5) and equity (w_6). Traditional portfolio optimization models such as Mean-Variance, Mean-VaR, and Mean-CVaR focus on stability by adopting more conservative allocations with low level of risks. On the other hand, fractal-based strategies hold more dynamic positions that build up over time with growing exposition to cryptocurrencies in longer-term strategies. The fractal-based approaches adopt a more dynamic and flexible strategies, increasing the allocations to cryptocurrencies in longer-term strategies while balancing out traditional assets like crude oil and equity. Crude oil stays consistently important across both types of strategies with its crucial role in portfolio diversification.

CHAPTER 5

CONCLUSION

Wealth allocation and portfolio investment has been one of the fundamental problem in finance researches. Several models under Modern Portfolio Theory have proposed an approach to the problem. However, the classical approaches lack to model and explain financial markets because the markets are complex systems with nonlinear dynamics. So, most of the approaches under MPT are insufficient to capture these characteristics of financial markets. The main disadvantage of classical portfolio models is that they rely on Efficient Market Hypothesis (EMH). On the other hand, the hypothesis bases itself on several controversial assumptions. For example, EMH assumes that (i) asset prices follow random walk, (ii) returns series are independent and have no memory, (iii) distributions of asset returns are Gaussian, (iv) all available information completely reflected to the prices, (v) stationarity in probability distribution functions, and (vi) linearity of the relationships and the market. On the other hand, these assumptions are mostly hypothetical and rarely fit the real world cases when it comes to financial markets.

Considering the inefficiency of EMH to explain financial market dynamics, Fractal Market Hypothesis (FMH) is proposed by Peters, founded on the studies of Mandelbrot on fractal geometry and fractal Brownian motions. FMH rejects EMH and its assumptions, and claims that financial market exhibit fractal behavior. According to FMH, financial markets are chaotic and fractal, meaning that the markets are not random walk, exhibit statistical self-similarity, non-Gaussian distribution of returns and contain long-range dependency.

Recently, some portfolio approaches are proposed in the literature based on fractal theory and multifractal structures of asset returns. These approaches utilizes multifractal methods, such as MF-DCCA, MF-DMA, MF-CCA, and models risk parameter by using detrended cross-covariance relationships. However, these studies, in general, lack in three aspects. First, these models have problems defining the covariance relationships between assets. Fractal-based portfolio models in the literature either ignore the sign of covariances and results in covariances with complex numbers or forces local covariances to be positive. So, it causes either problems in portfolio optimization or loss of information on the actual co-movements between assets. Second, some of these studies define minimum variance portfolios for only two assets. This approach restrict alternative objectives as well as multi-asset portfolios. Third, fractal-based portfolio models estimates detrended covariances through multifractal approaches. However, the multifractal methods used for this purpose can be improved in terms of detrending procedure. The current studies use global parameters for detrending and mostly suffer from a lack of eliminating local trends. Considering the importance of detrending in terms of estimating detrended covariances, using methods that more accurately estimate local trends will provide an advantage in terms of portfolio diversification.

In this study, we proposed a portfolio optimization framework Mean-MFTWXDFA model under fractal theory. By doing so, we offered a portfolio allocation scheme that considered fractal structures in financial market. Considering the unrealistic and problematic assumptions of EMH, our model relied on FMH and its perspective on financial markets. We modified Mean-Variance portfolio problem and offered using detrended cross-covariance obtained from a novel multifractal method called Multifractal Temporally Weighted Detrended Cross-Correlation Analysis (MF-TWXDFA). Recently proposed model by Wei et al. [67], MF-TWXDFA suggests using point-wise regression with weighted least square regression to eliminate local trends. The method is based on geographically weighted regression, suggesting the use of nearby points for detrending, unlike other methods that rely on fitting a global detrending algorithm. With this approach, local trends are eliminated better, leading to more accurate covariance estimation by avoiding spurious correlation relationships and therefore, more robust results are achieved in portfolio optimization process. Besides its robust detrending procedure, MF-TWXDFA also considers the sign issue of previous multifractal approaches and deals with the sign of local covariance functions successfully. We utilized this approach for portfolio optimization and solved three problems, aforementioned before, in the current literature of fractal-based portfolios. Mean-MFTWXDFA, unlike most of the previous models, offers a general framework for various assets. It robustly eliminates local trends and deals with the sign issue of detrended covariances. Lastly, since our model is based on FMH, we do not follow the strict constraints of EMH such as normality of asset returns, random walk of prices, independent return series, linearity etc.

We test our Mean-MFTWXDFA model on cryptocurrency portfolios, diversified with energy assets (crude oil and clean energy) and equity, and compared the results with other classical portfolio models such as Mean-Variance, Mean-Value at Risk, Mean-Conditional Value at Risk, and equally-weighted portfolios. We also include Equal Risk Contribution approach to achieve well-diversified portfolios and formulate Multiobjective Optimization problem since investors are generally have several objectives such as maximizing return, minimizing risk, and diversifying well.

In this study, we have reached the following results. According to multifractal analysis, it is found that research variables do not follow random walk. Instead, asset returns are fractal and exhibit statistical self-similarity. While Bitcoin, Ethereum, crude oil and clean energy assets show long-range dependency, Ripple and equity assets have short-term memory. The fractal behavior of asset returns vary over different q-moments. Ripple and equity returns exhibit anti-persistentcy in both small and large fluctuations in the market. On the other side, Bitcoin, Ethereum, and crude oil returns display persistentcy for both small and large fluctuations, but in the very-large fluctuations for large q-moments, the assets shift towards to anti-persistentcy. After showing that research variables are fractal, we continue with portfolio optimization application.

In applying the model, we adopt a dynamic and adaptive portfolio optimization approach. We use a six-year training dataset from 2017 to 2022 to estimate portfolio weights in each optimization. After every optimization, the first observation in the training set is removed, the tested observation from the test set is added, and the six-year observation set is maintained. We use a rolling window approach, reiterating the training dataset up until the end of 2023, estimating portfolio weights every day, and obtain out-of-sample results. The procedure results in 250 daily portfolios for every methodology.

According to the portfolio analysis results, none of the methods (Mean-Variance, Mean-VaR, Mean-CVaR, Mean-MFTWXDFA) outperformed an equally weighted portfolio based on the out-of-sample results. Excluding the equally-weighted portfolio, while analyzing the optimization-based methods, it can be found that the Mean-MFTWXDFA approach proposed in the study is offering the highest rate of return compared to the others. Contrarily, on a risk basis, the results showed that Mean-MFTWXDFA results in high-risk profile portfolios compared to other optimized portfolios. On the other hand, when risk-adjusted results are considered, it can be seen that the best performance is obtained by fractal portfolios. We also note that fractal-based portfolios have been assessed for different time scales. According to the research findings, the best performances are obtained for the $20 - 120 - 240$ days time scales. Therefore, we suggest that investors with relatively long investment horizons should consider $120-$day investment periods while a $20-$day periods would be more appropriate for investors with short-term goals.

There are some practical implications of this thesis as well. We confirm the presence of fractal structure within the crypto market and other assets, which investors can exploit in making their investment decisions. For instance, we show the various persistent and anti-persistent behavior in asset returns that financial market participants could use to help them in making their investment decisions. Further, this thesis proposes a new methodology for the optimization of portfolios, called Mean-MFTWXDFA, in the presence of fractality of financial time series; it can be applied for financial decision-making. Finally, we propose a way to diversify crypto assets with other financial assets. In addition, the optimized weights proposed in this study will definitely turn out to be helpful to investors, though one should consider that markets are dynamic and adaptive, and the dynamics of a portfolio will adjust accordingly.

This study also has some limitations and suggestions for further studies. Our multifractal portfolio analysis only considers medium fluctuations where $q = 2$. Our model is applicable for other q moments. However, the current literature lacks to offer a portfolio optimization where all q orders are considered at the same time. $q > 2$ exhibit large fluctuations while $q < 2$ display the dynamics in small fluctuations. On the other hand, our model, as well as the previous literature, can only present a decision based on one fluctuation. But it is known that financial markets are too complex to consider only one fluctuation in the decision making process. Developing an approach that consider all fluctuations, small and large at various q moments, would be a great progress for the literature. Further, this study analyzed cryptocurrency market and three diversifying assets from conventional markets. Further studies can examine other financial assets by using Mean-MFTXDFA. Lastly, we focus on portfolio studies in this thesis. Since fractality exists in financial markets, one can also consider utilizing multifractal methods for hedging purposes. Further studies could focus on the abovementioned topics.

REFERENCES

[1] M. Akhmet, M. O. Fen, and E. M. Alejaily, *Dynamics with chaos and fractals*, Springer, 2020.

[2] M. Akhtaruzzaman, A. Sensoy, and S. Corbet, The influence of bitcoin on portfolio diversification and design, Finance Research Letters, 37, p. 101344, 2020.

[3] F. Aslam, B. A. Memon, A. I. Hunjra, and E. Bouri, The dynamics of market efficiency of major cryptocurrencies, Global Finance Journal, 58, p. 100899, 2023.

[4] K. Bansal, S. Arora, K. S. Pritam, T. Mathur, and S. Agarwal, Dynamics of crime transmission using fractional-order differential equations, Fractals, 30(01), p. 2250012, 2022.

[5] S. Basak and A. Shapiro, Value-at-risk-based risk management: Optimal policies and asset prices, The Review of Financial Studies, 14(2), pp. 371–405, 2001.

[6] M. Batty and P. A. Longley, *Fractal cities: a geometry of form and function*, Academic press, 1994.

[7] T. Bollerslev, Generalized autoregressive conditional heteroskedasticity, Journal of Econometrics, 31(3), pp. 307–327, 1986.

[8] K. Boudt, P. Carl, and B. G. Peterson, Asset allocation with conditional value-at-risk budgets, Journal of Risk, 15(3), pp. 39–68, 2012.

[9] E. Bouri, B. Lucey, and D. Roubaud, Cryptocurrencies and the downside risk in equity investments, Finance Research Letters, 33, p. 101211, 2020.

[10] A. Brauneis and R. Mestel, Cryptocurrency-portfolios in a mean-variance framework, Finance Research Letters, 28, pp. 259–264, 2019.

[11] M. M. Carhart, On persistence in mutual fund performance, The Journal of Finance, 52(1), pp. 57–82, 1997.

[12] Q. Cheng, X. Liu, and X. Zhu, Cryptocurrency momentum effect: Dfa and mf-dfa analysis, Physica A: Statistical Mechanics and its Applications, 526, p. 120847, 2019.

[13] W. Chun, H. Li, and X. Wu, Portfolio model under fractal market based on mean-dcca, Fractals, 28(07), p. 2050142, 2020.

[14] S. Corbet, B. Lucey, and L. Yarovaya, Bitcoin-energy markets interrelationships - new evidence, Resources Policy, 70, p. 101916, 2021.

[15] A. C. da Silva Filho, N. D. Maganini, and E. F. de Almeida, Multifractal analysis of bitcoin market, Physica A: Statistical Mechanics and its Applications, 512, pp. 954–967, 2018.

[16] S. Demiralay and S. Bayracı, Should stock investors include cryptocurrencies in their portfolios after all? evidence from a conditional diversification benefits measure, International Journal of Finance Economics, 26(4), pp. 6188–6204, 2021.

[17] R. F. Engle, Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation, Econometrica, 50(4), pp. 987–1007, 1982.

[18] E. F. Fama, Efficient capital markets: A review of theory and empirical work, The Journal of Finance, 25(2), pp. 383–417, 1970.

[19] E. F. Fama and K. R. French, Common risk factors in the returns on stocks and bonds, Journal of Financial Economics, 33(1), pp. 3–56, 1993.

[20] E. F. Fama and K. R. French, A five-factor asset pricing model, Journal of Financial Economics, 116, pp. 1–22, 2015.

[21] G. Franceschetti and D. Riccio, *Scattering, natural surfaces, and fractals*, Elsevier, 2006.

[22] P. Frankhauser, The fractal approach. a new tool for the spatial analysis of urban agglomerations, Population: an english selection, 10(1), pp. 205–240, 1998.

[23] Z. Fu, H. Niu, and W. Wang, Market efficiency and cross-correlations of chinese new energy market with other assets: Evidence from multifractality analysis, Computational Economics, 62, pp. 1287–1311, 2023.

[24] A. Gaivoronski and G. Pflug, Value-at-risk in portfolio optimization: properties and computational approach, The Journal of Risk, 7(2), pp. 1–31, 2005.

[25] S. J. Grossman and J. E. Stiglitz, On the impossibility of informationally efficient markets, The American economic review, 70(3), pp. 393–408, 1980.

[26] G.-F. Gu and W.-X. Zhou, Detrending moving average algorithm for multifractals, Physical Review E, 82, p. 011136, 2010.

[27] W. G. Hallerbach, Decomposing portfolio value-at-risk: A general analysis, Journal of Risk, 5(2), pp. 1–18, 2003.

[28] L.-Y. He and S.-P. Chen, Nonlinear bivariate dependency of price–volume relationships in agricultural commodity futures markets: A perspective from multifractal detrended cross-correlation analysis, Physica A: Statistical Mechanics and its Applications, 390(2), pp. 297–308, 2011.

[29] H. E. Hurst, Long-term storage capacity of reservoirs, Transactions of the American society of civil engineers, 116(1), pp. 770–799, 1951.

[30] Z.-Q. Jiang, W.-J. Xie, W.-X. Zhou, and D. Sornette, Multifractal analysis of financial markets: a review, Reports on Progress in Physics, 82(12), p. 125901, 2019.

[31] Z.-Q. Jiang and W.-X. Zhou, Multifractal detrending moving-average cross-correlation analysis, Physical Review E, 84(1), p. 16106, 2011.

[32] S. Kakinaka and K. Umeno, Cryptocurrency market efficiency in short- and long-term horizons during covid-19: An asymmetric multifractal analysis approach, Finance Research Letters, 46, p. 102319, 2022.

[33] J. W. Kantelhardt, S. A. Zschiegner, E. Koscielny-Bunde, S. Havlin, A. Bunde, and H. Stanley, Multifractal detrended fluctuation analysis of nonstationary time series, Physica A: Statistical Mechanics and its Applications, 316(1), pp. 87–114, 2002.

[34] A. Karp and G. Van Vuuren, Investment implications of the fractal market hypothesis, Annals of Financial Economics, 14(01), p. 1950001, 2019.

[35] K. F. Kroner and V. K. Ng, Modeling asymmetric comovements of asset returns, The Review of Financial Studies, 11(4), pp. 817–844, 1998.

[36] J. Kwapień, P. Oświęcimka, and S. Drożdż, Detrended fluctuation analysis made flexible to detect range of cross-correlated fluctuations, Physical Review E, 92(5), p. 52815, 2015.

[37] M. Li, Fractal time series-a tutorial review, Mathematical Problems in Engineering, 2010, p. 157264, 2010.

[38] Z. Li and X. Lu, Cross-correlations between agricultural commodity futures markets in the us and china, Physica A: Statistical Mechanics and its Applications, 391(15), pp. 3930–3941, 2012.

[39] J. Lintern, The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets, Review of Economics and Statistics, 47(1), pp. 13–37, 1965.

[40] C. Liu, W. Zhang, Q. Xie, and C. Wang, Multiscale analysis of the complex interaction between uncertainty and the international commodity market, International Journal of Emerging Markets, 19(9), pp. 2499–2525, 2024.

[41] W. Liu, Portfolio diversification across cryptocurrencies, Finance Research Letters, 29, pp. 200–205, 2019.

[42] Y. Ma, F. Ahmad, M. Liu, and Z. Wang, Portfolio optimization in the era of digital financialization using cryptocurrencies, Technological Forecasting and Social Change, 161, p. 120265, 2020.

[43] M. A. Madani and Z. Ftiti, Is gold a hedge or safe haven against oil and currency market movements? a revisit using multifractal approach, Annals of Operations Research, 313(1), pp. 367–400, 2022.

[44] S. Maillard, T. Roncalli, and J. Teïletche, The properties of equally weighted risk contribution portfolios, Journal of Portfolio Management, 36(4), pp. 1–23, 2010.

[45] B. Mandelbrot, The variation of certain speculative prices, The Journal of Business, 36(4), pp. 394–419, 1963.

[46] B. Mandelbrot, How long is the coast of britain? statistical self-similarity and fractional dimension, science, 156(3775), pp. 636–638, 1967.

[47] B. B. Mandelbrot, Multifractal measures, especially for the geophysicist, Pure and Applied Geophysics, 131(1), pp. 5–42, 1989.

[48] B. B. Mandelbrot and B. B. Mandelbrot, *The fractal geometry of nature*, WH Freeman, New York, 1982.

[49] B. B. Mandelbrot and J. W. V. Ness, Fractional brownian motions, fractional noises and applications, SIAM Review, 10(4), pp. 422–437, 1968.

[50] H. Markowitz, Portfolio selection, The Journal of Finance, 7, pp. 77–91, 1952.

[51] M. Melgarejo and N. Obregon, Multifractal approach to the analysis of crime dynamics: Results for burglary in san francisco, Fractals, 25(05), p. 1750043, 2017.

[52] E. Mnif, A. Jarboui, and K. Mouakhar, How the cryptocurrency market has performed during covid 19? a multifractal analysis, Finance Research Letters, 36, p. 101647, 2020.

[53] J. Mossin, Equilibrium in a capital asset market, Econometrica, 34(4), pp. 768–783, 1966.

[54] P. Oświecimka, S. Drożdż, M. Forczek, S. Jadach, and J. Kwapień, Detrended cross-correlation analysis consistently extended to multifractality, Physical Review E, 89(2), p. 23305, 2014.

[55] C.-K. Peng, S. V. Buldyrev, S. Havlin, M. Simons, H. E. Stanley, and A. L. Goldberger, Mosaic organization of dna nucleotides, Physical Review E, 49(2), p. 1685, 1994.

[56] E. E. Peters, *Fractal Market Analysis: Applying Chaos Theory to Investment and Economics*, John Wiley Sons, New York, 1991.

[57] E. E. Peters, *Chaos and Order in the Capital Markets: A New View of Cycles, Prices, and Market Volatility*, John Wiley Sons, New York, 1993.

[58] R. T. Rockafellar and S. Uryasev, Optimization of conditional value-at-risk, The Journal of Risk, 2(3), pp. 21–41, 2000.

[59] S. A. Ross, The arbitrage theory of capital asset pricing, Journal of Economic Theory, 13(3), pp. 341–360, 1976.

[60] W. F. Sharpe, Capital asset prices: A theory of market equibilrium under conditions of risk, The Journal of Finance, 19(3), pp. 425–442, 1964.

[61] D. Stosic, D. Stosic, T. B. Ludermir, and T. Stosic, Multifractal behavior of price and volume changes in the cryptocurrency market, Physica A: Statistical Mechanics and its Applications, 520, pp. 54–61, 2019.

[62] X. Sun and Z. Liu, Optimal portfolio strategy with cross-correlation matrix composed by dcca coefficients: Evidence from the chinese stock market, Physica A: Statistical Mechanics and its Applications, 444, pp. 667–679, 2016.

[63] R. P. Taylor, A. P. Micolich, and D. Jonas, Fractal analysis of pollock's drip paintings, Nature, 399, p. 422, 1999.

[64] W. R. Tobler, A computer movie simulating urban growth in the detroit region, Economic geography, 46, pp. 234–240, 1970.

[65] Ö. Türkşen, *Optimizasyon Yöntemleri ve MATLAB, PYTHON, R Uygulamaları*, Nobel Publishing, Ankara, 2023.

[66] F. Wang, X. Ye, H. Chen, and C. Wu, A portfolio strategy of stock market based on mean-mf-x-dma model, Chaos, Solitons Fractals, 143, p. 110645, 2021.

[67] Y.-L. Wei, Z.-G. Yu, H.-L. Zou, and V. Anh, Multifractal temporally weighted detrended cross-correlation analysis to quantify power-law cross-correlation and its application to stock markets, Chaos: An Interdisciplinary Journal of Nonlinear Science, 27(6), 2017.

[68] B. J. West, G. Culbreth, R. I. Dunbar, and P. Grigolini, Fractal structure of human and primate social networks optimizes information flow, Proceedings of the Royal Society A, 479(2274), p. 20230028, 2023.

[69] T. Young and Z. Pengfei, Portfolio strategies of shanghai hong kong stock markets from the perspective of fractal theory, Systems Engineering—Theory Practice, 38(09), pp. 2188–2201, 2018.

[70] I. Yousaf and L. Yarovaya, Static and dynamic connectedness between nfts, defi and other assets: Portfolio implication, Global Finance Journal, 53, p. 100719, 2022.

[71] X. Zhang, X. Wu, L. Zhang, and Z. Chen, The evaluation of mean-detrended cross-correlation analysis portfolio strategy for multiple risk assets, Evaluation Review, 46(2), pp. 138–164, 2022.

[72] M. Zheng, G.-F. Feng, X. Zhao, and C.-P. Chang, The transaction behavior of cryptocurrency and electricity consumption, Financial Innovation, 9(1), p. 44, 2023.

[73] W.-X. Zhou, Multifractal detrended cross-correlation analysis for two nonstationary signals, Physical Review E, 77(6), p. 66211, 2008.

[74] Y. Zhou and Y. Leung, Multifractal temporally weighted detrended fluctuation analysis and its application in the analysis of scaling behavior in temperature series, Journal of Statistical Mechanics: Theory and Experiment, 2010(06), p. P06021, 2010.

[75] P. Zhu, Y. Tang, Y. Wei, and Y. Dai, Portfolio strategy of international crude oil markets: A study based on multiwavelet denoising-integration mf-dcca method, Physica A: Statistical Mechanics and its Applications, 535, p. 122515, 2019.

[76] S. Zou and T. Zhang, Multifractal detrended cross-correlation analysis of the relation between price and volume in european carbon futures markets, Physica A: Statistical Mechanics and Its Applications, 537, p. 122310, 2020.