# ON SPECIAL LINEAR SYSTEMS ON REAL TRIGONAL CURVES

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# ABSTRACT

## ON SPECIAL LINEAR SYSTEMS ON REAL TRIGONAL CURVES

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In this thesis we examine some of the topological properties of Brill-Noether varieties associated to real trigonal curves. More precisely, we aim to count the connected components of the real locus of the varieties parametrizing special linear systems of dimension at least 1 on a trigonal curve. We obtain lower and upper bounds when the degree $d$ of the linear systems and the Maroni invariant $m$ of the curve satisfy the relations $d + m = g$ and $g > d > (g + 2)/2$, where $g$ is the genus of the curve.

Keywords: Special linear systems, real trigonal curves, real Brill-Noether theory

# ÖZ

## REEL TRİGONAL EĞRİLERİN ÖZEL DOĞRUSAL SİSTEMLERİ ÜZERİNE

Akyar, Turgay
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Bu tezde reel trigonal eğrilerin Brill-Noether varyetelerinin bazı topolojik özelliklerini inceleyeceğiz. Trigonal eğriler üzerinde boyutu en az 1 olan özel lineer sistemleri parametrize eden varyetelerin reel mahallerinin bağlantılı bileşenlerini saymayı hedefliyoruz. Lineer sistemlerin derecesi $d$, eğrilerin Maroni değişmezi $m$ ve cinsi $g$ ise $d + m = g$ ve $g > d > (g + 2)/2$ bağlantıları sağlandığında alt ve üst sınırlar elde ediyoruz.

Anahtar Kelimeler: Özel lineer sistemler, reel trigonal eğriler, reel Brill-Noether teori

To Gamze

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## TABLE OF CONTENTS

# CHAPTER 1

# INTRODUCTION

Let $X$ be a smooth projective geometrically irreducible curve of genus $g$ defined over $\mathbb{R}$ such that $X(\mathbb{R}) \neq \emptyset$. By $X_{\mathbb{C}}$ we denote the complexification of $X$ obtained by base change from $\mathbb{R}$ to $\mathbb{C}$ . We want to understand under which conditions real complete linear systems on $X_{\mathbb{C}}$ exist, or equivalently the real locus of the variety

$$W_d^r(X_{\mathbb{C}}) = \left\{|D| : h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) \geq r+1, \deg(D) = d\right\} \subset Pic(X),$$

which can be defined over $\mathbb{R}$, is non-empty.

## 1.1 Motivation

Let us consider a non-singular plane curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ of degree $e$. Each connected component of $C(\mathbb{R})$ is a one dimensional compact, connected manifold and homeomorphic to circle. Harnack proved that the number $n(C)$ of connected components of $C(\mathbb{R})$ satisfies

$$0 \leq n(C) \leq \frac{(e-1)(e-2)}{2} + 1. \tag{1.1.1}$$

Furthermore, he constructed curves of maximum number of components allowed by the inequality for each $e$.

In 1900 Hilbert asked his famous questions. One part of his sixteenth problem is to study the relative positions of connected components of a real algebraic plane curve.

At that time full classification was known for $e \leq 5$. Together with the question, Hilbert constructed a degree $6$ maximal curve and he was sure that there was no other maximal curve except his and the one that Harnack had constructed.

In 1969 Gudkov constructed another maximal curve of degree $6$ and completed the classification for $e = 6$. Oleg Viro gave the full classification for $e = 7$ and the problem is still open when the degree is bigger [14].

Since the problem could be considered in a more general context, people have been studying the topology of real algebraic varieties in higher dimensions. Indeed, in this manner it is important to understand not only the existence of real points of $W^r_d(X_{\mathbb{C}})$ but also topological properties of the real locus as well.

**Lemma 1.** *If $g \geq 1$ and $p \in X_{\mathbb{C}}$ then $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(p)) = 1$.*

*Proof.* Let us take any $p \in X_{\mathbb{C}}$. Since $p$ is effective as a divisor, $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(p)) \geq 1$. Suppose that $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(p))$ is at least $2$. If the divisor $p$ has a base-point, say $q$, then we obtain

$$h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(p-q)) = h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(p)) \geq 2.$$

This means that the space of effective divisors linearly equivalent to $p - q$ is at least $1$-dimensional. This yields a contradiction as a degree $0$ divisor can not be linearly equivalent to an effective divisor other than the zero divisor. So, $p$ has no base-points and defines a degree $1$ map

$$\phi_p : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$

that is an isomorphism of curves. This contradicts with the assumption $g \geq 1$. Hence, $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(p)) = 1$. □

Note that when $g \geq 1$, by this lemma we can write

$$p \sim q \iff p = q$$

for any $p, q \in X_{\mathbb{C}}$. Then we get an isomorphism of varieties

$$\begin{aligned} X &\xrightarrow{\cong} W^0_1(X_{\mathbb{C}}) \subset Pic(X) \\ p &\longmapsto |p| \end{aligned}$$

and it induces a homeomorphism on the real locus $X(\mathbb{R}) \longrightarrow W^0_1(X_{\mathbb{C}})(\mathbb{R})$. This particular case $d = 1$ shows that the topology of $W^r_d(X_{\mathbb{C}})(\mathbb{R})$ could be highly depending on the topology of $X(\mathbb{R})$.

From the classical Brill-Noether theory for complex curves we know that $\rho(g,r,d) := g-(r+1)(g-d+r)$ is the expected dimension of $W^r_d(X_{\mathbb{C}})$. For the case $r=1$, we can say that $W^1_d(X_{\mathbb{C}}) \neq \emptyset$ if $\rho(g,1,d) = 2d-g-2 \geq 0$, i.e, if $d \geq \lfloor \frac{g+3}{2} \rfloor$. If $d < \lfloor \frac{g+3}{2} \rfloor$ and $X_{\mathbb{C}}$ is a general curve in $\mathcal{M}_g$, then $W^1_d(X_{\mathbb{C}}) = \emptyset$.

In [3], Chaudhary proves the real version of this. He shows that there exists an open set of a connected component of the moduli space $\mathcal{M}_{g/\mathbb{R}}$ of real curves of genus $g$ such that for each real curve $X$ in that open set, $W^1_d(X_{\mathbb{C}})(\mathbb{R}) \neq \emptyset$ if $d \geq \lfloor \frac{g+3}{2} \rfloor$ by showing that the minimal degree of a map from such curves to $\mathbb{P}^1_{\mathbb{R}}$ is $\lfloor \frac{g+3}{2} \rfloor$.

In [4], Huisman upgrades this result by considering not only the pencils that correspond to maps $\pi : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ but also the *topological types* of these maps, i.e, the degree information of $\pi|_{C_i} : C_i \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ for each connected component $C_i$ of $X(\mathbb{R})$ even for the case $X(\mathbb{R}) = \emptyset$.

Unlike the generality results above, we will follow a direct approach to the problem. We will fix our real curve $X$ according to the minimal degree map that it admits to $\mathbb{P}^1_{\mathbb{C}}$ for small degrees. Hyperelliptic and trigonal curves are the first ones to look at. Let $n(Y)$ be the number of connected components of the real locus $Y(\mathbb{R})$ for any variety $Y$ defined over $\mathbb{R}$. By the same analogy with Harnack's inequalities (1.1.1), a natural topological question for the real Brill-Noether theory is calculating $n(W^r_d(X_{\mathbb{C}}))$. The answer for the case $r=0$ is already known for any curve [6]:

$$n(W^0_d) = \sum_{s=0}^{\lfloor d/2 \rfloor} \binom{n(X)}{d-2s} \text{ when } n(X) > 0. \tag{1.1.2}$$

## 1.2 Hyperelliptic Case

**Definition 2.** A curve $X$ is called *hyperelliptic* if there is a degree 2 map $X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$.

When $g \geq 1$, there is a unique linear system $|E| \in W^1_2(X_{\mathbb{C}})$ of degree 2 such that $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(E)) = 2$. It is also known that if $d \leq g$, then any element $|D| \in W^r_d(X_{\mathbb{C}})$ can be written as

$$|D| = r|E| + D_0 \tag{1.2.1}$$

for some effective divisor $D_0$ [1, Page 13]. Furthermore, $|E| \in Pic(X)(\mathbb{R})$ [6,

Proposition 6.1]. Let $\phi : Pic(X) \longrightarrow J(X)$ be the usual Abel map and $\mathfrak{H} := \phi(|E|)$ be the image of the hyperelliptic class. Then inside the Jacobian we have

$$\phi(W_d^r(X_{\mathbb{C}})) = r\mathfrak{H} + \phi(W_{d-2r}^0(X_{\mathbb{C}})).$$

Note that $\phi$ is an injective morphism and $\mathfrak{H}$ is a real point. Then we can obtain the information $n(W_d^r(X_{\mathbb{C}}))$ from $n(W_{d-2r}^0(X_{\mathbb{C}}))$ which is already determined by (1.1.2).

### 1.3 Outline of the Results

Chapter 2 briefly covers the following preliminary topics.

- Real structures on a complex variety and some results on topology of real curves,
- Real linear systems, their invariants and corresponding points in $W_d^r(X_{\mathbb{C}})(\mathbb{R})$,
- Classical Brill-Noether results and the Abel-Jacobi map,
- Proof of (1.1.2) as it will play a huge role for our calculations,
- Maroni theory for trigonal curves.

Recall that the *Maroni invariant* of a given trigonal curve $\pi : X \to \mathbb{P}^1_{\mathbb{C}}$ is the unique integer $m$ satisfying the inequalities

$$\frac{g-4}{3} \leq m \leq \frac{g-2}{2}$$

such that

$$\pi_* K_X \cong \mathcal{O}_{\mathbb{P}^1}(g-2-m) \oplus \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-2),$$

where $K_X$ is the canonical bundle of $X$. This theory tells us that the canonical model of $X$ in $\mathbb{P}^{g-1}$ lies on a surface $S$ isomorphic to the Hirzebruch surface

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(g-2-m) \oplus \mathcal{O}_{\mathbb{P}^1}(m)) \longrightarrow \mathbb{P}^1_{\mathbb{C}}.$$

It is known that $W_d^r := W_d^r(X_{\mathbb{C}})$ has at most two irreducible components, namely $U_d^r$ and $V_d^r$. As in the hyperelliptic case, the number of connected components of real loci of these irreducible components can be calculated with the help of (1.1.2).

Let $\sigma$ be the anti-holomorphic involution on $X_{\mathbb{C}}$ induced by complex conjugation. In Chapter 3 we will make $n(U_d^r)$ and $n(V_d^r)$ precise. Then we focus on the case $r = 1$. In order to calculate or bound $n(W_d^1)$ we must find the conditions for which the intersection $U_d^1(\mathbb{R}) \cap V_d^1(\mathbb{R})$ is non-empty. If $T$ is a divisor of degree 3 on $X_{\mathbb{C}}$ such that $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(T)) = 2$, then this will be equivalent to finding divisors $D_1$ and $D_2$ such that

- $\deg(D_1) = g - 3$ and $\deg(D_2) = 2d - g - 2$,
- $\sigma(D_i) = D_i$ for each $i = 1, 2$, (1.3.1)
- $D_1 + D_2 \sim K_X - (g - d + 1)T$.

In the rest of Chapter 3 we find good conditions on $d, g$ and $m$ that provide us easier way to find such divisors $D_1$ and $D_2$.

In Chapter 4 we start with the natural bounds obtained by (1.1.2):

$$0 \leq n(W_d^r) \leq \sum_{s=0}^{\lfloor \frac{d-3r}{2} \rfloor} \binom{n(X)}{d-3r-2s} + \sum_{s=0}^{\lfloor \frac{s(d,r,g)}{2} \rfloor} \binom{n(X)}{s(d,r,g)-2s}, \tag{1.3.2}$$

where $s(d, r, g) = 2(d-1) - g - 3(r-1)$. Then we show that a connected component of $U_d^1(\mathbb{R})$ may intersect with at most one connected component of $V_d^1(\mathbb{R})$ and vice versa. This will allow us to have a non-trivial lower bound:

**Corollary 3.** *If* $0 < g - d \leq m$, *then*

$$n(W_d^1) \geq \sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s}.$$

Using the conditions obtained in Chapter 3 we upgrade the upper bound as well. Indeed, when $d = g - m$ and $d > (g + 2)/2$, we have the linear equivalence

$$K_X - (g - d + 1)T \sim (2d - g - 3)T + D,$$

where $D$ is a fixed effective divisor of degree $2g - 3d + 4$ so that we can make a great use of points in the support of the trigonal divisor $T$ while finding divisors $D_1$ and $D_2$ satisfying the conditions (1.3.1).

**Theorem 4.** *Let $X$ be a real trigonal curve of genus $g$ and Maroni invariant $m$ such that $m < \frac{g-3}{2}$ and $g \geq 5$. If $d = g - m$, then*

$$\sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s} \leq n(W_d^1) \leq \binom{n(X)-1}{2d-g-2} + \sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s}.$$

Finally, when we put another condition $2g - 3d + 4 = 1$, we minimize the effect of base locus $D$ and count the number exactly.

**Theorem 5.** *Let $X$ be any real trigonal curve of genus $g$ and Maroni invariant $m$ such that $g \equiv 0$ (mod 3), $g > 5$ and $m = \frac{g-3}{3}$. If $d = \frac{2g+3}{3}$, then we have*

$$n(W_d^1) = \binom{n(X)-1}{2d-g-2} + \sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s},$$

Note that in this case $n(W_d^1)$ attains its maximum.

In Chapter 5 we discuss the results and the existence of such real trigonal curves.

### 1.4 Notations and Conventions

We will use the following notations:

- $X$ is a smooth projective geometrically irreducible variety defined over $\mathbb{R}$.
- $\dim_{\mathbb{C}}(X) = 1$ after Section 2.2.
- $X(\mathbb{R}) \neq \emptyset$.
- $Pic(X)$ is the Picard variety of $X$ parametrizing linear systems.
- A linear system is assumed to be complete.
- $K_X$ is the canonical line bundle on $X$ or corresponding divisor class.

# CHAPTER 2

# PRELIMINIARIES ON REAL LINEAR SERIES AND TRIGONAL CURVES

In this chapter we recall basic definitions and facts about real linear series, the varieties parametrizing them and trigonal curves.

## 2.1 Real Structure on a Variety

We start with real structures on complex varieties.

**Definition 6.** An $\mathbb{R}$-variety is a variety $X$ defined over $\mathbb{C}$ together with a continuous involution $\sigma$ on $X$ such that for any open $U \in X$ and $f \in \mathcal{O}_X(U)$,

$$\overline{f} \circ \sigma \in \mathcal{O}_X(\sigma(U)),$$

where $\bar{}$ denotes the usual complex conjugation on $\mathbb{C}^n$.

The involution $\sigma$ is called *a real structure* on $X$.

**Lemma 7.** *If $Y \subset X$ is a subvariety, where $X$ is an $\mathbb{R}$-variety with a real structure $\sigma$, then the restriction $\tau = \sigma|_Y$ of $\sigma$ to $Y$ defines a real structure on $Y$ if $\sigma(Y) \subset Y$. In this case $Y$ is called an $\mathbb{R}$-subvariety of $X$.*

*Proof.* First note that $\tau$ is an involution on $Y$ by the assumption $\sigma(Y) \subset Y$. Take any open $U \subset Y$ and any $f \in \mathcal{O}_Y(U)$. We want to show

$$\overline{f} \circ \tau \in \mathcal{O}_Y(\tau(U)).$$

Let $y \in \tau(U)$ be any point. Then $\tau(y) \in \tau(\tau(U)) = U$. By definition of $f \in \mathcal{O}_Y$, there is an open neighborhood $V \subset X$ of $\tau(y)$ such that

$$g|_{U \cap V} = f|_{U \cap V}$$

for some $g \in \mathcal{O}_X(V)$. This yields

$$\overline{g}|_{\sigma(U\cap V)} = \overline{f}|_{\sigma(U\cap V)}. \tag{2.1.1}$$

Since $\sigma$ is a real structure on $X$ if we let $k = \overline{g} \circ \sigma$, we have $k \in \mathcal{O}_X(\sigma(V))$. As $\sigma$ is a continuous involution, $\sigma^{-1} = \sigma$ is a continuous inverse so that $\sigma$ is a homeomorphism. So we get $\sigma(U \cap V) = \sigma(U) \cap \sigma(V) = \tau(U) \cap \sigma(V)$. Then (2.1.1) yields

$$\overline{f} \circ \tau|_{\tau(U)\cap\sigma(V)} = \overline{f} \circ \sigma|_{\tau(U)\cap\sigma(V)} = \overline{g} \circ \sigma|_{\tau(U)\cap\sigma(V)} = k|_{\tau(U)\cap\sigma(V)}.$$

Summarizing, for each point $y \in \tau(U)$, there is an open neighborhood $\sigma(V) \subset X$ of $y$ such that $\overline{f} \circ \tau|_{\tau(U)\cap\sigma(V)} = k|_{\tau(U)\cap\sigma(V)}$ for some $k \in \mathcal{O}_X(\sigma(V))$ This shows that $\overline{f} \circ \tau \in \mathcal{O}_Y(\tau(U))$. □

**Definition 8.** If $X$ and $Y$ are two $\mathbb{R}$-varieties with real structures $\sigma_X$ and $\sigma_Y$, then a morphism of varieties $\phi : X \longrightarrow Y$ is called *real* if

$$\sigma_Y \circ \phi = \phi \circ \sigma_X.$$

**Definition 9.** Given an $\mathbb{R}$-variety with a real structure $\sigma$, by $X(\mathbb{R})$ we mean the $\sigma$-invariant points of $X$, i.e, $X(\mathbb{R}) := \{x \in X | \sigma(x) = x\}$. Then $n(X)$ stands for the number of connected components of the real locus $X(\mathbb{R})$.

**Remark 10.** Say $X$ and $Y$ are $\mathbb{R}$-varieties with real structures $\sigma_X$ and $\sigma_Y$, respectively, together with a real morphism $\phi : X \longrightarrow Y$.

1. Since $\phi$ is continuous and commutes with real structures, it induces a continuous map

   $$\phi_{\mathbb{R}} : X(\mathbb{R}) \longrightarrow Y(\mathbb{R})$$

   of topological spaces.

2. If $X$ is non-singular and $X(\mathbb{R}) \neq \emptyset$, then $X(\mathbb{R})$ has a differentiable manifold structure of real dimension equal to $\dim_{\mathbb{C}}(X)$ [8, Proposition 2.2.27].

**Remark 11.** 1. If $X$ is a projective variety defined over $\mathbb{R}$, then it is isomorphic to a variety $Y \subset \mathbb{P}^n_{\mathbb{C}}$ whose ideal has an $\mathbb{R}$-basis. Then we have $\sigma_{\mathbb{P}}(Y) \subset Y$,

where

$$\sigma_{\mathbb{P}} : \mathbb{P}^n_{\mathbb{C}} \longrightarrow \mathbb{P}^n_{\mathbb{C}}$$
$$[x_0 : ... : x_n] \longmapsto [\overline{x}_0 : ... : \overline{x}_n]$$

is the real structure on $\mathbb{P}^n_{\mathbb{C}}$ given by complex conjugation so that $\sigma_{\mathbb{P}}|_Y$ makes $Y$ an $\mathbb{R}$-variety.

2. Conversely, any $\mathbb{R}$-variety $X$ which is also projective is isomorphic to a variety $Y \subset \mathbb{P}^m_{\mathbb{C}}$ defined over $\mathbb{R}$. Furthermore, this isomorphism of complex varieties is given by a real morphism [8, Theorem 2.1.33].

We will be dealing with projective varieties and will not distinguish between $\mathbb{R}$-varieties and varieties defined over $\mathbb{R}$.

**Notation:** If we are given a real structure $\sigma_X$ on $X$, we will use $X_{\mathbb{C}}$ to emphasize that we are using the complex variety structure of $X$. By $X(\mathbb{C})$, we mean the set of complex points of $X$.

**Definition 12.** Let $X$ be a real, non-singular, projective curve of genus $g$. Then it is called *separating* if the complement $X(\mathbb{C}) \setminus X(\mathbb{R})$ is disconnected. We write $a(X) = 0$ in this case. Otherwise, $a(X) = 1$. The triple $\big(n(X), g, a(X)\big)$ is called *the topological type* of $X$.

We have the following theorem on the topological types of curves.

**Proposition 13.** *1.* $0 \leq n(X) \leq g + 1$.

2. *If* $n(X) = 0$, *then* $a(X) = 1$. *If* $n(X) = g + 1$, *then* $a(X) = 0$.

3. *If* $a(X) = 0$, *then* $n(X) \equiv g + 1$ *(mod 2).*

*Proof.* See [6, Proposition 3.1]. □

## 2.2 Real Linear Systems

From now on we assume that $X$ is a geometrically irreducible, non-singular, projective curve of genus $g$ defined over $\mathbb{R}$ with a real structure $\sigma$ such that $X(\mathbb{R}) \neq \emptyset$.

The action of $\sigma$ on $X$ can be linearly extended to the divisor group $\mathrm{Div}(X_{\mathbb{C}})$ by letting $\sigma(D) = \sum_{p \in X_{\mathbb{C}}} n_p[\sigma(p)]$ for any $D = \sum_{p \in X_{\mathbb{C}}} n_p[p] \in \mathrm{Div}(X_{\mathbb{C}})$.

**Definition 14.** A divisor $D$ on $X_{\mathbb{C}}$ is called *real* if it satisfies $\sigma(D) = D$. Such divisors together form the real divisor group $\mathrm{Div}(X)$ of $X$.

**Definition 15.** Let $d$ and $r$ be non-negative integers. A (complete) $g^r_d$ on $X_{\mathbb{C}}$ is a linear system

$$|D| = \mathbb{P}\big(H^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D))\big) \cong \mathbb{P}\big(\{f \in \mathbb{C}(X)^* | \ \mathrm{div}(f) + D \geq 0\} \cup \{0\}\big)$$

on $X_{\mathbb{C}}$, where $D \in \mathrm{Div}(X_{\mathbb{C}})$ such that $\deg(D) = d$ and $r(D) := h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) - 1 = r$ is its projective dimension.

If $f \in \mathbb{C}(X)^*$ and $z : U \longrightarrow \mathbb{C}$ is any local coordinate, where $U \subset X$ is open, then we let $\sigma\big(f(z)\big) = \overline{f(\sigma(z))} : \sigma(U) \longrightarrow \mathbb{C}$ so that $\sigma$ acts on the principal divisor $\mathrm{div}(f)$ by $\sigma\big(\mathrm{div}(f)\big) = \mathrm{div}\big(\sigma(f)\big)$. Using this we describe the action of $\sigma$ on $Pic(X)$ as well. Indeed, we have

$$\begin{aligned}
\sigma|D| =& \Big\{ \sigma(D) + \sigma\big(\mathrm{div}(f)\big) | f \in \mathbb{C}(X)^*, D + \mathrm{div}(f) \geq 0 \Big\} \\
=& \Big\{ \sigma(D) + \mathrm{div}\big(\sigma(f)\big) | f \in \mathbb{C}(X)^*, \sigma\big(D + \mathrm{div}(f)\big) \geq 0 \Big\} \\
=& \Big\{ \sigma(D) + \mathrm{div}(g) | g \in \mathbb{C}(X)^*, \sigma(D) + \mathrm{div}(g) \geq 0 \Big\} \\
=& |\sigma(D)|.
\end{aligned}$$

**Remark 16.** The equality $\sigma|D| = |\sigma(D)|$ in particular shows that the dimension of $|D|$ and $\sigma|D|$ are the same. So we may conclude that $\sigma$ takes a $g^r_d$ to a $g^r_d$.

We may wonder whether the image of a linear system under $\sigma$ corresponds to the same linear system. The following definitions connect this question to real divisors.

**Definition 17.** A linear system $|D|$ is called *$\sigma$-invariant* if $\sigma(E) \sim E$ for every $E \in |D|$.

Let us note that $\sigma$-invariant linear systems correspond to the real locus $Pic(X)(\mathbb{R})$, i.e, the points $|D| \in Pic(X)$ such that $\sigma|D| = |D|$.

**Definition 18.** A linear system is called *real* if it contains a real divisor.

**Remark 19.** 1. A real linear system $|D|$ is $\sigma$-invariant.

Without loss of generality, we may assume that $D$ is real. If $E \in |D|$, then $E = D+\text{div}(f)$ for some $f \in H^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D))$. Then $\sigma(f) \in H^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(\sigma(D)))$. Hence, we have

$$\begin{aligned}\sigma(E) = \sigma\big(D + \text{div}(f)\big) = \sigma(D) + \sigma\big(\text{div}(f)\big) &\sim \sigma(D) \\ &= D \\ &\sim E\end{aligned}$$

2. The converse also holds. If $X(\mathbb{R}) \neq \emptyset$, then any $\sigma$-invariant linear system must contain a real divisor [6, Proposition 2.2].

Therefore the notion of real linear systems will be used for both real and $\sigma$-invariant linear systems.

**Example 20.** The canonical class $K_X$ is a real $g^{g-1}_{2g-2}$ on $X_{\mathbb{C}}$, i.e, a real linear system that corresponds to a $g^{g-1}_{2g-2}$ on $X_{\mathbb{C}}$.

*Proof.* We know that $\deg(K_X) = 2g-2$ and $r(K_X) = g-1$. Let us take any effective divisor $D \in \text{Div}(X_{\mathbb{C}})$ of degree $2g-2$ satisfying $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) = g$. Then by the Riemann-Roch formula we have

$$g = h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) = 2g - 2 - g + 1 + h^0(X_{\mathbb{C}}, K_X(-D))$$

which leaves us $h^0(X_{\mathbb{C}}, K_X(-D)) = 1$. Since $\deg(K_X - D)$ is 0, we have $D \in K_X$. This shows that the class $K_X$ is the unique $g^{g-1}_{2g-2}$. Since $\sigma(K_X)$ is also a $g^{g-1}_{2g-2}$ on $X_{\mathbb{C}}$, we must have $K_X = \sigma(K_X)$. Hence, $K_X$ is real. □

Let us add a linear algebra fact that we will use in the future.

**Lemma 21.** *Let $E$ be a finite dimensional vector space over $\mathbb{C}$ together with a map $\sigma : E \longrightarrow E$ such that*

- $\sigma \circ \sigma = id_E$

- $\sigma(\lambda e) = \overline{\lambda}\sigma(e)$ *for every* $e \in E$ *and* $\lambda \in \mathbb{C}$.

*If* $E^\sigma = \{e \in E | \sigma(e) = e\}$ *is the set of* $\sigma$*-invariant points of* $E$*, then* $\dim_{\mathbb{R}} E^\sigma = \dim_{\mathbb{C}} E$*. Furthermore, any* $\mathbb{R}$*-basis for* $E^\sigma$ *corresponds to a* $\mathbb{C}$*-basis for* $E$*.*

*Proof.* See Lemma A.7.3 in [8]. □

**Remark 22.** Say $D \in \mathrm{Div}(X)$ defines a real $g^r_d$.

1. There exist real sections $s_0, ..., s_r$ generating the complex vector space $H^0(X_{\mathbb{C}}, \mathcal{O}(D))$ by Lemma 21.

2. If $|D|$ is base point-free, then the map

$$\begin{aligned} \phi_D : X &\longrightarrow \mathbb{P}^r_{\mathbb{C}} \\ x &\longmapsto [s_0(x) : ... : s_r(x)] \end{aligned}$$

   is real.

3. A real divisor in $|D|$ is of the form $D + \mathrm{div}(f)$ for some $f \in \mathbb{R}(X)^* \cap H^0(X_{\mathbb{C}}, \mathcal{O}(D))$. The set of all real divisors in $|D|$ then corresponds to $\mathbb{P}^r(\mathbb{R})$ in $\mathbb{P}^r_{\mathbb{C}} = |D| = \mathbb{P}\big(H^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D))\big)$.

**Definition 23.** [7, Page 23] For a real divisor $D = \sum_{p \in X} n_p[p]$, we define $\delta(D)$ to be the number of connected components $C$ of $X(\mathbb{R})$ such that $\deg_C(D) := \sum_{p \in C} n_p$ is odd.

**Proposition 24.** *If* $D$ *is any real effective divisor of degree* $d$*, then*

$$\delta(D) \leq \mathit{min}\{n(X), d\} \textit{ and } \delta(D) \equiv d \textit{ mod } 2.$$

*Proof.* Say $C_1, ..., C_{n(X)}$ are the connected components of $X(\mathbb{R})$. Let us write $P_{C_i}(D) = 0$ if $\deg_{C_i}(D)$ is even and $P_{C_i}(D) = 1$, otherwise. Then we have the relations

$$\begin{aligned} \delta(D) = \sum_{i=1}^{n(X)} P_{C_i}(D) \leq \sum_{i=1}^{n(X)} 1 = n(X), \\ \delta(D) = \sum_{i=1}^{n(X)} P_{C_i}(D) \leq \sum_{i=1}^{n(X)} \deg_{C_i}(D) = \deg(D) = d. \end{aligned}$$

Note that $\deg_{C_i}(D) - P_{C_i}(D)$ is by definition an even number for each $i$. This yields

$$\begin{aligned}\delta(D) = \sum_{i=1}^{n(X)} P_{C_i}(D) &\equiv \sum_{i=1}^{n(X)} \deg_{C_i}(D) \pmod 2 \\ &\equiv d \pmod 2.\end{aligned}$$

□

**Proposition 25.** *[6, Lemma 4.1] If $f \in \mathbb{R}(X)^*$, then $\delta(\mathit{div}(f)) = 0$.*

*Proof.* Say $C$ is any connected component of $X(\mathbb{R})$. Then $f|_C$ gives rise to a continuous map $f|_C : C \to \mathbb{P}^1_{\mathbb{R}}$. Note that $C$ is a smooth, compact, connected 1-manifold by Remark 10. Then, it must be homeomorphic to $\mathbb{P}^1_{\mathbb{R}}$. If $\phi : \mathbb{P}^1_{\mathbb{R}} \to C$ is a homeomorphism, consider the composition $f|_C \circ \phi : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^1_{\mathbb{R}}$. Once we go around the domain with respect to a chosen orientation, the total number of sign changes on the image is even. Since $\phi$ is a bijection, by definition of $f|_C$ this number also corresponds to the number of points of $C$ contained in the divisor $\mathrm{div}(f)$ with odd multiplicity. Points in $\mathrm{div}(f)$ with even multiplicity do not contribute to $\deg_C(\mathrm{div}(f))$ modulo 2. Therefore $\deg_C(\mathrm{div}(f))$ is even and $\mathrm{div}(f)$ has no pseudo-line. □

If $D$ is a real effective divisor, then $\delta(D) = \delta(D + \mathrm{div}(f)) =$ for any $f \in \mathbb{R}(X)^*$ by Lemma 25. So, we can extend the previous definition further to real linear systems. Given a real linear system $|E|$, we let $\delta(|E|) = \delta(D)$ for any real $D \in |E|$.

**Definition 26.** For a given real linear system $|E|$, any connected component of $X(\mathbb{R})$ contributing to $\delta(|E|)$ is called a *pseudo-line* for $|E|$.

**Example 27.** Consider a real plane curve $X \subset \mathbb{P}^2_{\mathbb{C}}$ of degree $d$. Traditionally, a connected component of $X(\mathbb{R})$ which intersects with a line $L \in \mathbb{P}^1(\mathbb{R})$ in an odd number of points is called a *pseudo-line* of $X$. Note that any effective divisor from the linear system $|\mathcal{O}_X(1)|$ is cut out on $X$ by the lines in $\mathbb{P}^2_{\mathbb{C}}$. According to the previous definition, a connected component is a pseudo-line of $X$ if and only if it is a pseudo-line for $|\mathcal{O}_X(1)|$.

A similar results to Proposition 25 holds for the canonical class $K_X$ as well.

**Proposition 28.** $\delta(K_X) = 0$.

*Proof.* See Proposition 2.1 in [4]. □

### 2.3 Brill-Noether Theory

Consider the *d-fold product* $X^d$ of $X$ and the symmetric group $S_d$ of all permutations on $d$ symbols. $S_d$ has a natural action on $X^d$:

$$X^d \times S_d \longrightarrow X^d$$
$$((p_1, ..., p_d), \alpha) \longmapsto (p_{\alpha(1)} ..., p_{\alpha(d)})$$

The quotient $X_d := X^d/S_d$ is called *the d-fold symmetric product* of $X$ and it is a smooth, projective variety of dimension $d$ [5, Page 236]. Its points by definition correspond to effective divisors of degree $d$ on $X_{\mathbb{C}}$. Let $s : X^d \longrightarrow X_d$ be the quotient map and we define the following map

$$u : X_d \longrightarrow Pic^d(X)$$
$$D \longmapsto |D|,$$

where $Pic^d(X)$ is the subvariety of $Pic(X)$ containing degree $d$ linear systems.

**Definition 29.** A divisor $D \in Div(X_{\mathbb{C}})$ is called *special* if $h^1\big(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)\big) > 0$.

**Proposition 30.** *If $D$ is special, then* $\deg(D) \leq 2g - 2$.

*Proof.* If $\deg(D) > 2g - 2$, then $\deg(K_X - D) < 0$. Since degree is invariant under linear equivalence, there can not be any effective divisor linearly equivalent to a divisor of negative degree. Hence, $h^0(X_{\mathbb{C}}, K_X(-D)) = 0 = h^1\big(X_{\mathbb{C}}, \mathcal{O}(D)\big)$. □

**Theorem 31.** *Let $D$ be a special divisor of degree $d$ on $X_{\mathbb{C}}$. Then*

$$r(D) \leq d/2.$$

*Moreover, if equality holds then $D$ is linearly equivalent to either zero divisor or $K_X$ or a multiple of a hyperelliptic divisor (a divisor that defines a $g^1_2$).*

*Proof.* See Clifford's Theorem in [1, Page 107]. □

Note that the same version of this theorem holds for line bundles and linear systems as well. In these cases instead of linear equivalence we have isomorphism of line bundles and equality of linear systems.

**Lemma 32.** *[1, Page 108] Let $r$ be any positive integer and $D \in Div(X_{\mathbb{C}})$. $r(D) \geq r$ if and only if for any $r$ points $p_1, ..., p_r$ in $X_{\mathbb{C}}$, there is a divisor in $|D|$ containing $p_1, ..., p_r$.*

*Proof.* For any divisor $D \in \text{Div}(X_{\mathbb{C}})$ and point $p \in X_{\mathbb{C}}$, we know that $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) - h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D-p)) = 0$ if $p$ is a base-point for $|D|$. Otherwise the difference is 1. Let us use induction on $r$: Say we have a point $p \in X_{\mathbb{C}}$.

$$r(D) \geq 1 \Longrightarrow h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) > 1 \Longrightarrow h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D-p)) > 0$$

Equivalently, the linear system $|D-p|$ is nonempty, i.e, there is an effective divisor $D' \in |D-p|$ so that $D'+p$ is in $|D|$ and it is effective.

Conversely, for any $p \in X_{\mathbb{C}}$ if there is a divisor $|D|$ containing $p$, then $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) > 0$. Let us choose $p$ so that it is not a base-point of $|D|$ (a linear system can admit finitely many base-points, otherwise we could obtain a divisor of negative degree and positive dimension by substracting enough points). Then there is a divisor $D' \in |D|$ such that $h^0(X, \mathcal{O}_{X_{\mathbb{C}}}(D'-p)) > 0$. Since $p$ is not a base point for $|D| = |D'|$, we get $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D')) = h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) > 1$ which gives us the inequality $r(D) \geq 1$.

Now we assume that the statement holds for $r = k$ for some $k \geq 1$. Say $r(D) \geq k \geq 1$ and we are given points $p_1, ..., p_k \in X_{\mathbb{C}}$. By the initial step, there is a divisor $D' \in |D|$ containing the first point $p_1$. In this case we have $r(D'-p_1) \geq k-1$. Then we can find another divisor $D'' \in |D'-p_1|$ containing the $k-1$ points $p_2, ..., p_k$ thanks to the inductive step. So we can write $D''+p_1 \sim D' \sim D$ so that $D''+p$ is effective and contains $p_1, ..., p_k$.

The converse similarly holds: Assume that given any $r$ points, there is a divisor in $|D|$ containing all of them. Choose $p_1, ..., p_k$ such that $p_1$ is not a base point of $|D|$ and the others are arbitrary. If there is $D' \in |D|$ containing $p_1, ..., p_k$, then $D' - (p_2 + ...p_k)$ is effective. Then $r(D') = r(D) \geq k-1$. Suppose that the strict equality $r(D') = r(D) = k-1$ holds. Since $p_2, ..., p_k$ are arbitrary and $D'-p_1$ is an effective divisor

containing these points, by the inductive step we have $r(D' - p) \geq k - 1$. This contradicts with the choice that $p_1$ is not not a base point for $|D'| = |D|$. We obtain $r(D) \geq k$. □

**Corollary 33.** *If $D, D'$ are any two effective divisors on $X_{\mathbb{C}}$, then*

$$r(D + D') \geq r(D) + r(D')$$

*Proof.* Say we have any $r(D) + r(D')$ points $p_i$ for $i = 1, ..., r(D) + r(D')$. Then by Lemma 32, there exist divisors $E \in |D|$ and $E' \in |D'|$ such that both of the divisors $E - (p_1 + ... + p_{r(D)})$ and $E' - (p_{r(D)+1} + ... + p_{r(D)+r(D')})$ are effective. This implies that $E + E'$ contains each $p_i$. Since these points are all arbitrary and $E + E' \in |D + D'|$, the corollary follows by Lemma 32. □

Let $W^r_d(X_{\mathbb{C}}) \subset Pic^d(X)$ be the Brill-Noether variety parametrizing special linear systems on $X_{\mathbb{C}}$ of projective dimension at least $r$. We state some of the classical Brill-Noether results.

**Theorem 34.** *[1, Chapter IV] Suppose that $d$ and $r$ are integers such that $d \geq 1$ and $r \geq 0$.*

1. *If $r \geq d - g$ and $\rho = g - (r + 1)(g - d + r) \geq 0$, then $W^r_d(X_{\mathbb{C}}) \neq \emptyset$ and every irreducible component of $W^r_d(X_{\mathbb{C}})$ has dimension at least equal to $\rho$.*
2. *If $\rho \geq 1$, then $W^r_d(X_{\mathbb{C}})$ is connected.*

When the curve is general, the theory goes much deeper in terms of smoothness and dimension results (even if $\rho$ is negative). However, we will be dealing with mostly trigonal curves and we can not interpret them as general curves in the moduli space $\mathcal{M}_g$.

**Remark 35.** 1. $u(X_d) = W^0_d(X_{\mathbb{C}})$, where $u : X_d \longrightarrow Pic^d(X)$ is the map defined by $u(D) = |D|$.

For every $D \in X_d$, $h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(D)) \geq 1$ by the effectiveness of $D$ so that $r(|D|) \geq 0$. Conversely, any linear series $|E| \in Pic^d(X)$ such that $r(|E|) = h^0(X_{\mathbb{C}}, \mathcal{O}_{X_{\mathbb{C}}}(E)) - 1 \geq 0$ must contain an effective divisor of degree $d$, say $D$. Then $u(D) = |D| = |E|$.

2. Both maps $s : X^d \longrightarrow X_d$ and $u : X_d \longrightarrow Pic^d(X)$ introduced above are real maps.

   For any points $p_1, p_2, ..., p_d \in X$, we have

$$\begin{aligned} s\big(\sigma(p_1, ..., p_d)\big) = s\big(\sigma(p_1), ..., \sigma(p_d)\big) &= \sigma(p_1) + ... + \sigma(p_d) \\ &= \sigma(p_1 + ... + p_d) \\ &= \sigma\big(s(p_1, ..., p_d)\big) \end{aligned}$$

   by the additivity of $\sigma$. This shows that $s$ is real.

   Recall by Remark 16 that $\sigma|D| = |\sigma(D)|$ for any $D \in X_d$. Equivalently, we may write $\sigma\big(u(D)\big) = u\big(\sigma(D)\big)$ so that $\sigma$ and $u$ commute.

3. $W_d^r(X_{\mathbb{C}})$ is a real subvariety of $Pic(X)$.

   By Remark 16, we have $\sigma\big(W_d^r(X_{\mathbb{C}})\big) = W_d^r(X_{\mathbb{C}})$. This equality makes $W_d^r(X_{\mathbb{C}})$ a real subvariety of $Pic(X)$ by Lemma 7.

Let $C_1, C_2, ..., C_{n(X)}$ be the connected components of $X(\mathbb{R})$. We recall an important proposition, together with its proof, as we will use it later.

**Proposition 36.** *[6, Proposition 3.2]*

1. $n(X_d) = \sum_{s=0}^{\lfloor d/2 \rfloor} \binom{n(X)}{d-2s}$.
2. *If* $n(X) > 0$, *then* $n(W_d^0) = n(X_d)$.

*Proof.* Let us define

$$Q = \{(p, \bar{p}) \in X(\mathbb{C}) \times X(\mathbb{C})\} \subset X^2$$

together with

$$U(i_1, ..., i_k) = C_{i_1} \times C_{i_2} \times ... \times C_{i_k} \times Q^s \subset X^d$$

when $d = k + 2s$. Since $s : X^d \longrightarrow X_d$ is the quotient map and these sets are connected, it is enough to consider the indices $i_1, ..., i_k$ in a non-decreasing order $i_1 \leq i_2 \leq ... \leq i_k$ so that the image will be covered.

The intersections occur only when we have an equality at any step, say $i_j = i_{j+1}$. In this case, the images of the sets $U(i_1, ..., i_j, i_{j+1}, ..., i_k)$ and $U(i_1, ..., i_{j-1}, i_{j+2}, ..., i_k)$ will have an intersection point $p_{i_1} + p_{i_2} + ... + p_{i_{j-1}} + p + p + p_{i_{j+2}} + ... + p_{i_k}$ where $p_{i_l} \in C_{i_l}$ for each $l = 1, ..., j-1, j+1, ..., k$ and $p \in C_{i_j}$.

Hence, in order to find $n(X_d)$ we need to count the number of distinct indices in increasing order $(i_1, ..., i_k)$ such that $d - k$ is non-negative, even and $1 \leq i_j \leq n(X)$ for each $j$ between 1 and $k$ (together with an empty set of indices standing for $U(0) = Q^{d/2}$ when $d$ is even). We obtain the following count:

$$\begin{aligned} n(X_d) =& \binom{n(X)}{d} + \binom{n(X)}{d-2} + ... + \binom{n(X)}{d - 2\lfloor d/2 \rfloor} \\ =& \sum_{s=0}^{\lfloor d/2 \rfloor} \binom{n(X)}{d-2s} \end{aligned}$$

We know that the image of $u : X_d \longrightarrow Pic^d(X)$ is $W_d^0(X_{\mathbb{C}})$ by Remark 35. Let us take any $|A| \in W_d^0(X_{\mathbb{C}})(\mathbb{R})$. Since the linear series is real, it must contain a real divisor, say $B$ with $u(B) = |B| = |A|$. This shows that we still have the surjectivity $u\big(X_d(\mathbb{R})\big) = W_d^0(X_{\mathbb{C}})(\mathbb{R})$ in the level of $u_{\mathbb{R}} : X_d(\mathbb{R}) \longrightarrow Pic^d(X)(\mathbb{R})$ so that $n(X_d) \geq n(W_d^0)$.

Let us show that $u$ takes connected components to connected components. Say we have two real divisors $D, E \in X_d(\mathbb{R})$ such that the images under $u$ coincide, i.e, $|D| = |E|$. Then $D = E + \text{div}(f)$ for some $f \in \mathbb{R}(X)^*$.

By Proposition 25, we know the equality $\delta(D) = \delta(E)$. Moreover, $D$ and $E$ have the same pseudo-lines. Notice by the above construction that the connected components of $X_d(\mathbb{R})$ are seperated exactly according to the pseudo-lines of their points. Therefore we may conclude that $D$ and $E$ are on the same component on $X_d(\mathbb{R})$. This proves the other inequality $n(X_d) \leq n(W_d^0)$. □

Let us choose real differentials $\omega_1, ..., \omega_g$ forming a basis for $H^0(X_{\mathbb{C}}, K_X)$ by Lemma 21 and a basis $\gamma_1, ..., \gamma_{2g}$ for $H_1(X, \mathbb{Z})$. Then we let $J(X) = \mathbb{C}^g/\Lambda$ be the Jacobian of $X$, where $\Lambda = \mathbb{Z}\Omega_1 + ... + \mathbb{Z}\Omega_{2g}$ is a rank-$2g$ period lattice generated in $\mathbb{C}^g$ by

$$\Omega_i = \Big( \int_{\gamma_i} \omega_1, ..., \int_{\gamma_i} \omega_g \Big)^t$$

for each $i = 1, ..., 2g$.

Choosing a real point $p_0 \in X(\mathbb{R})$ one version of the Abel-Jacobi map is defined by the holomorphic map

$$\begin{aligned} \varphi : X &\longrightarrow J(X) \\ p &\longrightarrow \Big( \int_p^{p_0} w_1, ..., \int_p^{p_0} w_g \Big). \end{aligned}$$

**Remark 37.** $\varphi$ is real.

We must first show that the lattice $\Lambda$ is closed under $\sigma$ operation. If $\alpha$ is any cycle in $H_1(X, \mathbb{Z})$, then the conjugate cycle $\sigma(\alpha)$ is also there. So, $\sigma \int_\alpha w_i = \int_{\sigma(\alpha)} \sigma(w_j) = \int_{\sigma(\alpha)} w_i \in \Lambda$ for each $i = 1, ..., g$. It is then enough to show that $\sigma$ and $\varphi$ commute.

$$\begin{aligned} \sigma\big(\varphi(p)\big) &= \Big( \int_{\sigma(p)}^{\sigma(p_0)} \sigma(w_1), ..., \int_{\sigma(p)}^{\sigma(p_0)} \sigma(w_g) \Big) \\ &= \Big( \int_{\sigma(p)}^{p_0} w_1, ..., \int_{\sigma(p)}^{p_0} w_g \Big) \\ &\varphi\big(\sigma(p)\big). \end{aligned}$$

We can extend this map to $X_d$ by linearity as follows: For any degree $d$ divisor $D = \sum_i p_i$ we let

$$\varphi(D) = \Big( \sum_i \int_{p_i}^{p_0} \omega_1, ..., \sum_i \int_{p_i}^{p_0} \omega_g \Big).$$

Let us state Abel's theorem now.

**Abel's Theorem.** *If $D, D' \in X_d$, then $D \sim D'$ if and only if $\varphi(D) = \varphi(D')$.*

*Proof.* See page 235 in [5]. □

As a concequence of Abel's Theorem, the map $\varphi : X_d \longrightarrow J(X)$ has a factorization $\varphi = \phi \circ u : X_d \xrightarrow{u} Pic(X) \xrightarrow{\phi} J(X)$. Furthermore, $\phi$ is real and injective.

**Remark 38.** If we identify the variety $W_d^r(X_\mathbb{C})$ with its holomorphic image $\phi\big(W_d^r(X_\mathbb{C})\big) \subset J(X)$ and use the proof of Proposition 36 we can denote the corresponding connected

component of $W_d^0(\mathbb{R})$ inside $J(X)$ containing the image of $U(i_1, ..., i_k)$ under $\phi \circ u$ by $V(i_1, ..., i_k)$ (and the one containing the image of $Q^{d/2}$ by $V(0)$ when $d$ is even).

Observe that $\delta(|D|) = k$ for each $\phi(|D|) \in V(i_1, ..., i_k)$ since any such effective divisor $D$ lies in the component containing the image of $C_{i_1} \times .. \times C_{i_k} \times Q^{d-2k}$ and all the $i'_j s$ are different from each other. Also given such a number $k$, there exist exactly $\binom{n(X)}{k}$ components of $W_d^0(\mathbb{R})$ whose points have their $\delta$ equal to $k$.

### 2.4 Trigonal Curves

Recall that a curve $X$ is called *trigonal* if it admits a degree 3 map $\pi : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$. In this case let $\mathcal{L} = \pi^*(\mathcal{O}(1))$ be the trigonal bundle and $T$ be the corresponding $g^1_3$. At this moment let us recall the following lemma.

**Lemma 39.** *Let $C$ be a smooth complex curve, $\mathcal{L}$ and $\mathcal{F}$ be invertible sheaves on $C$ and $s_1$ and $s_2$ be linearly independent sections of $\mathcal{L}$ with no common zeros such that they generate the space $V$. Then the kernel of the cup product map*

$$V \otimes H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{F} \otimes \mathcal{L})$$

*is isomorphic to $H^0(X, \mathcal{F} \otimes \mathcal{L}^{-1})$.*

*Proof.* See Base-Point-Free Pencil Trick in [1, Page 126]. □

**Proposition 40.** *If $g \geq 5$, then $T$ is unique.*

*Proof.* Let's say we have another divisor $L'$ that is not linearly equivalent to $L$ and corresponds to a $g^1_3$ on $X$, where $L$ is any divisor from the trigonal linear system $T$. Then by Corollary 33, we get

$$r(L + L') \geq r(L) + r(L') = 2.$$

Also by the Riemann-Roch formula, we have

$$h^1(X, \mathcal{O}_X(L+L')) = h^0(X, \mathcal{O}_X(L+L')) - \deg(L+L') + g - 1 \geq 3 - 6 + g - 1 = g - 4 > 0$$

for $g \geq 5$ so that $L + L'$ is special.

Note that $h^0(X, \mathcal{O}_X(L)) = 2$ and we can find sections $s_1$ and $s_2$ with no common zeros generating $H^0(X, \mathcal{O}_X(L))$ as $L$ corresponds to a non-constant map $\pi : X \longrightarrow \mathbb{P}^1$. Then by Lemma 39, we have a map

$$H^0(X, \mathcal{O}_X(L)) \otimes H^0(X, \mathcal{O}_X(L')) \longrightarrow H^0(X, \mathcal{O}_X(L + L'))$$

whose kernel has dimension equal to $h^0(X, \mathcal{O}_X(L' - L))$. By the assumption $L' \nsim L$ we have $h^0(X, \mathcal{O}_X(L' - L)) = 0$. Then $h^0(X, \mathcal{O}_X(L + L')) \geq 4$. Note that $\deg(L + L') = 6$ is equal to neither zero nor $2g - 2$ as $g \geq 5$. So, $L + L'$ can not be linearly equivalent to the zero divisor or $K_X$.

We also claim that there is no $g^1_2$ on $X$. Otherwise $T$ would be of the form

$$g^1_2 + p$$

by (1.2.1). Since $T$ has no base points, this is impossible. Therefore we have a contradiction by Theorem 31. □

We need the following theorem to make the necessary cohomological calculations.

**Theorem 41.** *[9, Chapter 1] Assume $g \geq 5$. Then*

*(1)* $\pi_* K_X \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, *where $a$ and $m$ are integers satisfying* $a \geq m$, $a + m = g - 2$ *and* $3a \leq 2g - 2$.

*(2) For each integer $k$, we have*

$$\begin{aligned} H^0(X, K_X \otimes \mathcal{L}^k) \cong & H^0(\mathbb{P}^1, \pi_* K_X \otimes \mathcal{O}_{\mathbb{P}^1}(k)) \\ \cong & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a + k)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m + k)) \\ & \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k - 2)). \end{aligned}$$

*Proof.* Note that $X \to \mathbb{P}^1$ is a constant degree map of smooth, projective curves. Then the direct image $\pi_* K_X$ must be a rank 3 locally free sheaf on $\mathbb{P}^1$. Also we know that any such sheaf on $\mathbb{P}^1$ splits into a direct sum of line bundles as follows

$$\pi_* K_X \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(e), \tag{2.4.1}$$

where $a, m$ and $e$ are the unique integers with $a \geq m \geq e$ [13, Theorem 18.5.6].

Now by the projection formula [13, 16.3.H], we get the isomorphisms

$$\pi_*(K_X) \otimes \mathcal{O}_{\mathbb{P}^1}(k) \cong \pi_*(K_X \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(k))) \cong \pi_*(K_X \otimes \mathcal{L}^k). \tag{2.4.2}$$

Also by Leray's theorem [13, 23.4.5], there is a spectral sequence with $E_2$ term is given by

$$E_2^{p,q} = H^q(\mathbb{P}^1, R^p\pi_* F)$$

converging to $H^{p+q} = H^{p+q}(X, F)$, where $F = K_X \otimes \mathcal{L}^k$.

Note that $H^{p+q}(X, F) = 0$ unless $0 \leq p+q \leq 1$ as $\dim X = 1$. Since $\pi$ is a morphsim of projective varieties with finite fibers, all the higher pushforwards $R^p(\pi_* F)$ vanish for each $p > 0$ [13, 18.8.5]. Therefore $E_2^{p,q}$ is trivial unless $0 = p \leq q \leq 1$.

So we may write a short exact sequence [15, Page 124]

$$0 \longrightarrow E^2_{0,q} \longrightarrow H^{0+q} = H^0(\mathbb{P}^1, \mathbb{R}^0\pi_* F) \longrightarrow E^2_{1,q-1} \longrightarrow 0$$

for each $q$. In particular, for $q = 0$ we get

$$H^0(\mathbb{P}^1, \pi_*(K_X \otimes \mathcal{L}^k)) \cong H^0(X, K_X \otimes \mathcal{L}^k).$$

Composing with (2.4.1) and (2.4.2) we get

$$\begin{aligned} H^0(X, K_X \otimes \mathcal{L}^k) \cong & H^0(\mathbb{P}^1, \pi_*(K_X) \otimes \mathcal{O}_{\mathbb{P}^1}(k)) \\ \cong & H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a+k)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m+k)) \\ & \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(e+k)). \end{aligned}$$

For $k > \max\{-e, 0\}$, using the Riemann-Roch formula we get

$$h^0(X, K_X \otimes \mathcal{L}^k) = 2g - 2 + 3k - g + 1 - h^0(X, \mathcal{L}^{-k}) = a + m + e + 3k + 3$$

which yields $a + m + e = g - 4$ since $h^0(X, \mathcal{L}^{-k})$ for $k > 0$.

When $k = 0$, we obtain

$$g = h^0(\mathbb{P}^1, \mathcal{O}(a)) + h^0(\mathbb{P}^1, \mathcal{O}(m)) + h^0(\mathbb{P}^1, \mathcal{O}(e)). \tag{2.4.3}$$

If $e \geq 0$, then this equation becomes $g = a + m + e + 3$ which would contradict with $a + m + e = g - 4$. So we must have $e < 0$. In this case $h^0(\mathbb{P}^1, \mathcal{O}(e)) = 0$ so that $a + m = g - 2$ by (2.4.3) and hence, $e = g - 4 - a - m = -2$.

Finally, for $k = -a$ we have by the duality

$$h^1(X, \mathcal{L}^a) = h^0(X, K \otimes \mathcal{L}^{-a}) = h^0(\mathbb{P}^1, \mathcal{O}) + h^0(\mathbb{P}^1, \mathcal{O}(m-a)) + h^0(\mathbb{P}^1, \mathcal{O}(-2-a)) \geq 1$$

as $h^0(\mathbb{P}^1, \mathcal{O}) = 1$. This means that $\mathcal{L}^a$ is special on $X$. In this case its degree can not exceed $2g - 2$. This proves the last inequality $3a \leq 2g - 2$. □

Here $m$ is called the *Maroni invariant* of the curve $X$ and it satisfies the following inequalities:

$$0 < \frac{g-4}{3} \leq m \leq \frac{g-2}{2}, \quad a = g - 2 - m. \tag{2.4.4}$$

Now let us fix $s(d, r, g) := 2(d-1) - g - 3(r-1)$, $\kappa := \phi(K_X)$, $\tau := \phi(T)$, where $\phi : Pic(X) \to J(X)$ is the Abel map. Let us use $W_d^r$ for $W_d^r(X_{\mathbb{C}})$. We can define the following sets:

$$U_d^r = \left\{ \begin{array}{cc} r\tau + W_{d-3r}^0, & \text{if} \quad d - 3r \geq 0 \\ \emptyset, & \text{otherwise} \end{array} \right\},$$

$$V_d^r = \left\{ \begin{array}{cc} \kappa - \big((g-d+r-1)\tau + W_{s(d,r,g)}^0\big), & \text{if} \quad s(d,r,g) \geq 0 \\ \emptyset, & \text{otherwise} \end{array} \right\}.$$

**Theorem 42.** *For $d < g$ and $r \geq 1$, we have*

*(1) $W_d^r = U_d^r \cup V_d^r$*

*(2) If $U_d^r \neq \emptyset$, then $U_d^r$ is an irreducible component of $W_d^r(X)$*

*(3) Suppose that $V_d^r \neq \emptyset$. Then also $U_d^r \neq \emptyset$ and $V_d^r$ is an irreducible component of $W_d^r$ different from $U_d^r$ if and only if $g - d + r - 1 \leq m$.*

*Proof.* See Proposition 5 in [9]. □

**Remark 43.** Suppose that $r = 1$ and $g - d \leq m$. Then the condition $g - d + r - 1 \leq m$

is satisfied. We have

$$\begin{aligned} s(d,1,g) = 2(d-1) - g - 3(1-1) =& 2(d-1) - g \\ \geq& 2(g-m-1) - g \\ =& g - 2 - 2m \\ =& a - m \\ \geq& 0 \end{aligned}$$

so that $V_d^1 \neq \emptyset$. Hence, if the inequality $g - d \leq m$ holds, then $W_d^1$ has two non-trivial irreducible components $U_d^1$ and $V_d^1$.

Let us use the main assumption that $X$ is a real curve. Since $T$ is the uniqe $g_3^1$, it must be real and $\pi$ is defined over $\mathbb{R}$. By Lemma 24 we have either $\delta(T) = 1$ or $\delta(T) = 3$. As usual, say $C_1, ..., C_{n(X)}$ are the connected components of $X(\mathbb{R})$.

**Proposition 44.** *[6, Page 129] If $\delta(T) = 3$, then $n(X) = 3$ and $g$ is even.*

*Proof.* Take a point $p \in C_i$ for any $i = 1, ..., n(X)$. Then $\pi(p) \in \mathbb{P}^1(\mathbb{R})$ as $\pi$ is defined over $\mathbb{R}$. $\pi^*(\pi(p))$ is an effective divisor of degree 3 in $T$. The claim here is that this divisor is real. Suppose to the contrary that it is not. Since it contains a real point $p$, it must contain a non-real point $p_0 \in X(\mathbb{C}) \setminus X(\mathbb{R})$ such that $\sigma(p_0) \notin \pi^*(\pi(p))$. Say we have $\pi^*(\pi(p)) = p + p_0 + p_1$ satisfying this condition. Since $p + p_0 + p_1 \in T$, using the fact that $T$ is real, we get

$$\begin{aligned} p + \sigma(p_0) + \sigma(p_1) = \sigma(p + p_0 + p_1) \sim \sigma(T) &\sim T \\ &\sim p + p_0 + p_1 \end{aligned}$$

so that $\sigma(p_0) + \sigma(p_1) \sim p_0 + p_1$. Note that $p_0$ is not real. Also $p_0$ and $p_1$ are not complex conjugates. Then $\sigma(p_0) + \sigma(p_1) \neq p_0 + p_1$ as divisors so that $|p_0 + p_1|$ contains an effective divisor other than $p_0 + p_1$ itself. This linear system is then actually a $g_2^1$. But we know that a trigonal curve of genus bigger than 4 can not possess a $g_2^1$. Therefore the assumption that $\pi^*(\pi(p))$ is not real can not be true.

Now that $\pi^*(\pi(p))$ is a real divisor in $T$, we get $\delta(\pi^*(\pi(p))) = 3$. This means that points of $\pi^*(\pi(p))$ must be from different components of $X(\mathbb{R})$. As $p \in \pi^*(\pi(p))$, $C_i$

is a pseudo-line for $\pi^*(\pi(p))$. Since $i$ is arbitrary, any component is a pseudo-line for $\pi^*(\pi(p))$. Therefore $n(X) = 3$.

Parity of $g$ comes from the fact that $n(X) \equiv g + 1 \pmod 2$ when $a(X) = 0$ by Proposition 13. To see the equality $a(X) = 0$, when $\delta(T) = 3$, the morphism $\pi : X(\mathbb{R}) \longrightarrow \mathbb{P}^1(\mathbb{R})$ induced by $T$ is a degree 3 real cover so that $\pi^{-1}(\mathbb{P}^1(\mathbb{R})) = X(\mathbb{R})$. Then $\pi\big(X(\mathbb{C}) \setminus X(\mathbb{R})\big) = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is disconnected, so is $X(\mathbb{C}) \setminus X(\mathbb{R})$. Hence $a(X) = 0$. □

# CHAPTER 3

## NECESSARY CONDITIONS FOR THE RESULTS

For this chapter assume that $X$ is a real trigonal curve of genus $g \geq 5$ and Maroni invariant $m$, $\pi : X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ is the trigonal map and $T$ is the trigonal linear system or any real effective divisor from this system, if there is no confusion. The aim in Chapter 4 will be to calculate $n(W^1_d)$, the number of connected components of the real locus of $W^r_d(X_{\mathbb{C}})$. For this purpose we want in this chapter to reveal the conditions on $d, g$ and $m$ for which we can do this calculation.

**Remark 45.** Recall that we identified $W^r_d = W^r_d(X_{\mathbb{C}})$ with its image under $\phi : Pic(X) \longrightarrow J(X)$. Since the canonical class $K_X$ and the trigonal class $T$ are real, we have $U^r_d(\mathbb{R}) = r\tau + W^0_{d-3r}(\mathbb{R})$ and $V^r_d(\mathbb{R}) = \kappa - \big((g-d+r-1)\tau + W^0_{s(d,r,g)}(\mathbb{R})\big)$ when they are non-empty, where $s(d,r,g) = 2(d-1) - g - 3(r-1)$. Note that $J(X)$ is a $g$-dimensional complex torus defined over $\mathbb{R}$. If we fix any $\eta \in J(X)$, then the morphism

$$\begin{aligned} J(X) &\longrightarrow J(X) \\ \alpha &\longmapsto \eta + \alpha \end{aligned}$$

is continuous. By Proposition 36 we may write

$$n(U^r_d) = n(W^0_{d-3r}) = \sum_{s=0}^{\lfloor \frac{d-3r}{2} \rfloor} \binom{n(X)}{d-3r-2s}, \text{ when } d-3r \geq 0$$

$$n(V^r_d) = n(W^0_{s(d,r,g)}) = \sum_{s=0}^{\lfloor \frac{s(d,r,g)}{2} \rfloor} \binom{n(X)}{s(d,r,g)-2s}, \text{ when } s(d,r,g) \geq 0.$$

**Lemma 46.** *If* $0 < g - d \leq m$, *then*

$$U^1_d(\mathbb{R}) \cap V^1_d(\mathbb{R}) \neq \emptyset \implies d > \frac{g+2}{2}$$

*Proof.* First note that degree of $K_X-(g-d+1)T$ is $2g-2-3(g-d+1) = 3d-g-5$. Since $a \geq m \geq g-d > 0$, by Theorem 42 we obtain

$$\begin{aligned} h^0(X_{\mathbb{C}}, K_X-(g-d+1)T) &= h^0(\mathbb{P}^1, \mathcal{O}(a-g+d-1)) \\ &\quad + h^0(\mathbb{P}^1, \mathcal{O}(m-g+d-1)) \\ &\quad + h^0(\mathbb{P}^1, \mathcal{O}(-2-g+d-1)) \\ &= a-g+d+m-g+d+0 \\ &= 2d-g-2. \end{aligned} \tag{3.0.1}$$

This shows that $K_X-(g-d+1)T$ is a real $g^{2d-g-3}_{3d-g-5}$. Using the descriptions of $U^1_d$ and $V^1_d$, we have

$$\begin{aligned} U^1_d(\mathbb{R}) \cap V^1_d(\mathbb{R}) \neq \emptyset &\iff T+D_1 \sim K_X-(g-d)T-D_2 \\ &\quad \text{for some } D_1 \in X_{d-3}(\mathbb{R}) \\ &\quad \text{and } D_2 \in X_{2(d-1)-g}(\mathbb{R}). \\ &\iff K_X-(g-d+1)T \sim D_1+D_2. \end{aligned} \tag{3.0.2}$$

The last line implies that $h^0(X_{\mathbb{C}}, K_X-(g-d+1)T) > 0$, i.e, $d > \dfrac{g+2}{2}$ by the above calculations (3.0.1).

□

**Remark 47.** Notice that the converse of Lemma 46 holds once we can find a real divisor $D \in K_X-(g-d+1)T$ such that $D$ can be written as $D = D_1+D_2$ with $D_1 \in X_{d-3}(\mathbb{R})$ and $D_2 \in X_{2(d-1)-g}(\mathbb{R})$. In most cases this will be possible and we will use this lemma in both ways. However, if $d-3$ is even, $D$ has an even degree and does not contain enough real points, then it may not be possible to decompose $D$ in this way.

**Lemma 48.** *If* $0 < g-d \leq m$ *and* $d > \dfrac{g+2}{2}$, *then*

$$\kappa-(g-d+1)\tau \in U^{2d-g-3}_{3d-g-5}(\mathbb{R}) \iff g-d = m$$

*Proof.* We know that $K_X-(g-d+1)T$ is a real $g^{2d-g-3}_{3d-g-5}$. Notice that

$$\kappa-(g-d+1)\tau \in U^{2d-g-3}_{3d-g-5}(\mathbb{R})$$

exactly when $K_X - (g-d+1)T \sim (2d-g-3)T + D$ for some real $D \geq 0$, i.e when

$$h^0(X_{\mathbb{C}}, K_X - (g-d+1+2d-g-3)T)) = h^0(X_{\mathbb{C}}, K_X - (d-2)) > 0.$$

We know that $m \leq \dfrac{g-2}{2}$ by (2.4.4) and $d-2 > \dfrac{g-2}{2}$ by assumption. Using Theorem 42, we have

$$\begin{aligned} h^0(X_{\mathbb{C}}, K_X - (d-2)T) &= h^0(\mathbb{P}^1, \mathcal{O}(a-d+2)) + h^0(\mathbb{P}^1, \mathcal{O}(m-d+2)) \\ &\quad + h^0(\mathbb{P}^1, \mathcal{O}(-2-d+2)) \\ &= h^0(\mathbb{P}^1, \mathcal{O}(a-d+2)) + 0 + 0 \end{aligned}$$

Summarizing, we may write

$$\kappa - (g-d+1)\tau \in U^{2d-g-3}_{3d-g-5}(\mathbb{R}) \iff a = g-2-m \geq d-2.$$

Under the hypothesis $g - d \leq m$ this is equivalent to saying that $g - d = m$. □

**Remark 49.** Assuming $g - d = m$ and $d > \dfrac{g+2}{2}$ we have

$$K_X - (g-d+1)T \sim (2d-g-3)T + D,$$

where $D \in X_{2g-3d+4}(\mathbb{R})$ is a fixed effective real divisor of degree $2g - 3d + 4$.

Composing with (3.0.2), in order to find the intersecting components of $U^1_d(\mathbb{R})$ and $V^1_d(\mathbb{R})$ it is enough to keep track of effective real divisors belonging to the class $(2d-g-3)T$ as long as $2g-3d+4$ is small.

**Lemma 50.** *If $g \equiv 0$ (mod 3) and $m = \dfrac{g-3}{3}$, then $\delta(T) = 1$.*

*Proof.* Letting $d = \dfrac{2g+3}{3}$, the conditions $d > \dfrac{g+2}{2}$ and $g - d = m$ of Lemma 48 are satisfied. Since the number of base points in $K_X - (g-d+1)T$ is $2g-3d+4 = 2g-2g-3+4 = 1$, we have the equivalence

$$K_X - (g-d+1)T = K_X - \frac{g}{3}T \sim (2d-g-3)T + p = \left(\frac{g}{3} - 1\right)T + p$$

for some fixed $p \in X$. As $K_X$ and $T$ are real, $p$ must be a real point of $X$. $\delta(K_X) = 0$ allows us to write

$$\delta\left(K_X - \frac{g}{3}T\right) = \begin{cases} \delta(T), & \text{if } g \text{ is odd} \\ 0, & \text{if } g \text{ is even} \end{cases}$$

and

$$\delta\left(\left(\frac{g}{3} - 1\right)T\right) = \begin{cases} \delta(T), & \text{if } g \text{ is even} \\ 0, & \text{if } g \text{ is odd} \end{cases}$$

so that

$$\delta\left(\left(\frac{g}{3} - 1\right)T + p\right) = \begin{cases} \delta(T) \pm 1, & \text{if } g \text{ is even} \\ 1, & \text{if } g \text{ is odd} \end{cases}$$

as adding one point to a divisor can change its $\delta$ by $+1$ or $-1$. By the equivalence $K_X - (\frac{g}{3})T \sim (\frac{g}{3} - 1)T + p$ we have

$$\delta\left(K_X - \left(\frac{g}{3}\right)T\right) = \delta\left(\left(\frac{g}{3} - 1\right)T + p\right).$$

This happens only if $\delta(T) = 1$. In this case $p$ and $T$ share the same pseudo-line. □

Now assume that $C_1$ is a pseudo-line for $T$.

**Lemma 51.** *Suppose that $\delta(T) = 1$. If $n(X) > 1$, then for any integer $i$ with $1 < i \leq n(X)$ there exists a real divisor $D = p_i + q_i + \tilde{q}_i \in T$ such that $p_i \in C_1$ and $q_i, \tilde{q}_i \in C_i$. Conversely, any real effective divisor $D \in T$ is either of this form (equality $i = 1$ is allowed) or contains a conjugate pair such that $D = p + q + \overline{q}$ where $p \in C_1, q \in X(\mathbb{C})\backslash X(\mathbb{R})$.*

*Proof.* If we choose a point $q \in C_i \subset X(\mathbb{R})$ for any $i \neq 1$ such that $\pi(q) \in \mathbb{P}^1(\mathbb{R})$, then $\pi^*(\pi(q)) \in T$ is an effective, degree 3 divisor. As in the proof of Proposition 44, it is also real. Any such divisor must contain a point $p \in C_1$ as $C_1$ is a pseudo-line for $T$. By the assumption $1 = \delta(T) = \delta(D)$ we have $D = p + q + \tilde{q}$ where $\tilde{q} \in C_i$.

Conversely, any real divisor $D \in T$ must also contain a point $p' \in C_1$. So, $D - p'$ is a real divisor of degree 2 such that $\delta(D) = 0$. If $D = p' + q_1 + q_2$, either we have $\overline{q_1} = q_2 \in X(\mathbb{C})\backslash X(\mathbb{R})$ or $q_1$ and $q_2$ are on the same component of $X(\mathbb{R})$. □

# CHAPTER 4

# COMPONENTS OF THE REAL BRILL-NOETHER LOCUS OF TRIGONAL CURVES

Recall by Remark 38 and Remark 45 that the connected components of $U_d^r(\mathbb{R}) = r\tau + W_{d-3r}^0(\mathbb{R})$ are of the form

$$\tau + V(i_1, ..., i_k),$$

where $1 \leq i_1 < i_2 < ...i_k \leq n(X)$, $d - 3r - k \geq 0$ and $d - 3r \equiv k \bmod 2$. Similarly the components of $V_d^r(\mathbb{R}) = \kappa - \big((g - d + r - 1)\tau + W_{s(g,r,d)}^0(\mathbb{R})\big)$ are

$$\kappa - \big((g - d + r - 1)\tau + V'(j_1, ..., j_l)\big)$$

for each $1 \leq j_1 < j_2 < ... < j_l \leq n(X)$, $s(d, r, g) - l \geq 0$ and $s(d, r, g) \equiv l \bmod 2$, where $s(d, r, g) = 2(d - 1) - g - 3(r - 1)$.

Since we have the equalities $W_d^r(\mathbb{R}) = U_d^r(\mathbb{R}) \cup V_d^r(\mathbb{R})$, $n(U_d^r) = n(W_{d-3r}^0)$ and $n(V_d^r) = n(W_{s(d,r,g)}^0)$, we obtain the immediate bounds

$$0 \leq n(W_d^r) \leq \sum_{s=0}^{\lfloor \frac{d-3r}{2} \rfloor} \binom{n(X)}{d - 3r - 2s} + \sum_{s=0}^{\lfloor \frac{s(d,r,g)}{2} \rfloor} \binom{n(X)}{s(d, r, g) - 2s}. \tag{4.0.1}$$

Now we focus on the case $r = 1$.

**Lemma 52.** *No component of $U_d^1(\mathbb{R})$ can intersect with two distinct components of $V_d^1(\mathbb{R})$ and vice versa.*

*Proof.* Assume to the contrary that $\tau + V(i_1, ..., i_k)$ intersects simultaneously with $\kappa - \big((g - d)\tau + V'(j_{1_1}, ..., j_{l_1})\big)$ and $\kappa - \big((g - d)\tau + V'(j_{1_2}, ..., j_{l_2})\big)$. Then there exist effective real divisors $D_1, D_2, D_3$ and $D_4$ such that $D_1, D_3 \in U(i_1, ..., i_k)$,

$D_2 \in U(j_{1_1}, ..., j_{l_1})), D_4 \in U(j_{1_2}, ..., j_{l_2}))$. Moreover, we have the following linear equivalence by (3.0.2)

$$D_1 + D_2 \sim D_3 + D_4 \sim K_X - (g - d + 1)T$$

such that it yields

$$D_1 - D_3 \sim D_4 - D_2.$$

Since $D_1$ and $D_3$ are on the same component $U(i_1, ..., i_k)$ of $X_{d-3}(\mathbb{R})$, their pseudo-lines are exactly the same. This implies

$$\delta(D_1 - D_3) = 0 = \delta(D_4 - D_2)$$

so that $\delta(D_4) = \delta(D_2)$. Indeed, $D_4$ and $D_2$ share the same pseudo-lines, i.e, they are on the same component of $X_{s(g,1,d)}(\mathbb{R}) = X_{2d-g-2}(\mathbb{R})$. Hence, we get

$$j_{1_1} = j_{1_2}, ..., j_{l_1} = j_{l_2}$$

such that two components $\kappa - \big((g-d)\tau + V'(j_{1_1}, ..., j_{l_1})\big)$ and $\kappa - \big((g-d)\tau + V'(j_{1_2}, ..., j_{l_2})\big)$ coincide.

Using the same method, we can say that no component of $V_d^1(\mathbb{R})$ can intersect with two distinct components of $U_d^1(\mathbb{R})$. □

Using Lemma 52 we can upgrade the lower bound in (4.0.1) for $n(W_d^1)$.

**Corollary 3.** *If* $0 < g - d \leq m$, *then*

$$n(W_d^1) \geq \sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s}.$$

*Proof.* The condition $d - g \leq m$ guarantees that $U_d^1(\mathbb{R})$ and $V_d^1(\mathbb{R})$ are non-empty by Remark 43. Also by Lemma 52 we may say that for each connected component $U$ of $U_d^1(\mathbb{R})$ there can be at most one connected component $V$ of $V_d^1(\mathbb{R})$ such that $V \cap U \neq \emptyset$ and $V$ has no non-empty intersection with other components of $U_d^1(\mathbb{R})$. So for each $U$ there is unique connected component of $W_d^1(\mathbb{R})$ containing $U$. The same holds for the connected components of $V_d^1(\mathbb{R})$ as well. This proves the formula

$$n(W_d^1) \geq \max\{n(U_d^1), n(V_d^1)\}.$$

By assumption $g-d$ is positive and $d-3 \geq 2(d-1)-g = s(d,1,g)$. This shows that

$$n(U_d^1) = \sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s} \geq \sum_{s=0}^{\lfloor \frac{s(d,1,g)}{2} \rfloor} \binom{n(X)}{s(d,1,g)-2s} = n(V_d^1).$$

The claim then follows. □

We also want to improve the upper bound in (4.0.1). Suppose that the conditions $d = g - m$ and $m < \frac{g-3}{2}$ hold. Then we have $d = g - m > g - \frac{g-3}{2} = \frac{g+3}{2}$ so we can use Remark 49. Our strategy is to find different components of $X_{d-3}(\mathbb{R})$ and $X_{2(d-1)-g}(\mathbb{R})$ on which there are divisors $D_1$ and $D_2$, respectively, such that

$$T + D_1 \sim K_X - (g-d)T - D_2,$$

equivalently,

$$D_1 + D_2 \sim K_X - (g-d+1)T \sim (2d-g-3)T + D_3 \tag{4.0.2}$$

for some fixed $D_3 \in X_{2g-3d+4}(\mathbb{R})$. Note that the value of $2g-3d+4$ is always non-negative due to the bound $\frac{g-4}{3} \leq m$ given in (2.4.4):

$$2g - 3d + 4 = 2g - 3(g-m) + 4 \geq 2g - 3\left(g - \frac{g-4}{3}\right) + 4 = 0.$$

**Remark 53.** The difference between $\deg(D_2) = 2(d-1)-g$ and $2d-g-3$ is just 1. Let us take any real effective divisor $D \in T$. It contains a real point, say $p$ and other points $q+q'$ such that $\sigma(q+q') = q+q'$. Then we can always make at least one such choice of divisors $D_1$ and $D_2$ in (4.0.2). For example

$$D_2 = (2d-g-3)p + q + q' \text{ and } D_1 = (2d-g-3)D - D_2 + D_3.$$

This in particular shows that $U_d^1(\mathbb{R}) \cap V_d^1(\mathbb{R}) \neq \emptyset$ whenever $2d-g-3 > 0$.

**Theorem 4.** *Let $X$ be a real trigonal curve of genus $g$ and Maroni invariant $m$ such that $m < \frac{g-3}{2}$ and $g \geq 5$. If $d = g - m$, then*

$$\sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s} \leq n(W_d^1) \leq \binom{n(X)-1}{2d-g-2} + \sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s}.$$

*Proof.* The lower bound is already given in Corollary 3. We will divide the proof of the upper bound into three cases.

**Case 1:** $\delta(T) = 1$ and $n(X) = 1$.

In this case we know that both of $U_d^1(\mathbb{R})$ and $V_d^1(\mathbb{R})$ are connected, because of the equalities

$$n(U_d^1) = \sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{1}{d-3-2s} = 1 = \sum_{s=0}^{\lfloor \frac{s(d,1,g)}{2} \rfloor} \binom{1}{s(d,1,g)-2s} = n(V_d^1).$$

Moreover, by Remark 53 they have a non-empty intersection and this implies

$$n(W_d^1) = 1 = 1 + \binom{n(X)-1}{2d-g-2}.$$

So the statement holds in this case.

**Case 2:** $\delta(T) = 1$ and $n(X) > 1$.

We want to show that any component $\kappa - \big((g-d)\tau + V'(j_1, ..., j_l)\big)$ of $V_d^1(\mathbb{R})$ has a non-empty intersection with $U_d^1(\mathbb{R})$ possibly except the ones with $l = 2d-g-2$ and $j_1 \neq 1$.

Let us use the points $p_i \in C_1$ and $q_i, \tilde{q}_i \in C_i$ of Lemma 51 such that $T \sim p_i + q_i + \tilde{q}_i$ for each $i = 2, ..., n(X)$.

If $j_1 \neq 1$ and $l < 2d-g-2$, then $j_l > 1$ for each $l$. Consider the equivalence

$$(2d-g-3)T + D_3 \sim (2d-g-3-l)(p_{j_l} + q_{j_l} + \tilde{q}_{j_l}) + \sum_{n=j_1}^{j_l} (p_n + q_n + \tilde{q}_n) + D_3.$$

Then the sum of the divisors

$$D_1 = D_3 + \left( \sum_{n=j_1}^{j_l} (p_n + \tilde{q}_n) \right) - p_{j_l} + (2d-g-3-l)(q_{j_l} + \tilde{q}_{j_l}) \text{ and}$$

$$D_2 = q_{j_1} + ... + q_{j_l} + (2d-g-2-l)p_{j_l}$$

lies in $(2d-g-3)T + D_3$. Note that $D_2 \in U(j_1, ..., j_l) \subset X_{2d-g-2}(\mathbb{R})$ since $2d-g-2-l$ is an even number by the very description of these components. Then $\kappa - ((g-d)\tau + V'(j_1, ..., j_l))$ intersects with $U_d^1(\mathbb{R})$.

If $j_1 = 1$ and $l \leq 2d - g - 2$, then we have another equivalence

$$(2d-g-3)T + D_3 \sim (2d-g-2-l)(p_{j_l} + q_{j_l} + \tilde{q}_{j_l}) + \sum_{n=j_2}^{j_l}(p_n + q_n + \tilde{q}_n) + D_3$$

such that the divisors

$$D_1 = D_3 + \left(\sum_{n=j_2}^{j_l}(p_n + \tilde{q}_n)\right) - p_{j_l} + (2d-g-2-l)(q_{j_l} + \tilde{q}_{j_l}),$$
$$D_2 = q_{j_2} + ... + q_{j_l} + (2d-g-1-l)p_{j_l}$$

add up to a divisor in $(2d-g-3)T + D_3$. As $2d-g-1-l$ is an odd number, we have $D_2 \in U(j_1, ..., j_l)$.

This shows that in addition to the number $n(U_d^1) = n(W_{d-3}^0)$, there could be at most $\binom{n(X)-1}{2d-g-2}$-many possible connected components of the form

$$\kappa - \big((g-d)\tau + V'(j_1, ..., j_{2d-g-2})\big)$$

coming from $V_d^1(\mathbb{R})$, where $2 \leq j_1 < ... < j_{2d-g-2} \leq n(X)$.

**Case 3:** $\delta(T) = 3$.

In this case we know that $n(X) = 3$ and $g$ is even by Proposition 44. This makes $g - 2m - 2$ an even number so that the connected components of $V_d^1(\mathbb{R})$ are just translates of $V'(0), V'(1,2), V'(1,3)$ and $V'(2,3)$. Since $\delta(T) = 3$, we have points $t_1 \in C_1, t_2 \in C_2, t_3 \in C_3$ such that $t_1 + t_2 + t_3 \sim T$. Consider the equivalence

$$(2d-g-3)T + D_3 \sim (2d-g-3)(t_1 + t_2 + t_3) + D_3.$$

For each $i$ and $j$ with $1 \leq i < j \leq 3$ we can choose $D_2 = (2d-g-3)t_i + t_j$ and $D_1$ as the rest of $(2d-g-3)(t_1+t_2+t_3) + D_3$ such that $D_1 + D_2 \in (2d-g-3)T + D_3$. This choice gives rise to a point of intersection of $U_d^1(\mathbb{R})$ and $\kappa - ((g-d)\tau + V'(i,j))$.

Also the choice $D_2 = (2d-g-4)t_1 + 2t_2$ lies in $U(0)$ and together with $D_1 = t_1 + (2d-g-5)t_2 + (2d-g-3)t_3 + D_3$ they add up to $(2d-g-3)(t_1+t_2+t_3) + D_3$. So, $\kappa - ((g-d)\tau + V'(0))$ intersects with $U_d^1(\mathbb{R})$ as well. Hence, $n(W_{d-3}^0)$ is an upper bound for $n(W_d^1)$ in the case $\delta(T) = 3$. □

**Theorem 5.** *Let $X$ be any real trigonal curve of genus $g$ and Maroni invariant $m$ such that $g \equiv 0$ (mod 3), $g > 5$ and $m = \frac{g-3}{3}$. If $d = \frac{2g+3}{3}$, then we have*

$$n(W^1_d) = \binom{n(X)-1}{2d-g-2} + \sum_{s=0}^{\lfloor \frac{d-3}{2} \rfloor} \binom{n(X)}{d-3-2s},$$

*Proof.* Note that by Lemma 50 $\delta(T) = 1$ in this case. Moreover, if $C_1$ is the pseudo-line for $T$, then we have

$$K_X - (g-d+1)T \sim (2d-g-3)T + p = \left(\frac{g}{3} - 1\right) T + p$$

for some fixed point $p \in C_1$. Suppose that an exceptional component $\kappa - \big((g-d)\tau + V'(j_1, ..., j_l)\big)$ such that $l = 2d-g-2 = g/3$ and $j_1 \neq 1$ has a non-empty intersection with $U^1_d(\mathbb{R})$. Then there exist real effective divisors $D_1$ and $D_2$ such that $D_2 \in U(j_1, ..., j_l) \subset X_{g/3}(\mathbb{R})$ and

$$D_1 + D_2 \sim \left(\frac{g}{3} - 1\right) T + p.$$

Recall that $p$ is the base point of this linear system. Since $\delta(D_2) = \deg(D_2) = g/3$ and $j_1 \neq 1$, $D_2$ can not contain $p$. Therefore $D := D_1 - p$ is effective. Then

$$D + D_2 \sim \left(\frac{g}{3} - 1\right) T$$

contains points from $C_{j_1}, ..., C_{j_{g/3}}$. This is impossible as the divisors from $T$ can not contain two points with different pseudo-lines other than $C_1$. So there is no such divisor in $(\frac{g}{3} - 1)T$. □

# CHAPTER 5

# DISCUSSION

Now let $X$ be any real curve and $C_1, ..., C_n$ be the connected components of $X(\mathbb{R})$. Consider the homomorphism probably first defined in [6]:

$$\begin{aligned} c : Pic(X)(\mathbb{R}) &\longrightarrow (\mathbb{Z}/2)^n \\ |D| &\longmapsto (..., P_{C_i}(D), ...), \end{aligned}$$

where, $P_{C_i}(D)$ is the parity of $\deg_{C_i}(D)$ modulo 2. Since there is no conditon on the degree of linear systems, the homomorphism is surjective.

Recovering the varieties $W_d^r$ on $Pic(X)$, once we restrict $c$ to $W_d^r(\mathbb{R})$ the map

$$c : W_d^r(\mathbb{R}) \longrightarrow (\mathbb{Z}/2)^n$$

is not surjective anymore. Notice in the case $r = 0$ that the connected components of $W_d^0(\mathbb{R})$ are given by the non-empty fibers of $c$. Indeed, if $(j_1, ..., j_n) \in (\mathbb{Z}/2)^n$ and $j_{l_1}, ..., j_{l_k}$ are the non-zero ones, then

$$c^{-1}(j_1, ..., j_n) = V(l_1, ..., l_k) \iff \sum_{i=1}^{n} j_i \equiv d \pmod 2,$$

where $V(l_1, ..., l_k)$ is the connected component of $W_d^0(\mathbb{R})$ that corresponds to the linear systems whose pseudo-lines are $C_{l_1}, ..., C_{l_k}$.

Let us define the following sublocus

$$W_{d,\delta}^r(X) := \{|D| \in W_d^r(X_{\mathbb{C}})(\mathbb{R}) : \delta(D) = \delta\} \tag{5.0.1}$$

for each $\delta$ satisfying

$$\delta \leq d \text{ and } \delta \equiv d \bmod 2.$$

In the language of the map $c|_{W_d^r} : W_d^r(\mathbb{R}) \to (\mathbb{Z}/2)^n$, we may write

$$W_{d,\delta}^r(X) = \bigcup_{\sum u_i = \delta} c^{-1}(u_1, ..., u_{n(X)}).$$

Since $c$ is constant on the connected components of $W_d^0(\mathbb{R})$ and

$$W_{d,\delta}^r(X) = W_{d,\delta}^0(X) \cap W_d^r(\mathbb{R}),$$

we may conclude that $W_{d,\delta}^r$ is union of some connected components of $W_d^r(\mathbb{R})$.

Even if $W_{d,\delta}^r := W_{d,\delta}^r(X)$ may not be a variety, with a slight abuse of notation, we will use $n(W_{d,\delta}^r)$ to indicate the number of connected components that $W_{d,\delta}^r$ corresponds in $W_d^r(\mathbb{R})$. By Proposition 36, we have

$$n(W_{d,\delta}^0) = \binom{n(X)}{\delta} \tag{5.0.2}$$

for each $\delta$ such that $\delta \leq d$ and $\delta \equiv d \bmod 2$.

When $r > 0$ it is very natural to have bounds on the number $n(W_d^r)$. Indeed, the inequalities found in Chapter 4 could be likely related to the intrinsic geometry of the curve itself. Let us see the following particular example.

**Example 54.** Consider a trigonal curve $X$ of genus 5. We have $m = 1$ by (2.4.4). We will calculate $n(W_4^1)$. Note that the condition $m < (g-3)/2$ in Theorem 4 is not satisfied. We only have the bounds given in (4.0.1) and Corollary 3 that yield

$$n(X) \leq n(W_4^1) \leq 2n(X).$$

Take any real $T \in g_3^1$. Then by the Riemann-Roch formula,

$$h^0(X_{\mathbb{C}}, K_X(-T)) = h^0(X_{\mathbb{C}}, \mathcal{O}(T)) + g - \deg(D) - 1 = 2 + 5 - 3 - 1 = 3$$

so that $K_X - T$ defines a real $g_5^2$. Note that this $g_5^2$ can not have a base point, otherwise we would get a $g_4^2$ but this contradicts with Theorem 31 since $X$ has no hyperelliptic divisor. Then it defines a real map

$$\phi : X \longrightarrow \mathbb{P}_{\mathbb{C}}^2$$

such that the image $\phi(X) \in \mathbb{P}_{\mathbb{C}}^2$ is a plane curve of degree 5. The equality $g = 5 = \frac{(5-1)(4-2)}{2} - 1$ implies by the degree-genus formula that $\phi(X)$ has only one singularity,

a cusp or a node [1, Page 208]. This means that there exists exactly one pair of points $p_0, q_0 \in X$ such that

$$r(K_X - T - p_0 - q_0) = r(K_X - T) - 1 = 1.$$

Since $g^1_3$ is unique and $\deg(K_X - T - p_0 - q_0) = 3$, we have $K_X - T - p_0 - q_0 \sim T$, i.e, $K_X - 2T \sim p_0 + q_0$. As this pair is the unique one such and both $K_X$ and $T$ are real, the divisor $p_0 + q_0$ is a real divisor. Moreover, $\delta(p_0 + q_0) = \delta(K_X - 2T) = 0$.

On the other hand, $W^1_4$ has two irreducible components $U^1_4 = \tau + W^0_1$ and $V^1_4 = \kappa - (\tau + W^0_1)$ both of which are isomorphic to $X$. Using the same method applied in (3.0.2), two connected components of $U^1_4(\mathbb{R})$ and $V^1_4(\mathbb{R})$ intersect if and only if there are points $p, q \in X(\mathbb{R})$ such that $T + p \sim K_X - T - q$. But we found above that there is only one pair satisfying such a linear equivalence. We may get at most two possible intersection points from $(p, q) = (p_0, q_0)$ and $(p, q) = (q_0, p_0)$.

Since $g$ is odd, $\delta(T) = 1$ by Lemma 44. We break the rest into two cases:

If the singularity above is a cusp or real node then $p_0, q_0 \in X(\mathbb{R})$ so that $U^1_4(\mathbb{R}) \cap V^1_4(\mathbb{R}) \neq \emptyset$. Since $\delta(p_0 + q_0) = 0$, we have $p_0, q_0 \in C_i$ for some $i = 1, ..., n(X)$. Then $\tau + V(i)$ intersects with $\kappa - (\tau + V'(i))$. In this case we get either

$$n(W^1_{4,0}) = 1 \text{ and } n(W^1_{4,2}) = 2n(X) - 2$$

or

$$n(W^1_{4,0}) = 2 \text{ and } n(W^1_{4,2}) = 2n(X) - 3$$

depending on $\delta(T + p_0)$ so that $n(W^1_4) = 2n(X) - 1$.

If the singularity of $\phi(X)$ is a complex node, then $p_0 = \overline{q_0} \notin X(\mathbb{R})$ and $U^1_4(\mathbb{R}) \cap V^1_d(\mathbb{R}) = \emptyset$. Hence, $n(W^1_{4,0}) = 2$, $n(W^1_{4,2}) = 2n(X) - 2$ and $n(W^1_4) = 2n(X)$.

Let us identify $X$ with its canonical image $\phi_{K_X}(X)$ under the embedding

$$\phi_{K_X} : X \longrightarrow \mathbb{P}^{g-1}_{\mathbb{C}}$$

given by $K_X$. The unique $g^1_3$ on $X$ generates a surface

$$S := \bigcup_{D \in g^1_3} \overline{D} \in \mathbb{P}^{g-1}$$

which is also a rational normal scroll over $\mathbb{P}^1$, where $\overline{D}$ denotes the linear span of $D$. Since the $g^1_3$ is real, we have $\sigma_{\mathbb{P}}(S) \subset S$ so that $S$ is a real surface by Lemma 7. By the geometric version of the Riemann-Roch theorem [1, Page 12] stating

$$r(E) = d - 1 - \dim \overline{E}$$

for any effective divisor $E$ and $g > 1$, we get $\dim \overline{D} = 1$ for any $D \in g^1_3$. Therefore the scroll is geometrically ruled and we know that any such surface is isomorphic to a Hirzebruch sufrace [2, Proposition III 15].

It is very well-known that this Hirzebruch surface is of the form

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(e_1) \oplus \mathcal{O}_{\mathbb{P}^1}(e_2)) \longrightarrow \mathbb{P}^1_{\mathbb{C}},$$

where $e_1 + 1$ and $e_2 + 1$ are the integers at which the function

$$d(j) = h^0(X_{\mathbb{C}}, K_X(-jT)) - h^0(X_{\mathbb{C}}, K_X((-j-1)T)), \text{ for } j = 0, 1, 2...$$

makes a jump [11, Theorem 2.5].

Recall by Theorem 41 that for each $k \in \mathbb{Z}$, we have

$$\begin{aligned} h^0(X_{\mathbb{C}}, K_X(kT)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(a+k)) + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m+k)) \\ + h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k-2)). \end{aligned}$$

So, we obtain

$$d(j) = \begin{cases} 2, & \text{if } 0 \leq j \leq m \\ 1, & \text{if } m \leq j \leq a \\ 0, & \text{if } a+1 \leq j. \end{cases}$$

Hence, $S$ is isomorphic to $\Sigma_{a-m} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(m))$. This is another and more geometric way to see the Maroni invariant $m$ of the trigonal curve $X$.

Let $\mathcal{M}_{g/\mathbb{R}}$ be the moduli space of real algebraic curves of genus $g$. Inside this variety we have the locus $\mathcal{T}_{g/\mathbb{R}}$ parametrizing the isomorphism classes of real trigonal curves of genus $g$. It is known that $\mathcal{T}_{g/\mathbb{R}}$ has a stratification given by the Maroni invariant of the curves.

Counting the isomorphism classes of such curves embedded in $\Sigma_{a-m}$, each strata $\mathcal{T}_{g/\mathbb{R}}(m)$ has dimension $g + 4 + 2m$ when $\frac{g-4}{3} \leq m < \frac{g-2}{2}$ [16]. The dimension is $2g + 1$ when $m = \frac{g-2}{2}$.

**Remark 55.** We need to remark that the curves of Maroni invariant $m = \frac{g-3}{3}$ in Theorem 5 form a $\frac{5g}{3}+2$-dimensional family in $\mathcal{T}_{g/\mathbb{R}}$ . When we remove the restrictions over the base points of $K_X - (g-d+1)T$, we talk about curves of almost arbitrary Maroni invariant in a more general setup in Theorem 4. However, the condition $m < \frac{g-3}{2}$ implies that these curves are not general when we treat them as complex curves in $M_g$.

**Remark 56.** We can nicely generalize Theorem 4 and Theorem 5 to the case $r > 1$ :

Let $X$ be a real trigonal curve of genus $g$ and Maroni invariant $m$. If $g-d+r-1 = m$ and $d > (g+3r)/2$, then

$$\sum_{s=0}^{\lfloor \frac{d-3r}{2} \rfloor} \binom{n(X)}{d-3r-2s} \leq n(W_d^r) \leq \binom{n(X)-1}{2d-g-3r+1} + \sum_{s=0}^{\lfloor \frac{d-3r}{2} \rfloor} \binom{n(X)}{d-3r-2s}.$$

If we further assume that $2g-3d+3r = 0$, then

$$n(W_d^r) = \binom{n(X)-1}{2d-g-3r+1} + \sum_{s=0}^{\lfloor \frac{d-3r}{2} \rfloor} \binom{n(X)}{d-3r-2s}.$$

# REFERENCES


[1] Arbarello, E., Cornalba, M., Griffiths, P. and Harris, J. “Geometry of Algebraic Curves, I", Grundlehren der Mathematischen Wissenschaften, 267, Springer-Verlag, New York, 1985.

[2] Beauville, A. “Complex Algebraic Surfaces”, No. 34. Cambridge University Press, 1996.

[3] Chaudhary, S. “The Brill-Noether Theorem for Real Algebraic Curves”, Ph. D. thesis, Duke University, 1996.

[4] Coppens, M. and Huisman, J. “Pencils on real curves”, Mathematische Nachrichten 286.8-9 (2013): 799-816.

[5] Griffiths, P. and Harris, J. “Principles of Algebraic Geometry”, Wiley-lnterscience, New York, 1978.

[6] Gross, B. H. and Harris, J. “Real algebraic curves”, Annales scientifiques de l’École Normale Supérieure. Vol. 14. No. 2. (1981), 157-182.

[7] Huisman, J. “Non-special divisors on real algebraic curves and embeddings into real projective spaces”, Annali di Matematica Pura ed Applicata 182 (2003): 21-35.

[8] Mangolte, F. “Real Algebraic Varieties”, Springer International Publishing, 2020.

[9] Martens, G. and Schreyer, F-O. “Line bundles and syzygies of trigonal curves”, Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg, Vol. 56. No. 1, Springer-Verlag, (1986), 169-189.

[10] Monnier, J. P. “Divisors on real curves”, Advances in Geometry (2003): 339-360.

[11] Schreyer, F. O. “Syzygies of canonical curves and special linear series”, Mathematische Annalen, 275(1), (1986), 105-137.

[12] Stöhr, K-O. and Viana, P. “Weierstrass gap sequences and moduli varieties of trigonal curves”, Journal of Pure and Applied Algebra 81.1 (1992): 63-82.

[13] Vakil, R. “The Rising Sea: Foundations of Algebraic Geometry", URL https://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf (Accessed 28.08.2024)

[14] Viro, O. “From the sixteenth Hilbert problem to tropical geometry”, Japanese Journal of Mathematics 3 (2008): 185-214.

[15] Weibel, C. A. “An Introduction to Homological Algebra”, Cambridge University Press, 1994.

[16] Zvonilov, V. I. “Isotopies of real trigonal curves on Hirzebruch surfaces”, Journal of Mathematical Sciences 113.6 (2003): 804-809.

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