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Constraints and time evolution in generic *f*(Riemann) gravity

Emel Altas[1] **and Bayram Tekin**[2,*]

[1] Department of Physics, Karamanoglu Mehmetbey University, 70100 Karaman, Turkey
[2] Department of Physics, Middle East Technical University, 06800 Ankara, Turkey

E-mail: btekin@metu.edu.tr and emelaltas@kmu.edu.tr

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Abstract
We give a detailed canonical analysis of the n-dimensional f(Riemann) gravity, correcting the earlier results in the literature. We also write the field equations in the Fischer–Marsden form which is amenable to identifying the non-stationary energy on a spacelike hypersurface. We give pure R^2 and $R_{\mu\nu}R^{\mu\nu}$ theories as examples.

Keywords: constraints, evolution, generic, gravity

1. Introduction

Within the paradigm of effective field theories, general relativity augmented with dark matter and dark energy is the lowest-order effective theory of gravity that works remarkably well at small and large scales. Of course, it is expected to be modified at extremely high energies, or extremely short distances, for example around the black hole or Big Bang singularities. The modifications can be computed from a microscopic theory, such as string theory, alternatively, one can study generic modifications consistent with symmetries (see, for example, [1] for more arguments of the raison d'être of modified gravity theories). A large subclass of modified

* Author to whom any correspondence should be addressed.

gravity theories is described by the action of the form[3]

$$S = \frac{1}{2}\int_{\mathcal{M}} d^n x\, \sqrt{-g}\, f(R_{\mu\nu\rho\sigma}), \tag{1}$$

where $f(R_{\mu\nu\rho\sigma})$ is a differentiable scalar invariant of the Riemann curvature tensor (we are taking the Newton's constant $\kappa = 1$). We consider the metric tensor to be the independent field. We shall work in the metric formulation and in n dimensions. Canonical analysis of this type of theory was given in the pioneering work [3], whose notation we shall adapt here. Our analysis mostly agrees with those of [3], however, we shall make some corrections and also recast the Hamiltonian formulation of the theory in the rather beautiful Fischer–Marsden form [4]. Such a construction easily allows one to define the Killing initial data (KID) and the approximate KIDs that are used in the definition of non-stationary energy contained in a spatial hypersurface [5–9]. These computations would be relevant to identifying the initial gravitational wave content. Many papers are dedicated to various aspects of the $f(R_{\mu\nu\rho\sigma})$ theory. For example in [10], the particle spectrum, the masses of the perturbative excitations of this generic theory with one massive spin-2, one massless spin-2, and a massive spin-0 particle was given around any one of its constant curvature vacua. In [11], conserved charges of the theory, such as energy-momentum and angular momentum, were constructed.

Here one of our goals is to expound upon the Arnowitt–Deser–Misner (ADM) analysis [12] and give sufficient details of the computations, so that the reader can follow all the details rather easily. We also apply our results to pure R^2 and $R_{\mu\nu}R^{\mu\nu}$ theories. We made a meticulous effort to write all the proofs of our statements in the appendices, so as not to significantly cut the flow of the discussion in the main text.

The layout of this paper is as follows: In section II, we introduce the action and the field equations of f(Riemann) theories. In section III, we summarize the ADM splitting [12] of the action, which yields the Hamiltonian, the constraint equations, and then the time-evolution equations of the initial data on the hypersurface. In section IV, we introduce the construction of the nonstationary energy for f(Riemann) theories. In section V, we consider the R^2 and $R_{\mu\nu}R^{\mu\nu}$ theories as examples. The computations are long, therefore, we give most of the details of the calculations in Appendices.

2. Field Equations of *f*(Riemann) theories

The field equations coming from variation of the action (1) in the presence of a minimally coupled matter field are[4]

$$-\frac{1}{2}g^{\mu\nu}f - R^{(\mu}{}_{\gamma\rho\sigma}\frac{\partial f}{\partial R_{\nu)\gamma\rho\sigma}} - 2\nabla_\sigma\nabla_\rho\frac{\partial f}{\partial R_{\sigma(\mu\nu)\rho}} = T^{\mu\nu}, \tag{2}$$

where the round brackets denote symmetrization with a factor of $1/2$. In appendix A, we gave the ADM decompositions of the necessary spacetime tensor fields, and then the proof of (2) is given in appendix B. This equation includes fourth-order derivatives of the metric.

[3] This manuscript is written for the volume 'Fields, Gravity, Strings and Beyond: In Memory of Stanley Deser' edited by M Henneaux, R I Nepomechie, and D Seminara. Deser (1931–2023) was very interested in modified theories of gravity: we dedicate this work to him. For personal reminiscence, the reader is invited to read [2].

[4] There is a sign difference in the second term of the field equations in [3].

One introduces auxiliary variables to simplify the ensuing discussion and lower the number of derivatives. Following [3], let us consider the 'mother action':

$$S = \frac{1}{2}\int_{\mathcal{M}} d^n x\,\sqrt{-g}\,\Big(f(\rho_{\mu\nu\rho\sigma}) + \varphi^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma})\Big), \tag{3}$$

where two auxiliary fields (ρ,φ) have been introduced. These rank-4 tensors have the same symmetries as the Riemann tensor and are assumed to be independent of each other and the metric $g_{\mu\nu}$ tensor. Therefore, variation of the action with respect to all fields can be written as

$$\begin{aligned}\delta S = \frac{1}{2}\int_{\mathcal{M}} d^n x\,\Bigg(&\delta\sqrt{-g}\,\Big(f(\rho_{\mu\nu\rho\sigma}) + \varphi^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma})\Big)\\ &+ \sqrt{-g}\,\Big(\delta f(\rho_{\mu\nu\rho\sigma}) + \delta\varphi^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}) + \varphi^{\mu\nu\rho\sigma}(\delta R_{\mu\nu\rho\sigma} - \delta\rho_{\mu\nu\rho\sigma})\Big)\Bigg).\end{aligned} \tag{4}$$

One has $\delta f(\rho_{\mu\nu\rho\sigma}) = \frac{\partial f}{\partial \rho_{\mu\nu\rho\sigma}}\delta\rho_{\mu\nu\rho\sigma}$, and the variation of the Riemann tensor reads

$$\begin{aligned}\delta R_{\mu\nu\rho\sigma} &= \delta g_{\mu\lambda}R^{\lambda}{}_{\nu\rho\sigma} + g_{\mu\lambda}\delta R^{\lambda}{}_{\nu\rho\sigma},\\ &= \delta g_{\mu\lambda}R^{\lambda}{}_{\nu\rho\sigma} + \nabla_\rho\nabla_{[\nu}\delta g_{\mu]\sigma} + \nabla_{[\rho}\nabla_{\sigma]}\delta g_{\mu\nu} + \nabla_\sigma\nabla_{[\mu}\delta g_{\nu]\rho}.\end{aligned} \tag{5}$$

Due to the symmetries of the tensor fields, one gets

$$\varphi^{\mu\nu\rho\sigma}(\delta R_{\mu\nu\rho\sigma} - \delta\rho_{\mu\nu\rho\sigma}) = \delta g_{\mu\nu}R^{(\mu}{}_{\lambda\rho\sigma}\varphi^{\nu)\lambda\rho\sigma} + 2\varphi^{\sigma(\mu\nu)\rho}\nabla_\rho\nabla_\sigma\delta g_{\mu\nu} - \varphi^{\mu\nu\rho\sigma}\delta\rho_{\mu\nu\rho\sigma}. \tag{6}$$

Using integration by parts and defining the following tensor field[5]

$$\mathcal{E}^{\mu\nu} := -R^{(\mu}{}_{\lambda\rho\sigma}\varphi^{\nu)\lambda\rho\sigma} - 2\nabla_\sigma\nabla_\rho\varphi^{\sigma(\mu\nu)\rho} - \frac{1}{2}g^{\mu\nu}\Big(f(\rho_{\lambda\gamma\rho\sigma}) + \varphi^{\lambda\gamma\rho\sigma}(R_{\lambda\gamma\rho\sigma} - \rho_{\lambda\gamma\rho\sigma})\Big), \tag{7}$$

one can express the variation of the total action as

$$\delta S = \frac{1}{2}\int_{\mathcal{M}} d^n x\,\sqrt{-g}\Bigg(-\mathcal{E}^{\mu\nu}\delta g_{\mu\nu} + \delta\varphi^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}) + \left(\frac{\partial f}{\partial\rho_{\mu\nu\rho\sigma}} - \varphi^{\mu\nu\rho\sigma}\right)\delta\rho_{\mu\nu\rho\sigma}\Bigg). \tag{8}$$

In appendix C, one can see more details of this computation including the variation of the 'mother action'. From the last expression, one gets a set of field equations.

[5] Note that in equation (2.3) of [3] there is an additional term involving the derivative of f with respect to ρ, which should not exist.

- Variation with respect to $\varphi^{\mu\nu\rho\sigma}$ gives

$$R_{\mu\nu\rho\sigma} = \rho_{\mu\nu\rho\sigma}. \tag{9}$$

- Variation with respect to $\rho_{\mu\nu\rho\sigma}$ yields

$$\varphi^{\mu\nu\rho\sigma} = \frac{\partial f}{\partial \rho_{\mu\nu\rho\sigma}} = \frac{\partial f}{\partial R_{\mu\nu\rho\sigma}}, \tag{10}$$

  where for the second equality we have used the previous result.
- The metric variation in the action yields

$$\mathcal{E}^{\mu\nu} = T^{\mu\nu}. \tag{11}$$

Substituting the additional equations in the explicit form of $\mathcal{E}^{\mu\nu}$, we recover the original fourth order derivative equation (2). We assumed minimal coupling of matter and the metric, and no direct coupling of the matter to the auxiliary fields ρ and φ.

3. ADM decomposition of the *f*(Riemann) theory

Let us assume that the topology of the spacetime manifold is $\mathscr{M} = \mathbb{R} \times \Sigma$, with the first factor being the time dimension and Σ being a spacelike hypersurface. In Einstein's gravity, the initial data constitute the Riemannian metric γ and the extrinsic curvature K on Σ together with the initial matter. The connection on the hypersurface satisfies the metric compatibility condition: $D_i\gamma_{jk} = 0$. See figure 1 for the slicing of the spacetime.

The metric in terms of the lapse function and the shift vector reads

$$\mathrm{d}s^2 = \left(N_iN^i - N^2\right)\mathrm{d}t^2 + 2N_i\,\mathrm{d}t\,\mathrm{d}x^i + \gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{12}$$

More explicitly, the components of the metric and the inverse metric are

$$g_{00} = -N^2 + N_iN^i, \qquad g_{0i} = N_i, \qquad g_{ij} = \gamma_{ij}, \tag{13}$$

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = \frac{1}{N^2}N^i, \quad g^{ij} = \gamma^{ij} - \frac{1}{N^2}N^iN^j. \tag{14}$$

We choose the future-pointing unit normal vector n^μ as

$$n^\mu = \left(\frac{1}{N}, -\frac{N^i}{N}\right), \qquad n_\mu = (-N, \vec{0}), \tag{15}$$

while the extrinsic curvature in terms of the unit-normal vector reads as

$$K_{ij} = \nabla_i n_j = \frac{1}{2N}\left(\dot{\gamma}_{ij} - D_iN_j - D_jN_i\right), \tag{16}$$

where $\dot{\gamma}_{ij} := \partial_0\gamma_{ij}$. To rewrite the action (3) in terms of the ADM fields, we start with the equality

$$\varphi^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}) = \varphi^{ijkl}(R_{ijkl} - \rho_{ijkl}) + 4\varphi^{ijk0}(R_{ijk0} - \rho_{ijk0}) + 4\varphi^{i0j0}(R_{i0j0} - \rho_{i0j0}). \tag{17}$$

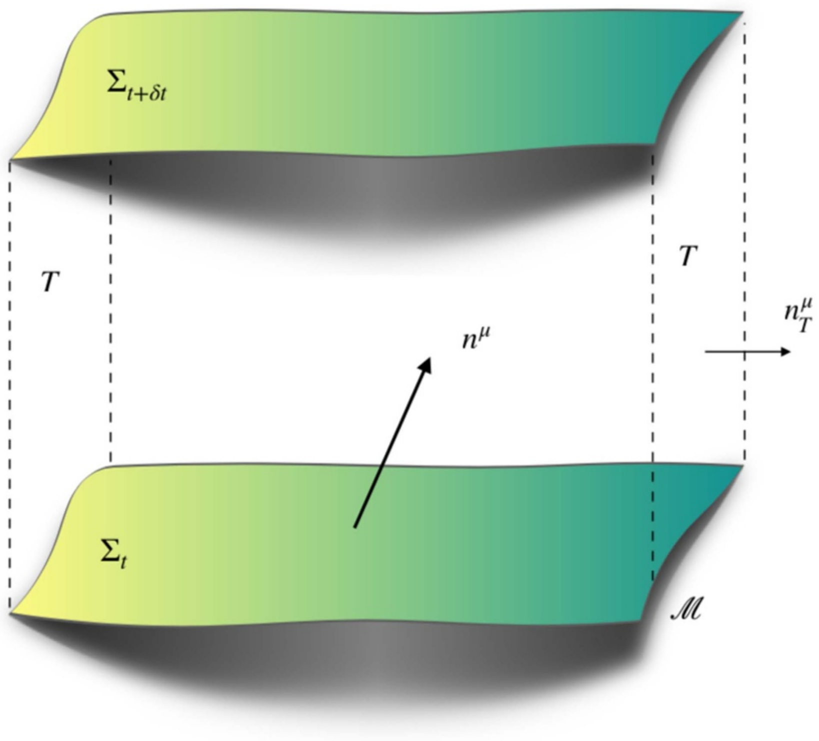

Figure 1. Slicing of the spacetime in terms of co-dimension one spatial hypersurface.

We introduce the ADM splitting of the corresponding components of the Riemann tensor in appendix A in detail. Using these results, one gets

$$\begin{aligned}\varphi^{\mu\nu\rho\sigma}(R_{\mu\nu\rho\sigma}-\rho_{\mu\nu\rho\sigma}) = \varphi^{ijkl}(R_{ijkl}-\rho_{ijkl}) + 4\varphi^{ijk0}\big(N(D_iK_{jk}-D_jK_{ik}) + N^lR_{ijkl}-\rho_{ijk0}\big) \\ + 4\varphi^{i0j0}\bigg(N^kN^l\,R_{ikjl}+N^2K_{ik}K^k_j+NN^k\,(D_iK_{jk}+D_jK_{ik}-2D_kK_{ij}) \\ + N(D_iD_jN-\dot{K}_{ij}+\mathcal{L}_{\mathcal{N}}K_{ij})-\rho_{i0j0}\bigg). \qquad (18)\end{aligned}$$

Here $\mathcal{L}_{\mathcal{N}}$ denotes the Lie derivative along the vector field N^i, which is defined as

$$\mathcal{L}_{\mathcal{N}}K_{ij} = N^kD_kK_{ij} + K_{ki}D_jN^k + K_{kj}D_iN^k. \qquad (19)$$

Following [3], let us introduce the once and twice hypersurface-projected spatial tensor fields, respectively as

$$^{\Sigma}\varphi^{ijk} := \gamma^{il}\gamma^{jm}\gamma^{kn}n^\mu\varphi_{lmn\mu} \equiv \varphi^{ijk}, \qquad (20)$$

$$^{\Sigma}\psi^{ij} := -2\gamma^{ik}\gamma^{jl}n^\mu n^\nu\varphi_{k\mu l\nu} \equiv \psi^{ij}, \qquad (21)$$

which can equivalently be written as

$$\varphi^{ijk} = -N\varphi^{ijk0}, \qquad (22)$$

$$\psi^{ij} = -2N^2 \varphi^{i0j0}. \tag{23}$$

Similarly, for the other auxiliary field, we introduce

$$^{\Sigma}\rho_{ijk} := n^{\mu} \rho_{ijk\mu} \equiv \rho_{ijk}, \tag{24}$$

$$^{\Sigma}\Omega_{ij} := n^{\mu} n^{\nu} \rho_{i\mu j\nu} \equiv \Omega_{ij}. \tag{25}$$

The last two expressions explicitly read

$$\rho_{ijk} = \frac{1}{N} \rho_{ijk0} - \frac{N^l}{N} \rho_{ijkl}, \tag{26}$$

$$\Omega_{ij} = \frac{1}{N^2} \rho_{i0j0} - \frac{N^k N^l}{N^2} \rho_{jkil} - \frac{N^k}{N} \left(\rho_{ikj} + \rho_{jki}\right), \tag{27}$$

where ρ_{ijkl} and φ^{ijkl} themselves are spatial tensor fields on the hypersurface by assumption: namely, one has $\rho_{ijkl} = {}^{\Sigma}\rho_{ijkl}$ and $\varphi^{ijkl} = {}^{\Sigma}\varphi^{ijkl}$, and for the sake of brevity, we dropped the index Σ. In other words, we can express the spacetime tensor components ρ_{ijk0} and ρ_{i0j0} in terms of purely spatial tensor fields:

$$\begin{aligned} &\rho_{ijk0} = N \rho_{ijk} + N^l \rho_{ijkl}, \\ &\rho_{i0j0} = N^2\, \Omega_{ij} + N^k N^l\, \rho_{ikjl} + N N^k \left(\rho_{ikj} + \rho_{jki}\right). \end{aligned} \tag{28}$$

So then, the contracted term (18) becomes[6]

$$\begin{aligned} \varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}\right) = {} & \varphi^{ijkl}\left(R_{ijkl} - \rho_{ijkl}\right) \\ & - 4\varphi^{ijk}\left(D_i K_{jk} - D_j K_{ik} - \rho_{ijk} + \frac{N^l}{N}\left(R_{ijkl} - \rho_{ijkl}\right)\right) \\ & - 2\psi^{ij}\left(\frac{N^k N^l}{N^2}\left(R_{ikjl} - \rho_{ikjl}\right) + \frac{1}{N}\left(D_i D_j N - \dot{K}_{ij} + \mathcal{L}_{\mathcal{N}} K_{ij}\right)\right. \\ & \left. + K_{ik} K^k_j - \Omega_{ij} + \frac{N^k}{N}\left(D_i K_{jk} + D_j K_{ik} - 2D_k K_{ij} - \rho_{ikj} - \rho_{jki}\right)\right). \end{aligned} \tag{29}$$

Inserting all of the variations to the generalized action (3), one arrives at

$$\begin{aligned} S = \int_{\mathcal{M}} d^n x\, N\sqrt{\gamma} \Bigg(& \frac{f(\rho, \Omega)}{2} + \frac{1}{2}\varphi^{ijkl}\left(R_{ijkl} - \rho_{ijkl}\right) \\ & - 2\varphi^{ijk}\left(D_i K_{jk} - D_j K_{ik} - \rho_{ijk} + \frac{N^l}{N}\left(R_{ijkl} - \rho_{ijkl}\right)\right) \\ & - \psi^{ij}\left(\frac{N^k N^l}{N^2}\left(R_{ikjl} - \rho_{ikjl}\right) + \frac{1}{N}\left(D_i D_j N - \dot{K}_{ij} + \mathcal{L}_{\mathcal{N}} K_{ij}\right)\right. \\ & \left. + K_{ik} K^k_j - \Omega_{ij} + \frac{N^k}{N}\left(D_i K_{jk} + D_j K_{ik} - 2D_k K_{ij} - \rho_{ikj} - \rho_{jki}\right)\right)\Bigg). \end{aligned} \tag{30}$$

Now let us eliminate the spatial tensors φ^{ijkl} and φ^{ijk}, which are nondynamical. The field equation for φ^{ijkl} is

$$\rho_{ijkl} = R_{ijkl} = {}^{\Sigma}R_{ijkl} + K_{ik} K_{jl} - K_{il} K_{jk}, \tag{31}$$

[6] In appendix D, we give more details of this computation.

while the field equation for φ^{ijk} is

$$D_iK_{jk} - D_jK_{ik} - \rho_{ijk} + \frac{N^l}{N}(R_{ijkl} - \rho_{ijkl}) = 0. \tag{32}$$

The last two terms inside the round brackets cancel each other due to the constraint and one arrives at the second constraint

$$\rho_{ijk} = D_iK_{jk} - D_jK_{ik}. \tag{33}$$

Using the constraints, the action (30) reduces to

$$S = \int_{\mathcal{M}} d^n x\, N\sqrt{\gamma}\left(\frac{f(\rho,\Omega)}{2} - \psi^{ij}\left(K_{ik}K^k_j - \Omega_{ij} + \frac{1}{N}D_iD_jN\right) + \frac{1}{N}\left(\psi^{ij}\dot{K}_{ij} - \psi^{ij}\mathcal{L}_{\mathcal{N}}K_{ij}\right)\right). \tag{34}$$

Using integration by parts, one can rewrite the last two terms, ignoring the boundary contributions, to arrive at the action $S = \int_{\mathcal{M}} d^n x\, \mathcal{L}$, where the Lagrangian density is

$$\mathcal{L} = N\sqrt{\gamma}\left(\frac{f(\rho,\Omega)}{2} - \psi^{ij}\left(KK_{ij} + K_{ik}K^k_j - \Omega_{ij} + \frac{1}{N}D_iD_jN\right) - \frac{1}{N}K_{ij}\left(\dot{\psi}^{ij} - \mathcal{L}_{\mathcal{N}}\psi^{ij}\right)\right). \tag{35}$$

Therefore, with the help of the auxiliary fields, we have managed to recast the higher derivative action as a lower derivative one as desired. It is now easier to find the Hamiltonian of the theory from this lower-derivative action.

3.1. Hamiltonian of the theory

The dynamical fields are $(\gamma_{ij},\ \psi^{ij})$, and hence one needs to introduce the two canonical momenta corresponding to these dynamical fields. These are

$$\Pi_{ij} := \frac{\delta\mathcal{L}}{\delta\partial_0\psi^{ij}}, \tag{36}$$

and

$$p^{ij} := \frac{\delta\mathcal{L}}{\delta\partial_0\gamma_{ij}} = \frac{\delta\mathcal{L}}{\delta K_{lm}}\frac{\delta K_{lm}}{\delta\dot{\gamma}_{ij}}. \tag{37}$$

After a straightforward computation, one ends up with

$$\Pi_{ij} = -\sqrt{\gamma}K_{ij}. \tag{38}$$

The constraints (31), (33) can now be recast in terms of the canonical momenta as

$$\rho_{ijk} = D_j\left(\frac{\Pi_{ik}}{\sqrt{\gamma}}\right) - D_i\left(\frac{\Pi_{jk}}{\sqrt{\gamma}}\right), \tag{39}$$

$$\rho_{ijkl} = R_{ijkl} = {}^{\Sigma}R_{ijkl} + \frac{1}{\gamma}\left(\Pi_{ik}\Pi_{jl} - \Pi_{il}\Pi_{jk}\right). \tag{40}$$

Similarly one arrives at

$$p^{ij} = \frac{\sqrt{\gamma}}{4}\frac{\delta f}{\delta K_{ij}} + \frac{\sqrt{\gamma}}{2}\left(\frac{1}{N}(\mathcal{L}_{\mathcal{N}}\psi^{ij} - \dot{\psi}^{ij}) - \gamma^{ij}\psi^{kl}K_{kl} - K\psi^{ij} - \psi^{ik}K^j_k - \psi^{jk}K^i_k\right). \tag{41}$$

Then the Hamiltonian density

$$\mathcal{H} = p^{ij}\dot{\gamma}_{ij} + \pi_{ij}\dot{\psi}^{ij} - \mathcal{L}, \tag{42}$$

becomes

$$\begin{aligned}\mathcal{H} = {} & 2NK_{ij}p^{ij} - \sqrt{\gamma}K_{ij}\mathcal{L}_{\mathcal{N}}\psi^{ij} + 2p^{ij}D_iN_j \\ & + \sqrt{\gamma}N\left(-\frac{f}{2} + \frac{1}{N}\psi^{ij}D_iD_jN - \psi^{ij}\Omega_{ij} + K\psi^{kl}K_{kl} + K_{ij}\psi^{jk}K^i_k\right).\end{aligned} \tag{43}$$

3.2. Constraint equations

Up to a boundary contribution, one can express the Hamiltonian density as a sum of constraint equations:

$$\mathcal{H} = N\Phi_0 + N^i\Phi_i. \tag{44}$$

Here Φ_0 denotes the Hamiltonian constraint and Φ_i denotes the momentum constraints. Using (43), one can rewrite the Hamiltonian density as

$$\begin{aligned}\mathcal{H} = {} & N\sqrt{\gamma}\left(\frac{2}{\sqrt{\gamma}}K_{ij}p^{ij} - \frac{f}{2} + D_iD_j\psi^{ij} - \psi^{ij}\Omega_{ij} + KK_{ij}\psi^{ij} + K_{ij}K^i_k\psi^{jk}\right) \\ & + N^i\sqrt{\gamma}\left(-2D_k\left(\frac{p^k_i}{\sqrt{\gamma}}\right) - K_{kl}D_i\psi^{kl} - 2D_k(\psi^{kl}K_{li})\right).\end{aligned} \tag{45}$$

Equivalently, in terms of the canonical momenta one obtains

$$\begin{aligned}\mathcal{H} = {} & N\sqrt{\gamma}\left(-\frac{f}{2} + D_iD_j\psi^{ij} - \psi^{ij}\Omega_{ij}\right) + \frac{N}{\sqrt{\gamma}}\left(-2\Pi_{ij}p^{ij} + \Pi\,\Pi_{ij}\psi^{ij} + \Pi_{ij}\Pi^i_k\psi^{jk}\right) \\ & + N^i\left(-2\sqrt{\gamma}D_k\left(\frac{p^k_i}{\sqrt{\gamma}}\right) + \Pi_{kl}D_i\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{li}}{\sqrt{\gamma}}\right)\right).\end{aligned} \tag{46}$$

Therefore the constraints are

$$\Phi_0 = \sqrt{\gamma}\left(\frac{2}{\sqrt{\gamma}}K_{ij}p^{ij} - \frac{f}{2} + D_iD_j\psi^{ij} - \psi^{ij}\Omega_{ij} + KK_{ij}\psi^{ij} + K_{ij}K^i_k\psi^{jk}\right), \tag{47}$$

$$\Phi_i = \sqrt{\gamma}\left(-2D_k\left(\frac{p^k_i}{\sqrt{\gamma}}\right) - K_{kl}D_i\psi^{kl} - 2D_k(\psi^{kl}K_{li})\right). \tag{48}$$

The constraints vanish in a vacuum; but if there is a non-zero energy momentum tensor, then they must be equal to the corresponding projection of the energy-momentum tensor onto the initial hypersurface:

$$\Phi_0 = 2T_{nn} = \frac{2}{N^2}\left(2N^iT_{0i} - T_{00} - N^iN^jT_{ij}\right), \tag{49}$$

$$\Phi_i = 2T_{ni} = \frac{2}{N}\left(N^j T_{ij} - T_{0i}\right). \tag{50}$$

In addition, the field equations of Ω_{ij} also are constraints: $\frac{\delta\mathcal{H}}{\delta\Omega_{ij}} = 0$. Finally, let us write all the constraints[7].

- The Hamiltonian constraint:

$$\Phi_0 = \sqrt{\gamma}\left(-\frac{f}{2} + D_i D_j \psi^{ij} - \Omega_{ij}\psi^{ij}\right) + \frac{1}{\sqrt{\gamma}}\left(-2\Pi_{ij}p^{ij} + \Pi\,\Pi_{ij}\psi^{ij} + \Pi_{ij}\Pi^i_k\psi^{jk}\right) = 0. \tag{51}$$

- The momentum constraint:

$$\Phi_i = -2\sqrt{\gamma}D_k\left(\frac{p^k_i}{\sqrt{\gamma}}\right) + \Pi_{kl}D_i\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{li}}{\sqrt{\gamma}}\right) = 0. \tag{52}$$

- The additional constraint of the auxiliary field:

$$2\psi^{ij} + \frac{\delta f}{\delta\Omega_{ij}} = 0. \tag{53}$$

3.3. Time evolution equations

3.3.1. The first set: $\dot{\gamma}_{ij}$, $\dot{\psi}^{ij}$. From now on we are going to construct the time evolution equations[8]. The phase space variables are (γ_{ij}, ψ^{ij}, p^{ij}, Π_{ij}). The canonical coordinates evolve via

$$\dot{\gamma}_{ij} = \frac{\delta\mathcal{H}}{\delta p^{ij}}, \qquad \dot{\psi}^{ij} = \frac{\delta\mathcal{H}}{\delta\Pi_{ij}}. \tag{54}$$

The definition of extrinsic curvature leads to

$$\dot{\gamma}_{ij} = 2NK_{ij} + D_i N_j + D_j N_i. \tag{55}$$

The relation (38) additionally yields

$$\dot{\gamma}_{ij} = -\frac{2N}{\sqrt{\gamma}}\Pi_{ij} + \mathcal{L}_N\gamma_{ij}, \tag{56}$$

and so we can write

[7] Note that the reader can study appendix E for more construction details.
[8] Since the computation is rather long, we delegate some details of this section to appendix F.

$$\dot{\gamma}_{ij} = 2NK_{ij} + D_i N_j + D_j N_i = -\frac{2N}{\sqrt{\gamma}}\Pi_{ij} + \mathcal{L}_N \gamma_{ij}. \tag{57}$$

On the other hand, one has

$$\dot{\psi}^{ij} = \frac{N}{\sqrt{\gamma}}\left(-2p^{ij} + \gamma^{ij}\Pi_{kl}\psi^{kl} + \Pi\psi^{ij} + \Pi^i_k\psi^{jk} + \Pi^j_k\psi^{ik}\right) + \mathcal{L}_{\mathcal{N}}\psi^{ij} - \frac{N\sqrt{\gamma}}{2}\frac{\delta f}{\delta \Pi_{ij}}. \tag{58}$$

3.3.2. The second set: $\dot{p}^{ij}$, $\dot{\Pi}_{ij}$. Next, we find the time-evolution equations for the canonical momenta:

$$\dot{p}^{ij} = -\frac{\delta \mathcal{H}}{\delta \gamma_{ij}}, \qquad \dot{\Pi}_{ij} = -\frac{\delta \mathcal{H}}{\delta \psi^{ij}}. \tag{59}$$

The second one is easier to obtain since we only focus on the variations with respect to ψ^{ij}. Using the Hamiltonian (46), we have

$$\begin{aligned}\dot{\Pi}_{ij} = &-N\sqrt{\gamma}\frac{\delta}{\delta\psi^{ij}}\left(D_k D_l \psi^{kl} - \psi^{kl}\Omega_{kl}\right) - \frac{N}{\sqrt{\gamma}}\frac{\delta}{\delta\psi^{ij}}\left(\Pi\,\Pi_{kl}\psi^{kl} + \Pi_{mn}\Pi^n_k\psi^{mk}\right)\\ &- N^m\frac{\delta}{\delta\psi^{ij}}\left(\Pi_{kl}D_m\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{lm}}{\sqrt{\gamma}}\right)\right),\end{aligned} \tag{60}$$

where

$$\frac{\delta\psi^{kl}}{\delta\psi^{ij}} = \frac{1}{2}\left(\delta^k_i\delta^l_j + \delta^l_i\delta^k_j\right). \tag{61}$$

After ignoring the total derivative terms, we get

$$N\frac{\delta}{\delta\psi^{ij}}D_k D_l\psi^{kl} = D_i D_j N, \tag{62}$$

and

$$N^m\Pi_{kl}\frac{\delta}{\delta\psi^{ij}}D_m\psi^{kl} = -N^m D_m\Pi_{ij} - \Pi_{ij}D_m N^m, \tag{63}$$

and also

$$N^m\frac{\delta}{\delta\psi^{ij}}D_k\left(\psi^{kl}\frac{\Pi_{lm}}{\sqrt{\gamma}}\right) = -\frac{1}{\sqrt{\gamma}}\Pi_{m(i}D_{j)}N^m. \tag{64}$$

Substituting all of these pieces, we end up with

$$\dot{\Pi}_{ij} = \sqrt{\gamma}\left(N\Omega_{ij} - D_i D_j N\right) - \frac{N}{\sqrt{\gamma}}\left(\Pi\,\Pi_{ij} + \Pi_{ik}\Pi^k_j\right) + \sqrt{\gamma}\mathcal{L}_{\mathcal{N}}\left(\frac{\Pi_{ij}}{\sqrt{\gamma}}\right) + \Pi_{ij}D_k N^k. \tag{65}$$

Similarly we can find $\dot{p}^{ij}$. Clearly, we can express

$$\begin{aligned}\dot{p}^{ij} = &\frac{N}{\sqrt{\gamma}}\left(\gamma^{ij}\left(-2\Pi_{kl}p^{kl} + \Pi\,\Pi_{kl}\psi^{kl} + \Pi_{km}\Pi^m_n\psi^{nk}\right) + \Pi^{ij}\Pi_{kl}\psi^{kl} + \Pi^i_l\Pi^j_k\psi^{kl}\right)\\ &- N\sqrt{\gamma}\frac{\delta}{\delta\gamma_{ij}}D_k D_l\psi^{kl} + \frac{N}{2}\sqrt{\gamma}\frac{\delta f}{\delta\gamma_{ij}}\\ &- N^m\frac{\delta}{\delta\gamma_{ij}}\left(-2\sqrt{\gamma}\gamma_{mn}D_k\left(\frac{p^{kn}}{\sqrt{\gamma}}\right) + \Pi_{kl}D_m\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{lm}}{\sqrt{\gamma}}\right)\right),\end{aligned} \tag{66}$$

where the last two terms cancel each other because of the variation with respect to the spatial metric. Taking into account the covariant derivatives correctly, we have[9]

$$\begin{aligned}\dot{p}^{ij} = &\frac{N}{\sqrt{\gamma}}\left(\gamma^{ij}\left(\Pi\,\Pi_{kl}\psi^{kl} + \Pi_{km}\Pi^m_n\psi^{nk} - 2\Pi_{kl}p^{kl}\right) + \Pi^{ij}\Pi_{kl}\psi^{kl} + \Pi^i_l\Pi^j_k\psi^{kl}\right)\\ &+ \frac{\sqrt{\gamma}}{2}\left(D_k\left(\psi^{ij}D^k N - 2\psi^{k(i}D^{j)}N\right) + \gamma^{ij}\left(ND_kD_l\psi^{kl} - \psi^{kl}D_kD_lN\right)\right)\\ &+ \sqrt{\gamma}\mathcal{L}_N\left(\frac{p^{ij}}{\sqrt{\gamma}}\right) + \frac{N}{2}\sqrt{\gamma}\frac{\delta f}{\delta\gamma_{ij}} + p^{ij}D_kN^k.\end{aligned} \tag{67}$$

In appendix G, we gave the construction of the constraint and time evolution equations of general relativity using the results we have obtained in this and in the previous sections.

As explained in detail in [7], the Hamiltonian form of the Einstein–Hilbert action, when extremized, leads to the Fischer–Marsden form [4] of the field equations. For the generic $f(Riemann)$ theory studied here, one can also recast the Hamiltonian flow in a concise form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\gamma\\ \psi\\ p\\ \Pi\end{pmatrix} = J\circ\mathbf{D\Phi}^*(\gamma,\psi,\pi,\Pi)(\mathcal{N}), \qquad J := \begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix}, \tag{68}$$

where the small circle represents the usual matrix product.
In this matrix equation, $\mathbf{D\Phi}^*(\gamma,\psi,\pi,\Pi)$ is the formal adjoint of the linearized constraint map $(D\Phi(\gamma,\psi,\pi,\Pi))$. Why the adjoint map appears in the Hamiltonian flow can be understood from the discussion in [7]. Here $\mathcal{N}$ is the lapse-shift vector with components (N,N^i). Observe

[9] Note that in both of these equations (65), (67), all the terms except the last term in each one are the same as those of [3]. Those two terms are missing in that work.

that there is no time-evolution when $\mathbf{D\Phi}^*(\gamma,\psi,\pi,\Pi)(\mathcal{N})=0$, and these points in the space of initial data yield Killing vectors in spacetime [13, 14]. Such a description of Killing symmetries is extremely useful in understanding the amount of non-stationary energy contained in a given initial data. Here is how: if $\mathcal{N}=\xi$ is a Killing vector, say a stationary Killing vector, then the time evolution is trivial. The failure of $\mathcal{N}$ to be a Killing vector field is given as

$$\mathbf{D\Phi}^*(\gamma,\psi,\pi,\Pi)(\mathcal{N})=J^{-1}\circ\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\gamma\\ \psi\\ p\\ \Pi\end{pmatrix}. \tag{69}$$

Next, we discuss the non-stationary energy in this generic theory based on the approximate Killing initial data.

4. Non-stationary energy in *f*(Riemann) Theories

Dain [5] introduced the concept of non-stationary energy for the time-symmetric initial data in general relativity for vacuum asymptotically flat spacetimes. That definition is based on the notion of approximate KID, which is to be defined below. Dain's invariant was extended to the time-asymmetric case in [6], and for asymptotically non-flat spacetimes in [7], where another definition based on the time-evolution equations was given. In [9] the construction was extended to non-vacuum spacetimes.

Let us briefly recap Dain's construction as it is not widely known and involves several subtle steps. Let the constraint covector be $\mathbf{\Phi}:=(\Phi_0,\Phi_i)$ and $\mathbf{D\Phi}$ be its linearization about a given solution initial solution. Then, $\mathbf{D\Phi}^*$ is the formal adjoint of the linearized constraint map that acts on the lapse and shift vector. A crucial tool in the construction of Dain's invariant is Bartnik's operator $\mathcal{P}$ defined as [15]

$$\mathcal{P}:=\mathbf{D\Phi}\circ\begin{pmatrix}1 & 0\\ 0 & -D^m\end{pmatrix}, \tag{70}$$

of which the formal adjoint is

$$\mathcal{P}^*(\mathcal{N}):=\begin{pmatrix}1 & 0\\ 0 & D_m\end{pmatrix}\circ\mathbf{D\Phi}^*(\mathcal{N}). \tag{71}$$

If one uses the densitized versions of the constraints, one must also rescale the Bartnik's operator as

$$\widetilde{\mathcal{P}}^*(\mathcal{N}):=\begin{pmatrix}\gamma^{-1/2} & 0\\ 0 & 1\end{pmatrix}\circ\mathcal{P}^*(\mathcal{N}). \tag{72}$$

Finally, we can write the Dain's invariant, $\mathscr{I}(\xi)$, that quantifies the amount of non-stationary energy in the initial data that solves the constraint equations:

$$\mathscr{I}(\xi):=\int_\Sigma \mathrm{d}V\,\mathcal{P}^*(\xi)\cdot\mathcal{P}^*(\xi), \tag{73}$$

where $\xi := (N, N^i)$, and $P^*(\xi) := P^* \begin{pmatrix} N \\ N^k \end{pmatrix}$. More explicitly, in (73) one has

$$\begin{pmatrix} N \\ N^i \end{pmatrix} \cdot \begin{pmatrix} A \\ B_i \end{pmatrix} := NA + N^i B_i. \tag{74}$$

The important step here is the following: in the integral (73), one considers only the lapse and shift functions that satisfy a fourth-order partial differential equation that arises in the integration by parts as

$$\mathcal{P} \circ \mathcal{P}^*(\xi) = 0. \tag{75}$$

This is called (by Dain) the 'approximate KID equation', which admit all the Killing initial data as solutions, but has more solutions than the Killing initial data.

Our formulation [7, 9] of Dain's invariant directly involves the time evolution equations since one can write the formal-adjoint of Bartnik's operator as

$$\mathcal{P}^*(\mathcal{N}) := \begin{pmatrix} 1 & 0 \\ 0 & D_m \end{pmatrix} \circ \mathbf{D\Phi}^*(\mathcal{N}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & D_m & 0 \\ 0 & 0 & 0 & D_m \end{pmatrix} \circ J^{-1} \circ \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} \gamma \\ \psi \\ p \\ \Pi \end{pmatrix}. \tag{76}$$

Then, Dain's invariant for generic f(Riemann) theories in the time evolution formulation reads as

$$\mathscr{I}(\mathcal{N}) = \int_\Sigma \mathrm{d}V \left(|D_m \dot{\gamma}_{ij}|^2 + |D_m \dot{\psi}^{ij}|^2 + \frac{1}{\gamma} \left(|\dot{p}^{ij}|^2 + |\dot{\Pi}_{ij}|^2 \right) \right), \tag{77}$$

where the time derivatives of the phase space fields appear. One must also be careful with the notation as one has $|D_m \dot{\psi}^{ij}|^2 \equiv \gamma_{ik}\gamma_{jl} D_m \dot{\psi}^{ij} D^m \dot{\psi}^{kl}$.

Let us remark on a possible use of the results of this section. Given initial data that solves the constraints, one can identify what fraction of that data will turn into gravitational waves using the expression (77). As a fully deterministic theory, this is what one expects in gravity. Unfortunately, it is generically hard to find analytical solutions to the constraints. Therefore, one needs to compute (77) for a numerical solution. Even in the simplest case, provided by Dain [5] for asymptotically flat time-symmetric initial data in Einstein's theory, a numerical evaluation of the related integral that gives the non-stationary energy has not been carried out. It is an outstanding problem[10].

5. Applications of the formalism

5.1. The R^2 theory

From now on, we shall adapt our results to the R^2 theory. We consider the following function

$$f(\rho_{\mu\nu\rho\sigma}) = \rho^2 = g^{\mu\rho} g^{\nu\sigma} g^{\alpha\beta} g^{\gamma\kappa} \rho_{\mu\nu\rho\sigma} \rho_{\alpha\gamma\beta\kappa} \tag{78}$$

[10] One reason this approach to the gravitational wave content of initial data has not received much attention could be the fact that Sergio Dain passed away at the age of 46 before he was able to expound upon his ideas on the topic [16].

to represent the R^2 theory, where $\rho = g^{\mu\rho}g^{\nu\sigma}\rho_{\mu\nu\rho\sigma}$ and so

$$\begin{aligned}\rho &= g^{\mu\rho}g^{\nu\sigma}\rho_{\mu\nu\rho\sigma}\\ &= g^{\nu\sigma}\left(g^{0\rho}\rho_{0\nu\rho\sigma}+g^{i\rho}\rho_{i\nu\rho\sigma}\right)\\ &= g^{\nu\sigma}\left(g^{00}\rho_{0\nu0\sigma}+g^{0i}\rho_{0\nu i\sigma}+g^{i0}\rho_{i\nu0\sigma}+g^{ij}\rho_{i\nu j\sigma}\right),\end{aligned} \tag{79}$$

which yields

$$\rho = g^{0\sigma}\left(g^{i0}\rho_{i00\sigma}+g^{ij}\rho_{i0j\sigma}\right)+g^{k\sigma}\left(g^{00}\rho_{0k0\sigma}+g^{0i}\rho_{0ki\sigma}+g^{i0}\rho_{ik0\sigma}+g^{ij}\rho_{ikj\sigma}\right), \tag{80}$$

and then

$$\begin{aligned}\rho =& g^{00}g^{ij}\rho_{i0j0}+g^{0k}\left(g^{i0}\rho_{i00k}+g^{ij}\rho_{i0jk}\right)+g^{k0}\left(g^{0i}\rho_{0ki0}+g^{ij}\rho_{ikj0}\right)\\ &+g^{kl}\left(g^{00}\rho_{0k0l}+g^{0i}\rho_{0kil}+g^{i0}\rho_{ik0l}+g^{ij}\rho_{ikjl}\right).\end{aligned} \tag{81}$$

The auxiliary field $\rho_{\mu\nu\rho\sigma}$ has the all symmetries of the Riemann tensor. Therefore, by renaming the indices we get

$$\rho = g^{ij}g^{kl}\rho_{ikjl}+2\rho_{i0j0}\left(g^{00}g^{ij}-g^{0i}g^{0j}\right)+4g^{ik}g^{0j}\rho_{0ijk}. \tag{82}$$

Inserting the corresponding components of the inverse spacetime metric, we arrive at

$$\rho = \gamma^{ij}\gamma^{kl}\rho_{ikjl}-2\gamma^{ij}\frac{1}{N^2}\left(N^kN^l\rho_{ikjl}+\rho_{i0j0}-N^k(\rho_{0ikj}+\rho_{0jki})\right), \tag{83}$$

where we have already introduced the hypersurface projected field $\Omega_{ij} = n^\mu n^\nu \rho_{i\mu j\nu}$ with the future pointing unit normal vector $n^\mu = \left(1/N_1 - N^i/N\right)$. Hence we get

$$\rho = \gamma^{ij}\gamma^{kl}\rho_{ikjl}-2\gamma^{ij}\Omega_{ij} = {}^{\Sigma}\rho - 2\Omega,$$

where we have used $\rho_{ikjl} = {}^{\Sigma}\rho_{ikjl}$ and $\Omega_{ij} = {}^{\Sigma}\Omega_{ij}$. Then

$$f = 4\gamma^{ij}\gamma^{kl}\Omega_{ij}\Omega_{kl}-4\gamma^{ij}\Omega_{ij}\gamma^{kl}\gamma^{mn}\,{}^{\Sigma}\rho_{kmln}+\gamma^{kl}\gamma^{mn}\,{}^{\Sigma}\rho_{kmln}\gamma^{ps}\gamma^{ij}\,{}^{\Sigma}\rho_{pisj}. \tag{84}$$

Here we have used $\gamma^{ik}\gamma^{jl}\rho_{ijkl} = {}^{\Sigma}\boldsymbol{\rho}$, to make the difference clear between the trace with the spacetime metric, $\rho = g^{\mu\rho}g^{\nu\sigma}\rho_{\mu\nu\rho\sigma}$. Then, we write

$$f(\rho_{\mu\nu\rho\sigma}) = \rho^2 = \left({}^{\Sigma}\boldsymbol{\rho}-2\Omega\right)^2. \tag{85}$$

To construct the primary constraint, $\partial f/\partial\Omega_{ij} = -2\psi^{ij}$, we need to calculate $\partial f/\partial\Omega_{ij}$. It is easy to prove that

$$\frac{\partial f}{\partial\Omega_{ij}} = 8\Omega\gamma^{ij}-4\gamma^{ij}\,{}^{\Sigma}\rho. \tag{86}$$

Therefore the primary constraint of the auxiliary field is

$$\psi^{ij} = \gamma^{ij}\left(2\,{}^{\Sigma}\rho-4\Omega\right). \tag{87}$$

Recall that in general relativity one has $\psi^{ij} = \gamma^{ij}$, and now we have $\gamma, {}^{\Sigma}\rho, \Omega$ dependence in the hypersurface field ψ^{ij}. The Hamiltonian constraint (51) reduces to

$$\Phi_0 = 2\sqrt{\gamma}\left(D_k D^k\, {}^{\Sigma}\boldsymbol{\rho} - 2D_k D^k\Omega + \Omega^2 - \frac{1}{4}{}^{\Sigma}\boldsymbol{\rho}^2\right) + \frac{2}{\sqrt{\gamma}}\left(\left(\Pi_{ij}^2 + \Pi^2\right)\left({}^{\Sigma}\boldsymbol{\rho} - 2\Omega\right) - p^{kl}\Pi_{kl}\right), \tag{88}$$

and the momentum constraint (52) becomes

$$\Phi_i = -2\sqrt{\gamma}D_k\left(\frac{p_i^k}{\sqrt{\gamma}}\right) + 2\Pi D_i\left({}^{\Sigma}\boldsymbol{\rho} - 2\Omega\right) + 4\sqrt{\gamma}D_k\left(\frac{\Pi_i^k}{\sqrt{\gamma}}\left({}^{\Sigma}\boldsymbol{\rho} - 2\Omega\right)\right). \tag{89}$$

5.2. *The $R_{\mu\nu}R^{\mu\nu}$ theory*

In this section, we are going to evaluate the $R_{\mu\nu}R^{\mu\nu}$ theory as an example. To be able to do this, first, we have to compute the space and time decomposition of the contraction $\rho_{\mu\nu}\rho^{\mu\nu}$. Clearly, we have

$$f(\rho_{\mu\nu\rho\sigma}) = \rho_{\mu\nu}\rho^{\mu\nu} = \rho_{00}\rho^{00} + \rho_{0i}\rho^{0i} + \rho_{i0}\rho^{i0} + \rho_{ij}\rho^{ij}, \tag{90}$$

and

$$f(\rho_{\mu\nu\rho\sigma}) = \rho_{00}\rho^{00} + 2\rho_{0i}\rho^{0i} + \rho_{ij}\rho^{ij}. \tag{91}$$

Now we should decompose the corresponding components into the ADM variables. We start with $\rho_{\mu\nu}$, which can be obtained as follows

$$\begin{aligned}\rho_{\mu\nu} &= g^{\alpha\beta}\rho_{\alpha\mu\beta\nu}\\ &= g^{0\beta}\rho_{0\mu\beta\nu} + g^{i\beta}\rho_{i\mu\beta\nu}\\ &= g^{00}\rho_{0\mu0\nu} + g^{0i}\rho_{0\mu i\nu} + g^{i0}\rho_{i\mu0\nu} + g^{ij}\rho_{i\mu j\nu}.\end{aligned} \tag{92}$$

Using the symmetries of $\rho_{\mu\nu g\sigma,}$ one has

$$\rho_{\mu\nu} = g^{00}\rho_{0\mu0\nu} + g^{0i}\left(\rho_{\nu i\mu0} + \rho_{\mu i\nu0}\right) + g^{ij}\rho_{i\mu j\nu}, \tag{93}$$

which yields

$$\rho_{00} = g^{ij}\rho_{i0j0}. \tag{94}$$

Recall that ρ_{0000} and ρ_{0i00} automatically vanish because of the symmetries. Similarly $\rho_{0\mathrm{i}}$ reads

$$\rho_{0i} = -g^{0k}\rho_{i0k0} + g^{kl}\rho_{kil0}. \tag{95}$$

Moreover, the spatial component ρ_{ij} can be written as

$$\rho_{ij} = g^{00}\rho_{0i0j} + g^{0k}\left(\rho_{ikj0} + \rho_{jki0}\right) + g^{kl}\rho_{kilj}. \tag{96}$$

Recall that, we have already introduced the hypersurface projected tensor fields ρ_{ijk} and Ω_{ij} via (28) and the inverse metric components. Let us reexpress ρ_{00}. Using (94) we can write

$$\rho_{00} = \left(\gamma^{mn} - \frac{N^m N^n}{N^2}\right)\left(N^2\Omega_{mn} + N^k N^l \rho_{mknl} + NN^k\left(\rho_{mkn} + \rho_{nkm}\right)\right), \tag{97}$$

which yields

$$\begin{aligned}\rho_{00} =& N^2\gamma^{mn}\Omega_{mn} + N^k N^l \gamma^{mn}\rho_{mknl} + 2NN^k\gamma^{mn}\rho_{mkn} - N^m N^n \Omega_{mn} \\ & - \frac{N^m N^n}{N^2} N^k N^l \rho_{mknl} - \frac{N^m N^n}{N} N^k \rho_{mkn},\end{aligned} \tag{98}$$

where the last two terms vanish because of the symmetries. For simplicity, we introduce

$$\Omega \equiv \gamma^{mn}\Omega_{mn}, {}^{\Sigma}\rho_{kl} \equiv \gamma^{mn}\rho_{mknl}, {}^{\Sigma}\rho_{k} \equiv \gamma^{mn}\rho_{mkn},$$

where ρ_{mknl} and ρ_{mkn} are purely spatial by assumption and therefore we removed the over Σ on these fields. Then, ρ_{00} reduces to

$$\rho_{00} = N^2\Omega + N^k N^l\left({}^{\Sigma}\rho_{kl} - \Omega_{kl}\right) + 2NN^{k\,\Sigma}\rho_k. \tag{99}$$

Now let us compute $\rho_{0\mathrm{i}}$. One has (95), which yields

$$\begin{aligned}\rho_{0i} =& -\frac{N^m}{N^2}\left(N^2\Omega_{mi} + N^k N^l \rho_{mkil} + NN^k\left(\rho_{mki} + \rho_{ikm}\right)\right) \\ & + \left(\gamma^{mn} - \frac{N^m N^n}{N^2}\right)\left(N\rho_{nim} + N^l \rho_{niml}\right),\end{aligned} \tag{100}$$

and

$$\begin{aligned}\rho_{0i} =& -N^m\Omega_{mi} - \frac{N^m N^k N^l}{N^2}\rho_{mkil} - \frac{N^m N^k}{N}\left(\rho_{mki} + \rho_{ikm}\right) \\ & + N\rho_{ni}{}^{n} + N^l \rho_{ni}{}^{n}{}_{l} - \frac{N^m N^n \rho_{nim}}{N} N_n - \frac{N^m N^n N^l \rho_{niml}}{N^2}.\end{aligned} \tag{101}$$

Then it reduces to

$$\rho_{0i} = -N^m\Omega_{mi} - \frac{N^m N^k \rho_{ikm}}{N^2} + N\,{}^{\Sigma}\rho_i + N^{m\Sigma}\rho_{im} - \frac{N^m N^k \rho_{kim}}{N}, \tag{102}$$

where $\rho_{ikm} = -\rho_{\mathrm{kim}}$. Then, one ends up with

$$\rho_{0i} = N\,{}^{\Sigma}\rho_i + N^k\left({}^{\Sigma}\rho_{ik} - \Omega_{ik}\right). \tag{103}$$

Similarly we can compute ρ_{ij}. Using (96), one has

$$\begin{aligned}\rho_{ij} =& -\frac{1}{N^2}\left(N^2\Omega_{ij} + N^k N^l \rho_{ikjl} + NN^k\left(\rho_{ikj} + \rho_{jki}\right)\right) \\ & + \frac{N^m}{N^2}\left(N\left(\rho_{jmi} + \rho_{imj}\right) + N^l\left(\rho_{jmil} + \rho_{imjl}\right)\right) + \left(\gamma^{mn} - \frac{N^m N^n}{N^2}\right)\rho_{minj}.\end{aligned} \tag{104}$$

Then, we obtain

$$\rho_{ij} = -\Omega_{ij} - \frac{N^k N^l}{N^2}\rho_{ikjl} - \frac{N^k}{N}\left(\rho_{ikj} + \rho_{jki}\right) + \frac{N^m}{N}\left(\rho_{jmi} + \rho_{imj}\right) + \frac{N^k N^l}{N^2}\left(\rho_{jkil} + \rho_{ikjl}\right) + \rho_{mi}{}^{m}{}_{j} - \frac{N^k N^l}{N^2}\rho_{kilj}, \tag{105}$$

which reduces to following compact form

$$\rho_{ij} = {}^{\Sigma}\rho_{ij} - \Omega_{ij}. \tag{106}$$

To compute the contraction $\rho_{\mu\nu}\rho^{\mu\nu}$, we have to compute the higher indices versions of the components. Let us start with ρ^{00} . Clearly one has

$$\rho^{00} = g^{\mu 0} g^{0\nu}\rho_{\mu\nu} = g^{00}g^{00}\rho_{00} + 2g^{m0}g^{00}\rho_{m0} + g^{n0}g^{m0}\rho_{nm}. \tag{107}$$

Inserting the inverse spacetime metric components, we have

$$\rho^{00} = \frac{1}{N^4}\rho_{00} - \frac{2N^m}{N^4}\rho_{0m} + \frac{N^n N^m \rho_{nm}}{N^4}, \tag{108}$$

and making use of (99), (103), (106) one arrives at

$$\rho^{00} = \frac{1}{N^4}\Big(N^2\Omega + N^k N^l\left({}^{\Sigma}\rho_{kl} - \Omega_{kl}\right) + 2NN^k\,{}^{\Sigma}\rho_k - 2NN^m\,{}^{\Sigma}\rho_m - 2N^m N^k\left({}^{\Sigma}\rho_{km} - \Omega_{km}\right) + N^m N^n\left({}^{\Sigma}\rho_{nm} - \Omega_{mn}\right)\Big), \tag{109}$$

where all the terms, except the first one on the right-hand side of the last equation cancels each other. Therefore we arrive at a simple result

$$\rho^{00} = \frac{\Omega}{N^2}. \tag{110}$$

Now we can compute the first piece in (91), that is $\rho_{00}\rho^{00}$. One has the following

$$\rho_{00}\rho^{00} = \Omega^2 + \frac{N^k N^l}{N}\Omega\left({}^{\Sigma}\rho_{kl} - \Omega_{kl}\right) + 2\frac{N^k}{N}\Omega\,{}^{\Sigma}\rho_k. \tag{111}$$

Similarly, we can evaluate ρ^{0i}:

$$\rho^{0i} = g^{\mu 0}g^{\nu i}\rho_{\mu\nu} = g^{00}g^{0i}\rho_{00} + g^{m0}g^{0i}\rho_{m0} + g^{00}g^{mi}\rho_{0m} + g^{n0}g^{mi}\rho_{nm}, \tag{112}$$

and we can write

$$\rho^{0i} = g^{00}g^{0i}\rho_{00} + \rho_{m0}\left(g^{00}g^{mi} + g^{m0}g^{0i}\right) + g^{n0}g^{mi}\rho_{nm}. \tag{113}$$

More explicitly, one obtains

$$\rho^{0i} = g^{00}g^{0i}N^2\Omega + \left({}^{\Sigma}\rho_{kl} - \Omega_{kl}\right)\left(N^k N^l g^{00}g^{0i} + g^{k0}g^{li} + N^k\left(g^{00}g^{li} + g^{l0}g^{0i}\right)\right) + {}^{\Sigma}\rho_k\left(2NN^k g^{00}g^{0i} + N\left(g^{00}g^{ki} + g^{k0}g^{0i}\right)\right). \tag{114}$$

Inserting the inverse metric components, one ends up with

$$\rho^{0i} = -\frac{1}{N}\left(N^i\,\Omega + {}^{\Sigma}\rho^i\right). \tag{115}$$

Then, the second piece in (91) becomes

$$\begin{aligned}\rho_{0i}\rho^{0i} = &-{}^{\Sigma}\rho_i\,{}^{\Sigma}\rho^i - \frac{1}{N}N^i\,{}^{\Sigma}\rho_i\,\Omega - \frac{N^iN^k}{N^2}\Omega\left({}^{\Sigma}\rho_{ik} - \Omega_{ik}\right)\\ &-\frac{N^k}{N}\,{}^{\Sigma}\rho^i\left({}^{\Sigma}\rho_{ik} - \Omega_{ik}\right).\end{aligned} \tag{116}$$

Note that $\rho_{ij} \neq {}^{\Sigma}\rho_{ij}$. Let's continue with ρ^{ij}. One can express

$$\begin{aligned}\rho^{ij} &= g^{i\mu}g^{j\nu}\rho_{\mu\nu}\\ &= g^{i0}g^{j0}\rho_{00} + \rho_{m0}\left(g^{im}g^{j0} + g^{i0}g^{jm}\right) + g^{in}g^{jm}\rho_{nm}.\end{aligned} \tag{117}$$

Since we will compute the contraction $\rho_{ij}\rho^{ij}$, we may use the symmetries of the indices at this step to simplify the construction from now on. Inserting the results (99), (103), (106), we obtain

$$\begin{aligned}\rho^{ij} =& N^2g^{i0}g^{j0}\,\Omega + 2N\,{}^{\Sigma}\rho_k\left(N^kg^{i0}g^{j0} + g^{jk}g^{i0}\right)\\ &+\left({}^{\Sigma}\rho_{kl} - \Omega_{kl}\right)\left(g^{i0}g^{j0}N^kN^l + 2N^kg^{i0}g^{jl} + g^{ik}g^{jl}\right).\end{aligned} \tag{118}$$

After using the inverse metric components, the last equation reduces to

$$\rho^{ij} = \frac{N^iN^j}{N^2}\Omega + {}^{\Sigma}\rho^{ij} - \Omega^{ij} + \frac{2}{N}N^i\,{}^{\Sigma}\rho^j. \tag{119}$$

The last term in (91) is easy to construct. We can easily obtain

$$\rho_{ij}\rho^{ij} = \left({}^{\Sigma}\rho_{ij} - \Omega_{ij}\right)^2 + \frac{N^iN^j}{N^2}\Omega\left({}^{\Sigma}\rho_{ij} - \Omega_{ij}\right) + \frac{2}{N}N^i\,{}^{\Sigma}\rho^j\left({}^{\Sigma}\rho_{ij} - \Omega_{ij}\right),$$

where

$$\left({}^{\Sigma}\rho_{ij} - \Omega_{ij}\right)^2 = \left({}^{\Sigma}\rho_{ij} - \Omega_{ij}\right)\left({}^{\Sigma}\rho^{ij} - \Omega^{ij}\right). \tag{120}$$

Collecting the pieces, $\rho_{\mu\nu}\rho^{\mu\nu}$ becomes

$$f(\rho_{\mu\nu\rho\sigma}) = \rho_{\mu\nu}\rho^{\mu\nu} = \Omega^2 - 2\,{}^{\Sigma}\rho_i\,{}^{\Sigma}\rho^i + {}^{\Sigma}\rho_{ij}\,{}^{\Sigma}\rho^{ij} - 2\Omega^{ij}\,{}^{\Sigma}\rho_{ij} + \Omega_{ij}\,\Omega^{ij}. \tag{121}$$

Recall that the constraint on the auxiliary field ψ was given in (53). In our case, differentiation of f with respect to Ω_{ij} yields

$$\begin{aligned}\frac{\partial f}{\partial\Omega_{ij}} = \frac{\partial}{\partial\Omega_{ij}}\Big(&\gamma^{kl}\gamma^{mn}\Omega_{k1}\Omega_{mn} - 2\,{}^{\Sigma}\rho_i\,{}^{\Sigma}\rho^i + {}^{\Sigma}\rho_{ij}\,{}^{\Sigma}\rho^{ij}\\ &-2\,{}^{\Sigma}\rho^{mn}\Omega_{mn} + \Omega_{mn}\Omega_{kl}\gamma^{km}\gamma^{ln}\Big).\end{aligned} \tag{122}$$

Working out the details, one has

$$\frac{\partial f}{\partial \Omega_{ij}} = \gamma^{kl}\gamma^{mn}\left(\Omega_{kl}\frac{\partial \Omega_{mn}}{\partial \Omega_{ij}} + \Omega_{mn}\frac{\partial \Omega_{k1}}{\partial \Omega_{ij}}\right) - 2\,{}^{\Sigma}\rho^{mn}\frac{\partial \Omega_{mn}}{\partial \Omega_{ij}} + \gamma^{km}\gamma^{ln}\left(\frac{\partial \Omega_{mn}}{\partial \Omega_{ij}}\Omega_{kl} + \frac{\partial \Omega_{kl}}{\partial \Omega_{ij}}\Omega_{mn}\right), \tag{123}$$

and using

$$\frac{\partial \Omega_{mn}}{\partial \Omega_{ij}} = \frac{1}{2}\left(\delta^i_m\delta^j_n + \delta^i_n\delta^j_m\right), \tag{124}$$

one arrives at

$$\frac{\partial f}{\partial \Omega_{ij}} = 2\left(\Omega^{ij} + \Omega\gamma^{ij} - {}^{\Sigma}\rho^{ij}\right). \tag{125}$$

Then, the constraint equation of the auxiliary field (53) yields

$$\psi^{ij} = {}^{\Sigma}\rho^{ij} - \Omega\gamma^{ij} - \Omega^{ij}. \tag{126}$$

We have already introduced the Hamiltonian and the momentum constraint Equations of generic f(Riemann) theories in (51), (52). Using these expressions, the momentum constraint reduces to

$$\Phi_i = -2\sqrt{\gamma}D_k\left(\frac{P_i{}^k}{\sqrt{\gamma}}\right) + \Pi_{kl}D_i\left({}^{\Sigma}\rho^{kl} - \Omega^{kl}\right) - \Pi D_i\Omega + 2\sqrt{\gamma}D_k\left(\frac{\Pi_{li}}{\sqrt{\gamma}}\left({}^{\Sigma}\rho^{kl} - \Omega^{kl}\right)\right) - 2\sqrt{\gamma}D^k\left(\frac{\Omega\Pi_{ki}}{\sqrt{\gamma}}\right). \tag{127}$$

Similarly, the Hamiltonian constraint becomes

$$\Phi_0 = \frac{\sqrt{\gamma}}{2}\left(\Omega^2 + 2{}^{\Sigma}\rho_i\,{}^{\Sigma}\rho^i - {}^{\Sigma}\rho^2_{ij} + \Omega^2_{ij} + 2D_iD_j\left({}^{\Sigma}\rho^{ij} - \Omega^{ij}\right) - 2D_iD^i\Omega\right) + \frac{1}{\sqrt{\gamma}}\left(-2\Pi_{ij}p^{ij} + \left({}^{\Sigma}\rho^{ij} - \Omega^{ij}\right)\left(\Pi_{ij} + \Pi_{ik}\Pi^k_j\right) - \Omega\left(\Pi^2 + \Pi^2_{ij}\right)\right). \tag{128}$$

6. Conclusions

We studied the time evolution and the constraint structure of f(*Riemann*)-type theories using the auxiliary fields as was done in [3] and recast the Hamiltonian flow in the compact Fischer–Marsden form [4]. This form of Einstein's equations can be considered to be a failure of the initial data to possess an exact time translation symmetry, a vantage point that led to a definition of non-stationary energy or Dain's invariant [5]. The type of theories we studied here represent a large class of theories that can be handled with two auxiliary fields, going beyond these and including the derivatives of the Riemann tensor is somewhat challenging which we shall do in a separate work. One of our motivations for this work was to give a detailed account of the computations leading to the final constraints and time evolution expressions as there are several important mistakes and omissions in the existing literature. We also gave two concrete examples: the R^2 and $R_{\mu\nu}R^{\mu\nu}$ theories.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. ADM decomposition

A.1. The metric and the connection

For the sake of completeness let us give here the ADM split of the Einstein's equations and all the relevant tensors. Using the $(n-1)+1$ dimensional splitting of the spacetime metric (12), we can express

$$g_{00} = -N^2 + N_i N^i, \qquad g_{0i} = N_i, \qquad g_{ij} = \gamma_{ij}, \tag{A1}$$

and the inverse metric components are

$$g^{00} = -\frac{1}{N^2}, \quad g^{0i} = \frac{1}{N^2}N^i, \quad g^{ij} = \gamma^{ij} - \frac{1}{N^2}N^i N^j. \tag{A2}$$

In generic n dimensions, the spacetime metric in a matrix form reads

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0i} \\ g_{0i} & g_{ij} \end{pmatrix} = \begin{pmatrix} N_i N_i - N^2 & N_i \\ N_i & \gamma_{ij} \end{pmatrix}. \tag{A3}$$

Taking the determinant of the spacetime metric, we can relate the determinants of the spacetime metric and the spatial metric as

$$\sqrt{-g} = N\sqrt{\gamma}, \tag{A4}$$

where we have used $g = \det g_{\mu\nu}$ and similarly $\gamma = \det \gamma_{ij}$. Let $\Gamma^{\mu}_{\nu\rho}$ denote the Christoffel symbol of the n dimensional spacetime

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\left(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}\right), \tag{A5}$$

and let ${}^{\Sigma}\Gamma^{k}_{ij}$ denote the Christoffel symbol of the $n-1$ dimensional hypersurface, which is compatible with the spatial metric γ_{ij}, $D_k \gamma_{ij} = 0$, as

$${}^{\Sigma}\Gamma^{k}_{ij} = \frac{1}{2}\gamma^{kl}\left(\partial_i \gamma_{jl} + \partial_j \gamma_{il} - \partial_l \gamma_{ij}\right). \tag{A6}$$

Then a simple computation gives one the following components of the connection

$$\Gamma^0_{00} = \frac{1}{N}\left(\dot{N} + N^k(\partial_k N + N^i K_{ik})\right), \tag{A7}$$

and

$$\Gamma^0_{0i} = \frac{1}{N}\left(\partial_i N + N^k K_{ik}\right), \quad \Gamma^0_{ij} = \frac{1}{N}K_{ij}, \quad \Gamma^k_{ij} = {}^\Sigma\Gamma^k_{ij} - \frac{N^k}{N}K_{ij}, \tag{A8}$$

and

$$\Gamma^i_{0j} = -\frac{1}{N}N^i\left(\partial_j N + K_{kj}N^k\right) + NK_j{}^i + D_j N^i, \tag{A9}$$

and also

$$\Gamma^i_{00} = -\frac{N^i}{N}\left(\dot{N} + N^k\left(\partial_k N + N^l K_{kl}\right)\right) + N\left(\partial^i N + 2N^k K_k{}^i\right) + \dot{N}^i + N^k D_k N^i. \tag{A10}$$

To compute the decomposition of the field equations, we need to express the additional tensor quantities such as the Riemann and the Ricci tensor components, the scalar curvature.

A.1.1. ADM splitting of the Riemann tensor. The Riemann tensor of the spacetime is defined as

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\nu\sigma} - \partial_\sigma\Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\gamma}\Gamma^\gamma_{\nu\sigma} - \Gamma^\mu_{\sigma\gamma}\Gamma^\gamma_{\nu\rho}. \tag{A11}$$

So, it is straightforward to compute the components given below

$$R^m{}_{jkl} = {}^\Sigma R^m{}_{jkl} + K_{jl}K^m_k - K_{jk}K^m_l + \frac{N^m}{N}\left(D_l K_{jk} - D_k K_{jl}\right), \tag{A12}$$

$$R^0{}_{jkl} = \frac{1}{N}\left(D_k K_{jl} - D_j K_{kl}\right), \tag{A13}$$

$$R^0{}_{j0i} = \frac{1}{N}\left(\dot{K}_{ij} - D_i D_j N - N^l D_i K_{jl} - 2K_{l(i}D_{j)}N^l\right) - K_{jk}K^k_i, \tag{A14}$$

where ${}^\Sigma R^m{}_{jkl}$ is the Riemann tensor of the hypersurface and it explicitly reads

$${}^\Sigma R^m{}_{jkl} = \partial_k{}^\Sigma\Gamma^m_{jl} - \partial_l{}^\Sigma\Gamma^m_{jk} + {}^\Sigma\Gamma^m_{ks}{}^\Sigma\Gamma^s_{jl} - {}^\Sigma\Gamma^m_{ls}{}^\Sigma\Gamma^s_{jk}. \tag{A15}$$

Also, we need to compute R_{i0j0}. Using the above results it becomes

$$\begin{aligned} R_{i0j0} = {} & N^k N^l\,{}^\Sigma R_{ikjl} + NN^k\left(D_i K_{jk} + D_j K_{ik} - 2D_k K_{ij}\right) \\ & + N\left(D_i D_j N - \dot{K}_{ij} + \mathcal{L}_\mathcal{N} K_{ij}\right) + N^2 K_{ik}K^k_j, \end{aligned} \tag{A16}$$

where $\mathcal{L}_\mathcal{N}$ denotes the Lie differentiation along the shift vector N^i and when it acts on the extrinsic curvature, one has

$$\mathcal{L}_\mathcal{N} K_{ij} = N^k D_k K_{ij} + K_{ki}D_j N^k + K_{kj}D_i N^k. \tag{A17}$$

Moreover, we can introduce the hypersurface projected components of the Riemann tensor as R_{ijkl}, $R_{ijk\vec{n}}$ and $R_{i\vec{n}j\vec{n}}$. Below we will prove the following three statements:

$$R_{ijkl} = {}^\Sigma R_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}, \tag{A18}$$

$$R_{ijk\vec{n}} = n^{\mu} R_{ijk\mu} = D_i K_{jk} - D_j K_{ik}, \tag{A19}$$
$$R_{i\vec{n}j\vec{n}} = n^{\mu} n^{\nu} R_{i\mu j\nu} = \frac{1}{N}\left(\mathcal{L}_{\mathcal{N}} K_{ij} + D_i D_j N - \dot{K}_{ij}\right) + K_{ik} K^k_j. \tag{A20}$$

Let us start with the proof of the first statement (A18). We have

$$R_{ijkl} = g_{i\mu} R^{\mu}{}_{jkl} = g_{i0} R^0{}_{jkl} + g_{im} R^m{}_{jkl} = N_i R^0{}_{jkl} + \gamma_{im} R^m{}_{jkl}, \tag{A21}$$

where one can express

$$\begin{aligned} R^0{}_{jkl} &= \partial_k \Gamma^0_{jl} - \partial_l \Gamma^0_{jk} + \Gamma^0_{k\mu}\Gamma^{\mu}_{jl} - \Gamma^0_{l\mu}\Gamma^{\mu}_{jk} \\ &= \partial_k \left(\frac{1}{N} K_{jl}\right) - \partial_l \left(\frac{1}{N} K_{jk}\right) + \frac{1}{N^2} K_{jl} \left(\partial_k N + N^m K_{mk}\right) \\ &\quad + \frac{1}{N} K_{km} \left({}^{\Sigma}\Gamma^m_{jl} - \frac{N^m}{N} K_{jl}\right) - \frac{1}{N^2} K_{jk} \left(\partial_l N + N^m K_{ml}\right) \\ &\quad - \frac{1}{N} K_{lm} \left({}^{\Sigma}\Gamma^m_{jk} - \frac{N^m}{N} K_{jk}\right), \end{aligned} \tag{A22}$$

which yields

$$R^0{}_{jkl} = \frac{1}{N}\left(\partial_k K_{jl} - \partial_l K_{jk} + K_{km}{}^{\Sigma}\Gamma^m_{jl} - K_{lm}{}^{\Sigma}\Gamma^m_{jk}\right), \tag{A23}$$

or in terms of the hypersurface covariant derivatives

$$R^0{}_{jkl} = \frac{1}{N}\left(D_k K_{jl} - D_l K_{jk}\right). \tag{A24}$$

Similarly we can compute $R^m{}_{jkl}$. By definition we have

$$R^m{}_{jkl} = \partial_k \Gamma^m_{jl} - \partial_l \Gamma^m_{jk} + \Gamma^m_{k\mu}\Gamma^{\mu}_{jl} - \Gamma^m_{l\mu}\Gamma^{\mu}_{jk}, \tag{A25}$$

which reduces to

$$R^m{}_{jkl} = {}^{\Sigma}R^m{}_{jkl} + K_{jl} K_k{}^m - K_{jk} K^m_l + \frac{N^m}{N}\left(D_l K_{jk} - D_k K_{jl}\right). \tag{A26}$$

Collecting the pieces we end up with (A18).

Let us prove (A19). By definition, projection once can be written as

$$R_{ijk\vec{n}} = n^{\mu} R_{ijk\mu} = n^0 R_{ijk0} + n^l R_{ijkl} = \frac{1}{N} R_{ijk0} - \frac{N^l}{N} R_{ijkl}, \tag{A27}$$

where

$$R_{ijk0} = g_{00} R^0{}_{kji} + g_{0l} R^l{}_{kji}, \tag{A28}$$

which reads

$$R_{ijk0} = N^l\,{}^{\Sigma}R_{lkji} + N\left(D_i K_{jk} - D_j K_{ik}\right) + N^l\left(K_{ki} K_{jl} - K_{kj} K_{il}\right). \tag{A29}$$

Using the last equation in (A27), one arrives at the desired result (A19).

We also need to construct $R_{i\vec{n}j\vec{n}}$, which reads

$$R_{i\vec{n}j\vec{n}} = n^{\mu} n^{\nu} R_{i\mu j\nu} = -R^0{}_{j0i} + N^k R^0{}_{jki}, \tag{A30}$$

where the nonvanishing components of the first term are

$$R^0{}_{j0i} = \frac{1}{N}\left[\dot{K}_{ij} - D_i D_j N - N^l D_i K_{lj} - K_{lj} D_i N^l - K_{ik} D_j N^k\right] - K_{ik} K^k_j. \tag{A31}$$

Substituting the results, we obtain the desired result (A20).

A.2. ADM splitting of the Ricci tensor and the scalar curvature

Starting with the definition of the spacetime Ricci tensor

$$R_{\rho\sigma} = \partial_\mu \Gamma^\mu_{\rho\sigma} - \partial_\rho \Gamma^\mu_{\mu\sigma} + \Gamma^\mu_{\mu\nu}\Gamma^\nu_{\rho\sigma} - \Gamma^\mu_{\sigma\nu}\Gamma^\nu_{\mu\rho}, \tag{A32}$$

one can express

$$\begin{aligned} R_{ij} = {} & \partial_0 \Gamma^0_{ij} + \partial_k \Gamma^k_{ij} - \partial_i(\Gamma^0_{0j} + \Gamma^k_{kj}) + \Gamma^0_{ij}(\Gamma^0_{00} + \Gamma^k_{k0}) \\ & + \Gamma^k_{ij}\Gamma^0_{0k} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^0_{0j}\Gamma^0_{0i} - \Gamma^0_{kj}\Gamma^k_{0i} - \Gamma^0_{ki}\Gamma^k_{0j} - \Gamma^k_{jl}\Gamma^l_{ki}, \end{aligned}$$

which yields

$$R_{ij} = {}^\Sigma R_{ij} + K K_{ij} - 2K_{ik}K^k_j + \frac{1}{N}\left(\dot{K}_{ij} - N^k D_k K_{ij} - D_i D_j N - K_{ki} D_j N^k - K_{kj} D_i N^k\right), \tag{A33}$$

where ${}^\Sigma R_{ij}$ denotes the ij component of the Ricci tensor on the hypersurface

$${}^\Sigma R_{ij} = \partial_k {}^\Sigma\Gamma^k_{ij} - \partial_i {}^\Sigma\Gamma^k_{kj} + {}^\Sigma\Gamma^k_{kl}\,{}^\Sigma\Gamma^l_{ij} - {}^\Sigma\Gamma^k_{jl}\,{}^\Sigma\Gamma^l_{ki}. \tag{A34}$$

The $0i$ component can be written as

$$R_{0i} = \partial_0 \Gamma^0_{0i} + \partial_k \Gamma^k_{0i} - \partial_i(\Gamma^0_{00} + \Gamma^k_{k0}) + \Gamma^0_{0i}\Gamma^k_{k0} + \Gamma^k_{kl}\Gamma^l_{i0} - \Gamma^k_{00}\Gamma^0_{ki} - \Gamma^k_{0l}\Gamma^l_{ki}, \tag{A35}$$

and this expression gives us the following simple result

$$R_{0i} = N^j R_{ij} + N\left(D_m K^m_i - D_i K\right). \tag{A36}$$

Similarly, the 00 component

$$R_{00} = \partial_k \Gamma^k_{00} - \partial_0 \Gamma^k_{0k} + \Gamma^0_{00}\Gamma^k_{k0} + \Gamma^k_{kl}\Gamma^l_{00} - \Gamma^k_{00}\Gamma^0_{k0} - \Gamma^k_{0l}\Gamma^l_{k0}, \tag{A37}$$

can be written in a compact form as

$$R_{00} = N^i N^j R_{ij} - N^2 K_{ij} K^{ij} + N\left(D_k D^k N - \dot{K} - N^k D_k K + 2N^k D_m K^m_k\right). \tag{A38}$$

Then, the scalar curvature of the spacetime, $R = g^{\mu\nu} R_{\mu\nu}$, can be expressed in terms of the scalar curvature of the spatial hypersurface, ${}^\Sigma R = \gamma^{ij}\,{}^\Sigma R_{ij}$, as

$$R = {}^\Sigma R + K^2 + K_{ij}K^{ij} + \frac{2}{N}\left(\dot{K} - D_k D^k N - N^k D_k K\right). \tag{A39}$$

Appendix B. Field equations of the *f*(Riemann) theory

Let us start with the action

$$S[g_{\mu\nu}] = \frac{1}{2}\int_{\mathcal{M}} d^n x\sqrt{-g}\, f(R_{\mu\nu\rho\sigma}), \tag{B1}$$

of which the first-order variation is

$$\delta S[g_{\mu\nu}] = \frac{1}{2}\int_{\mathcal{M}} d^n x\left(\delta\sqrt{-g}\, f(R_{\mu\nu\rho\sigma}) + \sqrt{-g}\,\delta f(R_{\mu\nu\rho\sigma})\right), \tag{B2}$$

where

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}, \tag{B3}$$

and

$$\delta f(R_{\mu\nu\rho\sigma}) = \frac{\partial f}{\partial R_{\lambda\gamma\rho\sigma}}\delta\left(g_{\lambda\tau}R^{\tau}{}_{\gamma\rho\sigma}\right) = \frac{\partial f}{\partial R_{\lambda\gamma\rho\sigma}}\left(R^{\tau}{}_{\gamma\rho\sigma}\delta g_{\lambda\tau} + g_{\lambda\tau}\delta R^{\tau}{}_{\gamma\rho\sigma}\right). \tag{B4}$$

Here, the linear order variation of the Riemann tensor is $\delta R^{\tau}{}_{\gamma\rho\sigma} = \nabla_\rho\delta\Gamma^{\tau}_{\gamma\sigma} - \nabla_\sigma\delta\Gamma^{\tau}_{\gamma\rho}$ and the variation of the spacetime connection is

$$\delta\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}\left(\nabla_\mu\delta g_{\nu\lambda} + \nabla_\nu\delta g_{\mu\lambda} - \nabla_\lambda\delta g_{\mu\nu}\right). \tag{B5}$$

So we have

$$\begin{aligned}\delta f(R_{\mu\nu\rho\sigma}) &= \frac{\partial f}{\partial R_{\lambda\gamma\rho\sigma}}\left(R^{\tau}{}_{\gamma\rho\sigma}\delta g_{\lambda\tau} + g_{\lambda\tau}\left(\nabla_\rho\delta\Gamma^{\tau}_{\gamma\sigma} - \nabla_\sigma\delta\Gamma^{\tau}_{\gamma\rho}\right)\right)\\ &= \frac{\partial f}{\partial R_{\lambda\gamma\rho\sigma}}\left(R^{\tau}{}_{\gamma\rho\sigma}\delta g_{\lambda\tau} + \nabla_\rho\nabla_{[\gamma}\delta g_{\lambda]\sigma} + \nabla_\sigma\nabla_{[\lambda}\delta g_{\gamma]\rho}\right).\end{aligned} \tag{B6}$$

By renaming the indices and using the antisymmetry of the Riemann tensor, the last equation can be written as

$$\delta f(R_{\mu\nu\rho\sigma}) = \frac{\partial f}{\partial R_{\lambda\gamma\rho\sigma}}\left(R^{\tau}{}_{\gamma\rho\sigma}\delta g_{\lambda\tau} + 2\nabla_\rho\nabla_\gamma\delta g_{\sigma\lambda}\right). \tag{B7}$$

Collecting the pieces, we arrive at

$$\delta S[g_{\mu\nu}] = \frac{1}{2}\int_{\mathcal{M}} d^n x\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\delta g_{\mu\nu}\, f(R_{\lambda\gamma\rho\sigma}) + \frac{\partial f}{\partial R_{\lambda\gamma\rho\sigma}}\left(R^{\tau}{}_{\gamma\rho\sigma}\delta g_{\lambda\tau} + 2\nabla_\rho\nabla_\gamma\delta g_{\sigma\lambda}\right)\right). \tag{B8}$$

Using integration by parts and ignoring the boundary terms, one has

$$\delta S[g_{\mu\nu}] = \frac{1}{4}\int_{\mathcal{M}} d^n x\sqrt{-g}\Bigg(g^{\mu\nu}\delta g_{\mu\nu} f(R_{\lambda\gamma\rho\sigma}) + \delta g_{\mu\nu}\left(\frac{\partial f}{\partial R_{\mu\gamma\rho\sigma}} R^{\nu}{}_{\gamma\rho\sigma} + \frac{\partial f}{\partial R_{\nu\gamma\rho\sigma}} R^{\mu}{}_{\gamma\rho\sigma}\right) + 2\delta g_{\mu\nu}\nabla_\sigma\nabla_\rho\left(\frac{\partial f}{\partial R_{\mu\sigma\rho\nu}} + \frac{\partial f}{\partial R_{\nu\sigma\rho\mu}}\right)\Bigg), \tag{B9}$$

and in a compact form it reads

$$\delta S[g_{\mu\nu}] = \frac{1}{2}\int_{\mathcal{M}} d^n x\sqrt{-g}\,\delta g_{\mu\nu}\left(\frac{1}{2}g^{\mu\nu} f(R_{\lambda\gamma\rho\sigma}) + R^{(\mu}{}_{\gamma\rho\sigma}\frac{\partial f}{\partial R_{\nu)\gamma\rho\sigma}} + 2\nabla_\sigma\nabla_\rho\frac{\partial f}{\partial R_{\sigma(\mu\nu)\rho}}\right), \tag{B10}$$

which yields the field equations

$$-\frac{1}{2}g^{\mu\nu} f(R_{\lambda\gamma\rho\sigma}) - R^{(\mu}{}_{\gamma\rho\sigma}\frac{\partial f}{\partial R_{\nu)\gamma\rho\sigma}} - 2\nabla_\sigma\nabla_\rho\frac{\partial f}{\partial R_{\sigma(\mu\nu)\rho}} = T_{\mu\nu}, \tag{B11}$$

where

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \tag{B12}$$

Appendix C. Introducing auxiliary fields

To turn the field equation (B11) into a set of first-order differential equations, we start with the augmented action

$$S[g_{\mu\nu}, \rho_{\mu\nu\rho\sigma}, \varphi^{\mu\nu\rho\sigma}] = \frac{1}{2}\int_{\mathcal{M}} d^n x\sqrt{-g}\Big(f(\rho_{\mu\nu\rho\sigma}) + \varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}\right)\Big), \tag{C1}$$

where the two auxiliary fields $\rho_{\mu\nu\rho\sigma}$ and $\varphi^{\mu\nu\rho\sigma}$ have all the algebraic symmetries of the Riemann tensor $R_{\mu\nu\rho\sigma}$.

Assuming that the matter couples minimally to the metric and not to the auxiliary fields, the variation of the action reads

$$\delta S[g,\rho,\varphi] = \frac{1}{2}\int_{\mathcal{M}} d^n x\Big(\delta\sqrt{-g}\Big(f(\rho_{\mu\nu\rho\sigma}) + \varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}\right)\Big) + \sqrt{-g}\Big(\delta f(\rho_{\mu\nu\rho\sigma}) + \delta\varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}\right) + \varphi^{\mu\nu\sigma}\left(\delta R_{\mu\nu\rho\sigma} - \delta\rho_{\mu\nu\rho\sigma}\right)\Big)\Big), \tag{C2}$$

where

$$\delta R_{\mu\nu\rho\sigma} = R^{\lambda}{}_{\nu\rho\sigma}\delta g_{\mu\lambda} + \frac{1}{2}\Big(\nabla_\rho\nabla_\nu\delta g_{\sigma\mu} + \nabla_\rho\nabla_\sigma\delta g_{\nu\mu} - \nabla_\rho\nabla_\mu\delta g_{\nu\sigma} - \nabla_\sigma\nabla_\nu\delta g_{\rho\mu} - \nabla_\sigma\nabla_\rho\delta g_{\nu\mu} + \nabla_\sigma\nabla_\mu\delta g_{\nu\rho}\Big). \tag{C3}$$

Using the symmetries of the fields, we have

$$\varphi^{\mu\nu\rho\sigma}\left(\delta R_{\mu\nu\rho\sigma} - \delta\rho_{\mu\nu\rho\sigma}\right) = \varphi^{\mu\nu\rho\sigma}R^{\lambda}{}_{\nu\rho\sigma}\delta g_{\mu\lambda} - \varphi^{\mu\nu\rho\sigma}\delta\rho_{\mu\nu\rho\sigma} + \frac{1}{2}\varphi^{\mu\nu\rho\sigma}\left(\nabla_\rho\nabla_\nu\delta g_{\sigma\mu} - \nabla_\rho\nabla_\mu\delta g_{\nu\sigma} - \nabla_\sigma\nabla_\nu\delta g_{\rho\mu} + \nabla_\sigma\nabla_\mu\delta g_{\nu\rho}\right), \tag{C4}$$

which can be rewritten as

$$\varphi^{\mu\nu\rho\sigma}\left(\delta R_{\mu\nu\rho\sigma} - \delta\rho_{\mu\nu\rho\sigma}\right) = \delta g_{\mu\nu}R^{(\mu}{}_{\lambda\rho\sigma}\varphi^{\nu)\lambda\rho\sigma} + 2\varphi^{\sigma(\mu\nu)\rho}\nabla_\rho\nabla_\sigma\delta g_{\mu\nu} - \varphi^{\mu\nu\rho\sigma}\delta\rho_{\mu\nu\rho\sigma}. \tag{C5}$$

Inserting these results in (C2), and integrating by parts, one ends up with

$$\delta S[g,\rho,\varphi] = \frac{1}{2}\int_{\mathcal{M}} d^n x\sqrt{-g}\Bigg(\delta g_{\mu\nu}\Big[R^{(\mu}{}_{\lambda\rho\sigma}\varphi^{\nu)\lambda\rho\sigma} + 2\nabla_\sigma\nabla_\rho\varphi^{\sigma(\mu\nu)\rho} \tag{C6}$$

$$+ \frac{1}{2}g^{\mu\nu}\left(f(\rho_{\lambda\gamma\rho\sigma}) + \varphi^{\lambda\gamma\rho\sigma}\left(R_{\lambda\gamma\rho\sigma} - \rho_{\lambda\gamma\rho\sigma}\right)\right)\Big] + \left(\frac{\partial f}{\partial\rho_{\mu\nu\rho\sigma}} - \varphi^{\mu\nu\rho\sigma}\right)\delta\rho_{\mu\nu\rho\sigma} + \left(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}\right)\delta\varphi^{\mu\nu\rho\sigma}\Bigg) + I_{\text{Boundary}}, \tag{C7}$$

where the boundary terms read

$$I_{\text{Boundary}} = \int_{\mathcal{M}} d^n x\sqrt{-g}\left(\nabla_\rho\left(\varphi^{\sigma(\mu\nu)\rho}\nabla_\sigma\delta g_{\mu\nu}\right) - \nabla_\sigma\left(\delta g_{\mu\nu}\nabla_\rho\varphi^{\sigma(\mu\nu)\rho}\right)\right). \tag{C8}$$

Introducing

$$\mathcal{E}^{\mu\nu} := -R^{(\mu}{}_{\lambda\rho\sigma}\varphi^{\nu)\lambda\rho\sigma} - 2\nabla_\sigma\nabla_\rho\varphi^{\sigma(\mu\nu)\rho} - \frac{1}{2}g^{\mu\nu}\Big(f(\rho_{\lambda\gamma\rho\sigma}) + \varphi^{\lambda\gamma\rho\sigma}\left(R_{\lambda\gamma\rho\sigma} - \rho_{\lambda\gamma\rho\sigma}\right)\Big), \tag{C9}$$

and dropping the boundary terms, one arrives at

$$\delta S = \frac{1}{2}\int d^n x\sqrt{-g}\left(-\mathcal{E}^{\mu\nu}\delta g_{\mu\nu} + \left(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}\right)\delta\varphi^{\mu\nu\rho\sigma} + \left(\frac{\partial f}{\partial\rho_{\mu\nu\rho\sigma}} - \varphi^{\mu\nu\rho\sigma}\right)\delta\rho_{\mu\nu\rho\sigma}\right). \tag{C10}$$

The field equations given in section II follow from the above variation. One can show that using the field equations for the auxiliary fields in the $\mathcal{E}^{\mu\nu} = T^{\mu\nu}$ equation, one gets back the correct second-order field equations (2), hence the consistency.

Appendix D. ADM splitting of the auxiliary fields

Let us give some details of the computations leading to the action (30). One has

$$\begin{aligned}\varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma}-\rho_{\mu\nu\rho\sigma}\right)&=\varphi^{0\nu\rho\sigma}\left(R_{0\nu\rho\sigma}-\rho_{0\nu\rho\sigma}\right)+\varphi^{i\nu\rho\sigma}\left(R_{i\nu\rho\sigma}-\rho_{i\nu\rho\sigma}\right)\\&=\varphi^{0i\rho\sigma}\left(R_{0i\rho\sigma}-\rho_{0i\rho\sigma}\right)+\varphi^{i0\rho\sigma}\left(R_{i0\rho\sigma}-\rho_{i0\rho\sigma}\right)+\varphi^{ij\rho\sigma}\left(R_{ij\rho\sigma}-\rho_{ij\rho\sigma}\right).\end{aligned}\tag{D1}$$

Due to the symmetries, it can be written as

$$\varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma}-\rho_{\mu\nu\rho\sigma}\right)=2\varphi^{0i\rho\sigma}\left(R_{0i\rho\sigma}-\rho_{0i\rho\sigma}\right)+\varphi^{ij\rho\sigma}\left(R_{ij\rho\sigma}-\rho_{ij\rho\sigma}\right),\tag{D2}$$

which yields

$$\begin{aligned}\varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma}-\rho_{\mu\nu\rho\sigma}\right)=&\,2\varphi^{0i0\sigma}\left(R_{0i0\sigma}-\rho_{0i0\sigma}\right)+2\varphi^{0ij\sigma}\left(R_{0ij\sigma}-\rho_{0ij\sigma}\right)\\&+\varphi^{ij0\sigma}\left(R_{ij0\sigma}-\rho_{ij0\sigma}\right)+\varphi^{ijk\sigma}\left(R_{ijk\sigma}-\rho_{ijk\sigma}\right),\end{aligned}\tag{D3}$$

and then

$$\begin{aligned}\varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma}-\rho_{\mu\nu\rho\sigma}\right)=&\,2\varphi^{0i0j}\left(R_{0i0j}-\rho_{0i0j}\right)+2\varphi^{0ij0}\left(R_{0ij0}-\rho_{0ij0}\right)+2\varphi^{0ijk}\left(R_{0ijk}-\rho_{0ijk}\right)\\&+\varphi^{ij0k}\left(R_{ij0k}-\rho_{ij0k}\right)+\varphi^{ijk0}\left(R_{ijk0}-\rho_{ijk0}\right)+\varphi^{ijkl}\left(R_{ijkl}-\rho_{ijkl}\right).\end{aligned}\tag{D4}$$

Using the symmetries, one ends up with
$\varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma}-\rho_{\mu\nu\rho\sigma}\right)=\varphi^{ijkl}\left(R_{ijkl}-\rho_{ijkl}\right)+4\varphi^{ijk0}\left(R_{ijk0}-\rho_{ijk0}\right)+4\varphi^{i0j0}\left(R_{i0j0}-\rho_{i0j0}\right)$.
Moreover, using the decomposition of the components of the Riemann tensor one obtains

$$\begin{aligned}\varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma}-\rho_{\mu\nu\rho\sigma}\right)=&\varphi^{ijkl}\left(R_{ijkl}-\rho_{ijkl}\right)\\&+4\varphi^{ijk0}\left(N\left(D_iK_{jk}-D_jK_{ik}\right)+N^lR_{ijkl}-\rho_{ijk0}\right)\\&+4\varphi^{i0j0}\Bigg(NN^k\left(D_iK_{jk}+D_jK_{ik}-2D_kK_{ij}\right)\\&+N\left(-\dot{K}_{ij}+\mathcal{L}_NK_{ij}+D_iD_jN\right)+N^2K_{ik}K^k{}_j+N^kN^lR_{ikjl}-\rho_{i0j0}\Bigg).\end{aligned}\tag{D5}$$

Defining

$$\begin{aligned}&\varphi^{ijk}\equiv\gamma^{il}\gamma^{jm}\gamma^{kn}n^{\mu}\varphi_{lmn\mu}, &&\psi^{ij}\equiv-2\gamma^{ik}\gamma^{jl}n^{\mu}n^{\nu}\varphi_{k\mu l\nu},\\&\rho_{ijk}\equiv n^{\mu}\rho_{ijk\mu}, &&\Omega_{ij}\equiv n^{\mu}n^{\nu}\rho_{i\mu j\nu},\end{aligned}\tag{D6}$$

one can express the corresponding components of the auxiliary field $\varphi^{\mu\nu\rho\sigma}$ in terms of the spatial tensor fields as follows

$$\varphi^{ijk0}=-\frac{\varphi^{ijk}}{N},\qquad\qquad\varphi^{i0j0}=-\frac{\psi^{ij}}{2N^2},\tag{D7}$$

and similarly, in terms of the spatial tensors, we can write the components of $\rho_{\mu\nu\rho\sigma}$ as

$$\rho_{ijk0}=N\rho_{ijk}+N^l\rho_{ijkl},$$

$$\rho_{ijk0} = N^2\Omega_{ij} + N^k N^l \rho_{ikjl} + NN^k\left(\rho_{ikj} + \rho_{jki}\right). \tag{D8}$$

Then, we arrive at

$$\begin{aligned}\varphi^{\mu\nu\rho\sigma}\left(R_{\mu\nu\rho\sigma} - \rho_{\mu\nu\rho\sigma}\right) =& \varphi^{ijkl}\left(R_{ijkl} - \rho_{ijkl}\right)\\ &- 4\varphi^{ijk}\left(D_iK_{jk} - D_jK_{ik} - \rho_{ijk} + \frac{N^l}{N}\left(R_{ijkl} - \rho_{ijkl}\right)\right)\\ &- 2\psi^{ij}\left(\frac{N^kN^l}{N^2}\left(R_{ikjl} - \rho_{ikjl}\right) + K_{ik}K^k{}_j - \Omega_{ij}\right.\\ &+ \frac{N^k}{N}\left(D_iK_{jk} + D_jK_{ik} - 2D_kK_{ij} - \rho_{ikj} - \rho_{jki}\right)\\ &\left.+ \frac{1}{N}\left(-\dot{K}_{ij} + \mathcal{L}_N K_{ij} + D_iD_jN\right)\right),\end{aligned} \tag{D9}$$

from which one obtains (30).

Appendix E. The constraints

By definition, the canonical momenta are defined as

$$\Pi_{ij} := \frac{\delta \mathcal{L}}{\delta \partial_0 \psi^{ij}}, \qquad p^{ij} := \frac{\delta \mathcal{L}}{\delta \partial_0 \gamma_{ij}} = \frac{\delta \mathcal{L}}{\delta K_{lm}}\frac{\delta K_{lm}}{\delta \dot{\gamma}_{ij}}, \tag{E1}$$

where the Lagrangian density is given in (35), and so we can express

$$\Pi_{ij} = -\sqrt{\gamma} K_{ij}. \tag{E2}$$

In the computation of p^{ij}, one needs to compute the variation of the extrinsic curvature with respect to $\dot{\gamma}_{ij}$. By definition, we have

$$\frac{\delta K_{lm}}{\delta \dot{\gamma}_{ij}} = \frac{1}{2N}\frac{\delta}{\delta \dot{\gamma}_{ij}}\left(\dot{\gamma}_{lm} - D_lN_m - D_mN_l\right) = \frac{1}{2N}\delta^i_l\delta^j_m. \tag{E3}$$

One also uses the explicit form of the Lagrangian (35) to arrive at

$$\frac{\delta \mathcal{L}}{\delta K_{lm}} = \frac{N\sqrt{\gamma}}{2}\frac{\delta f}{\delta K_{lm}} + N\sqrt{\gamma}\left(\frac{1}{N}\left(\mathcal{L}_N\psi^{lm} - \dot{\psi}^{lm}\right) - \gamma^{lm}\psi^{ps}K_{ps} - K\psi^{lm} - \psi^{ln}K^m_n - \psi^{mn}K^l_n\right).$$

Collecting the pieces we express the conjugate momenta as

$$p^{ij} = \frac{\sqrt{\gamma}}{4}\frac{\delta f}{\delta K_{ij}} + \frac{\sqrt{\gamma}}{2}\left(\frac{1}{N}\left(\mathcal{L}_N\psi^{ij} - \dot{\psi}^{ij}\right) - \gamma^{ij}\psi^{kl}K_{kl} - K\psi^{ij} - \psi^{ik}K_k^j - \psi^{jk}K_k^i\right).$$

The last expression directly yields the velocities as

$$\dot{\psi}^{ij} = \frac{N}{2}\frac{\delta f}{\delta K_{ij}} - \frac{2N}{\sqrt{\gamma}}p^{ij} + \mathcal{L}_{\mathcal{N}}\psi^{ij} + N\left(-\gamma^{ij}\psi^{kl}K_{kl} - K\psi^{ij} - \psi^{ik}K^j_k - \psi^{jk}K^i_k\right). \tag{E4}$$

The Hamiltonian density reads

$$\mathcal{H} = p^{ij}\dot{\gamma}_{ij} + \Pi_{ij}\dot{\psi}^{ij} - \mathcal{L}, \tag{E5}$$

and inserting the velocities together with the Lagrangian density (35) it can be expressed as follows

$$\mathcal{H} = 2NK_{ij}\,p^{ij} - \sqrt{\gamma}K_{ij}\mathcal{L}_N\psi^{ij} + 2p^{ij}D_iN_j + \sqrt{\gamma}N\left(-\frac{f}{2} + \frac{1}{N}\psi^{ij}D_iD_jN - \psi^{ij}\Omega_{ij} + K\psi^{kl}K_{kl} + K_{ij}\psi^{ik}K^j_k\right). \tag{E6}$$

Up to a boundary term, the Hamiltonian density can be written as a sum of the constraint equations. Namely, one has

$$\mathcal{H} = N\Phi_0 + N^i\Phi_i. \tag{E7}$$

After a straightforward computation, one obtains

$$\mathcal{H} = N\sqrt{\gamma}\left(\frac{2}{\sqrt{\gamma}}K_{ij}p^{ij} - \frac{1}{2}f(\rho_{\mu\nu\rho\sigma}) + D_iD_j\psi^{ij} - \psi^{ij}\Omega_{ij} + K\psi^{kl}K_{kl} + K_{ij}K^j_k\psi^{ik}\right) + N^i\sqrt{\gamma}\left(-2D_k\left(\frac{p^k_i}{\sqrt{\gamma}}\right) - K_{kl}D_i\psi^{kl} - 2D_k\left(\psi^{kl}K_{li}\right)\right). \tag{E8}$$

Since we have already obtained the relation between the extrinsic curvature and the conjugate momenta, $K_{ij} = -\Pi_{ij}/\sqrt{\gamma}$, we can equivalently write

$$\mathcal{H} = N\sqrt{\gamma}\left(D_iD_j\psi^{ij} - \psi^{ij}\Omega_{ij} - \frac{1}{2}f(\rho_{\mu\nu\rho\sigma})\right) + \frac{N}{\sqrt{\gamma}}\left(-2\Pi_{ij}p^{ij} + \Pi\,\Pi_{ij}\psi^{ij} + \Pi_{ij}\Pi^j_k\psi^{ik}\right) + N^i\left(-2\sqrt{\gamma}D_k\left(\frac{p^k_i}{\sqrt{\gamma}}\right) + \Pi_{kl}D_i\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{li}}{\sqrt{\gamma}}\right)\right), \tag{E9}$$

which yields the Hamiltonian and the momentum constraints as

$$\Phi_0 = \sqrt{\gamma}\left(D_iD_j\psi^{ij} - \psi^{ij}\Omega_{ij} - \frac{1}{2}f(\rho_{\mu\nu\rho\sigma})\right) + \frac{1}{\sqrt{\gamma}}\left(-2\Pi_{ij}p^{ij} + \Pi_{ij}\psi^{ij} + \Pi_{ij}\Pi^j_k\psi^{ik}\right), \tag{E10}$$

$$\Phi_i = -2\sqrt{\gamma}D_k\left(\frac{p^k_i}{\sqrt{\gamma}}\right) + \Pi_{kl}D_i\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{li}}{\sqrt{\gamma}}\right). \tag{E11}$$

Appendix F. Time evolution equations

As for the dynamical equations, the first set of the evolution equations is

$$\dot{\gamma}_{ij} = \frac{\delta\mathcal{H}}{\delta p^{ij}}, \qquad \dot{\psi}^{ij} = \frac{\delta\mathcal{H}}{\delta\Pi_{ij}}. \tag{F1}$$

This set gives the velocities in terms of canonical variables. Ignoring the total derivative terms, the Hamiltonian density can be expressed as

$$\mathcal{H} = N\sqrt{\gamma}\left(D_kD_l\psi^{kl} - \psi^{kl}\Omega_{kl} - \frac{1}{2}f(\rho_{\mu\nu\rho\sigma})\right) + \frac{N}{\sqrt{\gamma}}\left(-2\Pi_{kl}p^{kl} + \Pi\,\Pi_{kl}\psi^{kl} + \Pi_{kl}\Pi^k_m\psi^{ml}\right) + N^m\left(-2\sqrt{\gamma}D_k\left(\frac{p^k_m}{\sqrt{\gamma}}\right) + \Pi_{kl}D_m\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{lm}}{\sqrt{\gamma}}\right)\right). \tag{F2}$$

Then automatically we obtain

$$\dot{\gamma}_{ij} = -\frac{2N}{\sqrt{\gamma}}\Pi_{ij} + \mathcal{L}_N\gamma_{ij} = 2NK_{ij} + D_iN_j + D_jN_i, \tag{F3}$$

which is the expected result (57) and by definition Lie derivative yields $\mathcal{L}_N\gamma_{ij} = D_iN_j + D_jN_i$. Similarly $\dot{\psi}^{ij}$ can be written as

$$\dot{\psi}^{ij} = \frac{N}{\sqrt{\gamma}}\left(-2p^{ij} + \gamma^{ij}\Pi_{kl}\psi^{kl} + \Pi\,\psi^{ij} + \Pi^i_k\psi^{kj} + \Pi^j_k\psi^{ki}\right) + \mathcal{L}_N\psi^{ij} - \frac{N\sqrt{\gamma}}{2}\frac{\delta f}{\delta\Pi_{ij}}, \tag{F4}$$

where one can evaluate the last term easily for a given f.

Now, we can continue with the second set of the evolution equations. One has the following relations

$$\dot{p}^{ij} = -\frac{\delta\mathcal{H}}{\delta\gamma_{ij}}, \qquad \dot{\Pi}_{ij} = -\frac{\delta\mathcal{H}}{\delta\psi^{ij}}. \tag{F5}$$

Using the Hamiltonian density again let us construct $\dot{\Pi}_{ij}$. We have

$$\begin{aligned}\dot{\Pi}_{ij} =& N\sqrt{\gamma}\,\Omega_{ij} - \frac{N}{\sqrt{\gamma}}\Pi\,\Pi_{ij} - \frac{N}{\sqrt{\gamma}}\Pi_{ik}\Pi^k{}_j - N\sqrt{\gamma}\frac{\delta}{\delta\psi^{ij}}\left(D_kD_1\psi^{kl}\right)\\ &- N^m\Pi_{kl}\frac{\delta}{\delta\psi^{ij}}\left(D_m\psi^{kl}\right) - 2N^m\frac{\delta}{\delta\psi^{ij}}\left(D_k(\psi^{kl}\Pi_{lm})\right),\end{aligned} \tag{F6}$$

and we have to compute the last three terms. We have

$$D_m\psi^{kl} = \partial_m\psi^{kl} + {}^{\Sigma}\Gamma^k_{mn}\psi^{nl} + {}^{\Sigma}\Gamma^l_{mn}\psi^{kn}, \tag{F7}$$

which yields the following variation

$$\delta D_m\psi^{kl} = \partial_m\delta\psi^{kl} + {}^{\Sigma}\Gamma^k_{mn}\delta\psi^{n1} + {}^{\Sigma}\Gamma^l_{mn}\delta\psi^{kn} + \psi^{nl}\delta\,{}^{\Sigma}\Gamma^k_{mn} + \psi^{kn}\delta\,{}^{\Sigma}\Gamma^l_{mn}\ , \tag{F8}$$

and it can be written in a more compact form as

$$\delta D_m\psi^{kl} = D_m\delta\psi^{kl} + \psi^{nl}\delta\,{}^{\Sigma}\Gamma^k_{mn} + \psi^{kn}\delta\,{}^{\Sigma}\Gamma^l_{mn}\ . \tag{F9}$$

The variation of the hypersurface connection can be expressed as

$$\delta\,{}^{\Sigma}\Gamma^k_{mn} = \frac{1}{2}\gamma^{kp}\left(D_m\delta\gamma_{np} + D_n\delta\gamma_{mp} - D_p\delta\gamma_{mn}\right). \tag{F10}$$

Therefore in a more explicit form, one obtains

$$\begin{aligned}\delta D_m\psi^{kl} =& D_m\delta\psi^{kl} + \frac{1}{2}\psi^{nl}\gamma^{kp}\left(D_m\delta\gamma_{np} + D_n\delta\gamma_{mp} - D_p\delta\gamma_{mn}\right)\\ &+ \frac{1}{2}\psi^{kn}\gamma^{1p}\left(D_m\delta\gamma_{np} + D_n\delta\gamma_{mp} - D_p\delta\gamma_{mn}\right),\end{aligned} \tag{F11}$$

which can be reexpressed as

$$\delta D_m\psi^{kl} = D_m\delta\psi^{kl} + \psi^{n(k}\gamma^{l)p}\left(D_m\delta\gamma_{np} + D_n\delta\gamma_{mp} - D_p\delta\gamma_{mn}\right), \tag{F12}$$

and directly yields variation with respect to ψ^{ij} as

$$N^m \Pi_{kl} \frac{\delta \left(D_m \psi^{kl}\right)}{\delta \psi^{ij}} = -N^m D_m \Pi_{ij} - \Pi_{ij} D_m N^m. \tag{68}$$

Note that there is no contribution coming from the variations of the connection in the last expression. Similarly we can compute $\delta D_k D_l \psi^{kl}$ as

$$D_k D_l \psi^{kl} = \partial_k \left(\partial_l \psi^{kl} + {}^{\Sigma}\Gamma^k_{lm} \psi^{ml} + {}^{\Sigma}\Gamma^l_{lm} \psi^{km}\right) + {}^{\Sigma}\Gamma^k_{km} \left(\partial_l \psi^{ml} + {}^{\Sigma}\Gamma^m_{ln} \psi^{nl} + {}^{\Sigma}\Gamma^l_{ln} \psi^{mn}\right). \tag{F13}$$

Taking the variation, we write

$$\begin{aligned} \delta D_k D_l \psi^{kl} = {} & D_k D_l \delta \psi^{kl} + D^p \left(\psi^{ml} D_l \delta \gamma_{mp}\right) - \frac{1}{2} D^k \left(\psi^{ml} D_k \delta \gamma_{lm}\right) \\ & + \frac{1}{2} D_k \left(\psi^{km} \gamma^{lp} D_m \delta \gamma_{lp}\right) + \frac{1}{2} \gamma^{kp} D_l \psi^{ml} D_m \delta \gamma_{kp}. \end{aligned} \tag{F14}$$

Since we focus on variation of the spatial field ψ, we only consider the first term on the right-hand side. Hence, ignoring the total derivative terms, we get

$$N\sqrt{\gamma} \frac{\delta}{\delta \psi^{ij}} \left(D_k D_l \psi^{kl}\right) = \sqrt{\gamma} D_i D_j N. \tag{F15}$$

Now we should compute the last term: $\delta D_k \left(\psi^{kl} \Pi_{1m}\right)$. The variation of this term gives us

$$\delta D_k \left(\psi^{kl} \Pi_{lm}\right) = D_k \delta \left(\psi^{kl} \Pi_{lm}\right) - \psi^{kl} \Pi_{ln} \delta^{\Sigma} \Gamma^n_{km}. \tag{71}$$

Up to a boundary term we get

$$N^m \frac{\delta}{\delta \psi^{ij}} D_k \left(\psi^{kl} \Pi_{lm}\right) = -\Pi_{m(i} D_{j)} N^m. \tag{72}$$

Inserting them in $\dot{\Pi}_{ij}$, we end up with the desiring evolution Equation

$$\dot{\Pi}_{ij} = \sqrt{\gamma}\left(N \Omega_{ij} - D_i D_j N\right) - \frac{N}{\sqrt{\gamma}} \left(\Pi \Pi_{ij} + \Pi_{ik} \Pi^k_j\right) + \mathcal{L}_N \Pi_{ij} + \Pi_{ij} D_k N^k. \tag{F16}$$

Then, we can construct $\dot{p}^{ij} = -\delta \mathcal{H} / \delta \gamma_{ij}$. Since we have the variations

$$\frac{\delta \sqrt{\gamma}}{\delta \gamma_{ij}} = \frac{1}{2} \sqrt{\gamma} \gamma^{ij}, \qquad \frac{\delta \gamma^{-1/2}}{\delta \gamma_{ij}} = -\frac{1}{2} \gamma^{-1/2} \gamma^{ij}, \tag{F17}$$

from the Hamiltonian density (46) we directly obtain

$$\dot{p}^{ij} = -\frac{N}{2}\sqrt{\gamma}\gamma^{ij}\left(D_kD_l\psi^{kl} - \psi^{kl}\Omega_{kl} - \frac{1}{2}f(\rho_{\mu\nu\rho\sigma})\right) - N\sqrt{\gamma}\frac{\delta}{\delta\gamma_{ij}}\left(D_kD_l\psi^{kl} - \frac{1}{2}f(\rho_{\mu\nu\rho\sigma})\right)$$
$$+\frac{N}{2\sqrt{\gamma}}\gamma^{ij}\left(-2\Pi_{kl}p^{kl} + \Pi\,\Pi_{kl}\psi^{kl} + \Pi_{kl}\Pi^k_m\psi^{ml}\right)$$
$$-\frac{N}{\sqrt{\gamma}}\frac{\delta}{\delta\gamma_{ij}}\left(\Pi_{mn}\gamma^{mn}\,\Pi_{kl}\psi^{kl} + \Pi_{kl}\Pi_{mn}\gamma^{kn}\psi^{ml}\right)$$
$$-N^m\frac{\delta}{\delta\gamma_{ij}}\left(-2\gamma_{mn}\sqrt{\gamma}D_k\left(\frac{p^{kn}}{\sqrt{\gamma}}\right) + \Pi_{kl}D_m\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{lm}}{\sqrt{\gamma}}\right)\right). \tag{F18}$$

To simplify the last equation, we use the Hamiltonian constraint (51) together with

$$\frac{\delta\gamma^{mn}}{\delta\gamma_{ij}} = -\frac{1}{2}\left(\gamma^{mi}\gamma^{jn} + \gamma^{mj}\gamma^{in}\right), \tag{F19}$$

in (F18) to arrive at

$$\dot{p}^{ij} = \frac{N}{\sqrt{\gamma}}\left(\gamma^{ij}\left(\Pi\,\Pi_{kl}\psi^{kl} + \Pi_{kl}\Pi^k_m\psi^{ml} - 2\Pi_{kl}p^{kl}\right) + \Pi^{ij}\Pi_{kl}\psi^{kl} + \Pi^i_l\Pi^j_k\psi^{kl}\right)$$
$$-N\sqrt{\gamma}\frac{\delta}{\delta\gamma_{ij}}\left(D_kD_l\psi^{kl}\right) + \frac{N}{2}\sqrt{\gamma}\frac{\delta}{\delta\gamma_{ij}}f(\rho_{\mu\nu\rho\sigma})$$
$$-N^m\frac{\delta}{\delta\gamma_{ij}}\left(-2\gamma_{mn}\sqrt{\gamma}D_k\left(\frac{p^{kn}}{\sqrt{\gamma}}\right) + \Pi_{kl}D_m\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{lm}}{\sqrt{\gamma}}\right)\right). \tag{F20}$$

A straightforward calculation gives us

$$N\sqrt{\gamma}\frac{\delta}{\delta\gamma_{ij}}\left(D_kD_l\psi^{kl}\right) = -\frac{\sqrt{\gamma}}{2}\left(D_k\left(\psi^{ij}D^kN - 2\psi^{k(i}D^{j)}N\right) + \gamma^{ij}\left(ND_kD_l\psi^{kl} - \psi^{kl}D_kD_lN\right)\right), \tag{F21}$$

and

$$2N^m\frac{\delta}{\delta\gamma_{ij}}\left(\gamma_{mn}\sqrt{\gamma}D_k\left(\frac{p^{kn}}{\sqrt{\gamma}}\right)\right) = \sqrt{\gamma}\mathcal{L}_N\left(\frac{p^{ij}}{\sqrt{\gamma}}\right) + p^{ij}D_kN^k, \tag{F22}$$

and also

$$N^m\frac{\delta}{\delta\gamma_{ij}}\left(\Pi_{kl}D_m\psi^{kl} + 2\sqrt{\gamma}D_k\left(\psi^{kl}\frac{\Pi_{lm}}{\sqrt{\gamma}}\right)\right) = 0. \tag{F23}$$

Finally, after collecting the pieces, one ends up with the last evolution equation (67)

$$\dot{p}^{ij} = \frac{N}{\sqrt{\gamma}}\left(\gamma^{ij}\left(\Pi\,\Pi_{kl}\psi^{kl} + \Pi_{kl}\Pi^k_m\psi^{ml} - 2\Pi_{kl}p^{kl}\right) + \Pi^{ij}\Pi_{kl}\psi^{kl} + \Pi^i_l\Pi^j_k\psi^{kl}\right)$$
$$+\frac{\sqrt{\gamma}}{2}\left(D_k\left(\psi^{ij}D^kN - 2\psi^{k(i}D^{j)}N\right) + \gamma^{ij}\left(ND_kD_l\psi^{kl} - \psi^{kl}D_kD_lN\right)\right)$$
$$+\frac{N}{2}\sqrt{\gamma}\frac{\delta f}{\delta\gamma_{ij}} + \sqrt{\gamma}\mathcal{L}_N\left(\frac{p^{ij}}{\sqrt{\gamma}}\right) + p^{ij}D_kN^k. \tag{F24}$$

At this point, one needs to know the explicit form of the function f to proceed further.

Appendix G. Application to general relativity

In this section, we will use the generic results to evaluate Einstein's theory. Let us set the function f to

$$f(\rho_{\mu\nu\rho\sigma}) = g^{\mu\rho} g^{\nu\sigma} \rho_{\mu\nu\rho\sigma}. \tag{G1}$$

Since $\rho_{\mu\nu\rho\sigma}$ has the symmetries of the Riemann tensor by assumption, one can easily show that

$$f = g^{ik} g^{jl} \rho_{ijkl} + 4 g^{0j} g^{ik} \rho_{0ijk} + 2\rho_{0i0j} \left(g^{00} g^{ij} - g^{0i} g^{0j}\right). \tag{G2}$$

We can insert the components of the inverse spacetime metric to arrive at

$$f(\rho_{\mu\nu\rho\sigma}) = \gamma^{ik}\gamma^{jl}\rho_{ijkl} + \frac{2}{N^2}\gamma^{ij}\left(2N^k \rho_{0ikj} - \rho_{0i0j} - N^k N^l \rho_{ikjl}\right). \tag{G3}$$

Using the hypersurface projected tensor fields that we introduced before, we arrive at

$$f = \gamma^{ik}\gamma^{jl}\rho_{ijkl} - 2\gamma^{ij}\Omega_{ij}. \tag{G4}$$

Now we can construct the constraint on the auxiliary field ψ^{ij}, (53). It directly yields

$$\psi^{ij} = \gamma^{ij}. \tag{G5}$$

Therefore, the first set of evolution equations (57), (58) are related. Since $\psi^{ij} = \gamma^{ij}$, we can rewrite (58) as

$$\dot{\gamma}^{ij} = -D^i N^j - D^j N^i + \frac{2N}{\sqrt{\gamma}}\left(\Pi\gamma^{ij} + \Pi^{ij} - p^{ij} - \Pi_{kl}\frac{\partial f}{\partial \rho_{ikjl}}\right), \tag{G6}$$

where we can use

$$\frac{\partial f}{\partial \rho_{ikjl}} = \gamma^{pq}\gamma^{nm}\frac{\partial \rho_{pnqm}}{\partial \rho_{ikjl}}. \tag{G7}$$

We have to preserve the symmetries on both sides of the equation. So, we have to express it in a more correct form as

$$\frac{\partial f}{\partial \rho_{ikjl}} = \frac{1}{4}\gamma^{pq}\gamma^{nm}\frac{\partial}{\partial \rho_{ikjl}}\left(\rho_{pnqm} + \rho_{qmpn} - \rho_{npqm} - \rho_{pnmq}\right), \tag{G8}$$

which yields

$$\frac{\partial f}{\partial \rho_{ikjl}} = \frac{1}{2}\left(\gamma^{ij}\gamma^{kl} - \gamma^{il}\gamma^{kj}\right). \tag{G9}$$

Then we have

$$\dot{\gamma}^{ij} = -D^i N^j - D^j N^i + \frac{2N}{\sqrt{\gamma}}\left(\frac{1}{2}\Pi\gamma^{ij} + \frac{3}{2}\Pi^{ij} - p^{ij}\right), \tag{G10}$$

Using the basic relation $\dot{\gamma}_{ij} = -\gamma_{ik}\gamma_{jl}\dot{\gamma}^{kl}$, one can directly rewrite the last expression as

$$\dot{\gamma}_{ij} = D_i N_j + D_j N_i + \frac{2N}{\sqrt{\gamma}}\left(-\frac{1}{2}\Pi\gamma_{ij} + p_{ij} - \frac{3}{2}\Pi_{ij}\right). \tag{G11}$$

For consistency with the time evolution of the spatial metric

$$\dot{\gamma}_{ij} = -\frac{2N}{\sqrt{\gamma}}\Pi_{ij} + D_i N_j + D_j N_i, \tag{G12}$$

one needs

$$\Pi_{ij} = 2\left(p_{ij} - \frac{p}{n}\gamma_{ij}\right). \tag{G13}$$

G.1. Constraint equations

One can reexpress the Hamiltonian density of general relativity, using $\Psi^{ij} = \gamma^{ij}$ and (G13) as

$$\mathcal{H} = \left(-p^{ij} + \frac{2p}{n}\gamma^{ij}\right)\dot{\gamma}_{ij} - \mathcal{L}. \tag{G14}$$

We then introduce the new momentum

$$\pi^{ij} := -p^{ij} + \frac{2p}{n}\gamma^{ij}, \tag{G15}$$

which yields the trace

$$\pi = \frac{n-2}{n}p. \tag{G16}$$

Then, the Hamiltonian density becomes

$$\mathcal{H} = \pi^{ij}\dot{\gamma}_{ij} - \mathcal{L}. \tag{G17}$$

One has the inverse relations

$$\Pi^{ij} = -2\pi^{ij} + \frac{2}{n-2}\gamma^{ij}\pi, \tag{G18}$$

and $\Pi = 2\pi/(n-2)$. Additionally, we can express

$$p^{ij} = -\pi^{ij} + \frac{2}{n-2}\gamma^{ij}\pi, \qquad p = \frac{n}{n-2}\pi. \tag{G19}$$

Finally, the Hamiltonian and the momentum constraint equations (51) and (52) reduce to

$$\Phi_0(\gamma,\pi) = \frac{2}{\sqrt{\gamma}}(\pi_{ij}^2 - \frac{\pi^2}{n-2}) - \frac{\sqrt{\gamma}}{2}\,{}^{\Sigma}R, \tag{G20}$$

and

$$\Phi_i(\gamma,\pi) = -2\mathcal{D}_k\pi^k_i. \tag{G21}$$

ORCID iD

Bayram Tekin https://orcid.org/0000-0002-0792-9010

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