EXAMINING MIDDLE SCHOOL MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE IN THE CONTEXT OF FUNCTIONAL THINKING

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

RUMEYSA UZUN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS EDUCATION IN MATHEMATICS AND SCIENCE EDUCATION

AUGUST 2024

Approval of the thesis:

EXAMINING MIDDLE SCHOOL MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE IN THE CONTEXT OF FUNCTIONAL THINKING

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ABSTRACT

EXAMINING MIDDLE SCHOOL MATHEMATICS TEACHERS' PEDAGOGICAL CONTENT KNOWLEDGE IN THE CONTEXT OF FUNCTIONAL THINKING

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August 2024, 95 pages

The purpose of this study was to examine the pedagogical content knowledge of middle school mathematics teachers in the context of functional thinking. The participants of the study were five middle school mathematics teachers (MSMTs) working in a public middle school in Istanbul. Semi-structured interview questions related to two tasks were directed to the MSMTs. Individual semi-structured interviews lasting approximately 40 minutes were conducted with the participants. A qualitative design was used in this study**.** As a result of this study, when MSMTs were asked about students' possible correct solutions, they mostly expected students to find the rule to reach the general term without relating to the context or figure in the tasks. MSMTs mostly used the figures in the tasks only to find the difference between the steps, which showed that MSMTs focused on numerical relationships rather than the model. When MSMTs were asked about possible incorrect solutions from students, they were found to be aware of some common students' mistakes. When MSMTs were asked the reasons for students' mistakes, they gave superficial answers, which showed that they had limited knowledge on this subject. When MSMTs were asked how to overcome students' mistakes, they were found to have limited knowledge in helping students overcome their mistakes. Given different students' answers, when MSMTs were asked how the student might have thought, most MSMTs were able to explain how the student thought. When MSMTs were asked what they would do next with these students, MSMTs were found to have difficulty developing strategies for what to do next with the students who answered correctly.

Keywords: Functional thinking, Pedagogical content knowledge, Middle school mathematics teacher, Mathematical knowledge for teaching

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMENLERİNİN PEDAGOJİK ALAN BİLGİLERİNİN FONKSİYONEL DÜŞÜNME BAĞLAMINDA İNCELENMESİ

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Yüksek Lisans, Matematik Eğitimi, Matematik ve Fen Bilimleri Eğitimi
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Ağustos 2024, 95 sayfa

Bu çalışmanın amacı, ortaokul matematik öğretmenlerinin pedagojik alan bilgilerini işlevsel düşünme bağlamında incelemektir. Çalışmanın katılımcıları İstanbul'da bir devlet ortaokulunda görev yapan beş ortaokul matematik öğretmenidir. Ortaokul matematik öğretmenlerine iki görevle ilgili yarı yapılandırılmış görüşme soruları yöneltilmiştir. Katılımcılarla yaklaşık 40 dakika süren bireysel yarı yapılandırılmış görüşmeler yapılmıştır. Bu çalışmada nitel bir desen kullanılmıştır. Bu çalışmanın sonucunda, ortaokul matematik öğretmenlerine öğrencilerin olası doğru çözümleri sorulduğunda, çoğunlukla öğrencilerin görevlerdeki bağlam veya şekil ile ilişkilendirmeden genel terime ulaşmak için kuralı bulmalarını bekledikleri görülmüştür. Ortaokul matematik öğretmenleri görevlerdeki şekilleri çoğunlukla sadece adımlar arasındaki farkı bulmak için kullanmışlardır, bu da ortaokul matematik öğretmenlerinin modelden ziyade sayısal ilişkilere odaklandıklarını göstermektedir. Ortaokul matematik öğretmenlerine öğrencilerden gelen olası yanlış çözümler sorulduğunda, bazı yaygın öğrenci hatalarının farkında oldukları görülmüştür. Ortaokul matematik öğretmenlerine öğrencilerin hatalarının nedenleri sorulduğunda yüzeysel cevaplar vermeleri, bu konuda sınırlı bilgiye sahip olduklarını göstermiştir. Ortaokul matematik öğretmenlerine öğrencilerin hatalarının üstesinden nasıl gelebilecekleri sorulduğunda, öğrencilerin hatalarının üstesinden

gelmelerine yardımcı olma konusunda sınırlı bilgiye sahip oldukları görülmüştür. Farklı öğrencilerin cevapları göz önüne alındığında, ortaokul matematik öğretmenlerine öğrencinin nasıl düşünmüş olabileceği sorulduğunda, çoğu MSMT öğrencinin nasıl düşündüğünü açıklayabilmiştir. Bu öğrencilerle bir sonraki adımda ne yapacakları sorulduğunda, ortaokul matematik öğretmenlerine doğru cevap veren öğrencilerle bir sonraki adımda ne yapacaklarına dair strateji geliştirmekte zorlandıkları görülmüştür.

Anahtar Kelimeler: Fonksiyonel düşünme, Pedagojik alan bilgisi, Ortaokul matematik öğretmeni, Öğretim için gerekli matematiksel bilgi

To my brother

ACKNOWLEDGMENTS

First of all, my dear supervisor Assist. Prof. Dr. Işıl İşler Baykal thanks for contributing with her valuable feedback at every stage of the study, always encouraging and supporting me. Thanks to her, I learned much valuable information on this journey.

I would also like to thank the committee members. Assist. Prof. Dr. Ebru Aylar Çankaya and Assoc. Prof. Dr. Bülent Çetinkaya for their time, valuable comments and suggestions to improve my study.

I would also like to thank my family. Since I lost my brother, Mehmet Fatih UZUN, halfway through this process, I would have liked him to see me finish this study first. I am grateful to my mother, Gülfişan UZUN, my father, Hüseyin UZUN, and my sister, Nisa UZUN ŞEKERCİ, for always reminding me that I had the strength to finish this study at every stage of my difficulties. I want to thank my husband, Eren İŞSEVER, for supporting me and lightening my burden without complaining every time I was busy on this journey.

Finally, I would like to thank the five middle school mathematics teachers who volunteered to participate in this study for their time and valuable opinions.

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LIST OF ABBREVIATIONS

ABBREVIATIONS

MoNE Ministry of National Education

MSMT Middle School Math Teacher

PCK Pedagogical Content Knowledge

NCTM National Council of Teachers of Mathematics

CHAPTER 1

INTRODUCTION

The place of algebra in mathematics education is indisputable. According to Kieran (1992), algebra not only represents quantities with letters but also allows operations with these letters. As students learn algebra topics, the difficulties they experience in mathematics also increase (Erbaş & Ersoy, 2003). One of the reasons for the difficulties students experience might be their inability to think algebraically.

The basis of the development of algebraic thinking in mathematics education is based on variables and relationships between variables (Tanışlı & Kabael, 2019). Smith (2003) defined functional thinking as searching for relationships between variables. Stephens et al. (2007) defined functional thinking as "generalizing relationships between quantities; representing those relationships, or functions, in multiple ways using natural language, formal algebraic notation, tables, and graphs; and reasoning fluently with these representations in order to interpret and predict function behavior." (p. 144). Functional thinking is necessary to learn the concept of function, where the relationship between variables has an abstract meaning in the development of algebraic thinking.

Students are taught arithmetic in elementary school to gain fluency in operations, and in middle school, they encounter algebra (Blanton et al., 2007). Students have difficulty making sense of this abstract concept they encounter after arithmetic. One of the reasons for the difficulties students experience in algebra is the quality of teaching (Kieran, 2004). It is critical for teachers to manage the process well in this process. It is important for mathematics teachers to know mathematics and how to teach mathematics. There are views about what teachers need to know in order to teach effectively and one view is that the most important knowledge for teachers is the knowledge that is closest to the teaching practice (McCrory et al., 2012). McCrory et al. (2012) indicated pedagogical content knowledge (Shulman, 1986) and mathematical knowledge for teaching (Ball et al., 2008) are important for effective teaching. These knowledge domains are also important for teaching algebra.

It is important for students to be able to think functionally to overcome their difficulties in algebra so teachers need to guide students in functional thinking and help students learn conceptually. In this regard, the aim of this study was to examine middle school mathematics teachers' pedagogical content knowledge in the context of functional thinking

1.1 Purpose of the Study

The purpose of this study was to examine middle school mathematics teachers' pedagogical content knowledge in the context of functional thinking. The focus of this study is on the sub-categories of pedagogical content knowledge within the framework of Ball et al. (2008), which are knowledge of content and students and knowledge of content and teaching.

1.2 Research Questions

The research question of the study is as follows:

What is the pedagogical content knowledge of middle school mathematics teachers in the context of functional thinking?

- What is the MSMTs' knowledge of content and students in the context of functional thinking?
- What is the MSMTs' knowledge of content and teaching in the context of functional thinking?

1.3 Significance of The Study

Functional thinking is a critical subject in learning algebra (Kaput, 2008). When we look at the 2024 national mathematics curriculum, the objectives related to functional thinking which includes recognizing and understanding the relationship between quantities and linear functions. In this regard, it is important to examine middle school mathematics teachers' pedagogical content knowledge on this subject.

Teachers should anticipate how students will think about a topic and what they may have difficulty with (Ball et al., 2008). This requires knowledge of content and students. In addition, a teacher should know when to explain in class, when to give students a break, how to ask a question or how to prepare a task to advance students' learning (Ball et al., 2008). This requires knowledge of content and teaching. Examining the knowledge of teachers is important for good teaching. This study is important because it focuses on teachers' knowledge of content and students and knowledge of content and teaching.

Fennema and Franke (1992) stated that everyone accepts that teachers play a very important role in student learning. Teachers play a major role in students' algebra learning as well. For this reason, it is important to investigate teachers' pedagogical content knowledge in the context of functional thinking. There are few studies examining the functional thinking of teachers. Therefore, this study can contribute to the literature in this field. In addition, teaching functional thinking is very important for the professional development of teachers. Therefore, it is thought that this study may also contribute to the professional development of teachers.

1.4 Definition of Important Terms

Functional thinking: Blanton and Kaput (2011) defined functional thinking as "incorporating building and generalizing patterns and relationships using diverse linguistic and representational tools and treating generalized relationships, or functions, that result as mathematical objects useful in their own right" (p. 8).

Pedagogical content knowledge: Ball et al. (2008) described it as a blend of content knowledge and pedagogical knowledge, that is, how knowledge can be related to teaching.

Knowledge of content and students: Ball et al. (2008) defined as "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content'' (p. 375).

Knowledge of content and teaching: It is about how teachers choose examples and representations, how to overcome students' mistakes, and how to guide students' thinking (Ball et al., 2008).

Mathematical knowledge for teaching: It is the mathematical knowledge required for teaching mathematics. It includes the tasks involved in teaching and the mathematical demands of these tasks (Ball et al., 2008).

Middle school mathematics teachers: Middle school mathematics teachers are individuals who teach middle school students from fifth to eighth grade in middle school.

CHAPTER 2

LITERATURE REVIEW

This study aimed to investigate the pedagogical content knowledge of middle-school mathematics teachers in the context of functional thinking. This section describes teacher knowledge, pedagogical content knowledge, functional thinking, functional thinking studies with elementary and middle school teachers and preservice teachers, functional thinking studies with elementary and middle school students, and lastly, functional thinking in the 2018 National Mathematics Curriculum objectives.

2.1 Teacher Knowledge

Teachers are one of the most important factors in students' success (Silver, 1998). Teachers' awareness of students' mathematical learning and thinking enables effective teaching, and teacher education plays an important role in providing and developing this awareness (Even & Tirosh, 2002). The more knowledge teachers have about students' learning and their own teaching, the more effectively and accurately they can convey it to students (Darling-Hammond & Bransford, 2007). Although content knowledge is important for teaching, teaching a subject requires more than knowing the content (Shulman, 1986). The teacher should also know how to teach the subject to the students.

Lee Shulman (1986) proposed a specific area of teacher knowledge that he called pedagogical content knowledge. One of Shulman's purposes was to define teacher knowledge along with the role of content in teaching. Shulman also identified content knowledge as a type of specialized technique important to the teaching profession. Shulman's major categories of teacher knowledge included general pedagogical knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, knowledge of educational ends, purposes and values, curriculum knowledge, and pedagogical content knowledge (Shulman, 1987). The reason pedagogical content knowledge has become so prominent is that it provides the link between content knowledge and teaching practice. However, this link was not well understood, and Shulman's desired theoretical framework had not been developed, so Ball et al. (2008) proposed a framework to develop Shulman's categories.

2.2 Pedagogical Content Knowledge

Shulman (1986) proposed pedagogical content knowledge as a special domain of teacher knowledge. Pedagogical content knowledge is defined as the combination of subject and teaching (Ball et al., 2008). Shulman described pedagogical content knowledge as the most useful way to make the subject understandable to others. In addition, PCK includes understanding what would make it easier or harder for students to learn certain a subject and how students of different ages and backgrounds learn effectively. Ball et al. (2008) proposed a framework that would develop Shulman's categories.

Ball et al.'s framework (See Figure 2.1) explains the domains of Mathematical Knowledge for Teaching. In this framework, Ball et al. (2008) divided the domains of mathematical knowledge into subject matter knowledge and pedagogical content knowledge. Ball et al. (2008) divided subject matter knowledge into three categories, which are common content knowledge, specialized content knowledge, and horizon content knowledge. Common content knowledge is mathematical knowledge used in everyday life (Hill et al., 2008). Specialized content knowledge involves teachers being able to represent mathematical ideas appropriately and provide mathematical explanations for mathematical rules. Horizon content knowledge is concerned with having a broad understanding of the mathematical environment and being aware of issues that students may or may not encounter. This study focused on pedagogical content knowledge (PCK). PCK includes three parts: Knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum. This study focused on the two parts that are knowledge of content and students and knowledge of content and teaching.

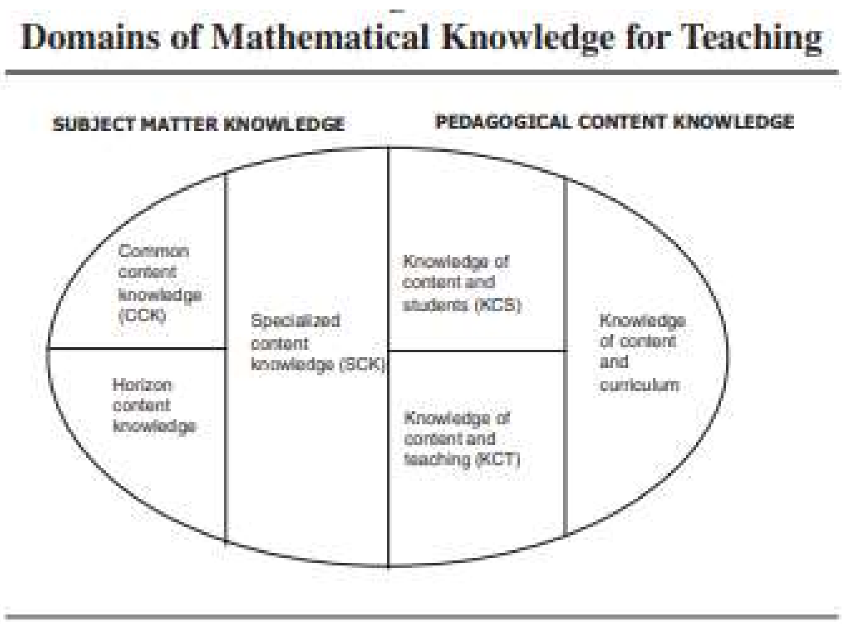


Figure 2.1. Domains of mathematical knowledge for teaching

(Taken from "Content knowledge for teaching: What makes it special?" by D. L. Ball, M. H. Thames, & G. Phelps, 2008, *Journal of Teacher Education*, *59*(5), p. 403.)

Ball et al. (2008) defined knowledge of content and curriculum as knowledge about the content of the curriculum. Knowledge of content and students as "content knowledge intertwined with knowledge of how students think about, know, or learn this particular content." (p. 375). This type of knowledge includes how students learn a concept, the mistakes students make about this concept, and where they have difficulty learning these concepts. For example, in this study, MSMTs were asked about the possible correct and incorrect answers they expected from students, and this question aimed to examine their knowledge of content and students. Knowledge of content and teaching includes teachers' teaching strategies (Ball et al., 2008). This type of knowledge includes how teachers will use representations and examples while teaching concepts, how to overcome student errors, and how to improve student thinking. For example, in this study, MSMTs were asked how to overcome possible students' mistakes, and this question aimed to examine their knowledge of content and teaching.

2.3 Functional Thinking

Blanton et al. (2011) defined functional thinking as generalizing the relationships among variables, representing these relationships with words, symbols, tables, or graphs, and reasoning using multiple representations to analyze the change of the function. According to Warren and Cooper (2005), the power of mathematics is based on relationships and transformations which includes patterns and generalizations. For this reason, it is necessary to encourage students to use important skills such as generalization, expression, and justification in mathematics teaching (Kaput & Blanton, 2001). Until recently, in the United States, functions were thought to be a subject mostly learned in high school. However, NCTM (2000) stated that functions need to be taught with rich content starting from elementary school. The objectives in the recent middle school national curriculum (MoNE, 2024) do not directly mention functional thinking, but the curriculum includes objectives related to functional thinking; therefore, teachers should create an environment that guides students to develop functional thinking.

Kaput (2008) defined functional thinking as an important part of algebraic thinking. Functional thinking provides an understanding of the relationships and inverse relationships between variables. In this way, it can be predicted that functional thinking makes it easier to discover arithmetic and understand the relationship between operations. According to research, the subject of functions is not an area that students generally understand, and the reason for this is that the subject of functions is taught abstractly (Chazan, 1996). In addition, although researchers think that it is important to teach functional thinking at elementary school age, the primary school curriculum is insufficient to enable students to think functionally (Blanton & Kaput, 2004). That is why functional thinking should be taught in a long and gradual way (Warren & Cooper, 2005).

Students use three types of functional thinking when generalizing relationships: recursive pattern, covariational relationship, and correspondence (Confrey & Smith, 1991). A recursive pattern expresses how a number can be obtained from the previous number in a number sequence, that is, the change in a single variable. A covariational relationship expresses how two variables change depending on each other. Correspondence is defined as the function rule between the dependent and independent variables (Confrey & Smith, 1991).

Stephens et al. (2017) identified levels of sophistication that represent students' generalization and representation of functional relationships based on students' written responses (See Figure 2.2).

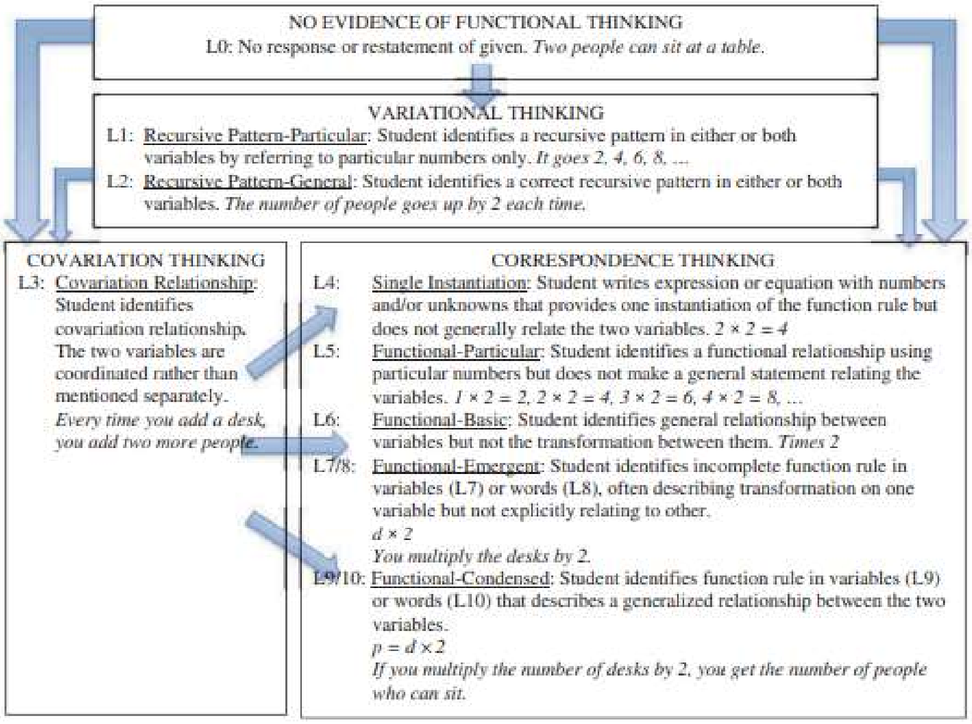


Figure 2.2. Levels of sophistication describing grades 3–5 students' generalization and representation of functional relationships

(Taken from: Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning*, *19*(3), 143-166.)

In the levels of sophistication, there are four categories: no evidence of functional thinking, variational thinking, covariational thinking, and correspondence thinking, and there are eleven levels from level 0 to level 10. No evidence of functional thinking includes level 0. At this level, the question could not be answered. Variational thinking includes level 1 and level 2. Covariational thinking includes level 3 (Covariation relationship). Correspondence thinking includes level 4 (Single instantiation), level 5 (Functional- particular), level 6 (Functional-basic), level 7/8 (Functional-emergent), and level 9/10 (Functional-condensed). In this study, three student responses were presented to MSMTs regarding Task 2. Two of these student responses were created based on levels of sophistication at level 3 (Covariation relationship) and level 9/10 (Functional-condensed).

2.4 Functional Thinking Studies Conducted with Elementary and Middle School Mathematics Teachers

In this section, studies conducted with elementary and middle school mathematics teachers and preservice mathematics teachers related to functional thinking were included.

Wilkie (2014) conducted a survey to 105 upper primary (8- to 12-year-olds) teachers to investigate their knowledge of teaching algebra. As a result of this survey, it was seen that two-thirds of the teachers had sufficient content knowledge about the pattern generalization task used as a data collection tool. However, it was concluded that more than half of the participants did not have sufficient pedagogical content knowledge. Although more than half of the teachers stated that they taught "Pattern and Algebra" content to students, less than half of them were able to provide appropriate examples, and in addition, more than two-thirds of these teachers stated that they were anxious about their skills teaching this content.

Girit's (2016) study investigated middle school mathematics teachers' mathematical knowledge of generalizing patterns and operations using algebraic expressions. In this study, data was collected during the time two middle school mathematics teachers taught the 7^{th}-grade algebra unit. Lesson plans prepared by teachers, lesson observations and pre- and post-observation interviews were used as data collection tools. The results of this study showed that teachers were inadequate in pattern generalization and in predicting the mistakes that students might make in discovering the relationship between the patterns. In addition, both teachers used tables and numerical reasoning to generalize the patterns. None of them used figural reasoning. The study showed that when teachers had strong content knowledge, they paid attention to students' thinking and used teaching methods effectively. When they had strong specialized content knowledge, they had strong pedagogical content knowledge. The reason for mathematics teachers' lack of pedagogical content knowledge was stated due to their lack of content knowledge.

Kutluk (2011) investigated to what extent teachers were aware of the difficulties experienced by students in the subject of pattern and the effect of this awareness on students' learning. The participants of this qualitative study consisted of 30 middle school mathematics teachers. Lesson observations and semi-structured interviews were used in the data collection. The findings of Kutluk's study (2011) showed that the participants perceived the figure only as a visual element. The participants did not use the figure to find the general rule. In addition, it was observed that teachers had deficiencies in predicting possible errors that students might make in generalizing number patterns. They also lacked knowledge about strategies to overcome student difficulties. When teachers were asked about the reasons for students' possible errors, they had difficulty in explaining the reasons. It was thought that the difficulty teachers had in explaining the reasons for students' errors affected their thoughts on how to overcome these difficulties.

Pang and Sunwoo (2022) investigated 119 elementary school teachers' knowledge in teaching functional thinking. A questionnaire was developed to examine the knowledge of the learners required to teach functional thinking. This questionnaire included three strands of knowledge: mathematical tasks, instructional strategies and mathematical discourse. As a result of this study, it was seen that elementary school mathematics teachers could create mathematical tasks for simple relationships involving two quantities. However, some teachers had difficulty creating tasks for $y=2x+2$. In addition, teachers were able to explain students' typical errors about functional thinking. Some teachers' explanations did not include a deep understanding of students' errors. In the study, teachers were asked to analyze students' mistakes in making associations and generalizations between two quantities. When teachers were shown the answer of the student who ignored the constant and focused on the increase while generalizing the pattern, approximately 48% of the teachers were able to give appropriate answers to how this student thought. When teachers were asked how to overcome students' errors, they offered strategies such as raising a question for the student to recognize the answer, reexplaining the concept, confirming by constructing a function table, and confirming by drawing a picture. It was observed that most of the teachers gave answers to have students realize their mistakes.

In another study by Wilkie (2016), research was conducted on the professional learning of upper primary school teachers to improve students' functional thinking in the context of generalizing patterns. A research project on design-based geometric pattern generalization was conducted with these teachers for 1 year**.** The participants of the research consisted of 10 teachers. The use of initial and final surveys investigated the changes in teachers' knowledge. The observation of their interactions during meetings and lessons also investigated how evidence of these changes was revealed. Detailed observation notes were written after each of the 10 teachers' three lessons and each team meeting. Both individual and group interviews were conducted with the teachers. The results of this study showed that improving teachers' knowledge of students' functional thinking processes was difficult. Teachers' experiences in analyzing their students' solutions together and learning how to interpret different levels of generalization gave the teachers confidence to recognize students' thinking. In addition, teachers showed an increase in their ability to overcome students' errors, use different strategies to encourage generalization, and use algebra-specific terminology in class discussions.

The purpose of Yılmaz Tığlı's study (2023) was to examine the middle school mathematics teachers' knowledge of students' algebraic thinking knowledge, to examine their interpretations of their students' algebra performance, and opinions about the reasons for the difficulties students experience in algebra. The participants of this research consisted of 5 MSMTs and 620 eighth-grade students from a public school in the Black Sea region of Turkey. Data collection tools included classroom observations and semi-structured interviews. MSMTs were able to analyze students' algebraic thinking in the algebra diagnostic test, but they could not explain the reasons for the students' difficulties. In addition, MSMTs were found not to give importance to covariational thinking. For example, one participant stated that no one had taught him covariational thinking in the past and that his students would learn this concept over time.

There are also studies conducted with preservice teachers regarding functional thinking. Öztürk's study (2021) examined the development of preservice elementary school teachers' content and pedagogical content knowledge regarding teaching early algebra including functional thinking. Participants, nine 3rd-year preservice teachers, participated in a 5-week intervention that was part of a methods course. This study used a qualitative design and conducted individual interviews with preservice teachers before and after the intervention using case discussions. The results of the study showed that preservice teachers may not have sufficient knowledge to generalize and represent functional relationships. It was observed that preservice teachers focused on a single variable, not two variables, in the pre-interviews. However, there was an increase in the number of preservice teachers who used covariational and correspondence thinking after the intervention. Preservice teachers were asked strategies for describing generalization and representing functional relationships in words. Two preservice teachers described the relationship at a functional condensed level in the pre-interviews. The number of preservice teachers who responded at this level increased to three in the post-interviews.

McAuliffe and Vermeulen (2018) examined preservice math teachers' knowledge of teaching functional thinking. The participants were 26 third-year preservice teachers enrolled in an early algebra course. Early algebra lessons were videotaped, and after each lesson, individual preservice teachers discussed and critiqued the course's teaching. Preservice teachers were asked to write reflections on the course. Participants were also administered a questionnaire regarding functional thinking. The findings of the study indicated that early algebra course improved preservice teachers' specialized content knowledge in using functions in different representations. In addition, one participant in this study had difficulty helping students generalize. It was observed that one of the participants had knowledge about different representations of functions and tried to make this knowledge understandable for the students. Since the participants did not have sufficient knowledge about the development of students' learning, their choice of course objectives, task designs, and the problems they asked the students were limited.

Some studies with preservice teachers have focused on preservice teachers' specialized content knowledge. For instance, Oliveira et al. (2021) investigated how prospective elementary Spanish and Portuguese mathematics teachers used functional thinking. The participants of this study were 94 Spanish and 70 Portuguese preservice elementary mathematics teachers. As a data collection tool, a questionnaire was developed by the researchers to improve preservice teachers' algebraic thinking. This paper analyzed the preservice teachers' responses to three tasks in the questionnaire. The results of this study showed that preservice teachers used different strategies to generalize functional relations, but most of these strategies were not successful. These preservice teachers had difficulty in understanding and relating different representations of functions. This showed that preservice teachers lacked important knowledge about functional thinking.

The study by Kabael and Barak (2019) conducted with preservice middle school mathematics teachers and investigated their functional thinking abilities. The participants of this study consisted of 10 preservice teachers who were enrolled in an elementary mathematics teaching program at a state university in Turkey and had completed their first two years of mathematics courses. In this qualitative study, data were collected through the clinical interview technique. The researchers prepared three problems requiring the use of functional relationships. The results of the study showed that only two of the preservice teachers were able to generalize the functional relationship in all three problems without guidance. Four preservice teachers identified the quantities in the problem and tried to understand the functional relationship, but they had difficulty and generalized with guidance. The other preservice teachers could not generalize. In addition, none of the participants tried to generalize the functional relationship from a graph; all participants tried to write algebraic equations.

Çatalkaya (2023) examined the ability of preservice elementary school mathematics teachers to use multiple representations in problems involving functional thinking. The participants of the study consisted of 105 third and fourth-year preservice teachers. Eight open-ended problems involving functional thinking were applied to these participants as a written test. The participants were asked to provide solutions involving as many different representations as they could for each problem. The results of the study showed that preservice teachers used algebraic representation the most. The least used representation was verbal representation. This situation showed that the participants' verbal representation knowledge and skills were insufficient. In addition, it was observed that preservice teachers had difficulty in switching between representations. Since functional thinking allows the use of different representations, it was concluded that the functional thinking skills of preservice teachers were not at a sufficient level.

To sum up, Wilkie (2014) stated that teachers were anxious about teaching the subject of pattern and algebra and had difficulty in giving appropriate examples for this subject. Girit (2016) and Kutluk (2011) concluded that teachers were inadequate in predicting students' errors. In addition, Kutluk (2011) and Yılmaz Tığlı (2023) stated that teachers also had difficulty in finding the reasons for students' errors, and in contrast, Pang and Sunwoo (2022) stated that teachers could find the reasons of students' errors and they mostly used realizing students' mistakes in overcoming strategies. When looking at studies conducted with preservice teachers, Çatalkaya (2023) and Oliveira et al. (2021) stated that preservice teachers had difficulty in switching between different representations. In addition, Öztürk (2021) stated that the number of preservice teachers using covariational and correspondence thinking increased after the intervention including functional thinking, while Yılmaz Tığlı (2023) stated that participants gave not much importance to focusing on covariance between variables. Girit (2016) stated that teachers and Kabael and Barak (2019) stated that preservice teachers had difficulty in pattern generalization, and McAuliffe and Vermeulen (2018) stated that preservice teachers had difficulty helping students generalize.

2.5 Functional Thinking Studies with Elementary and Middle School Students

This section will present studies conducted with elementary and middle school students related to functional thinking.

Blanton and Kaput (2004) examined how elementary grade students develop and explain functions. The data of this study were analyzed according to the forms of representation, the mathematical languages they used, the operations they performed, and how they used one or more variables. The findings of this study showed that students' functional thinking ability was greater than expected at an early age. As a result of this study, it was seen that as the grade level progressed, students needed fewer data values to indicate a functional relationship. In addition, the data showed that students at an early age began to think about the change of two variables. This data suggested that students' functional thinking should be encouraged as early as possible.

Arslandaş (2022) examined fifth-grade students' generalization and representation of functional relationships using a game-based learning activity tool. The participants of this study were four students studying at a middle school in Mardin, Turkey. A pre-test was given to participants that involved items that addressed functional thinking. Preliminary interviews were conducted to understand the answers given in the pre-test in more detail. Then, a game-based learning activity was conducted with the participants. Each player played the game individually under the supervision of the researcher. A game interview was conducted with the students. In this interview, six problems involving functional thinking were directed to the students. Finally, the same written test was applied as a post-test to examine the development of the students. As a result of the study, improvement was observed in the generalization and representation processes of functional relationships of all participants.

In Türkmen and Tanışlı's study (2019), the functional relations generalization levels of 3rd, 4th, and 5th-grade students were examined. The participants of the study consisted of 116 students studying in a school with a medium socio-economic level. During the data collection process, open-ended questions were asked to the participants. The results of this study showed that almost half of the third-grade students and more than half of the fourth and fifth-grade students are at a level that shows that they have functional thinking. It was observed that some students have covariational thinking by coordinating the changes in each variable with each other. Some students could not reach a higher level by focusing on the recursive pattern by focusing on only one variable. In addition, students were observed to have difficulty in generalizing relationships with the rule y=mx+n. About one-third of the third, fourth and fifth -grade students were found to ignore the constant term n when making the generalization with the rule y=mx+n.

Akın's (2020) study examined how functional thinking intervention affected the functional thinking of 5th-grade students in Ankara, Turkey. The experimental method was used in the study with 43 fifth-grade students. A Functional Thinking Test was applied to the control and experimental groups as a pre- and post-test. As a result of the study, half of the students in the experimental group and more than half of the control group who were asked to define patterns in the pre-test defined recursive pattern. Although the use of recursive patterns increased in the control group in the post-test, it decreased in the experimental group. It was seen that more than half of the students in the experimental group were able to define covariational and functional relationships in the post-test.

Stephens et al. (2017) investigated the characteristics of elementary school students' progress in generalizing and representing functional relationships. The participants in this study were approximately 100 elementary school students. This study investigated elementary school students' functional thinking with the instructional sequence. The instructional sequence was taught during the regular mathematics class hours. The results of the study showed that most students skipped covariational relationships (level 3) and moved to correspondence thinking (levels 4-10). This was attributed to the prioritization of correspondence thinking in the instructional sequence. In addition, it was observed that over time, some students made mathematically more complex definitions. It was observed that before the instructional sequence, students had difficulty structuring and representing functional relationships when writing function rules. It was observed that students' reasoning improved over the three years of early algebra courses and that the no response (level 0) response almost disappeared. It was also observed that more students were able to reach the functional-condensed level (level 9/10).

Panorkou et al. (2014) focused on students' early expression of covariation and correspondence (functional) relationships through instructional tasks. The participants in this study were 18 fifth-grade students in a North Carolina elementary school. A 6-day instructional experiment was conducted with these students. The students' expressions of covariation and functional relationships were examined. The task aimed to encourage students to distinguish between relationships. The results of this study showed that students were able to identify covariation and functional relationships. In addition, it was found that as students solved contextual problems, they were more likely to use covariation and functional relationship strategies to solve the problem.

Fonger et al. (2016) examined how six middle school students reasoned, their ability to symbolize the rule of a quadratic function, and how they made sense of the rule. A 15-day after-school teaching experiment was conducted with these students. The results of this study showed that students can begin to think of function rules as representations of covariation if covariational reasoning is encouraged.

Kulaç (2023) examined the functional thinking development of 7th grade students in her study. The focus of this study was on the hypothetical learning trajectory prepared on the basis of growing shape patterns. The participants of this study consisted of twenty-one 7th grade students studying at a middle school in Adana. The functional thinking test was applied as a pre-test and post-test as a data collection tool. During the teaching experiment, individual and group worksheets, observation notes and teacher diary were other data collection tools. The study's findings showed that the students showed development in their functional thinking skills between the pre-test and post-test. This showed that the hypothetical learning trajectory prepared with the theme of growing shape patterns had a positive effect on functional thinking.

The study of Akkaya and Durmuş (2006) aimed to determine the misconceptions of students in grades 6-8 about algebra. The participants of the study consisted of 280 students from 2 randomly selected classes, each from 6^{th}, 7^{th}, and 8^{th} grades, selected from three middle schools. A 30-question multiple-choice 'Algebra Test' was prepared to determine students' misconceptions. The first ten questions in the algebra test were administered to 6^{th} grade students, the first 20 questions to 7^{th} grade students, and all 30 questions to 8^{th} grade students. The findings of the study showed that students had difficulty perceiving the letters. It was observed that 54% of the 7^{th} and 8^{th}-grade students did not consider the order of operations when performing operations with algebraic expressions. Instead, students preferred to start with an operation that was easy for them. 46% of the students thought that the letters in an algebraic expression indicated the position in alphabetical order. 27% of the students considered an algebraic expression such as ab as a two-digit number.

The aim of Şahin and Soylu (2011) was to examine students' errors and misconceptions about the concept of variable. The participants of the study consisted of fifty 7^{th}-grade students in a middle school. A test consisting of eight open-ended questions was applied as a data collection tool. As a result of the analysis, nine different misconceptions were detected. These were overlooking the variables, processing the different units under the same unit, focusing on x, y variables, not being able to find the connection between the verbal expression and variables, reducing variables to constants, attributing digits to the variable in multiplication, confusing the x unknown with the multiplication sign and not using parenthesis.

When looking at the studies conducted with students, it has been seen that there were some interventions for students to develop think functionally. Panorkou (2014) stated that giving students context-related problems, Fonger et al. (2016) stated encouraging covariational reasoning, Kulaç (2013) stated hypothetical learning trajectory, Arslandaş (2022) stated that a game-based learning activity tool and Akın

(2020) stated that a functional thinking intervention was successful for helping students develop functional thinking. In addition, Stephens et al. (2017) stated that the instructional sequence increased students' reasoning and increased the number of students accessing functional-condensed answers. Akkaya and Durmuş (2006) and Şahin and Soylu (2011) stated that middle school students might have several difficulties such as thinking the letters as a digit.

2.6 Functional Thinking in the National Grades 5-8 Mathematics Curriculum

This section will present objectives related to functional thinking at different grade levels in the 2018 middle school mathematics curriculum. The terms function or functional thinking were not explicitly mentioned in the curriculum. The relevant objectives in the Grade 5-8 National Curriculum provided by the Ministry of National Curriculum (MoNE) published in 2018 are shown in Table 2.1.

Table 2.1. Objectives addressing functional thinking in Grades 5-8 (MoNE, 2018)

Grades	**Numbering in the curriculum**	**Objectives**
5th grade	M.5.1.1.3.	The student creates the desired steps of the number and shape patterns given the rule.
7th grade	M.7.2.1.3.	The student expresses the rule of number patterns with letters and finds the desired term of the pattern whose rule is expressed with letters.
8th grade	M.8.2.2.3.	The student expresses how one of two variables with a linear relationship between them changes depending on the other using tables and equations.

Table 2.1 (continued)

8th grade	M.8.2.2.4.	The student draws the graph of linear equations.
8th grade	M.8.2.2.5.	The student creates and interprets equations, tables, and graphs of real-life situations that involve linear relationships.
8th grade	M.8.2.2.6.	The student explains the slope of the line with models and relates linear equations and their graphs to the slope.

In the objective M.5.1.1.3., it is aimed for the student to find the desired steps in number and shape patterns, limited to patterns with a constant difference. In the objective M.7.2.1.3., it is aimed for the student to express the rule of patterns with a constant difference between the steps with variables and finds the desired term. In addition, the importance of using variables and understanding the necessity are emphasized in the subheadings. In the objective M.8.2.2.3., it is aimed for the student to express how one of two variables with a linear relationship change depending on the other. The dependent and independent variables are also emphasized. In the objective M.8.2.2.4., it is aimed for the student to transform linear equations into graphs. In the objective M.8.2.2.5., it is aimed for the student to create and interpret equations, tables and graphs, including the coordinate system, in real-life situations containing linear relationships. In the objective M.8.2.2.6., it is aimed that the student understands the slope of a line, the sign and magnitude of the slope, and relates linear graphs with the slope.

CHAPTER 3

METHODOLOGY

The aim of this study was to investigate middle school mathematics teachers' pedagogical content knowledge in the context of functional thinking. This part includes the research question of the study, the design of the study, the characteristics of the participants, the data collection tool, data analysis technique, the trustworthiness of the study and ethical issues.

3.1 Research Question

The research question of the study is as follows:

What is the pedagogical content knowledge of middle school mathematics teachers in the context of functional thinking?

- What is the MSMTs' knowledge of content and students in the context of functional thinking?
- What is the MSMTs' knowledge of content and teaching in the context of functional thinking?

3.2 Design of the study

This study aimed to examine pedagogical content knowledge, specifically knowledge of content and students, and knowledge of content and teaching of middle school mathematics teachers in the context of functional thinking. This research process was performed to understand teachers' knowledge of algebra in the context of functional thinking. The design of this research was qualitative design. Fraenkel et al. (2011) defined qualitative research as “the study of the quality of situations, relationships, or activities” (p. 426). One of the qualitative research methods is a basic qualitative study**.** Merriam (2009) stated, “Qualitative researchers conducting a basic qualitative study would be interested in (1) how people interpret their experiences, (2) how they construct their worlds, and (3) what meaning they attribute to their experiences’’ (p. 23). Basic qualitative research aims to examine how people make sense of their experiences and lives (Merriam, 2009).

In this study, the MSMTs were expected to make sense of the correct or incorrect answers from the students, to make sense of the possible reasons for incorrect answers, and to formulate strategies through their experiences on how to overcome incorrect answers. In this study, the pedagogical content knowledge in the context of functional thinking was examined through interviews with five mathematics teachers working in a public school.

3.3 Participants

The participants of this study were five mathematics teachers working in a public school. The convenience sampling method was used in this study. Participants were selected from the same school as the researcher who were willing to participate. The MSMTs participating in this study consisted of teachers teaching 7^{th} and 8^{th} grades. Four of the participants were female and one was male. Details of the participants in the research are shown in Table 3.1.

Table 3.1. The characteristics of participants

	Ms. Nisa	Ms. Lara	Ms. Buse	Ms. Azra	Mr. Eren
Age	25	28	25	30	32
Level of Education	Bachelor's degree	Bachelor's degree	Bachelor's degree	Master's degree	Bachelor's Degree
Graduated Program	Elementary mathematics education	Elementary mathematics education	Elementary mathematics education	Elementary mathematics education	Mathematics education
Teaching Experience	2 years	4 years	1 year	4 years	1 year

The first participant, Ms. Nisa, was 25 years old. She had a bachelor's degree. She graduated from the elementary mathematics education program. Her teaching experience was 2 years. Although she taught mathematics to all grade levels, she mostly taught 6^{th} and 7^{th} grade levels. The second participant, Ms. Lara, was 28 years old. She had a bachelor's degree. She graduated from the elementary mathematics education program. Her teaching experience was 4 years. She taught mathematics to all grade levels. She mostly taught 8^{th}-grade students. The third participant, Ms. Buse, was 25 years old. She had a bachelor's degree. She graduated from the elementary mathematics education program. Her teaching experience was one year. She taught mathematics to 7^{th}-grade students. The fourth participant, Ms. Azra, was 30 years old. She had a master's degree in mathematics education. She graduated from the elementary mathematics education program. Her teaching experience was 4 years. She taught mathematics to all grade levels but mostly taught 8^{th}-grade students. The last participant, Mr. Eren, was 32 years old. He had a bachelor's degree. He graduated from the mathematics education program. His teaching experience was one year. He taught 5^{th} and 7^{th}-grade students.

3.4 Context of the Study

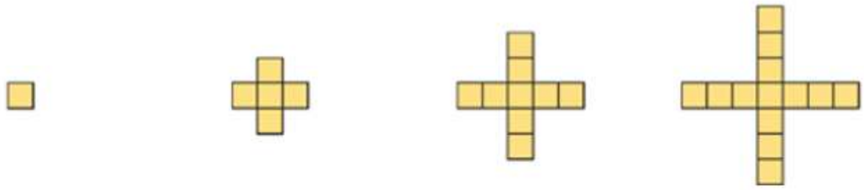
The data collected to answer the research questions was collected from a public school. There were ninety-two teachers in the public school, which had nearly two thousand students. Thirteen of these teachers were middle school mathematics teachers. The ages of teachers were generally between 25 and 35. The success level of this school, located in a district of a metropolitan city, was low. The school was in an area that received immigration and had a low socioeconomic level.

3.5 Data Collection Tool and Procedure

The data of this study was collected through individual semi-structured interviews. The purpose of collecting this data was to examine the pedagogical content knowledge of MSMTs in the context of functional thinking, specifically, the knowledge of content and teaching and knowledge of content and students. Individual interviews lasted approximately 40 minutes and were conducted face to face.

The interview protocol consisted of two tasks: Task 1 and Task 2. In each task, there were questions for MSMTs regarding knowledge of content and teaching and knowledge of content and students. The researcher developed Task 1 using the context in the textbook 7th-grade objective-focused activity book (Ceylan & Alptekin, 2020). Task 1 included three questions (See Figure 3.1).

A geometric pattern is given below.



1) How many square units will there be in step 5?

2) How many unit squares will there be in the 100th step?

3) How do you express the number of square units in any step? Express the rule expressing this relationship with words and variables.

Figure 3.1. Task 1

Table 3.2 shows the interview questions asked to the MSMTs about Task 1. There were five questions about Task 1. The MSMTs were asked about the purpose of the tasks related to Task 1, the possible correct answers they expected from the students, the possible incorrect answers they expected from the students, what they thought the purpose of using tables would be and what they thought the effect of using tables would be.

Table 3.2. Interview questions about Task 1

a) What could be the purpose of this task? Can you explain?

Table 3.2 (continued)

b)	What could be the possible correct solutions that may come from students?
c)	What could be the possible incorrect solutions that may come from students?
d)	What could be the reasons for students' mistakes? How can you overcome these mistakes that may come from your students?

In Task 1, MSMTs were shown a table in which the teacher asked the student to create a table to find the 15th step. Figure 3.2 shows the table question.

In Task 1, the teacher asked the student to create a table to find the number of unit squares in the 15th step.

The number of steps	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
The number of unit	1	5	9	13	17	21	25	29	33	37	41	45	49	53	57

Figure 3.2. The table question

MSMTs were asked two questions about this table. Table 3.3 shows the questions about this table question.

Table 3.3 Interview questions about table question

What do you think could be the purpose of the teacher asking the student to make a table?
What do you think would be the effect of using tables?

Task 2 was adapted from the task named The String Task by Isler et al. (2014/2015). Task 2 included three questions (See Figure 3.3).

a) This rope was cut as above. How many pieces of rope do you have?

b) Make 2 cuts and find out how many pieces the rope consists of. Repeat this for 3, 4, and 5 cuts, always remembering to use a new piece of string.

c) Express the number of pieces formed in any cutting using words and variables.

Figure 3.3. Task 2

Table 3.4 shows the interview questions asked to the MSMTs about Task 2. The MSMTs were asked about the purpose of Task 2, the correct answers they expected from the students, the incorrect answers they expected from the students, how the students thought about the three student solutions and what could be done with these students in the next step.

Table 3.4. Interview questions about Task 2

a) What could be the purpose of this task? Can you explain?
b) What could be the possible correct solutions that may come from students?
c) What could be the possible incorrect solutions that may come from students?
d1) The first student's solution: "Since y is the total number of parts and x is the number of cuts increasing by 2, we can express it as y = 2x." How do you think this student might have thought? What do you do next with this student?

Table 3.4. (continued)

d2) The second student's solution: "As the number of cuts increases by one, the number of parts formed increases by 2." How do you think this student might have thought? What do you do next with this student?
d3) The third student's solution: "In this question, since there are 2 pieces in each segment and there is 1 node at the beginning, we reach the number of segments when we multiply the number of segments by 2 and add 1. We can express the relationship as y= 2x+1, where y is the total number of parts and x is the number of cuts." How do you think this student might have thought? What do you do next with this student?

MSMTs were given an objective and asked about a lesson plan description (See Table 3.5). The chosen objective: "M.8.2.2.3. The student expresses in tables and equations how one of two variables that have a linear relationship between them changes depending on the other." (MoNE, 2018, p. 73). The reason for choosing M.8.2.2.3 as the objective was that the objective focuses on the relationship between two variables. In addition, this objective was chosen to reveal their functional thinking from a more holistic perspective. Table 3.5 shows the question about lesson plan description.

Table 3.5. The question about lesson plan description

3) You will describe a lesson for the 8th-grade objective below. What kind of a lesson plan would you prepare? What would you consider when preparing a lesson plan? What examples and materials would you use?

Ideas were taken from Yılmaz Tığlı's (2023) and Öztürk (2021) interview questions regarding the interview questions to be asked about the tasks. In the interview protocol, the questions asked to the MSMTs were categorized as knowledge of content and students and knowledge of content and teaching. Table 3.6 shows the categories of interview questions.

Table 3.6. The categorization of interview questions

Questions	Categories
What could be the purpose of this task? Can you explain? (1a, 2a)	Knowledge of content and teaching
What could be the possible correct solutions that may come from students? (1b, 2b)	Knowledge of content and students
What could be the possible incorrect solutions that may come from students? (1c, 2c)	Knowledge of content and students
What could be the reasons for students' mistakes? (1d/1)	Knowledge of content and students
How can you overcome these mistakes that may come from your students? (1d/2)	Knowledge of content and teaching
What do you think could be the purpose of the teacher asking the student to make a table? What do you think would be the effect of using tables? (1e)	Knowledge of content and teaching
How do you think this student might have thought? (2d)	Knowledge of content and students
What do you do next with this student? (2d)	Knowledge of content and teaching

Table 3.6. (continued)

You will describe a lesson for the 8th grade objective below. What kind of a lesson plan would you prepare? What would you consider when preparing a lesson plan? What examples and materials would you use? (3)	Knowledge of content and teaching
What do you think could be the purpose of the teacher asking the student to make a table? What do you think would be the effect of using tables? (1e)	Knowledge of content and teaching

Also, MSMTs were shown three sample student solutions related to Task 2 in the protocol and were asked how the students might have thought and what they could do next with these students. The first student's response was: "Since y is the total number of parts and x is the number of cuts by adding 2, we can express it as y = 2x." There is evidence in the literature that students tend to ignore the constant part in the rule (see Türkmen & Tanışlı, 2019); this response was added for that purpose. Table 3.7 shows the categorization of two students' solutions based on Stephens et al. (2017).

Table 3.7. The categorization of two students' solutions

Student solutions	**Level**
The second student's response: "As the number of cuts increases by one, the number of parts formed increases by 2."	L3: Covariation Thinking

Table 3.7. (continued)

The third student's response: "Since there are 2 pieces in each segment and there is 1 node at the beginning, we reach the number of segments when we multiply the number of segments by 2 and add 1. We can express the relationship as y= 2x+1, where y is the total number of parts and x is the number of cuts."	L9/10: Functional-Condensed

3.6 Data Analysis

This study aimed to examine middle school mathematics teachers' knowledge in the context of functional thinking. Since the data of this study were the verbal answers given by the participants to the interview questions, it was a qualitative study. Merriam (2009) defined that "content analysis is a systematic procedure for describing the content of communications" (p. 152). Content analysis was used to analyze the interviews conducted in this study. First, audio-recorded individual interviews were transcribed. Secondly, data analysis was carried out in two sections: knowledge of content and students and knowledge of content and teaching, which are categories of pedagogical content knowledge. Codes were created for MSMTs' responses to the questions. For instance, MSMTs were asked about the possible correct solutions of students in each task. Table 3.4 gives an example of the categories that emerged regarding this question. In this question, emphasizing covariational thinking, finding the general term of the pattern, drawing and relating math with daily life codes emerged based on the MSMTs' responses.

Table 3.8. An example of a coding table

Codes	Definitions	Participants
Emphasizing covariational thinking	Responses involving covariational thinking	Ms. Azra
Finding the general term of the pattern	Responses expressing finding the rule and general term of the pattern	Mr. Eren, Ms. Buse
Drawing	Responses solving the problem by drawing visuals	Ms. Lara
Relating with context	Responses solving the problem by relating it to the context	Ms. Nisa

Each table included codes and their definitions, which will be detailed in the findings.

3.7 The Trustworthiness of The Study

The reliability and validity of the study are important in qualitative studies, which include data collection and analysis, presentation, and interpretation of results (Merriam, 2009).

Credibility refers to internal validity. Internal validity is whether the findings of the research coincide with reality (Merriam, 2009). Methods to ensure internal validity are triangulation, member check, adequate engagement in data collection, and peer examination (Merriam, 2009). In this study, the peer examination was conducted. Peer debriefing is when a person with knowledge about the research topic examines the research from various dimensions (Creswell, 2003). Interrater agreement was done while coding answers. The interrater agreement is the investigation of how consistent the answers of two independent raters are (Gisev et al., 2013). Consistency or dependability is related to reliability. A mathematics education graduate student working on the same subject was asked to code the randomly selected 20% of the data, which was one interview. The agreement was over 90%, and the discussion was conducted until an agreement was reached and changes were reflected in the analysis.

Transferability is related to external validity. Transferability is the ability to generalize the results of the study to situations with similar participants and environments (Streubert & Carpenter, 2011). It is not intended to make generalizations in qualitative studies, but expressing the opinions of the participants in the study in detail allows other researchers to benefit from the results of the study. Therefore, to increase transferability in qualitative research, the way the participants were selected, the environment, and the characteristics of the participants should be clearly stated (Sharts-Hopko, 2002). For this reason, in this study, the selection of the sample, methodology, data collection process, and data analysis process were aimed to be explained in detail.

While developing the interview questions, expert opinion was taken from a mathematics education researcher working on teachers' mathematical knowledge for teaching in algebra. The expert was asked for her opinions on the suitability of the problems to measure the mathematical knowledge for teaching and their understandability in terms of language and visual appropriateness. In line with the expert opinion, necessary adjustments were made to the interview protocol. The expert stated that the language of some interview questions was inappropriate; these questions were corrected. The incorrectly categorized question from the categories of pedagogical content knowledge was corrected. Then, a pilot study was conducted. A mathematics teacher working in a different public school volunteered for the pilot interview. In the pilot study, the MSMT was also asked "What is functional thinking in your opinion?" and ''How would you prepare a question to encourage students to think functionally?'' but the MSMT could not answer the questions, saying that she did not know what functional thinking was. Therefore, these two questions were removed from the interview questions. This way, the interview protocol was finalized.

3.8 Ethical Issues

This research was conducted by paying attention to ethical issues. In the research, it was ensured that the participants were not harmed physically or psychologically. Before conducting the research, permission was obtained from the Human Research Ethics Committee (see Appendix B). Permission was also obtained from the Ministry of Education to interview middle school math teachers in the public school (see Appendix C). After permissions were obtained, five middle school mathematics teachers who were asked to participate in the study were invited to the interview and signed a voluntary participation form. While collecting data, interviews were audio-recorded with the consent of the participants. The researcher reminded participants that they were free not to respond to the questions during the interview and that the answers would be kept confidential. The researcher was careful not to be judgmental during the interview. In addition, participants were given pseudonyms to hide their identity in the findings of the study.

3.9 Limitations

The first limitation of the study is that the pedagogical content knowledge of the MSMTs was examined with individual semi-structured interview questions. Therefore, the study is limited with the questions asked and MSMTs' responses to the interview questions. The second limitation of this study was that the data was collected from five middle school math teachers in a public school in Istanbul. Participation from different schools or schools in different cities could have different results.

CHAPTER 4

FINDINGS

This chapter includes two main parts to address the research question. The first part includes findings related to middle school mathematics teachers' knowledge of content and teaching. The second part includes findings related to their knowledge of content and students.

4.1 Middle School Mathematics Teachers' Responses Regarding Knowledge of Content and Teaching

This part includes findings related to the questions which were categorized under the knowledge of content and teaching because they aimed to address teachers' knowledge of content and teaching.

4.1.1 Teachers' responses regarding the purpose of questions involving functional thinking

One of the aims of this study was to get teachers' opinions about the purpose of the tasks that address functional thinking. The question 'What could be the purpose of this task? Can you explain?' was posed to teachers for both tasks. The codes regarding teachers' responses are presented in Table 4.1 for Task 1. It is important to note that, across the coding, a participant's response was coded in more than one category, where relevant.

Table 4.1. Teachers' responses regarding the purpose of Task 1

Codes	Definition	Participants
Finding the general term of the pattern	Responses expressing finding the rule and general term of the pattern	All MSMTs
Emphasizing the relationships	Responses emphasizing the relationship between the variables	Ms. Lara
Emphasizing covariational thinking	Responses involving covariational thinking	Ms. Azra

In Task 1, they stated that the purpose of Task 1 was to find the general term of the pattern, to emphasize the relationship, and to emphasize covariational thinking. All MSMTs stated that the purpose of this task was to find the general term. For example, Mr. Eren said, "The aim of this task is teaching the child the concept of pattern and to enable them to formulate it in this way.'' Additionally, one MSMT stated that the purpose was to emphasize the relationship. Ms. Lara stated:

> It is necessary to relate the number of steps in the pattern with the number of squares in that step. First, students should find the relationship between numbers. In my opinion, students should find both a numerical relationship and a visual relationship between the shape and pattern.

In addition, one MSMT's response was interpreted to emphasize covariational thinking. Ms. Azra stated, "There is always a fixed square in the center, which allows him to see it and how much of an increase there is when each step changes. I think this is important.''

MSMTs were asked the purpose of Task 2. The codes regarding teachers' responses are presented in Table 4.2 for Task 2.

Table 4.2. Teachers' responses regarding the purpose of the task 2

Code	Definition	Participants
Finding the general term of the pattern	Responses expressing finding the rule and general term of the pattern	All MSMTs
Emphasizing the relationship	Responses emphasizing the relationship between the variables	Ms. Buse
Relating math with daily life	Responses relating math with daily life	Ms. Buse

In Task 2, MSMTs' responses regarding the purpose of the task were to find the general term, emphasize the relationship, and relate math with daily life. All MSMTs stated that the purpose of the task was to find the general term of the pattern. For example, Ms. Nisa stated:

> Now when I cut one, I have 3 pieces of string in the first cut. When I cut the second piece, I had three pieces of string, I will probably have 5 pieces of string. Each time I cut, the number of strings will increase by two. From there you will reach a generalization.

One MSMT stated that the purpose of this question was to emphasize the relationship between the number of cuts and the number of pieces; Ms. Buse stated, "The purpose of this question is to make people realize the relationship between the number of cuts and the number of parts."

One MSMT, Ms. Buse, also said, "The aim of this task was to relate math to daily life."

4.1.2. MSMTs' Responses on Using Tables in Task 1

MSMTs were shown in Figure 4.1 the student response regarding Task 1, in which the student was asked to create a table.

In Task 1, the teacher asked the student to create a table to find the number of unit squares in the 15th step.

The number of steps	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
The number of unit	1	5	9	13	17	21	25	29	33	37	41	45	49	53	57

Figure 4.1. Table question

Two questions were asked the MSMTs regarding the student response. The first question was, 'What do you think could be the purpose of the teacher asking the student to make a table? Can you explain?'. The codes for the first question are presented in Table 4.3.

Table 4.3. The codes about the purpose of using a table

Codes	**Definitions**	**Participants**
Emphasizing the relationship	Responses emphasizing the relationship between the variables	Ms. Lara
Facilitating finding the general term	Responses stating that the use of tables facilitates finding the rule and general term of the pattern	Ms. Buse, Ms. Azra, Mr. Eren

Table 4.3 (continued)

Showing students that using a table is a waste of time	Responses stating that using tables was a waste of time	Ms. Nisa

One MSMT emphasized the relationship. Ms. Lara said, “The teacher may have wanted such a table so that the students could better see the relationship between the number of unit squares and the number of steps.’’

Three MSMTs stated that the purpose of this question was to facilitate finding the general term. For example, Ms. Buse stated, “The teacher wanted such a table so that the students could more easily determine the rule of the pattern using this table when the student see the numerical values.”

Lastly, one MSMT said that showing students that using this table was a waste of time. Ms. Nisa stated, “Teachers’ aim may be that she/he wants to show that it will take a lot of time. 15^{th} step is not a very high step for us.”

The second question was, “What do you think would be the effect of using tables?” The codes for the second question are presented in Table 4.4.

Table 4.4. The codes about the effect of using a table

Codes	**Definition**	**Participants**
Useful in seeing the relationship	Responses emphasizing using the table is useful in seeing the relationship between the variables	Ms. Azra
Being systematic	Responses stating that the use of tables was systematic	Ms. Nisa

Table 4.4 (continued)

Making a transition from abstract to concrete	Responses stating that the use of tables enables the transition from abstract to concrete	Mr. Eren
Facilitating finding the general term	Responses stating that the use of tables facilitates finding the rule and general term of the pattern	Ms. Buse, Ms. Lara

One MSMT, Ms. Azra, said, "Using a table would be very useful in seeing the relationship between the number of steps and the number of unit squares."

One MSMT, Ms. Nisa, stated that the use of tables was systematic:

> It is systematic and orderly. Normally, students start from the top. He writes downwards, and, in the options, for example, he marks the answer directly. For example, he doesn't write down what the step is.…It becomes more organized, lowering the risk of making mistakes.

One MSMT stated that the effect of using a table was making a transition from abstract to concrete. Mr. Eren stated:

> In the first, because the student cannot think abstractly. First, he/she needs to see it as a concrete visual, then it will be much more comfortable for him/her to determine the rule. In the following questions, he/she has already thought abstractly in his/her mind without seeing this table, he/she will already determine his/her own rule.

Two MSMTs stated that the effect of using a table could be to facilitate finding the general term. For example, Ms. Lara stated that she thought of the table as a machine and that it could be useful in finding the general term:

The general term is a machine. I give two as an input to that general term. How can I get it as 5, or if I give three, how can I get it as 9? The table is very useful for this.

4.1.3. Lesson Plan Descriptions Created by MSMTs for the Given Objective

In this section, the findings regarding the question where MSMTs were asked to describe a lesson appropriate to the given objective will be provided. The objective was: "M.8.2.2.3. The student expresses in tables and equations how one of two variables that have a linear relationship between them changes depending on the other." (MoNE, 2018, p. 73). Table 4.5 shows the categories of the responses given.

Table 4.5. The categories based on the MSMTs' lesson description

	Context-based Task	Figural Pattern Task	Using a Graph	Using a Table	Using Materials	Using an Animation	Continuous	Discrete
Ms. Nisa	+		+	+				+
Ms. Lara		+		+				+
Ms. Buse	+			+		+	+	
Ms. Azra	+		+	+			+	
Mr. Eren	+				+			+

As seen in Table 4.4, most MSMTs used a context-based task. One MSMT used an example involving a figural pattern task. Also, most suggested using representations such as tables and graphs. One MSMT stated that she can use an animation involving the acceleration of a car. Three MSMTs gave discrete examples while two MSMTs gave continuous examples.

Their responses will be summarized next.

Ms. Nisa formed a context-based task. She stated that the example of the change in the holiday allowance given by a grandfather to his grandchild over the years would be suitable for this objective. Grandfather increases his holiday allowance by 4 TL every year. Ms. Nisa used an example related to real life. Her example was discrete. She stated that she would have students represent the change in the holiday allowance over the years, initially 10 TL, first with a table and then a graph.

Ms. Lara stated that she would process the gain with a task using a figural pattern. It was about the change in the number of circles over time. She asked the students to create a table for this example and find the general term:

> If we draw a circle, there will be 1 circle in the first step, 2 circles in the second step, and 3 circles in the third step. In the fifth step, I ask them to draw the relationship between the number of circles drawn and the number of steps in a table. Then, if he expresses this with a variable, how can he express it with a variable as x? He needs to establish an equation in the form of y=x.

Ms. Buse and Ms. Azra expressed the same context-based task. They mentioned changing the path of a constant-speed vehicle over time would be suitable for this objective. Ms. Azra stated that she cares about the use of graphs in this example:

> So initially, the vehicle is at zero point and it has spent zero minutes and there is a constant emphasis on constant speed. For example, this vehicle will travel at 60 km per hour. I would ask the student to express it verbally, such as traveling 60 km in one hour or 120 km in 2 hours. Then, I would ask it to create a table. I also attach great importance to the graph in this regard. So, the graph showing the linearity of that linear relationship is important. I would definitely use the graph, too… Finally, after using the graph and table, I would like to create the equation of this.

Mr. Eren chose a context-based task that established a relationship between the number of double-yolk eggs and the number of yolks. He stated that he would come to class with a double-yolk egg as a material:

> I walk into class with double yolk eggs. When we broke one egg, 2 yolks came out, and when we broke the second egg, there were 4 yolks in total. When we broke the third one, there were 6 egg yolks… Students will wonder why these eggs came. After creating curiosity, we select 1 or 2 students and start having them break the eggs, and we start collecting them in a transparent container. The broken eggs are put aside, and the total number of egg yolks inside is calculated and written down in a table in the classroom. If we go from the sum of 2 when an egg is broken, of course, it shows the total; we can write it as if there are 4 in the second egg and 6 in the third, and so on, we write it down, and it becomes a real-life example.

4.2. Middle School Mathematics Teachers' Responses Regarding Knowledge of Content and Students

This part includes findings related to the questions which were categorized under the knowledge of content and students.

4.2.1. MSMTs' Responses Regarding Possible Correct Solutions from Students

In Task 1, there were three questions, which asked the number of unit squares for the 5th step, the 100th step, and for any step. I asked the teachers to explain the possible correct answers they expected from the students regarding these three questions. For the three questions, MSMTs stated their expected correct student strategies.

The first question was, “How many unit squares are formed in the 5th step?” MSMTs’ responses were coded into three categories: rhythmic counting, drawing and finding a general term. The expected correct solution strategies of students for first question are listed in Table 4.6.

Table 4.6. MSMTs’ expected correct solution strategies of students for the first question

Codes	**Definitions**	**Participants**
Rhythmic counting	Responses using rhythmic counting in solving questions	Ms. Lara, Ms. Nisa
Drawing	Responses solving the problem by drawing visuals	Ms. Azra
Finding the general term of the pattern	Responses expressing finding the rule and general term of the pattern	Mr. Eren, Ms. Buse

Two MSMTs stated that students find solutions by doing rhythmic counting. For example, Ms. Lara stated, “'It can be found by rhythmic counting. 1,5,9,13,17. The answer is 17.”

One MSMT, Ms. Azra, stated that the student could reach the solution by drawing:

> He draws this. So, he puts the unit square in the center again. Then, in the fifth step, he draws four-unit squares on the right, left, and up and down. After that, she finds the fifth step through drawing. In other words, if a student wants to take the fifth step, he/she does not calculate the general term in this question. He definitely draws.

Two MSMTs stated that students find solutions by finding the general term. For example, Mr. Eren said: "When we find the general term as 4n-3 and give n a value of 5, 20-3=17."

The second question was, "How many unit squares are formed in the 100th step?" The expected correct solution strategies of students for second question are listed in Table 4.7.

Table 4.7. MSMTs' expected correct solution strategies of students for the second question

Codes	Definitions	Participants
Finding the general term of the pattern	Responses expressing finding the rule and general term of the pattern	All MSMTs
Relating with context	Responses solving the problem by relating it to the context	Ms. Azra

In this question, all MSMTs stated that students would find the answer by finding the general term. For example, Mr. Eren stated, "The student finds the general term 4n-3. He writes 100 instead of n. He reaches the result 400-3=397."

One MSMT, Ms. Azra, stated that the students can solve this problem by relating it to the context:

> Students can think like this: there are squares added horizontally as 1, 3, 5, 7, 9. 1, 2, 3 squares are added from above and below. For example, in the 4th step, how many squares were added from above and below, 3 squares were added from above and 3 squares were added from above. Then, for example, 99 squares are added in the 100th step. 99 times 2 finds the number of squares on the horizontal. In the 4th step, for example, there are 7 units, one in the center and three on the left and right. In the 100th step, 99 numbers come from

the bottom and top, again 99 times 2 + 1, so 4 numbers can be found by saying 99 + 1.

The third question was, “How do you express the number of unit squares in any step? Write the rule expressing this relationship with words and variables.” The expected correct solution strategies of students for third question are listed in Table 4.8.

Table 4.8. MSMTs’ expected correct solution strategies of students for the third question

Codes	**Definitions**	**Participants**
Finding the general term of the pattern	Responses expressing finding the rule and general term of the pattern	All MSMTs

In this question, all MSMTs stated that the student would reach the solution of this problem by finding the general term.

For example, Ms. Nisa stated, “It increases by four, we say 4n, and we substitute one for n again to find the first step. To get 1 from 4 times n, we subtract 3. We reach the formula 4n-3.”

In Task 2, there were also three questions. I asked the teachers to explain the possible correct answers they expected from the students regarding these three questions.

The first question was, “This rope has been cut as above. How many pieces of rope do you have?” The categories are provided in Table 4.9 for the first question.

Table 4.9. The MSMTs’ responses regarding possible correct solution strategies of students for the first question

Codes	**Definitions**	**Participants**
Finding by counting	Responses reaching a solution by counting	All MSMTs

In this question, all MSMTs stated that the students would solve the question by counting. For instance, Ms. Azra said: "Student says 3 pieces by counting."

The second question was, "Make 2 cuts and find out how many pieces the rope consists of. Repeat this for 3, 4, and 5 cuts, always remembering to use a new piece of string." The categories are provided in Table 4.10 for second question.

Table 4.10. The MSMTs' responses regarding possible correct solution of students for the second question

Codes	**Definitions**	**Participants**
Observing the change in a single variable	Responses expressing observing the change in a single variable	Ms. Nisa, Ms. Buse
Emphasizing covariational thinking	Responses involving covariational thinking	Mr. Eren
Drawing	Responses solving the problem by drawing visuals	Ms. Lara, Ms. Azra

In this question, MSMTs stated that students could answer the question by observing the change in a single variable, focusing on covariational thinking and drawing.

Two MSMTs stated that students can solve the question by observing the change in a single variable, that is, two pieces of rope come in each step. For instance, Ms. Nisa:

> Again, when I cut it, I have 3 pieces of rope, but at the beginning I had 2 pieces of rope, I have 5 pieces. So, at the end of the first step, I had 2 more pieces, I have 5 pieces. In total, I explained it a little complicated here. When I cut it again, I will do the same thing again. 4 of my pieces are on the edge, it was 5 pieces. I have a total of 4 pieces on the edge. When I cut that big rope

> again, I have 2 more pieces. I have 6 pieces. I also have a knotted rope, making 7 pieces. In the first step I had 3 strings, in the second step I had 5 strings. In the third step I have 7 strings. This goes on like this, in the fourth step I have 9 strings, in the fifth step I have 11 strings. I mean, I think he generalizes it in the first, second, third step after he sees that it increases two by two.

One MSMT, Mr. Eren, focused on covariational thinking. He stated, "if we cut ropes for 3, 4, 5 cuts, it increases by 2 each time. Then it will continue as 7, 9, 11."

Two MSMTs said that students can find solution by drawing. For example, Ms. Lara:

> In this one, we had separated 3 pieces at first. Then, when we cut and separated another one, it can be counted as 1 2 3 4 and one piece left below, it can be counted as 5. Similarly, when I cut 3 pieces, it can be counted as 1 2 3 4 5 6 7, and one piece left below, it can be counted as 7. ... in the fourth step, 2 more is 9, and in the fifth cut, it can be easily found as 11 pieces. You can do these by drawing.

The third question was, "Express the number of pieces formed in any segment using words and variables." The categories are provided in Table 4.11 for third question.

Table 4.11. The MSMTs' responses regarding possible correct solution of students for the third question

Codes	**Definitions**	**Participants**
Emphasizing covariational thinking	Responses involving covariational thinking	Ms. Azra
Finding the general term of the pattern	Responses expressing finding the rule and general term of the pattern	Mr. Eren, Ms. Buse

Table 4.11. (continued)

Drawing	Responses solving the problem by drawing visuals	Ms. Lara
Relating with context	Responses solving the problem by relating it to the context	Ms. Nisa

In this question, MSMTs stated that students can answer the question by focusing covariational thinking, finding the general term, drawing, and relating with the context.

One MSMT stated that students could solve this question by observing how two variables change together. Ms. Azra emphasized the 2 pieces increase in each cut. She focused on covariational thinking:

> In every cut, if we put it in words, it increases by 2 parts in each step. Because, let me do it in red, a piece is added from here and here, so it finds the amount of increase as 2n of our variable term. Then, how many pieces of rope did we have in the first step? In the 1^{st} step, there are 3 pieces, then 2n+1, I wrote 1 instead of n and tried it. When I write 2 to n, 5 comes out. I can express it as 2n+1. This is the correct answer we expect.

Two MSMTs stated that students by finding the general term. Mr. Eren stated, “We will write a formula according to the number of steps, let's write it, each time it increases by 2, 2n + 1. It can be expressed as 2 times one more.”

One MSMT stated that students find solutions by drawing. For instance, Ms. Lara:

> Without making it too difficult, we have had one cut, two cuts. For example, to ask the number of pieces in the sixth cut, we have them draw one by one. After drawing, we count those parts, we find the relationship between the

fifth step and the sixth step, find the difference there, and then we can go to coefficient.

One MSMT, Ms. Nisa, stated that students can find the solution by relating to context. She stated, “The student can say 2n+1. Because 2 pieces come out in each cut, and since +1 will give the number of remaining ropes, the student can express as 2n+1.”

4.2.2 MSMTs’ Responses Regarding Possible Incorrect Solutions or Errors from Students

In Task 1, there were three questions, asking the number of unit squares for the 5^{th} step, the 100^{th} step, and any step. I asked the teachers to explain the possible incorrect answers they expected from the students across the three questions.

The first question was, “How many squares are formed in the 5^{th} step?” The categories are listed in Table 4.12 for first question.

Table 4.12. MSMTs’ responses regarding incorrect student strategies for the first question in Task 1

Code	Definition	Participants
Inferring the general term from any step	Responses finding the general term of the pattern from any step of the pattern	Mr. Eren
Ignoring the constant	Responses focusing on ignoring the constant	Ms. Lara
Interpreting the amount of increase as additive	Responses considering the amount of increase in the pattern as the rule of the pattern	Ms. Azra

Table 4.12. (continued)

Interpreting the variable as a digit	Responses interpreting the variable as a digit	Ms. Nisa
Finding the general term incorrectly	Responses expressing finding the rule or general term of the pattern wrong	Ms. Buse
Finding by counting incorrectly	Responses that include finding by counting the steps of the pattern one by one incorrectly	Ms. Buse

In this question, MSMTs stated that students might give incorrect answers due to their methods of inferring the general term from any step, ignoring the constant, interpreting the amount of increase as an additive, interpreting the variable as a digit, finding the general term incorrectly and finding by counting incorrectly.

One MSMT stated that students can find an incorrect solution by inferring the question from any step. For example, Mr. Eren stated that students could make mistakes by inferring the number of squares in any step: “For example, if we go from 9 to 13, it is 2 times 9, 18, we subtracted 5, it is 13. For the 5th step, 2 times 13 is 26, and 5 less is 21.”

One MSMT stated students can ignore the constant. Ms. Lara stated:

> For example, the student cannot find the general term, for example, he says that it increases four by four. He says 4n, but he doesn't know what he needs to do to provide the first step. So, when I ask him to find the fifth step from 4 times n, he can say 4 times 5 equals 20.

One MSMT stated students can interpret the amount of increase as additive. Ms. Azra stated:

> A student who sees that it increases by 4 each time can say n+4. In other words, the student may confuse the unknown here, or rather the variable term, with the meaning of that amount of increase and fix that amount of increase by saying +4. He can say 5+4=9.

One MSMT stated that students can interpret the variable as a digit. Ms. Nisa stated that students might think of the variable in the general term as a digit and make incorrect calculations in the first question:

> The student knows that it increases four by four and he knows that he has to replace 4n with one and then do the first step. But when you ask them to find the 5th step, they may not multiply between 4 and n, but say 45-3=42 as if it were a 2-digit number.

One MSMT stated students can find general term incorrectly. Ms. Buse said:

> In other words, since it increases four by four, more precisely, the first step ignores and looks at the second step, it increases four by four. There is one square in the middle of 4n, 4n+1 can say one. The fifth step finds 21 from here.

One MSMT stated that students can find solutions by counting incorrectly. Ms. Buse also stated, "When he tries to count horizontally and vertically with his hand, he may ignore that the square in the middle has to be counted once."

The second question was, "How many squares are formed in the 100th step?" The categories for the second question are listed in Table 4.13.

Table 4.13. MSMTs' responses regarding incorrect student strategies for the second question in Task 1

Code	**Definition**	**Participants**
Inferring the general term from any step	Responses finding the general term of the pattern from any step of the pattern	Ms. Nisa
Finding the general term incorrectly	Responses expressing finding the rule or general term of the pattern wrong	Ms. Buse, Ms. Lara, Ms. Azra, Mr. Eren

In this question, MSMTs stated that students might make mistakes in the question due to their methods of making inferences from any step and finding the general term incorrectly.

One MSMT stated that student can make inferring the question from any step. Ms. Nisa:

> We are at the fourth step, the square of four is 16, and if I subtract 3 from this 16, I get 13. Then, if I take the square of 100 and subtract 3, the student can say 9997 at step 100 again.

Four MSMTs stated that students can find the general term incorrectly. For example, Ms. Buse stated:

> In the 100^{th} step, in the same way, since it will go in a pattern, it will increase by four as it counts. The student can find it as 401 from 4n+1 or as 402. Again, the student will count the middle square twice because s/he can count the middle twice, thinking that the number of squares vertically and horizontally will be the same.

The third question was, “How do you express the number of square units in any step? Write the rule expressing this relationship with words and variables.” The categories are listed in Table 4.14 for the third question.

Table 4.14. MSMTs’ responses regarding incorrect student strategies for the third question in Task 1

Code	**Definition**	**Participants**
Ignoring the constant	Responses focusing on ignoring the constant	Mr. Eren
Interpreting the amount of increase as additive	Responses considering the amount of increase in the pattern as the rule of the pattern	Ms. Azra
Finding the general term incorrectly	Responses expressing finding the rule or general term of the pattern wrong	Ms. Buse, Ms. Lara, Ms. Nisa

In this question, MSMTs stated that students may make mistakes due to the methods of ignoring constant, interpreting the amount of increase as additive and finding the general term incorrect.

One MSMT, Mr. Eren, stated that students may find incorrect answers by ignoring the constant: “It starts with 4x like this. It is correct to start with 4x because it increases by four, he says this correctly. He may not be able to find -3 in the rest of the problem.”

One MSMT stated that students can interpret the increase amount as additive. Ms. Azra stated that the student could solve the question incorrectly by interpreting the increase amount as additive: “The student who sees that it increases by 4 at each step can say n+4. The student can confuse the meaning of that increase amount with the variable term and fix that increase amount as if it were +4.”

Three MSMTs stated students can find the general term incorrectly. For example, Ms. Buse stated, “We put four squares around. Then we put another square in the middle, but in the first step there are no four squares around. 4n+1 can determine the rule as one.”

In Task 2, there were also three questions. I asked the teachers to explain the possible incorrect answers or errors they expected from the students regarding these questions. For the three questions, MSMTs stated their expected incorrect student strategies.

The first question was, “This rope has been cut as above. How many pieces of rope do you have?” The categories are provided in Table 4.15 for the first question.

Table 4.15. MSMTs’ responses regarding incorrect student strategies for the first question in Task 2

Code	**Definition**	**Participants**
Misinterpreting the problem	Responses that focus on students’ misunderstanding the problem	Ms. Buse, Mr. Eren, Ms. Lara, Ms. Nisa
Finding the general term incorrectly	Responses expressing finding the rule and general term of the pattern wrong	Ms. Azra

MSMTs stated that students could give the wrong solution by misinterpreting the problem and finding the general term incorrect.

Four MSMTs stated that students can find incorrect solution by misinterpreting the problem. For example, Ms. Buse stated:

> Because he thought that a cut had been made, it was divided into 2 equal parts vertically, not horizontally. Then, when it is cut again, he may think that these pieces will be cut again and he may think that 4 pieces can come out at once.

One MSMT, Ms. Azra, stated that the student may find the general term incorrect and give the wrong solution:

> So here, for example, the student can think of the untied version of the knot with the button on and think that in the first step there are 4 pieces. In the second step, 6 pieces are formed. It becomes 2n+2. Yes, he/she can say 2n+2, that is, the pattern rule.

The second question was, "Make 2 cuts and find out how many pieces the rope consists of. Repeat this for 3, 4 and 5 cuts, always remembering to use a new piece of string." The categories are provided in Table 4.16 for second question.

Table 4.16. MSMTs' responses regarding incorrect student strategies for the second question in Task 2

Code	**Definition**	**Participants**
Misinterpreting the problem	Responses that focus on students' misunderstanding the problem	All MSMTs

In this question, all MSMTs stated that the student would misinterpret the problem and find incorrect results. For example, Mr. Eren stated that students can find incorrect solution by misinterpreting the problem:

> The student may say that when I make 1 cut, 2 pieces are formed. The student continues in the same way. With this logic, the student says that if I divide 2

parts, it becomes 3, and if I know 3 parts, it becomes 4. If I divide by four, it becomes 5 and so on.

The third question was, “Express the number of pieces formed in any cutting using words and variables.” The categories are provided in Table 4.17 for the third question.

Table 4.17. MSMTs’ responses regarding incorrect student strategies for the third question in Task 2

Code	Definition	Participants
Misinterpreting the problem	Responses that focus on students’ misunderstanding the problem	Ms. Buse
Finding the general term incorrectly	Responses expressing finding the rule and general term of the pattern wrong	Ms. Azra
Interpreting the amount of increase as additive	Responses considering the amount of increase in the pattern as the rule of the pattern	Ms. Nisa
Ignoring the constant	Responses focusing on ignoring the constant	Ms. Lara, Mr. Eren

Three MSMTs stated respectively that students might find the wrong solution by misinterpreting the problem, finding the general term incorrectly and interpreting the increase as additive. Two MSMTs stated that students might find the wrong solution by ignoring the constant.

One MSMT stated that students can misinterpret the problem. Ms. Buse stated:

> I think of a student who continues in this way, the rope is always cut into one part. Then it was cut into 2 parts. He can say the direct answer as the number of pieces minus 1. In other words, he can say that one more than the number of cuts is the number of pieces.

One MSMT, Ms. Azra stated that students can find the general term incorrect:

> So, the student can get stuck in that knot here and think of the untied version of the knot, 4 pieces in the first step. In the second step, he can think that 6 pieces are formed. So again, the amount of increase is 2n+2 with an increase of 2.

One MSMT, Ms. Nisa, stated that the student could reach the wrong solution by interpreting the increase as additive:

> The question is visually in front of the student, but now you have only given the first step and did not show what was formed when cut, and you did not show how many ropes were formed in the second step. For example, the student has more difficulty in generalizing here. Since the rope increases by two with each step, the student can say x+2. He makes this mistake very easily.

Two MSMTs stated that students can ignore the constant. For instance, Ms. Lara said, “Its rule is that there are 2 pieces in each cut. He can set it as 2n. He can ignore the remaining rope because he doesn't cut horizontally.”

4.2.3. MSMTs’ Responses on the Reasons for Students’ Errors and How to Overcome

MSMTs were asked about possible incorrect answers students might give and what the reasons for these errors might be. Although the second question was related to the section on knowledge of content and teaching, it is provided in this section for the reader to follow easily with the first question results. In this section, MSMTs' opinions about the reasons for students' mistakes and how the mistakes could be overcome were obtained for Task 1. In the first question, MSMTs expressed the reasons for students’ mistakes which are having difficulty establishing the relationship between variables, having difficulty establishing the relationship between algebra and numbers, having difficulty describing variables and finding by counting incorrectly. The codes regarding the reasons for students’ mistakes are listed in Table 4.18.

Table 4.18. The reasons for students’ mistakes for Task 1

Codes	**Definition**	**Participants**
Having difficulty establishing the relationship between variables	Responses emphasizing difficulty in establishing the relationship between the variables	Ms. Azra
Having difficulty establishing the relationship between algebra and numbers	Responses emphasizing difficulty in establishing relationship between algebra and numbers	Ms. Lara
Having difficulty describing variables	Responses emphasizing difficulty what the variables in the general term and what they mean	Ms. Nisa, Mr. Eren
Finding by counting incorrectly	Responses that include finding by counting the steps of the pattern one by one incorrectly	Ms. Buse

Table 4.18 (continued)

One MSMT said that the possible wrong answers were because they could not establish the relationship between variables. Ms. Azra stated, “The students can make mistakes because they cannot establish a relationship between the number of squares and the number of steps.’’ One MSMT emphasized the relationship between algebra and numbers. Ms. Lara stated, “They might make a mistake because they cannot establish the relationship between algebra and numbers.’’

Two MSMTs said that students might have difficulty describing variables. For instance, Ms. Nisa stated, “The students may find it very difficult to find the formula $4n$-3, so what does the thing I call n define?’’

One MSMT said that students can make a mistake in finding by counting. Ms. Buse stated, “The student wants to find the steps of the pattern by counting them one by one instead of finding the general term.”

The second question was, “How can you overcome these incorrect responses that may come from your students?” In this question, MSMTs stated that students' mistakes can be overcome by explaining how to find general term, helping students finding general term by focusing on steps, helping students finding general term by discussing in the whole class and helping students finding general term by asking question. The codes for the second question are presented in Table 4.19.

Table 4.19. Methods to overcome students' mistakes for Task 1

Codes	**Definitions**	**Participants**
Helping students find the general term by focusing on the steps	Responses that focus on helping students find the general term by focusing on the steps	All MSMTs
Explaining how to find the general term	Responses that focus on explaining how to find the general term	Ms. Nisa, Ms. Azra

Table 4.19. (continued)

Helping students find the general term by discussing in the whole class	Responses that focus on helping students find the general term by discussing in the whole class	Ms. Lara
Helping students find the general term by asking questions	Responses that focus on helping students find the general term by asking questions	Ms. Nisa, Ms. Azra, Ms. Lara

All MSMTs stated that students' mistakes can be overcome by helping students find general term by focusing on the steps. For instance, Mr. Eren stated that students can solve the question incorrectly by inferring from any step. When asked how he would overcome this error, Mr. Eren focused on steps:

> One of the methods I use most in the lesson is that I write the steps one by one. I adapt what the children say under the steps and they see for themselves that they are wrong. We choose 2 or 3 of the best students of the class and write down what are your formulas friends and write the formulas like Ahmet, Mehmet, Cansu and so on, and after the first step, the second step, the third step, the most of all, some of the formulas come out quite close. After they are close to each other, we say that if we can make such a correction, you said 4x, but when we put one to x, it gives 4x here, but there is one square. Therefore, they can be encouraged to think about how we can decrease it.

Two MSMTs stated that students' mistakes can be overcome by explaining how to find general term. For example, Ms. Nisa stated that students can think of interpreting variables as a digit. When asked how to overcome this error, she said that she would explain to the student how to find the general term. Ms. Nisa stated:

> Now, for example, I can say that I will multiply this number by n if it increases by how many times. The multiple number is always at the beginning, I don't write n times 4, I say I'll put 4 directly at the beginning. Then 4 times n times 4 times 1 makes 4, but I need to find 1. Whatever you do to four to make it 1, I need to subtract 3. We can make it find the form 4n-3 in this way.

One MSMT, Ms. Lara, stated that student can find general term by inferring from any step. When asked how to overcome this error, she stated that students' mistakes can be overcome by helping students finding the general term by discussing in the whole class She stated, "For example, there is a student who finds the rule as n^2. I would take different answers from the class and ask them to discuss the answers. We would discuss in class which one might be correct and why."

Three MSMTs stated that students' mistakes can be overcome by helping students finding the general term by asking questions. For example, Ms. Azra stated that students can ignore constant when finding the general term. When asked how to overcome this error, she stated that she would overcome it by helping students finding general term by asking questions:

> How can we overcome this error? I focus on the difference between step one and step two. What changed from first step to second step? What changed from second step to third step? Then I have them draw five, six. I make them verbalize this change. Then I move on to the stage of finding the general rule of the pattern based on the constant, that is, the amount of increase being constant.

4.2.4. Middle School Mathematics Teachers' Responses Regarding Student Responses in Task 2

In this part, three student solutions were shown to MSMTs regarding Task 2. Teachers were asked, "How do you think this student might have thought?"

The first students' solution was: "Since y is the total number of parts and x is the number of cuts increasing by 2, we can express it as y = 2x." The codes about MSMTs' responses regarding first students' solution are listed in Table 4.20.

Table 4.20. The MSMTs' responses regarding the first student's solution

Code	**Definition**	**Participants**
Misinterpreting the problem	Responses that focus on students' misunderstanding the problem	Mr. Eren
Ignoring the constant	Responses focusing on ignoring the constant	Ms. Nisa, Ms. Buse
Relating with context	Responses solving the problem by relating it to the context	Ms. Nisa, Ms. Azra
Focusing on the amount of increase	Responses focusing on the amount of increase	Ms. Lara

Regarding the first student's solution, MSMTs stated that students might have misinterpreted the problem, ignored the constant and related with context.

One MSMT stated that students can misinterpret the problem. Mr. Eren stated that the student misunderstood the problem:

> He thought as I mentioned he said that we need to divide a whole once in order to divide it into 2 parts. That's why when we write 1 to variable *x*, I get 2. In this way, he continues his mistake.

Two MSMTs stated that students can ignore the constant. For example, Ms. Buse stated ''In the formula he ignored the remaining number of pieces. So yes, every time the number of pieces correctly expresses the number of cuts. However, in the end he ignored the number of strings left.’’

Two MSMTs stated that students established a relationship with the context. For example, Ms. Nisa said that the student was able to establish a relationship with the context but ignored the knot:

> The answer is 2*x*, she forgot +1. The student counted the knotted rope, he just took the knotted rope in his hand, always divided it, divided it and counted the ropes he created aside. Where the student said y=2x, the student probably put the knotted rope aside and did not consider it as a rope.

One MSMT stated that students can focus on the amount of increase. Ms. Lara said, “The student now says the number of pieces is 3, 5, 7. He may have said 2x because it increased by 2. That's probably why he said so, he said y=2x.”

The second student’s solution was: “As the number of cuts increases by one, the number of parts formed increases by 2.” The codes for MSMTs’ responses regarding the second student’s solutions are listed in Table 4.21.

Table 4.21. The MSMTs’ responses regarding the second student’s solution

Code	**Definition**	**Participants**
Having difficulty establishing the relationship	Responses emphasizing difficulty in the relationship between the variables	Ms. Lara, Mr. Eren
Emphasizing covariational thinking	Responses involving covariational thinking	Ms. Buse, Ms. Azra, Ms. Nisa
Ignoring the constant	Responses ignoring the constant	Ms. Buse

MSMTs were again asked how this student might have thought. They stated that the student could not establish a relationship and could think covariationally.

Two MSMTs stated that the student could not establish a relationship. For example, Ms. Lara stated that the student could not fully establish the relationship between the number of cuts and the number of pieces:

> This student did not think that he could establish the exact relationship between the number of cuts and the number of pieces. Because the number of cuts and the number of pieces increase separately. But what is the relationship between these? He didn't quite understand how to make a connection. I think there's a problem there.

Three MSMTs focused on the change of two variables. For instance, Ms. Buse's response emphasized covariation:

> The student took the right approach, the number of cuts increases by one and the number of ropes increases by two, but this student also ignores the knot. In fact, it is the correct statement, because the number of cuts that always increases by one, while the number of ropes increases by 2.

The third student's solution: "In this question, since there are 2 pieces in each segment and there is 1 knot at the beginning, we reach the number of segments when we multiply the number of segments by 2 and add 1. We can express the relationship as y= 2x+1, where y is the total number of parts and x is the number of cuts." The codes about MSMTs' responses regarding the third student's solution are listed in Table 4.22.

Table 4.22. The MSMTs' responses regarding the third student's solution

Code	**Definition**	**Participants**
Relating with context	Responses solving the problem by relating it to the context	Ms. Lara

Table 4.22. (continued)

Finding the general term of the pattern	Responses expressing finding the rule and general term of the pattern	Ms. Buse, Mr. Eren, Ms. Nisa

Regarding the third student's solution, MSMTs stated that the student finds the solution by finding the general term and relating to the context.

Three MSMTs stated that the student found the correct solution by finding the general term. For instance, Ms. Nisa stated, "This student found the general term and thought correctly. Two new pieces are added in each cut. At the very beginning, he says that there is only one knot, which is the correct answer."

One MSMT stated that the student related to the context. Ms. Lara stated, "I thought that this student is right. Because each of them increases two by two, he noticed that the difference increases two by two. He also realized that he needed to add the knotted rope initially."

One MSMT stated that she could not understand the student's solution. Ms. Azra stated:

> Since there is a knot at the beginning, we multiplied the number of cuts by 2. Well, at the beginning, a knot adds 1…what is the function of that knot? Does the student think of the 0^{th} step, that is, when n=0, one knot is a piece, for example, I mean, this one takes the step before the cut as the 0^{th} step, that's why it is 3 at the beginning. So, it seemed to me like…according to the rule here, when two are zero [one piece in the case where both zero]. He thinks he has one piece. So, why does he add 1 because there is a knot? I'm sorry, I couldn't understand that.

The second question was, "What do you do next with this student?" MSMTs' strategies to follow regarding student solutions are explained in this section, although they are related to knowledge of content and teaching to follow student responses.

The first student response was: “Since y is the total number of pieces and x is the number of cuts increasing by 2, we can express it as y=2x.” The codes regarding MSMTs’ strategies for the first student’s solution are listed in Table 4.23.

Table 4.23. The MSMTs’ strategies for the first student’s solution

Codes	**Definitions**	**Participants**
Helping students find the general term by relating to context	Responses that focus on helping students find the general term by relating to context	Ms. Nisa, Ms. Buse
Helping students find the general term by focusing on the steps	Responses that focus on helping students find the general term by focusing on steps	Mr. Eren, Ms. Azra, Ms. Lara
Helping students find the general term by asking questions	Responses that focus on helping students find the general term by asking questions	Ms. Nisa, Ms. Azra

MSMTs stated student’s mistakes can be overcome by helping students to realize their mistakes by relating to context, helping students find the general term by focusing on the steps, and helping students find the general term by asking questions.

Two MSMTs stated that this mistake can be overcome with helping students to realize their mistakes by relating to context. For example, Ms. Nisa stated:

One MSMT, Ms. Nisa, stated that she would have this student relate to the context in the next step:

> First, I ask the child, how many ropes did he have at first. The child will say, I had one knotted rope. Then in the first step, after making the first cut, I say how many pieces did you have? For example, I ask how many pieces you

have when you cut them. You know, after I cut it, I just cut 2 pieces, but I also had one knotted rope. I ask how many pieces there are in total. When I ask how many pieces there are in total, maybe he knows there should be 3 there. When you put 1 instead of x, it doesn't work. He thinks there are 2 ropes. He thinks that there are ropes that I cut and threw aside. He may realize that he needs to add 1, so he didn't count that knotted rope.

Three MSMTs stated that they helping student find general term by focusing on steps. For instance, Ms. Lara stated that she could help student find general term by focusing on the steps:

> As I said, when I make this student replace 1 with the substitution method, he finds 2. But I would find 3 on the table and have the student realize that there is one piece missing. How? Here he substituted it in the first step and said x = 1. She found 1 times 2 to 2, but when she specified x and y in the table, it became 3. We found it wrong here, let's go to the second step. When you replace 2 instead of x, 2 times 2 should find 4, but there is 5 on the table. Again, there is still one missing item. When we write three instead of x to find y, we get y= 6, but it says 7 in the table, we still have one missing item, so if we have one missing item at each step, we say we need to add 1 to each step and get y=2x+1 one. I try to express it.

Two MSMTs stated that they help student find general term by asking questions. For example, Ms. Azra stated:

> So, the number of cuts corresponds to x here, and when I write 1 instead of x, I cannot get 3, or when I write 2 instead of x, I cannot get 5. What else should I do to get 3 when I write 1 instead of x? So I can make the student question this by saying what other changes can I make on this y=2x.

The second student said, “As the number of cuts increases by one, the number of pieces formed increases by 2.” The codes regarding MSMTs’ strategies for the second students’ solution are listed in Table 4.24.

Table 4.24. The MSMTs' strategies for the second students' solutions

Codes	**Definitions**	**Participants**
Explaining how to find the general term	Responses that focus on how to find the general term	Ms. Nisa, Ms. Lara
Helping students find general term by using a table	Responses that focus on helping students find general term by using a table	Ms. Buse, Ms. Azra
Having the students find the general term	Responses that focus on having the students finding the general term	Mr. Eren
Helping students find the general term by asking questions	Responses that focus on helping students find the general term by asking questions	Ms. Nisa, Ms. Lara

Two MSMTs stated that they could explain how to find the general term to the students. For example, Ms. Nisa stated:

> It increases by two by two, then we say 2 times n. Here, n was the number of steps, so I say 2 times n. Then I had one in the first step. When I substitute 1 for n, it becomes 2, but he realizes that he has to subtract 1 to make it one. He says 2n-1.

Two MSMTs stated that they could help student find general term by using a table. For instance, Ms. Buse stated:

> The student can create a table according to the first 10 or first 15^{th} steps and then create a pattern with numerical values from the table? In other words,

> can he find a formula? We can make him do it, now this student can leave the material aside and go over the tables.

One MSMT, Mr. Eren, stated that he has the students find the general term. He stated "This student is ready to find the general term that we call this, we can now move on to this stage. Anyway, this is the first mistake it will make because y= 2x, 2, 2 increases."

Two MSMTs stated that they would help student find the general term by asking questions. For example, Ms. Lara stated that she could help this student find the general term by asking questions:

> What is the relationship between these? If we want to establish a relationship, we can ask questions such as how we would express it. Yes, the difference between the number of cuts and the number of pieces is 2 times each other, but we can add 2 times to get y = 2x. Afterwards, when we multiply 2 times each other, but instead of y, we get 2 times 1 is 2 plus 1, 3. As I said before, we could go from 2x plus 1.

The third student said: "In this question, since there are 2 parts in each cut and there is 1 knot at the beginning, we reach the number of parts formed when we multiply the number of cuts by 2 and add 1. We can express the relationship as y= 2x+1, where y is the total number of parts and x is the number of cuts." The codes regarding MSMTs' strategies for the third students' solution are listed in Table 4.25.

Table 4.25. The MSMTs' strategies for the third student's solution

Codes	**Definitions**	**Participants**
Asking a higher-level question	Responses that focus on asking a higher-level question	Mr. Eren
Asking a different version of the question	Responses that focus on asking a different version of the question	Ms. Buse

Table 4.25. (continued)

Asking students to create a graph	Responses that focus on asking students to create a graph	Ms. Lara
Asking students to justify the answer relating to the context	Responses that justifying the answer relating to the context	Ms. Nisa

MSMTs' strategies were asking a higher-level question, asking a different version of the question, asking students to create a graph, asking students to justify the answer relating to the context.

One MSMT, Mr. Eren, stated that he could ask higher-level question. He stated:

> First, I would give a feedback after something like "well done" and "congratulations". If we were in a one-on-one lesson, it would be like the first question (Task 1) you asked. I think it is a little harder to establish the pattern of this. I would give the first example you asked me; this is harder.

One MSMT, Ms. Lara, stated that she could ask different version of the question. She stated:

> For example, what would happen if the rope was cut not horizontally but vertically and each time it was cut vertically, or what would happen if it went horizontally and vertically? I mean, think in another dimension, the same question can be thought in another dimension. You know, this student already answers directly what he sees in the question. I mean, he didn't think of anything extra or different.

One MSMT, Ms. Lara, stated that she could ask student to create a graph. She stated:

> y=2x+1 is a linear equation. A student who thinks this at this stage has thought at a high level. Since they already have a high level of cognition,

> they can turn the relationship between the number of parts and the number of cuts into a graph and draw the graph of y= 2x+1 and then find the steps in it.

One MSMT, Ms. Nisa, stated that she could ask students to justify the answer relating to the context. She stated:

> For example, I questioned why, for example, he reached the generalization 2x+1, why didn't he say x+2, why 2x? Why didn't he write +2 next to x, but why did he say 2x because it increased by 2 parts by 2 parts? I ask this question. Does he know this? How is it related to the number of steps? For example, I can ask how it relates to the number of cuts. I ask him if he knows it by memorization or if he knows the logic of it.

One MSMT, Ms. Azra, said that she could not understand the solution of this student so she did not answer this question.

CHAPTER 5

DISCUSSION AND IMPLICATIONS

This research focused on knowledge of content and students and knowledge of content and teaching in the context of functional thinking. In this section, findings were discussed. Lastly, the implications of the findings were presented.

In this study, MSMTs were asked about the possible correct answers they expected from students for Task 1 and Task 2. In the first question of Task 1, two MSMTs stated that students would reach the correct solution by counting rhythmically, one MSMT stated drawing, and two MSMTs stated finding the general term of the pattern. In the second question of Task 1, all MSMTs stated that students would reach the correct solution by finding the general term of the pattern, and one MSMT also gave a response focusing on relating to the context. In the third question of Task 1, all MSMTs stated that students would reach the result by finding the general term of the pattern. One MSMT, Ms. Nisa, stated, "It increases by four, we say 4n, and we substitute one for n again to find the first step. To get 1 from 4 times n, we subtract 3. We reach the formula 4n-3." When the answers of other MSMTs who said finding general term were examined for both tasks, it was seen that teachers multiplied the difference by the position number and added a number to find the first term. In Girit's study (2016), it was seen that since the teacher explained finding the general term of the pattern as multiplying the difference by the position number and adding a number to find the first term, students wanted to memorize this rule and apply it to all pattern questions. This seemed to cause students some misunderstandings. In this study, similarly, the majority of MSMTs expected student responses to express the rule of the pattern in this way without relating it to the context or establishing any relationship with the variables.

In addition, this study found that MSMTs looked at the figure only for the difference in the pattern when expressing possible correct solutions in Task 1, and then they developed a solution by focusing on numerical relations. This shows that MSMTs focused more on numerical relations than on the model. Yeşildere and Akkoç (2010) stated that pre-service mathematics teachers and Kutluk (2011) middle school mathematics teachers focused more on numerical relations than on the model. The findings of this study were consistent with the findings of other studies.

As a result of the study conducted by Wilkie (2014) with 105 senior teachers, investigating the teachers' knowledge about teaching algebra, one-third of the teachers were able to make generalizations symbolically, but only two percent of these teachers wrote a full equation that included both variables. In this study, when MSMTs were asked about the possible correct answers they expected from students in two tasks, they expressed the possible correct answers of the students only as expression. For example, they stated that in Task 1, the students would correctly answer the question "How do you find the general term of the pattern?" as 4n-3. None of the teachers used the equation y = 4n-3. Stephens et al. (2017) defined this response at the functional emergent level of sophistication. At this level, students provide the incomplete function rule with variables but do not associate it with the other variable.

When MSMTs were asked about possible incorrect answers from students, in the first question of Task 1, one MSMT stated that students could make the wrong solution by inferring the general term from any step, one MSMT stated that students can ignore the constant, one MSMT stated that student can interpret the amount of increase as additive, one MSMT stated students can interpret the variable as a digit, and one MSMT stated that students can find the general term incorrectly and find by counting incorrectly. In the second question of Task 1, one MSMT stated students can infer the general term from any step, four MSMTs stated students can find the general term incorrectly. In the third question of Task 1, one MSMT stated that students can ignore the constant, one MSMT stated that students can interpret the amount of increase as additive and three MSMTs stated that students can find the general term incorrectly. In the first question of Task 2, four MSMTs stated that students can misinterpret the problem, one MSMT stated that students can find the general term incorrectly. In the second question of Task 2, all MSMTs stated that students can misinterpret the problem. In the third question of Task 2, one MSMT stated that students can misinterpret the problem, one MSMT stated that students can find the general term incorrectly, one MSMT stated students can interpret the amount of increase as additive and two MSMTs stated that students can ignore the constant. In the study by Türkmen and Tanışlı (2019), it was stated that students made generalizations by ignoring the constant n while determining the general term of the relationship y=mx+n. Additionally, in Pang and Sunwoo's study (2022), teachers were asked to analyze students' mistakes in making relations and generalizations between two quantities. 75.6% of teachers analyzed that the student's mistake was to focus only on the increase in the linear relationship and ignore the constant term. In Girit's study (2016), when the students were asked about the general term of the pattern going as 3, 4, 5,..., some of the students found the general term as n+1. Students focused on the difference between the terms. In this study, one teacher stated that they could make this mistake in both tasks. In Task 1, Ms. Azra stated, "A student who sees that it increases by 4 each time she can say n + 4. In other words, the student may confuse the unknown here, or rather the variable term, with the meaning of that amount of increase." In Task 2, Ms. Azra similarly stated ''Since the rope increases by two with each step, the student can say x+2. He makes this mistake very easily.'' In addition, one MSMT stated student can find incorrect solution by interpreting the variable as a digit. Akkaya and Durmuş (2006) observed in their study with middle school students that some students thought of an algebraic expression such as ab as a two-digit number. In their study, Şahin and Soylu (2011) observed that some students perceived the variable as a two-digit number. Therefore, it can be said that teachers could foresee some common students' mistakes in this study.

In Task 1, when the MSMTs were asked about the possible reasons for the incorrect answers they expected from the students, the MSMTs gave the following reasons:

having difficulty in establishing relationships between variables, having difficulty in establishing relationships between algebra and numbers, having difficulty in describing variables, and finding by counting incorrectly. Two MSMTs gave the answer “not describing variables.” For instance, Ms. Nisa stated, “The students may find it very difficult to find the formula 4*n*-3, so what does the thing I call *n* define?” In the study of Kutluk (2011), similarly, when the teachers were asked about the reason for the difficulties experienced by the students, teachers stated that the concept of *n* was not correctly introduced to the students and that the students had difficulty because they could not define *n*. It was observed that most MSMTs gave superficial answers about the reasons for students' mistakes and not detailed answers. This showed that they had limited knowledge about the reasons for students' mistakes.

In Task 1, the MSMTs were asked how to overcome these errors regarding the incorrect answers they expected from the students. MSMTs answered helping students find the general term by focusing on the steps, helping students find the general term discussing in the whole class, helping students find the general term by asking questions and explaining how to find general term. In the study of Pang and Sunwoo (2022), about 35% of the teachers stated that they would overcome the difficulties experienced by the students by raising a question to recognize their mistake. In this study, three MSMTs also used this method. In addition, in Pang and Sunwoo's (2022) study, about 25% of the teachers stated that they would overcome students' errors by re-explaining the problem context to students. Additionally, Yılmaz Tığlı (2023) stated that teachers suggested explaining the concept repeatedly to overcome students' mistakes. In this study, two MSMTs stated that they can overcome students' mistakes by explaining how to find general term. In addition, Yılmaz Tığlı (2023) stated in her study that teachers can overcome students' mistakes by making students active participant in learning. Similarly, in this study, one MSMT stated that she would help students to find the general term by discussing in the whole class. Although MSMTs used some common strategies, it can be said that their strategies were limited.

In Task 2, MSMTs were shown three student solutions and were asked how these students thought. The codes formed for the first student's solution were misinterpreting the problem, ignoring the constant and relating with context. This student made ignoring the constant mistake in his solution, which is also found in the literature (e.g., Pang & Sunwoo [2022]; Türkmen & Tanışlı [2019]). Two MSMTs stated this student ignored the constant. In the second student's answer, having difficulty in establishing the relationship and emphasizing covariational thinking were provided by the MSMTs. This student solution was at the covariation thinking level in levels of sophistication (Stephens et al., 2007). Three MSMTs noticed that this student focused on the change of two variables and emphasized covariational thinking. When the third student was asked how he thought, the codes formed from the answers of the MSMTs were relating with context and finding general term of the pattern. This student's solution was at the functional condensed level in the levels of sophistication (Stephens et al., 2007). At this level, the student reached the general term. Three MSMTs stated that this student found the general term and made the solution correctly. Most MSMTs were found to have an idea about how the student thought.

MSMTs were asked about the sample students' responses given, "What would you do next with this student?". Regarding the first students' response, MSMTs expressed the strategies of helping students find the general term by relating to context, helping students find the general term by focusing on the steps and helping students find the general term by asking questions. Regarding the second student's response, MSMTs expressed the strategies of explaining how to find the general term, having the students find the general term, helping students find the general term by asking questions, and helping students find the general term by using a table. Regarding the third student's response, MSMTs expressed the strategies of asking a higher-level question, asking a different version of the question, asking students to create a graph, and asking students to justify the answer relating to the context. MSMTs have been shown to have difficulty in extension deciding what to do next with students who answer correctly.

In this study, the MSMTs were given the M.8.2.2.3. objective and asked to explain how a lesson plan description would be related to this objective. Four MSMTs used context-based tasks, while one MSMT used a figural pattern task. Four MSMTs asked the students to create tables, while one of these MSMTs stated that she would also ask them to create graphs. One MSMT noted that he would use a double-yolk egg as a material. Two MSMTs chose continuous tasks, while three MSMTs chose discrete tasks. When asked about lesson plan descriptions, teachers did not give detailed responses as expected about their descriptions given the objective. The answers were rather superficial. Looking at the answers of the MSMTs, they stated that they would use representations such as graphs and tables. However, most MSMTs did not have any focus on the relationship between different representations. Moreover, although the objective emphasized the relationship between two variables, most MSMTs did not focus on this relationship.

Implications of the Study

When the knowledge of content and teaching of the MSMTs was examined, it was seen that they gave limited answers when asked how to overcome possible student mistakes. It can be said that MSMTs had difficulties overcoming student mistakes. In addition, it was observed that when MSMTs were asked about possible correct answers from students, they explained finding the general term by focusing on the difference. MSMTs expressed finding the general term as multiplying the difference by the position number and adding a number to find the number in the first step. MSMTs did not find the general term by establishing a relationship with the context or by establishing a relationship between variables. In order to encourage MMSMTs to focus on relationships and establish a relationship with the context regarding patterns, a professional development program can be organized that includes a pattern problem containing a context-based task and expressing the general term with as many different representations as possible and how to foster this approach in classroom can be examined and discussed as well.

When the knowledge of content and students of MSMTs were examined, it was seen that they could predict some common students' mistakes when asked about possible incorrect solutions from students, but mostly, each teacher could provide one possible wrong student answer. Regarding this, it can be said that MSMTs gave limited answers regarding possible incorrect solutions for students. An in-service training program can be organized where teachers are given a question and asked to write down as many possible correct and incorrect solutions as possible for students, and these solution methods and strategies to help teachers support their students regarding them can be shared with each other.

In this study, when MSMTs were asked to express students' possible correct answers, some teachers stated that they would try to find the general term of the pattern directly based on the rule without relating the answers to the context. However, when the Patterns and Algebra objectives in the 2024 national mathematics curriculum were examined, it was seen that more emphasis was given to students' noticing, interpreting, and reasoning with relationships in the themes. It can be said that the 2024 curriculum provides more opportunities for students to reason with linear functions. The issue of how teachers will adapt to this positive change will be important. For the transition between the two curricula to be carried out, teachers may need to be provided with effective in-service training on this subject.

The concepts of pattern and function should be taught to students in relation to each other (Türkmen & Tanışlı, 2019). When teaching the concept of patterns, the focus should be on the goal of having them comprehend functional relationships. When focusing on the functional relationships, different forms of representations should also be taken into consideration, such as the use of tables and graphs. In this study, it was seen that teachers, in general, did not mention different representations and connections between those. For this reason, mathematics teachers need to make sense of functional relationships and functional thinking as well as helping students understand functional relationships. This requires an in-service training program that focuses on fostering functional thinking in the classrooms.

Functional thinking is a route to developing students' algebraic thinking. Teachers play a key role in this development (Blanton & Kaput, 2005). Teacher educators need to pay attention to this issue. Professional development programs should be designed to encourage teachers to identify students' algebraic and functional thinking.

For future research, studies on the same topic can be conducted with more teachers using quantitative methods. Lesson observations can also be added to future studies. The effects of teachers' functional thinking on students' development of functional thinking can be investigated.

REFERENCES

Akın, G. (2020). *The effects of a functional thinking intervention on fifth-grade students' functional thinking skills* (Master's thesis, Middle East Technical University).

Akkaya, R., & Durmuş, S. (2006). İlköğretim 6-8. sınıf öğrencilerinin cebir öğrenme alanındaki kavram yanılgıları. Hacettepe Üniversitesi Eğitim Fakültesi Dergisi, 31(31), 1-12.

Arslandaş, T. (2022). *Investigating fifth-grade students' functional thinking processes through a game-based learning activity* (Master's thesis, Middle East Technical University).

Blanton, M. L., & Kaput, J. J. (2004). Elementary Grades Students' Capacity for Functional Thinking. *International Group For The Psychology Of Mathematics Education*.

Blanton, M. L., & Kaput, J. J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for research in mathematics education*, *36*(5), 412-446.

Blanton, M., Schifter, D., Inge, V., Lofgren, P., Willis, C., Davis, F. & Confrey, J. (2007). In Katz, V.J. (ed*). Algebra: Gateway to a technological future* (pp. 7-14). Washington: The Mathematical Association of America.

Blanton, M. L., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In *Early algebraization: A global dialogue from multiple perspectives* (pp. 5-23). Berlin, Heidelberg: Springer Berlin Heidelberg.

Blanton, M. L., Levi, L., Crites, T. & Dougherty, B. J. (2011). *Developing essential understandings of algebraic thinking*, Grades 3-5. Reston, VA: The National Council of Teachers of Mathematics.

Chazan, D. (1996). Algebra for all students?: The algebra policy debate. *The Journal of Mathematical Behavior*, *15*(4), 455-477.

Confrey, J., & Smith, E. (1991, October). A framework for functions: Prototypes, multiple representations, and transformations. In *Proceedings of the 13th annual meeting of the North American Chapter of The International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 57-63).

Ceylan D. & Alptekin A. (2020). *7. Sınıf kazanım odaklı etkinlik defteri.* Hız Yayınları.

Creswell, J. W. (2003). *Research design: qualitative, quantitative and mixed methods approaches*. California: Sage Publications.

Çatalkaya, Ş. (2023*). Investigation of the skills of elementary mathematics teacher to use multiple representations in solution of problems including functional thinking*. (Master's thesis, Erciyes University).

Darling-Hammond, L., & Bransford, J. (Eds.). (2007). *Preparing teachers for a changing world: What teachers should learn and be able to do*. John Wiley & Sons.

Erbaş, A. K, Ersoy,Y. (2003). Kassel projesi cebir testinde bir grup türk öğrencisinin başarısı ve öğrenme güçlükleri. *İlköğretim Online Dergisi*, 4 (1),18-39.

Even, R. & Tirosh, D. (2008). Teacher knowledge and understanding of students' mathematical learning and thinking. In English, L.D. (ed.). *Handbook of International Research in Mathematics Education* (2nd ed., pp. 202-223). London: Routledge.

Fonger, N. L., Ellis, A., & Dogan, M. F. (2016). Students' Conceptions Supporting Their Symbolization and Meaning of Function Rules. *North American Chapter of the International Group for the Psychology of Mathematics Education*.

Fraenkel, J. R., Wallen, N. E., & Hyun, H. H. (2011). How to design and evaluate research in education. New York: McGraw-Hill Humanities/Social Sciences/Languages.

Girit, D. (2016). Investigating middle school mathematics teachers' mathematical knowledge for teaching algebra: a multiple case study.

Gisev, N., Bell, J. S., & Chen, T. F. (2013). Interrater agreement and interrater reliability: key concepts, approaches, and applications. *Research in Social and Administrative Pharmacy*, *9*(3), 330-338. *arch*, 3(2), 95-108. https://doi.org/10.1177/1558689808330883

Hill, H. C., Ball, D. L., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for research in mathematics education*, *39*(4), 372-400.

Isler, I., Marum, T., Stephens, A., Blanton, M., Knuth, E., & Gardiner, A. M. (2014). The String Task Not Just for High School. *Teaching Children Mathematics*, *21*(5), 282-292.

Kabael, T., & Barak, (2017). Pre-service Middle School Mathematics Teachers' Functional Thinking Abilities in Solving Problems. *International Journal of Educational Sciences*, (13), 311-321.

Kaput, J., & Blanton, M. (2001). Algebrafying the elementary mathematics experience. In *The Future of the Teaching and Learning of Algebra. Proceedings of the 12th ICMI study conference* (Vol. 1, pp. 344-352).

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 5–17). Mahwah, NJ: Lawrence Erlbaum/Taylor & Francis Group; Reston, VA: National Council of Teachers of Mathematics.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 390- 419). New York: MacMillan.

Kieran, C. (2004). Algebraic thinking in the early grades: What is it. *The mathematics educator*, *8*(1), 139–151.

Kulaç, B. (2023). *Genişleyen şekil örüntüleri temalı hazırlanan bir varsayımsal öğrenme rotası ile 7. sınıf öğrencilerinin fonksiyonel düşünmelerinin gelişiminin incelenmesi* (Master Thesis, Çukurova University).

Kutluk, B. (2011). *The investigation of elementary mathematics teachers? knowledge of student difficulties related to pattern concept* (Doctoral dissertation, DEÜ Eğitim Bilimleri Enstitüsü).

Loewenberg Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. *Journal of teacher education*, *59*(5), 389-407.

McAuliffe, S., & Vermeulen, C. (2018). Preservice teachers' knowledge to teach functional thinking. In C. Kieran (Ed.), *Teaching and Learning Algebraic Thinking with 5- to 12-Year-Olds: The Global Evolution of an Emerging Field of Research and Practice* (pp. 403-426). Cham, Switzerland: Springer.

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M. D., & Senk, S. L. (2012). Knowledge of algebra for teaching: A framework of knowledge and practices. *Journal for Research in Mathematics Education*, *43*(5), 584-615.

Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation* (3rd ed). San Francisco, CA: Jossey-Bass.

Millî Eğitim Bakanlığı (MEB) (2018). *Matematik dersi öğretim programı 1-8.* sınıflar. (n.d.). Retrieved July 31, 2024, from http://mufredat.meb.gov.tr/ProgramDetay.aspx?PID=329

Millî Eğitim Bakanlığı (MEB) (2024). *Türkiye Yüzyılı Maarif Modeli 5-8.* sınıflar. (n.d.). Retrieved 2 August, 2024, from

https://tymm.meb.gov.tr/ogretim-programlari/ortaokul-matematik-dersi

National Council of Teachers of Mathematics [NCTM]. (2000*). Principles and Standards for School Mathematics*. Reston, VA: NCTM. United States of America.

Oliveira, H., Polo-Blanco, I., & Henriques, A. (2021). Exploring Prospective Elementary Mathematics Teachers' Knowledge: A Focus on Functional Thinking. *Journal on Mathematics Education*, *12*(2), 257-278.

Öztürk, N. (2021). *An investigation of development of prospective elementary teachers' knowledge to teach algebra in early grades through case discussions* (Master's thesis, Middle East Technical University).

Pang, J., & Sunwoo, J. (2022). An analysis of teacher knowledge for teaching functional thinking to elementary school students. *Asian Journal for Mathematics Education*, *1*(3), 306-322.

Panorkou, N.; Maloney, A. y Confrey, J. (2016). Expressing Covariation and Correspondence relationships in elementary schooling. Retrieved from:

https://nctm.confex.com/nctm/2014RP/ webprogram/ExtendedAbstract/Paper1940/ EQX_NCTM_040314%20.pdf

Sahin, Ö., & Soylu, Y. (2011). Mistakes and misconceptions of elementary school students about the concept of 'variable. *Procedia-Social and Behavioral Sciences*, *15*, 3322-3327.

Sharts-Hopko, N. C. (2002). Assessing rigor in qualitative research. *Journal of the Association of Nurses In Aids Care*, 13 (4), 84-86.

Stake, R. E. (2005). *Qualitative case studies*. In N. K. Denzin & Y. S. Lincoln (Eds.), *The Sage handbook of qualitative research* (3rd ed.) (pp. 443 – 466). Thousand Oaks, CA: Sage

Streubert, H. J., & Carpenter, D. R. (2011). *Qualitative research in nursing*. (5th ed.). Philadelphia: Lippincott Williams and Wilkins.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational researcher*, *15*(2), 4-14.

Silver, E. A. (1998). *Improving mathematics in middle school: Lessons from TIMSS and related research*. US Government Printing Office.

Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K12 curriculum. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A Research Companion to Principles and Standards of School Mathematics* (pp. 136–150). Reston, VA: National Council of Teachers of Mathematics.

Stephens, A. C., Fonger, N., Strachota, S., Isler, I., Blanton, M., Knuth, E., & Murphy Gardiner, A. (2017). A learning progression for elementary students' functional thinking. *Mathematical Thinking and Learning*, *19*(3), 143-166.

Türkmen, H., & Tanışlı, D. (2019). Cebir öncesi: 3, 4 ve 5. sınıf öğrencilerinin fonksiyonel ilişkileri genelleme düzeyleri. *Eğitimde Nitel Araştırmalar Dergisi*, *7*(1), 344-372.

Warren, E., & Cooper, T. (2005). Introducing Functional Thinking in Year 2: A Case Study of Early Algebra Teaching. *Contemporary Issues in Early Childhood*, 6(2), 150–162. https://doi.org/10.2304/ciec.2005.6.2.5

Wilkie, K. J. (2014). Upper primary school teachers' mathematical knowledge for teaching functional thinking in algebra. *Journal of Mathematics Teacher Education*, *17*, 397-428.

Wilkie, K. J. (2016). Learning to teach upper primary school algebra: Changes to teachers' mathematical knowledge for teaching functional thinking. *Mathematics Education Research Journal*, *28*, 245-275.

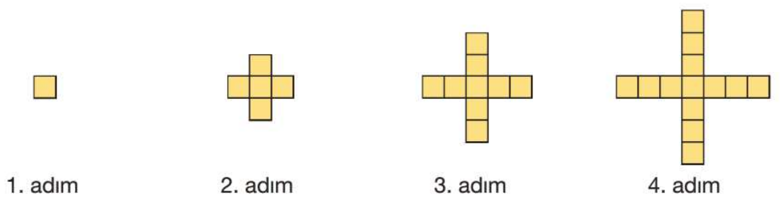
Yılmaz Tığlı, N. (2023). *Middle School Mathematics Teachers' Knowledge of Eighth-Grade Students' Algebraic Thinking*. (Doctoral thesis, Middle East Technical University).

Yin, R.K. (2003). *Applications of case study research*. (2nd ed.). Thousands Oak, CA: Sage.

APPENDICES

A. INTERVIEW PROTOCOL

1) Aşağıda bir geometrik örüntü verilmiştir.



1) 5. adımda kaç tane birim kare olur?

2) 100. adımda kaç tane birim kare olur?

3) Herhangi bir adımdaki birim kare sayısını nasıl ifade edersiniz? Bu ilişkiyi ifade eden kuralı sözcüklerle ve değişkenlerle ifade edin.

a) Bu sorunun amacı ne olabilir? Açıklar mısınız?

b) Yukarıdaki soruda öğrencilerden beklediğiniz olası doğru cevaplar nelerdir?

c) Yukarıdaki sorularda öğrencilerden beklediğiniz olası yanlış cevaplar nelerdir?

d) Öğrencilerin bu hataları/yanlışları neden kaynaklanıyor olabilir? Öğrencilerinizden gelebilecek bu hatalı/yanlış cevapları nasıl giderebilirsiniz?"

e) Yukarıdaki soruda Öğretmen öğrencisinden 15. adımdaki birim kare sayısını bulmak için aşağıdaki gibi bir tablo oluşturmasını istemiştir.

Adım sayısı	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Birim kare sayısı	1	5	9	13	17	21	25	29	33	37	41	45	49	53	57

Sizce, Öğretmenin öğrenciden tablo oluşturmasını istemesinin amacı ne olabilir? Açıklar mısınız?

2)

a) Bu ip yukarıdaki gibi 1 kesim yapılmıştır. Kaç parça ipiniz var?
b) 2 kesim yapın ve ipin kaç parçadan oluştuğunu bulun. Bunu her zaman yeni bir ip parçası kullanmayı hatırlayarak 3, 4 ve 5 kesim için tekrarlayın.
c) Herhangi bir kesimde oluşan parça sayısını sözcüklerle ve değişken kullanarak ifade edin.

a) Bu sorunun amacı ne olabilir? Açıklar mısınız?

b) Yukarıdaki soruda öğrencilerden beklediğiniz olası doğru cevaplar nelerdir? Niçin?

c) Yukarıdaki soruda öğrencilerden beklediğiniz olası yanlış cevaplar nelerdir? Niçin?

d) 4. soruda üç öğrencinin yapmış olduğu çözümler aşağıdaki gibidir;

1. öğrenci:

Kesim sayısı	Oluşan parça sayısı
1	3
2	5
3	7

‘’Burada y toplam parça sayısı x kesim sayısı olmak üzere 2 eklenerek gittiği için y= 2x olarak ifade edebiliriz.’’ Bu öğrenci sizce nasıl düşünmüştür? Bu öğrenciyle bir sonraki adımda ne yaparsınız?

2. öğrenci:

“Kesim sayısı birer artarken oluşan parça sayısı 2’şer artıyor” diye yorumlamıştır. Bu öğrenci sizce nasıl düşünmüştür? Bu öğrenciyle bir sonraki adımda ne yaparsınız?

3. öğrenci:

‘’ Bu soruda her kesimde 2 parça geldiği için ve en başta 1 düğüm olduğu için kesim sayısını 2 ile çarpıp 1 eklediğimizde oluşan parça sayısına ulaşıyoruz. Y toplam parça sayısı x kesim sayısı olmak üzere y= 2x+1 olarak ilişkiyi ifade edebiliriz.'' Bu öğrenci sizce nasıl düşünmüştür? Bu öğrenciyle bir sonraki adımda ne yaparsınız?

3)Aşağıdaki 8. Sınıf kazanımı için bir ders planı hazırlayacaksınız. Nasıl bir ders planı hazırlardınız? Ders planı hazırlarken neleri göz önüne alırdınız? Kullanacağız örnekler, materyaller neler olurdu?

''**M.8.2.2.3.** Aralarında doğrusal ilişki bulunan iki değişkenden birinin diğerine bağlı olarak nasıl değiştiğini tablo ve denklem ile ifade eder.''

B. APPROVAL OF THE UNIVERSITY HUMAN SUBJECTS ETHICS COMMITTEE

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ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

13 EKİM 2023

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Dr. Öğr. Üyesi Işıl İşler Baykal

Danışmanlığını yürüttüğünüz Rumeysa UZUN'un ***"Ortaokul Matematik Öğretmenlerinin Fonksiyonel Düşünme Bağlamında Pedagojik Alan Bilgilerinin İncelenmesi"*** başlıklı araştırmanız İnsan Araştırmaları Etik Kurulu tarafından uygun görülerek **0435-ODTUİAEK-2023** protokol numarası ile onaylanmıştır

Bilgilerinize saygılarımla sunarım.

Prof. Dr. Ş. Halil TURAN
Başkan

Prof.Dr. İ. Semih AKÇOMAK
Üye

Doç. Dr. Ali Emre Turgut
Üye

Doç. Dr. Şerife SEVİNÇ
Üye

Doç.Dr. Murat Perit ÇAKIR
Üye

Dr. Öğretim Üyesi Süreyya ÖZCAN KABASAKAL
Üye

Dr. Öğretim Üyesi Müge GÜNDÜZ
Üye

C. MoNE PERMISSION

T.C.
İSTANBUL VALİLİĞİ
İl Milli Eğitim Müdürlüğü

Sayı : E-59090411-20-93227493 29/12/2023
Konu : Anket ve Araştırma İzni (Rumeysa UZUN)

VALİLİK MAKAMINA

İlgi : a) Yenilik ve Eğitim Teknolojileri Genel Müdürlüğünün 21.01.2020 tarihli ve 2020/2 sayılı genelgesi.
b) Orta Doğu Teknik Üniversitesinin 06.11.2023 tarihli ve 54850036-605.01.-E.524 sayılı yazısı.
c) Müdürlüğümüz Araştırma ve Anket Komisyonunun 26.12.2023 tarihli tutanağı.

Araştırma Konusu	: Ortaokul Matematik Öğretmenlerinin Fonksiyonel Düşünme Bağlamında Pedagojik Alan Bilgileri
Araştırma Türü	: Anket / Görüşme
Araştırma Yeri	: [redacted]
Araştırma Kişiler	: Ortaokul Öğretmenleri
Araştırmanın Süresi	: 2023 - 2024 Eğitim - Öğretim Yılı

Yukarıda bilgileri verilen araştırmanın; 6698 sayılı Kişisel Verilerin Korunması Kanununa aykırı olarak kişisel veri istenmemesi, öğrenci velilerinden açık rıza onayı alınması, bilimsel amaç dışında kullanılmaması, bir örneği Müdürlüğümüzde muhafaza edilen mühürlü ve imzalı veri toplama araçlarının kurumlarımıza araştırmacı tarafından ulaştırılarak uygulanması, katılımcıların gönüllülük esasına göre seçilmesi, araştırma sonuç raporunun kamuoyuyla paylaşılmaması ve araştırma bittikten sonra 2 (iki) hafta içerisinde Müdürlüğümüze gönderilmesi, okul idarelerinin denetim, gözetim ve sorumluluğunda, eğitim ve öğretimi aksatmayacak şekilde, ilgi (a) genelge esasları dâhilinde uygulanması kaydıyla Müdürlüğümüzce uygun görülmektedir.

Makamınızca da uygun görüldüğü takdirde olurlarınıza arz ederim.

Doç. Dr. Murat Mücahit YENTÜR
İl Milli Eğitim Müdürü

OLUR

Mustafa KAYA
Vali a.
Vali Yardımcısı

Ek:
1- İlgi (b) Yazı ve Ekleri (9 Sayfa)
2- İlgi (c) Tutanak (1 Sayfa)