MIDDLE SCHOOL MATHEMATICS TEACHERS’ UNDERSTANDING OF THE SINGAPORE BAR MODEL METHOD AS A PROBLEM-SOLVING HEURISTIC IN ALGEBRA TEACHING

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submitted by **FEYZA ARİFE ÖZALP** in partial fulfillment of the requirements for the degree of **Master of Science** i**n Mathematics Education in Mathematics and Science Education, Middle East Technical University** by,

Prof. Dr. Naci Emre Altun
Dean, **Graduate School of Natural and Applied Sciences** ____________________

Prof. Dr. Mine Işıksal Bostan
Head of the Department, **Mathematics and Science Education** ____________________

Assoc. Prof. Dr. Şerife Sevinç
Supervisor, **Mathematics and Science Education, METU** ____________________

Examining Committee Members:

Assoc. Prof. Dr. Zeynep Sonay Ay
Mathematics and Science Education, Hacettepe University ____________________

Assoc. Prof. Dr. Şerife Sevinç
Mathematics and Science Education, METU ____________________

Assist. Prof. Dr. Işıl İşler Baykal
Mathematics and Science Education, METU ____________________

Date: 03.09.2024

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name Last name : Feyza Arife Özalp

Signature :

ABSTRACT

MIDDLE SCHOOL MATHEMATICS TEACHERS' UNDERSTANDING OF THE SINGAPORE BAR MODEL METHOD AS A PROBLEM-SOLVING HEURISTIC IN ALGEBRA TEACHING

Özalp, Feyza Arife
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This study aims to examine strategies of middle school mathematics teachers for solving algebra problems and their use of the Singapore Bar Model method. Data were collected from eighteen middle school mathematics teachers on an online platform. One-on-one interviews were recorded with the permission of the teachers. Each teacher was asked about three categories of algebra problems determined according to a previous study, and they were asked for solution strategies. The strategies used in these problems were analyzed by considering problem-solving heuristics. The frequencies of the strategies used by teachers were compared across the problems. In addition, the teachers examined the solutions in which students used the bar model method in similar problem styles. The study examined how teachers understand and use bar models in terms of the components and quantitative relations of the bar model. The findings showed that the teachers tended to use equations for the solutions of algebra problems. Also, based on the findings, the teachers did not know the terminology of the Bar Model method and they conceived the bar model as a representative object, not as a quantity. According to the results of the study, it was observed that the teachers perceived the importance of using the Bar Model method in algebra teaching due to its potential to provide students with a concrete space. This study suggests that curriculum developers and teacher educators should pay more attention to the use of the Bar Model in teaching so that teachers can be competent in teaching algebra.

Keywords: Algebra Teaching, Singapore Bar Model Method, Teachers' Understanding of Algebra, Middle School Mathematics Teachers

ÖZ

ORTAOKUL MATEMATİK ÖĞRETMENLERİNİN CEBİR ÖĞRETİMİNDE BİR PROBLEM ÇÖZME YÖNTEMİ OLARAK SİNGAPUR BAR MODELİ YÖNTEMİNİ ANLAYIŞLARI

Özalp, Feyza Arife
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Bu araştırmanın amacı, ortaokul matematik öğretmenlerinin cebir problemlerini çözme stratejilerini ve Singapur matematik müfredatında yaygın olarak kullanılan Bar Model yöntemini cebir öğretiminde kullanımlarını incelemektir. Veriler on sekiz ortaokul matematik öğretmeninden çevrimiçi bir platform üzerinden toplanmıştır. Öğretmenlerden izin alınarak bire bir görüşmeler kaydedilmiştir. Her öğretmene daha önceki bir çalışmaya göre belirlenen üç kategorideki cebir problemleri sorulmuş ve çözüm stratejileri istenmiştir. Bu problemlerde kullanılan stratejiler problem çözme yöntemleri dikkate alınarak analiz edilmiştir. Öğretmenlerin kullandıkları stratejilerin frekansları problemler arasında karşılaştırılmıştır. Ayrıca öğretmenler, öğrencilerin benzer problem tarzlarında bar model yöntemini kullandıkları çözümleri incelemişlerdir. Öğretmenlerin öğrenci çözümlerini kavramaları bar modelinin bileşenleri ve niceliksel ilişkileri açısından incelenmiş ve analiz edilmiştir. Bulgular, öğretmenlerin cebir problemlerinin çözümlerinde denklem kullanma eğiliminde olduklarını göstermiştir. Ayrıca bulgulara göre, öğretmenler Bar Model yöntemi terminolojisini bilmemekte ve bar modeli bir nicelik olarak değil, temsili bir nesne olarak algılamaktadırlar. Çalışmanın sonuçlarına göre, öğretmenlerin öğrencilere somut bir alan sağlama potansiyeli nedeniyle Bar Model yönteminin cebir öğretiminde kullanılmasının önemini algıladıkları görülmüştür. Bu çalışma, öğretmenlerin cebir öğretiminde yetkin olabilmeleri için program geliştiricilerin ve öğretmen eğitimcilerinin öğretimde Bar Model kullanımına daha fazla önem vermelerini önermektedir.

Anahtar Kelimeler: Cebir Öğretimi, Singapur Bar Model Yöntemi, Öğretmenlerin Cebir Bilgisi, Ortaokul Matematik Öğretmenleri

To my family for their endless love, support and encouragement…

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LIST OF ABBREVIATIONS

NCTM	National Council of Teachers of Mathematics
TIMSS	Trends in International Mathematics and Science Study
CPA	Concrete-Pictorial-Abstract
SMC	Singapore Mathematics Curriculum

CHAPTER 1

INTRODUCTION

School mathematics is clustered into five main learning domains, and algebra is one of them (NCTM, 2000). Algebra encompasses a realm of mathematics where equations substitute numbers with symbols, and mathematical operations are executed on these symbolized values (Saleh & Rahman, 2016). Algebra goes beyond just letters and numbers; it encompasses tables, graphs, number relationships, properties, and their applications. It serves as a language to articulate situations in numerical terms and embody numerical changes within a given scenario. Hence, algebra is defined as the language of mathematics (Lacampagne et al., 1995). This concept aligns with a broader interpretation of algebra as a discipline that deals with abstract structures and their operations, placing emphasis on the principles and patterns that govern mathematical expressions. The computational aspect of algebra not only encompasses arithmetic operations but also involves the examination and application of algebraic reasoning to comprehend the properties and behaviors of mathematical structures (Caspi & Sfard, 2012). Algebraic reasoning extends beyond mere algebra, encompassing a broader scope of mathematical thinking, so algebra learning has a significant place in mathematics learning.

Algebra is necessary for all areas and concepts of mathematics. Despite its important role in mathematics, algebra remains difficult for students to understand. Studies have been conducted showing that students' success rates in learning algebra is low (Bütüner & Güler, 2017; Çelik & Güneş, 2013; Yenilmez & Avcu, 2009). In this sense, studies conducted in Turkey have also revealed that TIMSS scores in the field of algebra are clearly lower than the international average (Bütüner & Güler, 2017). Because of these low performances, studies on teaching algebra still have a significant place in mathematics education.

In the light of the studies, many findings have surfaced about the reasons behind the difficulties experienced by students in algebra. Research shows that students often have difficulty understanding algebraic concepts and fail to develop necessary skills, which leads to poor academic performance (Knuth et al., 2005). In addition, it has been observed that students have difficulty understanding abstract concepts when moving from arithmetic to algebra (Kaminski & Sloutsky, 2012). The abstract nature of algebraic concepts, such as variables, equations, and functions, often creates a challenge for students striving to comprehend the subject. It is evident that students have difficulties in algebra and in order to overcome these difficulties, it is necessary to use different methods considering the abstract nature of algebra (Kayani & Ilyas, 2014).

In order to facilitate algebra learning, educators employ various effective teaching strategies, one of which involves the use of manipulatives and visual representations to tackle problems effectively. This approach supports students in transitioning from abstract concepts to concrete applications (Strickland & Maccini, 2012). Moreover, a specific strategy within this method entails the utilization of representations. In their publication, the National Council of Teachers of Mathematics emphasized the crucial role of representation in facilitating students' understanding of mathematical concepts (NCTM, 2000). According to the NCTM, using various forms of representation, such as visual models, diagrams, and concrete examples, is essential to help students comprehend, clarify, and extend mathematical ideas. By employing these representations, students become better equipped to organize their thoughts, explain complex concepts, and ultimately solve algebra word problems more effectively.

The role of teachers in algebra teaching is very important, and teachers’ algebra knowledge is also included in studies on algebra teaching. In this context, many studies have been conducted on the effects of teachers' algebra knowledge on student achievement (Hill et al., 2005; Osei & Kubi, 2022). As mentioned above, students need to make connections between concrete and abstract concepts in order to make sense of algebra and teachers have a crucial role in fostering students' algebraic thinking. They should develop learning environments and tools that encourage students to model, explore, discuss, predict, conjecture, and test ideas, while also honing their computational skills (Blanton & Kaput, 2004). Visualization is one of these tools that helps students better understand algebra problems and abstract concepts in algebra and develop a system to think about them (Osman et al., 2018). Many methods can be used for visualization, such as the "Bar Model Method" which is a diagram used in Singapore mathematics (Ng & Lee, 2009; Kaur, 2019).

Singapore Mathematics is a renowned approach to teaching mathematics that is internationally recognized for its effectiveness in fostering positive attitudes toward mathematics and nurturing a strong belief in the importance of mathematics for future careers (Kaur, 2019). This approach emphasizes mastery of concepts through a focused and coherent curriculum, which helps students build a deep understanding of mathematical principles. The Bar Model Method, a problem-solving technique that uses visual bar models to represent and solve mathematical problems, is central to the Singapore Mathematics curriculum (Ng & Lee, 2009). The bar model method is actually a problem-solving strategy. It is used to represent and solve mathematical problems, especially in algebra and fractions. It is a teaching method that uses bar diagrams to visually represent the structure of word problems given to students (Looi & Lim, 2009). Research has shown that using bar models in mathematics education effectively helps students tackle complex word problems that can be difficult without visual aids (Baysal & Sevinç, 2022; Low et al., 2020; Mahoney, 2012).

Considering the undeniable influence of teachers on student achievement and the success of teaching with models, it is essential for teachers to have sufficient knowledge and understanding on the use of these models in the algebra learning process. Numerous research studies have consistently demonstrated the efficacy of the bar model method in enhancing students' mathematical thinking abilities and facilitating the transition from abstract concepts to the application of concrete mathematical principles (Baysal & Sevinc, 2022; Cai & Moyer, 2008; Ng & Lee, 2009). This approach holds significant promise for advancing students' mathematical skills and comprehension. There are also studies on how teachers' algebra knowledge and understanding affects students' algebra achievement (Ball et al., 2008; Fennema & Franke, 1992; Hill et al., 2005). However, there are no studies in the existing literature on teachers' knowledge of algebra instruction and the bar model method.

Considering the effects of visualization through the bar model and teacher' understanding on student achievement, it can be thought that teachers' ability to visualize will also affect student achievement (Cai, 2005). For this reason, in this study, the teachers were asked about algebra problems and then interviewed about students' solutions to parallel questions using the bar model method. Subsequently, their opinions about the use of the stick model method in solving algebra problems were examined.

1.1 Purpose of the Study and Research Questions

The main aim of this study is to examine middle school mathematics teachers' problem-solving heuristics in algebra word problems and their understanding of the bar model method for algebra teaching. The research questions addressed in this study are as follows:

1- What are the problem-solving heuristics of middle school mathematics teachers in algebra word problems?

2- What do middle school mathematics teachers understand of the bar model method when examined students' bar model solutions in algebra problems?

3- What are the middle school mathematics teachers' conceptions about using the bar model method in algebra teaching?

1.2 Significance of the Study

Algebra is a fundamental branch of mathematics that introduces students to the concept of using letters and symbols to represent numbers and quantities in equations and formulas (NCTM, 2000). It is a crucial component of mathematics education, serving as a bridge to advanced mathematical concepts and their practical applications. Studying algebra equips students with the essential skills of critical thinking, problem-solving, and logical reasoning, all of which are fundamental for their academic and professional advancement. Understanding algebra is key to mastering higher-level mathematical concepts and applying them in various fields such as science, engineering, economics, and technology (Yenilmez & Avcu, 2009). The ability to manipulate algebraic expressions and solve equations is invaluable in analyzing real-world situations, making informed decisions, and addressing complex problems in different domains.

Despite its important role in mathematics, algebra remains a difficult topic not only for students but also for teachers to learn and teach (Drijvers et al., 2009). For this reason, many studies are still being conducted on algebra learning and algebra knowledge. There are studies in the literature on the low success rates of students in algebra compared to other subjects and the reasons behind this (Booth & Koedinger, 2008; Kieran, 2004; McNeil & Alibali, 2005). Studies conducted in Turkey also show that algebra success is quite low (Altun & Arslan, 2006; Aydın & Özgeldi, 2010; Boz, 2012; Işık & Kar, 2012). These studies highlight various factors contributing to low performance in algebra, such as insufficient conceptual understanding, challenges transitioning from arithmetic to algebra, and misunderstandings about algebraic operations. Research findings in literature on education indicate a strong relationship between teachers' depth of knowledge in algebra and the academic progress of their students (Hill et al., 2005; Rowan et al., 2001). These findings underscore the critical importance of educators possessing a robust understanding of algebraic concepts and their ability to address common student misconceptions effectively. This emphasizes the pivotal role that teachers play in facilitating effective instruction and improving student learning outcomes in algebra. Studies on teaching algebra show that teaching methods supporting the transition from abstract to concrete in algebra have a positive effect on student success (Cai & Moyer, 2008; Witzel, 2005). In the field of education, it is strongly advocated to incorporate visual representations, in the teaching of mathematics, especially in special education settings. This practice has been recommended to provide significant support for students' learning process (Garderen et al., 2016). One of the methods used to concretize the abstract nature of algebra is the Singapore Bar Model method. There are studies demonstrating that using the Singapore Bar Model has the potential to improve students' ability to learn different topics such as algebra, ratio, additive word problems, problem solving skills etc. (Baysal & Sevinc, 2022; Koleza, 2015; Mahoney, 2012; Ng & Lee, 2005; Kaur et al., 2004). However, studies on teachers' understanding of using the bar model in those topics is not sufficient (Sevinc & Lizano, 2022). No studies have been found in the literature in which teachers' understanding of teaching algebra with the bar model. Therefore, the aim of this study is to address this gap in the literature by exploring middle school mathematics teachers' understanding of the bar model in algebraic word problems.

The limited number of studies on how teachers use the bar model and how this model contributes to the teaching process increases the importance of our research. This study aims to examine the competencies of middle school mathematics teachers regarding the use of the bar model in algebra teaching and to understand their views on the use of the bar model in teaching. Furthermore, it is expected to provide significant findings on how teachers can effectively use the bar model in algebra instruction. This study is considered important not only for teachers but also for students and even curriculum developers. As a result, our study makes essential contributions to overcoming the difficulties encountered in teaching algebra, increasing the understanding of teachers, and promoting the use of the Singapore Bar Model as an effective teaching tool. These contributions are both theoretically and practically valuable and provide a foundation for future research in mathematics education. The findings of our study can help develop more effective strategies in both teacher training and algebra instruction.

1.3 Definitions of Terms

The definitions of important terms used in this study are given in this section.

Algebra: Algebra, a fundamental branch of mathematics, involves working with symbols and the rules for manipulating them (Kieran, 1992).

Algebra Word Problem: Word problems are typically defined as verbal explanations of problem situations, including numerical data and mathematical operations, are provided in the existing problem statement by asking one or more questions (Verschaffel et al. 2000). In this study, ‘algebra word problem’ refers to problems that include unknown values and require solving problems with algebraic calculations.

Problem-Solving Heuristic: Problem-solving heuristics are potent strategies, methods, or rules of thumb that individuals confidently use to approach and solve problems efficiently (Abel, 2003). In this research, the term ‘problem-solving heuristic’ refers to techniques or approaches that teachers use when they see an algebra problem.

Bar Model Method: A problem-solving approach that that utilizes rectangular bars to represent numbers, rather than using abstract letters to signify unknowns in word problems. This visual method can help students better understand and solve mathematical problems by providing a concrete representation of the quantities involved (Koleza, 2015).

Middle School Mathematics Teachers: In the context of middle school, mathematics teachers typically work with students in grades 6 through 8, though this can vary slightly depending on the country or educational system. In this study, middle school mathematics teachers work with grade 5, 6, 7, and 8 students. In grade 5, students are usually around 10-11 years old. In grade 6, students are usually around 11-12 years old. In grade 7, students are usually around 12-13 years old. In grade 8, students are usually around 13-14 years old (MoNE, 2018). Middle school teachers focus on teaching subjects appropriate for this age group, preparing students for more advanced topics in high school. In mathematics, this often includes topics like pre-algebra, algebra, geometry, and data analysis.

CHAPTER 2

LITERATURE REVIEW

The main goal of this study is to examine middle school mathematics teachers' problem-solving heuristics in algebra word problems and their understanding of the bar model method for algebra teaching. This chapter includes the related literature on algebra and algebra teaching, teachers' knowledge and understanding, teachers' problem-solving heuristics, the Singapore Bar model method, and related studies.

2.1 Algebra and Algebra Teaching

Most of the concepts in mathematics, especially in advanced mathematics, are generally abstract, so the transition to abstract thought is significant for learning and understanding mathematics (Tall, 2008). Abstract thought is the ability to think about complex concepts that are not tied to immediate sensory experiences and understand ideas, symbols, and relationships beyond the realm of concrete objects. It is a concept centered on reasoning with relationships (Green, 2017). Since algebraic thinking is also a mental process that involves reasoning with unknowns, analyzing patterns and relations, making generalizations about them, and comprehending the concept of variables, it is connected with abstract thought (Van Amerom, 2002). At the same time, algebraic thinking helps open the doors of abstract thought that is necessary in mathematics education.

Swafford and Langrall (2000) defined algebraic thinking and reasoning as "the ability to operate on unknown quantities as well as known quantities." Moreover, mastering algebra is necessary/essential to develop and improve algebraic thinking and skills. Algebra plays a crucial role in the development of algebraic thinking and is regarded as an entry point to thinking in algebraic and abstract thoughts (Levin &

Walkoe, 2022; Witzel et al., 2003). Research carried out by Blanton and colleagues (2015) state that comprehensive early algebra intervention significantly develops students' ability to use algebraic strategies and thinking to solve problems. Therefore, it's essential to introduce algebra at an early stage in childhood during primary education to foster the growth of algebraic thinking (Sibgatullin et al., 2022).

Algebra holds s significant place within the field of mathematics. When people think of mathematics as a language, they often think of algebra, which involves using symbols to express and manipulate general concepts in numerical contexts, especially in school settings. In the literature, there are different definitions and explanations for algebra. In most cases, algebra refers to generalization of arithmetic and uses symbols instead of numbers (Dekker & Dolk, 2011; Putri et al., 2020). Algebra allows generalization of arithmetic relations, operations, and properties. The generalization of arithmetic is defined as operating with numbers and making judgments about the relation of numbers and properties in general (Carpenter et al., 2003). For example, starting from the expression 3+5=8 and arriving at the statement the sum of odd numbers is equal to an even number" is a mathematical generalization and is based on algebraic thinking. Similarly, in subtraction, being able to state that the sum of minuend, subtrahend, and difference is equal to two minuends is also based on algebraic thinking. In other words, our ability to generalize the situations and properties we see in numbers and operations to larger sets of numbers or sets of operations is linked to the generalized meaning of algebra and arithmetic. The accurate processing of symbols during this generalization is also very important for the definition of algebra.

From a similar perspective, Sfard (1995) claimed that algebra is the science of calculation. This definition implies that algebra is a systematic way of performing mathematical operations and solving equations. In the sense of being "the science of computation," algebra involves not only the processing of numbers and symbols but also the study of structures and relationships within mathematical systems. With this concept, it is emphasized that algebra has a meaning that reveals abstract structures and mathematical rules and provides a broader view of mathematics.

Algebra is formally defined as a symbolic representation used to articulate the correlation between various quantities and the language of specific patterns and rules (O'Bannon et al., 2002). At this point, the importance of algebra also becomes apparent in terms of concretizing abstract concepts. Concrete materials used in algebra, which contain real-life objects that appeal to the sensory organs, are used to concretize the subject area (Van de Walle, 2007). In 1992, Kieran highlighted the dual role of algebra, emphasizing that it involves representing various quantities using symbols and facilitating mathematical calculations using these symbolic representations. In this way, it can be used as a tool for problem-solving as well as for finding different ways to solve problems. From all these definitions, it can be concluded that algebra has a different mission from mathematics. In essence, Lacampagne (1995) states that algebra is the language of mathematics. It means that a comprehensive understanding of algebra is imperative to attain proficiency in mathematics. Algebra is therefore regarded as an essential component of students' mathematical education (Wang, 2015).

Algebra holds an important place in mathematics education for several reasons. Usiskin (1995), in his article explaining the importance of algebra, states that algebra has a significant place in comprehending ideas about various fields, such as chemistry, physics, the earth sciences, etc., and in capturing different career and job opportunities. The National Council of Teachers of Mathematics also emphasizes that every student should learn algebra as it is important in mathematics education (NCTM, 2000). Mastery of algebra allows students to develop their skills in solving problems, improve their understanding of logical connections, and establish a structure for formulating and solving equations related to real-life problems such as age, work, motion, or currency (Usiskin, 1995; Rahmawati et al., 2019). These real-life problems serve as a bridge between abstract mathematics and daily life situations. Students who see the place of mathematics in daily life and use mathematics in these situations can easily internalize and comprehend mathematics meaningfully (NCTM, 2000). Moreover, word problems have a crucial place in the sense of helping students use mathematics in their everyday lives, so it is also important to include algebraic word problems for algebra teaching (Baysal & Sevinç, 2022; Chang, 2010).

2.1.1 Algebra Teaching for Students

Despite being so important, the abstract nature of algebra and the variety of algebraic symbols make it quite challenging to learn algebra for students (Çelik & Güler, 2013; Cousins-Cooper et al., 2017; Ferretti et al., 2018; Rakes et al., 2010; Witzel et al., 2003). As teachers face challenges in finding and applying most conceptual approaches to teach algebra conceptually, students encounter difficulties in comprehending algebraic concepts (Akkan et al., 2012). There are several studies demonstrating the difficulties of algebra for students (Cai & Moyer, 2008; Kaput, 2008; Kilpatrick et al., 2001; Ng & Lee, 2009; Radford, 2007; Sadovsky & Sessa, 2005; Welder, 2012).

Wang (2015), who also conducted a literature review on the difficulties students face in learning algebra, examined these studies under five categories. According to Wang's literature review (2015), these five categories, which can also be used as a framework to understand students' difficulties in learning algebra, are algebra content, cognitive gap, teaching issues, learning matters, and transition knowledge. According to studies in the algebra content category, students experience difficulties in subjects such as creating equations, symbolic representations and solving equations (Kaput, 2008; Radford, 2007). According to studies examined in the cognitive gap category, students have difficulties in understanding non-semantic symbolic representations, performing operations with them, expressing word problems symbolically, and operating with the unknown (Banerjee & Submaniam, 2012; Kieran, 1992). Studies in the category of teaching issues indicate that the lack of connection between primary school arithmetic and algebra, teachers' attitudes towards algebra, and teaching methods cause students to have difficulties in algebra (Cai & Moyer, 2008). Studies examined in the learning matters category show that the lack of knowledge of operational symbols, simplified expressions, the concept of equality, and the stages in word problems (establishing relationships between given quantities, establishing equations expressing relationships, and solving equations) are the reasons why students have difficulty in algebra (Ng & Lee, 2009; Welder, 2012). According to studies in the transition knowledge category, students have difficulty transitioning from arithmetic to algebra, which arises from issues such as adaptation and social interaction (Kilpatrick et al., 2001; Sadovsky & Sessa, 2005).

Apart from these studies, Welder (2012) stated that students also face some misconceptions in the use of brackets (parenthesis). This issue often emerges when students are introduced to more complex expressions that require careful consideration of the order of operations. The lack of complete understanding of bracket usage, especially in terms of order of operations, leads to confusion in algebra learning (Christou & Vosniadou, 2012). As students attempt to solve algebraic problems, they may misinterpret the role of brackets, leading to incorrect solutions. These difficulties can also be attributed to students' lack of proficiency in arithmetic operations, which forms the foundational skills necessary for algebra.

Additionally, challenges in symbolizing and modeling problems make it more complicated for students to grasp abstract concepts, as these skills are critical for understanding the structural aspects of algebra. Another common issue is the difficulties students face in applying the concept of variables across different scenarios (Welder, 2012). Variables represent unknown quantities that can change, and without a solid grasp of their purpose, students often struggle to use them correctly. Furthermore, the tendency to employ consistent problem types and solution approaches in teaching can reinforce a limited understanding of algebraic concepts. This approach encourages rote memorization rather than genuine comprehension, thereby complicating the shift from arithmetic to algebra (Chow, 2011). To foster more profound understanding, educators need to diversify problem types and encourage exploratory learning, which can bridge the gap between concrete arithmetic operations and abstract algebraic reasoning.

The success of students in algebra is directly linked to the proficiency of teachers in teaching algebra (Hill et al., 2005). Understanding the role of teachers in the field of algebra is crucial for effective learning. However, the abstract nature of algebra and the variety of algebraic symbols make algebra quite challenging not only for students but also for teachers (Cousins-Cooper et al., 2017; Çelik & Güler, 2013; Ferretti et al., 2018; Rakes et al., 2010; Witzel et al., 2003). Teachers can also face multifaceted difficulties in algebra related to many different topics and have difficulties helping students understand the initial algebra instruction (Witzel et al., 2003).

2.1.2 Teachers' Difficulties in Algebra

The most challenging aspect of teaching algebra for educators is conveying abstract concepts such as algebraic expressions and equations to students (Booth & Koedinger, 2008). The transition from arithmetic to algebra is crucial, and it's important to make prior concepts more tangible so that students can better understand abstract ideas (Kaput, 2008). Teachers should use strategies that make previous ideas easier to understand, helping students grasp new and complex concepts. Teachers can help students comprehend abstract concepts by incorporating concrete manipulatives and visual representations in the teaching process. To address these challenges, teachers can utilize effective teaching strategies such as the Concrete-Representational-Abstract Integration Strategy (Strickland & Maccini, 2012). Additionally, the use of concrete manipulatives in a structured instructional sequence, such as the Concrete-Representational-Abstract (CRA) approach, has proven effective in making algebra instruction more accessible to students, especially those with learning difficulties (Witzel et al., 2003).

Another challenge for teachers in teaching algebra is to address students' misconceptions and errors (Knuth et al., 2005). Understanding the difference between students' misconceptions and errors is also crucial for teachers in teaching algebra. Misconceptions often reflect more fundamental problems in students' understanding of algebraic concepts, making it essential to address them (Welder, 2012). Misconceptions can hinder students' progress in algebra learning and algebra achievement, so it is essential for teachers to identify and respond to these misconceptions (Yıldız & Akyüz, 2019). Teachers need to have a deep and flexible comprehension of the mathematics topics that they teach to understand how algebraic structures can be effectively elicited when teaching arithmetic and to avoid causing misconceptions in students (Ball et al., 2008).

Middle school educators confront significant challenges when teaching algebra, including facilitating the transition from arithmetic to algebra, establishing connections between abstract and concrete concepts, and identifying and addressing students' misunderstandings and errors. Overcoming these challenges requires a thorough understanding of algebraic principles, targeted interventions, and pedagogical strategies that effectively meet the diverse learning needs of students.

2.2 Teachers’ Knowledge and Understanding for Algebra Teaching

Learning and teaching are connected to each other, so teachers play a significant role in shaping students' understanding of mathematics (Van de Walle, 2018). The effectiveness of mathematics education depends on the teacher's proficiency, which includes what they know about subjects and students and how they can use their knowledge (Ball, 2003). The history of thoughts and studies on teachers’ proficiency goes back a long way (Shulman, 1986). Reports on professional development in teaching indicate that having standards for evaluating teacher education and performance is necessary for improving the profession's respect and professional status (Shulman, 1987). Professional reform supporters believe that there is a codified knowledge base for teaching that can be represented and communicated. In Shulman's (1986) article on professional reform and reports, he points out a knowledge base that is considered more extensive than the rhetorical knowledge base cited, deeming it insufficient for assessing teaching. Shulman delineates this more intricate teacher knowledge base into three classifications: subject matter content knowledge, pedagogical content knowledge, and curriculum knowledge. Teachers' competencies are deeply intertwined with their knowledge and understanding.

Many studies have been conducted on teachers' knowledge, and many theoretical frameworks have been put forward in this sense. Special frameworks were also needed for some subject. In recent years, there has been an increasing number of studies examining teacher knowledge in a particular subject area (Li, 2007; McCrory et al., 2012). One of the specific topics addressed in studies that examine teacher knowledge is the learning domain of algebra. Since algebra is such a fundamental component of school mathematics, researchers have proposed different models that specifically examine or evaluate teachers' algebra knowledge for teaching, describing, or classifying this knowledge (Doerr, 2004; Even, 1993; Floden & McCrory, 2007; Li, 2007). Briefly, teachers’ algebra knowledge is essential for quality algebra teaching.

The quality of algebra teaching is directly linked to teachers' mathematical understanding as well, which ultimately shapes student achievement in this critical area of mathematics (Knut et al., 2005). Mathematics teachers must possess a robust conceptual understanding of algebra to teach effectively and inspire confidence in their students. This foundational knowledge is essential, as teachers are often the primary source of algebraic understanding for their students, and any gaps in their knowledge and understanding can lead to poor student performance in algebraic assessments (Fennema & Franke, 1992). Moreover, teachers' pedagogical approaches are closely linked to their understanding of algebra (Hill et al., 2008).

A significant aspect of teachers’ understanding of students is their ability to perceive and interpret students’ mathematical thinking. It is important to emphasize that teachers’ understanding of students’ mathematical thinking directly influences their instructional practices and student learning outcomes (Hill et al., 2005). Teachers’ understanding is crucial for teachers when it comes to addressing the prevalent misconceptions among students. Kaput (2008), emphasizes the importance of educators modifying their teaching techniques to help students comprehend the significance of algebraic symbols and expressions, as they form the bedrock of success in algebra. Teachers' choices in teaching techniques are often rooted in their own mathematical understanding. Studies suggest that when teachers have a deep understanding of mathematical concepts, they are more likely to use effective, student-centered instructional methods that promote conceptual understanding (Hill et al., 2008).

In summary, the importance of teachers' knowledge and understanding in mathematics and algebra learning cannot be overstated. A robust knowledge base enables teachers to employ effective and different teaching strategies and address student misconceptions. As research consistently shows, the quality of algebra instruction is directly linked to teachers' mathematical understanding, which ultimately shapes student achievement in this critical area of mathematics (Ball et al., 2008; Fennema & Franke, 1992; Ma, 1999).

2.3 Singapore Mathematics and The Bar Model Method

When examining studies on the learning and teaching of algebra, it is very important to convey concepts and relationships using different representations for the best understanding of algebra (Chu et al., 2017; Strickland & Maccini, 2012). Especially for younger age groups, different representations and visualizations make learning easier when examining the relationship between numbers and operations (Clements & Sarama, 2011). Studies support the idea that using multiple representations, such as physical, verbal, and written symbols, in mathematics education enhances mathematical thinking, understanding of concepts, and problem-solving skills (Bakar & Karim, 2019). Teachers face a significant challenge in incorporating various representations into mathematics education.

Singapore Mathematics is a highly regarded pedagogical approach to teaching mathematical concepts. This method is acclaimed for its emphasis on utilizing various visual and tactile tools, as well as different representations, to enrich students' comprehension of mathematical principles. In the Singapore Mathematics Curriculum (SMC), there are teacher guides that provide ideas to teachers, multiple representations of mathematical concepts, and the Concrete-Pictorial-Abstract (CPA) approach, which helps gain a deeper understanding of mathematics and is particularly useful for students with special needs/language barriers (Kaur, 2011). The Concrete-Pictorial-Abstract (CPA) approach is a well-established teaching strategy that progresses from the concrete stage to the pictorial stage and finally to the abstract stage to facilitate students' understanding of mathematical concepts

(Kim, 2020). Using multiple representations and the CPA approach in Singapore Mathematics contributes to improving learning by providing visual and tactile stimulation. The Bar Model Method is a prominent technique employed in Singapore Mathematics education for visual representation and problem-solving (Kho, 1987). The Bar Model Method allows for different representations and concrete, representational, and abstract learning stages in the teaching process, as stated in the CPA approach used in the Singapore mathematics curriculum. This method uses bar models to help students understand mathematical concepts and solve complex problems by visually representing the relationships between quantities. The bar model is an educational tool employing rectangular bars to visualize and solve mathematical problems (Kaur, 2019). This method transfers algebraic symbols to visual representations, making it particularly useful for learners who benefit from a more visual approach to understanding mathematical concepts.

In the model method, students visually represent the information in word problems as pictorial equations, allowing them to solve the problems by considering all the information as a connected whole rather than separate parts (Cai et al., 2005). To acquire a deeper grasp of the bar model, we can explore illustrative examples rooted in the fundamental problem structures commonly taught to primary school students in Singapore. These problem structures encompass part-whole, comparison, and change scenarios providing a comprehensive foundation for understanding the bar model (Kaur, 2019).

The part-whole model represents a quantitative relationship between the whole and its parts. This model is particularly useful for helping students solve word problems that involve understanding the relationships between the whole and its constituent parts. In Figure 2.2, these relationships are visually depicted to provide a clearer understanding of the concept.

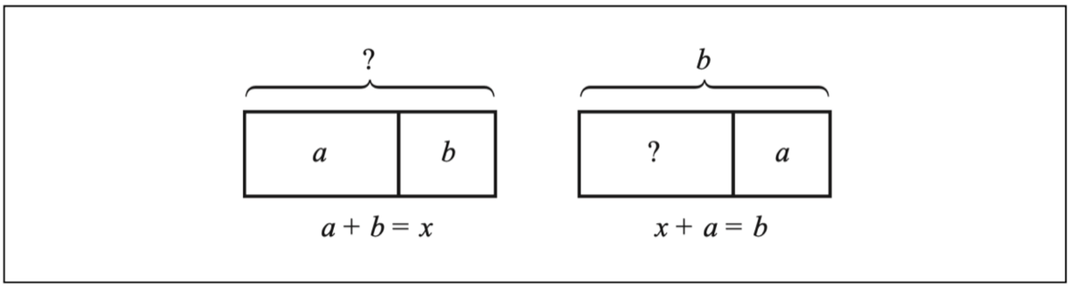


Figure 2.1 Part-whole models: Arithmetic model (on the left) and algebraic model (on the right) (Ng & Lee, 2009, p.286)

In Figure 2.2, it is illustrated that part-whole models can be expressed in both arithmetic and algebraic forms. When the quantity of whole is unspecified, the representation reflects the arithmetic model. Conversely, when the quantity of one part is unknown, it signifies the algebraic model. This demonstrates the flexibility and applicability of part-whole models in various mathematical contexts. In the part-whole model, if we know both parts, we need to add them together to find the whole. If we know one part and the whole, we need to subtract the known part from the whole to find the unknown part.

In the comparative model, two or more unknowns are compared, and the relationship between these unknowns is shown by this comparison. Without a model, students may only rely on clue words such as ‟more″ or ‟less″ and immediately perform operations to solve the problem without realizing whether they are right or wrong. In Figure 2.3, these relationships are visually depicted to provide a clearer understanding of quantitative relationships among three quantities: the larger quantity, the smaller quantity, and the difference.

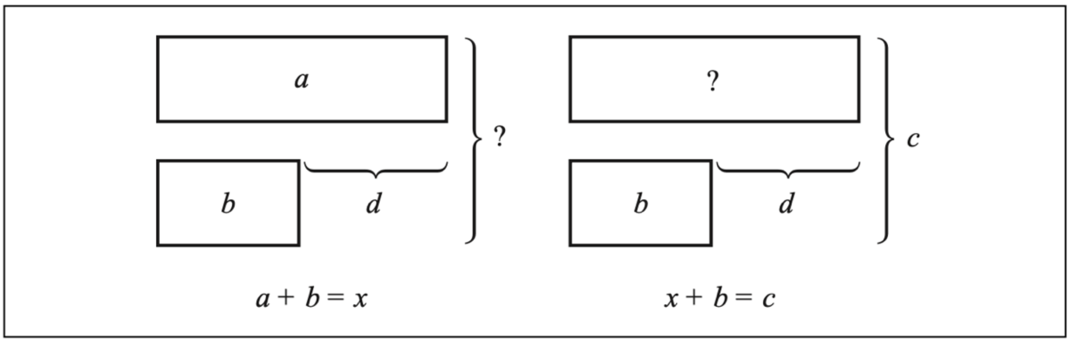


Figure 2.2 Comparison Models: Arithmetic model (on the left) and algebraic model (on the right) (Ng & Lee, 2009, p. 287)

As shown in Figure 2.3, the comparison model can be shown for both an arithmetic and an algebraic problem. The bars' lengths are also different to show that the values of the unknowns are different: the larger quantity, the smaller quantity. In this way, the difference between the rods also represents the difference between the quantities. If the larger quantity and the difference are known, subtraction is done to find the smaller quantity. If the smaller quantity and the difference are known, we perform addition to find the larger quantity. Finally, subtraction is used to find the difference if the larger and smaller quantities are known.

The change model illustrates how the initial value of a quantity relates to its new value after an increase or decrease. By understanding the magnitude of the change, we can calculate the new value based on the initial value, and conversely, determine the initial value from the new value. This model allows for efficient determination of the new value from the initial value, and vice versa. This makes it particularly useful for solving word problems involving comparing two quantities. In Figure 2.4, these relationships are visually depicted to provide a clearer understanding of the relation between quantities.

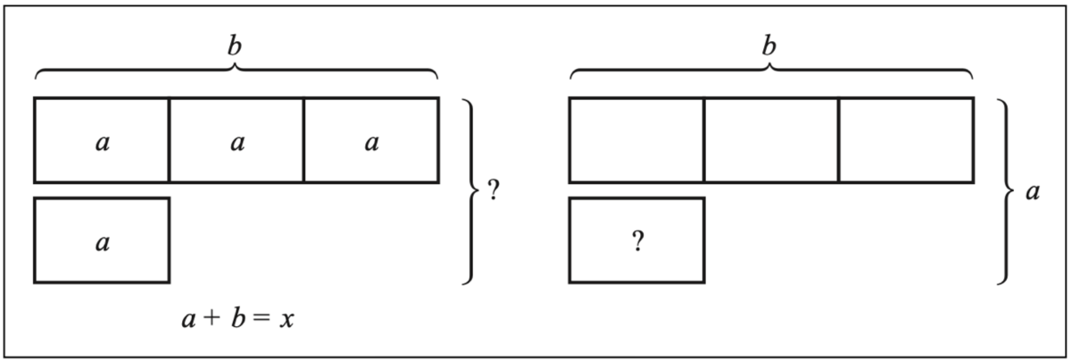


Figure 2.3 Change model for an arithmetic word problem (on the left) and an algebraic word problem (on the right) (Ng & Lee, 2009, p. 289)

In the illustration provided in Figure 2.4, it is evident that the change model encompasses both arithmetic and algebraic aspects. In this model, all the bars have the same length, and one quantity is shown as a multiple of the other variable in the model. When the ratio between the quantities is known, you can use multiplication or division to find the desired quantity.

The Bar Model method is an effective and visually intuitive approach employed in mathematics education to amplify students' proficiency in problem-solving, particularly in the domain of algebra. By leveraging this methodology, students can gain a visual representation of mathematical problems, rendering abstract concepts more tangible and simplifying the comprehension process (Booth & Davenport, 2019; Ng & Lee, 2009).

Studies have shown that implementing the Bar Model method leads to a marked improvement in students’ capacity to solve mathematical word problems. This is achieved by offering a structured approach that allows students to visualize and comprehend the relationships between different quantities (Clements & Sarama, 2009; Van de Walle, 2007). Utilizing bar models during problem-solving simplifies the process of teaching algebra for both teachers and students. For instance, when word problems are transformed into bar diagrams, students can gain a better understanding of the mathematical relationships involved, which in turn helps them comprehend and remember algebraic concepts (Booth & Koedinger, 2012). The use of visual aids helps to make abstract algebraic concepts more tangible and understandable, enabling students to translate verbal information into mathematical representations more effectively.

Shah and colleagues' research findings indicate that the implementation of the bar model visualization technique has a noteworthy influence on enhancing students' mathematical word problem-solving abilities, while also sparking a greater interest in the subject among students (Shah et al., 2021). This is in line with the findings of Ramasamy and Puteh (2019), who highlighted that the Bar Model not only helps in solving standard mathematical problems but also improves higher-order thinking skills (HOTS) among students. Similarly, in a study conducted by Osman and his colleagues, it was found that the introduction of the Bar Model led to a substantial improvement in students' mathematical problem-solving abilities. This finding reinforces the concept that utilizing visual aids can effectively boost cognitive engagement and motivation among students (Osman et al., 2018).

The collaborative nature of incorporating the Bar Model into mathematics instruction creates an environment where individual understanding is deepened, and peer learning is encouraged. This approach allows students to gain insights from each other's methods of utilizing the Bar Model, thereby transforming the educational experience to align with current goals of nurturing critical thinking and problem-solving skills in students (Bulac, 2019; Gani et al., 2019).

Additionally, the Bar Model method encourages students to engage in critical thinking and creativity as they interpret problems and determine how to visually represent them (Osman et al., 2018). This method enhances cognitive development during the problem-solving process and fosters a deeper understanding of mathematical principles, ultimately leading to improved performance in algebra (Goldin & Shteingold, 2001). In short, the Bar Model method is an effective pedagogical approach that empowers students to confidently and clearly tackle complex problems.

2.4 Teachers' Problem-Solving Heuristics

Problem-solving in mathematics education is a vital component that fosters students' analytical thinking skills and prepares them for real-world challenges (Schoenfeld, 1985). Problem-solving heuristics are strategies or techniques used to approach and solve problems effectively (Polya, 1945). They are practical methods that help individuals simplify complex problems, navigate challenges, and find solutions efficiently. Engaging students in problem-solving activities encourages them to develop critical thinking and reasoning abilities, which are essential not only in mathematics but also in everyday decision-making. These heuristics act as mental shortcuts or rules of thumb that streamline the problem-solving process (Schoenfeld, 1987).

Polya (1945) states that teachers' problem-solving heuristics refer to the strategies and approaches they use to help students tackle complex problems. These heuristics often include techniques such as breaking down a problem into smaller, more manageable parts, encouraging students to look for patterns, and prompting them to make educated guesses or hypotheses. Teachers may also model working backward from a solution or use visual representations like diagrams or graphs to understand a problem better (Polya, 1945). They employ these heuristics to support students in finding solutions to specific problems and to foster critical thinking and independent problem-solving skills. The importance of problem-solving heuristics lies in their ability to enhance critical thinking and foster a deeper understanding of mathematical concepts. Effective use of problem-solving heuristics in the classroom helps students develop a systematic approach to challenges, enabling them to apply these strategies across various contexts and disciplines (Schoenfeld, 1985). These strategies foster critical thinking and adaptability, allowing students to transfer their problem-solving skills to real-world situations (National Research Council, 2001).

CHAPTER 3

METHODOLOGY

This research aims to examine middle school mathematics teachers' problem-solving heuristics in algebra word problems and their understanding of the bar model method, which is used and well-known method in the Singapore mathematics curriculum for algebra teaching. Therefore, the following research question was addressed in this study:

1. What are the problem-solving heuristics of middle school mathematics teachers in algebra word problems?
2. What do middle school mathematics teachers understand about the bar model method when examined students' bar model solutions in algebra problems?
3. What are middle school mathematics teachers' conceptions about using the bar model method in algebra teaching?

In this chapter, the methodology of the study is described. This chapter includes information about the research design of the study, the participants, the data collection and analysis procedures, the role of the researcher, and the trustworthiness and credibility of the research.

3.1 Research Design

Semi-structured interviews are a qualitative research technique that combines the flexibility of unstructured interviews with the focus of structured interviews. This method allows researchers to delve deeply into a topic while following a predetermined set of questions. The semi-structured format enables interviewers to ask follow-up questions based on the interviewee's responses, creating a more conversational and interactive dialogue (Wengraf, 2001). In this research, semi-structured interview techniques were employed to examine middle school mathematics teachers' problem-solving heuristics in algebra word problems and their understanding of the bar model method, well-known method in the Singapore mathematics curriculum for algebra teaching.

After obtaining the necessary permissions from the ethics committee and participants, the researcher conducted an interview study to conduct an in-dept investigation of teachers' comprehension of students' solutions involving the bar model method in algebra word problems and their thoughts about the use of the bar model method in algebra teaching. The participants and detail of study are described in detail in the next section.

3.2 Participants

A purposive sample was used to achieve what this study intended, and the study was performed with 18 middle school mathematics teachers. According to this criterion, 22 middle school mathematics teachers were selected, all of whom graduated from the Bachelor Program of Elementary Mathematics Education at a faculty of education. However, due to a lack of suitable conditions, the study was conducted on a voluntary basis with 18 teachers (7 male and 11 female).

Participants are distributed as shown in Table 3.1 according to their educational background, place of employment, and duration of teaching experience.

Table 3.1 *Information About Participants*

Participant Teachers	Gender	University Degree	Type of School S/he Works	Experience
T1	Female	Master’s degree	Public	<5
T2	Female	PhD (ongoing)	Public	<5
T3	Female	Master’s degree	Public	<5
T4	Male	Undergraduate	Public	<5
T5	Female	Master’s degree	Private	<5
T6	Male	Undergraduate	Public	<5
T7	Male	Undergraduate	Public	>5
T8	Male	Master’s degree	Private	<5
T9	Male	Master’s degree	Public	>5
T10	Female	Undergraduate	Public	<5
T11	Female	PhD (ongoing)	Public	<5
T12	Female	Master’s degree	Public	<5
T13	Female	Master Degree	Private	<5
T14	Female	Master Degree	Private	<5
T15	Male	Undergraduate	Public	>5
T16	Female	Master Degree	Private	<5
T17	Male	Undergraduate	Private	>5
T18	Female	Undergraduate	Public	>5

Among the participants, there are those who are currently pursuing a master's degree or doctorate and some teachers have already completed their studies. While 6 (33,3%) of the teachers participating in the study work in private schools, 12 (66,6%) work in public schools. While 4 of the public-school teachers have more than 5 years of experience, only 1 of the private school teachers has been working for more than 5 years. 9 (50%) of the teachers who participated in the study have already completed or are currently pursuing a master's degree, whereas the rest are undergraduate graduates.

3.3 Data Collection Procedures

Yin (2009) categorizes interview types into three groups: open-ended, focused, and survey interviews. In this research, the focused interview type was utilized. In focused interviews, respondents are asked a specific set of questions, which can also be supplemented with open-ended questions. While collecting data, online interviews were conducted with teachers via Zoom. These interviews were audio and screen recorded with permission. Since the teachers were asked to write their answers to the questions on the screen, screenshots were also collected as data. The questions planned to be asked in the interview and their targeted understanding and purpose are shown in Table 3.2. Apart from these questions, unplanned questions were also asked during the interviews.

Table 3.2 *Interview Questions and Targeted Understandings of Questions*

Interview Questions	**Targeted understanding**
How would you solve the problem?	To understand their problem-solving preferences while trying to connect algebra with a real-life situation.

Table 3.2 (continued)

If you had to solve it in a different way, how would it be?	To see whether they can think of the bar model or any visual method as a second solution for problem solution.
Can you evaluate student’s' solution? What kind of thinking did they use?	To understand teachers are aware of students’ understanding and students’ misconceptions about algebra.
How do you convey this solution to the class? How do you connect it to the topic? How do you lead them to the right solution?	To understand if teachers have the knowledge of the concept and how to explain this concept meaningfully. That is, whether they have what is required to make the algebra subject meaningful to students.
What do you think about using the Bar Model in solving algebra word problems?	To take the general opinions of the teachers about the use and pros and cons of the bar model.
Can the Bar Model be used in algebra teaching to analyze students’ misconceptions? Can you give an example?	To obtain teachers' views on whether the use of models will affect students' learning
Is the Bar Model included in the curriculum, and if not, should it be? Is it appropriate?	To find out whether teachers find the use of models necessary according to their curriculum knowledge.

At the beginning of the interview, first, teachers were asked to solve three algebra problems that were determined based on their characteristics and prepared in this context. These questions were determined based on the problem sets on the study of Baysal and Sevinc (2022). In the Baysal and Sevinç's study, problems were divided into problem sets according to the seventh-grade mathematics curriculum, textbooks and learning objectives. The Ministry of Education stated the learning objectives of the topics as (MoNE, 2018):

- Recognizing algebraic equations with one unknown.
- Writing the algebraic equation representing the given real-life situation.
- Solving algebraic equations with one unknown.
- Solving algebraic word problems that require writing algebraic equations.

The characteristics of these problems are divided into problem sets as shown Table 3.3 below.

Table 3.3 *Problems That Were Asked to Teachers and Their Problem Sets*

Problem Set	**Problem**
Decontextualized Problems Involving Quantitative Relations	1- If 12 less than 3 times a number is equal to 2 times 8 more than the same number, what is this number?
Problems Involving Quantitative Relationships between Consecutive Numbers	2- The sum of the ages of Türkan, Seda and Derya is 55. If Türkan is 13 years older than Seda and Derya is 3 years younger than Seda, how old is Türkan?
Contextualized Problems with Two Unknown Quantities, One of Which Could be Described by the Other One	3- The total number of legs of chicken and sheep on a farm is 122. If the total number of chicken and sheep on this farm is 42, how many sheep are there?

Decontextualized problems involving quantitative relations (Problem Set 1) refer to problems involve finding the value of an unknown quantity using words like "more than," "less than," "equal to", and "addition" to describe the relationships between quantities. Problems involving quantitative relationships between consecutive numbers (Problem Set 2) refer to problems that have multiple unknowns with a defined consecutive relationship as provided in the problem statement. Contextualized problems with two unknown quantities, one of which could be described by the other one (Problem Set 3) refer to problems include two unknown quantities, one of which could be described by the other one. First, they were asked to express these three algebra problems algebraically and explain the solution they would use in class for such problems. The aim of this question was to understand their problem-solving preferences while trying to connect algebra with a real-life situation. After that, they were asked for an alternative method to solve these problems to see whether they could think of a bar model or any visual method as a second solution.

Then, the teachers were shown students' solutions to some problems parallel to the problems they were asked to solve previously. The students' solutions shown were taken from the research on the students' use of the bar model (Baysal & Sevinç, 2022). The six students' solutions showed during the clinical interviews were also divided into the same three problem sets for the analysis as shown in Table 3.4 (Baysal & Serife, 2022). In Problem set 1 (P1 and P3), the focus was on quantitative relations without contextual situations. Problem set 2 (P7 and P10) explored quantitative relationships among consecutive numbers. Lastly, in Problem set 3 (P4 and P5), the problems involved contextual situations with two unknown quantities, one of which could be described in terms of the other. While there must be continuity in bar model drawings, students used discrete drawings in these solutions. Therefore, using the bar model shown to the teachers in this study were not completely correct.

Table 3.4 *Problems Classification*

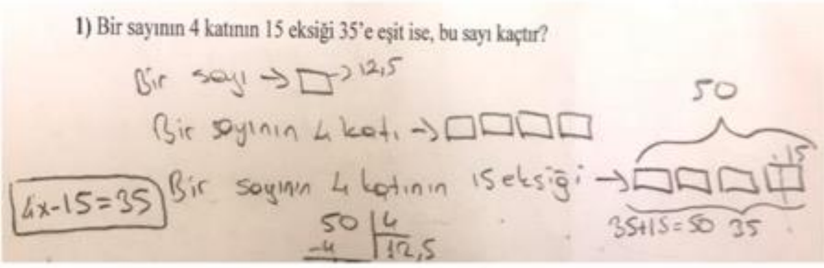
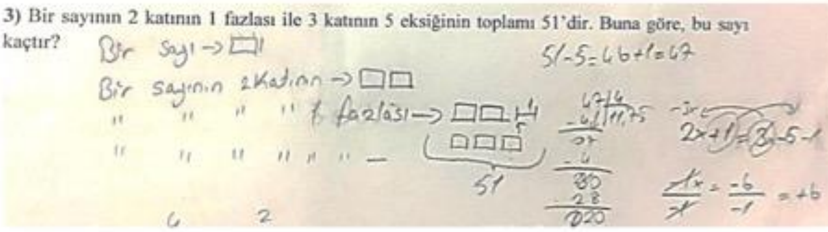
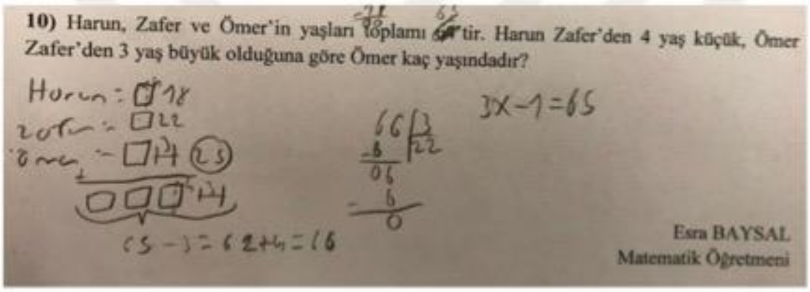
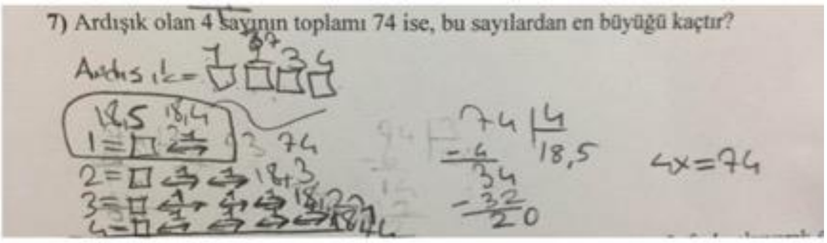
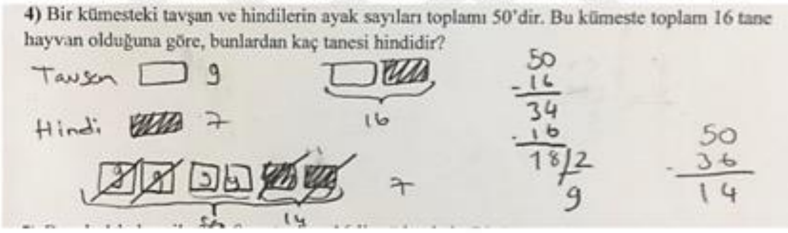
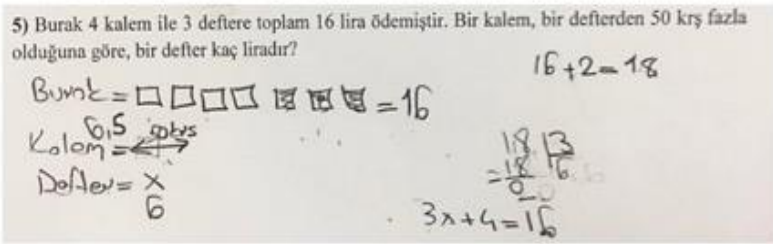
Problem Set	Sample Student Responses with Bar Model Method
Decontextualized Problems Involving Quantitative Relations	 
Problems Involving Quantitative Relationships between Consecutive Numbers	 

Table 3.4 (continued)

Contextualized Problems with Two Unknown Quantities, One of Which Could be Described by the Other One	 

They were asked to evaluate these students' solutions as true or false. So that, it can be understand if teachers were aware of students' understanding and misconceptions about algebra. Then, they were asked about how they would lead students to correct reasoning if the answer was wrong or how they would convey the solution to the class if the answer was right. The aim was to determine whether they had an adequate understanding of the concept to explain it to students meaningfully, that is, whether they had what it takes to make algebra meaningful to students. Finally, they were asked about their opinions on the Singapore Bar Model and its use in teaching algebra and its place in the curriculum.

3.4 Data Analysis

The data are analyzed and presented according to predetermined themes, taking into account the questions and dimensions used in the research process (Yıldırım & Şimşek, 2005). Yıldırım and Şimşek (2005) emphasize that findings obtained through descriptive analysis are presented in an organized and interpreted manner. In this context, descriptive analysis consists of four stages: determining the analysis themes, processing the data according to the thematic framework, defining the findings, and interpreting the findings. In this study, middle school mathematics teachers' use of and their opinions on the Singapore Bar Model in teaching algebra were examined.

For the analysis, I first transcribed the data I collected as video footage and audio recordings, and then imported these documents into MAXQDA, where I created my codes as shown in the Table 3.5.

Table 3.5 *Code System*

Codes	**Subcodes**	**Memo**
Correct Answer	identify	Teachers can understand and explain what the student think
	cannot identify	Teachers cannot understand and explain what the student think
Incorrect Answer	identify	Teachers can understand and explain what the student think
	cannot identify	Teachers cannot understand and explain what the student think
	use bar model component incorrectly for explain solution	Teachers try to overcome the mistakes of students with bar model method but use incorrectly
	use bar model component correctly for explain solution	Teachers try to overcome the mistakes of students with bar model method and use correctly
	use different way for explain solution	Teachers try to overcome the mistakes of students with different ways like arithmetic thinking, with algebraic equations etc.

Table 3.5 (continued)

	Verbal Explanation	Teachers explained the solution verbally (e.g., using layer logic).
	Using Manipulatives (Algebra Tiles, scales vs.)	Teachers used a physical manipulative such as algebra tiles or a visual model like the balance scale analogy.
	Working Backwards	Teachers used the concept of inverse operations.
	No answer	Teachers didn't answer the problem.
Problem-Solving Heuristics	Guess and Check	Teachers solved the problem by trying numerical values for unknown.
	Using Shapes	Teachers solved the problem by drawing shapes on the screen (like square, heart, star etc.).
	Using Equation	Teachers solved the problem using an unknown and an equation with traditional way.
	No other way	Teachers couldn't find second way to solve problem.

Table 3.5 (continued)

Bar model component	can identify	Teachers can identify bar model component
	cannot identify	Teachers cannot identify bar model component
Quantitative relation in the visual part of the bar model	can identify	Teachers can identify quantitative relation in the visual part of the bar model
	cannot identify	Teachers cannot identify quantitative relation in the visual part of the bar model
Quantitative relations in the visual and the procedural/operational part of the bar model	cannot identify	Teachers cannot identify parallel quantitative relations in the visual and the procedural/operational part of the bar model
	can identify	Teachers can identify parallel quantitative relations in the visual and the procedural/operational part of the bar model
Appropriateness for mathematics teaching program	appropriate for problem solving	Teachers found bar model appropriate for problem solving
	appropriate for teaching	Teachers found bar model appropriate for teaching
	unappropriated	Teachers found bar model unappropriated in general
	complicated	Teachers found bar model complicated in general

Table 3.5 (continued)

Functions of the bar model in solving algebra problems	different ways	Teachers wanted to use bar model to ensure different ways to solve problems in algebra
	visualization arithmetic relation	Teachers wanted to use bar model to see arithmetic relation more visual in algebra
	confusing	Teachers didn’t want to use bar model since its possibility of confusing the student
	connection between concrete and abstract	Teachers wanted to use bar model to connect concrete concepts with abstract nature of algebra
	visualization of algebra	Teachers wanted to use bar model to understand algebra more visual way
	overcoming misconceptions	Teachers wanted to use bar model to overcome the misconceptions

3.5 The Role of the Researcher

The researcher comes from a background in mathematics education with experience. The researcher's interest in this topic stems from her observations of the difficulties experienced by students in the algebra learning process. The researcher's ideas about the importance of the topic were reinforced by the observation that there was generally only one method of algebra instruction provided by teachers. These experiences provided the researcher with a unique lens through which she views and interprets the data, but they also bring certain biases that the researcher must continuously acknowledge and mitigate. In qualitative research, the researcher is often considered the research instrument, and the researcher's interactions, observations, interpretations, and insights shape the data collection and influence the study results (Creswell, 2009).

In this study, the researcher benefited from the advantage of being in the same field as all participants and being their colleague. In addition, the fact that the researcher graduated from the same university as the twelve participants made it easier to understand what they wanted to say. The researcher also had the opportunity to observe the effects of the education they received. The researcher avoided interfering with the participants and influencing their opinions during data collection. After presenting the questions on the screen, the researcher asked the participants questions and gave them the time they needed to produce solutions.

The role of the researcher is crucial in qualitative research so they should be transparent and provide information about their research collection process (Creswell, 2009). In this study, the researcher provided the participants with the information that she could about the study and ensured that the images/videos recorded within the scope of the confidentiality of the study would not be used anywhere. After asking the questions, the researcher left the comments until the end in order not to influence the opinions of the participants. When they solved the questions incorrectly or when they could not understand the student' solutions, they did not talk about it until the end of the process.

Lincoln and Guba (1985) emphasized the importance of researchers being transparent about their roles and maintaining reflective research diaries to enhance the trustworthiness of qualitative research findings. These explanations about the role of researcher in the study were made to ensure the validity of the study and increase the reliability.

3.6 Trustworthiness and Credibility

Examining the study's reliability and validity is very important because these two topics are linked to the collection and analysis of data (Merriam, 1998). During data collection, the researcher avoided intervening with the participants. After presenting the questions on the screen, the researcher asked the participants questions and gave them the time they needed to formulate their solutions. While the questions asked to the teachers and their answers were audio/video recorded, what they wrote on the screen was also recorded. Additionally, for data analysis triangulation, the researcher's consultant played an essential role in the data collection and analysis phases. In the realm of qualitative research, to ensure the reliability and validity of research, several key factors need to be considered including credibility, confirmability, and dependability (Lincoln & Guba, 1985).

In qualitative research, credibility is the extent to which the findings accurately represent the phenomena under investigation, that is internal validity. It involves ensuring that the study effectively measures its intended constructs and that the data, interpretations, and conclusions are trustworthy (Shenton, 2004). This means that researchers need to establish that the methods used in the study are appropriate, the data is rich and comprehensive, and the interpretations are grounded in the data. Essentially, credibility in qualitative research is about ensuring that the study's findings are convincing and valid. This study triangulation for internal validity. The researcher also coded audio and video recordings of the clinical interviews. During the process of creating the codes, the researcher ensured that the codes underwent a thorough review by both the thesis advisor and the entire research group involving a group of three other masters' students in mathematics education. Codes and subcodes were explained to the advisor and research group by giving examples. The codes determined and checked on the examples were discussed. An agreement was ensured regarding the codes and subcodes. Then, in the final case, coding was done in line with this agreement.

Triangulation, on the other hand, means using different methods to collect data and having different researchers analyze the same data set independently (Shenton, 2004). Triangulation can provide multiple perspectives, validate findings, and enhance the depth of understanding by integrating different perspectives. For this purpose, the researcher incorporated a range of data sources, including video and audio recordings, interviews, and observations. This comprehensive approach allowed for the derivation of more nuanced and robust conclusions. Furthermore, to enhance the credibility of the findings, the codes used in the analysis were compared with those generated by another researcher, contributing to a triangulated and substantiated interpretation of the data.

In qualitative research, to ensure internal validity, confirmability plays a crucial role in decreasing the researcher's bias. Triangulation through the use of multiple methods, sources, or perspectives to cross-check the data and findings can help reduce researcher bias and increase confirmability (Shenton, 2004). In the preceding section, the researcher's role in the present study was thoroughly detailed. Therefore, the study employed triangulation allowing for the comprehensive validation of findings by multiple data sources, methods, and perspectives, reducing researcher bias, and increasing confirmability.

In qualitative research, dependability, which is also known as reliability, refers to the stability of data over time and across conditions (Lincoln & Guba, 1985). It means that similar results can be obtained when the study is repeated in a similar context with the same type of participants and methods. To ensure the study's reliability, it is important to provide a detailed explanation of the research process, including the conditions under which the results were obtained, data collection methods, and analysis. For this reason, the selection of case participants, data collection tools, and the data analysis process are discussed in detail in this section. The questions asked to the participants, the clinical interviews, the characteristics of the participants, and the observation notes were recorded. In addition, the triangulation method is an important strategy used to increase reliability (Merriam, 1998). As mentioned above, triangulation was carried out by using different data sources and the opinions of more than one researcher in the data analysis, thus ensuring the reliability of the study.

CHAPTER 4

RESULTS

The main aim of this study is to examine middle school teachers’ understanding of the Singapore Bar Model Method as a problem-solving heuristic in algebra teaching. In this chapter, the research findings gathered from the analysis of the collected data are categorized and explained under three categories. The first category is the problem-solving heuristics of middle school mathematics teachers in algebra word problems. The second one is middle school mathematics teachers’ understanding of students’ solutions. This category consists of three aspects: identifying the components of the bar model, identifying quantitative relations in the visual part of the bar model, and identifying parallel quantitative relations in the visual and the procedural/operational parts of the bar model. The third category concerns with middle school mathematics teachers’ conceptions about using the bar model method in algebra. This consists of two aspects: thoughts about the functions of the bar model in solving algebra problems and thoughts about its appropriateness for mathematics teaching programs. In this section, the findings are presented according to the research questions.

4.1 What are the Problem-Solving Heuristics of Middle School Mathematics Teachers in Algebra Word Problems?

The problem-solving approaches employed by teachers directly influence students' grasp of mathematical concepts and their application in practical contexts. Teachers utilize a variety of strategies and heuristics to enhance students' problem-solving prowess. Therefore, it is crucial to analyze teachers' problem-solving approaches to evaluate the effectiveness of teaching methods and determine the best practices in mathematics education. This section will delve into teachers' methodologies and preferred strategies in the problem-solving process.

In the interviews, the teachers were asked to solve three algebraic word problems using the first method that came to their mind. They were then asked to solve the problems using different methods. When the methods used by the teachers in algebra word problems were analyzed, it was seen that the most commonly used method was using equations with unknowns. When the answers given by the teachers to the questions were analyzed, the findings shown in Figure 4.1 were obtained.

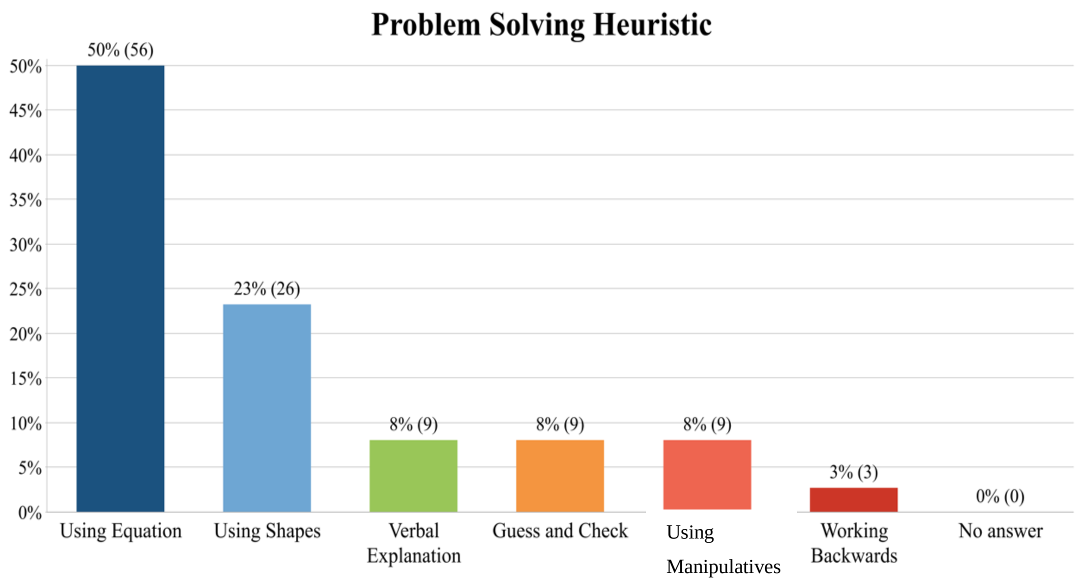


Figure 4.1 Problem-Solving Heuristic of Teachers in Algebra Word Problem

The teachers approached the algebra problems by solving them with the equation construction method at a rate of 50%. The problem-solving heuristics we encountered during the data collection process were grouped under separate headings.

4.1.1 Teachers' Use of Equations as a Problem-Solving Heuristic

The algebraic equation method is one that utilizes mathematical expressions to solve problems. With this method, the problem situation can be modeled, or the unknown value in the problem can be found.

In Problem 1, which is "If 12 less than 3 times the number is equal to 2 times 8 more than the same number, what is this number?" 15 out of 18 teachers preferred using the equation method as the first problem-solving heuristic.

T15, one of the participant teachers, wrote the equation for Problem 1, as shown in Figure 4.2. Teacher T15 responded as follows for the solution of the first question:

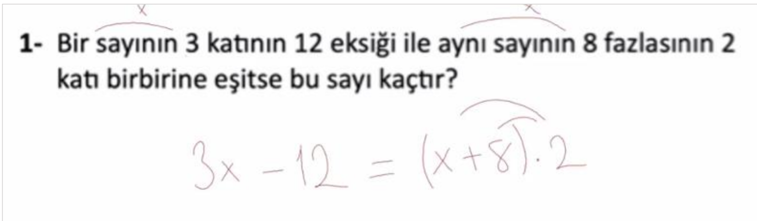
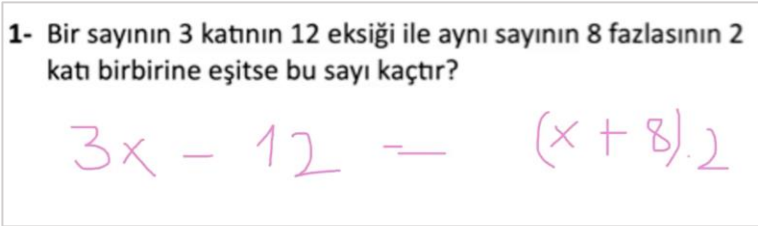


Figure 4.2 The Solution of Teacher T15 for Problem 1

> Now, since I don't know the number here, I'm going to do it using algebra. I called the number 'x'. Then, I multiplied it by three and subtracted 12. This is on the left side of my equation, and on the right side, since the same number is still called, x, I add 8 to x, and now it's a new number. So, I put it in parentheses, multiplied it by two, and wrote it on the right side of the equation. That way, I have an equation.

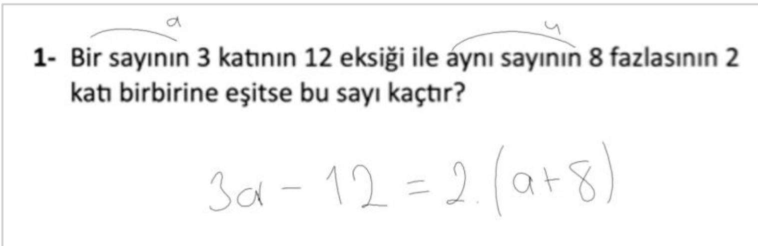
While constructing the equation, the teachers' approach was first to identify the unknown and then determine how to transfer the given verbal expressions algebraically. Similarly, teacher T6 also approached the problem in this way and explained that "I solved this problem with an equation. I wrote 'x' for the unknown. Then, I wrote the equation required for the solution along with the operations mentioned." (in Figure 4.3)


1- Bir sayının 3 katının 12 eksiği ile aynı sayının 8 fazlasının 2 katı birbirine eşitse bu sayı kaçtır?

$3x - 12 = (x+8).2$

Figure 4.3 The Solution of Teacher T6 for Problem 1

Just like participants T15 and T6, other teachers tended to use ‘x’ or another letter directly in algebraic problems. Due to the solution system that teachers are used to and their efforts to mathematize solutions, the first method that comes to mind is to form equations by giving unknowns. There were teachers who thought of a different method but did not prefer to use it. For example, teacher T5 said, “The use of shapes was long and laborious.” Therefore, he did not prefer to use it; instead, he wrote the equation shown in Figure 4.4 and explained his approach to the question as follows:


1- Bir sayının 3 katının 12 eksiği ile aynı sayının 8 fazlasının 2 katı birbirine eşitse bu sayı kaçtır?

$3a - 12 = 2.(a+8)$

Figure 4.4 The Solution of Teacher T5 for Problem 1

> If I want to use a shape here, it will be difficult to show 12 minus, and it will take a long time. That is why I ask them to give a letter instead of a shape for the unknown number, usually ‘a’ or ‘x’.

Participant teacher T16, who proceeded with the equation even though he came up with other solutions, stated that the grade level of the students was also important in determining the solution. T16 said that if it were a younger age level, the approach might be different. Then, he started to solve the question as shown in Figure 4.5 and explained as follows:

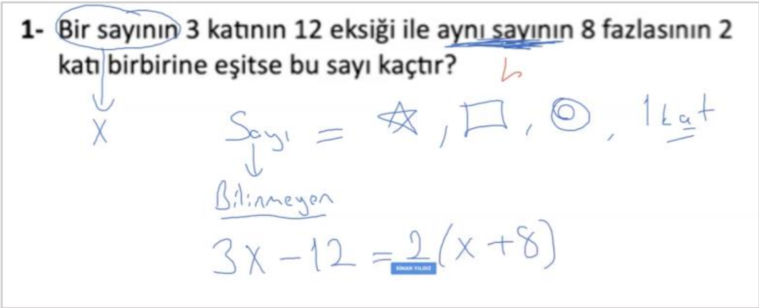


Figure 4.5 The Solution of Teacher T16 for Problem 1

> Since student see algebraic expressions in 6th grade, I will proceed in that way. However, in the past, that is, when students were in the younger age group, children would progress by using stars, boxes, circles instead of numbers or by saying 'one level' because they did not know the number. In 6th grade and beyond, they have learned to use the unknown instead of these representations for the number. So, for the number expression, a letter is usually chosen, which is usually an 'x'.

In the answers given in Problem 2, which is "The sum of the ages of Türkan, Seda, and Derya is 55. If Türkan is 13 years older than Seda and Derya is 3 years younger than Seda, how old is Türkan?" 16 teachers again first chose the method of using equations.

In this problem, the method of forming equations by determining the unknown was used more commonly as in the first problem. The point that distinguished the teachers from each other in the solutions was that there were different approaches as well as the same ideas when determining the unknown. For example, teacher T3 (See Figure 4.6), explained the process of determining the reference point for the solution as follows: "It says that Türkan is 13 years older than Seda, and Derya is 3 years younger than Seda. Here, we write Türkan as Seda and Derya as Seda. Therefore, I take Seda as unknown. I use x for Seda."

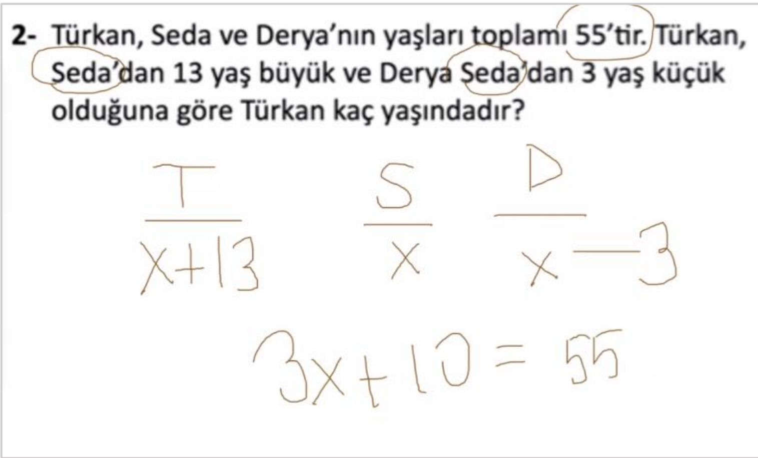


Figure 4.6 The Solution of Teacher T3 for Problem 2

Teacher T3 stated that it would be easier to take the amount related to the other two numbers as the unknown when determining the reference point as shown in Figure 4.12. Similarly, teachers T9, T12, T13, T14, and T15 stated that both they and their students tended to assign the unknown to Seda in such questions since she comes up twice in the problem and is related to both sides. It was observed that the teachers assigned the unknown to the person or object that was more prominent in the questions because they thought that students would understand better in this way.

Another response of the teachers was to identify the smaller one and take it as a reference point. T10, one of the participant teachers, explained the reason for this as follows:

> Since Türkan is older than Seda and Derya is younger than Seda, Derya is the youngest in this question. I try to take the smaller one as a reference in such questions. In this way, we do not need to subtract, that is, we do not need to subtract when determining relationships. I think addition is easier to understand and healthier, so Derya is x.

Similar to teacher T10, teachers T6, T17, and T18 stated that it would be more understandable to identify the smaller one and assign the unknown to it. The common reason behind assigning 'x' to the smaller one was their observation that students understood addition more easily than subtraction. Since the teachers thought that it would be more difficult to convey subtraction or negative numbers to students in the algebraic approach, that they determined the smallest amount and assigned the unknown to it.

In some cases, the teachers mentioned that it is typical to directly designate the person or data as unknown in such types of problems. For instance, participant teacher T16 mentioned that:

> The problem is about Türkan, so we can initiate the solution by assigning Türkan as the unknown. Consequently, the result we obtain will be the direct answer to the question, eliminating the need for further interpretation. If we assign the unknown to Seda or Derya, the value we find for x will not provide the answer to the question. This is a common mistake made by students.

In Problem 3, which is "The total number of legs of chickens and sheep on a farm is 122. If the total number of chickens and sheep on this farm is 42, how many sheep are there?" using equations was one of the most frequently used problem-solving heuristics. 16 of 18 teachers started the solution using equations.

There were also different approaches to this question in terms of formulation of equations. For example, T4, one of the participant teachers, approached the first solution with an equation with two unknowns, as shown in Figure 4.7, and explained this solution as follows:

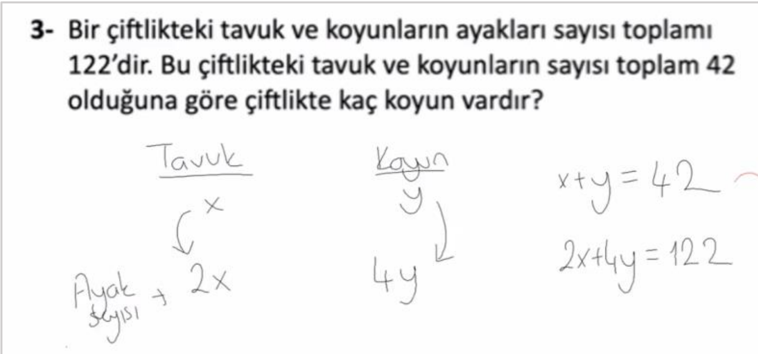


Figure 4.7 The Solution of Teacher T4 for Problem 3

> Here, we call the chicken x and the sheep y and set their sum equal to 42. Thus, an equation is written where (x + y) equals 42. Then, since chickens have two feet, their total number of feet equals 2x. Likewise, since sheep have 4y feet, their total number of feet equals 4y. Adding these two together, we see that (2x+4y) equals 122. Then, x and y are easily found by solving two equations with two unknowns.

When Teacher T4 was asked about the second method, he stated that another way to solve this problem could be to use only one unknown. "If we say 'x' for chickens, the number of sheep will be (42-x). For the number of feet, we multiply the number of chickens by 2 and multiply the number of sheep by 4. After this point, the problem can be solved with a single unknown." Similarly, teacher T15 mentioned using an equation with one unknown as the second method after solving the equation with two unknowns as in the first method. Teacher T16 stated that a single unknown would facilitate the solution of the question. He made the following explanation to overcome the confusion about the use of unknowns while solving the problem:

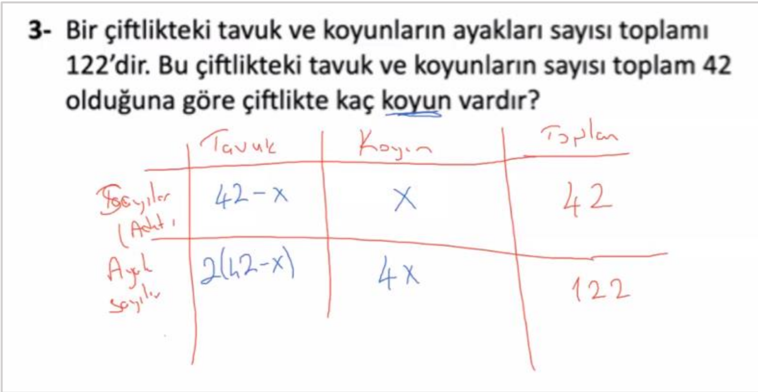


Figure 4.8 The Solution of Teacher T16 for Problem 3

> In these kinds of problems, I feel that I need a table to clarify (shown in Figure 4.8). After filling in the table, since we were asked about sheep, I want to call it 'x'. From this point on, we need to find the chickens. This is a part where we talk a lot with students. For example, we get stuck, we cannot figure out how to write algebraically, here, instead of giving a second unknown, immediately think of a natural number. For example, if there were 10 sheep, how would we find the chickens? We would subtract 10 from 42, then algebraically we need to subtract x from 42.

Although they have different approaches in determining the unknown, the teachers who proceeded with the method of using equations approached the problem in the same way when determining the unknown and accepted the value 'a number' as the unknown, calling it 'x'. This approach was the first solution that came to the teachers' minds in general, so we can say that the teachers were more prone to traditional solutions because it was a situation that we encounter in traditional solutions.

However, there were different approaches when solving the equation. For example, participant teacher T8 responded to the question, "How do you convey the equation you created to students while solving the equation?" as follows:

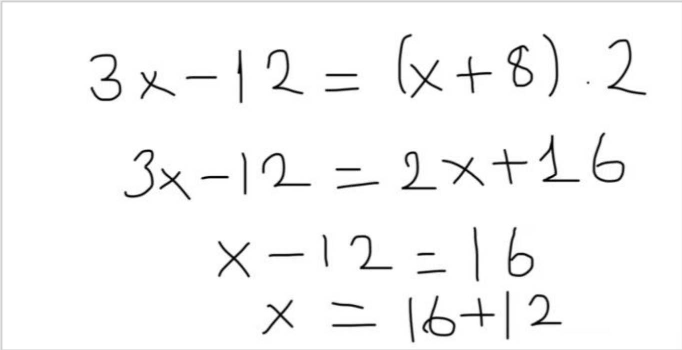


Figure 4.9 Teacher’s Solution for Equation (T8)

> I prefer to focus on the scale analogy rather than the traditional ‘let’s throw it across’ way. Let’s say there is a scale here. On the left side, there are 3 shapes whose weights I don't know, and on the right side, there are 2 shapes whose weights I don't know. Since these shapes are identical, I remove two shapes from each side, and the balance is not changed in this way. When I take these two unknowns, we can show the situation in the equation as ‘x-12 = 16’. Then again, if we add 12 kg to both sides, we can think that x can be left alone. Thus, we can find that x is 28.

While explaining his solution process, T8 wrote and conveyed the equation as shown in Figure 4.9. Although he used the perspectives of the scales, his progression of the solution was in the form of an equation. When I asked teacher T16 the same question, “How do you convey the equation you created to students while solving it?” he explained as follows:

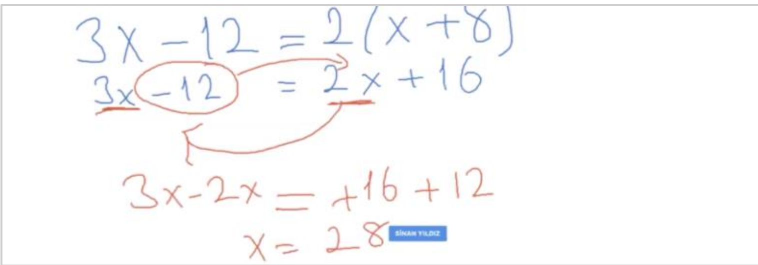


Figure 4.10 Teacher’s Solution for Equation (T16)

> My classic move is to look at the unknowns first to avoid dealing with the negative. I give an example to my students: on holidays, the younger ones kiss the elders' hands, so we send the small unknown to the larger one. This way, we don't have to deal with the minus because children get very confused at this point. When the unknowns are on one side, the unknowns should be on the other side. Here, I use the inverse operation; minus 12 goes across as plus 12. In this way, we can see that x is equal to 28.

While explaining his solution method, T16 wrote this process as shown in Figure 4.10. T6, one of the participant teachers, similarly advanced the solution of the equation by using the approach 'knowns on one side and unknowns on the other.' Knowns on one side and unknowns on the other approach, reverse operation approach, and the scale analogy logic were among the most common solution techniques used by the teachers in solving equations. The differences among teachers at this point stem from their experiences and their interpretation of how their students can best understand. Since each teacher grounds their solutions from their own point of view and since the understanding of each teacher may differ, the solution methods also vary. Even if they use the same method, the way they explain their methods to students may vary. Teachers' solution approaches can also vary depending on the circumstances brought by the education system. While solving equations, some teachers used the scale analogy when they wanted to express the rational of the equation, while others used the method of leaving 'x' alone because practicality and speed were important.

4.1.2 Teachers' Use of Shapes for The Unknown as a Problem-Solving Heuristic

Using shapes for algebra word problems is a visual strategy to help students understand and solve algebraic problems, particularly those involving relationships between quantities. This visual method only replaces numerals with images.

In Problem 1, which is "If 12 less than 3 times the number is equal to 2 times 8 more than the same number, what is this number?" only two of the teachers used shapes to solve the problem as the first problem-solving heuristic. Participant teacher T7 started the solution with the use of shapes and explained it as follows:

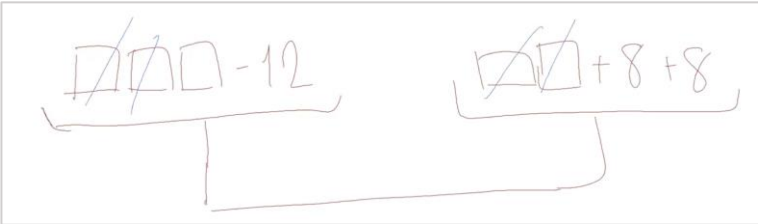


Figure 4.11 The Solution of Teacher T7 for Problem 1

> We can also solve these kinds of problems by using scales, analogy, and shapes. For this problem, for example, I draw 3 squares on one side of the scale and write minus 12 next to it. Minus 12 can confuse students since it is not in the form of addition; it is important to convey it carefully while explaining. In the same way, I first draw a square on the other side of the scale and add 8 next to it. Then I redraw what I drew because it says, "2 times this." By talking about the principle of equality on the scale, I subtract squares from both sides of the scale, two at a time. Then I ask the students, 'From which number should I subtract 12 to get 16?'. In other words, I wait for the students to think about the reverse operation. Thus, we reach the solution.

At this point, although he used the scales analogy, he provided an explanation different from Participant T8 without using the unknown, as shown in Figure 4.11. Teacher T7 responded to the question, "Do you solve all the questions in this way?" as follows:

> This is the technique I use at the beginning of the unit so that it is meaningful while conveying the subject. Of course, if I am solving problems with an older age group, the use of the unknown is more common.

Teachers also use problem solutions to explain the subject. Although it was not same with the first method, when the teachers were asked about the second method for Problem 1, the first thing that came to their minds was drawing shapes. Eight teachers chose to use shapes for solving Problem 1. For example, T6, one of the participant teachers, responded as follows when asked for a second method:

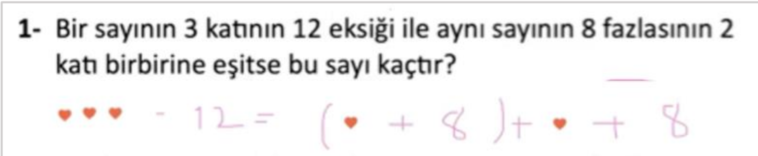


Figure 4.12 The Solution of Teacher T6 for Problem 1

> When it says the number, I immediately put a symbol for that number. Thus, I provide a more concrete representation for the student. If he/she is younger or cannot understand the first solution, it will be more meaningful this way.

As in the solution of participant T6, shown in Figure 4.12, the teachers preferred a more visual method when asked for a second option. The difficulty in understanding the traditional method for students can be attributed to the necessity to visualize it.

In the answers given in Problem 2, which is "The sum of the ages of Türkan, Seda, and Derya is 55. If Türkan is 13 years older than Seda and Derya is 3 years younger than Seda, how old is Türkan?" only one teacher used shapes as the first problem-solving strategy. Participant teacher T7 also employed this method as shown in Figure 4.13. Teacher T7, who utilized shapes as a solution method, provided the following response to the question; "What do you pay attention to when determining which one to give the shape to?": "In this kind of question, I first look at what is asked of us; in this question, Turkan’s age was asked. Then, Turkan is my reference point, and I will give a box for Turkan's age.”

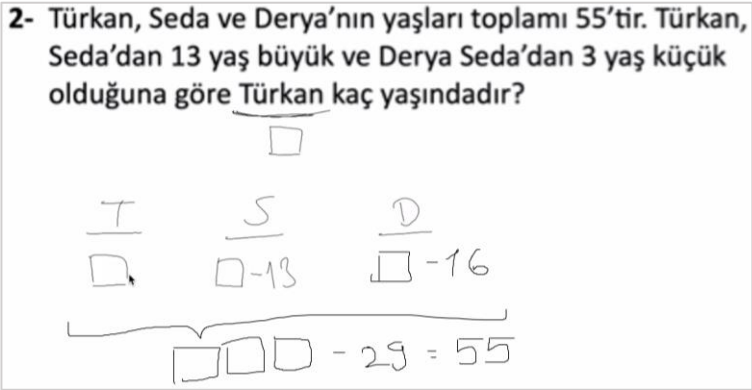


Figure 4.13 The Solution of Teacher T7 for Problem 2

There were more teachers who used shapes as the second solution than those who used shapes as the first solution. Eight of the teachers preferred to use shapes for the second solution for Problem 2. Teacher T12, who preferred to use equations as the first solution, preferred to use shapes as the second method (shown in Figure 4.14). She gave the following answer to the problem solution and to the question of what to pay attention to when using shapes as asked to T7:

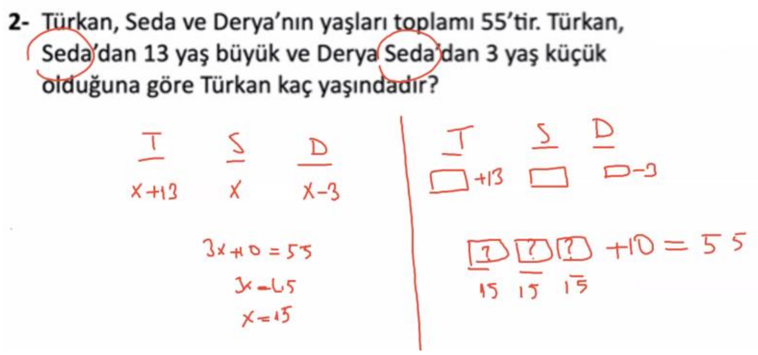


Figure 4.14 The Solution of Teacher T12 for Problem 2

When determining the unknown, students can usually assign it to Türkan, but since both are related to Seda, I direct my students to assign Seda the unknown, that is, the shape. At this point, it does not make sense to me to put a number next to the square because there is a number on one side and a shape on the other. This is why students can get confused. Therefore, when using a shape, I push my students to think ‘There is a number under the shape here and it covers it’ so that they can understand the concept of using shapes.

In Problem 3, which is "The total number of legs of chickens and sheep on a farm is 122. If the total number of chickens and sheep on this farm is 42, how many sheep are there?" it was observed that shapes were used less than the other two problems. There were only two teachers that used shapes to solve problems. The reason for this was that the teachers had difficulty in using figures for solutions when there were two unknowns. Participant teacher T14, who used the method of using shapes as the first solution method, expressed her solution approach to the problem as follows:

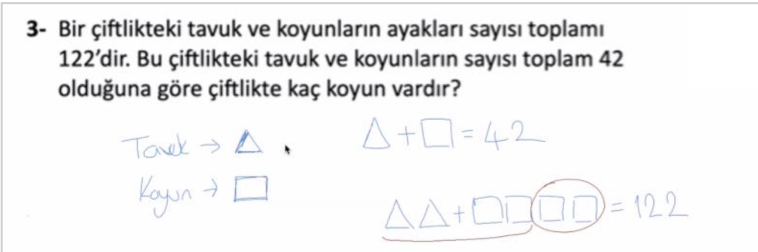


Figure 4.15 The Solution of Teacher T14 for Problem 3

For the numbers of chickens and sheep, I used two different shapes, a triangle and a square, for example (given in Figure 4.15). I draw the shapes as the sum of the number of chickens and the number of sheep and it equals 42. Then, since chickens have two feet, I must multiply the number of chickens by two, which means drawing two triangles to represent the number of the feet of the chickens. Similarly, since sheep have 4 feet, I draw 4 squares to represent the number of the feet of the sheep.

Teacher T14's solution was quite close to the expected shape solution for this question, but the teachers did not provide this type of approach in general. For

Problem 3, the use of shapes was not preferred as the second method, only one teacher preferred the use of shapes as the second method. Participant teacher T13, whose first preferred method was to construct an equation with one unknown, preferred the use of shapes as the second method (in Figure 4.16). While explaining her solution method, she mentioned the following:

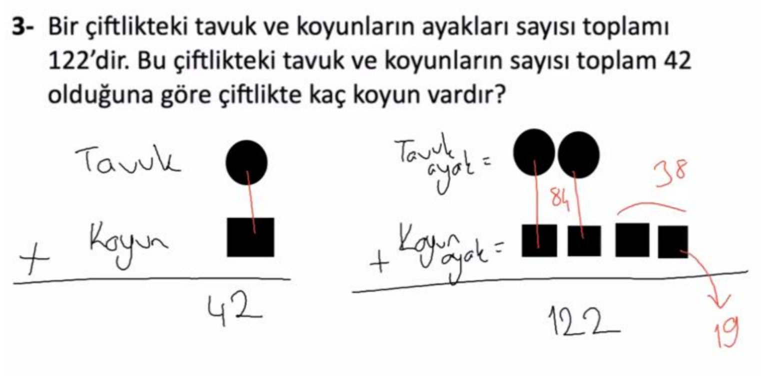


Figure 4.16 The Solution of Teacher T13 for Problem 3

> We can start by assigning one of the quantities a circle and the other one a square. After determining the shapes according to the number of feet, I would give a number value so that the sum of the circle and the square would be 42. Firstly, I would start with the idea where they were both 21. I would try this with close numbers. In this way, students can also try until reaching the answer where the square is 19.

Although participant teacher T13 started with the use of shapes, she continued this solution with the guess & check method using shapes. Although this solution was initially quite close to the expected shape solution for this problem, its finalization was not suitable for the components of the bar model.

4.1.3 Teachers' Use of Other Methods as a Problem-Solving Heuristic

In addition to the use of equations and shapes, the teachers also used other problem-solving methods including verbal explanations, modeling, guess & check and working backwards.

In Problem 1, which is "If 12 less than 3 times the number is equal to 2 times 8 more than the same number, what is this number?" participant teacher T9 gave the following verbal answer:

> Normally, I prefer to work with colored cartons when I give these cartoons to 7th graders as well as 6th. First, we use cartons with basic colors such as yellow, red, green. For example, we call one of them x and when we say 3 times this, the student takes 3 yellow cartons. We also show quantities such as more or less than unknown with small materials. For example, we can use number stamps or soda caps or even backgammon stamps to make numbers. Obviously, we need to use these to give abstract concepts so that, after working with these materials a few times, they can better understand abstract concepts.

Participant teacher T9 explained the solution verbally, but the solution he mentioned was to make use of modelling in problem-solving. He thought that physical models would make it easier to explain abstract concepts to students. However, in general, the tendency of teachers to use physical materials and models was low. In addition to physical modeling, some teachers said that they could benefit from drawing models. For Problem 2, which is "The sum of the ages of Türkan, Seda, and Derya is 55. If Türkan is 13 years older than Seda and Derya is 3 years younger than Seda, how old is Türkan?" participant teacher T15 said that she could benefit from the use of models as a solution method and provided the answer shown in Figure 4.17:

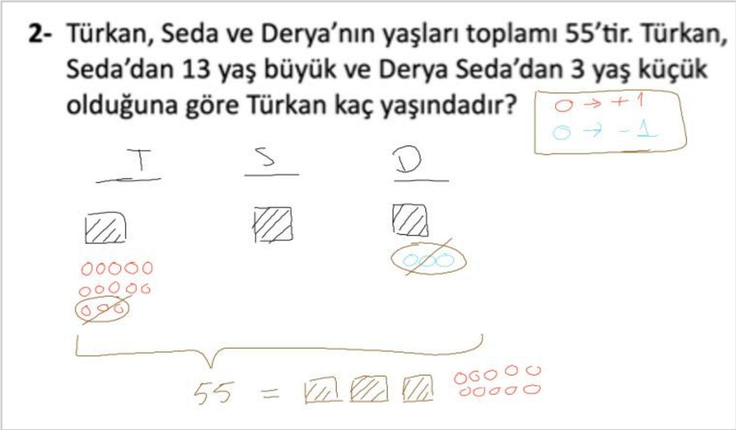


Figure 4.17 The Solution of Teacher T15 for Problem 2

> I think of using algebra tiles. First, I identify Seda as the unknown because of the common point and assign her the 'x' tile. Since Türkan is 13 years older than Seda, I assign Türkan 13 tiles next to the square that represent a positive one. Since Derya is 3 years younger than Seda, I assign Derya 3 negative ones next to her tile. Adding them all together gives us 55, and the problem can be solved from there.

Participant teacher T17 said that he could benefit from verbal explanations as a solution method for Problem 2, as shown in Figure 4.18, and provided the following answer: "In a way that would be verbally understandable, I would say that Derya is one time, so Seda is one time plus 3, and Türkan is one time plus 16. I can also reach the result when I put them all together and equate them to 55."

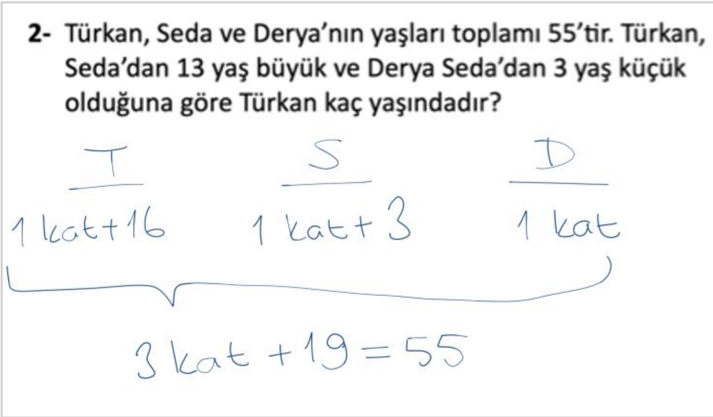


Figure 4.18 The Solution of Teacher T17 for Problem 2

Participant teacher S17 stated that he could solve the question with a verbal explanation using the concept of time. However, since the concept of time is not valid for this type of problem, the progression is incorrect.

Problem 3, which is "The total number of legs of chickens and sheep on a farm is 122. If the total number of chickens and sheep on this farm is 42, how many sheep are there?" was the question with the least use of shapes among the solution methods, while the 'Guess & Check' was used the most in this question. For example, while T12 first used an equation in the solution of this question, she answered the question, "How would you solve it with another method?" as follows:

> In questions like this, assigning natural numbers to the number of animals can be useful. For example, the number of animals is 42; here, I prefer to use larger numbers rather than smaller numbers to make it faster. Suppose there are 21 chickens and 21 sheep; let's find the number of feet. I think that if my number of feet is too high, I should reduce the number of animals with more feet. Then, for example, if I reduce the number of sheep to 19 and increase the number of chickens to 23, yes, this way, I get 122 feet.

Since the Guess & Check method can be one of the first options for students in this kind of problem, it was observed that the teachers were used to it and allowed for students in this sense. Compared to the other problems, this question style was the one in which the teachers had the most difficulty in finding different methods. For example, one of the participant teachers, T1, gave the following answer when asked for another method after solving the problem using equations:

> These are the types of questions we usually encounter in 7th grade. They solve them with such things; they solve them by using equations. They always do it with unknowns, so I honestly couldn't think of any other method right now.

Similarly, eight of the teachers stated that they did not have an answer to this question when asked about a second method. The teachers' inability to give an answer when asked for a second solution shows that they teach with a single method. In this way, students are prevented from approaching the question with different solutions. At the same time, the learning process will become difficult for students with different learning styles.

When the collected data were analyzed in terms of teachers' problem-solving heuristical tendencies in algebra word problems, it was observed that their tendency was to use equations for three different types of problems, as shown in Table 4.1.

Table 4.1 *Problem-Solving Preferences of Teachers as The First Option*

	Problem 1	Problem 2	Problem 3
Using Equations	15	16	16
Using Shapes	2	1	2
Using Manipulatives	1	1	-

The use of equations was the first solution method that came to the teachers' minds. The findings that differed in terms of the teachers' preferences for using equations were the ways of solving equations. 'Knowns on one side and unknowns on the other' and 'scale analogy' were the commonly used methods. In addition to the traditional approach of 'knowns on one side and unknowns on the other,' they proceeded with the scale model because the scale also represents the concept of equality in the equation. Teachers try to express the logic of the traditional approach in this way.

When the teachers were asked for the second option, the use of shapes was the most preferred heuristic in the first two problem types (as shown in Table 4.2). However, when they were asked for a second option in the third problem, the teachers mostly could not think of a solution.

Table 4.2 *Problem-Solving Preferences of Teachers as The Second Option*

	Problem 1	Problem 2	Problem 3
Using Equations	3	2	4
Using Shapes	8	8	1
Using Manipulatives	2	1	-
Verbal Explanations	2	2	-
Guess & Check	1	1	5
No other way	2	4	8

The teachers who preferred to use shapes only used them as symbols instead of numerals. There were no teachers who approached in a quantitative way, but two of the teachers stated that using shapes as symbols in this way was not very conceptual. Nevertheless, since they since they did not know how to approach this method in a way that is different from this use, they continued to use it only as a visual symbol. In the third problem, most of the teachers who answered the question related to a second method also preferred the guess & check approach. While applying this approach, they stated that this is also a valid method instead of making up numbers.

Considering all these data, it can be said that teachers' main problem-solving heuristic in algebra problems is the use of the equations in all three problem types. In decontextualized problems involving quantitative relations and problems involving quantitative relations between consecutive numbers, the most commonly used problem-solving heuristic was the use of shapes. In contextualized problems with two unknown quantities, one of which could be described by the other one, the teachers had difficulty finding a second solution.

4.2 What Do Middle School Mathematics Teachers Understand about the Bar Model Method When Examined Students' Bar Model Solutions in Algebra Problems?

The Bar Model Method is a visual tool used primarily in mathematics education to help students solve word problems, including algebra problems. It involves using bars or rectangles to represent quantities and their relationships, making abstract concepts more tangible and easier to understand for students. This method can be taught to students to enable them to use it in questions. In this study, the teachers were asked to analyze the responses of the students, who were taught the Bar Model Method, to some algebra questions. This section will delve into how teachers can distinguish between correct and incorrect solutions in student' responses and how they can make sense of students' use of the bar model in solutions.

During the interviews, the teachers were requested to assess students' responses to six distinct algebra word problems. They were asked to evaluate the correctness of these solutions and provide interpretations of these solutions. They were further prompted to suggest potential improvements if they deemed any of the solutions incorrect. Three of the questions that were shown to the teachers for review included correct solutions, while three included incorrect solutions. While analyzing the teachers' answers, their approaches are examined under two sub-headings.

4.2.1 Teachers' Understanding of Students' Correct Solutions

When the teachers' responses to the students' answers were analyzed in detail, it was observed that they mostly had no problems understanding the correct answers. In total, 43 of the 54 correct answers shown to the teachers were understood and classified as correct by the teachers. It was noticed that the students followed different ways while checking their solutions for correctness. For example, participant teacher T11 followed the student's way of checking the first solution (Figure 4.19) and made the following explanation:

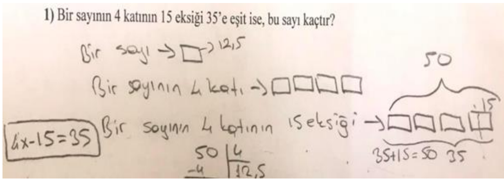
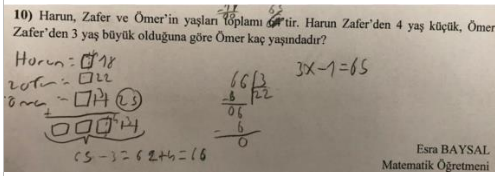
The student’s bar model solution	*The teacher’s understanding of the solution*
	He drew a box instead of the unknown. Then, because it said 4 times, he draw 4 squares and he indicated 15 less by drawing a dotted line over the square. He realizes that it's not 4 squares, 4 squares minus 15 equals 35. That's why he added 15, then divided it by 4 since it would be 4 squares. Yes, the student's solution is correct.

Figure 4.19 Teacher T11’s understanding of the first students’ solution

Similarly, T13 followed the same student’s solution correctly. It was observed that the teachers could make sense of the drawings in the student’s solution. In this context, although it is not the first solution that comes to their minds, their understanding of its use shows that they are sufficient in terms of student knowledge and content knowledge.

To understand some of the solutions, the teachers solved the problem themselves and compared their solutions with the student's solution. Participant teacher T11 solved the sixth problem (Figure 4.20) because she had difficulty understanding the student’s solution. After she solved the problem herself, she refocused on the student's solution based on her solution:

The student's bar model solution



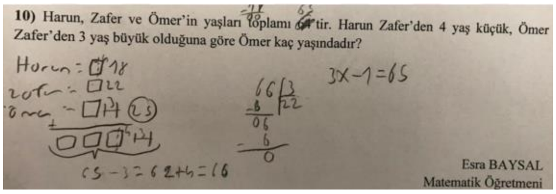
The teacher's understanding of the solution

I would call Zafer x, Harun x minus 4, and Ömer x plus 3. When I add them up, 3x minus 1 equals 65. When I solved the equation, I reached the conclusion that the student had reached the same result. When I looked at it, I saw that the student had solved it well. I can say that the student's solution is correct.

Figure 4.20 Teacher T11's understanding of the sixth students' solution

Similarly, teacher T13 solved the question herself before evaluating the student's solution by using boxes (shown in Figure 4.21) and added the following comment:

The student's bar model solution



The teacher's understanding of the solution

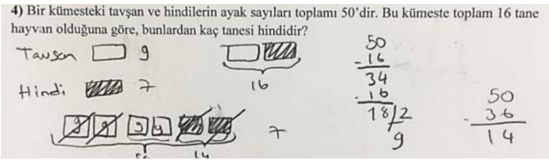
Let me solve the question myself first. Let me use boxes for Zafer just like the student. When I specify what to add and what to subtract in this way, yes, I have 3 boxes and minus 1, which equals 65. Ok, the student's solution is correct. Since I cannot make sense of the student's representations at first glance, I can realize it better when I see it myself.

Figure 4.21 Teacher T13's understanding of the sixth students' solution

When teachers have difficulty understanding students' solutions, they solve the problem themselves and make true-false evaluations based on their own answers, which shows their inadequacy in assessment. The path that students follow is as important as the result they reach; therefore, ignoring this may disrupt the teaching process. In this sense, the teachers who could not make use of the second method properly had more difficulty understanding different approaches employed by the students.

Among the correct solutions, the teachers had the most difficulty understanding the fifth solution and determining whether it was correct. While seven teachers determined that the solution was correct and conveyed the solution, eleven teachers had difficulty understanding how the solution progressed. For example, T17, one of the participant teachers, could not understand what the student did in the question and solved the question himself to check the correctness (shown in Figure 4.22). Then, when he saw that the student had given the correct answer, he explained that the student could have solved the question in the following way:

The student's bar model solution



The teacher's understanding of the solution

I honestly couldn't understand why he did repeated subtraction. I mean, he also wrote 9 and 7; is that correct? Let me check. Yes, 9 and 7 are correct. However, the solution makes no sense. I think he saw the answer somewhere and tried to create his own solution.

Figure 4.22 Teacher T17's understanding of the third students' solution

When the solutions of the teacher and the student coincide, teachers sometimes attribute this to cheating etc. The teacher's knowledge about the student is also very important here because a teacher who knows and recognizes the student will make inferences at this point according to the proficiency of the student. To understand this better, the teacher was asked the question "Would you accept this solution (in Figure 4.22) if your student had done it?" T17 answered the question as follows:

> I mean, first, because I couldn't understand, I would call the student and ask him to explain, because I couldn't understand what he was thinking from here. I would accept the result because it was correct, but I would also talk to him to find out what he was thinking.

In cases where the teacher cannot understand the student's solution, they should first ask the student to explain it. In this way, instead of making a direct judgment, it is possible to proceed in a more reliable way. Similarly, teacher T1 could not make sense of the third student's solution and stated that "I think he used the guess & check method to solve this problem." The main reason why the teachers thought that the third question was solved using the guess and check method was that the students wrote the answer on the boxes. The teachers who thought that the solution was reached using guess & check also stated that they could not understand where the student started and progressed from. At this point, what the teachers found unclear was the comprehensibility of the solution rather than the accuracy of the solution.

Seven of the participant teachers were able to understand and evaluate the solution of the third problem correctly (shown in Figure 4.23). Teacher T14 provided the following explanation regarding the solution:

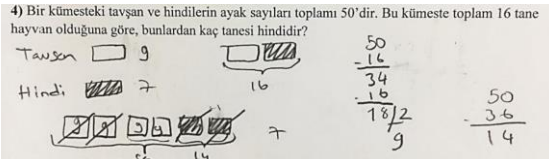
The student's bar model solution	*The teacher's understanding of the solution*
	In fact, the student solved the question in the same way I solved the question about the number of feet at the beginning. He made two different representations and wrote them as 4 empty boxes and 2 full boxes according to the number of feet. I mean, he wrote it in a little complicated way, but yes, I understand it now. It was a correct and good solution.

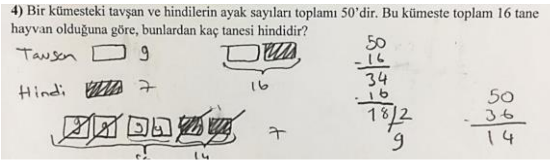
Figure 4.23 Teacher T14's understanding of the third students' solution

T14, who had no difficulty understanding this solution because she had used it herself, explained the reason for this as follows:

> This situation is a little bit due to students. When I used algebra with x for the first time in 6th grade, the children were very confused and had difficulty understanding. Because of this, I tried to add a little visualization, a box, a heart, a star, whatever. And I tried to use different symbols for each one so that the students wouldn't generalize the shape.

It was observed that the teachers who made sense of the fifth solution used or could use this method in their lessons. They found it logical to use different notations instead of two unknowns. Similarly, teacher T4 found the problem's solution very logical (shown in Figure 4.24) and added:

The student’s bar model solution



The teacher’s understanding of the solution

He assigned the rabbit a hollow square and the turkey a filled square. Since he referred to the number of feet, he put 4 squares for the rabbit and 2 squares for the turkey. The solution is very good, very clean. There is nothing I want to change. It seems to be useful to give two different symbols. So instead of x and y, this might be more useful.

Figure 4.24 Teacher T14’s understanding of the third students’ solution

While the teachers were interpreting the solutions, the ideas about the student's use of the bar model were also observed.

Although they could not name the bar model, they successfully recognized and identified which unknown it was used for. For example, participant teacher T1 said the following (given in Figure 4.25) for the first bar model solution:

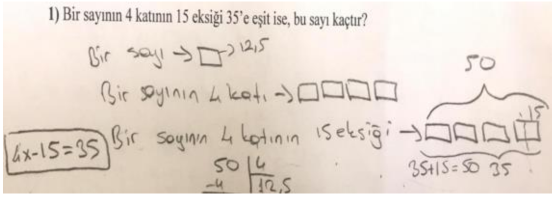
The student's bar model solution	*The teacher's understanding of the solution*
	He assigned the number a box and drew 4 boxes for 4 times. In fact, he did the same thing setting up the equation; he did not do anything very different there; he gave boxes instead of writing x. I mean, it's not a very creative solution, but it's good, it's good that he can think about it. At least he modeled it. For example, when it is in the form of 4x - 15 = 35, it may remain abstract. But now the question is more concrete. So, I think he understood the logic of that equation.

Figure 4.25 Teacher T1's understanding of the first students' solution

Although the teachers did not know the name of the bar model and could not make sense of its components, they were able to understand which model the student used for which unknown, especially in correct solutions.

Similarly, participant teacher T2 made this comment (shown in Figure 4.26) about the third students' solution:

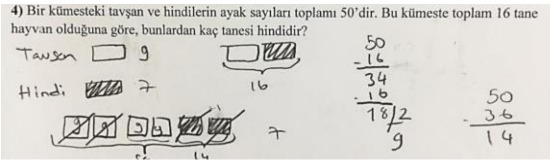
The student's bar model solution	*The teacher's understanding of the solution*
	He assigned the rabbit and the turkey different boxes. If I interpret this drawing in my own way, I interpret it as X and Y. The sum of x and y is 16. I didn't understand much after that, so let me analyze it a little more. So yes, he wrote the boxes according to the number of feet. Then, he proceeded by subtracting from the sum of two different boxes. He established a very good relationship in there.

Figure 4.26 Teacher T2's understanding of the third students' solution

The teachers appreciated that the students solved the problem by modeling it in their solutions. They were also pleased that using a concrete method was more understandable for them. It was observed that the teachers were generally not accustomed to these types of solutions involving drawings. Although they could not name them as bar models, it was observed that they found these solutions very effective for teaching.

Another point of interest in the use of the bar model while interpreting the solutions was whether they could identify quantitative relations in the visual parts of the bar model. The teachers had different approaches to the representations of "more" and "less" in bar model representations. For example, participant teacher T11 provided the following explanation (given in Figure 4.27) about the student's representation in the first students' solution:

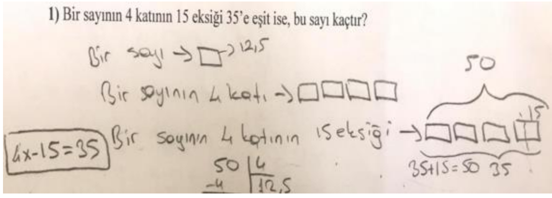
The student's bar model solution	*The teacher's understanding of the solution*
	The student showed 15 less here by dividing the square in the middle. So, it looks as if one of the boxes is 30. That's why I think it would be confusing. Instead of doing this, it would be more accurate if he added it directly, even off the top of his head, and if this is what is missing, I will add it. Or he could show it as -15.

Figure 4.27 Teacher T11's understanding of the first students' solution

Most of the teachers could not make sense of the representation of subtraction using a dashed line on the Bar Model solution. In this sense, we can understand that the teachers did not see the boxes as quantities but as representing objects because they said that it would be better to write the number directly as + or - instead of using this notation. Participant teacher T13 (given in Figure 4.28), who thought that the boxes used by the student were a representation but still considered it wrong, said that:

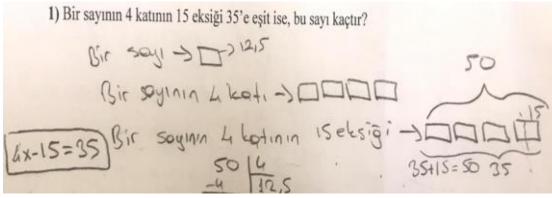
The student's bar model solution	*The teacher's understanding of the solution*
	There is something wrong with the drawing. He made it look as if he cut the square into pieces. I don't understand the logic there. If I think of it as subtracting from 4 squares instead of one square, I think he tried to make a small representation. However, since one of the squares is equal 12.5, I think that representation is wrong.

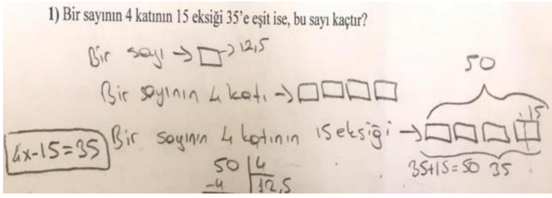
Figure 4.28 Teacher T13's understanding of the first students' solution

Teacher T12, on the other hand, made the following comment about using numbers in the form of + and - when representing the numbers with boxes with boxes in the questions that she was asked about at the very beginning (Problem 2: The sum of the ages of Türkan, Seda, and Derya is 55. If Türkan is 13 years older than Seda and Derya is 3 years younger than Seda, how old is Türkan?):

> I assign Seda's age as a square, and since her mother is 13 years older than her, I write her mother as a square +13. I am a little doubtful that my writing here is correct, so it can be confusing. Students ask, "How is it that one is a square and one is a number?" I think they are right; it seems strange to write a number directly there, so I indicate that there is a number under the square. However, I would like to learn if there is a more correct way.

T12, who correctly evaluated the components of the bar model in the solution, made the following comment (given in Figure 4.29) about the use of the bar model in the first students' solution:

The student's bar model solution



The teacher's understanding of the solution

In this question, instead of writing minus 15, it made a lot of sense to indicate that there is a decrease in a using dashes, and I can use it, for example. In other words, I think it allows us to understand better and transfer the notation there.

Figure 4.29 Teacher T12's understanding of the first students' solution

At this point, it can be said that teacher T12 did not consider the bar model only as a symbol replacing a number and was able to make sense of its quantitative meaning. However, the teachers who thought in this way were in the minority according to the collected data. Nevertheless, T9 also found that it was very logical to use these representations, which are dashed squares for subtraction and units for addition, in the solutions:

> I think it makes sense to indicate a decrease with a dashed line and a decrease with a unit. Since the subject is abstract, this approach concretizes it, making it easier for students to understand.

The fact that the teachers found this approach correct and said that they would use it themselves reveals their understanding that the bar model representation does not only serve as a symbol for numerals but shows a quantity.

In addition, the analysis focused on whether the teachers identify the connection between the bar model and the solution of the equation, that is, parallel quantitative relations in the visual and the operational parts of the bar model. The teachers generally interpreted the equation they saw in students' solutions as a separate solution. For example, participant teacher T18 (given in Figure 4.30) made the following comment in the first students' solution:

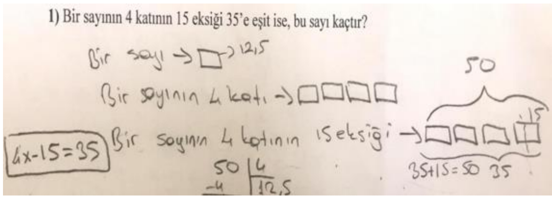
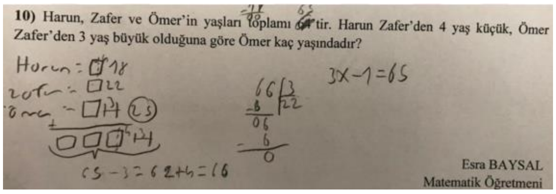
The student's bar model solution	*The teacher's understanding of the solution*
	First, he wrote the equation he wanted to solve using x. After writing the equation, he drew a figure. I think he should have chosen one of the two. His drawings are correct, but I think he mixed it up a bit by adding the equation.

Figure 4.30 Teacher T18's understanding of the first students' solution

Teacher T18 stated that the student had to choose one of the two, which means that these two solutions could not be connected. From this point of view, it can be concluded that teacher T18 could not understand the parallelism between the use of the bar model and the equation. Teacher T17, who had similar thoughts for the same question, provided the following explanation about this issue: "When I look at the equation, I can see that he solved it correctly, and the result is correct. He drew 4 boxes and added 35 and 15 below it, but I couldn't understand whether he did it because of the dashed part in the notation." Based on this comment, it can be said that teacher T17 sees the model and the equation as different things.

Noticing the parallelism between this quantity and procedure, participant teacher T12 interpreted the sixth solution as follows (given in Figure 4.31):

The student's bar model solution



The teacher's understanding of the solution

He solved it correctly and I found the same result. Here, he assigned Zafer a square because Zafer is connected to both. Since Ömer is 3 years older than Zafer, he added the representation of 3 units next to the square. Since Harun is 4 years younger than Zafer, he showed 4 subtractions from the square with a line.

Figure 4.31 Teacher T12's understanding of the sixth students' solution

Teachers' inability to realize the parallelism between the equation and the bar model or to create this parallelism shows that they cannot make sense of the model representation. It can be said that they are still dependent on the equation at some point.

4.2.2 Teachers' Understanding of Students' Incorrect Solutions

When the teachers' responses to the students' solutions were analyzed in detail, it was observed that they did not have any problems understanding the incorrect answers. In total, all 54 incorrect answers shown to the teachers were understood and categorized as incorrect by the teachers. The teachers followed different methods while checking the incorrectness in student solutions. For example, participant teacher T2 wanted to solve the problem from second students' solution while checking the solution and made the following explanation (given in Figure 4.32):

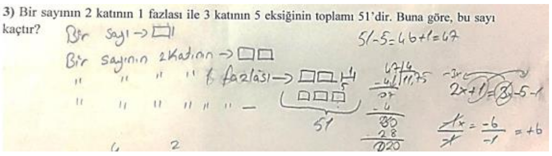
The student's bar model solution	*The teacher's understanding of the solution*
	I didn't quite understand what she did after the boxes. If we write it as an equation, there are 2x + 1 and 3x - 5 equal to 51. From here, 5x - 4 = 51. I added 4 to both sides and got 55. She got it wrong. It was difficult to make sense of the boxes, but this is the solution.

Figure 4.32 Teacher T2's understanding of the second students' solution

The teacher's interpretation that it is not very meaningful to solve the solution without an equation shows his deficiency in terms of understanding students' possible solutions and the use of the bar model in algebra problems. Moreover, determining the correctness of the solution directly using the equation shows the teacher's dependence on a single method.

What the teachers realized about the components of the bar model in incorrect solutions proceeded in parallel with what they realized in correct solutions. It was observed that they showed deficiency in understanding the components of the bar model in correct solutions as well as in incorrect solutions. For example, participant teacher T2 made the following comment regarding the fourth students' solution (given in Figure 4.33):

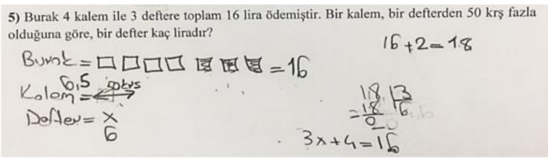
The student's bar model solution	*The teacher's understanding of the solution*
	How nice; he assigns a scanned box to one of them and an unscanned box to the other one. He understood the difference here and was trying to write something with two unknowns. He supported it with a visual. In other words, he has established the logic of two unknowns in the notation part, which is nice.

Figure 4.33 Teacher T2's understanding of the fourth students' solution

It can be said that the teachers did not have any problems understanding how the unknown was represented with the bar model. Similar to the case in correct solutions, although they could not name the bar model, they were successful in recognizing and identifying which unknown it was used for. Similarly, participant teacher T3 made the following comment for the fifth students' solution (given in Figure 4.34):

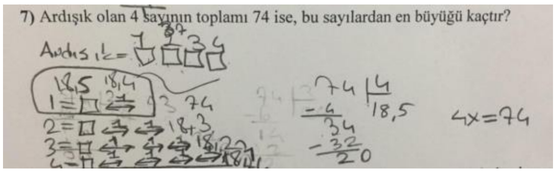
The student's bar model solution	*The teacher's understanding of the solution*
	If the sum of 4 numbers is 74, what is the largest of them? This child is aware that the difference between consecutive numbers is 1. He called the first number square plus 1, the second number square plus 2, the third number square plus 3 and so on. This is a nice demonstration.

Figure 4.34 Teacher T3's understanding of the fifth students' solution

While the teachers were interpreting the students' solutions, another point of interest in the use of the bar model was the visuality of the bar model and whether they could make sense of the relationships between quantities. As was the case in correct solutions, the teachers had different thoughts about using dashed squares for subtraction and units for addition. For example, participant teacher T6 provided the following explanation about the representation in the second students' solution (given in Figure 4.35):

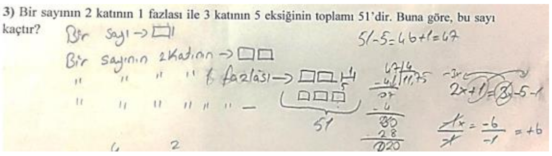
The student's bar model solution	*The teacher's understanding of the solution*
	For 5 less than 3 times, he did not write -5 but divided the square again. I think he should have written one as +1 and the other as -5 as separate natural numbers. It would have been more accurate if he had specified it this way.

Figure 4.35 Teacher T6's understanding of the second students' solution

Similarly, T13, while interpreting the second students' solution, attributed the student's mistake to his confusion and misconception about the use of dashed squares for subtraction and units for addition (given in Figure 4.36):

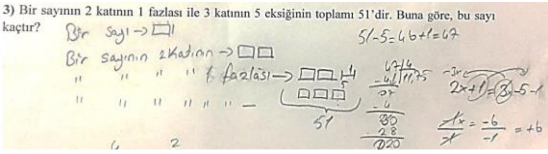
The student's bar model solution	*The teacher's understanding of the solution*
	The student divided the sum by 4 when he should have divided by 5. While showing the decrease from a box with the dashed representation, he thinks it is as if a box is missing. So, the notation confused the student.

Figure 4.36 Teacher T13's understanding of the second students' solution

On the contrary, there were teachers who argued that these errors were not due to the use of the bar model but originated from not understanding the inverse operation and misidentifying the unknown. For example, participant teacher T9 made the following comment about the second students' solution (given in Figure 4.37):

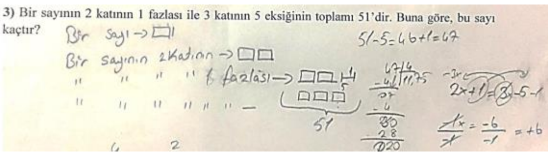
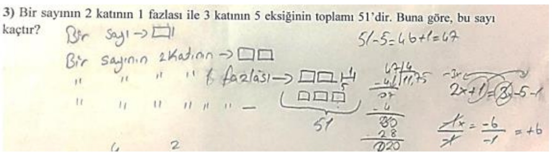
The student's bar model solution	*The teacher's understanding of the solution*
	The student accurately represented the given data using a box for the unknown, units for "more," and a dashed line for "less." However, he didn't realize he needed to use the inverse operation afterward, leading to an incorrect answer.

Figure 4.37 Teacher T9's understanding of the second students' solution

We focused on how the teachers evaluated the relations in the bar model's visual and procedural/operational parts. In this context, the teachers had difficulty explaining and seeing this relationship in incorrect solutions. It was more difficult for them to follow incorrect solutions. However, with further examination, they realized and interpreted that the error in the operation was due to the fact that it mismatch with the visual. For example, participant teacher T17 commented on the general solutions: "We can decide whether the question is right or wrong when we see an equation that does not fit the drawing." According to this approach, teacher T18 is aware that the bar model should have a visual and procedural relationship but sees them as different solutions. Teachers generally interpreted the equations they saw in student solutions as separate solutions. They had difficulty making a connection between the equation and the bar model notation in incorrect solutions as well as in correct solutions. For example, participant teacher T18 made the following comment in the second students' solution (given in Figure 4.38):

The student's bar model solution



The teacher's understanding of the solution

There is no mistake in the drawing as a shape, but I think he also created an equation on the right side, but that equation was wrong in a different way. So, he got different results on both sides. Either he should not have written the equation, or he should have provided a control with the figure, or he should have thought the opposite.

Figure 4.38 Teacher T18's understanding of the second students' solution

Teachers who interpreted incorrect solutions were also asked how they could correct the error. In this sense, the teachers were more inclined to use the bar model.

However, it was also observed that they could not express the quantities correctly when using the bar model. For example, participant teacher T17 used the following approach to correct the student's mistake in the fourth students’ solution (given in Figure 4.39):

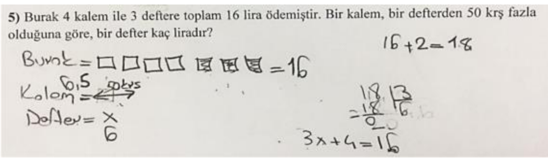
The student’s bar model solution	*The teacher’s overcome way to the mistake*
	It could be done like this: we have three scanned boxes for notebooks. Since the pencils are 50 cents more than the notebooks, we could write one scanned box plus 50 cents to represent the pencils so that we would have a total of 7 scanned boxes plus 2. From here, we have converted them all to the same genus and we have a surplus of 2. I would emphasize that she must subtract because we must do the reverse operation.

Figure 4.39 Teacher T17’s method of correcting the mistake in the fourth students’ solution

This was also the explanation provided by 17 teachers who could not understand the quantitative relationship and the components of the bar model accurately. All of them started to use the bar model correctly, but they described the quantities as + and -. Teacher T16, who tried to correct the error by using the bar model (but explained this verbally) and was able to use quantities correctly, stated that he could correct the second students’ solution as follows (given in Figure 4.40):

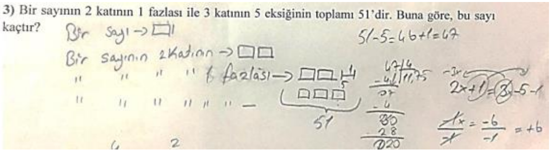
The student's bar model solution	*The teacher's overcome way to the mistake*
	I try to work through his solution first. I even draw again using his method. Here, we have 5 boxes, we subtract 5 from the box and then we add 1 to it. The total here is 51. Then, I proceed in the form of question and answer. What would have happened without that 1 excess? Then, if the missing 5 is 50, what is the total? I would ask questions like that.

Figure 4.40 Teacher T16's method of correcting the mistake in the second students' solution.

Teacher T16 preferred to use the bar method to correct the student's mistake. After placing the components correctly, he also used the quantities correctly. Similarly, 16 of the participant teachers approached the solutions in this way.

In addition to these, there were approaches other than the bar model to correct errors. Instead of the bar model, 21 of the teachers tried to solve these errors by using equations, verbal expressions, or simpler shapes. For example, participant teachers T17 and T18 argued that using equations would be more understandable for students. T6, on the other hand, stated that he could solve the problem easier by drawing shapes. On the other hand, T16 stated that he could do this with more verbal expressions and that he would guide the student in a way that he would discover his own mistake using questions. He argued that going through the students' solution is the best way.

4.3 What Are the Middle School Mathematics Teachers' Conceptions About Using the Bar Model Method in Algebra Teaching?

Understanding the perspectives of middle school mathematics teachers on the utilization of the Bar Model Method in teaching algebra is essential for gaining insights into how this visual tool is both perceived and employed within educational environments. Further, by comprehending the depth of teachers' knowledge, confidence, and beliefs in relation to the incorporation of visual aids such as the Bar Model Method, educators can refine their pedagogical methods and provide enhanced support to students in cultivating a profound comprehension of algebraic principles. This section will thoroughly examine educators' perspectives on implementing the Bar Model Method in the instruction of algebra.

In this context, the teachers provided insights during the interviews regarding the applicability of the bar model method in solving algebraic word problems and instructing algebra. These can be examined under the heading of teachers' conceptions about the functions of the bar model method in algebra. Additionally, the instructors expressed their perspectives on the suitability of the bar model for being incorporated into the curriculum for instructional purposes. It can be expressed under the heading of teachers' conceptions about the appropriateness of the bar model method in algebra teaching curriculum.

4.3.1 Teachers' Conceptions about the Role of The Bar Model Method in Algebra

Teachers' views on the use of bar models were observed both indirectly while examining students' bar model solutions and directly through the questions posed about the bar model. In this context, the first question asked to the teachers was "What do you think about using the Bar Model in solving algebra word problems?" When the answers to this question were analysed, it was seen that there was no teacher who found the use of the model unnecessary. The teachers agreed that this representation is useful. For example, participant teacher T2 said, "Actually, it is a useful method for transition from known to unknown. It can be used at the beginning to help students make better sense of algebra." Similarly, participant T8 thought in the same way: "Students can be afraid when they encounter the unknown directly. This prevents the provision of an effective learning environment. In this sense, if it is shown before or during the transition to algebra, I think it is a method that can eliminate students' reservations". It can be said that teachers are aware that students have difficulty in algebra and are in search of a way to help them understand these subjects more easily. Although they are aware that it is difficult to use a different method for each student, it can be concluded that they want to reach every student.

Among the ideas about the use of bar models in algebra problems, the most common one among the teachers was the advantage of connecting concrete concepts with abstract concepts. In this context, participant T3 stated the following:

> It is quite abstract for students to see a symbol called ‘x’ when algebra is introduced. For this reason, giving this context before moving on to x allows the student to approach the subject from a more concrete point.

Simiarly, participants T7, T12, and T16 also stated that the use of the bar model enables students to see algebra problems more concretely and understand their logic more easily. In the light of these comments, it can be concluded that teachers think that algebra is a subject that needs to be concretised. According to participant teacher T12, students are always looking for something concrete. Although it is difficult to concretise a subject in terms of mathematics, abstract concepts remain elusive for students, especially at a younger age. It can be said that teachers agree on this view and therefore, find the bar model useful for its advantage of concretisation.

In addition to making algebra concrete, there were teachers who stated that the bar model method enabled them to understand algebra in a more visual way. For example, T18 made the following statement in this regard:

> Students see algebraic relationships more easily through visuals. For example, when they see what we call three times as three figures, it is

> easier for them to visualise the concept. And it is faster for them to remember it later.

T13, T8 and T4 made parallel comments in terms of understanding algebra visually. These comments show that teachers find it more appropriate to teach algebra by visualising it. The general perception of teachers is that students learn better in a visual way. Likewise, there were teachers who thought that conveying the arithmetic relationship more visually in this way facilitated expression and understanding. For example, participant teacher T14 made the following comment on this issue:

> By using this bar model, the child can observe how much space a multiplicity represents, how it changes when it is decreased and how it changes when it is increased. In this way, the child can discover arithmetic relationships and generalise them to algebra.

It was observed that teachers generally did not see algebra as a generalisation. Only one teacher interpreted the connection of algebra with arithmetic as a generalisation.

In addition to teaching algebra, the teachers also mentioned the advantages of problem solving. For example, T16 provided the following explanation about showing such methods to students: "Showing such methods to students in problem solving provides the message that they can reach the result using different methods." In parallel with participant T16, T5 stated the following:

> Sometimes students do not understand the first method no matter how many times it is explained. In such cases, including different representations, even as a second method, helps students understand the problem and the subject.

Based on these comments of the teachers about the use of the bar model, it can be said that they are actually prone to use the model, but they do not prefer to use it because they have conventional-traditional solution approaches. Although they tend to give students what is ready, they are actually ready to use bar models as an alternative method or to provide ease of transition from abstract to concrete.

There were also teachers who thought that the use of the Bar Model would confuse students at some point. Participant teacher T6 provided the the following explanation in this sense:

> Using the bar model method is actually logical for the transition from abstract to concrete. However, the representation of arithmetic relations in the model may confuse students. At this point, it may be better to show increases and decreases directly with numbers.

Similarly, T18 made the following comments: "I think the dashed lines used in the model are something that will confuse students. This can lead to other mistakes such as deleting that box completely." It can be inferred from their responses that their perceived cause of the confusionwas actually due to their inability to conceptualise the bar model. There were also ideas in the opposite direction of this interpretation. Regarding the use of the Bar Model in analysing and resolving students' misconceptions in algebra learning, teacher T3 stated the following:

> I think it would be a nice and simple method to solve students' misconceptions with this model method. Since it is concrete, it is useful for us in terms of eliminating misconceptions when we realise them.

According to these comments, it can be said that the teachers were divided into two categories, with one group thinking that representation of addition and subtraction operations on the bar model would lead to misconceptions and the other group thinking that these representations would be good for solving misconceptions. Among the causes of misconceptions, the teachers firstly stated that algebra is an abstract subject. T1 stated that he could benefit from the bar model while using the 'inverse operation' method while solving equations. T13 explained the effects of using the bar model by providing an example of another student misconception, as follows:

> For example, students make mistakes and misconceptions while explaining the 8 times of 3 times and 3 times of 8 times of a number.

> Showing these expressions with the model method and perhaps solving the problem in this way will resolve their confusion.

Participant teacher T1 stated that the bar model would not only eliminate misconceptions, but also prevent them from developing such misconceptions in the first place and added:

> While solving problems, students may fall into the misconception of adding where subtraction should be done and subtracting where addition should be done. And in such questions, saying that the 'inverse operation' should be done may not always work. However, we can prevent these misconceptions if these models are used first when writing algebraic expressions mathematically.

Considering these comments, it can be said that teachers think that the bar model will not only be a solution to students' misconceptions but also facilitate teaching for teachers.

4.3.2 Teachers' Conceptions about The Appropriateness of The Bar Model Method in Algebra Teaching Curriculum

The teachers were first asked whether they had encountered the bar model in the curriculum. They said that they had not encountered this model as a teaching method. In this sense, it cannot be claimed that teachers lack knowledge about the use of the bar model in the curriculum since there is no precedent of this. In response to the question asked to the teachers about the integration of the bar model into the curriculum, only one of the 18 participant teachers stated that it was not appropriate to include the bar model in the curriculum. T10 explained his reasoning for not finding it appropriate as follows:

> This is really a good method for comprehensibility, but students do not use it after they learn ‘x’, that is, the unknown. Therefore, I do not think it is very necessary to integrate it into the curriculum.

In this sense, it can be said that teacher T10 did not find it very important for students to learn algebra cenceptually. It can be said that the teacher prefers to employ faster and more frequently used methods rather than those facilitating the learning process for students. Other teachers stated that it would be useful to have it in the curriculum and that it was appropriate to be given. Nevertheless, there were some teachers whose opinions were similar to T10. For example, although participant teacher T16 found it appropriate to include it in the curriculum, he added the following comment:

> I think it is appropriate and can be integrated into the 6th-grade curriculum, even into the 5th-grade curriculum. However, I don't know how useful it will be since students think that why bother with this, why is it necessary, let's do it in the short way etc.

At this point, it can be said that teachers generalize students and hesitate to make them discover a new process. Although teachers think that students are not open to innovations, it can be claimed that it is it is the teachers that cause this.

The teachers mostly provided opinions in favor of including the bar model in the curriculum. T3 explained one of these opinions as follows:

> I would like to see models that will facilitate this kind of learning directly in the book to be used in the classroom. Thus, teaching this to students in the classroom will be a requirement rather than the teacher's initiative. For example, I remember this method from university, but I can't think of having used it in my class. It would be reasonable to have this directly available in our resources rather than having it as an option.

Teachers think that the use of the bar model in the curriculum will affect their teaching techniques. However, they do not think that it is their role to make this change on their own. Teachers tend to teach their students based on the materials and methods they are instructed to use. Even if they have knowledge of alternative techniques, , they mostly do not implement them in the classroom. Teacher T6 expressed another common idea as follows:

> It would be good to have it in the curriculum, that is, I think that teaching this should not be an option and should be included in the curriculum. As children come from primary school where they learn about subjects using objects and then we try to replace them with letters directly, I think it doesn't work. So, I think it would be better for them to make a connection for the transition between them. It would also be a good transition for the difference between numbers and objects.

Teachers generally advocated for including the bar model directly in the curriculum for teaching as they felt that there was little use of teaching techniques that were not incorporated into the teaching process. This shows that teachers actually think that the curriculum needs to be revised in terms of teaching techniques. There were also common ideas about the grade level at which the Bar Model should be included in the curriculum. For example, T11 explained as follows:

> In addition to being included in the curriculum, the timing is also very important. For example, it would be effective if it is included before the student is introduced to the unknown, that is, in the 5th grade. The 6^{th} grade would also be appropriate, but they may not have enough time to reinforce this method.

Based on these comments, it can be said that teachers have a good command of the curriculum as well as a good command of which needs are more important at which age level. It can be said that there is a point where teachers' curriculum knowledge and student knowledge are intertwined.

The findings were analyzed to answer the research questions, and a summary of the findings is presented here. It was observed that teachers' problem-solving heurustic tendency in algebra word problems was the use of equations. While the first solution method that came to the teachers’ minds was mostly equation-based solutions, they were followed by the use of shapes. In addition, it was observed that teachers succeeded in identifying correct and incorrect answers as a result of examining students' bar model solutions, but their identification methods were insufficient within the scope of the components of the bar model. The teachers compared the bar model solution with the use of shapes and said that they found the components meaningless. They approached both the use of shapes and the use of the bar model only as a visual symbol and could not comprehend quantitative meaning of the bar model. After the explanation of the bar model, they made positive comments about its place in the educational process. The teachers stated that the bar model can be used to reduce misconceptions and facilitate the transition from concrete to abstract. They also stated that its direct introduction in the curriculum would have a positive effect on algebra learning.

CHAPTER 5

CONCLUSION AND DISCUSSION

The main purpose of this study is to examine the problem-solving heuristics used by middle school mathematics teachers in algebra word problems and teachers' views on the use of the bar model in algebra teaching. In the first section, teachers' problem-solving heuristic in algebra word problems is presented. The second section includes teachers' opinions about the Singapore Bar Model in algebra topic. The last section includes the limitations of the study, as well as implications and suggestions for future studies.

5.1 Teachers' Problem-Solving Heuristic in Algebra Word Problems

The findings indicated that teachers tended to use the method of using equations as the first solution method in algebra problems (see Table 4.1). This result overlaps with previous studies (Fuchs et al., 2012; Koedinger & Nathan, 2004). The method of "using equations" for problem-solving involves employing mathematical equations to represent and solve various types of problems. This involves identifying unknown quantities, representing them with variables, and setting up equations based on the relationships described in the problem (Koedinger & Nathan, 2004). Consequently, the method of "using equations" serves as a fundamental approach to problem solving in a variety of disciplines and provides a mathematical framework for analyzing, interpreting and solving complex problems through application of mathematical equations and techniques (Schoenfeld, 1992). This approach is pivotal in fields such as physics, engineering, and economics, where equations are utilized to model real-world scenarios and predict outcomes. In this study, the teachers were asked to solve three types of problems which decontextualized problems involving quantitative relations, problems involving quantitative relations between consecutive numbers, and contextualized problems with two unknown quantities. The fact that the teachers primarily preferred to use equations to reach the solution in all three problem types may indicate that they only see algebra as questions solved by using "unknowns" at the middle school level. The fact that the teachers had no difficulty setting up equations and that they correctly solved the equations may indicate that they did not lack algebraic knowledge. However, the fact that they proceeded in a single way and used traditional methods shows that they needed to gain deeper knowledge in teaching algebra. Traditional methods often focus on procedural calculations without emphasizing the fundamental relationships between different algebraic representations (Kilpatrick et al., 2001). This focus can be an obstacle for students in learning algebra effectively because effective teaching and learning require addressing different learning styles. Differentiated instructional approaches positively affect students' academic achievement in algebra (Bal, 2016).

As the second method, most teachers used shapes in the first two problem types (see Table 4.1). The first problem related decontextualized problems involving quantitative relations (If 12 less than 3 times the number is equal to 2 times 8 more than the same number, what is this number?) and the second problem related involving quantitative relations between consecutive numbers (The sum of the ages of Türkan, Seda, and Derya is 55. If Türkan is 13 years older than Seda and Derya is 3 years younger than Seda, how old is Türkan?) were considered more suitable for using shapes because these were in the style of a problem that teachers used in the first teaching of algebra. This made these problems more suitable for using shapes according to the teachers. Of the teachers who used shapes as the second method in the problems, most of them preferred drawing boxes. In comparison, the others used shapes such as hearts and stars, indicating that they were recommended for the younger age group. Even this difference in methodology was quite traditional for the teachers. They tried to create a more visual solution by covering the unknown with a box/shape. However, it is difficult for this to be meaningful for students on its own because, in the teachers' solutions using shapes, there were shapes in one place and numbers (for addition and subtraction) in the other. When students see numbers and shapes together, they do not fully encounter a concrete structure.

In this context, teachers think that visualizing algebra will facilitate learning, but they do not know how to convey this visualization.

On the other hand, the third problem related contextualized problems with two unknown quantities (The total number of legs of chickens and sheep on a farm is 122. If the total number of chickens and sheep on this farm is 42, how many sheep are there?) was the problem where the teachers had the most difficulty in finding an alternative method to using equations. According to the results of the Baysal and Sevinç's (2022) study in which the problem sets were determined, the problem type that student had the most difficulty with was problem-related contextualized problems with two unknown quantities. In this sense, parallelism was observed in the approaches of teachers and students between the studies. Moreover, they did not prefer to use shapes in this problem (see Table 4.2). Most of the teachers said they could not find any other method, while some stated that they could use the guess & check method. In cases where there were two independent unknowns, it was seen that it is easier to reach the answer by guessing compared to other methods. For teachers, such problems were considered unsolvable without using unknowns/or constructing equations. In this sense, it would be difficult for them to give students the subtext that there are different ways to solve such algebra problems. Various problem-solving techniques play an important role in determining students' success in mathematics (Polya, 1945). For example, the guess & check method effectively develops students' problem-solving skills and algebraic reasoning (Kaminski & Sloutsky, 2012).

The use of physical models was very rarely observed among the methods used as a solution by the teachers. This is due to the fact that most of the teachers have recently started to work as the teachers who preferred to use physical materials had been on duty for a longer period of time. In this sense, the findings obtained in this study are in line with the study that found that teachers had difficulty using algebra tiles and other physical models effectively (Yıldız & Akyüz, 2019). Teachers preferred model and shape representations such as drawings rather than physical representations. However, these visualization methods did not include a conceptual background. These findings are in line with previous studies. Studies show that many educators lack the necessary mathematical expertise and pedagogical knowledge to use various representations effectively, which can hinder students' understanding of algebraic concepts (Hill et al., 2008; Even, 1993). However, the cognitive processes involved in solving algebraic problems are enhanced using visual aids. For example, discussions on how students' misunderstanding of algebraic notation can hinder their ability to accurately represent word problems suggest that visual representations can help fill this gap. Visual tools such as diagrams, number lines, and bar models can make abstract algebraic concepts more concrete, providing students with a more intuitive understanding of mathematical relationships (Clement, 1982; Kieran, 2007). Therefore, teachers' competence in using representations in teaching is crucial for effective mathematics teaching, especially in algebra. The ability to use a variety of representations, such as visual, symbolic and concrete forms, enables teachers to appeal to different learning styles and improve students' understanding of complex concepts. The findings of the study show that teachers are deficient in this sense and further research is necessary on this issue. These results coincide with the findings in the literature (Hill et al., 2008; Even, 1993).

5.2 Teachers' Conceptions about the Singapore Bar Model in Algebra

The findings indicate that teachers did not have much difficulty identifying correct solutions in general. However, it was observed that in cases where teachers could not make sense of the bar model solutions, they made judgments by comparing their own results with the students' results. In this context, it was observed that they focused on the result rather than the process. In the field of algebraic problem solving, both the process students follow in their solutions and the results they obtain are essential elements. The process of solving algebraic problems involves cognitive processes that are necessary to effectively understand and apply mathematical concepts (Schoenfeld, 1985). Teachers who see the process and the outcome as a whole in problem-solving positively affect student achievement (Lester, 1994; Silver, 1987). For this reason, considering the process and the result independent of each other will hinder effective teaching and assessment. It is seen that teachers are also deficient in this sense. In the case of incorrect solutions, they had difficulty understanding and following students' thought processes. It is common for teachers to have difficulty analyzing students' incorrect solutions. This challenge stems from the difficulty of determining what information students know incompletely or incorrectly (Welder, 2012).

Teachers were often unable to provide effective interventions to address student errors when they did not have an accurate understanding of their errors. However, when they were able to correctly identify the source of the error, they usually had sufficient knowledge about how to intervene. After identifying the errors, the teachers often tried to solve the problem using their own methods, ignoring the student's individual learning needs. This leads to using generalized approaches instead of personalized instruction needed to correct the student's errors. As a result, the student's difficulties in the learning process may not be fully addressed, and the learning process may not be effectively supported. Therefore, teachers' ability to better analyze student errors and develop student-specific interventions plays a critical role in improving the quality of education. While most of the teachers (17 participants) moved towards a direct teaching technique, one of the teachers stated that he had a philosophy of pushing students to the answer instead of giving them the answers. In this sense, the importance of recognizing students' mistakes and determining the reasons behind them is undeniable for effective learning. Addressing these misconceptions not only corrects errors in thinking but also encourages students to engage more deeply with the material, developing critical thinking and analytical skills (Carpenter et al., 1989; Swan, 2001).

The bar model method is a visual technique that helps solve algebraic problems by using bar diagrams to represent and visualize mathematical relationships.

Furthermore, this methodology has been developed not only to enrich students' comprehension but also to refine pedagogical strategies, with the aim of fostering confidence among pre-service educators. The fact that teachers first solve the problem themselves and then compare their solutions with students' answers shows that they do not understand bar model solutions. The inability of teachers to check the answer through the bar model solution and try methods other than the students' bar model solution to correct the wrong answers shows that they are deficient in terms of the components of the bar model. Although the teachers preferred drawing boxes when they used shapes to solve problems, they were not familiar with the Singapore Bar Model because they had not used the components of the bar model. It can be concluded that teachers are inadequate in terms of understanding the bar model because they think that the bar model and the use of shapes are the same, and they approach the use of shapes only as visual symbols.

The bar model method is a commonly used visual representation technique in mathematics, especially in algebra, to aid students in understanding and solving problems. Teachers' beliefs about the method can greatly influence how they use it in their teaching. Teachers' comments about the use of the bar model while examining student solutions show that their knowledge of the Singapore Bar Model is not comprehensive and that they are not aware of the use of this model. Teachers stated that they also used these solutions in their lessons but interpreted them as using boxes in the solution. In fact, although the teachers do not know the name "Bar Model," they said that they had seen this model before. However, while they could understand the use of units used for addition, they did not understand the use of boxes with dashed lines for subtraction. They stated that dashed representation needed to be conceptualized and suggested using numbers instead. Since the students' use of the bar model was also incorrect in the solutions shown to the teachers, the teacher commented that the use of the bar model and the use of shapes were the same. For these reasons, teachers interpreted the model only as shape, not quantity. Using shapes and models in algebra problems both serve as visual aids, but they differ in their approach and purpose. Shapes are often used as symbolic placeholders for unknown values, helping students visualize relationships between variables without providing a structured method for solving the problem (Kilpatrick et al., 2001). Using shapes is commonly used in early algebra, where students learn to think abstractly about equations and variables. In contrast, models like the Singapore Bar Model offer a systematic, step-by-step representation of algebraic relationships. They provide a concrete, visual way to break down and solve problems by mapping out quantities and their relationships, guiding students through a logical problem-solving process (Ng & Lee, 2009). While shapes help in understanding abstract relationships, models are tools that aid in developing problem-solving strategies by making the structure of algebraic problems more explicit. At this point, it can be said that even if the teachers had no difficulty identifying the bar model, they could not comprehend the components of the bar model and determine the quantitative relationships in the visual parts of the bar model. It can be concluded that they did not see the bar model as a quantity but rather as a representative object. If the bar model cannot be understood as a whole in this way, teachers' use of this model in the solutions of questions or algebra instruction will constitute an obstacle to effective learning. Unconscious use of the bar model by teachers may mislead students, lead to misconceptions, and therefore, may not produce the intended learning outcomes (Ng & Lee, 2009). However, the bar model method is a valuable component in algebra instruction and offers several benefits that enhance students' learning experiences. Research by Baysal and Sevinç (2022) shows that the bar model method serves as a visual tool that increases students' motivation and plays a vital role in solving algebra problems.

The teachers generally provided positive opinions regarding the use of the bar model. A few teachers (3 participants) stated that they did not find it appropriate to integrate it into educational programs because they thought that the representation of quantitative relationships (addition/subtraction) would confuse students. On the other hand, there were some teachers (2 participants) who thought that it would be good to have it in the curriculum, but they thought that it would not be very useful because they thought that students would find this model meaningless even if they saw it. On the contrary, most of the teachers (15 participants) stated that it would make sense for both teachers and students to have the Bar Model directly in the curriculum. Teachers are aware that abstractness in algebra teaching distracts students and causes them to struggle. For this reason, they think that approaches such as the Bar Model that help students visualize and concretize algebra should be included in the curriculum. Research by Jahudin (2024) highlights the positive effects of using Polya's problem-solving approach with the digital bar model in developing students' algebraic thinking skills. In addition, Osman et al. (2018) highlight how the bar model visualization technique improves students' mathematical problem-solving skills by helping them construct their knowledge. The Bar Model method is a valuable tool that should be integrated into the curriculum due to its significant impact on students' learning experiences and mathematical problem-solving skills.

As a result, when we examined the teachers' perspectives on the use of the Singapore Bar Model in algebra teaching, it was seen that the teachers were not familiar with the Bar Model notation. However, studies stated that the bar model serves as an effective heuristic method that combines Polya's problem-solving approach with digital tools (Jahudin & Siew, 2023). This integration not only helps students visualize algebraic problems but also fosters cognitive development during the problem-solving process. Therefore, teachers' familiarity with this model is very important. Another issue was related to whether teachers wanted to include the Bar Model Method in the curriculum. Some of the teachers wanted to integrate the Bar Model into the curriculum because they considered it useful for the transition from abstract to concrete, and some of them considered it useful for the transition from arithmetic to algebra. In addition, they thought that it should be included in the curriculum because it would be effective in eliminating and even preventing students' mistakes and misconceptions. The bar model has been shown to effectively address misconceptions in algebra, enhance higher-order thinking skills in mathematics for younger students and helps bridge the gap between concrete problem situations and abstract algebraic representations (Baysal & Sevinç, 2022;

Kaur, 2019; Ng & Lee, 2009). Accordingly, it is an understandable result that teachers want to see the bar model method in the curriculum.

Following the completion of this study, the participating teachers gained awareness of the term "Bar Model Method" through exposure to the literature. Similarly, the method attracted the interest of many teachers and led them to express their intention to conduct further research on this topic after the interview. The study increased the teachers' awareness of the Singapore Bar Model and algebra teaching.

5.3 Limitations, Implications, and Recommendations

The primary objective of this study was to examine the problem-solving heuristics used by middle school mathematics teachers for algebra word problems, as well as their perspectives on the application of the Singapore Bar Model in algebra problem-solving and algebra instruction. Within this context, the study recognizes specific limitations based on its findings and offers recommendations for future research and implications for educational practices.

In this study, eighteen middle school mathematics teachers were first presented with three algebra problems from three different sets and then with six questions, with one correct and one incorrect solution, which are parallel to each of the three sets and solved with the Bar Model method, through one-on-one online interviews, each of which took approximately 45 minutes. A semi-structured interview format was used. A potential limitation lies in the interview questions, which could have been more comprehensive to capture a broader range of insights. The study was conducted in a limited geographical area and with a convenience sample, which restricted the diversity of the participant pool. A more comprehensive and heterogeneous selection of participants covering different geographical locations could offer a broader range of strategies. As a suggestion for future research, it is recommended to include a more diverse and more extensive group of participants for a more comprehensive understanding of teachers' approaches to the use of the Bar Model in algebra problems.

In the study, teachers' bar model understandings developed and were affected by students' answers. Since the use of bar models in students' answers was not correct, teachers' bar model understanding could not be entirely observed in this sense. For this reason, as a suggestion for future research, it is recommended that teachers should be given the correct bar model drawing and usage in various problem situations.

Based on the limitations and implications of this study, several recommendations are presented for future research and educational practice. The study acknowledges that there is a limitation in terms of time constraints and lack of observation opportunities as the research was conducted in a limited time frame. The researcher suggests that a more extended observation period, possibly through face-to-face interactions and in-class observations, could have provided richer data. For future research, it is recommended that a more extended study period be considered to allow for a comprehensive examination of teachers' experiences with the Bar Model Method.

The research findings are quite exciting, indicating that introducing the bar model to early ages can have a positive impact (Belecina & Ocampo, 2016). Therefore, the researcher suggest that studies should be focused on supporting our dedicated primary teachers in implementing this valuable learning tool for early grades. Integrating these techniques into the primary school curriculum and classroom teachers' teaching processes benefits individual learning and supports broader educational reforms aimed at improving students' mathematical competence.

Incorporating the Bar Model into the curriculum may provide teachers with a structured framework to introduce algebraic concepts in a more accessible and visual way. Moreover, as a suggestion for future research, the researcher suggests working with both teachers and students. In addition, longitudinal studies could provide deeper insights into the long-term impact of using the Bar Model in teaching algebra, especially in relation to students' conceptual understanding and problem-solving abilities.

As a result, it is anticipated that by addressing these recommendations, both teachers and students may benefit from more robust and effective algebra instruction, which will ultimately lead to improved mathematical outcomes.

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APPENDICES

A. Ethical Permission from Middle East Technical University

UYGULAMALI ETİK ARAŞTIRMA MERKEZİ
APPLIED ETHICS RESEARCH CENTER

ORTA DOĞU TEKNİK ÜNİVERSİTESİ
MIDDLE EAST TECHNICAL UNIVERSITY

DUMLUPINAR BULVARI 06800
ÇANKAYA ANKARA/TURKEY
T: +90 312 210 22 91
F: +90 312 210 79 59
ueam@metu.edu.tr
www.ueam.metu.edu.tr

16 AĞUSTOS 2023

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Doç.Dr. Şerife SEVİNÇ

Danışmanlığını yürüttüğünüz Feyza Arife ÖZALP'ın ***"İlköğretim Matematik Öğretmenlerinin Cebir Öğretiminde Singapur Şerit Modeli Kullanımı Hakkında Bilgi ve Görüşleri"*** başlıklı araştırmanız İnsan Araştırmaları Etik Kurulu tarafından uygun görülerek **0345-ODTUİAEK-2023** protokol numarası ile onaylanmıştır.

Bilgilerinize saygılarımla sunarım.

Prof. Dr. Ş. Halil TURAN
Başkan

Prof.Dr. İ. Semih AKÇOMAK
Üye

Doç. Dr. Ali Emre Turgut
Üye

Doç. Dr. Şerife SEVİNÇ
Üye

Doç.Dr. Murat Perit ÇAKIR
Üye

Dr. Öğretim Üyesi Süreyya ÖZCAN KABASAKAL
Üye

Dr. Öğretim Üyesi Müge GÜNDÜZ
Üye