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A blow up of solutions for a system of Klein-Gordon equations with variable exponent. Theoretical and Numerical Results

Abstract In this paper, we consider a system of Klein-Gordon equations with variable exponents. The first part of the manuscript is devoted to the proof of the blow up of solutions with negative initial energy under suitable conditions on variable exponents and initial data. The theoretical part is supported by numerical experiments based on $P1$-finite element method in space and the BDF and the Generalized-alpha methods in time illustrated in the second part. The numerical and analytical results of the blow up solutions agree with each other.

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1. Introduction. In this work, we consider the following initial boundary problem

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + |u_t|^{p(x)-1} u_t = f_1(u,v), & in\ \Omega,\ t>0, \\ v_{tt} - \Delta v + m_2^2 v + |v_t|^{r(x)-1} v_t = f_2(u,v), & in\ \Omega,\ t>0, \\ u(x,t) = v(x,t) = 0, & on\ \partial\Omega,\ t\geq 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & \\ u_t(x,0) = u_1(x),\ v_t(x,0) = v_1(x) & in\ \Omega, \end{cases} \tag{1}$$

where Ω is a bounded and regular domain of R^n, $(n = 1, 2, 3)$, with a smooth boundary $\partial\Omega$, $f_1(u,v)$ and $f_2(u,v)$ are the source terms defined by

$$\begin{cases} f_1(u,v) = a\,|u+v|^{2(q(\cdot)+1)}(u+v) + b\,|u|^{q(\cdot)}\,u\,|v|^{q(\cdot)+2}, \\ f_2(u,v) = a\,|u+v|^{2(q(\cdot)+1)}(u+v) + b\,|u|^{q(\cdot)+2}\,|v|^{q(\cdot)}\,v, \end{cases} \tag{2}$$

where m_1 and m_2 are positive constants, a and b are non-negative constants; and $p(\cdot)$, $r(\cdot)$ and $q(\cdot)$ are given continuous functions on $\overline{\Omega}$ satisfying some conditions to be specified later. We recall that the log-Hölder continuity condition for any function $m(\cdot)$ is

$$|m(x) - m(y)| \leq -\frac{A}{\log|x-y|}, \; for\ all\ x, y \in \Omega, \; with\ |x-y| < \delta, \tag{3}$$

where $0 < \delta < 1$ and $A > 0$. By the definition of $f_1(u,v)$ and $f_2(u,v)$, one can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(q(x)+2)F(u,v), \; \forall (u,v) \in R^2, \tag{4}$$

where

$$F(u,v) = \frac{1}{2(q(x)+2)}\left[a|u+v|^{2(q(x)+2)} + 2b|uv|^{q(x)+2}\right].$$

The exponents $p(\cdot), r(\cdot)$ and $q(\cdot)$ are measurable functions on Ω satisfiying

$$\begin{cases} 2 \leq p_1 \leq p(x) \leq p_2 \leq p^*, \\ 2 \leq r_1 \leq r(x) \leq r_2 \leq r^*, \\ 2 \leq q_1 \leq q(x) \leq q_2 \leq q^*, \end{cases}$$

where

$$\begin{cases} p_1 = ess\inf_{x\in\Omega} p(x), \; p_2 = ess\sup_{x\in\Omega} p(x), \\ r_1 = ess\inf_{x\in\Omega} r(x), \; r_2 = ess\sup_{x\in\Omega} r(x), \\ q_1 = ess\inf_{x\in\Omega} q(x), \; q_2 = ess\sup_{x\in\Omega} q(x), \end{cases}$$

and

$$\begin{cases} 2 \leq p^*, r^*, q^* < \infty \quad if\ n = 1, 2, \\ 2 \leq p^*, r^*, q^* \leq 6 \quad if\ n = 3. \end{cases}$$

The system with variable exponents we deal with is a very general system:

- The problems with variable exponents arises in many branches in sciences such as nonlinear elasticity theory, image processing and electrorheological fluids [5, 7, 21].
- The coupled nonlinear Klein-Gordon equation which models the motion of charged mesons in an electromagnetic field is investigated [22].

Consider a problem of a single Klein-Gordon equation of the form

$$u_{tt} - \Delta u + m^2 u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u. \tag{5}$$

Pişkin [19] proved the global nonexistence of solutions. When $m = 0$, (5) is reduced to the following wave equation

$$u_{tt} - \Delta u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u.$$

Messaoudi et al. [10] studied the existence and blow up of solutions.

In the absence of the $m_1^2 u$ and $m_2^2 v$ terms ($m_1 = m_2 = 0$) the problem (1) reduces to the following form

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{p(x)-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + |v_t|^{r(x)-1} v_t = f_2(u,v). \end{cases} \tag{6}$$

Messaoudi and Talahmeh [9] studied the blow up of solutions of the system (6).

In [17], Pişkin studied the following system of nonlinear Klein-Gordon equations with constant exponents

$$\begin{cases} u_{tt} - \Delta u + m_1^2 u + |u_t|^{p-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + m_2^2 v + |v_t|^{r-1} v_t = f_2(u,v). \end{cases}$$

He proved the blow up of the solutions within finite time with negative initial energy. Also, in [18] he studied the lower bounds for blow up time.

Recently, problems with variable exponents have been handled carefully in several papers. In particular, some results relating to the local existence, global existence, blow up and stability have been found in [1, 2, 4, 15, 20, 23].

On the other, numerical studies of the blow up of the solutions for nonlinear models of hyperbolic and elliptic equations have been paid much attention in recent years. The finite time blow-up solutions for the nonlinear Klein-Gordon equation is considered by Korpusov et al. in [8] where they proved that numerical analysis of the blow-up of the solution with an initial positive energy enables to ameliorate the analytical estimate. Besides, Messaoudi and his colleagues have recent works with numerical approaches: the existence and blow up for nonlinear damped equation [10], blow up and numerical analysis of biharmonic coupled system with variable exponents [11] and the coupled systems of nonlinear hyperbolic equations with variable exponents [12] are the most relevant articles to the current work.

In this paper, we prove the blow up of the solutions of the system (1). The rest of our work is organized as follows: In section 2, we present some lemmas, definition and theorem. In section 3, we state and prove our main result. Section 4 is devoted to the numerical study of the governing system with particular initial data in accordance with the first part. Numerical solutions and the energy of the system at the blow-up time are analyzed in the final part of the paper.

2. Preliminaries. In this part, we state some results about the variable exponents Lebesgue spaces ($L^{p(x)}(\Omega)$) and Sobolev spaces ($W^{1,p(x)}(\Omega)$), (see [3, 7, 16]).

Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a domain of R^n. We define the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u : \Omega \longrightarrow R;\ u \text{ measurable in } \Omega : \varrho_{p(\cdot)}(\lambda u) < \infty, \text{for some } \lambda > 0 \right\},$$

where

$$\varrho_{p(\cdot)}(u) = \int_\Omega \frac{1}{p(x)} |u(x)|^{p(x)} \, dx$$

is a modular. Equipped with the following Luxembourg-type norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

$L^{p(\cdot)}(\Omega)$ is a Banach space.

We also define the variable-exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ as

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) \ \textit{such that } \nabla u \textit{ exists and } |\nabla u| \in L^{p(\cdot)}(\Omega) \right\}.$$

LEMMA 2.1 *(Poincaré Inequality) Let Ω be a bounded domain of R^n and $q(\cdot)$ satisfies (3), and $1 \leq q_1 \leq q(x) \leq q_2 < \infty$, where*

$$q_1 = ess \inf_{x \in \Omega} q(x), \ q_2 = ess \sup_{x \in \Omega} q(x).$$

Then

$$\|u\|_{q(\cdot)} \leq C \|\nabla u\|_{q(\cdot)}, \ \textit{for all } u \in W_0^{1,q(\cdot)}(\Omega),$$

where the positive constant C depending on q_1, q_2 and Ω only.

LEMMA 2.2 *If $1 < q_1 \leq q(x) \leq q_2 < \infty$ holds, then*

$$\min \left\{ \|w\|_{q(\cdot)}^{q_1}, \|w\|_{q(\cdot)}^{q_2} \right\} \leq \varrho_{q(\cdot)}(w) \leq \max \left\{ \|w\|_{q(\cdot)}^{q_1}, \|w\|_{q(\cdot)}^{q_2} \right\},$$

for any $w \in L^{q(\cdot)}(\Omega)$.

LEMMA 2.3 *[9]. There exist two constants c_0 and c_1 such that*

$$\frac{c_0}{2(q(x)+2)} \left[|u|^{2(q(x)+2)} + |v|^{2(q(x)+2)} \right] \leq F(u,v) \leq \frac{c_1}{2(q(x)+2)} \left[|u|^{2(q(x)+2)} + |v|^{2(q(x)+2)} \right].$$

COROLLARY 2.4 *[9]. There exist two constants c_0 and c_1 such that*

$$c_0 [\varrho(u) + \varrho(v)] \leq \int_\Omega F(u,v) \, dx \leq c_1 [\varrho(u) + \varrho(v)]. \tag{7}$$

Definition 2.5 (Weak solution). A pair of functions (u,v) is said to be weak solution of (1) on $[0,T]$, $T>0$, if

$$\begin{aligned}(u,v) &\in L^\infty\left((0,T),H_0^1(\Omega)\right),\\ u_t &\in L^\infty\left((0,T),L^2(\Omega)\right)\cap L^{p(\cdot)+1}(\Omega\times(0,T)),\\ v_t &\in L^\infty\left((0,T),L^2(\Omega)\right)\cap L^{r(\cdot)+1}(\Omega\times(0,T))\end{aligned}$$

with

$$u(\cdot,0)=u_0,\ v(\cdot,0)=v_0,\ u_t(\cdot,0)=u_1,\ v_t(\cdot,0)=v_1$$

and (u,v) satisfies

$$\begin{aligned}\int_\Omega u_t\phi+\int_0^t\int_\Omega \nabla u\nabla\phi+\int_0^t\int_\Omega m_1^2u\phi+\int_0^t\int_\Omega |u_t|^{p(\cdot)-1}u_t\phi &= \int_0^t\int_\Omega f_1\phi,\\ \int_\Omega v_t\psi+\int_0^t\int_\Omega \nabla v\nabla\psi+\int_0^t\int_\Omega m_2^2v\psi+\int_0^t\int_\Omega |v_t|^{r(\cdot)-1}u_t\psi &= \int_0^t\int_\Omega f_2\psi,\end{aligned}$$

for all $\phi\in H_0^1(\Omega)\cap L^{p(\cdot)+1}(\Omega)$, $\psi\in H_0^1(\Omega)\cap L^{r(\cdot)+1}(\Omega)$, and all $t\in[0,T]$.

We state the following theorem which can be obtained by exploiting the Faedo-Galerkin method and using the similar arguments as in [2, 13, 14].

Theorem 2.6 *(Local existence). Assume that $p(\cdot)$, $r(\cdot)$, $q(\cdot)\in C\left(\overline{\Omega}\right)$,satisfy (3) and, for all $x\in\Omega$,*

$$\begin{cases} p(x)\geq 0, & \text{if } n=1,2,\\ p(x)=0, & \text{if } n=3,\end{cases} \tag{8}$$

$$\begin{cases} r(x)\geq 0, & \text{if } n=1,2\\ 2\leq r(x)\leq 6, & \text{if } n=3,\end{cases} \tag{9}$$

$$\begin{cases} q(x)\geq 2, & \text{if } n=1,2,\\ 2\leq q(x)\leq 6, & \text{if } n=3,\end{cases} \tag{10}$$

and $(u_0,u_1),(v_0,v_1)\in H_0^1(\Omega)\times L^2(\Omega)$. Then (1) has a unique weak local solution

$$(u,v)\in L^\infty\left((0,T),H_0^1(\Omega)\right),$$

$$u_t\in L^\infty\left((0,T),L^2(\Omega)\right)\cap L^{p(\cdot)+1}(\Omega\times(0,T)),$$

$$v_t\in L^\infty\left((0,T),L^2(\Omega)\right)\cap L^{r(\cdot)+1}(\Omega\times(0,T)),$$

for $T>0$.

3. Blow up

In this part, we state and prove our main result. For this purpose, we define energy functional of (1) as

$$E(t) = \frac{1}{2}\left(\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2\right) + \frac{1}{2}\left(m_1^2\|u\|^2 + m_2^2\|v\|^2\right) - \int_\Omega F(u,v)\,dx. \tag{11}$$

LEMMA 3.1 *$E(t)$ energy functional is nonincreasing function.*

PROOF Multiplying the first equation of (1) by u_t and the second equation by v_t, integrating over Ω, using integration by parts and summing up the product results, we obtain

$$E'(t) = -\int_\Omega |u_t|^{p(x)+1}\,dx - \int_\Omega |v_t|^{r(x)+1}\,dx \le 0. \tag{12}$$

LEMMA 3.2 *[9]. Suppose that (8) holds. Then, we have the following inequalities:*

$$[\varrho(u)+\varrho(v)]^{\frac{s}{2(q_1+2)}} \le C\left[\|\nabla u\|^2 + \|\nabla v\|^2 + \varrho(u) + \varrho(v)\right], \tag{13}$$

$$\|u\|_{2(q_1+2)}^s \le C\left[\|\nabla u\|^2 + \|\nabla v\|^2 + \|u_t\|_{2(q_1+2)}^{2(q_1+2)} + \|v_t\|_{2(q_1+2)}^{2(q_1+2)}\right], \tag{14}$$

$$\|v\|_{2(q_1+2)}^s \le C\left[\|\nabla u\|^2 + \|\nabla v\|^2 + \|u_t\|_{2(q_1+2)}^{2(q_1+2)} + \|v_t\|_{2(q_1+2)}^{2(p_1+2)}\right], \tag{15}$$

$$\int_\Omega |u|^{p(x)+1}\,dx \le c_1\left[(\varrho(u)+\varrho(v))^{\frac{p_1+1}{2(q_1+2)}} + (\varrho(u)+\varrho(v))^{\frac{p_2+1}{2(q_1+2)}}\right], \tag{16}$$

$$\int_\Omega |u|^{r(x)+1}\,dx \le c_2\left[(\varrho(u)+\varrho(v))^{\frac{r_1+1}{2(q_1+2)}} + (\varrho(u)+\varrho(v))^{\frac{r_2+1}{2(q_1+2)}}\right], \tag{17}$$

for any $u, v \in H_0^1(\Omega)$ and $2 \le s \le 2(q_1+2)$, with $C > 1$, $c_1 > 0$ and $c_2 > 0$ are constants.

THEOREM 3.3 *Suppose that (3),(8), (9) and (10) hold. Assume further that*

$$2(q_1+1) \ge \max\{p_2+1, r_2+1\} \tag{18}$$

and

$$E(0) < 0.$$

Then the solution of problem (1) blows up in finite time.

Proof Set $H(t) := -E(t)$, then $E(0) < 0$ and (12) gives $H(t) \geq H(0) > 0$. By the definition of $H(t)$ and (7), we have

$$\begin{aligned} H(t) &= -\frac{1}{2}\left[\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2\right] \\ &\quad -\frac{1}{2}\left[m_1^2\|u\|^2 + m_2^2\|v\|^2\right] + \int_\Omega F(u,v)\,dx \\ &\leq \int_\Omega F(u,v)\,dx \\ &\leq c_1\left[\varrho(u) + \varrho(v)\right]. \end{aligned} \tag{19}$$

We define

$$\Psi(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega (uu_t + vv_t)\,dx \tag{20}$$

for ε small to be chosen later and

$$0 < \alpha \leq \min\left\{\frac{q_1+1}{2(q_1+2)}, \frac{2(q_1+2)-(p_2+1)}{2m_2(q_1+2)}, \frac{2(p_1+2)-(r_2+1)}{2r_2(q_1+2)}\right\}. \tag{21}$$

Differentiating $\Psi(t)$ with respect to t, and using (1) and (4), we obtain

$$\begin{aligned} \Psi'(t) &= (1-\alpha)H^{-\alpha}(t)H'(t) + \varepsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad -\varepsilon\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) - \varepsilon\left(m_1^2\|u\|^2 + m_2^2\|v\|^2\right) \\ &\quad +2\varepsilon\int_\Omega (q(x)+2)F(u,v)\,dx \\ &\quad -\varepsilon\int_\Omega u|u_t|^{p(x)-1}u_t dx - \varepsilon\int_\Omega v|v_t|^{r(x)-1}v_t dx. \end{aligned} \tag{22}$$

By using the definition of $H(t)$, it follows that

$$\begin{aligned} -\varepsilon q_1(1-\xi)H(t) &= \frac{\varepsilon q_1(1-\xi)}{2}\left(\|u_t\|^2 + \|v_t\|^2\right) \\ &\quad +\frac{\varepsilon q_1(1-\xi)}{2}\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &\quad +\frac{\varepsilon q_1(1-\xi)}{2}\left(m_1^2\|u\|^2 + m_2^2\|v\|^2\right) \\ &\quad -\varepsilon q_1(1-\xi)\int_\Omega F(u,v)\,dx, \end{aligned}$$

where $2 < \eta < 2(q_1+2)$. Adding and subtracting $-\varepsilon q_1(1-\xi)H(t)$ from the right-hand side of (22), we obtain

$$\begin{aligned}\Psi'(t) \geq{}& (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon\left(\frac{q_1(1-\xi)}{2}+1\right)\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ &+\varepsilon\left(\frac{q_1(1-\xi)}{2}-1\right)\left(\|\nabla u\|^2+\|\nabla v\|^2\right)+\varepsilon q_1(1-\xi)H(t)\\ &+\varepsilon\left(\frac{q_1(1-\xi)}{2}-1\right)\left[m_1^2\|u\|^2+m_2^2\|v\|^2\right]\\ &+\varepsilon\left(2(q_1+2)-q_1(1-\xi)\right)\int_\Omega F(u,v)\,dx\\ &-\varepsilon\int_\Omega\left[u|u_t|^{p(x)-1}u_t+v|v_t|^{r(x)-1}v_t\right]dx.\end{aligned} \tag{23}$$

By using (7), we have

$$\begin{aligned}\Psi'(t) \geq{}& (1-\alpha) H^{-\alpha}(t) H'(t)+\varepsilon\beta\Big[H(t)+\|u_t\|_2^2+\|v_t\|_2^2\\ &+\|\nabla u\|^2+\|\nabla v\|^2+m_1^2\|u\|^2+m_2^2\|v\|^2+\varrho(u)+\varrho(v)\Big]\\ &-\varepsilon\int_\Omega\left[u|u_t|^{p(x)-1}u_t+v|v_t|^{r(x)-1}v_t\right]dx\end{aligned} \tag{24}$$

where

$$\beta=\min\left\{\begin{array}{c} q_1(1-\xi),\left(\frac{q_1(1-\xi)}{2}+1\right),\left(\frac{q_1(1-\xi)}{2}-1\right),\\ c_0\left(2(q_1+2)-q_1(1-\xi)\right)\end{array}\right\}>0.$$

To estimate the last term in (24), we use the Young inequality. It follows that

$$\begin{aligned}&\int_\Omega |u_t|^{p(x)}|u|\,dx\\ &\leq \frac{1}{p_1+1}\int_\Omega \delta_1^{p(x)+1}|u|^{p(x)+1}dx+\frac{p_2}{p_1+1}\int_\Omega \delta_1^{-\frac{p(x)+1}{p(x)}}|u_t|^{p(x)+1}dx.\end{aligned} \tag{25}$$

Similarly, we have

$$\begin{aligned}&\int_\Omega |v_t|^{r(x)}|v|\,dx\\ &\leq \frac{1}{r_1+1}\int_\Omega \delta_2^{r(x)+1}|v|^{r(x)+1}dx+\frac{r_2}{r_1+1}\int_\Omega \delta_2^{-\frac{r(x)+1}{r(x)}}|v_t|^{r(x)+1}dx,\end{aligned} \tag{26}$$

where $\delta_1,\delta_2>0$ are constants depending on the time t and specified later. Let us choose δ_1 and δ_2 so that

$$\delta_1^{-\frac{p(x)+1}{p(x)}}=k_1H^{-\alpha}(t)\ \ \textit{and}\ \ \delta_2^{-\frac{r(x)+1}{r(x)}}=k_2H^{-\alpha}(t),$$

for a large constant k_1 and k_2 to be specified later, and substituting in (25) and (26), respectively, we get

$$\int_\Omega |u_t|^{p(x)} |u| \, dx \leq \frac{k_1^{-p_1}}{p_1+1} \int_\Omega |u|^{p(x)+1} H^{\alpha p(x)}(t) \, dx + \frac{p_2 k_1}{p_1+1} H^{-\alpha}(t) H'(t), \tag{27}$$

and

$$\int_\Omega |v_t|^{r(x)} |v| \, dx \leq \frac{k_2^{-r_1}}{r_1+1} \int_\Omega |v|^{r(x)+1} H^{\alpha r(x)}(t) \, dx + \frac{r_2 k_1}{r_1+1} H^{-\alpha}(t) H'(t). \tag{28}$$

Combining (24), (27) and (28) gives

$$\begin{aligned} \Psi'(t) \geq & \left[(1-\alpha) - \varepsilon \frac{p_2 k_1}{p_1+1} - \varepsilon \frac{r_2 k_1}{p_1+1}\right] H^{-\alpha}(t) H'(t) \\ & + \varepsilon\beta \Big[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|^2 + \|\nabla v\|^2 \\ & + m_1^2 \|u\|^2 + m_2^2 \|v\|^2 + \varrho(u) + \varrho(v) \Big] \\ & - \frac{\varepsilon k_1^{-p_1}}{p_1+1} \int_\Omega |u|^{p(x)+1} H^{\alpha p(x)}(t) \, dx \\ & - \frac{\varepsilon k_2^{-r_1}}{r_1+1} \int_\Omega |v|^{r(x)+1} H^{\alpha r(x)}(t) \, dx. \end{aligned} \tag{29}$$

From (16), (17) and (19), we have

$$\int_\Omega |u|^{p(x)+1} H^{\alpha p(x)}(t) \, dx \leq C' \left[(\varrho(u) + \varrho(v))^{\frac{p_1+1}{2(q_1+2)} + \alpha p_2} + (\varrho(u) + \varrho(v))^{\frac{p_2+1}{2(q_2+2)} + \alpha p_2}\right]. \tag{30}$$

We then use Lemma 3.2 for

$$s = (p_2+1) + 2\alpha p_2 (q_1+2) \leq 2(q_1+2),$$

and

$$s = (p_1+1) + 2\alpha p_2 (q_1+2) \leq 2(q_1+2),$$

to deduce from (30) that

$$\int_\Omega |u|^{p(x)+1} H^{\alpha p(x)}(t) \, dx \leq C \left[\|\nabla u\|^2 + \|\nabla v\|^2 + \varrho(u) + \varrho(v)\right]. \tag{31}$$

Similarly

$$\int_{\Omega} |v|^{r(x)+1} H^{\alpha r(x)}(t)\, dx \leq C\left[\|\nabla u\|^2 + \|\nabla v\|^2 + \varrho(u) + \varrho(v)\right]. \tag{32}$$

Combining (31), (32) and (29), it follows that

$$\begin{aligned} \Psi'(t) \geq & \left[(1-\alpha) - \varepsilon\frac{p_2 k_1}{p_1+1} - \varepsilon\frac{r_2 k_2}{r_1+1}\right] H^{-\alpha}(t) H'(t) \\ & + \varepsilon\left(\beta - \frac{k_1^{-p_1}}{p_1+1}C - \frac{k_2^{-r_2}}{r_2+1}C\right) \\ & \times \left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|^2 \right. \\ & \left. + \|\nabla v\|^2 + m_1^2\|u\|^2 + m_2^2\|v\|^2 + \varrho(u) + \varrho(v)\right]. \end{aligned} \tag{33}$$

Let us choose k_1, k_2 large enough so that

$$\gamma = \beta - \frac{k_1^{-p_1}}{p_1+1}C - \frac{k_2^{-r_2}}{r_2+1}C > 0,$$

and picking ε small enough such that

$$(1-\alpha) - \varepsilon\frac{p_2 k_1}{p_1+1} - \varepsilon\frac{r_2 k_2}{r_1+1} \geq 0,$$

and

$$\Psi(0) = H^{1-\alpha}(0) + \varepsilon\int_{\Omega}(u_0 u_1 + v_0 v_1)\, dx + \frac{\varepsilon}{2}\left[\|\nabla u_0\|^2 + \|\nabla v_0\|^2\right] > 0.$$

Hence (33) takes the form

$$\begin{aligned} & \Psi'(t) \\ \geq & \gamma\varepsilon\left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + \varrho(u) + \varrho(v)\right] \\ \geq & \gamma\varepsilon\left[H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|\nabla u\|^2 + \|\nabla v\|^2 + m_1^2\|u\|^2 + m_2^2\|v\|^2\right]. \end{aligned} \tag{34}$$

Consequently, we get

$$\Psi(t) \geq \Psi(0) > 0, \ \textit{for all } t \geq 0.$$

On the other hand, thanks to the Hölder inequality and the embedding $L^{2(q_1+2)}(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\left|\int_{\Omega} u u_t dx\right| \leq \|u\|_2 \|u_t\|_2 \leq C\|u\|_{2(q_1+2)} \|u_t\|_2,$$

which implies

$$\left|\int_{\Omega} uu_t dx\right|^{\frac{1}{1-\alpha}} \leq C \left\|u\right\|_{2(q_1+2)}^{\frac{1}{1-\alpha}} \left\|u_t\right\|_2^{\frac{1}{1-\alpha}}.$$

Similarly

$$\left|\int_{\Omega} vv_t dx\right|^{\frac{1}{1-\alpha}} \leq C \left\|v\right\|_{2(q_1+2)}^{\frac{1}{1-\alpha}} \left\|v_t\right\|_2^{\frac{1}{1-\alpha}}.$$

Young's inequality gives

$$\begin{aligned}\left|\int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx\right|^{\frac{1}{1-\alpha}} \\ \leq C\left[\left\|u\right\|_{2(q_1+2)}^{\frac{\mu}{1-\alpha}} \left\|u_t\right\|_2^{\frac{\theta}{1-\alpha}} + \left\|v\right\|_{2(q_1+2)}^{\frac{\mu}{1-\alpha}} \left\|v_t\right\|_2^{\frac{\theta}{1-\alpha}}\right],\end{aligned} \tag{35}$$

with the condition

$$\frac{1}{\mu} + \frac{1}{\theta} = 1.$$

We take $\theta = 2(1-\alpha)$, to get

$$\frac{\mu}{1-\alpha} = \frac{2}{1-2\alpha} \leq 2(q_1+2),$$

by (30). Therefore (35) becomes

$$\left|\int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx\right|^{\frac{1}{1-\alpha}} \leq C\left[\left\|u\right\|_{2(q_1+2)}^{s} + \left\|v\right\|_{2(q_1+2)}^{\frac{\mu}{1-\alpha}} + \left\|u_t\right\|_2^2 + \left\|v_t\right\|_2^2\right],$$

where

$$s = \frac{2}{1-2\alpha} \leq 2(q_1+2).$$

By using (14) and (15), we have

$$\begin{aligned}\left|\int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx\right|^{\frac{1}{1-\alpha}} \\ \leq C\left[H(t) + \left\|u_t\right\|_2^2 + \left\|v_t\right\|_2^2 + \left\|\nabla u\right\|^2 + \left\|\nabla v\right\|^2 + \varrho(u) + \varrho(v)\right],\end{aligned} \tag{36}$$

for all $t \geq 0$. Thus,

$$\begin{aligned}\Psi^{\frac{1}{1-\alpha}}(t) &= \left[H^{1-\alpha}(t) + \varepsilon\int_{\Omega} uu_t + vv_t dx\right]^{\frac{1}{1-\alpha}} \\ &\leq 2^{\frac{1}{1-\alpha}}\left[H(t) + \left|\int_{\Omega} uu_t dx + \int_{\Omega} vv_t dx\right|^{\frac{1}{1-\alpha}}\right] \\ &\leq C\left[H(t) + \left\|u_t\right\|_2^2 + \left\|v_t\right\|_2^2 + \left\|\nabla u\right\|^2 + \left\|\nabla v\right\|^2 + \varrho(u) + \varrho(v)\right]\end{aligned} \tag{37}$$

where

$$(a+b)^p \leq 2^{p-1}\left(a^p + b^p\right)$$

is used. By combining (34) and (37), we arrive

$$\Psi'(t) \geq \xi \Psi^{\frac{1}{1-\sigma}}(t), \tag{38}$$

where ξ is a positive constant. A simple integration of (38) over $(0,t)$ yields

$$\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}},$$

which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}. \tag{39}$$

This completes the proof of the theorem. ■

4. Numerical Results.

This section is devoted to numerical illustrations for the solutions of the system in Theorem 3.3. The numerical implementation is based on two approaches for the time discretization, mainly the BDF (Backward Differentiation Formula) and the Generalized-α (GA) methods. Besides, a $P1$-finite element method is carried out for the space discretization. For further details on BDF and GA methods, we address the reader to the article [6] and the reference therein. A simple introduction for Time-Dependent solvers can also be found in the COMSOL Multiphysics Reference Manual. All of the numerical illustrations in that work are implemented by the COMSOL Multiphysics software.

4.1. Mesh Domain and Numerical Implementation Consider the system on a two-dimensional elliptical domain

$$\Omega = \{(x,y) : \frac{x^2}{4} + y^2 < 1\}$$

with the parameters $a=1, b=1$ with the initial-boundary conditions satisfying

$$\begin{aligned}
u_0(x,y) &= 3\Big(1 - \frac{x^2}{4} - y^2\Big),\\
v_0(x,y) &= 2\Big(1 - \frac{x^2}{4} - y^2\Big),\\
u_1(x,y) &= 0,\\
v_1(x,y) &= 0.
\end{aligned}$$

(a) Domain Ω (b) Triangulation for Ω

Figure 1: Mesh Domain

We take the exponent functions $p(x, y) = 2, q(x, y) = 2$ and $r(x, y) = 2$. It can be easily checked that they satisfy the conditions given in Theorem 2.6.

Next, divide the time interval $[0, T]$ into N equal subintervals $[t_n, t_{n+1}]$ such that

$$t_n = n\Delta t, \quad n = 1, 2, \cdots, N + 1,$$

where Δt and n be the time step and the iteration number, respectively.

Let $U^n(x, y)$ and $V^n(x, y)$ be approximate solutions at $t = t_n$. Using the backward and the center difference formulas for the time-derivatives $U_t^n(x, y)$ and $U_{tt}^n(x, y)$, respectively by

$$U_t^n(x, y) = \frac{U^n(x, y) - U^{n-1}(x, y)}{\Delta t}$$

and

$$U_{tt}^n(x, y) = \frac{U^{n+1}(x, y) - 2U^n(x, y) + U^{n-1}(x, y)}{\Delta t^2},$$

it follows that the discretized system takes the form

$$\begin{aligned} -\Delta W^{n+1} + \mu W^{n+1} &= F \quad \text{in} \quad \Omega, \\ W^{n+1} &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where

$$\begin{aligned} W^{n+1} &= [U^{n+1}, V^{n+1}]^T, \\ \Delta W^{n+1} &= [\Delta U^{n+1}, \Delta V^{n+1}]^T, \\ \mu &= \frac{1}{\Delta t^2}. \end{aligned}$$

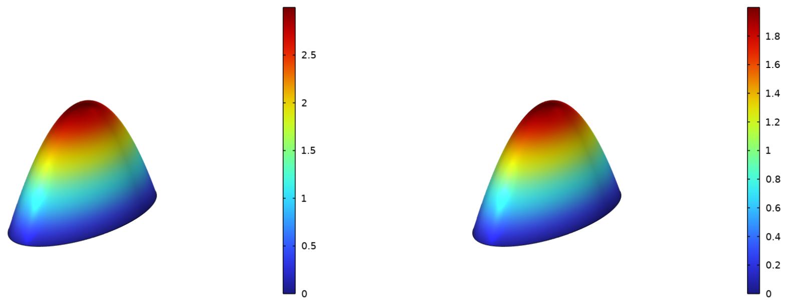
We solve the coupled system by using linear finite elements as follows:

1. The time step $\Delta t = 10^{-4}$ is small enough to obtain finite-time blow-up behaviour.

2. The triangulation of Ω consists of 12890 degrees of freedom with 3160 number of triangles. See Figure 1.

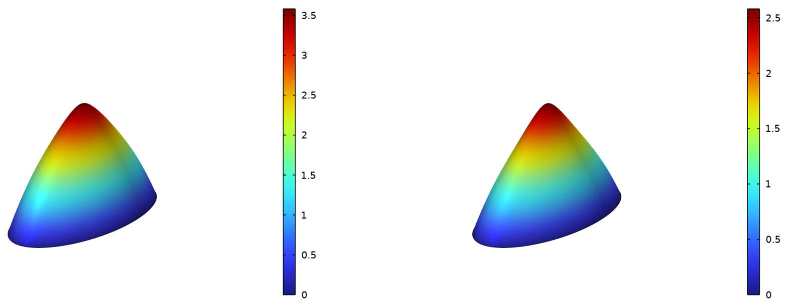
4.2. Solutions by the BDF Solver. The BDF method is a time-dependent solver that uses backward differentiation formulas. The numerical method in that part are based on BDF method in time and $P1$-finite element method in space. The implementation of numerical tests demonstrate that the solutions of the system blows up in a finite time.

Figure 2 shows the initial data u_0 and v_0. Figure 3, 4, 5, and 6 illustrate the numerical solutions at the iteration $n = 32, 46, 53$, and 55, respectively. It can be observed that at $n = 56$ (or at time $t = 0.0056$) the numerical solution blows-up.



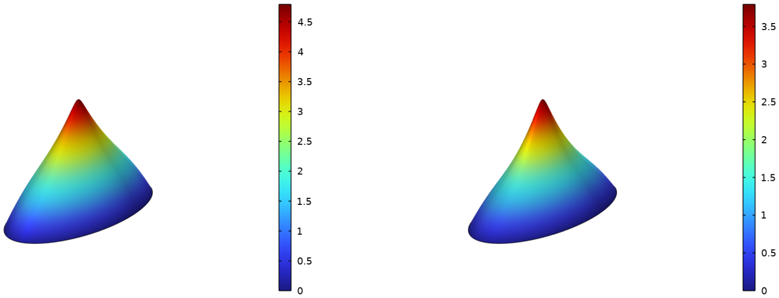
(a) $u_0(x, y) = 3\left(1 - \frac{x^2}{4} - y^2\right)$ (b) $v_0(x, y) = 2\left(1 - \frac{x^2}{4} - y^2\right)$

Figure 2: Initial data at $t = 0$



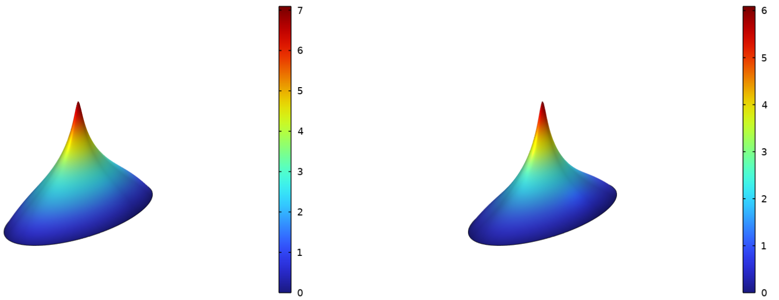
(a) U^{32} (b) V^{32}

Figure 3: Numerical solutions at $t = 0.0032$



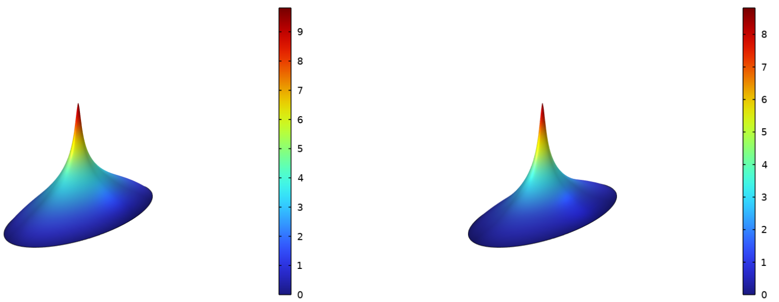
(a) U^{46} (b) V^{46}

Figure 4: Numerical solutions at $t = 0.0046$



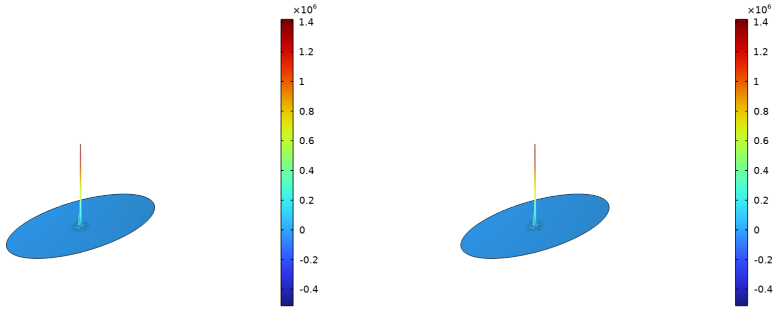
(a) U^{53} (b) V^{53}

Figure 5: Numerical solutions at $t = 0.0053$



(a) U^{55} (b) V^{55}

Figure 6: Numerical solutions at $t = 0.0055$



(a) U^{56} (b) V^{56}

Figure 7: Numerical solutions at $t = 0.0056$–Blow-up Time

4.3. Solutions by the Generalized-α (GA) Solver. The Generalized-α method is also a time dependent solver. The parameter α is the term that controls the degree of damping of high frequencies. For further details see [6].

We repeat the numerical tests by considering the GA method in time and $P1$-finite element method in space. Since the figures before and at the blow up time for the GA method are almost the same as the BDF method, we display only the numerical illustrations for the BDF solver. The blow-up time of the numerical solution occurs at $t = 0.0062$ in the GA method whereas in the BDF method it happens at $t = 0.0056$. In other words, the BDF method captures blow-up behaviour faster than the GA method.

4.4. Energy of the system

In that part we examine the relation between the numerical solutions and the energy of the system (1). More particularly, we observe the behaviour of the energy function $E(t)$ defined by

$$\begin{aligned} E(t) = &\frac{1}{2}\Big(\|u_t\|^2 + \|v_t\|^2 + \|\nabla u\|^2 + \|\nabla v\|^2\Big) \\ &+\frac{1}{2}\Big(m_1^2\|u\|^2 + m_2^2\|v\|^2\Big) - \int_\Omega F(u,v)\,dx \end{aligned} \tag{40}$$

before and at blow up time by numerical approximations. In the Table 1, the infinity norms of the numerical solutions and the energy of the system are listed at the iteration time $n = 1, 32, 46, 53, 55$ and finally at $n = 56$ where blow up takes place. The iteration times are chosen in accordance with the numerical results. Notice that the initial energy is negative; that is $E(t_1) = -34249.9$. We observe that before the blow up occurs, the energy decreases slowly at the first iterations; then after the time $t = 0.0032$ it decreases faster. This process proceeds until the blow up time $t = 0.0056$. This feature can also be viewed by the Figure 8 where the graph of the energy with respect to the time before the blow up and at the instant of the blow

t_n	$\|U\|_\infty$	$\|V\|_\infty$	$E(t_n)$
0.0001	2.00	3.00	-34249.9
0.0032	2.58	3.58	-39458.4
0.0046	3.85	4.85	$-1.5428E+5$
0.0053	6.23	7.23	$-1.7630E+6$
0.0055	18.02	19.03	$-1.134E+7$
0.0056	$1.4E+6$	$1.4E+6$	$-1.386E+47$

Table 1: Numerical values of $\|U\|_\infty$, $\|V\|_\infty$ and $E(t_n)$. Blow up time $t = 0.0056$

up is illustrated. In order to observe energy of the system during the blow up time, we restrict the time interval between $t = 0.0040$ and $t = 0.0056$. A sharp decay is observed after $t = 0.0050$ until $t = 0.0056$. According to the Table 1 and the Figure 8, just before the blow up time and during the blow up time, a sharp drop of energy appears. In other words, at the blow up time 0.0056 a burst of the energy takes place. Hence, the results in Table 1 with the numerical results shown in the Figures 3, 4, 5, 6, 7 are suitable with the blow up result of the Theorem 3.3.

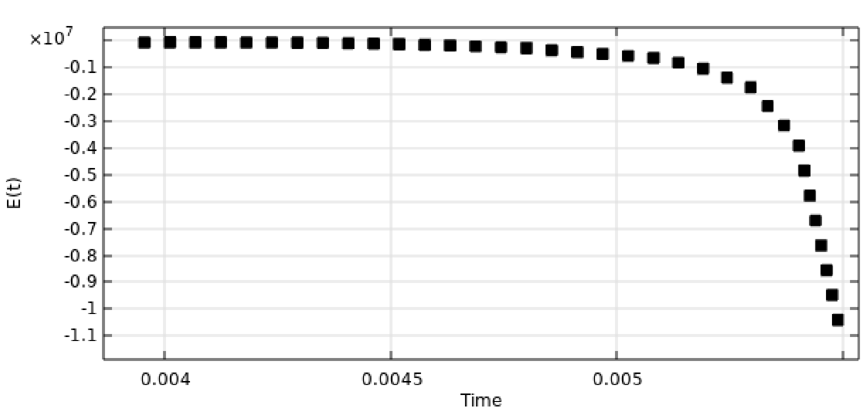


Figure 8: Energy between the iteration $n = 40$ and $n = 56$

4.5. An Example comparing the theoretical and the numerical blow-up time. As a result of Theorem 3.3, we proved that the solution blows up in a finite time T^* where an upper bound for this relation is displayed in the inequality

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

Here σ satisfies the inequality (21), ξ is a positive constant small enough and Ψ satisfies

$$\Psi^{\frac{\sigma}{1-\sigma}(0)} = H^\alpha(0) = (-E(0))^\alpha$$

where $E(0)$ is the initial energy.

On the other hand we have numerically illustrated that based on the BDF and GA methods, the solution blows up at a finite time $t = 0.0056$ and $t = 0.0062$, respectively, under certain conditions and initial data.

In can be confirm that the numerical blow up time are satisfied also theoretically.

Choose $\sigma = 0.3125$ and $\xi = 14.9$ in accordance with the relation (21). Substituting these values in (39), it follows that an upper bound for T^* is 0.0056; i.e. $T^* \leq 0.0056$.

Similarly taking $\sigma = 0.3125$ with $\xi = 13.43$ compatible with (21), the inequality (39) result in the numerical blow up time 0.0062 as an upper bound for T^*; i.e. $T^* \leq 0.0062$.

Hence we obtained the blow up times of the BDF and GA methods both theoretically and numerically, which implies that the numerical results are consistent with the theoretical results of Theorem 3.3.

5. Summary. We take into account a system of the Klein-Gordon equations with variable exponents given by (1). Under suitable conditions on variable exponents with a negative initial energy and initial data, we proved analytically that the solution blows up in finite time. The second part of that manuscript is devoted to the numerical illustration of the blow up solutions with some initial data in accordance with the theoretical part. The numerical methods implemented here are based on $P1-$ finite element method in space and on the BDF and the GA methods in time. As a result of these implementation, we observe that the numerical solutions blow up at a finite time. The numerical and analytical results of the blow up solutions agree with each other.

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References

[1] L. Alkhalifa, H. Dridi, and K. Zennir. Blow-up of certain solutions to nonlinear wave equations in the Kirchhoff-type equation with variable exponents and positive initial energy. *J. Funct. Spaces*, 2021:9, 2021. ISSN 2314-8896. doi: 10.1155/2021/5592918. Id/No 5592918; Zbl 1464.35040. Cited on p. 145.

[2] S. Antontsev and J. E. P. Ferreira. Existence and blow up of petrovsky equation solutions with strong damping and variable exponents. *Electronic J. Diff. Equations*, 06:1–18, 2021. ISSN 2314-8896. URL https://ejde.math.txstate.edu/Volumes/2021/06/antontsev.pdf. Cited on pp. 145 and 147.

[3] S. Antontsev and S. Shmarev. *Evolution PDEs with nonstandard growth conditions*, volume 4 of *Atlantis Studies in Differential Equations*. Atlantis Press, Paris, 2015. ISBN 978-94-6239-111-6; 978-94-6239-112-3. doi: 10.2991/978-94-6239-112-3. Existence, uniqueness, localization, blow-up. MR 3328376. Cited on p. 145.

[4] O. Bouhoufani and I. Hamchi. Coupled system of nonlinear hyperbolic equations with variable-exponents: global existence and stability. *Mediterr. J. Math.*, 17(5):Paper No. 166, 15, 2020. ISSN 1660-5446. doi: 10.1007/s00009-020-01589-1. MR 4149754. Cited on p. 145.

[5] Y. Chen, S. Levine, and M. Rao. Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math.*, 66(4):1383–1406, 2006. ISSN 0036-1399. doi: 10.1137/050624522. URL `semanticscholar.org/paper/98d86a353e36ea1b591e9298cd7ca4428a7d5ec0`. Zbl 1102.49010. Cited on p. 144.

[6] J. Chung and G. M. Hulbert. A time integration algorithm for structural dynamics with improved numerical dissipation: The generalized-α method. *J. Appl. Mech.*, 60(2):371–375, 1993. ISSN 0021-8936. doi: 10.1115/1.2900803. Zbl 0775.73337. Cited on pp. 154 and 158.

[7] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. *Lebesgue and Sobolev spaces with variable exponents*, volume 2017 of *Lect. Notes Math.* Berlin: Springer, 2011. ISBN 978-3-642-18362-1; 978-3-642-18363-8. doi: 10.1007/978-3-642-18363-8. Zbl 1222.46002. Cited on pp. 144 and 145.

[8] M. O. Korpusov, A. N. Levashov, and D. V. Lukyanenko. Analytical-numerical study of finite-time blow-up of the solution to the initial-boundary value problem for the nonlinear Klein-Gordon equation. *Comput. Math. Math. Phys.*, 60(9):1452–1460, 2020. ISSN 0965-5425. doi: 10.1134/S0965542520090109. MR 4169394. Cited on p. 145.

[9] S. A. Messaoudi and A. A. Talahmeh. Blow up of negative initial-energy solutions of a system of nonlinear wave equations with variable-exponent nonlinearities. *Discrete Contin. Dyn. Syst. Ser. S*, 15(5):1233–1245, 2022. ISSN 1937-1632. doi: 10.3934/dcdss.2021107. MR 4405343. Cited on pp. 145, 146, and 148.

[10] S. A. Messaoudi, A. A. Talahmeh, and J. H. Al-Smail. Nonlinear damped wave equation: existence and blow-up. *Comput. Math. Appl.*, 74(12): 3024–3041, 2017. ISSN 0898-1221. doi: 10.1016/j.camwa.2017.07.048. MR 3725935. Cited on pp. 144 and 145.

[11] S. A. Messaoudi, O. Bouhoufani, I. Hamchi, and M. Alahyane. Existence and blow up in a system of wave equations with nonstandard nonlinearities. *Electron. J. Differential Equations*, pages Paper No. 91, 33, 2021. doi: 10.58997/ejde.2021.91. MR 4340425. Cited on p. 145.

[12] S. A. Messaoudi, O. Bouhoufani, I. Hamchi, and M. Alahyane. Existence

and blow up in a system of wave equations with nonstandard nonlinearities. *Electron. J. Differential Equations*, pages Paper No. 91, 33, 2021. doi: 10.58997/ejde.2021.91. MR 4340425. Cited on p. 145.

[13] S. A. Messaoudi, O. Bouhoufani, I. Hamchi, and M. Alahyane. Existence and blow up in a system of wave equations with nonstandard nonlinearities. *Electron. J. Differential Equations*, pages Paper No. 91, 33, 2021. doi: 10.58997/ejde.2021.91. MR 4340425. Cited on p. 147.

[14] S. A. Messaoudi, A. A. Talahmeh, M. M. Al-Gharabli, and M. Alahyane. On the existence and stability of a nonlinear wave system with variable exponents. *Asymptot. Anal.*, 128(2):211–238, 2022. ISSN 0921-7134. doi: 10.3233/asy-211704. MR 4400916. Cited on p. 147.

[15] S.-H. Park and J.-R. Kang. Blow-up of solutions for a viscoelastic wave equation with variable exponents. *Math. Methods Appl. Sci.*, 42(6):2083–2097, 2019. ISSN 0170-4214. doi: 10.1002/mma.5501. MR 3937654. Cited on p. 145.

[16] E. Pişkin and B. Okutmuştur. *An introduction to Sobolev spaces*. Singapore: Bentham Science Publishers, 2021. ISBN 978-1-68108-914-0; 978-1-68108-913-3. doi: 10.2174/97816810891331210101. Zbl 7675018. Cited on p. 145.

[17] E. Pişkin. Uniform decay and blow-up of solutions for coupled nonlinear Klein-Gordon equations with nonlinear damping terms. *Math. Methods Appl. Sci.*, 37(18):3036–3047, 2014. ISSN 0170-4214. doi: 10.1002/mma.3042. Zbl 1311.35146. Cited on p. 145.

[18] E. Pişkin. Lower bounds for blow-up time of coupled nonlinear Klein-Gordon equations. *Gulf J. Math.*, 5(2):56–61, 2017. MR 3661735. Cited on p. 145.

[19] E. Pişkin. Global nonexistence of solutions for a nonlinear klein-gordon equation with variable exponents. *Appl. Math. E-Notes*, 19:315–323, 2019. ISSN 1607-2510. URL `http://emis.icm.edu.pl/journals/AMEN/2019/AMEN-180713.pdf`. Cited on p. 144.

[20] E. Pişkin and N. Yılmaz. Blow up of solutions for a system of strongly damped Petrovsky equations with variable exponents. *Acta Universitatis Apulensis*, 71(06):87–99, 2022. ISSN 1582-5329. doi: 10.17114/j.aua.2022.71.06. Cited on p. 145.

[21] M. Růžička. *Electrorheological fluids: modeling and mathematical theory*, volume 1748 of *Lect. Notes Math.* Berlin: Springer, 2000. ISBN 3-540-41385-5. doi: 10.1007/BFb0104029. Zbl 0962.76001. Cited on p. 144.

[22] I. E. Segal. Nonlinear partial differential equations in quantum field theory. In *Proc. Sympos. Appl. Math., Vol. XVII*, pages 210–226. Amer. Math. Soc., Providence, RI, 1965. MR 202406. Cited on p. 144.

[23] M. Shahrouzi. On behaviour of solutions for a nonlinear viscoelastic equation with variable-exponent nonlinearities. *Comput. Math. Appl.*, 75(11): 3946–3956, 2018. ISSN 0898-1221. doi: 10.1016/j.camwa.2018.03.005. MR 3797035. Cited on p. 145.

Wybuch rozwiązań układu równań Kleina-Gordona ze zmiennym wykładnikiem. Wyniki teoretyczne i numeryczne.

Nebi Yılmaz, Sedanur Mazı Gözen, Erhan Pişkin, Baver Okutmuştur

Streszczenie Praca poświęcona jest układowi równań Kleina-Gordona ze zmiennymi wykładnikami. W pierwszej części pokazano, że rozwiązania o ujemnej energii początkowej uciekają do nieskończoności przy odpowiednich warunkach na wykładniki oraz dane początkowe. Część teoretyczną uzupełniają obliczenia numeryczne oparte na metodzie elementu skończonego dla zmiennych przestrzennych oraz metodzie różniczkowania wstecz (Backward Differentiation Formula, BDF). Wyniki numeryczne i analityczne dotyczące wybuchowego charakteru rozwiązań wzajemnie potwierdzają się.

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