A DESIGN-BASED INVESTIGATION ON ADVANCED MATHEMATICAL KNOWLEDGE FOR MATHEMATICS TEACHERS IN THE CONTEXT OF LIMIT

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ABSTRACT

A DESIGN-BASED INVESTIGATION ON ADVANCED MATHEMATICAL KNOWLEDGE FOR MATHEMATICS TEACHERS IN THE CONTEXT OF LIMIT

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The study had two principal aims. The first was to explore the essential advanced mathematical knowledge of limits for mathematics teachers from the perspective of experts. The second was to document the characteristics of the difficulties experienced by mathematics teachers in a teaching sequence designed to promote their advanced mathematical knowledge of the concept of limit. A design-based research approach was adopted to fulfil the objectives of the study. In this context, a need analysis phase of the design-based research comprised an interview study, conducted with five mathematicians and mathematics educators. Subsequently, the main study was held with six secondary school mathematics teachers in Alanya as an exploration phase. A variety of data collection instruments were employed, including interviews, pre- and post-tests, questionnaires, observations, written in-class works, and video data. These data were analyzed using qualitative methods.

The results of the interview study demonstrated how the concept of equality is interpreted differently in the context of limits, as opposed to other mathematical contexts and revealed essential advanced mathematical knowledge for limit. Furthermore, the findings explicitly revealed covariational relationships in the definition of limit. The data from the main study indicated that despite the presentation of a variety of arguments and approaches throughout the teaching sequence, the teachers demonstrated a certain degree of difficulty in comprehensively understanding the concept of limit, constructing covariational relationships among variables within limit, and employing equality and symbols appropriately. In light of these results, the study offered several key implications, limitations and suggestions for further research.

Keywords: Advanced Mathematical Knowledge, Limit, Design-Based Research, Teaching Sequence, Need Analysis Phase

ÖZ

MATEMATİK ÖĞRETMENLERİNİN LİMİT BAĞLAMINDA İLERİ MATEMATİK BİLGİLERİ ÜZERİNE TASARIM TEMELLİ BİR ARAŞTIRMA

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Bu çalışmanın iki temel amacı vardır. Birincisi, matematik öğretmenleri için limit konusunda gerekli olan ileri matematik bilgisini uzmanların bakış açısından ortaya koymaktır. İkinci amacı ise matematik öğretmenlerinin limit kavramına ilişkin ileri matematik bilgilerini geliştirmek için tasarlanmış bir öğretim dizisinde yaşadıkları zorlukları ortaya koymaktır. Çalışmanın amaçlarını gerçekleştirmek için tasarım tabanlı bir araştırma yaklaşımı kullanılmıştır. Bu bağlamda, ihtiyaç analizi aşaması olarak beş matematikçi ve matematik eğitimcisi ile bir mülakat çalışması yürütülmüştür. Ana çalışma daha sonra Alanya'da altı orta öğretim matematik öğretmeni ile yürütülmüştür. Çeşitli veri toplama araçları kullanılmış ve veriler nitel yöntemler kullanılarak analiz edilmiştir.

Uzmanlarla yapılan mülakat çalışmasının sonuçları, eşitlik kavramının limit bağlamındaki kullanımın ve anlamının matematikte diğer kullanıldığı bağlamlara göre nasıl farklılaştığını ve matematik öğretmenleri için gerekli limite dair ileri matematik bilgisini ortaya çıkarmıştır. Ayrıca, bulgular limit tanımına dair bilinmesi gereken kovaryasyonel ilişkileri açık bir şekilde ortaya koymuştur. Ana çalışmadan elde edilen veriler, öğretim dizisi boyunca öğretmenlere çeşitli argümanlar ve yaklaşımlar sunulmasına rağmen, öğretmenlerin limit kavramını kapsamlı bir şekilde anlamada, limit kavramında yer alan değişkenler arasında kovaryasyonel ilişkiler kurmada ve eşitlik ve sembolleri uygun şekilde kullanmada belirli bir zorluk yaşadıkalrını göstermiştir. Bu sonuçlar ışığında, çalışma aynı araştırma alanında çalışan araştırmacılar için önemli çıkarımlar, kısıtlılıklar ve öneriler sunmuştur.

Anahtar Kelimeler: İleri Matematik Bilgisi, Limit, Tasarım Tabanlı Araştırma, Öğretim Dizisi, İhtiyaç Analizi Çalışması

To my mother S and motherhood

To my beloved S, my lovely son İ, and my baby in the womb

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ABBREVIATIONS

AMK: Advanced mathematical knowledge

AMT: Advanced mathematical thinking

PCK: Pedagogical content knowledge

CK: Content knowledge

MTs: Mathematics teachers

LIST OF SYMBOLS

SYMBOLS

ε: epsilon

δ: delta

∀: for every

∋: there exists

CHAPTER 1

INTRODUCTION

Teachers should have well-grounded knowledge of content of all kinds such as subject matter knowledge and pedagogical content knowledge (Shulman, 1986). However, the question of how much and what kind of content knowledge is sufficient for essential mathematics teaching remains a topic of debate (Ball et al., 2005; Ball et al., 2008; Dreher et al., 2018; Hill et al., 2004). In light of these ongoing debates, a number of theoretical frameworks have emerged which vary in their focus on content, pedagogy, or a combination of these two areas. The relationship between the amount of teachers' content knowledge and its applicability in practice is clarified to a certain extent with the introduction of these refined frameworks. However, there is still a discrepancy between mathematics teachers' subject matter knowledge and their use of it in the teaching process (e.g., Ball et al., 2005).

In teacher training programs, mathematics teachers are required to enroll in a number of content courses (Stacey, 2008). While the exact number of these mathematics courses varies considerably, both primary and secondary school mathematics teachers are expected to complete at least a certain number of pure mathematics courses during their undergraduate education. Through these courses, teacher education programs aim to ensure that mathematics teacher candidates are equipped with the mathematical knowledge necessary for teaching secondary or primary school mathematics.

Some researchers highlight the discrepancy between the mathematical knowledge acquired by mathematics teachers at the university level and the mathematical content they are expected to teach in elementary or secondary school classrooms (e.g., Dreher et al., 2018). In other words, the argument is that there is a disconnection or gap between the mathematics taught to mathematics teachers in university mathematics courses and the mathematics knowledge that they are then responsible for teaching in elementary or secondary schools. Indeed, this discrepancy was initially addressed by Felix Klein (1932) under the term "double discontinuity." The study provides a concise overview of the two contradictory transitions experienced by mathematics teachers from their secondary school education to university education and from university to teaching mathematics in schools. On the one hand, the mathematical knowledge acquired by mathematics teachers at the secondary school level and at the university level exhibit distinct characteristics. On the other hand, there is a lack of connection between the mathematical content that a mathematics teacher learns during their undergraduate education and the mathematical knowledge that they will subsequently teach. Klein proposed a new perspective, which viewed mathematics from a more advanced standpoint, with the aim of integrating university-level mathematical knowledge into mathematics teachers' professional knowledge. Researchers who have adopted Klein's perspective hypothesize that this approach could facilitate a more sophisticated level of mathematical thinking and knowledge among mathematics teachers.

International platforms (e.g., the Conference Board of the Mathematical Sciences [CBMS], 2021) emphasize the significance of mathematics teachers' coherent and deep-rooted knowledge of the mathematics they will teach. In this regard, one of their recommendations is to encourage mathematics teachers to perceive and establish links between the content of university-level mathematics and that taught at the school level. CBMS (2001, 2012) posits that collaborative efforts between mathematicians and mathematics educators to equip mathematics teachers with the intended knowledge could be advantageous. The prevailing approach to the education of mathematics teachers in universities is to foster a continuous development of mathematical knowledge (Wasserman, Weber, et al., 2017; Wu, 2011). However, CBMS reports emphasize that mathematics teachers' education must also involve a reconsideration of the mathematics content taught in schools, with an appreciation of all mathematical knowledge gained in university mathematics courses.

Although the terminology differs, numerous researchers have identified analogous issues (e.g., Harel et al., 2006; Wasserman, Fukawa-Connelly, et al., 2017; Zazkis & Leikin, 2010). For example, Zazkis and Leikin (2010) and Wasserman (2016) employed the term of advanced mathematical knowledge (AMK), which is defined as the knowledge acquired in university mathematics courses or the knowledge of mathematics beyond the elementary or high school curriculum. Albeit the aforementioned discussion going on in the field, advanced mathematical knowledge becomes important for mathematics teachers in terms of its potential to affect their "understanding about and perception of the content they teach in ways that influence their teaching" (Wasserman, 2016, p. 30). Advanced mathematics courses may assist mathematics teachers in expanding their understanding of advanced topics. However, failure to present these connections with school mathematics in a clear manner may result in the loss of potential positive contributions to instructional practices (Cuoco, 2001). In order to shed light on the contribution and role of advanced mathematics courses on teaching activities of mathematics teachers, researchers conducted interviews with both pre-service and in-service mathematics teachers and mathematicians (Hoffman & Even, 2018; Yan et al., 2022; Zazkis & Leikin, 2010). The studies conducted with both pre-service and in-service mathematics teachers demonstrated that they did not attach significance to the advanced mathematical content covered during their undergraduate education in relation to their teaching activities (e.g., Wasserman et al., 2018; Zazkis & Leikin, 2010). It is encouraging to see that studies performed with mathematicians have provided valuable insight into the potential benefits of engaging with advanced mathematical studies for mathematics teachers (e.g., Leikin et al, 2018; Yan et al., 2022). To illustrate, Suominen (2015) examined the interrelationships between abstract algebra and school-level algebra through collaboration with mathematicians and mathematics educators. Similarly, Yan et al. (2022) surveyed mathematicians to ascertain the value of calculus courses for mathematics teachers.

But it has been reported that even mathematics teachers with a strong foundation in advanced mathematical knowledge are unable to demonstrate a clear understanding of how their knowledge relates to the school mathematics curriculum (Crisan, 2019). It is therefore recommended that student teachers should be provided with opportunities and guidance on how advanced mathematical knowledge can be relevant to mathematics teaching, regardless of their academic success (Crisan, 2019). In order to enhance the potential of AMK in teaching school-level mathematics, the literature has called upon mathematicians and mathematics educators to develop mathematics courses that are tailored to the specific requirements of mathematics teachers in relation to AMK (e.g., Cuoco, 2001; Hill & Senk, 2004; Leikin et al., 2018). It should be noted that the design of a teacher-specific course does not imply a reduction in the course's complexity or an overall simplification of its content. Instead, it entails “emphasizing the values of formalism, proof, and rigor”, similar to the approach taken in mathematics departments with regard to courses such as real analysis (Yan et al., 2020, p. 372).

In this study, I sought to contribute to and expand the existing literature on how AMK could be made more relevant for mathematics teachers. To this end, I focused my attention on an issue within the concept of limit. The concept of limit is a central topic in the field of calculus and other branches of mathematics, such as complex analysis (Tall, 2002). The historical quest for a solution to the conflicts arising from the infinite and infinitesimally small quantities led to the rigorous definition of the concept of limit via symbols, thus in turn, to the formulation of the fundamental tenets of mathematical analysis (Tall, 2009). With its formal definition, it represents a transition to "more formal, rigorous mathematics" (Swinyard & Larsen, 2012, p. 466) and serves as a reference point for future calculus topics, including convergence, derivatives, and integrals (Cornu, 2002). In particular, the concept of limit provides pre-service teachers with the opportunity to gain not only an understanding of single-variable calculus but also an appreciation of multiple-variable calculus. This, in turn, facilitates their comprehension of mathematical modelling (CBMS, 2012) and of the ideas inherent to STEM-related disciplines

(Larsen et al., 2017). Despite its significance, prior research has identified challenges learners face in understanding the concept of limit. These include a lack of awareness about the role of quantifiers in interpreting the formal definition of limit (Juter, 2007), the function of the equal sign in identifying a limit (Páez & Hitt, 2004), and the role of limit in grasping diverse mathematical concepts, such as those related to a circle (Kajander & Lovric, 2017).

The following section delineates the objectives of the study and the pertinent research questions.

1.1 Purpose of the Study and Research Questions

The purposes of the present study were (1) to explore the essential advanced mathematical knowledge of limit for mathematics teachers from the experts' points of view, and (2) to document the characteristics of the difficulties experienced by mathematics teachers in a teaching sequence designed to promote their advanced mathematical knowledge of the concept of limit. In other words, the first aim of the study deal with determining the needs in the field about necessary advanced mathematical knowledge for teachers in the context of limit. The second aim of the study was designing a teaching sequence and analyzing the aspects of teachers' progress in terms of advanced mathematical knowledge through the teaching sequence in the context of limit.

To manage these purposes, the research questions of this study are determined as follows:

1. What aspects of advanced mathematical knowledge of limit do experts (mathematicians and mathematics educators) identify as essential for mathematics teachers?
2. What are the characteristics of the difficulties experienced by mathematics teachers in a teaching sequence designed with principles promoting advanced mathematical knowledge of limit?

1.2 Significance of the Study

In-service and pre-service mathematics teachers demonstrate unappreciation towards the necessity of enrolling in advanced mathematics courses for their teaching (e.g., Cuoco, 2001) and confusion about how to use advanced mathematical knowledge in teaching-related situations (e.g., Zazkis & Leikin, 2010). While some mathematics teachers acknowledge the value of advanced mathematical knowledge, only a few are able to clearly articulate the specific reasons of why this knowledge is essential and how it is utilized in school-level teaching activities through specific examples (e.g., Zazkis & Leikin, 2010).

Researchers often interviewed pre-service and in-service mathematics teachers about the contribution and role of advanced mathematics courses on teaching activities of mathematics teachers (e.g., Zazkis & Leikin, 2010). Recently, there have been some studies with mathematicians inquiring about the value of calculus courses for mathematics teachers (e.g., Hoffman & Even, 2018; Yan et al., 2022). Hoffman and Even (2018) study showed that mathematicians had a consensus on the primary purpose of the given mathematical education as to elaborate on teachers' general knowledge about the nature and epistemology of mathematics. Similarly, Yan et al. (2022) reported that mathematicians pointed out that drawing connections across mathematical domains, gaining mathematical experience for developing problem-solving abilities, and increasing epistemological awareness of the subject are the specific aspects of advanced mathematical knowledge that mathematics teachers need to have. In addition, CBMS (2012) further notified the need for the (pre-service) mathematics teachers' knowledge on mathematics from an advanced stand of point stating that "knowing mathematics as a mathematician is important for pre-service high school teachers (in fact, for any mathematics major)." (p. 65).

Advanced mathematics courses are taught to pre-service mathematics teachers in higher education institutions, often by mathematicians. Pointing to the importance of the collaboration between mathematicians and mathematics educators and their role in the education of mathematics teachers, CBMS (2012) stated "mathematicians have an essential role to play in fulfilling this potential in teacher education, curriculum, and assessment" (p.16). Although the solutions that can be adopted to overcome the problems about the gap between advanced mathematical knowledge and its informing of teachers' teaching practices are still under discussion, there is a rising call for making collaboration with the field experts such as mathematicians and mathematics educators and some intervention to reduce this problem of disregard towards advanced mathematical knowledge. It is advocated that specially designed courses could promote teachers' acquisition of elegant experiences with mathematics in universities (e.g., Wasserman et al., 2019). Additionally, such courses could give teachers opportunities to revisit their advanced mathematical knowledge from the perspective of school mathematics (Larsen et al., 2018). Even if the necessity to apply instructional design interventions is emphasized (Crisan, 2019), what kind of intervention should be done is still an unanswered question in the literature.

In light of the aforementioned considerations, this study will present a way for bridging the gap between the mathematics that mathematics teachers learn at the university level and the mathematics they teach in schools. It is recommended that methods supported by theory be adopted in parallel with suggestions from the literature, rather than relying on the assumption that mathematics teachers will naturally connect their advanced mathematical knowledge with related school mathematics topics. At this juncture, this study is significant in that it documents the essential advanced mathematical knowledge for limit from the perspectives of field experts including mathematicians and mathematics educators. Apart from that, it offers a teaching sequence designed with theoretically and empirically driven principles to promote mathematics teachers' advanced mathematical knowledge of limit and limit-related ideas. Furthermore, the activities devised within the current study is conjectured to assist mathematics teachers in leveraging AMK to enhance their teaching knowledge within the context of limit. Lastly, this study is of great importance in contributing to the field of advanced mathematical knowledge, which is currently lacking in substantial literature. Most importantly, as this study employs a design-based approach, it is anticipated to propose a theoretical framework for advanced mathematics courses specifically designed for teachers.

1.3 Definition of Terms

Advanced mathematical knowledge is the knowledge gained in university mathematics courses (Zazkis & Leikin, 2010). In this study, advanced mathematical knowledge at the most general level is defined as the knowledge acquired through the university-level mathematics courses. Similarly, aligned with the perspective on advanced mathematical knowledge, the given the established connections with advanced mathematical thinking are adopted as essential features of advanced mathematical knowledge.

Design principles are theoretically generated products of the design-based research process that are ready for use by practitioners and educational researchers (Reeves, 2000; van den Akker, 2006). This study employs two-phased design research where the design principles are driven by related literature and the interview study in the first phase. They further provide insight into the development of the teaching sequence in the second phase.

A teaching sequence is designed to facilitate the construction of disciplinary concepts (Ruthven et al., 2009, p. 330). The teaching sequence in this study comprises two consequential and related sessions with a shared end purpose. The steps in the teaching sequence are informed by the related literature and the findings of the first phase of the study.

Concept of limit is an essential mathematical concept particularly within the fields of calculus and analysis, as well as other mathematical disciplines such as complex analysis (Bezuidenhout, 2001; Cornu, 2002; Tall, 2002). This study examines the concept of limit from the perspective of advanced mathematical knowledge.

Experts in mathematics and mathematics education comprise mathematicians and mathematics educators from different state universities in Turkey. They all have an undergraduate degree and a doctoral qualification and are currently engaged in research in the fields of mathematics and/or mathematics education with a minimum of twenty years of experience.

A mathematics teacher is an individual who has completed their undergraduate education in mathematics and subsequently undertaken further pedagogical training or completed a secondary school mathematics education program, before assuming a teaching role in a secondary school in Turkey. The participating secondary school mathematics teachers in this study are all engaged in the active instruction of mathematics to secondary school students (Grades 9-12).

CHAPTER 2

LITERATURE REVIEW

This study was conducted in accordance with a design-based approach. Firstly, the need analysis phase of the design-based research explored the essential advanced mathematical knowledge of limit for mathematics teachers from the experts' points of view. Secondly, the exploration phase of the design-based research documented the evolution of mathematics teachers' advanced mathematical knowledge within the teaching sequence, with a particular emphasis on the concept of limit and associated topics. The design principles that guided the development of the teaching sequence were derived from an analysis of the relevant literature and the results of the needs analysis study conducted as part of this dissertation study. The literature review initially presented a synthesis of studies related to teacher knowledge. After this, the study's central argument, which is advanced mathematical knowledge and its relevancy with teaching was inspected. Within the scope of the study, the subject of limit was put into the center, and so the subsequent part of the literature review was dedicated to it.

2.1 A Brief History of the Classifications of Teacher Knowledge

Throughout history, various implementations have been made to identify the qualifications of teachers. When teachers' examinations belonging to different periods are compared analytically, radical changes in the content of these examinations could be observed. For example, Shulman (1986) analyzed changing focuses on teachers' certification exams from 1875 to 1980 and caught an exciting result. In the article, Shulman reported that while a considerable emphasis in the exams was on the content in the previous years, in the 1980s it was discarded hugely from the examinations. Calling this problematic issue a missing paradigm, he pointed out the necessity of making a general theoretical framework to determine what a teacher should know to be accepted as sufficiently qualified to teach. In this aspect, Shulman (1986) divided teacher knowledge into three as: content knowledge (CK), pedagogical content knowledge (PCK), and curricular knowledge.

After this pioneering classification of Shulman (1986), researchers continued to question various requirements for teachers' knowledge and to characterize what teachers should know (e.g., Ball et al., 2008). In recent years, the stimulating questions related to the content knowledge aspect of teacher knowledge have mostly evolved into what degree and what kind of mathematics content knowledge a teacher should have (e.g., Ball et al., 2008; Rowland et al., 2005). Even though a positive relationship between teachers' content knowledge and the students' mathematical capability was found in some studies, the horizons of this relationship still have not been theoretically confirmed (Zazkis & Leikin, 2010). For example, while the number of mathematics courses that mathematics teachers take in universities does not promise accomplished teaching (Monk, 1994), teaching activity could not occur without it (Wiley, 2014).

In this context, the concept of mathematical knowledge for teaching was accepted as revolutionary (Ball et al., 2008; Hill et al., 2004). The researchers were able to ascertain not only the extent of mathematics teachers' content knowledge, but also how they transfer this knowledge into their practices for effective teaching, thanks to this term. Their classification has embodied PCK and CK of Shulman (1986) in a more detailed form with three subdomains for each. While they partitioned PCK into the knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum, CK consisted of common content knowledge, specialized content knowledge, and horizon content knowledge (Ball et al., 2008).

Among the subdomains of content or subject matter knowledge, specialized content knowledge and horizon content knowledge deserve attention for this study. The specialized content knowledge aspect of CK underlines mathematics knowledge, which is unique to teachers and mathematics teaching. In this respect, it includes knowledge that distinguishes mathematics teachers from others. It involves, for example, the ability to comprehend students' different ways of finding solutions and whether these solutions could be carried out in more general settings. Improving this knowledge type is considered crucial since it affects teachers' making quick decisions in teaching moments. Horizon content knowledge is about knowing the interrelations among the spans of mathematics. It includes both teachers' awareness of how mathematics in different grade levels is related and how they are all connected in the broader context. The power of this knowledge type for mathematics teachers lies in catching all the mathematical ideas hidden in students' statements and assessing their value for students' current or further mathematics teaching (Jakobsen et al., 2012).

Though theoretical background has not been fully developed and empirical studies are rare, advanced mathematical knowledge situates between specialized and horizon content knowledge (Zazkis & Leikin, 2010). In this study, the researcher adopts Zazkis and Leikin's (2010) definition of AMK, "knowledge of the subject matter acquired in mathematics courses taken as part of a degree from a university or college" (p. 264), and similarly, the theoretical approach in this study stems from the discussion of advanced mathematical thinking. For this reason, the next part is devoted to the definition and components of advanced mathematical thinking.

2.2 Advanced Mathematical Thinking

Since advanced mathematical knowledge is considered relevant to advanced mathematical thinking (Zazkis & Leikin, 2010), understanding this notion and situating AMK based on advanced mathematical thinking (AMT) seems crucial for this study. Although there are different conceptualizations regarding advanced mathematical thinking in the literature, a synthesis of various conceptualizations proposed by researchers can help to determine the components of AMT. For this reason, this section will discuss how researchers define advanced mathematical thinking, how they differentiate AMT from elementary mathematical thinking, and activities considered in connection with AMT.

While defining AMT, Tall (1988) initially differentiated the meaning of the qualifier advanced in the term of AMT as “advanced forms of mathematical thinking” or as “thinking related to advanced mathematics”. Researchers focused on the thinking concept in the usages of advanced as the qualifier of mathematical thinking (e.g., Edwards et al., 2005; Selden & Selden, 2005; Tall, 1995, 2002). At this point, to support the meaning of advanced mathematical thinking, it can be helpful initially to mention what is considered to be the opposite side of this term.

Tall (1995) located elementary mathematical thinking on the opposite side and stated that, in the transition from elementary to advanced, mainly the actions in the processes are changing. While elementary mathematical thinking involves explaining and characterizing, advanced mathematical thinking relates to persuading and justifying (Tall, 2002). Additionally, contrary to elementary thinking, which mainly depends on physical or tangible feelings, advanced mathematical thinking leans primarily on internalized structures. Tall (1992) noted, "In taking students through the transition to advanced mathematical thinking, we should realize that the formalizing and systematizing of mathematics is the final stage of mathematical thinking, not the total activity” (pp. 508–509). In other words, according to Tall’s (1992) conception, advanced mathematical thinking cannot merely relate to advanced topics or concepts.

Edwards et al. (2005) put forth the view that there is no definitive stage at which one can identify the transition from elementary mathematical thinking to the beginning of AMT. In this aspect, “AMT resides on a continuum of mathematical thought that seems to transcend, but does not ignore, the procedural experiences or intuitions of elementary mathematical thinking” (Edwards et al., 2005, p. 18). They described the significant characteristic of AMT as “the combination of the need for deductive and rigorous reasoning about concepts and the fact that these concepts are not accessible to the individual through the five senses" (p. 18). For example, although the limit as a mathematical concept in real analysis necessitates heavily thinking deductively and rigorously, it could not be claimed that all students working on limit problems are engaged in AMT. On the other hand, although the concepts or topics in high school geometry are presented in a way that is accessible to students' five senses, it does not mean that high school geometry does not require any rigorous thinking.

Advanced mathematical thinking is mainly identified with crucial mathematical abilities such as communicating between words and symbols, using representations, justifying, and discovering (Herlina & Batusangkar, 2015), and using intuition and visualization (Krussel, 1994). Advanced mathematical thinking involves all the thought processes, from a problem situation with a context to the context-free mathematical form, from offering mathematical explanations with justifications to revising them to acquire an elegant proof (Tall, 2002). Stockton and Wasserman (2017, p. 582) stated, "That is, for any idea in advanced mathematics, there are many different ways that it can be known to and cognitively structured within an individual." Therefore, students must engage in activities related to AMT from the beginning of their educational lives (Selden & Selden, 2005).

Regarding the differentiation of AMT from elementary mathematical thinking, Edwards et al. (2005) indicated some points namely "the size of the problem, the problems involving infinity, rigorous and deductive reasoning, and the use of models" (p. 15). The size of the problem is about the way how a question is posed to students. Whether the problem could be solved by only contemplating on some models or if the students would need to think rigorously. Following that, components of AMT will be generated using the definitions of AMT and activities focused on as necessary to develop AMT. For example, the problems involving infinity are related to the act of reasoning about an infinite process through the lens of finite iterations. Edwards et al. (2005) provide the example of the comparison of the cardinality of the natural numbers and the even natural numbers as a means of illustrating the problems involving infinity. The underlying mathematical process is to establish a one-to-one correspondence between the two sets and thereby conclude that they have the same cardinality. This line of reasoning requires the construction of an iterative finite process of matching among elements in the sets, with justifications provided for the endpoint, which is inaccessible in the physical world. Harel and Sowder (2005) discussed obstacles hindering someone from developing a way of thinking and characterized it as AMT if “its development necessarily involves at least one of the three necessary conditions for epistemological obstacles” (p. 47).

The mathematical focus of the study is the concept of limit, which encompasses a number of theories and proofs related to rigorous reasoning. Furthermore, it requires consideration of repeating decimals, which comprise an infinite number of repeating digits. So, the ability to justify through theorems and proofs, reasoning from finite processes to understand infinite processes, supporting learners through activities involving various representations and intuition, transferring verbalization to formalization seemed crucial components for the concept in the center of this study. In conclusion, I consider advanced mathematical thinking as advanced forms of mathematical thinking, as described by Edwards et al. (2005), Selden and Selden (2005), and Tall (2002). Correspondingly advanced mathematical knowledge is regarded as “an advanced perspective on elementary knowledge” (Zazkis & Mamolo, 2011, p. 13).

As a summary, in this study, advanced mathematical knowledge is considered to be mathematical knowledge that is beyond the scope of the curriculum, but is related to mathematical terms, concepts, or disciplinary principles at the university level that have the potential to influence the interpretation of mathematical ideas in the school curriculum. The following section will present a summary of the definitions of advanced mathematical knowledge and its relation to teaching.

2.3 Advanced Mathematical Knowledge and Its Relation to Teaching

In the existing literature, there is an initiative to promote teachers' use of advanced mathematical knowledge for teaching and instructional purposes. Advanced mathematical knowledge gains importance in terms of its possible contributions for the teaching purposes. However, there is a lack of clarity regarding the extent to which mathematics teachers or candidates utilize their AMK for teaching purposes. This section presents a synthesis of the ways in which AMK is examined by researchers and shows how the evaluation of the efficiency of this phenomenon is problematic because of the scarcity of theoretical and empirical studies in the field.

The researchers described AMK as the knowledge beyond the school curriculum or gained through university mathematics courses (Zazkis & Leikin, 2010). In their work, Usiskin et al. (2003) posit that advanced mathematical knowledge encompasses an understanding of the multiple definitions of a given concept, an awareness of the broader applications of well-known theorems, and an ability to discern the different applications of the school-level mathematics content. As examples for advanced mathematical knowledge, the Conference Board of the Mathematical Sciences' (CBMS, 2001) report provides examples which demand a more comprehensive and profound understanding than that which is typically taught in the context of school-level.

Stockton and Wasserman (2017) divided the connections of AMK to teaching into five categories: peripheral, evolutionary, axiomatic, logical, and inferential. They refer to how mathematical knowledge gets complex, how mathematical ideas mature gradually, how to develop mathematical structures based on axioms, how mathematical reasoning adopts the rules of logic, and how reasoning derives from facts. It is advocated that exploring advanced mathematics from these aspects makes it more practical for teachers and their instruction. Besides, the researchers linked these forms of knowledge to some specific teaching practices for mathematics teachers to inform them regarding the applicability of undergraduate mathematics courses in teaching.

Zazkis and Leikin (2010) conducted a study with secondary mathematics teachers to analyze their perceptions about using AMK in teaching practices through semi-structured interviews. The value of AMK was perceived in terms of fostering confidence, enhancing competence in responding to students' queries, and establishing interconnections between mathematical concepts. However, despite this recognition, the participants were unable to provide concrete illustrations of these perceived benefits. Wiley (2014) investigated a similar research problem in his dissertation, which is about middle school mathematics teachers' perceptions of the contribution of AMK to their teaching practices. He questioned how teachers' AMK experiences affect their noticing of student errors, instructional strategies, and the depth of the content to conclude the effect of AMK knowledge of content and students and knowledge of content and teaching. The findings indicated that the participants perceived AMK to be beyond the scope of requisite knowledge for their teaching.

In advanced mathematics literature, apart from the ones that specifically used the term advanced mathematical thinking and advanced mathematical knowledge, some studies aim to develop the connection between university and school mathematics and increase the effectiveness of content courses. Even though researchers preferred different terms, these studies investigate the issue of how the mathematics knowledge acquired in university mathematics courses informs teachers' further instructional activities. Considering that researchers focus on mathematics teachers' conceptions or perceptions of the impact of AMK in their practice (Baldinger, 2014; Zazkis & Leikin, 2010), ability to use that knowledge in tasks (Dilberoğlu, 2015) or designing learning environments and specific courses to promote their uses of AMK in teaching (Wasserman, Fukawa-Connelly, et al., 2017). To comprehend the meaning of advanced mathematical knowledge in depth and uncover its probable aspects, I review the studies about teachers' conceptions and competence in using AMK in teaching in this part.

In her master thesis, Dilberoğlu (2015) investigated pre-service middle school mathematics teachers' views about connecting mathematics knowledge covered in university courses with their further teaching activities. In the study, the researcher generated tasks imitating a hypothetical classroom situation on the topic of Basic Mathematical Structures. For example, one of the dialogues was about Prime

Numbers, asking participating pre-service teachers what their action would be if a student questioned the rationale behind excluding 1 from prime numbers. Questions such as how they give an explanation, whether they remember it from their content courses, and whether they can get an answer by using the textbook of that course followed the task of uncovering existing but hidden connections in student teachers' minds. Study results indicated that pre-service mathematics teachers are confused regarding the role and place of mathematical knowledge taught in pure mathematics courses in their future teaching, similar to Zazkis and Leikin's (2010) study.

Although it is scarce, some studies in the literature aimed to examine how mathematics teachers use their advanced mathematical knowledge in problems at the elementary or high school mathematics level. For example, Buchholtz et al. (2013) investigated pre-service mathematics teachers' competence in analyzing elementary-level mathematics problems from an advanced standpoint. They tried to generate a valid instrument to measure mathematics teachers' ability to notice the academic knowledge behind elementary mathematics problems. For example, one of the questions was a task about 0.999… =1 and asked how participating teachers would respond to a student being unconvinced about this equality. Researchers intended to determine how participants relate this task to advanced mathematics content through test items and their reasoning. The findings of this task showed that only less than half of pre-service mathematics teachers justify their answers using geometric series and limit appropriately, which is the desired association to build with advanced mathematics for this question. Moreover, only a few students based their rationale on the completeness theorem of real numbers, accepted as a more advanced argument for this context. Researchers interpreted these results as indicating pre-service mathematics teachers' lack of recalling their advanced mathematical knowledge when necessary for teaching-related issues. This study is essential for us in explaining how a school-level mathematics problem could be connected to advanced mathematical knowledge and investigated empirically.

Crisan (2019) investigated how dealing with AMK relevant to a school-level mathematics problem promotes teachers in conceptual and pedagogical aspects. In this regard, the researcher designed a workshop to encourage mathematics teachers' awareness of the links existing between school mathematics and related advanced mathematics. In the workshop aiming to increase teachers' familiarity with various representations of functions, he included school-level mathematics tasks requiring teamwork. A function in the same rule (such as $f(x) = x^2$), but with different continuous or discrete domains in real numbers, was presented to teachers, and they were asked to draw the graphs of the functions as a group. In the study, the groups did not know the tasks of other groups. On each group's sharing their answers and seeing different graphs, a classroom discussion was stimulated by the research regarding the reasons for differences. Different graphs for changing domains were the expected mathematical action in the activity. But here, the important thing was teachers need to remember the definition of a function from advanced mathematics to solve this ambiguity. It is reported at the end of the study that participants regarded the activities as valuable, which urges them to revisit their AMK and become aware of its place in the school mathematics curriculum. On the other hand, it also revealed that teachers' AMK must be activated through purposeful interventions.

Apart from that, some researchers with similar purposes were inclined to the documentation of mathematical connections between advanced mathematics courses and school mathematics. Suominen (2015) tried to reveal all the connections between abstract algebra and secondary school mathematics under three essential steps. First, she generated a theoretical classification in the related literature to outline different types of connections: alternative representation, comparison through common features, equivalent representation, generalization, hierarchical relationship, logical implication, procedural, and real-world application. After that, all the mathematical connections appearing in the primary abstract algebra textbooks were listed. After that, the previously configured connection list was refined by interviewing mathematics faculty members. This ended the process of reporting the connections between abstract algebra and secondary school mathematics content. Documentation of connections can guide teacher educators and mathematicians in focusing on courses designed for mathematics teachers to make AMK more relevant for them.

Studies showed that although mathematics teachers had a positive attitude about the influence of AMK on teaching practices, they could not present specific examples for that (e.g., Zazkis & Leikin, 2010). When a scarce number of studies were analyzed in terms of the branches of mathematics, it was seen that the examples provided were mainly from calculus and abstract algebra (Zazkis & Leikin, 2010). Such studies are valuable in presenting essential aspects of advanced mathematical knowledge for secondary mathematics teachers.

2.4 Developing Advanced Mathematical Knowledge and Attempts to Connect Advanced Mathematics and School Mathematics

In this section, the meaning of connecting advanced and school-level mathematics will be discussed through course designs, theoretical frameworks, task designs, etc. in the related studies. Besides, specific examples will be presented to give an idea for course and task designs to build relevance between advanced and secondary or primary school mathematics for teachers. In these designs, researchers centered their studies mostly on different subfields of mathematics, such as abstract algebra (e.g., Alvarez et al., 2022) and real analysis or calculus (e.g., Wasserman, 2016), rather than a specific concept.

Engagement with advanced mathematical knowledge or thinking is advocated to empower mathematics teachers in pedagogical ways, but a few studies could present empirical justifications for that. To promote pre-service mathematics teachers' ability to transfer their existing advanced mathematical knowledge into future classrooms, some courses are offered in undergraduate education. For example, Crisan (2019) claimed that some contexts and environments must be created to awaken mathematics teachers' AMK and their awareness about the connections between the mathematical knowledge. Regarding the same problem, Murray and Star (2013) stated that many teaching methods courses have begun integrating content knowledge, especially related to the school curriculum. In parallel with this, they figured out that the content courses given by mathematicians could also be revisited apart from designing purposeful workshops or courses for teachers and analyzed these types of connection courses existing in some universities. They categorized such courses into two: “secondary mathematics from an advanced standpoint” and “tertiary mathematics with connections” (p. 1298). The examples from the study were included here for the sake of clarity. In the first one, student teachers look at secondary school topics from an advanced perspective and try to capture the implicit relations between secondary school topics and tertiary mathematics. For example, essential theorems and how they appear in the secondary school curriculum and unfamiliar approaches to familiar generalizations in mathematics are discussed in these courses. For example, discussions based on the solution spaces of $a + x = b$ or $ax = b$, secondary school level knowledge could be leveraged to the advanced level. In the second type of connection course, tertiary mathematics is normally covered in its rigorous nature. It is followed by classroom discussions and lesson materials on how they would be connected to secondary school mathematics. For example, while proving why real or integer numbers have no zero divisors, they could have a chance to understand the reasoning behind “if $(x - 1)(x - 3) = 0$, then $x = 1$ and $x = 3$ are the solutions” from secondary school mathematics content.

Although it rarely appears in the literature, an instructional model exists to provide the use of advanced mathematical knowledge in the instructional setting explicitly (Wasserman, et al., 2017; Wasserman et al., 2019). Using design-based research methods, they integrated “advanced mathematics, secondary mathematics, and teaching secondary mathematics” (Wasserman et al., 2019, p. 385) in their model. Since they aimed to enhance the use of the knowledge gained in advanced mathematics courses in natural instructional settings, they supposed that this kind of model must rise from and end up with practice-based teaching situations. For this reason, they generated their model based on the following three steps: *building up from (teaching) practice, the advanced mathematics course,* and *stepping down to (teaching) practice* (see Figure 2.1).

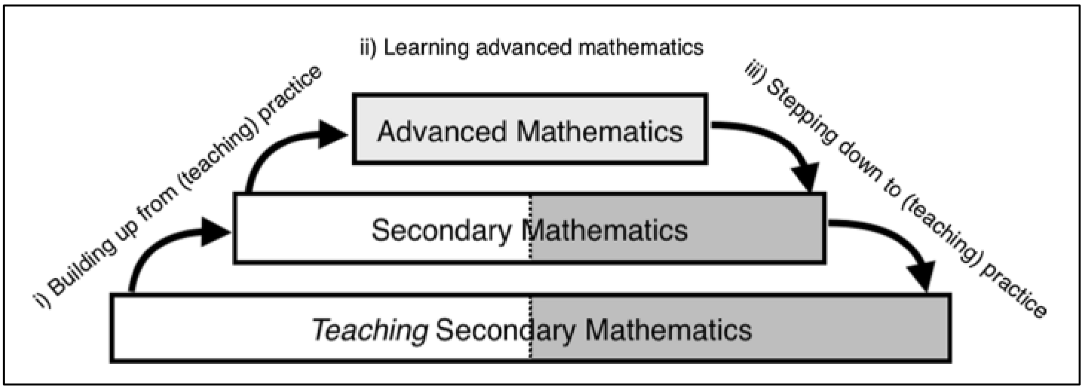


Figure 2.1. The instructional model proposed by Wasserman, Fukawa-Connelly, et al. (2017)

The building-up part seen in the figure refers to the idea that the process must start with an actual classroom teaching scenario in which real analysis content in focus is situated in a secondary school context. This initial step makes mathematics teachers think about these scenarios and determine their actions in such situations. The second step serves mathematics teachers to revisit advanced mathematics topics related to the teaching situations presented in the previous step. The theorems or concepts are discussed without simplifying their advanced characteristics, including their explicit connections with secondary school mathematics content and their probable uses in mathematics teaching (Weber et al., 2020). Lastly, stepping down to (teaching) practice necessitates analyzing how mathematics teachers adapt their new understanding gained in the intermediate step to other classroom situations in the related content (Wasserman, Fukawa-Connelly, et al., 2017; Wasserman, Weber, et al., 2017). In the model, through teaching-centered situations, researchers want to uncover how teachers detect the advanced mathematical ideas behind the scenarios and explore how teachers’ knowledge of advanced mathematics affects their future in-class actions. As evidenced in the studies conducted by Murray and Star (2013) and Wasserman et al. (2019), prospective mathematics educators are provided with opportunities to engage with more advanced mathematical knowledge through supplementary tasks or instructional designs.

Wasserman (2016) proposed an instructional model containing content-related connections and instructional connections between advanced and secondary school mathematics. These are *content connections, disciplinary practice connections, classroom teaching connections,* and *modeled instruction connections* (Wasserman, 2018, p. 6). While the first two connections are directly related to the content of advanced mathematics courses, the third and fourth refer to the probable applications of this knowledge in the classroom setting. In the figure, these connections can be seen within a scale that is ranked based on their dominant type of contributions to the process of mathematics instruction.

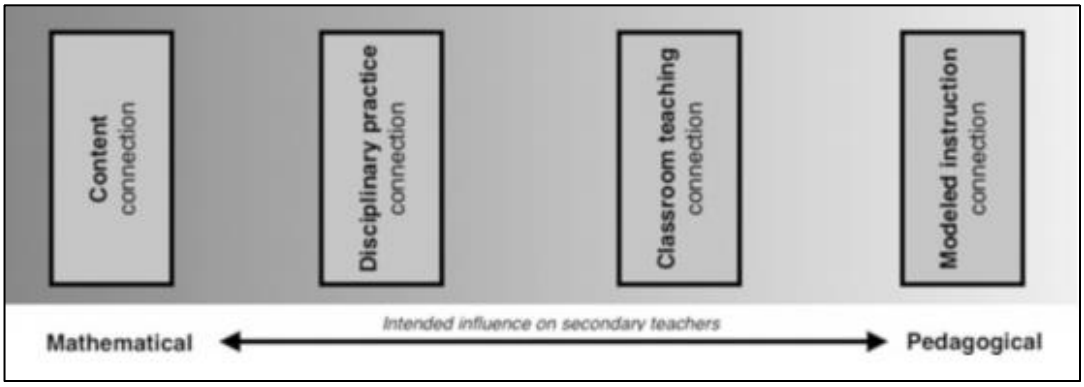


Figure 2.2. The indicated connections between advanced and secondary mathematics in the study of Wasserman (2018)

It can be insightful to look at their meanings and the improvement they could make to mathematics teaching in detail. For example, through content connections, mathematics teachers could gain awareness about how the mathematical terms, theorems, and structures appear in various forms in the school mathematics spectrum (e.g., the group structure in advanced algebra course versus invertible functions in school mathematics). The other connection is the disciplinary practice connection, which is mainly related to the mathematical practices within advanced mathematics. Namely, the thinking processes and abilities that teachers gain while involved in activities regarding the essence and epistemology of mathematics could support their students in a similar way. The researcher associates disciplinary practice connection with mathematical practice standards (CCSSM, 2010), and Cuoco et al.'s (1996) mathematical habits of mind. As a third, classroom teaching connection is about the probable applications of advanced level mathematical ideas into the mathematical content at the school level. It is advocated that teachers' building such connections in classroom could shape the flow of the lesson by affecting their in-class mathematical decisions and actions. Lastly, the modeled instruction connection arises from the idea that how someone learns mathematics guides how s/he teaches it. It is stated that being aware of these connections may inform mathematics teaching by letting teachers project their advanced mathematical knowledge into the classroom and so, by building more explicit relevance between them.

In a recently published book by Wasserman et al. (2022), six design principles were mentioned, and examples of the content based on these principles were explained. These design principles guided their study in connecting "content from real analysis to situations in teaching secondary mathematics" (Wasserman et al., 2022, p.1). The principles are generated through a combination of mathematical practices, including the disciplinary actions employed by mathematicians such as "conjecturing, generalizing, defining, formalizing, and proving," and pedagogical practices indicating the teaching actions applied by mathematics teachers such as "explaining, eliciting, facilitating, designing, and interpreting" (p. 2). The first principle in their study is "acknowledge and revisit assumptions and mathematical constraints or limitations" (p.3). In the field of mathematics, some conditions are attached to mathematical statements in definitions, or theorems. For example, the slight differences in the statements of "Let $f: A \rightarrow R$" or "Let $f: [a, b] \rightarrow R$ be continuous" state too many things. They indicate two different functions with two different domains (Wasserman et al., 2022, p. 3). Besides, it must be inferred that the first function is not continuous since it does not have such an additional statement. Comprehension and teaching of such important nuances are important practices for mathematics teachers. When a student proposes a solution strategy for a problem, controlling under which conditions this student-generated problem-solving strategy works and discussing it with students are essential teaching principles. They are examples of the teaching principle of acknowledging and revisiting assumptions and limitations. Their second principle is “consider and use special cases to test and illustrate mathematical ideas” (p.3). In mathematics, another important practice is checking the special cases to reach a conclusive statement. Besides, working on the counter and non-examples is crucial to build the borders of properties belonging to a mathematical statement. Wasserman et al.’s (2022) this second principle focuses on these two practices generally. In pedagogical practices, this principle must be implemented in various ways. It is essential to have a comprehensive understanding of the concept. The next principle is “expose logic as underpinning mathematical interpretation” (p. 4). To avoid possible ambiguity, while using abstract and formal language of mathematics, everyday usages of words or symbols must also be considered. For this reason, it is good practice to visit the background meaning of the symbols in their contexts and to differentiate different meanings of them in different contexts. The fourth principle is “use simpler objects to study more complex objects” (p. 5). This principle is like the meaning of a component of AMT, namely ‘inaccessible to five senses’ which indicates to apply simpler methods to understand the idea behind a hard problem. Some mathematical problems have an infinite nature or require thinking on more dimensional spaces. To have a better understanding about these types of problems, it is appropriate to model them with finite numbers or in fewer dimensional spaces. The fifth principle is “avoid giving rules without accompanying mathematical explanations” (p. 5). This principle pedagogically appears as giving rationalization with the theorems, facts, rules, or procedures so that learners could catch the reasons behind them and have more profound understanding. The last principle is “seek out and give multiple explanations”. This principle indeed arises from the nature of mathematics. In the fields of mathematics, it is possible and natural to achieve the same conclusion even if following different roads. Welcoming different approaches in problem solving provides to meet all learners’ needs of different learning styles. Besides, Wasserman et al. (2022) stated that learners could expand their creativity and mathematical intuition skills thanks to different problem-solving approaches. It should be noted that not all of their principles are entirely novel, particularly in light of the literature review presented thus far. The use of simpler objects or the avoidance of providing direct rules without their accompanying rationale are also encompassed within the AMT component.

2.5 Mathematicians’ Views About Advanced Mathematics for Secondary School Teachers

How advanced mathematical knowledge could contribute to and relate to secondary school mathematics teaching is a question tried to answer through various methods, some of which were explained in previous sections. Since mathematicians teach mathematics courses in many education faculties, it is an important issue to investigate how mathematicians perceive the role of advanced mathematics courses for mathematics teachers’ instructional activities. In this part, mathematicians’ views about the contribution of advanced mathematics courses for mathematics teachers and what kind of mathematical knowledge must be known will be discussed through the related literature.

Yan et al. (2022) reported that mathematicians appreciate advanced mathematics studies for secondary mathematics teachers in terms of (1) *drawing connections across mathematical domains*, (2) *gaining mathematical experience for developing problem-solving abilities*, and (3) *increasing epistemological awareness of the subject* (p. 570). It was stated that mathematicians firstly give value to studying advanced mathematics issues in terms of giving an opportunity to secondary mathematic teachers to view mathematics from a broader perspective. As an example, a mathematician drew attention to the generation of the formula $\cos(2\theta) = cos^2\theta - sin^2\theta$ from secondary school mathematics through rotation matrices in linear algebra and complex numbers from advanced mathematics. Through these various approaches, it is claimed to demonstrate the connected and encompassing aspect of mathematics within different branches. Secondly, rich mathematical experience gained in advanced mathematical courses could support mathematics teachers’ ability to manage mathematical problems. This aspect is exemplified through a mathematical puzzle in which, rather than adoption of trial-and-error or case analysis method, the use of the properties of number sets leads to a more sophisticated and easier solution. The third aspect emphasized by mathematicians regarding the value of mathematics for secondary mathematic teachers is acquisition of epistemological awareness. For this aspect, it is stated that mathematics teachers' involvement in advanced mathematical issues make them attentive about restraints and affordances of mathematical knowledge.

Hoffman and Even (2018) reported that mathematicians all agree that the primary purpose of the given mathematical education is to elaborate on teachers' general knowledge about the nature and epistemology of mathematics. In this scope, three dimensions were discovered at the end of the study: *the essence of mathematics, doing mathematics,* and *the worth of mathematics* (p. 102) as in the figure below.

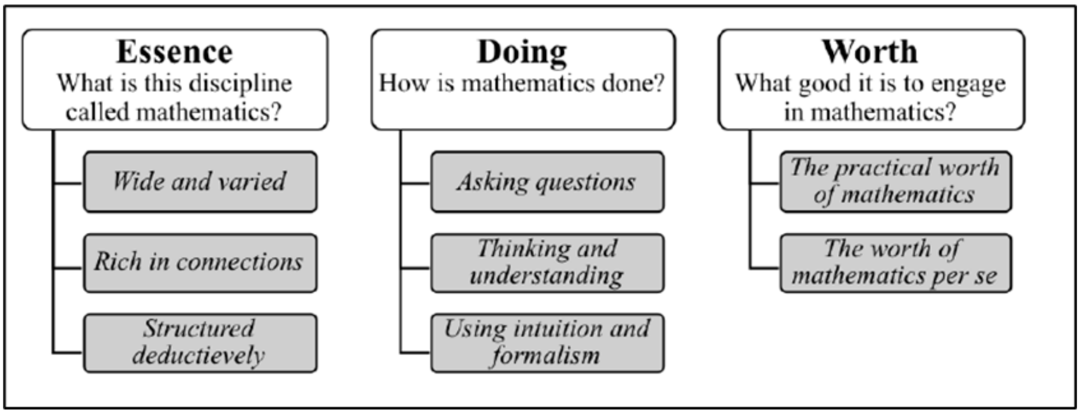


Figure 2.3. What mathematicians wish to teach teachers about the nature of mathematics (Hoffman & Even, 2018, p. 105)

The first component is related to the epistemology of mathematics. It includes the abundance of branches in mathematics as a science, the connectedness of mathematical concepts or topics among them, and the deductive structure of mathematics. Another crucial aspect for mathematics teachers is dealing with mathematical activities and skills. About this issue, mathematics teachers can ask the right questions, reasonably build all their thinking, and employ an intuitive and formal way of mathematical thinking. Lastly, mathematicians think that mathematics teachers must appreciate the value of mathematics (Yan et al., 2020). The studies conducted in collaboration with mathematicians, which yielded insights into the AMK for teachers based on the perspectives of field experts, appear to constitute a valuable contribution to the AMK literature, particularly in terms of their potential to elucidate the theoretical limitations of AMK and its relationship to teaching knowledge.

The following section is dedicated to a comprehensive examination of the mathematical concept of limit, which forms the basis of this study. In addition, the literature review will investigate the nature of the relationship between 0.999... and 1.

2.6 Limit

The concept of limit has a crucial role especially in the branches of calculus and analysis and in the other branches of mathematics such as complex analysis (Bezuidenhout, 2001; Cornu, 2002; Tall, 2002). Calculus, also known as analysis, is regarded as the discipline concerned with “the study and applications of limits” (Grant et al., 2016, p. 825). In the progression towards more advanced mathematical concepts (Swinyard, 2011), the formal definition of limit, which is characterized by a high degree of rigor, plays a pivotal role for further calculus concepts such as convergence, derivative, integral (Cornu, 2002).

The evolution of the concept of limits of functions has resulted in the current degree of precision and rigor over an extended period of time, as evidenced by Katz and Katz (2010). In some studies, the historical development of the concept is compared with the conceptual development within students' understanding of limits (e.g., Juter, 2006). However, history and psychology are two distinct research areas, and thus a comparison between the historical development of a concept and the process of acquiring knowledge of the concept presents numerous challenges (Furinghetti & Radford, 2002). Bagni (2005) elucidates the lack of parallelism between these two through his investigation of the dynamic and static aspects of the concept of limit, which developed across different historical periods, and showed how they involved different semiotic registers. The concept of the limit of a function is typically introduced in a dynamic manner, whereby as x approaches a, $f(x)$ approaches c (Tall & Vinner, 1981). This emphasizes a motion, rather than a focus on the quantification of approaching. Subsequently, it transitions to a quantified mode, as evidenced by the statement, "We can make $f(x)$ as close as we like to c, provided that we take x sufficiently close to a" (Tall & Vinner, 1981, p. 155), or as expressed in the formal definition of limit.

In defining the concept of limit and communicating the properties associated with it, a range of terminologies are employed, including tend to, approaching, to reach and converge (Oehrtman, 2009). However, these phrases possess their own denotation in everyday usage although their carried meaning in the context of limit is the same. Consequently, an essential challenge arises in comprehending the concept of limit accurately, given the diverse usage of such phrases (Monaghan, 1991). The terminology preferred by teachers or students when articulating limits offers valuable insights into their conceptualization of limits. For instance, for a sequence of $s_n = \begin{cases} 0 \text{ (n odd)} \\ \frac{1}{2n} \text{ (n even)} \end{cases}$, mathematics students mentioned that there were two different sequences here, the odd terms equal to 0, but the even terms tend to 0 (Tall & Vinner, 1981). Here, it could be claimed that such students had a concept image of 's_n approaches to s as n gets large, but it actually never reaches there' for the representation of $s_n \to s$.

A number of intricate conceptual challenges lie at the heart of the concept of limit. Cornu (2002) defines epistemological obstacles as the obstacles caused by "the nature of the mathematical concepts themselves" (p. 158). Any potential challenges that students or teachers may face in the context of limit was evaluated as a matter for epistemological obstacles in this study. Regarding the limit concept, teachers and students have difficulties due to the infinite nature of the concept, deficiency in perceiving limit as a product beyond a process (Conner, 2013; Tall, 1992), and epistemological reasons in developing the limit concept. For example, concentrating on the process beyond the limit causes a perception of always remaining a tiny hole at the end of the limiting process and students' disapproval of using the equal sign while identifying the limit (Páez & Hitt, 2004). Specifically, Conner (2013) reported on pre-service secondary mathematics teachers' lack of awareness of the equality of 0.999... vs. 1, with the limit's role in understanding this fact. Furthermore, the difference between the everyday meanings of quantifiers and their meanings in limit creates obstacles for understanding the limit concept (Cornu, 2002; Juter, 2007).

Some other epistemological obstacles regarding limit are related to understanding infinity (Sierpinska, 1987). In the historical development of the limit concept, the terms infinitely small and infinitely large have generated an epistemological obstacle (Cornu, 2002). There is still no consensus about using infinitesimals in introducing the concept of limit (Tall & Katz, 2014). The attainability of the limit value also creates an obstacle in understanding the concept of limit (Cornu, 2002). Apart from that, the conceptualization of limit process involves various challenges such as the potential infinity involved in limit process (Kidron & Tall, 2015), the algebraic aspect to compute the value of infinite decimal expansions (Tall, 1992).

Another complex and challenging issue is the interpretation of the role of quantifiers, symbols or notations in the formal definition of limits. Understanding formal definition involves understanding inequalities, absolute value, and interpreting the choice and temporal order of epsilon and delta, comprehending the concepts of neighborhood and intervals (Lee, 1992). The other one is the integration of formal language generated from symbols, notations and quantifiers with the verbal forms of limits. It is regrettable that the definition of limit is presented in a variety of ways in mathematics textbooks. To illustrate, most of the times it is merely conveyed in verbal statements. In other instances, there is a clear preference for the use of symbolic language (Durand-Guerrier & Arsac, 2003). Additionally, there are instances where the connection between ε and δ is explicitly identified.

In light of the presented summary of the literature on limits, the literature on the equality of 0.999... and 1 demands particular attention, given that it constitutes the specific mathematical focus of the present study. Understanding the equality of 0.999... and 1 is related to various concepts in mathematics, and so it necessitates conceptually grounded knowledge of different concepts in mathematics such as infinity (Richman, 1999), the nature of real numbers, limit, geometric series, etc. Proving this equality is also possible with the help of algebraic operations, the sum of geometric series, and the limit. In the report of Mathematical Education of Teachers II (CBMS, 2012), the Conference Board of the Mathematical Sciences highlights the significance of knowing and being able to explain the reasons for equality in multiple ways for teachers. In short, the mathematical reasoning behind the equality of 0.999... and 1 is important to understand the concept of limit certainly, but it also helps to develop a coherent understanding of other topics in mathematics.

Most of the time, although students and teachers state their equality, they cannot convince themselves completely (Wasserman, Weber, et al., 2018). They feel like there is a very tiny, small space between these two numbers. However, such a belief, though rising from a deficiency in understanding the limit concept, could reason for inconsistency in teaching activities and erroneous beliefs regarding fractions (Yopp et al., 2011). In misunderstanding this equality, various reasons can have a role. For example, a weak understanding of the limit subject might also be incorrectly transferred to this issue. 0.9, 0.99, …, 0.999... is a sequence of having all the terms smaller than 1. For this reason, a student could build a rationale like that since all the terms of this sequence have a character of less than 1, then the limit of this sequence must also be less than 1 (Tall, 2002).

There is a real contradiction in teachers' and students' minds about the relation of these two numbers. While students, and even mathematics teachers, are hardly convinced of the equality of two numbers without self-confidence in some cases, they firmly deny it in a substantial number of cases. Even so, students and teachers could not find any numbers between them; they stuck to their belief about not equality (Juter, 2022). Various studies were conducted regarding teachers' conception of the relation between 0.999... and 1. An extremely general conception could be outlined with the following three statements: (i) *The number 0.99…, where the number of nines after the comma is infinite, is equal to 1 since there is no number between 0.99… and 1,* (ii) *there are no numbers between 0.99… and 1, but 0.99… and 1 are different numbers anyway, and* (iii) *there are numbers between 0.99… and 1, so they are different numbers* (Juter, 2022, p. 6). Although at first glance, the first two statements seem to indicate the same thing, no numbers between 0.999... and 1, the second one states that these two numbers are not equal but very close. On the other hand, the last statement directly claims that there are different numbers between the two numbers.

Besides, some studies have reported that mathematics teachers' knowledge of this elementary school's mathematical content is mostly veiled. For example, in a study held with pre-service teachers (Buchholtz et al., 2013), results showed that although student teachers were reminded that there was a proof of the equality of these two numbers through geometric series and limit, they could not derive the correct connection between the equality and their academic knowledge of real analysis.

The equality of 0.999... and 1 is also an issue that is encountered in studies aiming to make advanced mathematics knowledge more meaningful for mathematics teachers. For example, Wasserman et al. (2018) designed a connection lesson and focused on this phenomenon in their study. When working with pre-service and in-service mathematics teachers, they revise the concept of limit and then discuss how it can be used to grasp 0.999... =1.

Tall (1977) distinguished between two interpretations of the number 0.999..., namely, "0.99...9 to a finite number of places and the infinite expansion" (p. 11). A discrepancy in understanding this issue was reported in the study, whereby while students answered $\lim_{n\to\infty}\left(1+\frac{9}{10}+\frac{9}{100}+\cdots+\frac{9}{10^n}\right)^n=2$, same students could also write $0.999\ldots<1$. Tall posited that one of the reasons to consider 0.999... <1 is the notion of "getting close to" in the verbal or informal definition of limit (p. 13). This notion conveys the impression that the two phenomena, 0.999... and 1, are in close proximity to one another, yet they are not identical. Especially the interpretation of the relationship between 0.999... and 1 through geometric series leads to an interesting contradictory argument, which is “0.999... converges to 1, or that it is equal to 1 in the limit, but is not equal to 1” (Richman, 1999, p. 396).

A review of the relevant literature revealed studies conducted with pre-service mathematics educators to enhance their content or pedagogical content knowledge about the concept of limit. To illustrate, Savuran (2022) explored the means of cultivating pre-service mathematics teachers' specialized content knowledge, a facet of content knowledge deemed indispensable for teachers (Ball et al., 2008), in the context of limit through the lens of the lesson study method. The study revealed the evolution of one teacher's approach to her teaching methodology, adopting an analytical perspective to examine her own pedagogical knowledge and identify strategies for effectively conveying the abstract concept of limit for students' comprehension. Furthermore, in the study, Savuran (2022) put forth a pedagogical model for teachers, encompassing the method of lesson planning.

Aliustaoğlu and Tuna (2019) examined the pedagogical content knowledge of pre-service mathematics teachers with regard to their ability to identify and address student errors in problems pertaining to the concept of limit. The researchers reported that although the participants were able to identify errors made by students, they were unable to propose potential explanations for these misunderstandings. Moreover, the study demonstrated that pre-service teachers lacked the capacity to delineate a comprehensive approach to teaching that would address the specific errors identified in the students' work. The descriptions of pedagogical actions displayed a deficiency in specificity, failing to address the concept of limit in a comprehensive manner. Furthermore, the study concluded that pre-service teachers exhibited analogous misconceptions and difficulties to students regarding the concept of limit. The researchers attributed these findings to the inadequate content knowledge of pre-service teachers regarding the concept of limit. In a study conducted by Biber and Argün (2012), the ability of pre-service mathematics teachers to generalize and abstract knowledge regarding the limits of single-valued functions was investigated. But the participants demonstrated a lack of understanding of the concept of limits of functions of one variable, which indicated that the pre-service teachers had not yet developed the ability to generalize their knowledge.

Teachers experience difficulties with the concept of limits in a number of different aspects of the concept, as previously outlined in this section. Articulating the definition of the limit through the variables, symbols, and inequalities within the formal definition of the limit represents a significant obstacle for pre-service teachers (Baki & Çekmez, 2012). For instance, rather than demonstrating the existence of an appropriate delta across a specified epsilon value in a mathematical problem, pre-service teachers attempted to focus on the uniqueness of this delta value, which is not a concern for a learner who comprehensively understands the role of variables within the formal definition (Baki & Çekmez, 2012).

Moreover, teachers face challenges in distinguishing between the concepts of limit, continuity, and definability of a function. It is a fallacy to assume that if a function has a limit at a point, it must be continuous and definable at that point. Nevertheless, research has demonstrated that this is a prevalent misunderstanding among pre-service mathematics teachers (Baştürk & Dönmez, 2011). Additionally, they may inadvertently focus on flawed assumptions related to the concept of limits. Bansilal and Mkhwanazi (2022) identified this as a significant challenge in understanding the concept of limit values for functions, noting that a limit value must be a value that can be accessed. This is a recurring issue in the literature on limits (Cottrill et al., 1996). In considering the concept of a limit, some pre-service teachers may view it as a maximum value for a function (Monaghan, 1991). Additionally, they may perceive the limit in a dynamic manner, which could lead to the acceptance of infinities as limit values for functions such as $f(x) = \frac{1}{x^2}$ at the point $x = 0$ (Bansilal & Mkhwanazi, 2022).

The aforementioned difficulties can be attributed to a number of factors. For instance, Petty (1996) suggests that students encounter challenges when attempting to grasp the formal epsilon-delta limit definition, largely due to their difficulty in integrating the discrete components of the definition into a unified statement. Petty (1996) also highlights the prevalence of underqualified activities during undergraduate calculus courses, particularly in relation to the concept of limits. These activities, which are often rote and technically oriented, include substituting numbers into functions to determine limits or attempting to simplify complex functions like trigonometric ones to facilitate straightforward substitution. In order to gain an understanding of the concept of limit in the context of the formal definition of limit, it is recommended that the dependence relationship between ε and δ, whereby δ is determined on the consideration of ε, be fully understood (Swinyard & Larsen, 2012).

As a summary, despite the significance and fundamental nature of the limit concept for numerous mathematical disciplines (e.g., Ervynck, 1981), a thorough understanding of this concept is reported as rare (e.g., Cornu, 1981; Davis & Vinner, 1986; Tall & Vinner, 1981) for some reasons such as regarding limit like as a bound for a function or a sequence, inability to differentiate between the dynamicity of the limit process and being a static object, the obscure of the process of the conception of the limit with issue of the attainability of the limit value, and inability to switch or link the various terminology adopted in the description of the limit etc.

2.7 Conceptual framework

The study had two objectives: firstly, to explore the essential advanced mathematical knowledge of limit for mathematics teachers from the experts’ points of view (i.e., need analysis phase of the design-based research), and secondly, to document the development of mathematics teachers' advanced mathematical knowledge within the teaching sequence, with a particular focus on the concept of limit and related topics (i.e., exploration phase of the design-based research). In order to achieve these purposes, based on previous research, first, I define advanced mathematical knowledge adopted in this study with its general characteristics linked with advanced mathematical thinking. Subsequently, I summarize the value of advanced mathematical knowledge for mathematics teachers through a review of the literature, outlining how I investigated the value of advanced mathematical knowledge in the context of limits in line with the employed framework. I then detail the characteristics of advanced mathematical knowledge within the design principles I used in the teaching sequence of this study, further pointing to how I conceptualize the concept of limits from the point of view of advanced mathematical knowledge. As a summary, the outline of presenting conceptual framework of the study contains the following issues respectively,

- The research on advanced mathematical knowledge and how it was defined in this study
- The value of advanced mathematical knowledge for mathematics teachers was synthesized and the framework adopted in the data collection and analysis of the need analysis phase was connected with the concept of limit
- The design principles leading the design of the teaching sequence were introduced in line with the literature and the early findings of the need analysis phase

As indicated in the preceding bullets, the initial section of the conceptual framework commences with the introduction of advanced mathematical knowledge. The term "advanced mathematical knowledge" has been defined in various ways in previous sections of this study. In a similar manner to that proposed by Zazkis and Leikin (2010), advanced mathematical knowledge at the most general level is defined in this study as the knowledge acquired through the university-level mathematics courses. Similarly, aligned with my perspective on advanced mathematical knowledge I adopted in this study and given the established connections with advanced mathematical thinking, McCrory et al. (2012) defined advanced knowledge for teachers as "knowing more advanced mathematics that is relevant to what they will teach" (p. 595). Notably, advanced mathematical thinking is regarded as advanced forms of mathematical thinking (Edwards et al., 2005; Selden & Selden, 2005; Tall, 2002). Correspondingly advanced mathematical knowledge regarded as “an advanced perspective on elementary knowledge” (Zazkis & Mamolo, 2011, p. 13). In this regard, the particular constituents of advanced mathematical thinking such as using theorems, rigor and formalism (Edwards et al., 2005; Tall, 2002), visualization, dealing with infinity and involving with the epistemological obstacles (Harel & Sowder, 2005) were of great importance in determining the conceptual skeleton of advanced mathematical knowledge for the concept of limit in this study.

Although the empirical justifications are scarce, it is claimed that dealing with advanced mathematical knowledge develops mathematics teachers pedagogically (Hoffman & Even, 2018, 2024; Wasserman, Fukawa-Connelly, et al., 2017; Yan et al., 2022; Ziegler & Loos, 2017). Particularly, Wasserman, Fukawa-Connelly, et al. (2017) study showed that involving the logic behind proofs could enhance teachers in teaching secondary mathematics content, rather than merely attaining positive values towards mathematics as a discipline. In addition, Ziegler and Loos (2017) focused on the content-related contributions of advanced mathematics courses and classified them into two aspects. While one relates to knowing the content specific to some topics or concepts, the other refers to cumulative knowledge about epistemology and mathematical skills.

Following Ziegler and Loos’ (2017) categorization, Hoffman and Even (2018) focused further on the second part, which comprises the nature of mathematics and the sense of dealing with mathematics. They reported that mathematicians somewhat had a consensus that the primary purpose of the given mathematical education is to elaborate teachers’ general knowledge about the nature and epistemology of mathematics. In this scope, three dimensions were reported: the essence of mathematics, doing mathematics, and the worth of mathematics. The first dimension is about the description of mathematics as a discipline (Hoffman & Even, 2024). In this aspect, the perception of mathematics as encompassing an abundance of branches, including interrelated mathematical concepts, and having a deductive structure were indicated as crucial for mathematics teachers. The second aspect of doing mathematics refers to essential mathematical activities and skills for teachers, such as directing suitable questions and adopting a formal way of mathematical thinking. The last aspect means mathematics teachers' recognition of the value of mathematics in practice and by its nature (Hoffman & Even, 2018).

Additionally, Yan, Marmur, and Zazkis (2022) reported that mathematicians appreciated advanced mathematics studies for secondary mathematics teachers in terms of (1) drawing connections across mathematical domains, (2) gaining mathematical experience for developing problem-solving abilities, and (3) increasing epistemological awareness of the subject. The first component is related to advanced mathematics that helps secondary mathematics teachers to view mathematics from a broader perspective. An example of this component is the generation of a secondary-school level trigonometric formula such as $\cos(2\theta) = cos^2\theta - sin^2\theta$ through rotation matrices in linear algebra. Although such an explanation through linear algebra is far from secondary school-level mathematics, a mathematician in Yan et al.'s (2022) study pointed out the necessity of such advanced mathematical knowledge for teachers. The second component suggested that rich mathematical experience gained in advanced mathematical courses could support mathematics teachers' ability to manage problem-solving processes. This aspect was exemplified by a mathematical puzzle in which, rather than the trial-and-error or case analysis method, the use of the properties of number sets was used to lead to a more sophisticated and accessible solution. The third aspect mathematicians emphasized regarding the value of mathematics for secondary mathematic teachers was the acquisition of epistemological awareness. It was stated that mathematics teachers' involvement in advanced mathematical issues could make them attentive to restraints and affordances of mathematical knowledge.

In sum, albeit scarce in number, all the aforementioned studies pointed to the importance of having advanced mathematics knowledge for mathematics teachers. Specifically presenting frameworks regarding what mathematics teachers must know from the point of mathematicians, while Hoffman and Even's (2018) study focused on the discipline of mathematics in general, Yan et al. (2022) concentrated on mathematics taught in universities and provided a framework for how this knowledge could contribute to mathematics teachers. Yan et al.'s framework arose from questioning the necessity of advanced mathematics studies for secondary mathematics teachers at a macro level. However, this detailed framework also allows for a topic-by-topic analysis, providing insight into what undergraduate courses for mathematics teachers should focus on. Thus, I used their framework as a starting point for this study, aiming to zoom into the viable connections of advanced mathematics studies for secondary mathematics teachers in a narrower context. In this respect, using Yan et al. (2022) framework, the first part of the study provides a report on the advanced mathematical knowledge for secondary mathematics teachers in a particular mathematical concept. Thus, in the following paragraphs, I elaborate on how I acknowledge the concept of limit based on previous research with regards to the components of Yan et al. (2022) framework.

Drawing connections across mathematical domains in the concept of limit

In Yan et al.'s framework, this component indicates how studying advanced mathematics could help mathematics teachers to see the connections between mathematical issues, concepts, or topics at both micro and macro levels. For this study, the researcher interpreted this component as the connections between the concept of limit and other mathematical ideas that teachers should be aware of at the advanced mathematics level. Researchers pointed that limit, as a central topic in calculus and other branches of mathematics (Cornu, 2002; Tall, 2002), notifies “a move to a higher plane of mathematical thinking” (Tall, 1992, p.11). However, the advanced nature in limit concept, which necessitates thinking rigorously, does not indicate that all activities about this concept are advanced (Edwards et al., 2005).

The concept of limit contains various issues that teachers must face and think about. For example, a sequence's formal limit definition comprises many mathematical concepts such as quantifiers, ε and δ symbols, distance, neighborhoods, in/equality symbols, a number N, and logical statements. On the one hand, teachers should clearly understand the roles of these symbols in the definition of limit (Lee, 1992). On the other hand, focusing only on formal definitions is not enough to fully understand the concept (Tsamir & Tirosh, 1992). However, previous research indicated to the difficulties of undergraduate students in the use of such as quantifiers in making sense of the definition of limit (Juter, 2007) and the equal sign while describing the limit (Páez & Hitt, 2004).

Gaining mathematical experience to develop problem-solving abilities for the concept of limit

The second component of Yan et.al. (2022) framework explores how advanced mathematics can provide teachers with diverse experiences, enabling them to develop flexibility in mathematical thinking and to create new problem-solving strategies. In this part of the study, I embraced this component as identifying the mathematical concepts taught in the context of limit that might allow teachers to solve limit-related problems more elegantly. For example, while deriving the area formula for a circle, a process of drawing triangular segments inside it in addition to various steps is followed. However, students often struggle to comprehend the process of obtaining a formula for a circle through triangles, which could raise a question of "How does the limit process eliminate corners?" (Kajander, & Lovric, 2017, p. 1035). Under this component, such issues related to limit that could be problematic to comprehend were particularly considered. When interviewed with the experts, the researcher scrutinized how experts suggested teachers to focus on the reasons why the limiting process could be involved in such events. Specifically, in deriving the area formula for a circle, the experts explained the rationale behind the limiting process, which is how to fill an area inside a circle by drawing polygons. In this process, as the number of corners of the polygons is increased and the area of the enclosed polygons is gradually filled, it becomes necessary to determine the limit of the areas of the enclosed polygons, rather than the polygons themselves.

Increasing epistemological awareness of the concept of limit

In Yan et al.'s framework (2022), epistemological awareness refers to teachers' ability of recognizing the efficient and insufficient parts of their own mathematical knowledge. For this part of the study, the development process of the concept of limit and the problems experienced in this process, i.e. epistemological obstacles (Moru, 2009), were considered under this component. Cornu (2002) defines epistemological obstacles as the obstacles caused by "the nature of the mathematical concepts themselves" (p. 158). The researcher considered any potential challenges that students or teachers may face in the context of limit as a matter of this component.

Regarding the limit concept, teachers and students have difficulties due to the infinite nature of the concept, deficiency in perceiving limit as a product beyond a process (Conner, 2013; Tall, 1992), and epistemological reasons in developing the limit concept. For example, concentrating on the process beyond the limit causes a perception of always remaining a tiny hole at the end of the limiting process and students' disapproval of using the equal sign while identifying the limit (Páez & Hitt, 2004). Specifically, Conner (2013) reported on pre-service secondary mathematics teachers' lack of awareness of the equality of 0.999... vs. 1, with the limit's role in understanding this fact. Furthermore, the difference between the everyday meanings of quantifiers and their meanings in limit creates obstacles to understand the limit concept (Cornu, 2002; Juter, 2007). Some other epistemological obstacles regarding limit are related to understanding infinity (Sierpinska, 1987). In the historical development of the limit concept, the terms infinitely small and infinitely large have generated an epistemological obstacle (Cornu, 2002). There is still no consensus about using infinitesimals in introducing the limit concept (Tall & Katz, 2014). Lastly, the attainability of the limit value creates an obstacle in understanding the concept of limit (Cornu, 2002).

The remaining parts of the conceptual framework were set aside in favor of an introduction to the design principles, which guided the teaching sequence. In consideration of the aforementioned points, the initial design principle from the perspective of advanced mathematical knowledge that informed the preparation of the content and flow of the design process was as follows:

> **Design Principle 1:** To promote advanced mathematical knowledge, the activities involving rigorous and deductive reasoning (Edwards et al., 2005),

> necessitating intuition and visualization (Krussel, 1994), touching epistemological obstacles (Harel & Sowder, 2005), necessitating communication between words and symbols (Herlina & Batusangkar, 2015), necessitating justifications (Tall, 2002) and the problems involving infinity (Edwards et al., 2005) must be included in a teaching sequence.

Based on this design principle focusing on advanced mathematical knowledge, I characterized the concept of limit from the point of view of an advanced mathematical knowledge as follows:

> **Design Principle 2:** Regarding the limit concept, teachers and students have difficulties due to the infinite nature of the concept (Sierpinska, 1987), deficiency in perceiving limit as a product beyond a process (Conner, 2013; Tall, 1992), and epistemological reasons in developing the limit concept such as the attainability of the limit value (Cornu, 2002) and the meaning of infinitesimals (Tall & Katz, 2014). So, dealing with practices gaining mathematical experience in limit gains importance. In this regard, some essential practices can be identified as presenting examples and non-examples of theorems and definitions regarding the concept, utilizing graphs, and playing with formal definition by changing the place of quantifiers (from the findings of the need analysis study in the preliminary phase). Additionally, understanding the formal definition of limit, which is a crucial element for AMK, involves understanding inequalities, absolute value, and interpreting the choice and temporal order of epsilon and delta, comprehending the concepts of neighborhood and intervals (Lee, 1992). Lastly, in the context of limit, some symbols such as equality has different meanings as compared to other contexts in mathematics. Focusing different meanings of symbols in the related concept and different contexts is also focused as important (from the findings of the need analysis study).

Design principle 2 based on the design principle 1 was considered viable because of the following. Notably, the literature indicates that although students and teachers affirm the equality of 0.999… and 1, they remain unable to fully convince themselves of this assertion (Choi & Do, 2005; Wasserman, Weber et al., 2018). The reasons for this weak belief are twofold: firstly, there is an inadequate understanding of the limit concept, and secondly, there is a perception that the infinite number of nines in 0.999… should be regarded as a finite number (Tall & Schwarzenberger, 1978). It is recommended that teachers adopt a multifaceted approach to comprehend the mathematical rationale behind the equality of 0.999… and 1 (CBMS, 2012). This necessitates a comprehensive understanding of the concept of limit, along with other related topics. It was therefore essential to incorporate the concept of the limit into the design of the teaching sequence. Moreover, studies conducted in collaboration with mathematicians have yielded valuable insights into the value of advanced mathematical knowledge for mathematics teachers (e.g., Leikin et al., 2018; Yan et al., 2020; Yan et al., 2022). Consequently, an interview study was conducted as part of the preliminary phase with experts including mathematicians and mathematics educators as a needs analysis (Plomp, 2013, p. 19) to ascertain the needs within the field and the essential mathematical knowledge for mathematics teachers regarding the concept of limit. Thus, design principle 2 was established, based on expressions for understanding the concept of limit from the perspective of advanced mathematical knowledge and rising from the related literature, in addition to the findings of the aforementioned need analysis study with experts. Having such advanced view of the concept of limit, I further built the design principle 3 based on the aforementioned ones in the following way.

> **Design Principle 3**: It is recommended that special cases be incorporated into teaching activities by revisiting the affordances and power of mathematical statements throughout the process. Furthermore, complex matters should be presented in a simplified form to facilitate comprehension. Finally, the learning process should be supported through the presentation of a variety of explanations, accompanied by an articulation of the underlying principles that underpin mathematical theorems or rules (Wasserman et al., 2022).

In the following paragraphs, I provide an elaboration on the ways in which these design principles influenced the progression of the study. First of all, in accordance with the third design principle, I proceeded in a sequential manner, beginning with a relatively simple and elementary approach to facilitate comprehension of a complex matter. As previously stated, there were multiple approaches to the equality of the numbers such as 0.999... and 1. The term ‘elementary’ (Zazkis & Mamolo, 2011) pertains to the approach most suitable for students at the lowest grade level. Thus, situating this as one main principle in the design of the teaching sequence, the teaching sequence started with decimal representations, which represent the most elementary perspective on the relationship between such as 1.5999… and 1.6 and went through touching to the more advanced issues. In this regard, following the discussion of the decimal representation, a theorem concerning the equality of two real numbers and its proof were presented for discussion at the implementation stage. Subsequently, the reasoning employed in the theorem on the real number was interpreted in the context of the equality of two infinite decimal representations. Thereafter, the formal definition of limit with its associated symbols and quantifiers was included in the sequence.

With this goal, in subsequent stages of the teaching sequence, a theorem concerning the equality of two real numbers was examined and analyzed in depth, with particular attention paid to its statements and logical structure. An additional step involved applying the reasoning and logic employed in the proof of the theorem on real numbers to a problem regarding the equality of two infinite decimal representations such as a = 0.2500... and b = 0.24999... In addition, with this task, in accordance with the design principle 1, a situation involving infinity that was "not accessible through the five senses" (Edwards et al., 2005, p. 18) was also made more accessible to learners by following a path of reasoning through finite numbers. To provide a more detailed explanation, in this step, the objective is to adopt a particular way of thinking, namely, to consider the potential distance between 0. 24999.. and 0.2500... It is therefore hypothesized that the distance in question may be 0.001. An alternative possibility is that the distance in question could be 0.0001. It is evident that the distance in question cannot be one of the aforementioned values, given that 0. 2499.. is in closer proximity to 0. 2500.. than 0.249, and similarly, 0.24999... is in closer proximity to 0. 2500.. than 0.2499. This pattern continues in a similar fashion for all subsequent values. The participants then established a connection between the numbers 0.001, 0.0001, and so forth and the phrase "for every epsilon" in the theorem, which indicated that the value of $|a-b|$ was smaller than every epsilon. This allowed them to conclude that $a=b$ through the theorem. Consequently, in order to reach the intended solution, the participants were encouraged to engage in a thinking process, whereby they would consider the distances involved and utilize individual cases in a finite manner as a means of achieving the desired outcome in an infinite manner which is not accessible through the five senses.

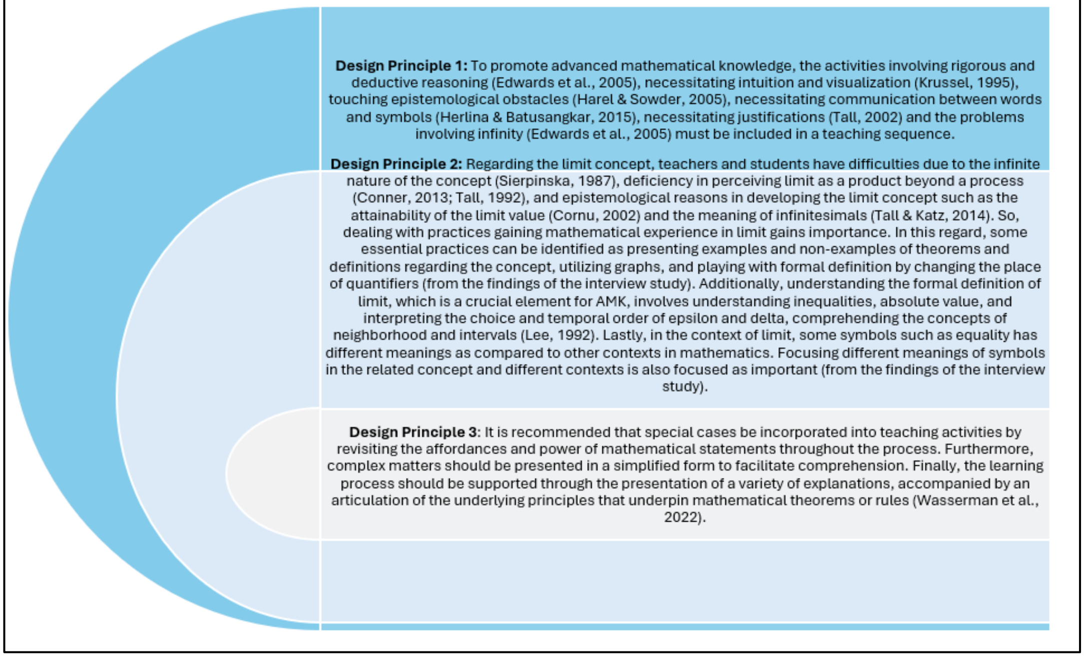
Considering the abovementioned design principles, I further elaborated on the remaining steps in the design of the teaching sequence. In accordance with the design principle 3, which recommends a reconsideration of the affordances and limitations of mathematical statements, the formal and informal definitions of limit was shared with the participating teachers during the implementation phase. In this context, the participants were asked to compare and contrast the informal and formal definitions of limit in order to assess their relative adequacy in conveying the complete meaning of the concept of limit. This practice revealed how they linked the approaching terms in the informal definition with the absolute value, symbols, and quantifiers in the formal definition of limit.

Subsequently, the formal definition of limit was subjected to a comprehensive examination in accordance with design principle 2. The dependence relationship between epsilon and delta was emphasized in the literature (Adiredja, 2021; Juter, 2007) and was also identified in the data from the need analysis study. Furthermore, the ordering of variables in the formal definition, and the manner in which they affect one another, was another topic addressed in the teaching sequence derived from the third design principle (Adiredja, 2021; Davis & Vinner, 1986).

In the final part of the teaching sequence, the concept of equality in the context of limit was considered, with the aid of illustrative examples, in accordance with design principle 2. The distinct meaning of equality as compared to other contexts in mathematics was a recurring theme in the interview data set. When viewed through the lens of the three design principles introduced thus far, this activity was also guided by design principle 3, which recommends the incorporation of special cases into the activities and the provision of a range of explanations to facilitate learning.

During the implementation, I also made reference to some epistemological problems identified in the literature regarding the conceptualization of infinitesimals (Tall & Katz, 2014), once more in alignment with the joint direction of design principles 2 and 3. In this regard, an analysis of a theorem within session 1 of the teaching sequence was conducted in order to examine its affordances and constraints in establishing a mathematically comprehensible assertion (design principle 3) through the lens of the epsilon criterion (design principle 2). This refers to the aforementioned theorem concerning the equality of two real numbers. One side of the if-and-only-if statement was comprised of an assertion, "for every epsilon," and teachers were encouraged to engage in debate surrounding the function of the quantifier "for every" and the epsilon criteria in the theorem. A comprehensive account of the steps undertaken will be provided in the methods chapter and the empirical evaluation concerning the teaching sequence will be presented in the findings chapter.

In conclusion, although the design principles were documented in three distinct categories, their boundaries were not as clearly delineated as might have been expected. There were instances where the principles appeared to overlap or intertwine with one another. The following figure illustrates the relative inclusivity among the design principles.



Design Principle 1: To promote advanced mathematical knowledge, the activities involving rigorous and deductive reasoning (Edwards et al., 2005), necessitating intuition and visualization (Krussel, 1995), touching epistemological obstacles (Harel & Sowder, 2005), necessitating communication between words and symbols (Herlina & Batusangkar, 2015), necessitating justifications (Tall, 2002) and the problems involving infinity (Edwards et al., 2005) must be included in a teaching sequence.

Design Principle 2: Regarding the limit concept, teachers and students have difficulties due to the infinite nature of the concept (Sierpinska, 1987), deficiency in perceiving limit as a product beyond a process (Conner, 2013; Tall, 1992), and epistemological reasons in developing the limit concept such as the attainability of the limit value (Cornu, 2002) and the meaning of infinitesimals (Tall & Katz, 2014). So, dealing with practices gaining mathematical experience in limit gains importance. In this regard, some essential practices can be identified as presenting examples and non-examples of theorems and definitions regarding the concept, utilizing graphs, and playing with formal definition by changing the place of quantifiers (from the findings of the interview study). Additionally, understanding the formal definition of limit, which is a crucial element for AMK, involves understanding inequalities, absolute value, and interpreting the choice and temporal order of epsilon and delta, comprehending the concepts of neighborhood and intervals (Lee, 1992). Lastly, in the context of limit, some symbols such as equality has different meanings as compared to other contexts in mathematics. Focusing different meanings of symbols in the related concept and different contexts is also focused as important (from the findings of the interview study).

Design Principle 3: It is recommended that special cases be incorporated into teaching activities by revisiting the affordances and power of mathematical statements throughout the process. Furthermore, complex matters should be presented in a simplified form to facilitate comprehension. Finally, the learning process should be supported through the presentation of a variety of explanations, accompanied by an articulation of the underlying principles that underpin mathematical theorems or rules (Wasserman et al., 2022).

Figure 2.4. Design principles informing the teaching sequence

The relatively inclusive nature of the design principles is illustrated in Figure 2.4. It was interpreted as follows: while design principle 1 comprises rigorous and deductive reasoning, design principle 2 incorporates examples and non-examples of theorems and definitions pertaining to the concept. Furthermore, visualization was identified as one of the key elements in design principle 1, with the use of graphs incorporated into design principle 2. Furthermore, visualization seemed to be one of the key elements in design principle 1, with the use of graphs incorporated into design principle 2. Additionally, design principle 1 highlighted the role of epistemological obstacles and symbols, while design principle 2 delved deeper into this concept, particularly focusing on the attainability of the limit value as an epistemological obstacle and the interpretation of epsilon and delta, with a specific emphasis on the significance of equality in the context of limits. Furthermore, when the design principles were examined from a broader standpoint, it became evident that design principle 1 encompasses the fundamental mathematical activities associated with advanced mathematical knowledge, whereas design principle 2 provides an overview of the concept of limit from an advanced mathematical knowledge perspective. Conversely, design principle 3 proposes a methodology whereby one progresses from fundamental concepts towards more sophisticated ones in a pedagogical process. Given its role in guiding the preceding two principles, it was evaluated as occupying a central position in relation to them.

CHAPTER 3

METHODOLOGY

The purposes of the present study were (1) to explore the essential advanced mathematical knowledge of limit for mathematics teachers from the experts' points of view, and (2) to document the characteristics of the difficulties experienced by mathematics teachers in a teaching sequence designed to promote their advanced mathematical knowledge of the concept of limit.

In the following parts, the design and the developmental process of the study, the stages of the proposed teaching sequence, the context, and the data collection process were explained in detail.

3.1 Design of the Study

In this study, a design-based research approach was adopted to attain the purposes of the study and to answer the research questions. A sequence of instructional activities is devised, subjected to evaluation, and refined throughout the course of the teaching experiment (Gravemeijer & van Eerde, 2009). The teaching experiment is designed to facilitate researchers' investigation of the process by which participants individually rearrange their "mathematical ways of knowing" (Cobb et al., 2003, p. 4). This section provides an overview of the structure of design research and an account of the procedures followed in the process.

Design research allows researchers to investigate the general properties of the instructional setting to attain a targeted learning outcome (Gravemeijer & van Eerde, 2009). Integrating and informing theory and practice in education is an important characteristic of design research studies (Bakker, 2018). All design research studies include some essential steps such as creating new materials with a theoretically supported basement, implementing them in natural classroom settings with simultaneous modifications and revisions, evaluating the designs in an iterative process, and ending with domain-specific statements to inform the theory (Cobb et al., 2003; Cobb & Gravemeijer, 2008). The steps of analysis, design, evaluation, and revisions are adopted to offer an appropriate design idea for the related educational problem (Bakker, 2018). Although there is an agreement on the necessity that design-based studies should include some systematic phases within themselves, the details of the process may vary in line with the characteristics of the research.

Plomp (2013) classify the phases of design research into three: preliminary research, development or prototyping phase, and assessment phase. Preliminary research is like a preparatory stage in which a literature review, needs analysis, or the building of a theoretical framework is held (Bakker, 2018). At this stage of the process, researchers attempt to identify key elements that could potentially address the issues at hand. These elements, which are referred to as design principles, are then articulated in a systematic manner. It is imperative that design principles reflect an answer to the question of what kind of instructional setting is necessary for the intended learning to be attained. In other words, provisional solutions are established on the basis of empirical evidence. This step comprises “an analysis of the user practice (needs and context analysis) and an exploration of the scientific knowledge base (literature review and expert appraisal)” (Nieveen & Folmer, 2013, p.154). As a summary, at the end of the preliminary stage, a researcher must have some tentative guiding principles and a design outline which is ready for action. In the development or prototyping phase, the intervention includes a few or more cycles, applied iteratively, and revised considering the formative evaluations during the process. This phase entails a series of iterations aimed at enhancing the intervention under development and the associated design principles. Accordingly, it is recommended that a preliminary version of the intended intervention be constructed and evaluated. Lastly, the assessment phase evaluates to what extent the planned prototype compensates for the point at issue. The evaluation in the assessment phase is of a more summative nature (Prediger et al., 2015).

Although design researchers intend to propose outputs for practice and theory (Bakker, 2018), Prediger et al. (2015) differentiate design research into two forms in terms of what they focus on primarily and what they spend most of the time in studies. While the first is dominant in generating "curriculum products and design principles" that can be directly used in teaching environments by practitioners, the second type principally involves creating "local theories and paradigm cases" to be updated by the researchers before use (Prediger et al., 2015, p. 880). The present study is aligned with the first style, as defined by Prediger et al., through its provision of design principles and a teaching sequence that is based on these principles.

In the current study, Plomp's (2013) classification regarding the phases of design research was followed in expressing the design process: preliminary research, development or prototyping phase, and assessment phase. In the context of the preliminary research, a needs analysis with experts was conducted as a part of the current study. The prototyping phase comprised two cycles: an initial pilot study and the main study, which constituted the first iteration. The preliminary design principles, which were derived from a synthesis of the relevant literature and the findings of the online interviews with experts, underwent a process of revision through the pilot study. The revised design principles and the teaching sequence resulted in the generation of the main study. The research process involved the formulation of a revised framework based on the outcomes of the preceding cycle. This framework then served as the basis for the subsequent cycle, and so forth, until the final design principles and the teaching sequence were attained. The assessment phase was about the evaluation of the extent to which the proposed design framework fulfilled the objectives of the current study. The final version of the design principles and teaching sequence were produced as a consequence of the collective contribution of all actions undertaken throughout the entire process.

In summary, the process of the current study involved the generation of design principles derived from theory, the development of these principles through the views of experts, the design of a teaching sequence based on these principles, the testing of the principles in a natural classroom setting, including a pilot study, and the subsequent refinement of the theory in accordance with the results. In consideration of the aforementioned factors, the methodological approach of design research appeared to be the most appropriate methodology for the current study.

3.2 General Structure of the Study

In this study, parallel to the nature of the design research approaches, some essential steps were followed (see Table 3.1). As mentioned previously, Plomp's (2013) perspective was adopted for the study. For this reason, an interview study as need analysis phase was conducted in the first step to determine the needs for the related research area. Secondly, the main study following a pilot study was managed as an exploration phase. The table below shows the periods and other information about the related steps.

Table 3.1 Timeline of the study

Period	Phase	Participants	Place
April-May 2022	Need Analysis Study as Preliminary Phase	Experts (Mathematicians and mathematics educators)	Zoom Platform
March-April 2023	Pilot Study Phase	Pre-service secondary school mathematics teachers	İstanbul
September 2023	Main Study as Exploration Phase	In-service secondary school mathematics teachers	Alanya

As indicated in Table 3.1, the study was conducted in three distinct stages. The following figure presents a summary of the principal characteristics of the two main phases of the study, namely the need analysis phase and the exploration phase.

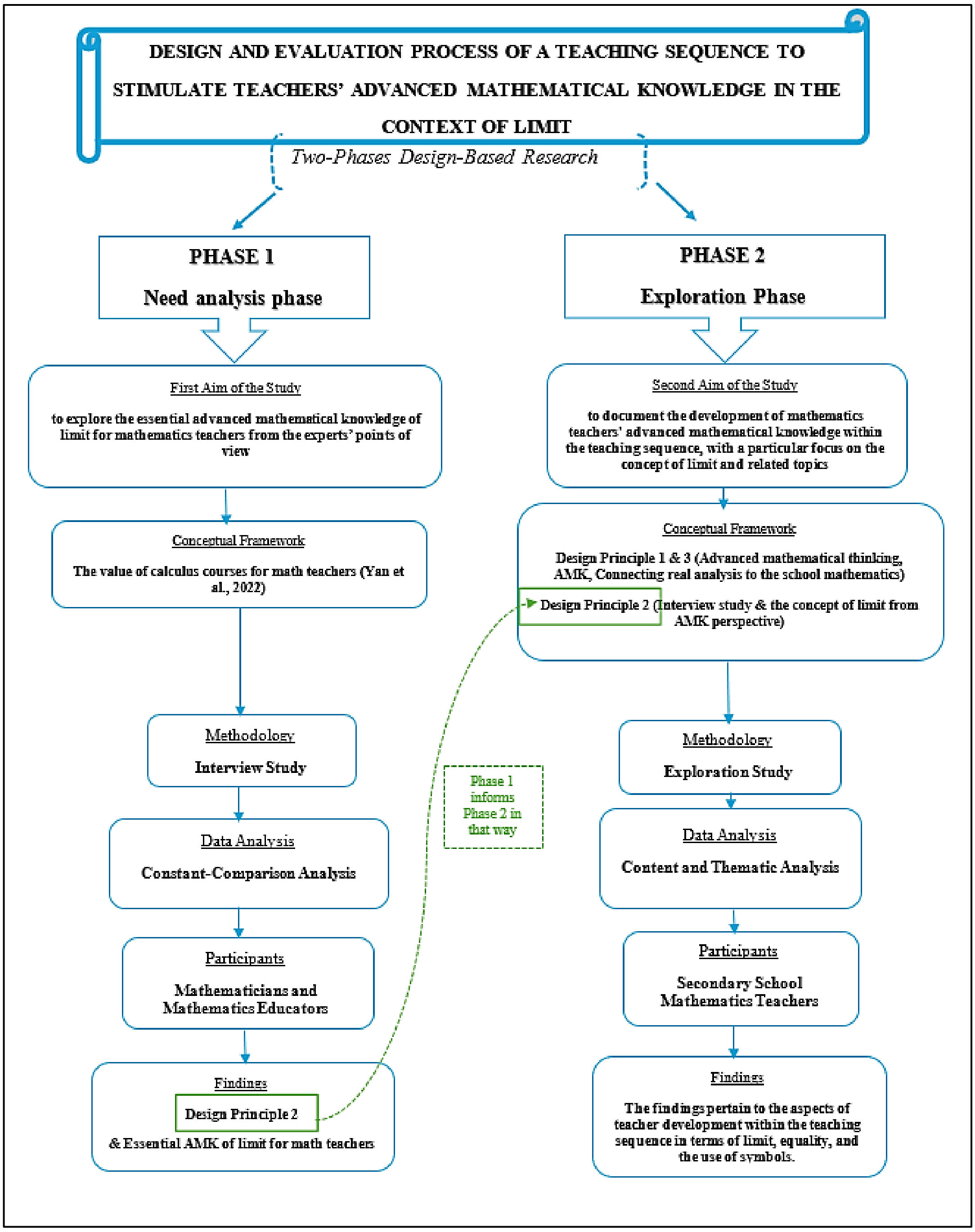


Figure 3.1. The outline of the two phases of the study

The figure included the purposes of the phases, the conceptual framework adopted in these phases, participants, methodology, data collection process, findings and how need analysis phase informed exploration phase superficially. The details of all these were explained in detail under the heading devoted to each separate stage. The following section will continue with an elaboration of the interview study, which constitutes the initial stage.

3.3 Need Analysis Phase: Interview Study

In the preliminary phase, researchers propose a review of related designs and design principles that have been previously documented in the literature (Plomp, 2013). As previously stated, the primary challenge in this study is to identify how advanced mathematical knowledge for teaching could be promoted among mathematics teachers in the context of limits. A comprehensive review of the relevant literature was conducted during the design and development of the teaching sequences. This review, along with the synthesis of findings and the associated design principles that could guide the teaching sequence, has been presented in the conceptual framework section before.

Subsequently, an interview study was conducted as part of the preliminary phase with mathematicians and mathematics educators as a needs analysis (Plomp, 2013). Semi-structured interviews (Clement, 2000; Goldin, 2000) were conducted to ascertain the needs within the field and the essential mathematical knowledge for mathematics teachers regarding limits. This step played a supplementary role in determining the design principles for subsequent teaching sequences. However, its most significant contribution was in identifying the essential advanced mathematical knowledge required of mathematics teachers regarding the concept of limit from experts' perspective. To achieve these aims, interviews were conducted with mathematicians and mathematics educators. The next subsections present information about participants and the data collection process for this part.

3.3.1 Participants of the need analysis study

The participants of the need analysis study were five experts, including mathematicians and mathematics educators from different state universities in Turkey. They all had over twenty years of experience in mathematics and/or mathematics education and varied in their undergraduate, Ph.D., and current research areas. A summary of information about the characteristics of the participants is in the table below.

Table 3.2 The information about participants of the need analysis study

Participant Id	Undergraduate area	Ph.D. area	Current research area	Experience in the field of M and ME
Expert 1	ME	M	ME	20 years (M), 5 years (ME)
Expert 2	M, ME	ME	ME	22 years (ME)
Expert 3	M	M	M	33 years (M)
Expert 4	M	M	ME	17 years (M), 21 years (ME)
Expert 5	M	M	M	23 years (M)

Note. In the table, M stands for mathematics, and ME represents mathematics education.

They were purposefully selected (Merriam & Tisdell, 2015) with the intention of understanding the thoughts of experts from different academic backgrounds and providing variability according to their field of studies in university education and their current research area. Particularly, as seen in Table 3.2, the participants had more than twenty years of experience in mathematics and/or mathematics education fields, and all were involved in teaching calculus or mathematics teaching methods courses and had experience in doing research in advanced mathematical ideas.

3.3.2 Data collection process of the need analysis study

The data collection instrument employed in the need analysis study was structured interviews. The interviews were conducted via the Zoom platform during the spring semester of the 2021-2022 academic year. The need analysis study's data collection process spanned nearly one month, with interviews lasting approximately 40 to 70 minutes. Ten questions were posed to participants. The framework components developed by Yan et al. (2022) informed the preparation of the interview questions. These components, along with their rationale, are outlined in the table below.

Table 3.3 Interview questions in the interviews in need analysis study

The components of the framework	**Interview Questions**	**The rationale behind the question**
drawing connections of limit across mathematical domains	Question 1: Which terms or key ideas in limit do you consider crucial for mathematics teachers?	To investigate the role of understanding elements in the concept of limit for mathematics teachers such as **quantifiers, ε and δ symbols, inequalities, absolute value, topological terms**
	Question 2: What do you think about the importance of visualization in mathematics? While teaching limit, do you adopt visual or graphical representations? For which purposes do you require the use of visual representations? (e.g., to make students gain an intuitive understanding of the limit or to teach it formally, etc.)	To investigate the place of adopting various representations in limit such as **graphics, visuals, tables**
	Question 3: When you think of the place of limit in the span of mathematics subjects, to what extent could limit be important and indispensable for mathematics teachers?	To investigate which mathematical topics that limit is evaluated as connected
	Question 4: Limit has two different definitions: formal and informal. But in the secondary school curriculum, it is strictly noted that "*The limit of a function at a point is not given through ε-δ technique, which is more of a concern to mathematicians*" (MoNE, 2018). So, why do you think a mathematics teacher should learn the formal definition of limit? In teaching which topics, a mathematics teacher could benefit from the idea behind the formal definition of limit?	To investigate the importance of understanding **the formal definition** of limit

Table 3.3 (cont'd)

	Question 5: In which way do you think the formal definition of limit could be taught to mathematics teachers efficiently?	To reveal the mathematical practices efficient for mathematics teachers regarding **the formal definition** of limit
	Question 6: What do you think about the importance of knowing the proofs of theorems in the context of limit for mathematics teachers? Which theorems do you think they must know definitely? Why?	To investigate to what extent and in which ways teachers must know the **theorems and proofs** in the context of limit
mathematical experience for developing problem-solving abilities	Question 7: Could you define a mathematical problem in which if someone borrows ideas from the concept of limit, the solution could be more sophisticated, meaningful, and shorter? (such as maximizing area problems)	To investigate limit-related issues over the branches of mathematics
	Question 8: In the generation of formula for the area of a circle, the limit process is used. But suppose that a mathematics teacher is asked, "*A polygon of any number of sides has corners, but the circle does not. How does the limit process eliminate corners*?" (Kajander & Lovric, 2017, p. 1035). How would this situation be related to the concepts, theorems, or definitions that a mathematics teacher has learned in calculus courses regarding limit?	To investigate how the idea of limit is involved in an example of a limit-related issue, the area of a circle
epistemological awareness of the subject	Question 9: *"There is a possible parallelism between the historical development of ideas about infinitesimals and developments within a student's understanding" (Bagni, 2005, p. 461).* This part is taken from an article about the historical developments of limit notion. What do you think about this statement? Could knowing more of history help an undergraduate student? In which ways could it be helpful for mathematics teachers especially?	To investigate the role of **historical developments** in the concept of limit
	Question 10: The notion of infinitely small and infinitely large is hard to conceptualize for undergraduate students. While some consider an infinitesimal as simply a variable which tends to zero, some view the symbol ε as a number not zero, but smaller than any positive real numbers. As a result of this understanding, 0. 999... could be believed as "*the last number before 1*" (Cornu, 2002, p.161). How could you evaluate the importance of understanding infinitesimals correctly?	To investigate to what extent a mathematics teachers must know about **infinitesimals**

As seen in Table 3.3, regarding the component of *drawing connections of limit across mathematical domains,* six questions were asked to participants about critical ideas or terms in the concept of limit. Our rationale was that limit as a subject intertwined with various essential concepts in calculus and other advanced branches of mathematics; it is crucial to analyze how experts connect it to different domains of mathematics. In this aspect, quantifiers, absolute value, inequalities, the role of visualization and other types of representations in teaching limit, the place of limit considering the span of mathematics subjects in mathematics teachers' professional knowledge, the significance of knowing formal definition and the proofs of theorems in the concept of limit were targeted. Under the second component of the framework, namely *gaining mathematical experience for developing problem-solving abilities,* which mathematical problems can be solved more meaningfully with the help of the ideas that lie beneath the limit was investigated. For example, as also having connections back to the school mathematics, the involvement of the limit process in the formula generation for the area of a circle was asked. Lastly, for the last aspect of the framework, namely *increasing epistemological awareness of limit,* the experts were asked about infinitesimals and the contribution of knowing the history in teaching limit. Besides, a fundamental but mostly misunderstood issue, the equality of 0.999... vs. 1, with the limit's role in understanding this fact, was asked to participants.

The interview data informed the implementation process in two ways. Initially, the data was subjected to analysis in order to inform the design principles that would guide the further development of teaching sequences. The second function of this stage was to provide guidance on the preparation of the content for the teaching sequences. The pertinent findings were presented in the findings section in a comprehensive format (see Section 4.1).

3.3.3 Data analysis of the need analysis study

A constant comparison analysis was adopted in the data analysis process of the need analysis study (Clement, 2000; Savin-Baden & Major, 2013). First, the recordings of interviews were transcribed verbatim at the beginning of the analysis process. Also, at this phase, memos emphasizing the essential statements were written for a sound interpretation of the data. These steps were evaluated as the first-cycle coding of the data. Then, I imported all transcripts to a qualitative data analysis program, MAXQDA 2022 (VERBI Software, 2021) and read the transcript data from each expert line by line. I followed a process of open coding to create a general picture of the data provided. The unit of analysis was a statement or a set of statements the experts uttered in the interviews. All the concepts related to limits, ways of thinking, mathematical practices, difficulties, etc. were coded separately without having any categories in the mind. The initial codes created in the MAXQDA software were reviewed repeatedly to ensure that all critical points were included in the codes.

After this stage, the analysis proceeded with the help of an Excel file: the codes to the columns, the participants' names to the rows, and all relevant parts of the data were transferred to the cells. The analysis here focused on determining what the experts valued as important for mathematics teachers to know in terms of 1) the connections of the limit concept across mathematical domains, 2) the problem-solving processes, and 3) the epistemological awareness related to it.

This process was followed by a second stage comparing the codes among all participants and categorizing the compared codes through the components of the framework. At this stage, two other researchers examined the data excerpts, and I included them when agreed on their importance although not all the codes from the first coding process explained the data from all participants. In doing so, as well as the commonalities, the differences pointing to significant aspects of reasoning could also be specified about the concept of limit pointed by different participating experts. The second stage of analysis process was repeated until my categorization in the first round was reviewed by the two researchers, and conflicting points were discussed.

For example, in our meetings collectively working together on the data, I noted from the first round of analysis that one of the experts emphasized covariation of the quantifiers such as δ and ε, and coded this as “covariational reasoning”. I had included covariational reasoning in the first component of the framework (drawing connections) but one of the researchers in the data analysis process advocated that it must be under the second component (problem-solving) as related to the ways of thinking (Tallman & Frank, 2020). Discussing these controversies until reaching a consensus, I finalized the analysis and wrote narratives about the advanced mathematical knowledge experts considered important for teachers about the concept of limit. The following table shows a brief description of the theory-driven themes and data-driven sub-themes created in the need analysis phase.

Table 3.4 Themes and sub-themes generated in the interview study

Themes	Subthemes	Description
Drawing connections of limit across mathematical domains	Limit vs topology	Making relations between limit and topological terms
	Different meanings of symbols in different contexts	Identifying the differentiating role of symbols in the context of limit
Gaining mathematical experience for developing problem-solving abilities in limit	(In which concepts) understanding the definition of limit conceptually	The concepts emphasizing important to comprehend limit
	(In which concepts) exploring the elements within the formal definition critically	Focusing the components within the formal definition specifically to grasp
	(In which concepts) scrutinizing the covariational reasoning behind the definition of limit and thinking quantitatively about quantifiers	Identifying covariational relationships within the context of limit
	(Through what kind of activities) to use graphs and make interpretations from graphs about limit	Essential disciplinary practices such as graphs to adopt in limit
	(Through what kind of activities) to give examples and non-examples	Focusing the use of examples as essential

Table 3.4 (cont'd)

	(Through what kind of activities) to build the correspondence between the intuitive understanding of limit and the formalization	Making a connection between intuition and rigor
	(Through what kind of activities) to manipulate formal definition by changing the place of quantifiers	Identifying the role of quantifiers as essential
	(Through what kind of activities) to use analogies	Emphasizing the use of analogy
Increasing epistemological awareness of the limit	the obstacles caused by the difference in using the terms in daily life	Differentiating the differences in meaning of the terms
	misunderstanding of the limiting process	Some misunderstanding related to limit
	making improper sense of ε and δ	Focusing the obstacles risen from unhealthy grasp of ε and δ
	the unnecessary use of infinitesimals in limit	Discussing the role of infinitesimals

Following the completion of the need analysis study, a pilot study was initiated with pre-service secondary mathematics teachers. The following section presents the context of the pilot study, and its evaluations associated with the subsequent implementation of the teaching sequence.

3.4 Pilot Study: Testing Teaching Sequence with Pre-Service Teachers

Within this dissertation study, a pilot study was conducted with the intent of evaluating the suitability of the type of pre-test instruments for the purpose of the study, assessing the applicability of teaching sequences in real classroom settings, and determining the data collection procedure for the implementation of the actual design study.

3.4.1 The context of the pilot study

The pilot study was conducted with a sample of twelve secondary school pre-service mathematics teachers during the spring semester of the 2022-2023 academic year within the "Teaching Methods" course at a state university in Istanbul. The pilot study spanned a period of four weeks, with each session lasting between 50 and 85 minutes.

The first week focused on the relationship between 0.999... and 1. In this context, the first week was dedicated to the study of decimal representations, a theorem on the equality of real numbers and its proof, the interpretation of this theorem in the context of infinite decimal representations, and the interpretation of a mathematician's statement about the relationship between 0.999... and 1. The second week focused on sequences, limit of sequences, and specifically limit of a constant sequence. In this regard, the second week presented two distinct definitions of the limit of a sequence. It also discussed the relationship between the limit of a function and the limit of a sequence through the lens of a theorem. Furthermore, it examined the nuances of frequently used terms, such as "approaching," "tending to," "attaining or not," and "converging," in the context of limits.

3.4.2 Evaluation of the pilot study and revisions

This section assesses the pilot study in terms of its influence on the research process. The pilot study was conducted in order to test the tasks that were subsequently employed in the main study, as well as the implementation process itself. In the pilot study, a classroom scenario was presented as a pre-test, and some questions were asked in a worksheet to analyze pre-service teachers’ advanced mathematical knowledge by how they complete the scenarios, which points they focus on their work. In this process, most participants had trouble completing the worksheets. For this reason, another type of questionnaire was preferred for the main study as a pre-test.

During the first week of the pilot study, in some parts, students were required to think about the questions for a while and then state their ideas in the classroom as a short discussion. However, the discussion sessions could have been more productive. At this point, I realized that distributing papers for individual work could support participants' thinking processes and enable them to participate more actively. Besides, these in-class works could enrich data collection methods.

The pilot study also made a significant contribution in the decision of the central focus of the teaching sequence. The aim of the pilot study was to develop lessons on the concept of the limit in a comprehensive way, rather than focusing on one specific aspect of the limit. However, what was expected to be covered in a single lesson was in fact only covered in two weeks. Consequently, following the pilot study, as a result of the retrospective analysis with the advisors, it was decided to focus on one of the limit-related topics for the main study, specifically the relationship between 0.999... and 1.

The need analysis study conducted in the preliminary phase had included a specific question about the equality of 0.999... and 1. Following the pilot study, the interview data were re-examined, focusing on the relationship between 0.999... and 1 as the core of the teaching sequences. The data showed that in explaining the equality of 0.999... and 1, experts demonstrated a comprehensive understanding of the real numbers, the concept of limit and the notion of infinity. They also distinguished between the meaning of equality in the context of limit and other contexts in mathematics, which guided the main implementation of the teaching sequence and were chosen as some of the target concepts in the main study.

The need analysis study provided insights into fundamental mathematical concepts, including limit definition, equality, and limit perception. Consequently, the context of 0.999.... proved conducive to the effective navigation of these concepts. In the pilot study, it became evident that this construct lends itself to the application of design principles and offers a meaningful context for teacher candidates. Consequently, it was hypothesized that this task could serve the multiple purposes of the study. Additionally, the concept of 0.999.... and 1 is a suitable vehicle for exploring a range of mathematical terms from the lens of advanced mathematical knowledge, as detailed in the literature, and is represented in two distinct subjects within the high school curriculum.

3.5 Exploration Phase: Participants and the Context of the Main Study

In this section, the main study's participants and context were introduced. The participants of this study were six in-service secondary school mathematics teachers working in different governmental secondary schools in Alanya. Volunteers were selected to participate in this study through a convenient sampling method. Secondary school mathematics teachers graduated from the departments of mathematics education from education faculties, and the departments of mathematics from faculties of art and science were invited for the study. Besides, teachers working in schools of different achievement levels were included in the study to support the heterogeneity of participants. The table below presents the demographic information of the two male and four female secondary school mathematics teachers who participated in the study.

Table 3.5 Participating teachers’ demographic information

Teacher	Gender	Year of experience	Department of graduation	Professional experience after graduation
Merve	Female	17	math	no
Zerrin	Female	Not applicable	Not applicable	no
Ali	Male	24	math education	no
Hale	Female	15	math	no
Emre	Male	22	math	no
Cansu	Female	22	math	A math workshop

As demonstrated in Table 3.5, the four participants had obtained their degrees in mathematics and subsequently engaged in further training in pedagogical formation. This was regarded as an encouraging indication of their proficiency in pure mathematics. Teacher Zerrin did not submit the requisite demographic information form, resulting in a lack of data regarding their graduation and experience. Additionally, despite the fact that all six teachers participated in the sessions, only three (Merve, Ali and Emre) completed both the pre-test and the post-test. In presenting the findings regarding the pre- and post-tests, only the data from the three aforementioned teachers was utilized. However, the names of other teachers could be included in the data pertaining to the implementation, allowing for an examination of their potential influence during the discussions. It is noteworthy that Teacher Ali exhibited distinctive characteristics compared to other participants. When I approached the teachers to present my study and persuade them to take part, Teacher Ali was the most enthusiastic to participate and to recommend his colleagues to participate in the study. When I visited the school to meet him, he provided a detailed account of his mathematical experiences during his undergraduate education, displaying remarkable enthusiasm. Furthermore, throughout the course of the study, he demonstrated a continued willingness and enthusiasm. He was the first participant to return the tests, and his responses were more detailed than those of the others. Additionally, he was eager to answer the questions and participate in discussions during the implementation phase. In parallel with all of this, as will be seen in the findings chapter, this is the reason for the abundance of data on him in the study.

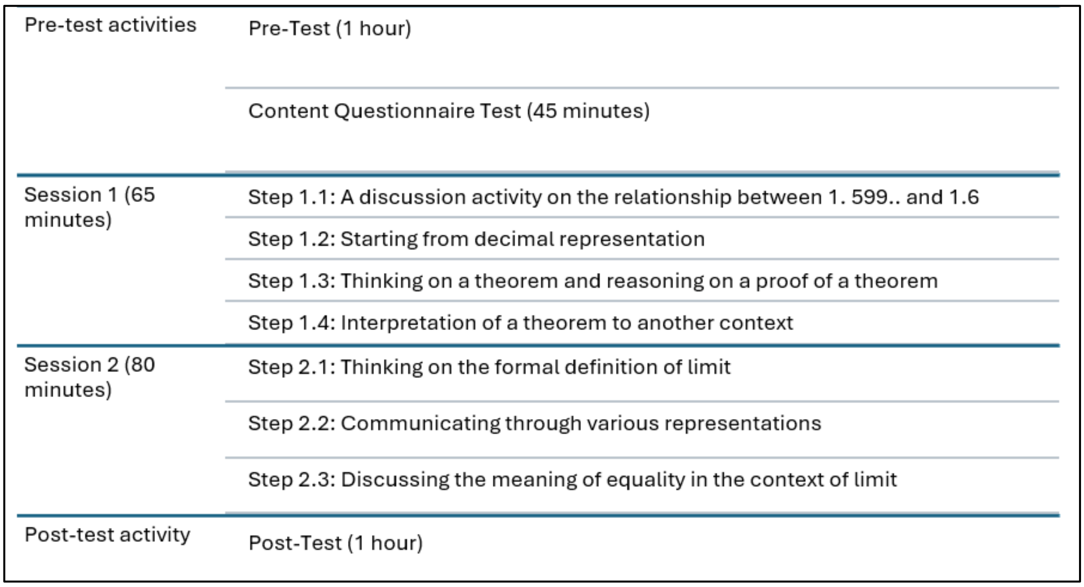
The implementation and the data collection process were conducted before the beginning of the 2023-2024 fall semester. Three participants were affiliated with the same educational institution which was also in an easily accessible location for other teachers. The study was therefore conducted at that educational establishment. Prior to the initiation of the study, the participating teachers were provided with a form containing demographic information, a pre-test, a content questionnaire, and consent forms. The aforementioned tests were completed independently by the participants prior to the commencement of the teaching sequence. I provided verbal instructions to the participants regarding the completion of the tests, and also included a note at the beginning of the test papers. These were about completing an individual test in one go without getting assistance from any source and about the time allocated for the tests. I have requested that the test sheets with the written answers be sent to me as a photo or pdf document at least one day prior to the starting date of the lessons. The administration of the post-tests proceeded in a similar manner. The post-tests were delivered to the participants one week after implementation, with a two-day deadline for completion and return. The implementation of the teaching sequence spanned a period of two days, comprising two sessions. As previously stated, the sessions were supplementary to one another. Despite the differing activities, the sessions together constituted the teaching sequence, serving the same purpose. The first session lasted approximately 65 minutes, while the second session lasted approximately 80 minutes. For the completion of the content questionnaire test, the participants were allocated 45 minutes for the content questionnaire test and one hour for the pre- and post-tests.

3.5.1 Description of the Proposed Teaching Sequence

The teaching sequence focused on the equality of 0.999... and 1, which was linked to the concept of limit. However, this is a multifaceted issue that extends to other concepts, such as real number systems and decimal representations, as detailed in the literature review. For this reason, this issue was identified as a key focus of the teaching sequence, with links to the concept of limit and other concepts indicated as related in the literature review and the interview study.

The main study consists of three steps: a pre-test activity aiming to measure mathematics teachers' advanced mathematical knowledge regarding the concept of limit before the implementation, the implementation of the teaching sequence in the classroom setting, and a post-test activity aiming to measure mathematics teachers' advanced mathematical knowledge regarding the concept of limit after the implementation.

The main study started with two pre-test activities. First, the Content Questionnaire Test (see Appendix A) was generated from some content questions measuring the preparedness of participants about the points to visit in the lesson. The second pre-test, the Pre-Test (see Appendix B), was aimed at measuring participants' understanding of the relationship between 0.999... and 1, the central focus of the teaching sequence. The process continued with the implementation of the teaching sequence and ended with the administration of the post-test. During the implementation of the teaching sequence, in-class assessments were also applied (e.g. how participants apply a theorem in another context). The figure below displays the general sketch of the process of the teaching sequence.



Pre-test activities	Pre-Test (1 hour)
	Content Questionnaire Test (45 minutes)
Session 1 (65 minutes)	Step 1.1: A discussion activity on the relationship between 1. 599.. and 1.6
	Step 1.2: Starting from decimal representation
	Step 1.3: Thinking on a theorem and reasoning on a proof of a theorem
	Step 1.4: Interpretation of a theorem to another context
Session 2 (80 minutes)	Step 2.1: Thinking on the formal definition of limit
	Step 2.2: Communicating through various representations
	Step 2.3: Discussing the meaning of equality in the context of limit
Post-test activity	Post-Test (1 hour)

Figure 3.2. The process of teaching sequence

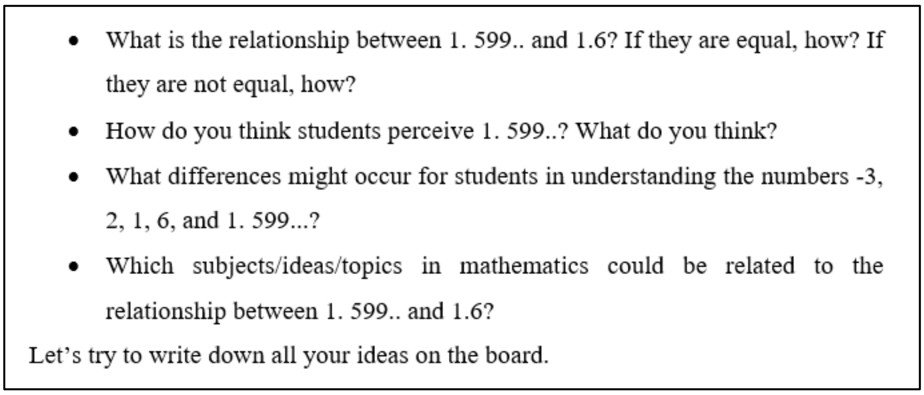
As illustrated in Figure 3.2, the teaching sequence comprised two consecutive, discrete sessions that were designed to reinforce one another. The first and second sessions are hereafter referred to as "Session 1" and "Session 2," respectively. The activities undertaken in Session 1 included a discussion on the relationship between the numbers 1.5999… and 1.6, a flow starting from the decimal representation, an activity centered on a theorem and reasoning on a proof of a theorem, an activity exploring the interpretation of a theorem in another context and the formulation of a conclusion. Followingly, the second session centered on contemplating the components of the formal definition of limit, communicating through multiple representations, and analyzing the role of equality in the context of limits. Despite the differing focus of the sessions, they primarily sought to encourage mathematics teachers advanced mathematical knowledge concerning a particular limit-related issue. As the design principles that underpin the teaching sequence were previously articulated in the conceptual framework, this section provides a detailed account of the individual steps accompanying with references to the design principles as shown in Table 3.6.

Table 3.6 The steps in the teaching sequence and the related design principles

Number of steps	Steps in the sessions	Design principles leading the steps
Step 1.1	A discussion activity on the relationship between 1.5999... and 1.6	Design Principle 1 & 2
Step 1.2	Starting from decimal representation	Design Principle 3
Step 1.3	Thinking on a theorem and reasoning on a proof of a theorem	Design Principle 1 & 2
Step 1.4	Interpretation of a theorem to another context	Design Principle 1 & 2
Step 2.1	Thinking on the formal definition of limit	Design Principle 1 & 2
Step 2.2	Communicating through various representations (tables, graph etc.)	Design Principle 2 & 3
Step 2.3	Discussing the meaning of equality	Design Principle 2

As demonstrated in Table 3.6, the primary activities in Session 1 were the articulation of teachers' perspectives on the relationship between 1.5999... and 1.6, a visit to the topic from the standpoint of infinite decimal representation through a theorem on the equality of two real numbers, and the interpretation of the theorem in an alternatively contextualized manner for infinite decimal representations.

By situating a problem involving infinity at the core of the teaching sequence, as recommended by design principle 1, and to gain insight into the challenges and misconceptions that teachers face regarding the concept of limit from the perspective of the advanced mathematical knowledge indicated in design principle 2, Step 1.1 involved an investigation of the relationship between the numbers 1.599… and 1.6, which constituted the theme of the pre-test. This was evaluated as an introductory step for the subsequent activities. The numbers analogous to 0.999… and 1 were selected purposefully for this step because this mathematical problem was identified as the focus of the teaching sequence. The discussion questions are appended to the figure below.



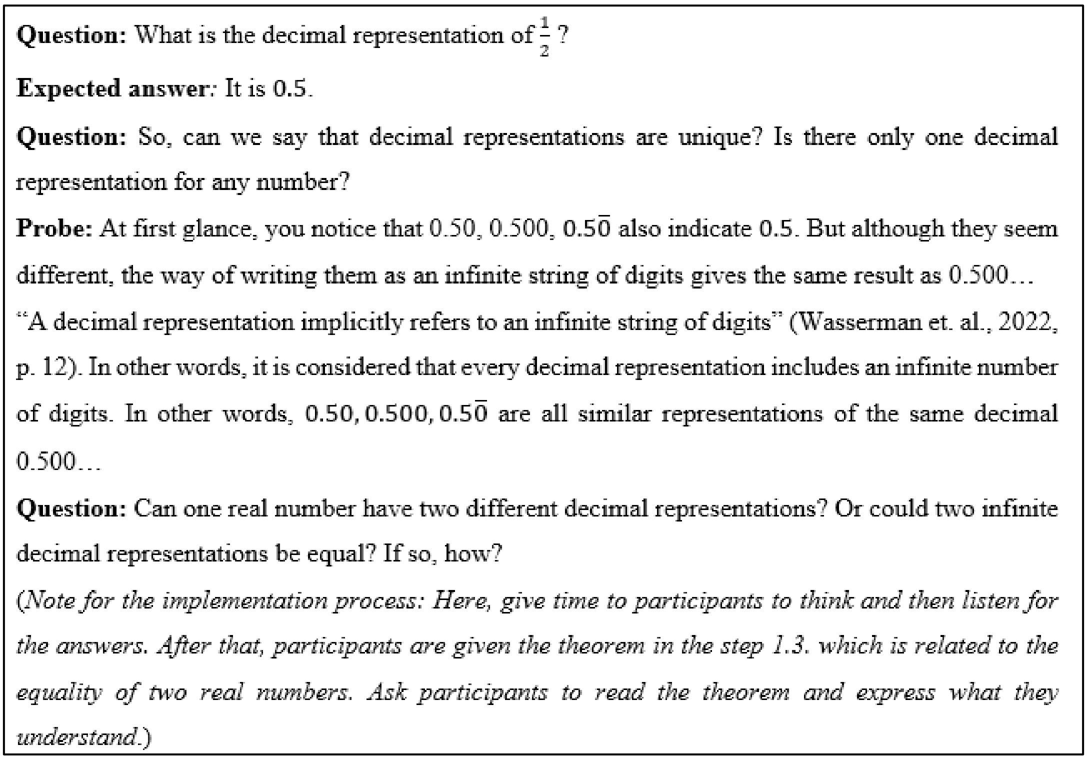
- What is the relationship between 1. 599.. and 1.6? If they are equal, how? If they are not equal, how?
- How do you think students perceive 1. 599..? What do you think?
- What differences might occur for students in understanding the numbers -3, 2, 1, 6, and 1. 599...?
- Which subjects/ideas/topics in mathematics could be related to the relationship between 1. 599.. and 1.6?

Let’s try to write down all your ideas on the board.

Figure 3.3. The discussion questions in Step 1.1.

The discussion questions were designed to ascertain mathematics teachers' comprehension of the relationship between 1.5999... and 1.6 and their familiarity with students' understanding of the subject matter. In addition, the objective is to examine how they establish the interconnections between 1.5999... and 1.6 and other mathematical concepts. During the course of the interview study and in the search of the literature, it was observed that a number of different approaches to explaining the relationship between 0.999... and 1 were put forth, including the real numbers, the limit, the sum of a series, and so on. Subsequently, activities are conducted pertaining to the fundamental manner of conceptualizing the relationship between 1.5999... and 1.6, as indicated in design principle 3, which is through decimal representation.

In Step 1.2, a progression from the simple to the complex was initiated as suggested by design principle 3. Consequently, the principal content of this session started with decimal representations, which represent the most elementary perspective on the relationship between 1.5999... and 1.6. The term "elementary" pertains to the approach most suitable for students at the lowest grade level. The content addressed in this step is attached to the figure below.



Question: What is the decimal representation of $\frac{1}{2}$?

Expected answer: It is 0.5.

Question: So, can we say that decimal representations are unique? Is there only one decimal representation for any number?

Probe: At first glance, you notice that 0.50, 0.500, $0.5\bar{0}$ also indicate 0.5. But although they seem different, the way of writing them as an infinite string of digits gives the same result as 0.500…

"A decimal representation implicitly refers to an infinite string of digits" (Wasserman et. al., 2022, p. 12). In other words, it is considered that every decimal representation includes an infinite number of digits. In other words, $0.50, 0.500, 0.5\bar{0}$ are all similar representations of the same decimal 0.500…

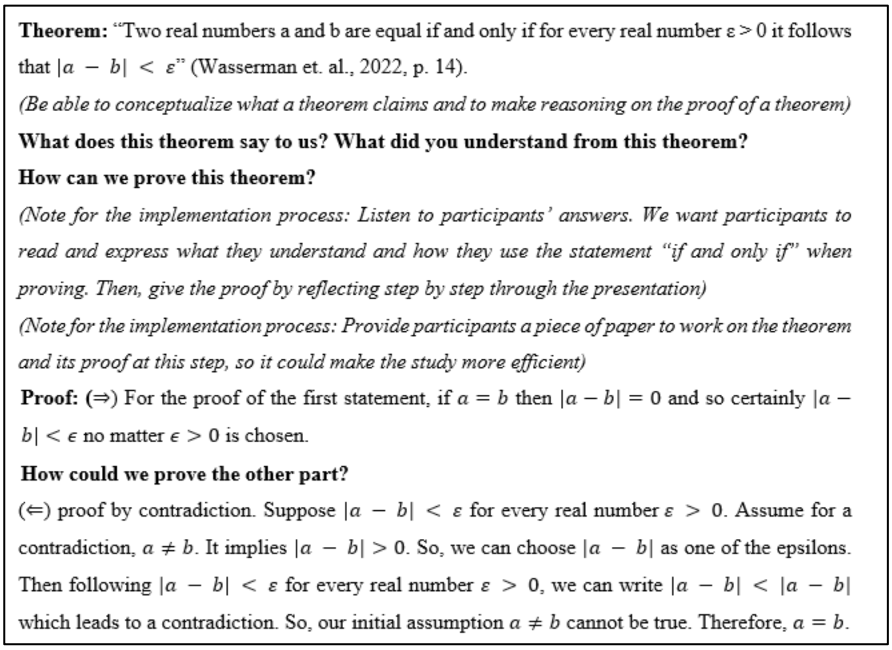
Question: Can one real number have two different decimal representations? Or could two infinite decimal representations be equal? If so, how?

(*Note for the implementation process: Here, give time to participants to think and then listen for the answers. After that, participants are given the theorem in the step 1.3. which is related to the equality of two real numbers. Ask participants to read the theorem and express what they understand.*)

Figure 3.4. The activities in Step 1.2

The activities in Step 1.2 were adapted from a section of a book chapter devoted to the topic of Equivalent Real Numbers and Infinite Decimals by Wasserman et. al. (2022). Initially, the presentation addressed the question of whether decimal representations have a unique way of being written and subsequently the formal definition of decimal representations in mathematics. It was deemed essential to allocate sufficient time for participants to express their ideas.

In Step 1.3 of the study, the mathematics teachers (MTs) were initially prompted to consider a theorem that asserts the equality of two real numbers. Subsequently, they were encouraged to engage in reasoning process to envision the proof for the aforementioned theorem. An insight into the nature of the theorem facilitated a more favorable progression in the reasoning employed in its proof. The activities undertaken in this phase are depicted in the figure below.



Theorem: "Two real numbers a and b are equal if and only if for every real number ε > 0 it follows that $|a - b| < \varepsilon$" (Wasserman et. al., 2022, p. 14).

(Be able to conceptualize what a theorem claims and to make reasoning on the proof of a theorem)

What does this theorem say to us? What did you understand from this theorem?

How can we prove this theorem?

(Note for the implementation process: Listen to participants' answers. We want participants to read and express what they understand and how they use the statement "if and only if" when proving. Then, give the proof by reflecting step by step through the presentation)

(Note for the implementation process: Provide participants a piece of paper to work on the theorem and its proof at this step, so it could make the study more efficient)

Proof: (⇒) For the proof of the first statement, if $a = b$ then $|a - b| = 0$ and so certainly $|a - b| < \epsilon$ no matter $\epsilon > 0$ is chosen.

How could we prove the other part?

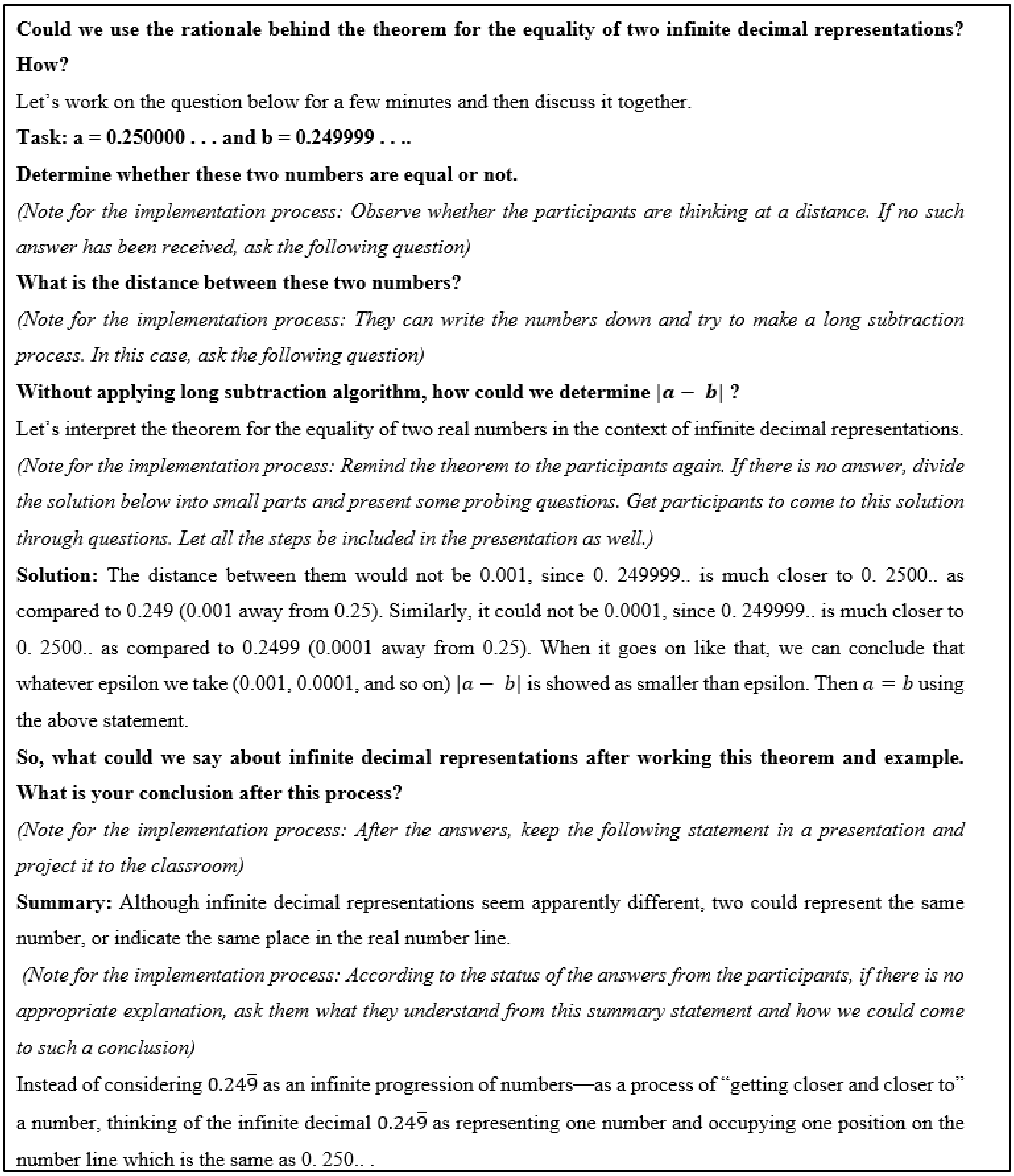
(⇐) proof by contradiction. Suppose $|a - b| < \varepsilon$ for every real number $\varepsilon > 0$. Assume for a contradiction, $a \neq b$. It implies $|a - b| > 0$. So, we can choose $|a - b|$ as one of the epsilons. Then following $|a - b| < \varepsilon$ for every real number $\varepsilon > 0$, we can write $|a - b| < |a - b|$ which leads to a contradiction. So, our initial assumption $a \neq b$ cannot be true. Therefore, $a = b$.

Figure 3.5. The theorem and following activity regarding its proof in Step 1.3

This theorem was also derived from the work of Wasserman et al. (2022), who proposed the 'ε criteria' as a means of determining the equivalence of real numbers. Furthermore, Expert 1 highlighted the significance of this theorem while elucidating the differentiating role of the equal sign in the context of limits and in other mathematical contexts. In the initial phase of the activity, the MTs were first permitted to read and reflect upon the theorem individually. The objective of this activity was to ascertain the essence of the theorem and to identify the underlying idea.

Although there are multiple approaches to proving this theorem, the most direct method was selected in the implementation. As illustrated in the figure above, during the execution of Step 1.3, the MTs were encouraged to dedicate a considerable amount of time to working on their respective papers. Subsequently, the participants disseminated their ideas to the other participants. This stage was pivotal in terms of encouraging teachers to engage with rigorous and deductive reasoning, as well as justifications, which were identified in design principles 1 and 2 as essential for the development of advanced mathematical knowledge.

The following step, Step 1.4, was an application of the idea behind the theorem and its proof on the real numbers to another context on the infinite decimal representations. The proposed task as shown in figure below necessitated to make reasoning in a situation involving infinity, as design principle 1 suggested. Besides, MTs were motivated to follow a thinking path through finite numbers or situations ‘accessible to five senses’ as described in design principle 2.



Could we use the rationale behind the theorem for the equality of two infinite decimal representations? How?

Let's work on the question below for a few minutes and then discuss it together.

Task: a = 0.250000 . . . and b = 0.249999 . . ..

Determine whether these two numbers are equal or not.

(Note for the implementation process: Observe whether the participants are thinking at a distance. If no such answer has been received, ask the following question)

What is the distance between these two numbers?

(Note for the implementation process: They can write the numbers down and try to make a long subtraction process. In this case, ask the following question)

Without applying long subtraction algorithm, how could we determine $|a - b|$?

Let's interpret the theorem for the equality of two real numbers in the context of infinite decimal representations.

(Note for the implementation process: Remind the theorem to the participants again. If there is no answer, divide the solution below into small parts and present some probing questions. Get participants to come to this solution through questions. Let all the steps be included in the presentation as well.)

Solution: The distance between them would not be 0.001, since 0. 249999.. is much closer to 0. 2500.. as compared to 0.249 (0.001 away from 0.25). Similarly, it could not be 0.0001, since 0. 249999.. is much closer to 0. 2500.. as compared to 0.2499 (0.0001 away from 0.25). When it goes on like that, we can conclude that whatever epsilon we take (0.001, 0.0001, and so on) $|a - b|$ is showed as smaller than epsilon. Then $a = b$ using the above statement.

So, what could we say about infinite decimal representations after working this theorem and example. What is your conclusion after this process?

(Note for the implementation process: After the answers, keep the following statement in a presentation and project it to the classroom)

Summary: Although infinite decimal representations seem apparently different, two could represent the same number, or indicate the same place in the real number line.

(Note for the implementation process: According to the status of the answers from the participants, if there is no appropriate explanation, ask them what they understand from this summary statement and how we could come to such a conclusion)

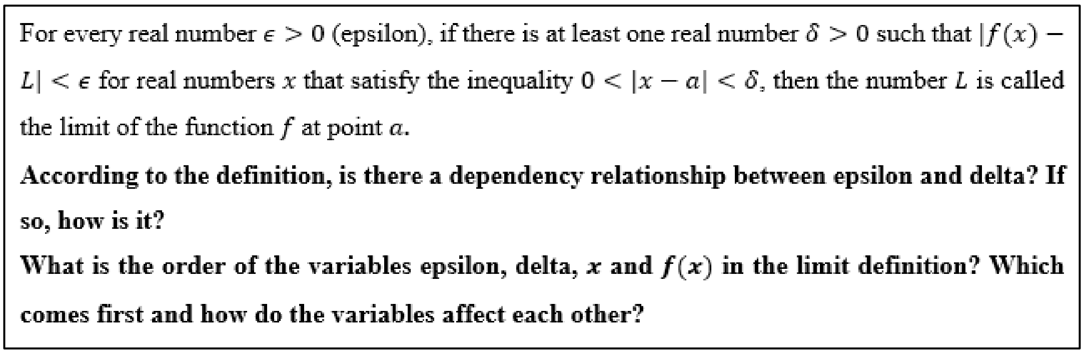
Instead of considering $0.24\overline{9}$ as an infinite progression of numbers—as a process of "getting closer and closer to" a number, thinking of the infinite decimal $0.24\overline{9}$ as representing one number and occupying one position on the number line which is the same as 0. 250.. .

Figure 3.6. The process of Step 1.4

This stage as illustrated in Figure 3.6 involved the application of a reasoning process, originally developed to prove a theorem, to an alternate context. It was therefore recommended that the reasoning inherent in the theorem be employed, rather than a novel solution to the task being proposed. To make more detail, in this step, a way of thinking like "What could be the distance between 0.24999... and 0.2500...? Could it be 0.001? Or could it be 0.0001?”. The fact that the distance could not be one of them, since 0.24999... is much closer to 0.2500... as compared to 0.249 and much closer to 0.2500... as compared to 0.2499 and so on. Then the participants made a connection with these numbers 0.001, 0.0001, etc. to the “for every epsilon” in the theorem as showing $|a - b|$ smaller than every epsilon and got the conclusion of $a = b$ through the theorem. As a result, in order to arrive at the desired solution, the participants were motivated to engage in a process of thinking process, whereby they would consider the distances involved and utilize individual cases as a means of achieving the desired outcome.

The preceding paragraphs provided a comprehensive overview of the specifics of Session 1. This section will now present the details of Session 2. The essential steps in Session 2 and the corresponding design principles were previously presented in Table 3.6. For the sake of recall, the sequence of the activities that constituted Session 2 was as follows: first, the formal definition of limit was considered (Step 2.1); then, communication through various representations, including tables and graphs, was addressed (Step 2.2); finally, the meaning of equality in the context of limits was contemplated (Step 2.3). The following paragraphs present the details about these steps.

Step 2.1 contains a detailed examination of the formal definition of limit. The dependence relationship between epsilon and delta was emphasized both through the literature and appeared in the data of interview study (Expert 1 and 3). Besides, how the variables are ordered in the formal definition, or which one affects the choice of others and in which ways, was the second issue discussed in Step 2.1. The design principles, namely 1 and 2, informed the generation of the activities in Step 2.1. While principle 2 detailed the essential issues regarding the formal definition, principle 1 highlighted communication between words and symbols in addition to a focus on rigor. The ordering relationship between epsilon and delta appears in the formal definition of the limit with the statement, “for every number $\varepsilon > 0$, there exists a number $\delta > 0$”. Davis and Vinner (1986) refer to it as the “temporal order” (p. 295) and Adiredja (2021) generated four different questions asking the students' understanding of the temporal order. While preparing the sub-questions in Steps 2.1 and 2.2, I mainly adapted Adiredja's questions. The shared formal definition of limit and the related questions about it in Step 2.1 are presented in Figure 3.7 below.



For every real number $\epsilon > 0$ (epsilon), if there is at least one real number $\delta > 0$ such that $|f(x) - L| < \epsilon$ for real numbers x that satisfy the inequality $0 < |x - a| < \delta$, then the number L is called the limit of the function f at point a.

According to the definition, is there a dependency relationship between epsilon and delta? If so, how is it?

What is the order of the variables epsilon, delta, x and $f(x)$ in the limit definition? Which comes first and how do the variables affect each other?

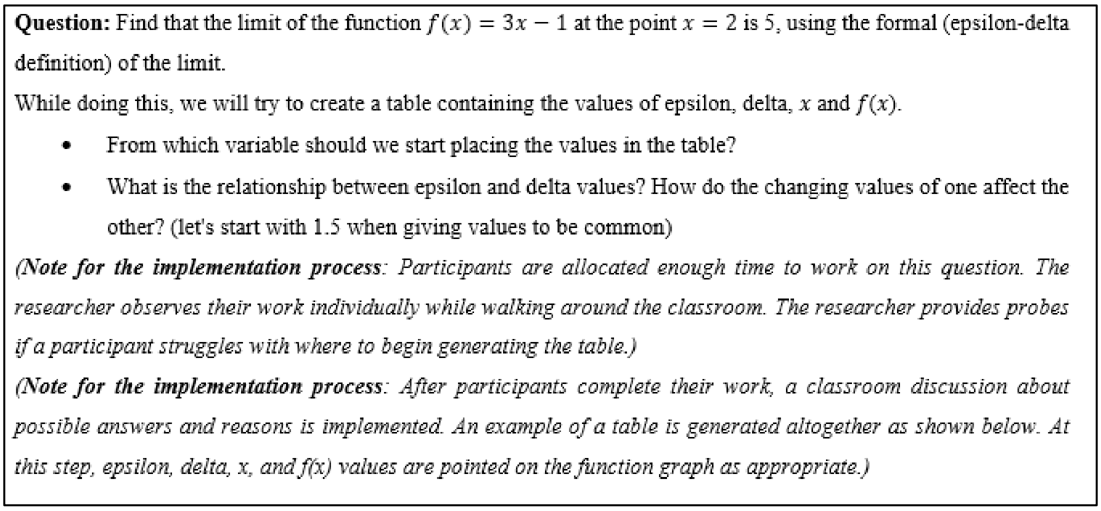
Figure 3.7. The formal definition and accompanying activities in Step 2.1

In this phase, prior to analyzing the dependence relationship between variables, the informal definition of limit, as previously presented in the content question test, was shared with the participants. They were then asked to compare the informal and formal definitions of limit in terms of their comprehensiveness and the necessary conditions for a definition of limit to be considered sound. This practice served to shed light on the participants' understanding of the affordances and limitations of mathematical knowledge, as well as their understanding of the concept of limit as a double win.

During the course of the discussions at Step 2.1, on occasion, probes were directed towards the teachers. For example, when I inquired as to whether there existed a dependence relationship between epsilon and delta in the formal definition of the limit, Teacher Cansu made a comment indicating that they were both greater than zero. In order to ensure that the discussion remained focused on the topic at hand and to facilitate a productive exchange of ideas, I probed further to ascertain whether there was a dependence relationship between epsilon and delta in the formal definition of the limit. To elucidate the matter further, an additional inquiry was posed, "Let us examine the elements situated between these two points. As you can discern, the definition employs quantifiers, encompassing two distinct distances and referencing both $x - a$ and $f(x) - L$."

The subsequent phase, labelled Step 2.2, involved the utilization of diverse representations, including tables and graphs, to facilitate an examination of the dependences between the variables within the formal definition and the manner in which they are related. Therefore, this step was still related to the issues addressed in the preceding step. In this step, design principle 2 was also considered to be a principal contributor to the design process. As the step remained connected to the comprehension of the formal definition of the limit, the other design principle that was pertinent to this step was principle 3, which posits that the teaching sequence must be supported through the presentation of a variety of explanations to facilitate learning.

An ordinary limit problem asking the limit of a function at a point was the mediating task for Step 2.2. In other words, the dependence relationship between the variables in the formal definition of limit was worked in detail through this task. But here, participants were motivated to work at a table. The aim was to support participants build links among various representations for the same question and conclude how the choice of epsilon and delta depends on each other. Juter (2007) advocated the necessity of thinking about the dependence relationship. Also, in the interview, Expert 1 remarked on the importance of being aware of this relationship for mathematics teachers in the context of limits. The task used in this step (see Figure 3.8) was modified from Expert 1’s associated comments.



Question: Find that the limit of the function $f(x) = 3x - 1$ at the point $x = 2$ is 5, using the formal (epsilon-delta definition) of the limit.

While doing this, we will try to create a table containing the values of epsilon, delta, x and $f(x)$.

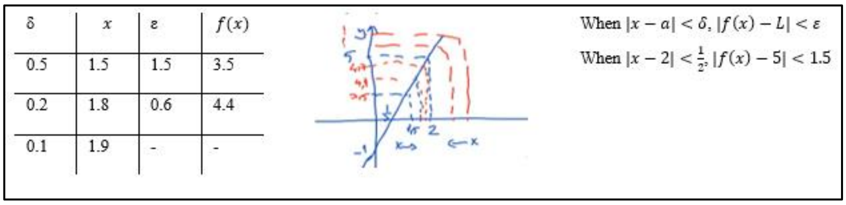
- From which variable should we start placing the values in the table?
- What is the relationship between epsilon and delta values? How do the changing values of one affect the other? (let's start with 1.5 when giving values to be common)

*(**Note for the implementation process**: Participants are allocated enough time to work on this question. The researcher observes their work individually while walking around the classroom. The researcher provides probes if a participant struggles with where to begin generating the table.)*

*(**Note for the implementation process**: After participants complete their work, a classroom discussion about possible answers and reasons is implemented. An example of a table is generated altogether as shown below. At this step, epsilon, delta, x, and f(x) values are pointed on the function graph as appropriate.)*

Figure 3.8. The activity related to the use of various representations in Step 2.2

As indicated in the note accompanying Figure 3.8, the participants were allowed time to work on their papers in order to generate a table containing the variables of epsilon, delta, x, $f(x)$, and the values for them. Once the requisite time for study and observation had been allotted to each participant, a sample table was generated as shown in the figure below.



δ	x	ε	$f(x)$
0.5	1.5	1.5	3.5
0.2	1.8	0.6	4.4
0.1	1.9	-	-

When $|x - a| < \delta$, $|f(x) - L| < \varepsilon$

When $|x - 2| < \frac{1}{2}$, $|f(x) - 5| < 1.5$

Figure 3.9. Different representations practiced in Step 2.2

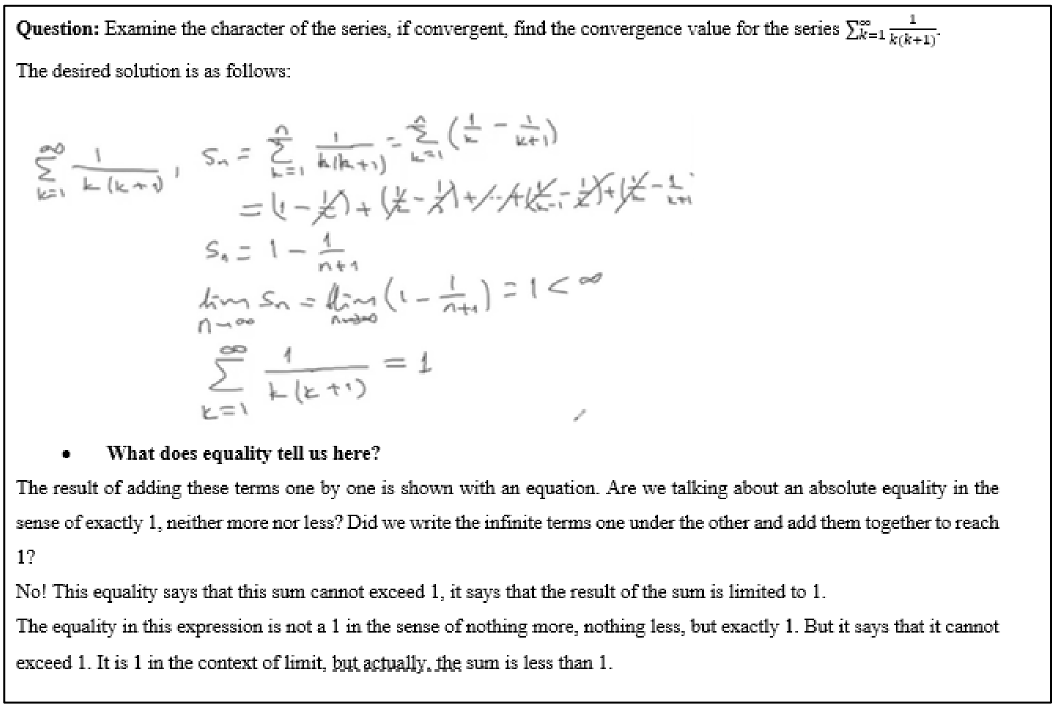
Furthermore, the graph of the function was used to illustrate the relationships between the variables, as shown in the middle of Figure 3.9. Finally, the other two representations were compiled as depicted on the far right of the figure above. In other words, a connection was made between all these variables and representations with the formal definition of limit. Apart from the ordering relationship between epsilon and delta, there is also an emphasis on the relationship between their ranges in the literature (Boester, 2008). So, a discussion of their range values was also included in this step by using the expert's example from the interview.

The last step of Session 2, Step 2.3, was devoted to an examination of the concept of equality in the context of limits, with the aid of illustrative examples. The distinct meaning of equality as compared to other contexts in mathematics was a recurrent theme in the interview data, as articulated by Experts 1, 2 and 5. When viewed through the lens of the design principles introduced in the conceptual framework before, the principal design principle guiding the activities in Step 2.3 was Design Principle 2, which recommends focusing to the different meaning of equality in the context of the concept of limit. The activities conducted in Step 2.3 were as follows:

- Consideration of equality in the context of the limit.
- Presentation of an illustration of equality in a mathematical problem pertaining to the limit
- Presentation of a second example concerning series, demonstrating the application of equality in the context of the limit.

At the outset of the process in Step 2.3, the MTs were first asked to consider the manner in which the concept of equality manifests itself in the context of limits. Subsequently, the participants were presented with an example involving the equality sign. The example was "Does the function $f(x) = \frac{1}{(x-1)^2}$ have a limit at $x = 1$?". The graph of the function was constructed in which a vertical asymptote arising at the point $x = 1$. In this context, it was conveyed to the MTs that, in an ordinary secondary school setting, an individual would encounter a scenario where the right and left limits are equal to each other and are infinite. In accordance with the conventional notion of the concept of limit, which is "if the left-hand and right-hand limits are equal, then this function has a limit at that point", it was determined that the function must have a limit at $x = 1$. However, in order to demonstrate the existence of the limit, it is necessary to establish an epsilon-delta relation such that when $|x - 1| < \delta$, $|f(x) - L| < \varepsilon$, but it is not feasible to define such an interval for $f(x)$ in this context.

Subsequent to the aforementioned activities, the participants were reminded of the series and the convergence of series as a continuous segment of Step 2.3. Then asked, "Examine the character of the series, if convergent, find the convergence value for the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$". Rather than expecting a solution from the participants, it was shared with them as illustrated in the figure below with the accompanying activities.



Question: Examine the character of the series, if convergent, find the convergence value for the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$.

The desired solution is as follows:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}, \quad S_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1}\right)$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \cdots + \left(\cancel{\frac{1}{k-1}} - \cancel{\frac{1}{k}}\right) + \left(\cancel{\frac{1}{k}} - \frac{1}{k+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{n+1}\right) = 1 < \infty$$

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = 1$$

- **What does equality tell us here?**

The result of adding these terms one by one is shown with an equation. Are we talking about an absolute equality in the sense of exactly 1, neither more nor less? Did we write the infinite terms one under the other and add them together to reach 1?

No! This equality says that this sum cannot exceed 1, it says that the result of the sum is limited to 1.

The equality in this expression is not a 1 in the sense of nothing more, nothing less, but exactly 1. But it says that it cannot exceed 1. It is 1 in the context of limit, but actually, the sum is less than 1.

Figure 3.10. An activity from Step 2.3

The activity depicted in the figure above constituted the second example provided to MTs for consideration of changing meaning of equality in the context of the notion of limit. In the example, although a number 1 was produced as a result of the prescribed procedures, the sum of these numbers could not be 1. Rather, it represents a boundary for this process. This concludes the description of the teaching sequence. The subsequent section provides a concise overview of the ongoing analysis conducted in the transition from Session 1 to Session 2. Thereafter, the methods section proceeds with an examination of the data collection instruments.

3.5.1.1 Ongoing analysis from Session 1 to Session 2

Upon introduction to a theorem proving the equality of 1.5999... and 1.6, the teachers demonstrated rationality in their assessment and accepted the truth of the steps presented. Nevertheless, they remained unconvinced. The theorem stated that the number of ε must be constrained to an extremely small value. Nevertheless, at the conclusion of the discussion, they asserted that the utilization of such an ε in the definition could be highly ambiguous. The meaning of epsilon was elucidated as being "closer but bigger than zero." Consequently, at the conclusion of the initial session, it was my intention to prioritize an examination of the significance of epsilon and its utilization in the formal definition of the limit, as a preliminary step. Furthermore, it proved challenging to persuade participants about the concept of equality. Consequently, I opted to discuss the notion of equality in accordance with the design principles. The interpretation of equality in this context would be elucidated through illustrative examples. In the interviews, there had already been some emphasis on this issue. Therefore, the examples could be derived from these interviews. The next part explains the data collection tools of the study.

3.5.2 Data collection tools

The data collection of the need analysis study in the preliminary phase was described in the related section previously. This part only devoted to the data collection tools of the main study in the exploration phase, which were questionnaire, content questionnaire test, pre and post-tests, classroom observations, video, and audio recordings of classroom implementation, written in-class works of MTs, and fieldnotes.

3.5.2.1 The demographic information questionnaire

At the very beginning of implementation, a questionnaire was administered to participants. One week before the first meeting, the researcher delivered the questionnaire, in addition to the content questionnaire test and pre-test, to participants in printed format. The participants submitted the documents that they had completed as a photo or a pdf document at least one day prior to the starting date of the lessons and presented the written versions of these documents at the first meeting. So, the researcher had enough time to view their responses on the questionnaire before the implementation process began.

The Demographic Information Questionnaire (see Appendix C) consisted of five questions regarding the professional experience of participants. Since mathematics teachers working currently in secondary schools were welcomed for the main study, they were asked about the university and departments that they graduated from, the grade levels that they taught before, and whether they attended to extra professional mathematical activities after their graduation.

3.5.2.2 Content questionnaire test

The Content Questionnaire Test (see Appendix A) comprised four questions which were designed to elicit information on the most essential issues regarding the content covered in the teaching sequence. The participants were asked to define the concept of a limit, to evaluate the clarity and quality of an informal definition of limit (which was "as x approaches c, the value of $f(x)$ approaches L") and to explain the meaning of equality. The rationale behind inquiring about the meaning of equality was the emphasis placed on its significance in the context of limits during the interview study. Lastly, a question about real numbers was included due to the multifaceted nature of the relationship between 0.999... and 1, as previously discussed in the literature and further elaborated upon by the experts during the interviews.

3.5.2.3 Pre- and post-tests

The pre-test comprised five questions, which are presented in Appendix B for reference. The questions were designed to gain insight into the participating mathematics teachers' comprehension of the concept of limit in relation to the relationship between 1.5999... and 1.6. The participants were asked to indicate whether they considered the two numbers to be equal and, if so, to identify the mathematical concepts with which this issue was associated. Additionally, their difficulties or misunderstandings were investigated through questioning on their understanding of students' understanding. Lastly, their pedagogical approaches were examined in relation to their attempts to convince a student of the equality of these two numbers or their actions towards a student claiming the inequality of the numbers.

The post-test was administered one week after the implementation of the teaching sequence to prevent the influence of the teaching sequence on the respondents' answers. In the post-test (see Appendix D), as in the pre-test, a few questions were posed, with the numbers 7.999… and 8 at the center. In some parts of the post-test, respondents were required to construct classroom dialogues between a teacher and a student. The questions in the pre- and post-test were open-ended. Both the pre-test and post-test questions were developed by the researcher, with the central focus being the relationship between 0.999... and 1. The construction of the questions in the pre- and post-tests was informed by the related literature on 0.999..., and the concept of limits, and the findings of the need analysis study.

3.5.2.4 Video recordings of the sessions

The implementation of the teaching sequence was video recorded with two cameras since there was no observer other than the researcher herself. Despite the deployment of two cameras, one of the cameras was unable to record due to a technical problem. In the initial session, the camera was oriented towards the classroom board. In the subsequent session, however, the camera was adjusted to point towards the participants, and photographs were taken of the work on the board at key points. After the implementation, the researcher transcribed the video recordings by tracking the whole classroom dialogues second by second and by monitoring the participants' voices person by person. In total, approximately two and a half hours of recordings were transcribed for use in the data analysis process.

3.5.2.5 Written in-class work and field notes

Participants were required to work on a paper for not all, but some certain steps in the teaching sequences. The researcher had a need for this tool during the pilot study. Working on a paper was considered helpful in stimulating participants' mathematical thinking process and solution strategies, supporting their engagement in the tasks, and having a productive classroom discussion. Participants' written in-class works were collected and used as a supportive data collection tool in addition to audio/video records.

Furthermore, the researcher kept a record of their observations as field notes at the end of each instructional phase. The field notes included detailed observations of the participants' engagement and attitude throughout the teaching sequence, their level of involvement in tasks and discussions, and any instances from participants' work or statements that could be crucial for subsequent data analysis.

3.5.3 Data analysis

The methodology employed for the data analysis of the preliminary need analysis study has been previously delineated in the related section. This section presents the data analysis for the main study conducted during the exploration phase. The objective of this part of the study was to document the characteristics of the difficulties experienced by mathematics teachers in a teaching sequence designed to promote their advanced mathematical knowledge of the concept of limit. For this reason, a coding frame was generated to describe the nature of teachers' difficulties through pre- and post-test results and their performance during the implementation. This was done through content and thematic analysis. Content analysis is a method of identifying patterns within the content of the data sets (Creswell, 2013). In order to ensure the acquisition of a "valid" coding frame, Schreier (2012) posits that the categories must be sufficiently reflective of the concepts employed in the research question (p. 7).

In accordance with the data-driven approach to developing the coding frame through content analysis, some dimensions were initially identified based on Design Principle 2 of the conceptual framework (see Section 2.7). These included concepts such as "limit" and "equality." The remaining dimensions were derived through the researcher's active engagement with the data. So, in the initial stages of the data analysis process, all the video data was transcribed verbatim. The transcriptions and written works produced by the teachers during sessions one and two were transferred to the MAXQDA 2022 (VERBI Software, 2021) as a single project file. The project file also included the participants' responses to the questionnaire, the pre-test, the content questionnaire test and the post-test.

After that, the data was documented in a descriptive manner through verbal explanations, including a brief interpretation of the data, with the objective of facilitating a comprehensive understanding of the context of the data and characterizing it. However, the process necessitates a more thorough examination and interpretation of the available data. Consequently, the data was subjected to analytical coding in accordance with the advanced mathematical knowledge of limit and limit-related math ideas mentioned in the conceptual framework section. This approach entailed a process of coding that originated from the descriptive analysis of the data, rather than from the raw data itself. This methodology was employed to enhance the researcher's engagement with the data on a repetitive basis and to prevent the inadvertent omission of crucial elements. Upon completion of the coding process, a coding table was generated, comprising the relevant codes, including those related to the concepts of limit and equality, and the corresponding points focused in the analysis of the data.

Once the researcher had completed the coding process of the data in accordance with the limits and limit-related ideas from the perspective of advanced mathematical knowledge, a meeting was held with a mathematics educator who was conversant with the literature on mathematics education field and had obtained a master's degree in mathematics. The aim of this meeting was to discuss the process of the coding process, to gain insight from the educator's expertise on the subject matter, and to identify all the ideas within the data related to limits and limit-related ideas. The coding table and illustrative examples from the data were initially presented by the researcher. Subsequently, the mathematics educator was requested to undertake the coding of the pertinent data in accordance with the specified coding table, addressing certain inconsistencies that had arisen during the researcher's own coding process.

Upon completion of the coding process by the mathematics educator, a meeting was arranged to address any discrepancies that had arisen. One such discrepancy pertained to the classification of data segments, specifically whether the findings related to the concept of equality must be presented under the findings related to the concept of limit or individually. Some of the evidence relating to the concept of equality was found to be inextricably linked with that specifically presented for the concept of limit. Following a discussion of the relevant literature and the findings of the need analysis study, the researcher and the mathematics educator were able to reach a consensus on this matter, agreeing that the findings of these two concepts should be presented separately. This was due to the strong emphasis on the role of equality in the context of limits that was identified in the need analysis study. The themes emerged from this part of the study was given in the table below:

Table 3.7 Themes emerged in the exploration phase and their descriptions

Themes	Description

Table 3.7 (cont'd)

Understanding limit as "approach" and "getting as close as possible"	The interpretation and comparison in terms of the affordances and limitations of the different definitions of limit The un/attainability of the limit value
Building the relationship among variables within limit incorrectly	Understanding of the relationship between the quantifiers "for every" and "there exists" in the definition of the limit Understanding of the relationship between epsilon and delta in the definition of the limit
Misconceived covariational relationship between variables in the definition of limit	Describing the covariational relationships between variables
Challenges in employing equality in the context of limit	The interpretation of the relationship between the numbers such as 0.999… and 1 The use of equal sign while identifying the limit The interpretation of the equality while referring to the concept of limit
Inadequate use of symbols in the context of limit	The adoption of symbols in the teachers' utterances and statements

As illustrated in Table 3.7, the themes in the exploration phase were characterized under five aspects, focusing on the concept of limit from the perspective of advanced mathematical knowledge. In this context, the difficulties experienced by MTs regarding the concept of limit and equality, and their use of symbols throughout the study, were subjected to analysis.

3.6 Researcher Role and Trustworthiness

In the study, the researcher served as the implementer and the observer of the implementation of the teaching sequence. Lincoln and Guba (1985) posit that credibility, which corresponds to internal validity as a scientific term, is one of the means of ensuring the trustworthiness of a study. The term 'credibility' refers to the extent to which a researcher can be considered to have presented the findings of a study in an accurate and truthful manner (p. 79). A method for assuring the credibility of the study is facilitated through the utilization of an approach known as triangulation. In the study, the principle of triangulation, whereby the consistency of findings across different data sources is established, was supported through the use of a range of data sources, including teachers' in-class written works, transcripts, field notes, and classroom discussions.

During the course of the discussions, the researcher endeavored to ascertain the thoughts of all participants. Subsequently, the researcher prompted the participants to engage in reflection on the questions or issues presented. In certain phases (e.g., Step 1.3, Step 2.2), participants used a sheet of paper to focus their thoughts and ideas, which was also used as a data collection tool in the study. When a teacher sought to provide a hasty response to a question, I intervened to allow other participants an opportunity to reflect on the matter. Upon the necessity of probes, the researcher posed supplementary questions or encouraged reflection through the introduction of contradictory statements, including the phrases "what else" or "could it be."

The reliability of the research, which is another means of establishing trustworthiness, is contingent upon the reproducibility of the results under identical conditions (Creswell, 2013). To this end, field notes and video recordings were employed. Although the researcher was the sole observer during the implementation, video recordings of the sessions were provided. Furthermore, the selection process of the participants and the preliminary and exploration phases of the design process were explained in detail. Researchers interested in comparing the study data and findings with their own work will find this easily achievable.

3.7 Ethical Considerations

In the course of educational research, ethical considerations may arise, particularly with regard to the manner in which data is collected and subsequently analyzed and interpreted (Creswell & Creswell, 2018). One of the ethical issues that could arise during the data collection process is deception. This occurs when the researcher's true purpose for the study is not revealed to the participants. Such a deception did not occur in any phase of the current study. The primary objective of the study was explicitly delineated for the participants, who were also furnished with the opportunity to review it through the informed consent forms. Furthermore, researchers are obliged to treat participants with respect and ensure their safety and well-being. In this regard, the study was submitted to and approved by the institutional review board, and the METU Human Subjects Ethics Committee granted approval (see Appendix E), thereby affirming the risk-free nature of the study in terms of the rights of participants as a human (Creswell & Creswell, 2018). Furthermore, confidentiality is an additional ethical consideration that must be taken into account during the data collection process. The data collected from participants was not shared with any third parties, and pseudonyms were used in place of the participants' real names to ensure confidentiality. In addition, the study presented findings and evidence derived from the utilization of disparate data collection instruments (Creswell & Creswell, 2018).

CHAPTER 4

FINDINGS

The purposes of the present study were (1) to explore the essential advanced mathematical knowledge of limit for mathematics teachers from the experts' points of view, and (2) to document the characteristics of the difficulties experienced by mathematics teachers in a teaching sequence designed to promote their advanced mathematical knowledge of the concept of limit.

As evidenced by the study's stated objectives and research questions, the study was designed with two distinct but complementary aims. Firstly, an interview study, for the questions see Appendix F, with experts was conducted to ascertain the advanced mathematical knowledge required by mathematics teachers in the context of limits. The findings section initially presented the findings related to this topic. Secondly, a teaching sequence was devised and implemented with principles promoting advanced mathematical knowledge in the context of limits. The findings related to the characteristics of the difficulties experienced by mathematics teachers in a teaching sequence were reported through the data obtained from the pre-test, content questionnaire test, post-test, and data collected during the implementation phase.

4.1 Findings of Need Analysis Phase

The findings of this part of the study were presented under three sections considering the components of the framework that directed the study.

4.1.1 Drawing connections of limit across mathematical domains

In the first place, the captured connections of limit across mathematical domains were summarized in Figure 4.1 below and then provide examples from experts' statements.

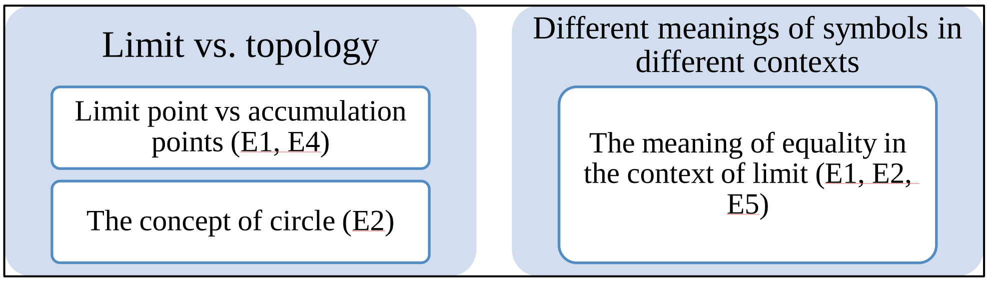


Figure 4.1. The identified categories for the drawing connections of limit across mathematical domains

As shown in Figure 4.1, Experts 1, 2 and 4 mentioned limit, mainly known as a calculus term, with some topological terminologies. Specifically, Expert 1 distinguished between limit and accumulation points in topology and noted that a sequence could have some accumulation points but not a limit. Besides, Expert 4 acknowledged that a teacher must know that "for a sequence to have a limit, it must first have an accumulation point". Lastly Expert 2 used the circle as a topological term and expressed that "having a limit is like that after a certain point you stay completely inside in a circle". Expert 2 further stated:

> There is a point, and there is a circle around it. How big or small you make this circle depends on your goals. This will specify our matter of error. If all the values you get after a point stay within this circle and if it is a function, you can think of it as the values that the function takes. If it's just a sequence of numbers, you can say this is the limit of the sequence. Staying completely inside this circle after a point... It's like being trapped inside the circle. If you provide this, it's ok. The radius of the circle will be the error. From here we go to the epsilon-delta conversation.

He also related this term to the quantifiers in the formal definition of limit as follows:

> The part we call 'for every' is actually the radius of the circle. You can get it as small or as big as you want. However, once you set the radius, after a certain point, which is the ‘there exists’ part, all those values have to stay inside the circle, they cannot exceed its boundaries. No matter what you get, because we are saying 'for every' here, it's not a problem if you take it big enough anyway, the main problem here is to provide same thing when you take it small enough. No matter how big you take, at some point all the values will stay inside the circle. When we talk about epsilon-delta, the range we give represents that circle. Of course, when we say circle, we are in a sense talking about two dimensions in a sense. It is a range for one dimension.

Here, the comments of these three participating experts could be associated in the following way. While Expert 1 drew attention to the fact that an accumulation point does not necessarily imply a limit, Expert 4 highlighted that an accumulation point is necessary for the sequence to have a limit. Lastly, Expert 2, in addition to supporting these two ideas, claimed that this idea could be reinforced by the term circle. His statement “for every epsilon”, referring to the radius of the circle he considered, suggested that he was thinking about the boundaries of the inequality, $|f(x) - L| < \varepsilon$ with links to the inequality of $0 < |x - a| < \delta$ for which the values of delta would be dependent on the values of epsilon. In this regard, as Expert 4 pointed out, all those accumulation points would have to lie in within the boundaries of the inequality based on which one could determine whether a sequence would have limit or not. Expert 1’s ideas, though, are still in line with Expert 2 and 4’s explanations where Expert 2 stated specifically that “...after a certain point…” in the sense that not all accumulation points could point to the identification of the limit value, yet after some certain point one can consider its existence.

Moreover, Experts 1, 2, and 5 distinguished the meaning of mathematical symbols such as equality, less than or equal, addition, and division in the context of limit and other contexts of mathematics. For example, Experts 1 and 2 exemplified how the meaning of equality changes in limit compared to its ordinary meanings. Expert 1 claimed that a teacher must be able to think limit out of the ordinary arithmetic. He provided the rationale as that it is not feasible to account for these phenomena through the lens of four algebraic operations alone. Otherwise, s/he may encounter paradoxical situations. Besides, he gave an example for changing meaning of equality in the context of limit using a series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$ and an attached question like "examine the character of the series, and if it converges, find its converged value". For such a question, the respondent must find the limit of the sequence of the partial sums and, for this question it is found as 1 that is a finite number and indicates the convergence of the series. Here, the limit of the sequence of the partial sums corresponds to $\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1$ in number and Expert 1 attended to this expression. He noted that:

> The equality in this expression is not an absolute 1 in the sense of exactly 1, neither more nor less. But it says that it cannot exceed 1. In fact, this sum is less than 1. But we take it as 1 in the sense of limit. I have taken infinite terms and written them one under the other. It does not mean that it made exactly 1. The sum of these terms does not mean that it made exactly 1 in the sense of nothing more, nothing less, but exactly 1. But it says that it cannot exceed 1. It is 1 in the context of limit, but actually, the sum is less than 1.

Expert 2 pointed out the same issue in response to the equality of 0.999... and 1, as seen in the following excerpt:

> Is it equal to 1 or approximately 1? It is exactly 1. This is also a situation related to the understanding of real numbers. Here, 1 is not the limiting value or limiting number for 0.999... So, it didn't become a trap after a certain thing (referring to the circle analogy). It is equal to 1, we do not use the approximate 1 sign... But you can also see it as a limiting value. I mean this is one of the explanations. If you consider 0.9, 0.99, 0.999 etc. as a series of numbers, you

> say the last one is supposed to be 1. But this is again equal to 1, not 1 which is an approximation.

As seen from the excerpts, both Expert 1 and Expert 2 made a distinction between the equality to 1 as a numerical value and as a limiting value. Additionally, Expert 2 differentiated the meaning of ‘less than or equal to’ in drawing graphs and in limit. He exemplified how we consider the points where equality is achieved in graphs, but it does not necessarily mean that they will be equal in limit.

4.1.2 Gaining mathematical experience for developing problem-solving abilities in limit

Two categories that teachers need to gain mathematical experience for developing problem-solving abilities were identified in the context of limit as in Figure 4.2:

in which concepts,	**through what kind of activities,**
• understanding the definition of limit conceptually (E3, E4, E5) • exploring the elements within the formal definition critically (E2, E3, E4) • scrutinizing the covariational reasoning behind the definition of limit (E1) and thinking quantitatively about quantifiers	• to use graphs and make interpretations from graphs about limit (E1, E3, E5), • to give examples and non-examples (E1, E2, E5), • to build the correspondence between the intuitive understanding of limit and the formalization (E1, E2, E3, E4), • to manipulating formal definition by changing the place of quantifiers (E5), and • to use analogies (E2, E4).

Figure 4.2. The obtained categories regarding gaining mathematical experience for developing problem-solving abilities in limit

I explained these categories in the following sections.

Gaining mathematical experience in limit: In which concepts?

The experts mainly emphasized understanding the definition of limit conceptually, exploring the elements within the formal definition critically and scrutinizing the covariational reasoning behind the definition of limit. At this stage, experts' first focus was on the fact that although the limit was defined at a point, it necessitated looking at the behavior of the function near that point (e.g., E3, E4, E5). Secondly, some experts (e.g., E2, E3, E4) emphasized the role of quantifiers in the formal definition. For example, they reminded that the definition of limit involves a claim for every ε, but it should be checked if it works sufficiently for small values. Besides, Experts 2 and 3 claimed that students and even teachers sometimes attempted to find the best δ for each ε exactly, although it is sufficient to show the existence of a good enough solution for δ. Apart from that, Experts 1 and 5 pointed out a few issues regarding the role of ε-δ in defining limit as crucial for teachers. These were the use of ε-δ symbols properly in showing the existence or inexistence of limit and awareness of the falsifiability of the common belief that "if right -left limits are equal to each other, then it must be the limit".

As shown in Figure 4.2, one expert (Expert 1) strongly emphasized the importance of covariational reasoning in understanding the formal definition of limit. He highlighted its necessity in three aspects. First, while articulating how the limit of a function at a point "a" can be said to be L, he asserted a way of thinking about the relationship between $x - a$ and $f(x) - L$ in limit as "In case of $x - a$ remains smaller than δ, if $f(x) - L$ can be shown to be smaller than ε, then we say that the limit at that point is L". Additionally, he uttered the following statements:

> The closer the x values get to the variable a, the closer the $f(x)$ values get to a constant value on the y axis, such as L. The relationship between the two must be established. Of course, mathematics teachers do not need to explain a relationship between convergence rates, such as $f(x)$ approaches L at the same rate as x approaches a. But they should understand that there is a relationship between them.

The second context emphasized by Expert 1 was to build the covariational relationship between ε and δ in the limit definition. Here, he reminded the behavior of searching for δ based on a chosen ε and added the importance of establishing a thought process about "what kind of change is expected for δ in response to changing ε values". Expert 1 stated:

> We say, for example, if you choose the epsilon 1/50, what range should the delta be? Since we call it a delta dependent on the epsilon, s/he could find the delta as an exact numeric value depending on 1/50. What happens if it is 1/100, as a second case? Does the range of delta enlarge or get smaller? Or what happens if it is 1/10? Accordingly, s/he would compare what the range of delta should be. I also think that s/he would have compared epsilon and delta in both a relational sense and a numerical sense, so he would understand more.

Lastly Expert 1 took attention to follow the varying values of $f(x)$ across x values:

> When we ask what the limit is, we must specify a point (where does x go). The statement that the limit is 1 is meaningless. A function can have different limits at different points. Therefore, it is important to follow the changes of x versus $f(x)$ and state that they are not independent of each other.

In the excerpts above, although not limited, Expert 1's mention of the relationship between x versus $f(x)$ as well as specifying values of epsilon and delta as dependent on each other pointed out that he considered the changes in the values of independent variables such as epsilon in the change of dependent variables such as delta. His mention of the relationship between how fast $f(x)$ approaches to L dependent on how fast x values get closer to the variable "a" further pointed that Expert 1's consideration of limit was based on the covariance of quantities within intervals.

Gaining mathematical experience in limit: Through what kind of activities?

The experts interviewed in this study identified essential practices involving utilizing graphs and making interpretations from them about limit (E1, E3, E5), giving examples and non-examples in interpreting the existence of limit (E1, E2, E5), building the correspondence between the intuitive understanding of limit and the formalization (E1, E2, E3, E4), playing with formal definition by changing the place of quantifiers (E5) and using analogies (E2, E4).

Experts 3 and 5 initially asserted that even without any technological tools, only the graphs themselves can be helpful for a clear understanding of the concept of limit. Moreover, Expert 3 stated that drawing graphs of univariate functions can help pre-service teachers reason about limits and continuity and can also help them make smooth transitions to the concepts of limit and continuity in multivariate functions. He justified his idea: “Because symbols are insufficient to explain exactly what is happening. It is crucial to draw the function's graph and understand what the limit means there and how it can be guessed just by looking at the graph.” Besides, in response to a hypothetical student who believed that for a function to have a limit at a point, it must be monotonically decreasing or increasing around it, Expert 5 suggested to present an oscillating function of $\sin\frac{1}{x}$ as a non-example.

Another mathematical practice was to relate intuition and informal understanding to rigor. Here, experts focused on transferring verbalization and visualization to formalization. For example, Expert 3 highlighted the ambiguity behind using a statement such as ‘when x is close to a, $f(x)$ is close to L’ for limit. He detailed this issue: “There is no objective measure of closeness; it is a subjective evaluation. Cauchy compensates for this deficiency with numerical values.” Accordingly, he related the term closeness to the absolute value and so distance, in the formal definition, and explained how conceptualizing distances in limit enables us to extend the concept of limit to all metric spaces such as bivariate, functions, function spaces where distance can be defined. Regarding visualization and formalization, Expert 4 expressed,

> Algebra explains everything, but we need to use geometry to make sense. Using geometry means the employment of the metric system. For the number line, the area encompassing the selected point becomes an interval; if you

> take the plane as a basis, the area encompassing the selected point becomes an open disc. As such, you need to illustrate the aforementioned open disc or interval, to make sense of it.

He emphasized the importance of visualization through geometry and then formalization of the sensed things through algebra. Besides, Expert 1 remarked that, instead of an understanding like "What else can the limit of $3x - 1$ at $x = 2$ be other than 5?", the process should be visualized in a table that contains changing values of variables like ε, δ, x and $f(x)$. Based on this argument, he further added that some further discussions must hold with students how these variables correspond to the statements in the formal definition of limit. Apart from that, Expert 5 explained how he manipulated the formal definition of limit in his lessons as follows: “In my lessons, I practice by changing the order of the quantifiers. I wrote the sentences worse and asked the students to find counterexamples for them.”

As a last mathematical activity, experts used the analogies in expressing the intuitive meaning of limit (E2) in explaining the issue of whether the limit was attainable or not (E2, E4) and about the narrowness of neighborhoods in the definition of limit (E4). For example, Expert 2 expressed that there is no concern regarding the attainability of limit through the term of the ‘margin of error’ in measurements from daily life. He stated that:

> The aim of the limit is not to be attained. In packaged foods, the value of the weight is expressed by a margin of error + and -. There may be a difference of -5 + 5 grams in one kilo of bulgur. However, when buying gold from the jeweler, let alone a gram, even a tenth error becomes many. In the case of the limit, it may be equal to or less. However, it does not have to be equal in many cases.

Expert 4 preferred making an analogy while expressing the narrowness of neighborhoods.

> Consider searching for a fight in a marketplace. The whole town is a neighborhood of this fight, the strict is a neighborhood, and the houses surrounding the market are neighbors. Now, if we look at a neighborhood as a set, it has a manner of narrowing and expanding. The narrower the neighborhood, the stronger your focus becomes. As you expand the neighborhood, your focus becomes troublesome. Since we work locally, we need to focus as close to the point of interest as possible.

His analogy reminded a significant anecdote regarding limit such that the definition of limit demands for every ε, but especially checks if it works for sufficiently small values.

4.1.3 Increasing epistemological awareness of the limit

Finding also showed that the experts drew attention to four issues about epistemological awareness in limit.

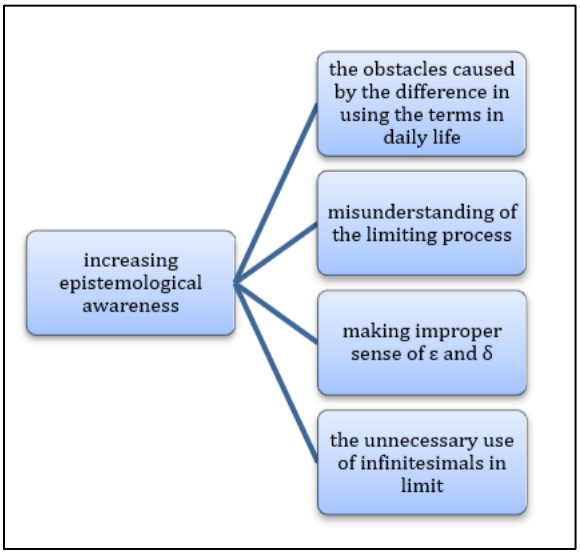


Figure 4.3. The identified categories to increase epistemological awareness in limit

As shown in Figure 4.3, these were the obstacles caused by the difference in using the terms in daily life, misunderstanding of the limiting process, making improper sense of ε and δ, and the unnecessary use of infinitesimals in limit.

Expert 3 referred to two things about the first issue. He claimed that the daily life meaning of limit could be tricky since it has a meaning of a boundary value, and the quantifier “there exists” was sometimes used verbally as “some,” but “at least one” could be more reasonable to prevent misunderstandings.

Regarding misunderstanding of the limiting process, experts remarked how the never-ending number of 9s in 0.999… prevents teachers from accepting the equality of 0.999… to 1. For this reason, they advocated the necessity of discussing such controversial issues with mathematics teachers through different branches of mathematics. Here, in addition to clarification of the limiting process behind 0.999... =1 with the concept of series, Expert 2 remarked that equality should be expressed via the construction of real numbers. About this issue, while differentiating the meaning of equality in limit from the ordinary meaning in algebra, Expert 2 previously stated:

> Is it (referring to 0.999...) equal to 1 or approximately 1? It is exactly 1. This is also a situation related to the understanding of real numbers. Here, 1 is not the limiting value or limiting number for 0.999... So, it didn't become a trap after a certain thing (referring to the circle analogy). It is equal to 1, we do not use the approximate 1 sign... but you can also see it as a limiting value. I mean this is one of the explanations. If you consider 0.9, 0.99, 0.999, … as a series of numbers, you say the last one is supposed to be 1. But this is again equal to 1, not 1 which is an approximation.

As evidenced in the excerpt, Expert 2 put forth two methods for approaching the number 0.999... One of these ultimately yielded a result of 1 in an exact form, while the other approach resulted in 1 as a consequence of a limiting process.

Moreover, Expert 5 mentioned that a teacher can answer a question like “We are saying undefined for 1/0; but not for 0/0. Why is it like that?” The teacher must state that “There are not two different 0s here, but two different series of numbers leading to 0”. These data indicated that although Expert 5 did not specify the type of numbers he mentioned, the questions he asked and the explanations he provided suggested that, like Expert 2, he was also contemplating the concept of series involving real numbers as a possible avenue for teachers to consider while understanding the meaning and role of equality in the learning of limit.

Additionally, Expert 4 focused on understanding the symbols of ε and δ in the definition of limit. He noted that teachers should comprehend them as a variable, not a constant number. Otherwise, they could develop an illegitimate idea such as perceiving 0.999... as the last number before 1. His point of view with regards to considering the quantifiers as variables supported Expert 1’s explanations regarding the simultaneous variance of the quantifiers earlier. Yet, his remarks showed he considered this as an epistemological obstacle for teachers to overcome.

Lastly, Experts 3, 4 and 5 claimed that the use of infinitesimal was unnecessary to introduce the limit concept and difficult to understand even by itself. This was interesting considering that the notion of infinitesimal is a crucial step in the historical development of the concept of limit. About infinitesimals, Expert 3 expressed:

> It's both not zero and behaving like a zero. Infinitely small. That is, it is smaller than all the positive numbers, but not zero. There is no such real number, it cannot be (something contrary to the ordering property of the real numbers). It is very easy to prove that such a number will not exist …The concept of limit has overcome the problem without the need for such a problematic concept.

How the concept of limit overcame the aforementioned problems was elucidated by Expert 3 in another section of his statements. By pointing out some of the pivotal periods in the historical development of the concept of limit, namely how Augustin

Louis Cauchy provided a verbal definition of limit without using symbolic language, and how Karl Weierstrass made it slightly more abstract by replacing words with symbols, Expert 3 expressed that the concept of limit has had a completely abstract, seamless definition.

Similarly, Expert 4 stated:

> Just as we obtain complex numbers from real numbers by putting i number, they (referring to mathematicians) also added something called dx to the real numbers and obtained an extended field. By using them, by using their algebraic properties, they are trying to determine the limit, the limit of functions. Since they are introducing something that does not exist, I do not think that there will be a parallelism between the perceptions of children or young people, and these (referring to the indicated developmental process in history).

Expert 5, by also explaining how the number system was expanded though the addition of infinitesimal, stated that “There are those who teach calculus in this way, but they are in the minority, and I think they are very wrong pedagogically. Because it requires maturity to understand those number systems.”

Data suggested that although it has a huge place in the historical development of the limit concept, experts evaluated dealing with infinitesimals for teachers as unnecessary. Expert 1 and 3’s explanations suggested that their focus was purely mathematical such that ontologically it would be impossible to identify the existence of such number.

4.2 Findings of Exploration Phase

The second part of the study examined the difficulties experienced by mathematics teachers throughout the pre-, during and post- processes of a teaching sequence (see Appendix G) designed to promote their advanced mathematical knowledge of the concept of limit. In other words, the reported findings explored the difficulties about the participants' knowledge of the concept of limit from the perspective of advanced mathematical knowledge at the beginning, during and at the end of the teaching sequence. In general, the results indicated that the teachers faced persistent and consistent difficulties throughout the pre-, post-testing and implementation phases.

These difficulties were grouped under five themes, namely: (i) understanding limit as “approach” and “getting as close as possible” in a limited way; (ii) building the relationship among variables within the concept of limit incorrectly; (iii) misconceived understanding of the covariational relationships between variables in the definition of limit; (iv) challenges in employing equality in the context of limit; and (v) inadequate utilization of symbols in limit. The findings were subsequently discussed in greater detail under the related themes in the following sections.

4.2.1 Understanding limit as “approach” and “getting as close as possible”

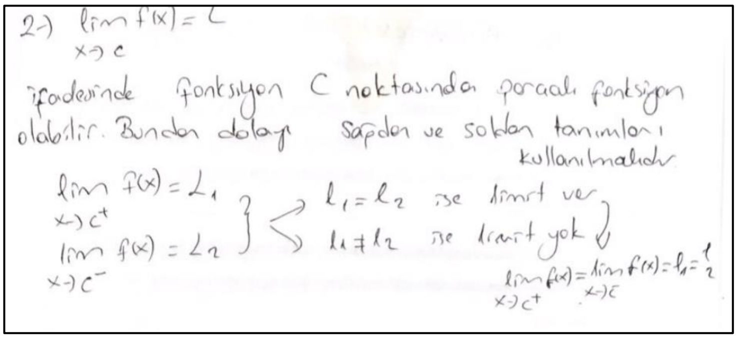
The first apparent theme related to the concept of limit in the participating teachers' data was to regard limit as approaching or getting as close as possible. As a result of this understanding, the teachers showed a strong inclination to the method of left- and right- hand side approaching the limit in describing the definition of limit. The related findings were presented for discussion followingly.

The first evidence of this theme emerged from the classroom discourse focusing on the formal definition of limit, which constituted Step 2.1 of Session 2. In this segment of the session, teachers were posed a question to evaluate their understanding of the roles of the variables within the formal definition, in particular the dependence-relationship between the variables within the formal definition. Teacher Zerrin stated that the formal definition of limit was distinct from the informal definition of limit presented them in the test before, which was “as x approaches c, the value of $f(x)$ approaches L”. After her comment, a discussion started regarding the comparison of the formal and informal definitions of limit in terms of their affordances and limitations as potential definitions of limit. Teachers Zerrin, Merve and Ali signified the formal definition of limit is more explicit than the informal version, due to its clarity in implying the right-hand and left-hand approaches to the limit. The following classroom dialogues represent the continuation of this discussion:

> Researcher: Let's say point c is 2 and I come from 100. Is it accepted?
>
> Merve: No no no... It is much closer...
>
> Ali: Neighboring...
>
> Hale: There is also the concept of right and left to consider.
>
> Ali: It is from our neighborhood concept.
>
> Merve: Exactly.

As evidenced by the excerpt, the discussion centered on the degree of approximation to a point in expressing the limit at that point and the extent to which we were approaching the point c. Teacher Zerrin stated that the formal definition was more expressive than the informal one, as it includes an approach from both the right and left sides with the absolute value symbol. Teacher Merve proposed that the formal definition was more precise than the informal one yet did not provide a rationale for her assertion. Upon noting the inclusion of the term 'approaching' in the informal definition, Teacher Merve maintained an emphasis on the right and left sides as a determining criterion for accepting a definition suitable for the concept of limit. Furthermore, the teachers put forth the suggestion that an approximation of the point c should be evaluated in close proximity by referencing the concept of neighborhood. In a subsequent stage of the teaching sequence, Teacher Merve also stated, "For a function to have a limit at a point, it is necessary that the values taken by the function when approaching the point from the right and left be equal to each other." In her response, there was no mention of the necessity for that value to be a real number. Furthermore, her statements lacked any nuances regarding the components of the formal definition of limit.

Additional evidence substantiating the participating teachers' predominant association of limit with the right and left approaches was derived from Teacher Emre's data. In the content questionnaire test, Teacher Emre defined the term limit as "to get closer, as close as possible." In his conceptualization of the limit, the investigation of limits for the right- and left-hand sides approaching the value was a significant criterion for determining the existence of the limit at that point. He articulated this conceptualisation through the exemplification of piecewise functions.



2-) $\lim_{x \to c} f(x) = L$

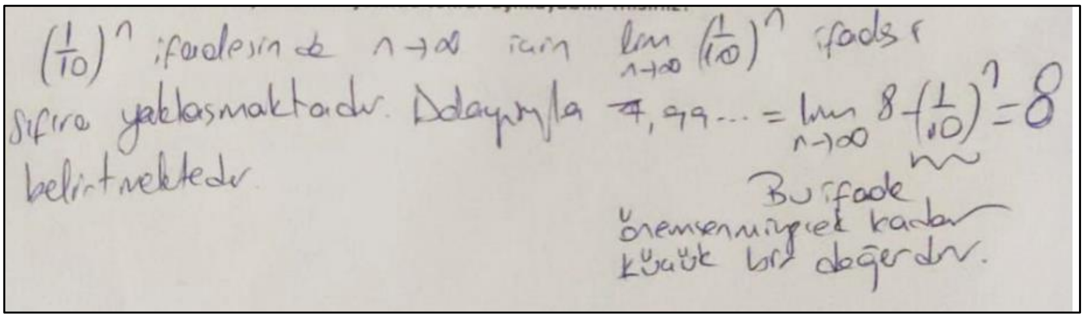
ifadesinde fonksiyon C noktasında parçalı fonksiyon olabilir. Bundan dolayı sağdan ve soldan tanımları kullanılmalıdır

$\left.\begin{array}{l} \lim_{x \to c^+} f(x) = L_1 \\ \lim_{x \to c^-} f(x) = L_2 \end{array}\right\} \begin{array}{l} \nearrow l_1 = l_2 \text{ ise limit var} \\ \searrow l_1 \neq l_2 \text{ ise limit yok} \end{array}$

$\lim_{x \to c^+} f(x) = \lim_{x \to c} f(x) = l_1 = \frac{1}{2}$

Figure 4.4. Teacher Emre’s response to the second question in the content questionnaire test

As illustrated in the figure, he proposed that in cases where the function is piecewise, the inspection of the limit from the right- and left-hand side approaching must be employed. Moreover, in one of the responses to the post-test, Teacher Emre employed the phrase "approaching to zero" in preference to writing directly as "equal to zero" in identifying a limit, as illustrated in the figure below.



$(\frac{1}{10})^n$ ifadesinde $n \to \infty$ için $\lim_{n \to \infty} (\frac{1}{10})^n$ ifadesi sıfıra yaklaşmaktadır. Dolayısıyla $7{,}99\ldots = \lim_{n \to \infty} 8 - (\frac{1}{10})^n = 8$ belirtmektedir

Bu ifade önemsenmeyecek kadar küçük bir değerdir.

Figure 4.5. Teacher Emre’s response for question 1 at the post-test

As n approaches infinity, the value of $\left(\frac{1}{10}\right)^n$ approaches zero. Consequently, $\lim_{n\to\infty}\left(\frac{1}{10}\right)^n$ is equal to zero. However, the teacher referred to this limit as "going to 0," which is not technically correct. This demonstrated how, in the context of Teacher Emre's understanding of limits, the notion of 'tending to' is analogous to that of a limit.

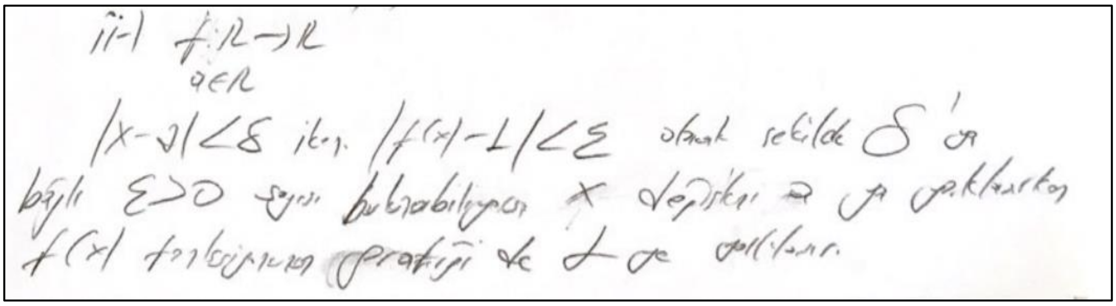
According to Teacher Merve, limit means approaching. Teacher Merve stated at the beginning of the teaching sequence, “For a function to have a limit at a point, it is necessary that the values taken by the function when approaching the point from the right and left be equal to each other.” In her answer, there was no emphasis for that value to be a real number. Also, in her statements, there were no nuances about the components of the formal definition of limit. Furthermore, during the pre-test, she indicated that the distance between 1.5999... and 1.6 was insignificant representing an ignorable, tiny, small difference. Accordingly, she proposed that the two numbers in question could be regarded as equal. Nevertheless, an examination of Teacher Merve's statements revealed no justifications for their equality on the basis of a rationale centred on the concept of the limit. The second question of the content questionnaire test examined how teachers evaluate an informal definition of limit that was inadequate in terms of meeting all the requirements as a definition of limit as compared to the formal definition. The informal definition in question was "as x approaches c, the value of $f(x)$ approaches L." Teacher Merve reiterated the significance of scrutinising the equality of limit values as x approaches the point a from the right- and left-hand sides. Additionally, Teacher Ali proposed that the definition of limit should emphasise the approach from the right and left sides of x in response to the second question of the content questionnaire test. He argued that it was crucial to emphasise the equality of the values that $f(x)$ approaches as x approaches a point c.

In the post-test, Teacher Merve's conception of limit appeared to align with the meaning of 'boundary'. In response to the question of how she would proceed if a student remained unconvinced about the equality of 7.999... and 8, even after the presentation of the two arguments in the post-test, she replied that she would attempt to clarify that the nearest number to 7.999... is 8, utilizing the concept of limit.

In summary, the data from the questionnaire and the pre- and post-tests suggested that the three teachers had a consistent and limited understanding of the definition of limit. In particular, teachers consistently interpreted the concept of limit as "approaching" and "getting closer," failing to recognize the equality of such numbers as 7.999….and 8.

4.2.2 Building the relationship among variables within limit incorrectly

The second theme identified was teachers' difficulties in establishing relationships between variables within the concept of limit incorrectly. This was evidenced by their responses to the pre-tests and statements given during the teaching sequence. As an example, an erroneous understanding of the relationship between the quantifiers in the definition of the limit, as evidenced by their responses to the content questionnaire test, was identified. In response to the question regarding the description of limits through various representations, Teacher Ali provided a verbal description through symbols, as illustrated in the figure below.



iii-) $f: R \to R$
$a \in R$
$|x-a| < \delta$ iken $|f(x)-L| < \varepsilon$ olacak şekilde δ'ya bağlı $\varepsilon > 0$ sayısı bulunabiliyorsa x değişkeni a ya yaklaşırken $f(x)$ fonksiyonunun grafiği de L ye yaklaşır.

Figure 4.6. Teacher Ali's one of the definitions for limit

In this definition, Teacher Ali defined the concept of limit as follows: "When $|x-a| < \delta$, if there is an $\varepsilon > 0$ connected to δ such that $|f(x)-L| < \varepsilon$, then the graph of the function $f(x)$ approaches L as the variable x approaches a." It was observed that Teacher Ali’s description of the dependence-relationship was in fact the other way around, with ε dependent on δ. Furthermore, his definition lacked any indication of the use of quantifiers such as “for every” (∀) and “there exists” (∋). The data thus indicated that Ali had a misunderstanding, namely that he was unaware of the dependency of δ on ε. This further suggested that he had an inadequate understanding of the meaning of quantifiers such as ∀ and ∋ in the formal definition of limit. In addition, Ali’s assertion that the limit is "approaching" in the previous section, despite his recognition of left- and right-sided limits, indicates that he lacked a mathematically legitimate understanding of the definition of limit.

Additionally, Teacher Emre presented an erroneous connection regarding the relationship among variables within the formal definition of limit during the teaching sequence. In a segment of the teaching sequence, the teachers were asked: "According to the definition, is there a dependence-relationship between epsilon and delta? If so, how is it manifested?". The following excerpt was from teacher Emre’s statements about this issue:

> It has already given in the sentence (referring to the given formal definition). We are trying to find a delta greater than 0 while epsilon is greater than 0. Therefore, you accept the existence of one and try to prove the existence of the other. So, there is a relationship in that sense. Anyway, delta is generally related to the domain set, and epsilon is related to the image set. You are trying to capture the range in the image set by starting from a certain range in the domain.

As the excerpt illustrates, Teacher Emre established an inappropriate connection between the epsilon and delta variables and the image and domain sets, similar to the way in the concept of function. The teacher's approach to the limit is akin to a function definition, whereby y is dependent on x. However, the relationship in the context of limit is defined in opposition to this; x is dependent on y. This implies that delta is dependent on epsilon.

Another finding related to the relationships among variables within the concept of limit was about covariational reasoning. However, due to its significant importance as identified by experts in the field, this topic will be addressed in greater detail in the subsequent section.

4.2.3 Misconceived covariational relationship between variables in the definition of limit

The third theme, which pertains to the difficulties associated with the notion of limit, is related to the covariational relationship between variables. The data collected from the teaching sequence revealed a deficiency in teachers' covariational reasoning with respect to the variables included in the definition of limit.

In the need analysis phase, experts identified three distinct relationships within the context of limits. These were described as follows: the relationship between $x - a$ and $f(x) - L$; the covariational relationship between ε and δ in the limit definition; and the relationship between x versus $f(x)$, as well as the specification of values for epsilon and delta. Nevertheless, while the data provided by the teachers offered some insights into the relationship between the variables, the precise nature of this relationship remained unclear in their instances. To illustrate, the following excerpt from Teacher Ali demonstrates his approach to presenting a comparison between the neighborhoods of ε and δ, utilising the concepts of dependent and independent variables during the teaching sequence.

> Ali: While $|x - a| < \delta$, how the change a change in the delta neighbourhood of x creates a change in the system. This is actually the intuitive approach of the limit. That is, the changes in the independent variables which may be an increase or a decrease. We can call this a right-hand or left-hand approach of limit. So, What kind of change does this change create in the system, that is, in the image?

From this excerpt, one can infer that Teacher Ali is referencing the covariational reasoning between $|x-a|$ and $|f(x)-L|$ in relation to the concept of limit. However, although there is an implication of a covariational relationship between the independent and dependent variables, no specification is given regarding the quantities involved in this relationship.

Additionally, Teacher Emre displayed an inclination to express a covariational relationship among the variables within the definition of limit. During the course of the teaching sequence, the teachers were questioned on the following matter: "In accordance with the formal definition of limit, is there a dependence relationship between epsilon and delta? If such a relationship exists, how is it manifested?". Teacher Emre responded to this question as follows:

> It has already given in the sentence (referring to the given formal definition). We are trying to find a delta greater than 0 while epsilon is greater than 0. Therefore, you accept the existence of one and try to prove the existence of the other. So, there is a relationship in that sense. Anyway, delta is generally related to the domain set, and epsilon is related to the image set. You are trying to capture the range in the image set by starting from a certain range in the domain.

As the excerpt illustrates, he began to establish a connection between the epsilon and delta variables versus the image and domain sets. Subsequently, he emphasised the interrelationship between the ranges in the image and domain set. An insight into the covariational relationship underlying the definition of limit can be gained, particularly from Teacher Emre's assertion in the final sentence of the excerpt; however, no explicit reference was made to the variables.

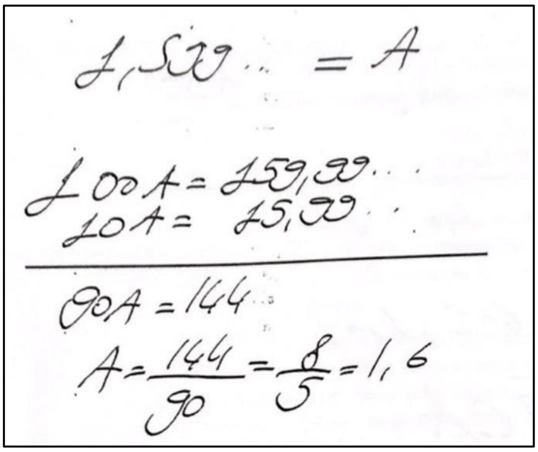
Lastly, an additional observation regarding the teachers' understanding of covariational relationships in relation to the concept of limit was derived from the following dialogue from the teaching sequence at Session 2:

> Zerrin: A narrowing of the interval of $x - a$ will result in a corresponding narrowing of the other interval. Both are of a very slight magnitude within the specified domain and image set.
>
> Merve: Are we going to think like approaching at the same rate.
>
> Zerrin: The rates are not the same, but only if I can get close enough in the domain, that is, if I am talking about a negligibly small number. For example, I'm talking about 2.49 and 2.51 at the domain set and I'm talking about 7.1 and 7 on the image set (referring to a point 2.5 on the x-axis and a limiting value 7).
>
> Emre: Actually, it is a bit similar to the definition you gave yesterday (referring to the theorem in Step 1.3). When we said that for every $\varepsilon > 0$ if $|x - y| < \varepsilon$, then we said x equals y. In fact, this is a bit similar to that. Because when you take delta greater than 0, it is seen that x is different from a under suitable conditions. The smaller you choose that delta range, the smaller x goes around point a. Therefore, the more you expand the domain, the smaller the image set you will get. The result you get in the domain set takes you to the result in the image set.

As can be observed, the discussion centred on the interrelationships between the variables associated with the points a and L in the informal and formal definitions of a limit. Although an emphasis on the covariational relationship between the degrees of convergence of x and $f(x)$ values versus a point a on the x-axis and a constant value L on the y-axis could be inferred, no such clear reference is made by the teachers in their statements.

4.2.4 Challenges in employing equality in the context of limit

Another challenge encountered by teachers in understanding the concept of limit was their difficulty in grasping the use of the equality sign within the context of limit. One of the instances demonstrated that the teachers do not did not anticipate the necessity of utilizing the concept of equality when confronted with a mathematical scenario involving infinity. For example, when questioning how a teacher might demonstrate to students that the numbers 1.5999… and 1.6 are equal, as the fourth question on the pre-test, Teacher Ali proposed a method based on repeating decimals.



$$1{,}599\ldots = A$$

$$100A = 159{,}99\ldots$$

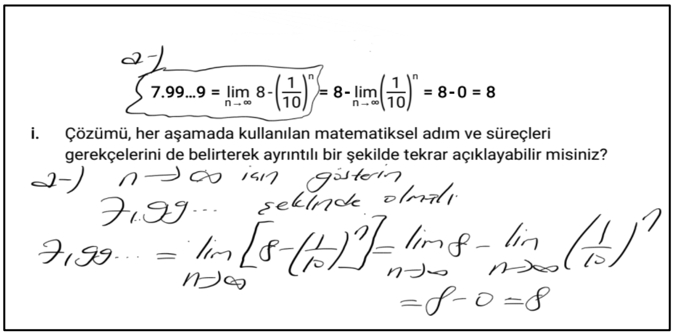
$$10A = 15{,}99\ldots$$

$$90A = 144$$

$$A = \frac{144}{90} = \frac{8}{5} = 1{,}6$$

Figure 4.7. Teacher Ali’s suggested method to show the equality

He expanded upon his response, stating, "The infinite number of 9s is being discussed here. If the matter is approached in a quantitative manner, it will never be possible to reach the final digit. There is no end in infinity. But we act as if they were equal." In other words, despite his presentation of a mathematical argument on the equality of the two numbers and use of an equal sign, he asserted that they are not equal because of the infinite involved in "point nines". Similarly, Teacher Ali posited that when the topic of the limit is issued, the two numbers in question are accepted as unequal, but as being in close approximation to each other. It was therefore reasonable to assume that he would never employ the equality symbol in the context of limits. Nevertheless, his response to the question posed in the post-test exhibited a contradictory stance on this matter, as seen below:



2-)

$$7.99...9 = \lim_{n\to\infty} 8 - \left(\frac{1}{10}\right)^n = 8 - \lim_{n\to\infty}\left(\frac{1}{10}\right)^n = 8 - 0 = 8$$

i. Çözümü, her aşamada kullanılan matematiksel adım ve süreçleri gerekçelerini de belirterek ayrıntılı bir şekilde tekrar açıklayabilir misiniz?

2-) $n \to \infty$ için gösterin
7,99... şeklinde olmalı

$$7{,}99\ldots = \lim_{n\to\infty}\left[8-\left(\frac{1}{10}\right)^n\right] = \lim_{n\to\infty} 8 - \lim_{n\to\infty}\left(\frac{1}{10}\right)^n$$
$$= 8 - 0 = 8$$

Figure 4.8. Teacher Ali’s response for the question 1 at the post-test

In the question in Figure 4.8, the teachers were required to provide a rationale for the mathematical operations presented. It was observed that Teacher Ali expressed satisfaction with the utilization of the equality sign in this argumentative process, even providing supplementary justifications. Despite the issue being related to limits and involving infinity, Teacher Ali demonstrated satisfaction with the steps in the process. This indicated a contradiction in his understanding of the equality sign.

4.2.5 Inadequate use of symbols in the context of limit

The final theme to emerge from the analysis of teachers' difficulties with the concept of limit was their inadequate use of mathematical symbols. The analysis revealed that the use of symbols by MTs was limited, as evidenced by their responses to the pre- and post-tests.

Teacher Emre and Merve employed the limit notation in a few of their responses to the pre-tests. For instance, in one instance where the notation was used, Teacher Emre defined the concept of a limit by focusing on the left- and right-hand sides approaching the limit. However, no additional symbolic representations were observed beyond those present in the responses of either teacher.

Indeed, an investigation of the data revealed the use of symbols such as epsilon in Teacher Ali's responses. In his explanation of the relationship between 1.5999… and

1.6 at the pre-test, Teacher Ali posited that 1.5999… is in the epsilon neighborhood of 1.6 and demonstrated this on a number line, as illustrated in the figure below.

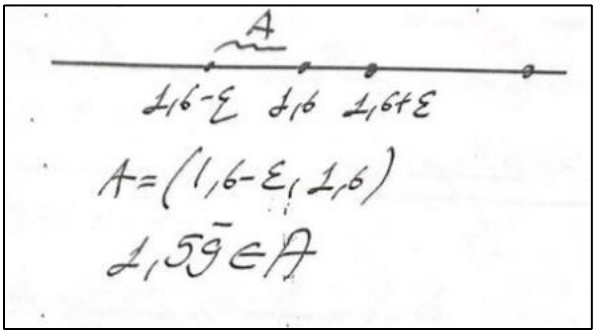


Figure 4.9. Teacher Ali’s response to the first question at the pre-test

He linked his response to the figure with the concept of neighborhood. The concept of neighborhood forms a fundamental basis for the field of topology. Therefore, Teacher Ali's association between the symbol and this term may be regarded as an encouraging sign in relation to his AMK. However, his representation of a neighborhood was erroneous and incomplete. While in topology, a neighborhood is typically depicted through a disk, he showed the neighborhood of 1.6 on a number line. This demonstrated a lack of comprehension of the symbol epsilon in relation to the term neighborhood.

CHAPTER 5

DISCUSSION AND CONCLUSION

The present study had two main objectives: firstly, to gain insight into the essential advanced mathematical knowledge of limits from the perspective of experts in the field, and secondly, to document the characteristics of the difficulties experienced by mathematics teachers in a teaching sequence designed to promote their advanced mathematical knowledge of the concept of limit.

The essential advanced mathematical knowledge of limit for mathematics teachers was explored through a framework developed by Yan et al. (2022), which comprises three components: drawing connections between limit concepts across mathematical domains, developing mathematical experience for problem-solving abilities, and increasing epistemological awareness. In the design of the teaching sequence, both the essential AMK of limit, derived from the interview study, and the existing literature on AMK were employed. Following the teaching sequence, it became evident that MTs faced various difficulties in the context of limit. Accordingly, the aforementioned findings related to the characteristics of difficulties experienced by mathematics teachers were presented under the overarching themes described in the preceding section. This section presents a discussion of the presented findings and the underlying reasons for the lack of progress and consistent reported difficulties in teachers' AMK for limit, with reference to the relevant literature.

The findings of the studies indicated that mathematics teachers encountered difficulties in certain aspects that experts had identified as crucial for those teaching mathematics. For instance, experts had highlighted the significance of covariational reasoning in comprehending the concept of limits, delineating the specific variables that required covariational reasoning to be established. However, despite the observation of nuances in some covariational reasoning between variables, mathematics teachers exhibited deficiencies in identifying and articulating these relationships with precision.

In the following parts, the findings for the need analysis and exploration phases will be discussed in more detail respectively.

5.1 Aspects of Advanced Mathematical Knowledge of Limit Essential for Mathematics Teachers

In the first part of the study, advanced mathematical knowledge that mathematics teachers should possess in the context of limit from the perspective of experts was investigated. For this purpose, a framework (Yan et al., 2022) that encapsulates the value of advanced mathematical knowledge for mathematics teachers from mathematicians' viewpoint was adopted to the concept of limit particularly.

To draw connections of limit across other mathematical domains, experts notified the meaning of equality in the context of limit compared to its meaning in arithmetic, and the relevance of limit with topology. Experts identified the meaning of the symbols and signs with their links to topology and the meaning of equality in the context of limit. While teaching limit, some lecturers prefer to follow a road passing through analysis, and some adopt a way of teaching visiting topology. Especially introducing convergence for a sequence of functions, the definition of a topology on the related space is claimed as necessary (Dreyfus, 2002). Interestingly, although there is an emphasis on pre-service teachers' having advanced mathematical knowledge (CBMS, 2012), there is no mention of topology courses. Similarly, in most undergraduate education programs for secondary school mathematics teachers, topology courses are not compulsory in Turkey. On the other hand, three experts, even though they have been working in the department of mathematics education, reminded the importance of adopting topological terms in the context of limit. Findings showed that experts' focus was on topological ideas such as accumulation points with links to the inequalities and making sense of the quantifiers "for every" and “there exist” in the formal definition of limit. These suggested that it might be beneficial for especially teacher educators to consider as a way of teaching limit in education faculties by including topological side of the topic.

Experts in this study also considered the meaning of equality sign both as an important aspect to connect to different domains in mathematics but also as an epistemological obstacle to be overcome by mathematics teachers. First, the examples of equality symbols in the context of limits seemed crucial. Because the symbol of equality in the experts’ explanation was not in the sense of ‘sameness’ or ‘do something to get the answer,’ but rather an equality obtained at the end of a process. Their responses reminded that the symbol of equality must be reconsidered through the terms of process and product in the context of limit. In particular, experts identified some epistemological obstacles to understanding the 0.999… and 1 equality. Similar to the experts’ ideas in this study, Choi and Do (2005) made a comparison of different meanings of equal sign involved in 0.999… = 1. Choi and Do (2005) explain that, from the point of the convergence of a sequence, 0.999… becomes equal to 1 through the concept of limit. In addition to Choi and Do (2005), the experts in our study stated that a sound explanation of equality regarding 0.999… = 1 in its ordinary meaning in arithmetic can be explained through the construction of real numbers (Belin & Karagöz Akar, 2020a, 2020b).

Interestingly, although proofs or justifications for the equality of 0.999... and 1 are various and include clear mathematical facts, undergraduate mathematics students or mathematics teachers still have problems thoroughly convincing themselves (Choi & Do, 2005; Conner, 2013). This misinterpretation leads students to state the relationship between the two as “a nonzero but infinitely small difference” (Dawson, 2016, p.5) or “infinitely close but not equal” (Tall & Schwarzenberger, 1978, p.6). Some reasons for this weak belief are inadequate understanding of the limit concept and perceiving the infinite number of 9s in 0.999… as finitely many (Tall & Schwarzenberger, 1978). The data further showed that experts regarded being aware of the epistemological obstacles related to limits as crucial practice for teachers. They pointed out that conflicting meanings of some elements or statements in the formal definition with their everyday usage might lead teachers and students to have difficulties in comprehending the concept of limit. In parallel to this finding, Cornu (2002) named such a difficulty for the quantifiers as "a conceptual obstacle which may cause serious difficulties" (p. 153). Similarly, Cottrill et al. (1996) assessed the unsound qualification of ε and δ as a barrier to understanding of limit.

The issues related to limit that a mathematics teacher must have experienced from the perspective of experts were mainly centered on the conceptual meaning of limit, the roles of the elements in the formal definition of limit, and some aspects of limit as requiring covariational reasoning. At this point, roughly breaking the definition into its small pieces and working on them seemed crucial for experts. Similarly, Lee (1992) drew attention to understanding epsilon, the meaning of existence in the definition, and what absolute value represents. Otherwise, a weak understanding of the definition of limit may result in stating the definition literally but using ineffectively in further proofs. Furthermore, experts in this study paid attention to the situations in calculus in which intuition must be formalized. A similar finding appeared in the study of Yan et al. (2020) on the underlying meaning of doing mathematics in calculus. The current study presents concrete ways and examples for how intuition is supported by formalization in the context of limits as repeatedly uttered graphs and visuals with formal definitions and theorems, so emphasizing the importance of formalization.

In addition, as Antonini et al. (2007) have noted, visual representations may generate an obstacle in front of learning concepts unless they end up with theoretical knowledge. Besides, Wasserman (2023) advocates the necessity of visualization, focusing on the non-existence and abstractness of mathematical concepts in the real world. The experts' interviews also highlighted the use of graphs in limit. Because of the infinite process lying in the limit, the support of this topic by various tools, from tables of function values and graphs of functions to symbolic expressions, is advocated in the related literature (Juter, 2007).

The activities identified by experts as fundamental for developing mathematical experience in limit-related problems were found to exhibit notable similarities to the components of advanced mathematical thinking. Before formalization or rigor, gaining some intuition about the concept of limit was highlighted in the findings. Additionally, the findings revealed that, although school curricula do not include the formal definition of limit (MoNE, 2018), experts emphasised the necessity for mathematics teachers to engage in more practices pertaining to the components of the formal definition and to gain insights into the necessary and sufficient conditions associated with the concept of limits.

The findings of the current study showed that one expert described the place of covariational reasoning when thinking about limit in detail. In the literature, covariational reasoning is mostly featured with the concept of function (Oehrtman et al., 2008) and regarded as a fundamental mathematical way of thinking (Tallman & Frank, 2020) for such as rate and ratio, rate of change, and derivative (Thompson & Carlson, 2017). The concept of limit, in its nature, also involves the behaviors of functions. Although the literature emphasized the importance of covariational reasoning in understanding variables changing concurrently (e.g., Carlson et al., 2002; Thompson & Carlson, 2017), its role in the context of limit is rarely mentioned in empirical evidence (Carlson et al., 2001). Among a few, for example, Jensen (2009) speculated that improvement in thinking about covariation might be helpful for understanding limit, referring to how it is demanded to construct genetic decomposition (Cottrill et al., 1996) of limit understanding. Additionally, Dixon et al. (2020) argued that higher covariational reasoning skills could help learners to develop more sophisticated understanding of limit. They included covariational reasoning as a component in their study on reasoning about limits and compared students' reasoning skills before and after the implementation of a unit. In their study, covariational reasoning in limit appears only as the relationship between x and y values, which was one of the aspects mentioned by experts in the current study.

Carlson et al. (2001) conducted a study with first-semester calculus students to observe how students used covariational reasoning in some calculus-based concepts including limit and accumulation. For example, one of the required thinking processes necessitating covariational reasoning in limit appeared in their data was “the limit is what it gets close to…I mean what the y-value gets close to as x gets close to some value from both sides.” (Carlson et al., 2001, p. 148). As seen from the findings of the current study, one expert (Expert 1) highlighted the importance of covariational reasoning in the context of limit in detail. Particularly, Expert 1’s explanations suggested that he was at least at the chunky continuous covariation as he envisioned both variables varying within an interval (Thompson & Carlson, 2017). His statements further suggested a direction to Cottrill et al.’s (1996) recommendation for grounding the limit concept on the understanding of "values of a function approaching a limiting value as the values in the domain approach some quantity” (Cottrill et al., 1996, p. 6).

These results point out the importance of opening a new page for covariational reasoning including the concept of limit. While these relationships may be addressed in calculus lessons and are likely to be known to mathematics majors and mathematics teachers, it is uncertain whether these covariational relationships in limit are specifically highlighted and conveyed explicitly. Thus, this study finding offers an empiric and elaborated way from where teachers and teacher educators can start to engage learners in covariational reasoning in the context of limit.

The subsequent section presents a discussion of the findings regarding the difficulties experienced by mathematics teachers throughout a teaching sequence designed to promote their advanced mathematical knowledge of the concept of limit.

5.2 Nature of Mathematics Teachers' Advanced Mathematical Knowledge of Limit

The second part of the study aimed to document the nature of mathematics teachers' advanced mathematical knowledge at the beginning and at the end of the teaching sequence, with a particular focus on the concept of limit and related topics. The teaching sequence introduced the teachers to the concept of equality in real numbers and demonstrated how this concept was transferred to the context of infinite decimal expansions through the application of a theorem. Additionally, the post-test presented two arguments for the equality of real numbers, such as 0.999... and 1. However, the teachers in this study demonstrated persistent knowledge about the inequality of these two real numbers, a finding that is similarly reported in Buchholtz et al. (2013).

The concept of limit that they espoused was strikingly consistent with the conventional notions of right- and left-hand limits. When they were tasked with evaluating the suitability of informal and formal definitions of limit during the teaching sequence, they were focused on whether these definitions included the x values approaching the point a from the right- and left-hand side. None of them addressed whether the function f is defined in a neighborhood of the point a as a crucial aspect which is fundamental to define a limit. Bansilal and Mkhwanazi (2022) referred to this approach from the left- and right-hand side as an intuitive definition of the limit. It is employed as a preliminary step to facilitate students' understanding of the formal definition of the limit.

Participating mathematics teachers' reasoning about equality or inequality of 1.5999... and 1.6 differed from each other. While one of them emphasized the existence of ignorable, small distance between them so that they could be accepted as equal, another one claimed that 1.5999... was at the epsilon neighborhood of 1.6. But the common point in their statements were their use of the phrases such as “very very close,” “close but not zero,” and “ignorable, tiny, small difference” while identifying the distance between them. Their employment of such phrases was interpreted that they considered 1.5999... as the infinite process involving in the non-terminating decimal number. A discrepancy in understanding this issue was reported in a study by Tall (1977). The notion of "getting close to" in the verbal or informal definition of limit (p. 13) conveys the impression that the two phenomena, 0.999... and 1, are in close proximity to one another, yet they are not identical. However, the responses to a related question in the Content Questionnaire Test did not provide any indication about this in the current study. Rather, the participating teachers identified the lack of emphasis on the two sides of approaching as the only issue with the informal definition of limit.

Furthermore, the data indicated that participating mathematics teachers employed vague terminology frequently in their explanations. The literature made a distinction between the meanings of the phrases "limit" and "tend to." In contrast to the concept of limit, the phrase "tends to" is understood to convey a certain degree of vagueness (Cornu, 2002). In the context of elucidating the relationship between 0.999… and 1, it is asserted that "the sequence '0.9, 0.99, 0.999, 0.9999, …' has a limit of 1" or "tends to 0.999…". (Cornu, 2002, p. 325). In the related data provided by the participating teachers, a comparable observation was also documented on some occasions by teachers who made a differentiation in their use of these phases. To illustrate, while the teachers referred to the limiting process with the phrase "tend to," they refrained from placing an equal sign at the conclusion of the process.

One of the intriguing insights derived from the concept of equality was the assertion by Teacher Ali that when confronted with the concept of infinity, the notion of equality becomes inapplicable. Consequently, he forcefully argued that 0.999... and 1 are inherently unequal due to the involvement of infinity in the former. A comparable scenario was documented in the study conducted by Szydlik (2000). In a similar vein, one of the participants in Szydlik's study posited that it is not possible to provide a definitive answer to the question of the relationship between 0.999... and 1, given that 0.999… is an infinite decimal and therefore lacks a definitive terminus.

A review of the findings related to the use of symbols revealed that teachers were more inclined to utilize verbal representations in comparison to symbolic ones. While both teachers, Ali and Emre, demonstrated the use of symbolic language in the teaching sequence, Ali exhibited a distinctive manner in his use of symbols in responses to the pre- and post-tests. This was particularly evident in the adoption of the epsilon in his statements. However, an analysis of the data revealed that even Teacher Ali's responses to the post-test did not demonstrate any discernible improvement in comparison to their pre-test performances. Therefore, no progression could be attributed to the teaching sequence with regard to the adoption of symbols from the pre- to post-test. A further observation regarding the utilization of symbols was that teachers Emre and Merve did not endeavor to employ the epsilon-delta symbols in order to describe the concept of limit. Teacher Ali was the only teacher who attempted to define the concept of limit using the infinitesimals epsilon and delta. This approach may have been motivated by two factors: the challenge of employing rigorous mathematical language and the pervasiveness of conventional practices related to limits, which often involve substituting numbers for variables and making automatic calculations. Harel and Kaput (1990) proposed that conceptualizing the concept of limit as a routine process of substitution, rather than as an operator in real numbers, could be a potential outcome of such customized activities. While the teachers were encouraged to employ symbolic registers in certain stages, such as Steps 1.3, 1.4, 2.1 and 2.2 of the teaching sequence, it is possible that the activities could have been extended in duration and frequency.

The advanced mathematical practices essential for tertiary education was identified by Gueudet (2008) as syntactic knowledge and the integration of semantic and syntactic understanding. The utilization of symbols and quantifiers facilitated the transition to abstraction and formalization, thereby enabling effective communication within the field (Iannone & Nardi, 2007). The observations in the current study revealed that only one teacher, namely Teacher Ali, demonstrated an intense inclination and willingness to employ symbols and syntactic language as facilitators in his responses. Consequently, he was evaluated at a higher level in terms of the adoption of symbols. In the course of the teaching sequence, even it was the converse of the real dependence, Teacher Ali highlighted the dependence between the epsilon and delta in the formal definition of the limit, noting that some textbooks identify the dependence of ε on δ as ε_{δ}. It is regrettable that the definition of limit is presented in a variety of ways in mathematics textbooks. To illustrate, most of the times it is merely conveyed in verbal statements. In other instances, there is a clear preference for the use of symbolic language (Durand-Guerrier & Arsac, 2003). Additionally, there are instances where the connection between ε and δ is explicitly identified, as Teacher Ali stated.

In the need analysis phase of the study, the experts emphasized the significance of grasping the underlying covariational reasoning embedded in the definition of limit. This was deemed crucial for acquiring the requisite mathematical expertise to develop problem-solving abilities in limit. The experts proceeded to categorize these relationships as follows: the covariational relationship between epsilon and delta, between the values of $f(x)$ across x values, and the relationship between $x - a$ and $f(x) - L$. Nevertheless, the findings of the exploration phase indicated that the mathematics teachers lacked a comprehensive understanding behind covariational relationship associated with the definition of the limit.

5.3 Evaluating the Teaching Sequence and Design Process

In this study, an essential mathematical controversial issue of the equality of 0.999... and 1 was put on the center. This issue was selected intendedly because it had an aspect that relates to different mathematical concepts and there was strong advice in the related literature to work such comprehensive issues for the sake of advanced mathematical knowledge (CBMS, 2012). As previously stated, although the term "advanced mathematical knowledge" was defined in accordance with the concept of "advanced mathematical thinking" (Zazkis & Leikin, 2010), the latter has been subject to a multitude of interpretations and perspectives. In designing the activities within the teaching sequence, three design principles derived from the relevant literature and the data collected from the needs analysis phase informed the subsequent flow of activities. In other words, the design process of the teaching sequence was developed in accordance with established theoretical principles and the empirical results concerning its efficacy were duly documented. With respect to these attributes, this study represents a significant contribution to the existing body of literature on the subject.

The teaching sequence comprised a number of activities designed to facilitate a progression in the teachers' understanding of the concept of a limit. To illustrate, the relationship between the numbers 1.5999... and 1.6 was examined in Step 1.1 of the teaching sequence. In Step 2.1, the formal definition of a limit was investigated through a comparison with the informal definition. Additionally, in Step 2.2 of the teaching sequence, the communication of variables within the formal definition was conveyed through the use of various representations.

Furthermore, the teaching sequence encompassed a series of activities aimed at enhancing teachers' understanding of the concept of equality in the context of limit. The first of these was a discussion activity on the relationship between 1.5999... and 1.6, which constituted Step 1.1 of the teaching sequence. The uniqueness of infinite decimal representations was studied, and the equality of two different versions of infinite decimal representation was demonstrated through a theorem on the equality of real numbers, and the interpretation of this theorem in infinite decimal representations was examined as Step 1.2, 1.3, and 1.4 of the teaching sequence. Subsequently, in Step 2.3, a discussion activity on the interpretation of equality in the context of limit was conducted through the presentation of several examples. While the teaching sequence does not explicitly focus on the use of symbols, they are nevertheless integral to the process, particularly in relation to the formal definition of limit as in Step 2.1 and the work of a theorem with its statements, including the epsilon symbol as in Step 1.3. The aforementioned explanations provided a synthesis of the stages comprising the teaching sequence.

The studies aiming to promote mathematics teachers in terms of their ability to advance their university mathematics knowledge were predominantly conducted with pre-service teachers within the context of their undergraduate education (e.g., Wasserman, Fukawa-Connelly, et al., 2017; Weber et al., 2020). This meant that participants of such studies have recently learned their real analysis or algebra courses, and even so researcher provided them some background information (e.g., Wasserman et al., 2018). In this study, although the pilot study was held with pre-service teachers, the main study held with in-service mathematics teachers whose experience was far more than ten years. For this reason, the participating teachers in this study are considered to have less knowledge of university-level mathematics than both newly graduated and pre-service teachers. So, in addition to the inherent difficulties in the central concept of the study, the distance between the participants and the advanced mathematics courses presents an additional challenge in reporting on the development of teachers' AMK of limit in the teaching sequence. Re-examining the study with pre-service mathematics teachers, who may be more familiar with AMK, could facilitate a more efficient result in reporting on the proposed teaching sequence.

5.4 Implications, Suggestions, and Limitations

This study was an attempt to the question of which aspects of advanced mathematical knowledge are essential for mathematics teachers, as conceptualized by experts, and to determine the nature of advanced mathematical knowledge of limits and limit-related ideas among secondary school mathematics teachers throughout a teaching sequence designed based on essential advanced mathematical knowledge. In order to achieve this aim, a preliminary phase was conducted in accordance with the recommendations of the relevant literature, comprising a need analysis study with field experts. This was undertaken with the objective of exploring the aforementioned essential advanced mathematical knowledge for mathematics teachers in the context of limits. Subsequently, the design process of a teaching sequence was initiated as an exploration phase in this design-based research study and the mathematics teachers' difficulties experienced in the teaching sequence was investigated.

This study is noteworthy for its inclusion of a diverse range of participant profiles throughout the research process. The interviews were conducted in the preliminary phase with experts to gain insight into the needs of the problem situation. Furthermore, while the need analysis phase of the study was conducted with pre-service teachers, the exploration phase was carried out with in-service teachers.

The findings of need analysis study in the preliminary phase revealed essential issues in the context of limit, which may lead mathematics educators to revise the content of undergraduate mathematics education courses. Furthermore, the emerging items identified within the three components of the framework employed in the need analysis phase may assist interested researchers in the design of specialised courses with the objective of promoting mathematics teachers' use of advanced mathematical knowledge for teaching, as recommended by the CBMS report (2001). In this study, the researcher was not directly interested in teachers’ classroom practices; rather, it was aimed to shed light on the advanced mathematical knowledge regarding limits that teachers require. Nevertheless, teachers may draw upon the findings of the need analysis study for the pedagogical purposes, either explicitly or implicitly. To illustrate, it is crucial for mathematics teachers to consider the covariational relationships identified by experts and integrate them into their classroom practices.

Conducted as an interview study, the need analysis in the preliminary phase of the study did not document all the advanced mathematical knowledge essential for mathematics teachers regarding limit. The analysis discussed in this part was limited to experts’ opinions and reflected their verbal statements only. Therefore, to gain a deeper insight into the issue, in addition to the interview, it may be helpful to observe the experts’ teaching and interview pre-service teachers who have completed the course. It is also possible to look at secondary school textbooks covering the limits and compare the concepts there with the themes highlighted by the experts in this part of study. In addition, the need analysis study only presents a limited number of experts’ concept-specific opinions. It is recommended that such studies be carried out on different topics with more participants and even by including experts from other countries.

In line with the recommendations of Wu (2011) and other researchers, mathematicians should play a more prominent role in supporting mathematics teachers in developing their mathematical knowledge. As a preliminary phase, a collaboration with mathematicians was initiated prior to the determination and preparation of the content of the teaching sequences. However, integrating this collaboration into the process of lesson preparation could facilitate the creation of more effective teaching sequences. It is recommended that those pursuing further research in this field extend their collaboration with mathematicians to encompass the entirety of the design process.

The teaching sequence was developed in accordance with the design principles that promote teachers' advanced mathematical knowledge in the context of limit. However, additional design principles guiding the implementation of the teaching sequence may yield varying outcomes compared to the current study. It is therefore recommended that further research be conducted with an increased focus on developing a theoretically based design framework for the implementation process of the teaching sequences.

The tasks included in the teaching sequence enabled the identification of the challenges that mathematics teachers face in understanding the concept of limits. It is therefore recommended that the tasks in the teaching sequence be employed by mathematics educators to overcome the difficulties encountered by teachers in relation to the concept of limit.

To sum up, a review of the findings across the teaching sequence revealed that mathematics teachers exhibited a range of challenges in understanding the concept of limit. The concept of limit is inherently complex, with a profound history of development and significant epistemological challenges. Furthermore, the teaching of the concept of limit to undergraduate students presents a number of challenging issues, including for example the choice of whether to adopt an approach based on infinitesimals and the extent to which the formal definition of limit should be included in the teaching process. It is noteworthy that even in the interview study, the experts identified disparate opinions on these matters. It can be reasonably inferred that the difficulties encountered throughout the teaching sequence may be attributed to the inherently complex nature of the concept under examination. Furthermore, the relatively limited number of participants in the study may be a contributing factor to the absence of observed improvements. The involvement of a greater number of participants would facilitate more efficacious discussion sessions during the implementation phase.

REFERENCES

Adiredja, A. P. (2021). Students' struggles with temporal order in the limit definition: Uncovering resources using knowledge in pieces. *International Journal of Mathematical Education in Science and Technology*, *52*(9), 1295–1321, https://doi.org/10.1080/0020739X.2020.1754477

Aliustaoğlu, F., & Tuna, A. (2021). Examining the pedagogical content knowledge of prospective mathematics teachers on the subject of limits. *International Journal of Mathematical Education in Science and Technology*, *52*(6), 833–856. https://doi.org/10.1080/0020739X.2019.1703148

Alvarez, J. A., Kercher, A., Turner, K., Arnold, E. G., Burroughs, E. A., & Fulton, E. W. (2022). Including school mathematics teaching applications in an undergraduate abstract algebra course. *Primus*, *32*(6), 685–703.

Antonini, S., Furinghetti, F., Morselli, F., & Tosetto, E. (2007). University students generating examples in real analysis: where is the definition? In D. Pitta-Pantazi & G. Philippou (Eds.), *Proceedings of the 5th congress of the European Society for Research in Mathematics Education*, (pp. 2241–2249). ERME. http://ermeweb.free.fr/CERME5b/.

Bagni, G.T. (2005). The historical roots of the limit notion: Cognitive development and the development of representation registers. *Canadian Journal of Science, Mathematics and Technology Education*, *5*, 453–468. https://doi.org/10.1080/14926150509556675

Baki, M., & Çekmez, E. (2012). İlköğretim matematik öğretmeni adaylarının limit kavramının formal tanımına yönelik anlamalarının incelenmesi. *Turkish Journal of Computer and Mathematics Education*, *3*(2), 81–98.

Bakker, A. (2018). *Design research in education: A practical guide for early career researchers* (1st ed.). Routledge.

Baldinger, E. E. (2014). *Studying abstract algebra to teach high school algebra: Investigating future teachers' development of mathematical knowledge for teaching* (Unpublished doctoral dissertation). Stanford University, Stanford, CA.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, *29*(1), 14–46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Bansilal, S. & Mkhwanazi, T. W. (2022). Pre-service student teachers' conceptions of the notion of limit. *International Journal of Mathematical Education in Science and Technology*, *53*(8), 2083–2101. https://doi.org/10.1080/0020739X.2020.1864488

Baştürk, S. & Dönmez, G. (2011). Mathematics student teachers' misconceptions on the limit and continuity concepts. *Necatibey Faculty of Education Electronic Journal of Science and Mathematics Education*, *5*(1), 225–249.

Belin, M. & Karagöz Akar, G. (2020a). The effect of quantitative reasoning on pre-service mathematics teachers' proof comprehension: The case of real numbers. *Journal of Mathematical Behavior, 57*. https://doi.org/10.1016/j.jmathb.2020.100757.

Belin, M., & Karagöz Akar, G. (2020b). Exploring real numbers as rational number sequences with pre-service mathematics teachers. *Mathematics Teacher Educator*, *9*(1), 63–87. https://doi.org/10.5951/MTE.2020.9999

Bezuidenhout, J. (2001). Limits and continuity: Some conceptions of first year students. *International Journal of Mathematical Education in Science and Technology*, *32*(4), 487–500. https://doi.org/10.1080/00207390010022590

Boester, T. (2008). A design-based case study of undergraduates' conceptions of limits [Doctoral dissertation]. Retrieved from ProQuest Dissertations and Theses database. (UMI No. 3314378). University of Wisconsin--Madison.

Buchholtz, N., Leung, F., Ding, L., Kaiser, G., Park, K., & Schwarz, B. (2013). Future mathematics teachers' professional knowledge of elementary mathematics from an advanced standpoint. *ZDM*, *45*, 107-120. https://doi.org/10.1007/s11858-012-0462-6

Carlson, M. P., Larsen, S., & Jacobs, S. (2001). An investigation of covariational reasoning and its role in learning the concepts of limit and accumulation. *Proceedings of the 23rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, *1*, 145–153.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, *33*(5), 352–378.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics (CCSSM). Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. https://www.thecorestandards.org/wp-content/uploads/Math_Standards1.pdf

Choi, Y., & Do, J. (2005). Equality involved in 0.9999... and $(-8)^{\frac{1}{3}}$. *For the Learning of Mathematics*, *25*(3), 13-15.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. Lesh (Eds), *Handbook of research design in mathematics and science education* (pp. 517–545). Lawrence Erlbaum Associates.

Cobb, P., & Gravemeijer, K. (2008). Experimenting to support and understand learning processes. In A. E. Kelly, R. A. Lesh & J. Y. Baek (Eds.), *Handbook of design research methods in education* (pp. 68–95). Routledge.

Cobb, P., Confrey, J., DiSessa, A., Lehrer, R., & Schauble, L. (2003). Design Experiments in Educational Research. *Educational Researcher*, *32*(1), 9–13. http://doi.org/10.3102/0013189X032001009

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers*. Providence, R.I. and Washington, D.C.: American Mathematical Society and Mathematical Association of America. Retrieved from: http://cbmsweb.org/MET_Document/index.htm

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. Providence, R.I. and Washington, D.C.: American Mathematical Society and Mathematical Association of America. Retrieved from: https://www.cbmsweb.org/archive/MET2/met2.pdf

Conner, (2013). Authentic argumentation with prospective secondary teachers: The case of 0.999… *Mathematics Teacher Educator*, *1*(2), 172–180

Cornu, B. (2002). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 153–166). Springer. https://doi.org/10.1007/0-306-47203-1_10

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. *Journal of Mathematical Behavior*, *15*, 167–192. https://doi.org/10.1016/S0732-3123(96)90015-2

Creswell, J. W., & Creswell, J. D. (2018). Research design: Qualitative, quantitative, and mixed methods approaches (5th ed.). Sage Publications.

Creswell, J. W. (2013). *Qualitative inquiry and research design: Choosing among five approaches*, (3rd Edition). Thousand Oaks, CA: Sage Publication.

Crisan C. (2019, February). *Empowering teachers conceptually and pedagogically through supporting them in seeing connections between school mathematics and relevant advanced mathematics knowledge*. Paper presented at the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11), Utrecht, Netherlands. http://www.mathematik.uni-dortmund.de/~prediger/ERME/CERME11_Proceedings_2019.pdf

Cuoco, A. (2001). Mathematics for teaching. *Notices of the American Mathematical Society*, *48*(2), 168–174.

Cuoco, A., Goldenberg, E. P., & Mark, J. (1996). Habits of mind: An organizing principle for mathematics curricula. *Journal of Mathematical Behavior*, *15*(4), 375–402.

Davis, R. B., & Vinner, S. (1986). The notion of limit: Some seemingly unavoidable misconception stages. *Journal of Mathematical Behavior*, *5*, 281–303

Dawson, B. (2016). 0.99..= 1: An infinitesimal explanation. *Math Horizons*, *24*(2), 5–7. https://doi.org/10.4169/mathhorizons.24.2.5

Dilberoğlu, M. (2015). *An investigation of pre-service middle school mathematics teachers' ability to connect the mathematics in content courses with the middle school mathematics* [Unpublished doctoral dissertation]. Middle East Technical University, Ankara.

Dixon, N., Carroll, E., & Teuscher, D. (2020). Influence of curriculum on college students' understanding and reasoning about limits. In Karunakaran, S. S., Reed, Z., & Higgins, A. (Eds), *Proceedings of the 23rd annual conference for Research in Undergraduate Mathematics Education*, (pp. 115–123).

Dreher, A., Lindmeier, A., Heinze, A., & Niemand, C. (2018). What kind of content knowledge do secondary mathematics teachers need? A conceptualization taking into account academic and school mathematics. *Journal Für Mathematik - Didaktik*, *39*, 319–341. https://doi.org/10.1007/s13138-018-0127-2

Dreyfus, T. (2002). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25–41). Springer. https://doi.org/10.1007/0-306-47203-1_2

Durand-Guerrier, V., & Arsac, G. (2003). Méthodes de raisonnement et leurs modélisations logiques. Spécificité de l'analyse. Quelles implications didactiques? (Reasoning methods and their logical models. Specificity of calculus. Which didactical implications?). *Recherches en Didactique des Mathématiques*, *23*(3), 295–342.

Edwards, B. S., Dubinsky, E., & McDonald, M. A. (2005). Advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 15–25. https://doi.org/10.1207/s15327833mtl0701_2

Ervynck, G. (1981). Conceptual difficulties for first year university students in the acquisition of the notion of limit of a function. In *Proceedings of the Fifth Conference of the International Group for the Psychology of Mathematics Education* (pp. 330-333).

Furinghetti, F., & Radford, L. (2002). Historical conceptual developments and the teaching of mathematics: From philogenesis and ontogenesis theory to classroom practice. In *Handbook of international research in mathematics education* (pp. 643-666). Routledge.

Goldin, G. (2000). A scientific perspective on structures, task-based interviews in mathematics education research. In A. E. Kelly & R. Lesh (Eds), *Handbook of research design in mathematics and science education* (pp. 517–545). Lawrence Erlbaum Associates.

Grant, M. R., Crombie, W., Enderson, M., & Cobb, N. (2016). Polynomial calculus: rethinking the role of calculus in high schools. *International Journal of Mathematical Education in Science and Technology*, *47*(6), 823–836.

Gravemeijer, K., & Van Eerde, D. (2009). Design research as a means for building a knowledge base for teachers and teaching in mathematics education. *Elementary School Journal*, *109*(5), 510–524.

Gueudet, G. (2008). Investigating the secondary–tertiary transition. *Educational studies in mathematics*, *67*(3), 237–254.

Harel, G., & Sowder, L. (2005). Advanced mathematical-thinking at any age: Its nature and its development. *Mathematical Thinking and Learning*, *7*(1), 27–50. https://doi.org/10.1080/0020739X.2015.1133851

Harel, G., & Kaput, J. (1990). The role of conceptual entities in learning mathematical concepts at the undergraduate level. In *Proceedings of the XIVth PME conference* (Vol. 1, pp. 53–60).

Harel, G., Selden, A., & Selden, J. (2006). Advanced mathematical thinking. In A. Gutierrez & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education* (pp. 147–172). Rotterdam, The Netherlands: Sense.

Herlina E., & Batusangkar, S. (2015). Advanced mathematical thinking and the way to enhance it. *Journal of Education and Practice*, *6*(5), 79–88.

Hill, R., & Senk, S. L. (2004). A capstone course for prospective high school mathematics teachers. *Mathematicians and Education Reform Newsletter*, *16*(2), 8–11.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, *105*(1), 11–30.

Hoffmann, A., & Even, R. (2018). What do mathematicians wish to teach teachers in secondary school about mathematics? *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (pp. 99–106). PME.

Hoffmann, A., & Even, R. (2024). What do mathematicians wish to teach teachers about the discipline of mathematics? *Journal of Mathematics Teacher Education*, *27*, 479–497. https://doi.org/10.1007/s10857-023-09577-4

Iannone, P., & Nardi, E. (2007, February). The interplay between syntactic and semantic knowledge in proof production: mathematicians' perspectives.

Proceedings of The Fifth Congress of the European Society for Research in Mathematics Education (pp. 2300–2309).

Jakobsen, A., Thames, M. H., Ribeiro, C. M., & Delaney, S. (2012, July). Using practice to define and distinguish horizon content knowledge. *12th International Congress in Mathematics Education (12th ICME)* (pp. 4635–4644).

Jensen, T. A. (2009). *A study of the relationship between introductory calculus students' understanding of function and their understanding of limit* [Doctoral dissertation, Montana State University]. Montana State University ProQuest Dissertations & Theses. https://www.proquest.com/dissertations-theses/study-relationship-between-introductory-calculus/docview/304941712/se-2

Juter, K. (2006). *Limits of functions: University students' concept development* [Doctoral dissertation, Luleå Tekniska Universitet]. https://urn.kb.se/resolve?urn=urn:nbn:se:ltu:diva-25675

Juter, K. (2007). Students' conceptions of limits, high achievers versus low achievers. *The Mathematics Enthusiast*, *4*(1), 53–65. https://doi.org/10.54870/1551-3440.1058

Juter, K. (2022). Pre-service teachers' general and specific arguments in real number contexts. *Mathematics Teacher Education and Development*, *24*(1), 1–22.

Kajander, A., & Lovric, M. (2017) Understanding and supporting teacher horizon knowledge around limits: A framework for evaluating textbooks for teachers. *International Journal of Mathematical Education in Science and Technology*, *48*(7), 1023-1042. https://doi.org/10.1080/0020739X.2017.1301583

Katz, K., & Katz, M. (2010b). Zooming in on infinitesimal 1–0.9… in a post-triumvirate era. *Educational Studies in Mathematics*, *74*(3), 259–273.

Kidron, I., & Tall, D. (2015). The roles of visualization and symbolism in the potential and actual infinity of the limit process. *Educational Studies in Mathematics*, *88*, 183–199.

Klein, F. (1932). *Elementary mathematics from an advanced standpoint: Arithmetic, algebra, analysis* (E. R. Hendrick & C. A. Noble, Trans., 3rd ed.). Macmillan. (Original work published 1924).

Krussel, C. E. S. (1994). *Visualization and reification of concepts in advanced mathematical thinking* (Publication No. 9517901) [Doctoral dissertation, Oregon State University]. ProQuest Dissertations & Theses Global.

Larsen, S., Glover, E., Bergman, A. M., & Caughman, J. (2018). What kind of opportunities do abstract algebra courses provide for strengthening future teachers' mathematical knowledge for teaching? *Connecting Abstract*

Algebra to Secondary Mathematics for Secondary Mathematics Teachers (pp. 71–84). Springer.

Larsen, S., Marrongelle, K., Bressoud, D. M., & Graham, K. (2017). Understanding the concepts of calculus: Frameworks and roadmaps emerging from educational research. In J. Cai (Ed.), *Compendium for research in mathematics education*. (pp. 526–550). National Council of Teachers of Mathematics.

Lee, B. S. (1992). *An investigation of pre-service secondary mathematics teachers' understanding of the mathematical limit concept* [Unpublished doctoral dissertation]. Michigan State University.

Leikin, R., Zazkis, R., & Meller, M. (2018). Research mathematicians as teacher educators: Focusing on mathematics for secondary mathematics teachers. *Journal of Mathematics Teacher Education*, *21*(5), 451–473.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Sage Publications.

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M. D., & Senk, S. L. (2012). Knowledge of algebra for teaching: A framework of knowledge and practices. *Journal for Research in Mathematics Education*, *43*(5), 584–615.

Merriam S. B., & Tisdell E. J. (2015). *Qualitative research: A guide to design and implementation* (4th ed.). Jossey-Bass.

Ministry of National Education [MoNE] (2018). Mathematics curriculum for secondary school education: grades 9-12. Ankara: Ministry of National Education.

Monaghan, J. (1991). Problems with the language of limits. *For the Learning of Mathematics*, *11*(3), 20–24.

Monk, D. H. (1994). Subject area preparation of secondary mathematics and science teachers and student achievement. *Economics of Education Review*, *13*(2), 125-145

Moru, E. K. (2009). Epistemological obstacles in coming to understand the limit of a function at undergraduate level: A case from the national university of Lesotho. *International Journal of Science and Mathematics Education*, *7*(3), 431–454. https://doi.org/10.1007/s10763-008-9143-x

Murray, E., & Star, J. R. (2013). What do secondary preservice mathematics teachers need to know? Content courses connecting secondary and tertiary mathematics. *Notices of the American Mathematical Society*, *60*(10), 1297–1298.

Nieveen, N., & Folmer, E. (2013). Formative evaluation in educational design research. *Design Research*, *153*(1), 152–169.

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, *40*(4), 396–426. https://doi.org/10.5951/jresematheduc.40.4.0396

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' understandings of function. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics*. (pp. 27–42). Mathematical Association of America.

Páez, R., & Hitt, F. (2004, October). *On the limit concept in a cooperative learning environment: A case study* [Paper presentation]. North American Chapter of the International Group for the Psychology of Mathematics Education, Toronto, Ontario, Canada.

Plomp, T. (2013). Educational design research: An introduction. In T. Plomp, & N. Nieveen (Eds.), *Educational design research – part A: An introduction* (pp. 10-51). Enschede, the Netherlands: SLO.

Prediger, S., Gravemeijer, K., & Confrey, J. (2015). Design research with a focus on learning processes: An overview on achievements and challenges. *ZDM*, *47*, 877–891.

Reeves, T. C. (2000). Enhancing the worth of instructional technology research through "design experiments" and other development research strategies. *International Perspectives on Instructional Technology Research for the 21st Century*, *27*(1), 1–15.

Richman, F. (1999). Is 0.999… = 1? *Mathematics Magazine*, *72*(5), 396–400. https://doi.org/10.1080/0025570X.1999.11996777

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*, 255–281. https://doi.org/10.1007/s10857-005-0853-5

Ruthven, K., Laborde, C., Leach, J., & Tiberghien, A. (2009). Design tools in didactical research: Instrumenting the epistemological and cognitive aspects of the design of teaching sequences. *Educational Researcher*, *38*(5), 329–342. https://doi.org/10.3102/0013189X093385

Sacristan, A. I. (1997). *Windows on the infinite: Constructing Meanings in a Logo-based Microworld*. [Doctoral dissertation, University of London] https://discovery.ucl.ac.uk/id/eprint/10021817/1/149009.pdf

Savin-Baden, M. & Major, C. (2013). *Qualitative research: The essential guide to theory and practice*. Routledge, London.

Savuran, R. (2022). *Development of prospective secondary school mathematics teachers' specialized knowledge of limit concept through lesson study*. [Doctoral dissertation, Middle East Technical University] https://hdl.handle.net/11511/97349

Schreier, M. (2012). *Qualitative content analysis in practice.* London: Sage Publications.

Selden, A., & Selden, J. (Guest Eds.). (2005). Perspectives on advanced mathematical thinking. *Mathematical Thinking and Learning*, *7*(1), 1–13. https://doi.org/10.1207/s15327833mtl0701_1

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4–14. https://doi.org/10.3102/0013189X015002004

Sierpinska, A. (1987). Humanities students and epistemological obstacles related to limits. *Educational Studies in Mathematics*, *18*, 371–397. https://doi.org/10.1007/BF00240986

Stacey, K. (2008). Mathematics for secondary teaching: Four components of discipline knowledge for a changing teacher workforce. In P. Sullivan & T. Wood (Eds.), *The international handbook of mathematics teacher education* (pp. 87–113). Sense Publishers.

Stockton, J. C. & Wasserman, N. H. (2017). Forms of knowledge of advanced mathematics for teaching. *The Mathematics Enthusiast*, *14*(1), 575–606. https://doi.org/10.54870/1551-3440.1412

Suominen, A. L. (2015). *Abstract algebra and secondary school mathematics: Identifying and classifying mathematical connections* [Doctoral dissertation, University of Georgia]. https://getd.libs.uga.edu/pdfs/suominen_ashley_l_201505_phd.pdf

Swinyard, C. (2011). Reinventing the formal definition of limit: The case of Amy and Mike. *The Journal of Mathematical Behavior*, *30*(2), 93–114.

Swinyard, C., & Larsen, S. (2012). Coming to understand the formal definition of limit: Insights gained from engaging students in reinvention. *Journal for Research in Mathematics Education*, *43*(4), 465–493.

Szydlik, J. E. (2000). Mathematical beliefs and conceptual understanding of the limit of a function. *Journal for Research in Mathematics Education*, *31*(3), 258–276. https://doi.org/10.2307/749807

Tall, D. (1977). Conflicts and catastrophes in the learning of mathematics. *Mathematical Education for Teaching*, *2*(4), 2–18.

Tall, D. (1988). The nature of advanced mathematical thinking. In *Proceedings of Psychology of Mathematics Education Conference, Hungary*.

Tall, D. (1992). The transition to advanced mathematical thinking: functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495-511). New York: Macmillan.

Tall, D. (1995). Cognitive growth in elementary and advanced mathematical thinking. In L. Meira & D. Carraher (Eds.), *Proceedings of the 19th International Conference for the Psychology of Mathematics Education* (Vol. 1, pp. 61–75). Recife, Brazil: Universidade Federal de Pernambuco.

Tall, D. (2002). The psychology of advanced mathematical thinking. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 3–21). Springer. https://doi.org/10.1007/0-306-47203-1_1

Tall, D., & Katz, M. (2014). A cognitive analysis of Cauchy's conceptions of function, continuity, limit and infinitesimal, with implications for teaching the calculus. *Educational Studies in Mathematics*, *86*, 97–124. https://doi.org/10.1007/s10649-014-9531-9

Tall, D., & Schwarzenberger, R. L. E. (1978). Conflicts in the learning of real numbers and limits. *Mathematics Teaching*, *82*, 44–49.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.

Tall, D.O. (2009). Dynamic mathematics and the blending of knowledge structures in the calculus. *ZDM*, *41*, 481–492. https://doi.org/10.1007/s11858-009-0192-6

Tallman, M. A., & Frank, K. M. (2020). Angle measure, quantitative reasoning, and instructional coherence: an examination of the role of mathematical ways of thinking as a component of teachers' knowledge base. *Journal of Mathematics Teacher Education*, *23*(1), 69–95.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education*. (pp. 421–456). National Council of Teachers of Mathematics. https://doi.org/10.1007/s11858-021-01270-1

Tsamir, P., & Tirosh, D. (1992). Students' awareness of inconsistent ideas about actual infinity. *Proceedings of the Sixteenth PME Conference*, *3*, 90–97. University of New Hampshire.

Usiskin, Z., Peressini, A., Marchisotto, E. A., & Stanley, D. (2003). *Mathematics for high school teachers: An advanced perspective*. Pearson.

Van den Akker, J., Gravemeijer, K., & McKenney, S. (2006). Introducing educational design research. In *Educational design research* (pp. 15–19). Routledge.

VERBI Software. (2021). *MAXQDA 2022* [computer software]. Berlin, Germany: VERBI Software. https://maxqda.com

Wasserman, N. H. (2016). Abstract algebra for algebra teaching: Influencing school mathematics instruction. *Canadian Journal of Science Mathematics and Technology Education*, *16*(1), 28–47. https://doi.org/10.1080/14926156.2015.1093200

Wasserman, N. H. (2018). Knowledge of nonlocal mathematics for teaching. *The Journal of Mathematical Behavior*, *49*, 116–128. https://doi.org/10.1016/j.jmathb.2017.11.003

Wasserman, N. H., Weber, K., Fukawa-Connelly, T., & McGuffey, W. (2019). Designing advanced mathematics courses to influence secondary teaching: Fostering mathematics teachers' "attention to scope". *Journal of Mathematics Teacher Education*, *22*, 379–406. https://doi.org/10.1007/s10857-019-09431-6

Wasserman, N., Fukawa-Connelly, T., Villanueva, M., Mejia-Ramos, J. P., & Weber, K. (2017). Making real analysis relevant to secondary teachers: Building up from and stepping down to practice. *PRIMUS*, *7*(6), 559–578.

Wasserman, N., Weber, K., & McGuffey, W. (2017). Leveraging real analysis to foster pedagogical practices. In A. Weinberg, C. Rasmussen, J. Rabin, M. Wawro, & S. Brown (Eds.). *Proceedings of the 20th Annual Conference on Research in Undergraduate Mathematics Education (RUME)* (pp. 1–15). RUME.

Wasserman, N., Weber, K., Villanueva, M., & Mejia-Ramos, J. P. (2018). Mathematics teachers' views about the limited utility of real analysis: A transport model hypothesis. *The Journal of Mathematical Behavior*, *50*, 74–89.

Wasserman, N.H. (2023). Investigating a teacher-perspective on pedagogical mathematical practices: Possibilities for using mathematical practice to develop pedagogy in mathematical coursework. *ZDM*, *55*, 807–821. https://doi.org/10.1007/s11858-023-01468-5

Wasserman, N.H., Fukawa-Connelly, T., Weber, K., Pablo Mejia-Ramos, J., Abbott, S. (2022). Algebraic Limit Theorems and Error Accumulation. In: Understanding Analysis and its Connections to Secondary Mathematics Teaching. Springer Texts in Education. Springer, Cham. https://doi.org/10.1007/978-3-030-89198-5_4

Weber, K., Mejía-Ramos, J. P., Fukawa-Connelly, T., & Wasserman, N. (2020). Connecting the learning of advanced mathematics with the teaching of secondary mathematics: Inverse functions, domain restrictions, and the arcsine function. *The Journal of Mathematical Behavior*, *57*. https://doi.org/10.1016/j.jmathb.2019.100752

Wiley, K. E. (2014). *Middle school mathematics teachers' use of advanced mathematical knowledge in practice: An interpretative phenomenological analysis* (Order No. 3665237). Available from ProQuest Dissertations & Theses Global. (1637646591). Retrieved from http://search.proquest.com/docview/1637646591?accountid=13014

Wu, H. (2011). Phoenix rising: Bringing the Common Core state mathematics standards to life. *American Educator*, *35*(3), 3–13.

Yan, X., Marmur, O. & Zazkis, R (2022). Advanced mathematics for secondary school teachers: Mathematicians' perspective. *International Journal of Science and Mathematics Education*, *20*, 553–573. https://doi.org/10.1007/s10763-020-10146-x

Yan, X., Marmur, O., & Zazkis, R. (2020). Calculus for teachers: Perspectives and considerations of mathematicians. *Canadian Journal of Science, Mathematics and Technology Education*, *20*(2), 355–374. https://doi.org/10.1007/s42330-020-00090-x

Yopp, D. A., Burroughs, E. A., & Lindaman, B. J. (2011). Why it is important for in-service elementary mathematics teachers to understand the equality. 999…=1. *The Journal of Mathematical Behavior*, *30*(4), 304–318. https://doi.org/10.1016/j.jmathb.2011.07.007

Zazkis, R. & Leikin, R. (2010). Advanced mathematical knowledge in teaching practice: perceptions of secondary mathematics teachers. *Mathematical Thinking and Learning*, *12*(4), 263–281. https://doi.org/10.1080/10986061003786349

Zazkis, R., & Mamolo, A. (2011). Reconceptualizing knowledge at the mathematical horizon. *For the Learning of Mathematics*, *31*(2), 8–13.

Ziegler, G. M., & Loos, A. (2017). "What is Mathematics?" and why we should ask, where one should experience and learn that, and how to teach it. In G. Kaiser (Ed.), *Proceedings of the 13th International Congress on Mathematical Education* (pp. 63–77). Springer International Publishing

APPENDICES

A. Content Questionnaire Test

Herkese merhaba! Bu test, başarınızı değerlendirmek için değil, bugünkü oturum öncesinde ilgilenilen konular hakkındaki ön bilginizi gözlemlemek için hazırlanmış bir testtir. Lütfen aşağıdaki soruları, soru hakkındaki fikirlerinizi olabildiğince farklı yollardan ifade ederek ve detaylı bir şekilde cevaplandırmaya çalışınız.

1) Kendi cümlelerinizle limit kavramının matematiksel anlamını tarif ediniz. Semboller, tablolar, grafikler ve / veya sözel anlatım gibi olabildiğince farklı sayıda aracı tanımlamanızda kullanınız.

2) Limiti tanımlarken kullandığımız “x, c ye yaklaşırken f(x) değeri de L ye yaklaşır” ifadesi ne derecede açık ve anlaşılırdır? İfadenin varsa eksik noktalarını belirterek; limitin tanımı olarak daha kabul edilebilir hale getirmek için yukarıdaki ifadede ne gibi düzenlemeler yapılması gerektiğini belirtiniz.

3) Matematikte eşitlik ne demektir? Farklı örneklerle eşitliğin ne anlama geldiğini açıklayınız.

4) Reel sayıların inşası size neleri anımsatıyor, reel sayıların inşası dendiğinde aklınıza gelen kavramları aşağıya yazınız.

B. Pre-Test

Bu test toplam 5 sorudan ve 4 sayfadan oluşmaktadır. Soruları tek seferde çözmeniz ve fikirlerinizi detaylı bir şekilde açıklamanız çalışmanın etkiliği açısından çok önemlidir. Şimdiden ayırdığınız zaman ve emeğiniz için teşekkür ederim.

a) 1.5999... ile 1.6 arasındaki ilişki nedir? Eşitlerse/eşit değillerse nasıl?

b) a şıkkındaki matematiksel durum matematikte hangi konularla ve fikirlerle ilişkili olabilir? Lütfen aklınıza gelen tüm fikirlerinizi, ilişkili olarak düşündüğünüz konuları yazınız.

c) Öğrenciler bu durumu anlamakta ne gibi zorluklar ve kavram yanılgıları yaşayabilirler? Hangi kavramları anlamamış olmaları 1.5999... ile 1.6 arasındaki ilişkiyi de anlamalarını zorlaştırıyordur? Başka bir deyişle, bu konuyu anlamakta güçlük çeken bir öğrencinin başka hangi konu veya kavramlarla ilgili sorun yaşıyor olabileceğini düşünürsünüz?

d) Bu iki sayı eğer birbirine eşitse, bir öğretmen olarak öğrencileri buna nasıl ikna edersiniz? (Farklı sınıf seviyeleri için ayrı ayrı düşünebilirsiniz). Kullanabileceğiniz her türlü aracı (modeller, figürler, gösterimler, teoremler, görseller) detaylı bir şekilde açıklayınız.

e) Bu iki sayının birbirine eşit olmadığını iddia eden bir öğrenciye nasıl karşılık verirsiniz? Matematiksel ve pedagojik yönden böyle bir sınıf ortamında nasıl davranacağınızı ayrıntılı olarak açıklamaya çalışınız.

C. Demographic Information Questionnaire

Bu testte mesleki deneyiminizi öğrenmek için sorulmuş toplam 5 soru bulunmaktadır.

1) Hangi üniversite ve hangi bölümden mezunsunuz?

2) Üniversite eğitiminiz boyunca aldığınız matematik derslerinden özellikle ilginizi çeken, sevdiğiniz ve zorlandığınız dersler hangileriydi?

3) Kaç senedir öğretmenlik yapmaktasınız?

4) Hangi okul türlerinde ve hangi sınıf seviyelerinde öğretmenlik yaptınız?

5) Üniversite eğitiminiz sonrası matematik ya da matematik eğitimi alanı ile ilgili herhangi bir eğitim ya da faaliyete katıldınız mı?

D. Post-Test

Bu test toplam 6 sorudan oluşmaktadır. Soruları tek seferde çözmeniz ve fikirlerinizi DETAYLI bir şekilde açıklamanız çalışmanın sonuçları açısından çok önemlidir. Şimdiden ayırdığınız zaman ve emeğiniz için teşekkür ederim.

Habibe Toker Kekik

1) Aşağıda yer alan her bir durum için altında yer alan soruları cevaplayınız.

<u>Gösterim 1:</u>

$$7.\underbrace{999\ldots9}_{n} = 8 - \left(\frac{1}{10}\right)^{n}$$

$$7.99\ldots9 = \lim_{n\to\infty} 8 - \left(\frac{1}{10}\right)^{n} = 8 - \lim_{n\to\infty}\left(\frac{1}{10}\right)^{n} = 8 - 0 = 8$$

i. Çözümü, her aşamada kullanılan matematiksel adım ve süreçleri gerekçelerini de belirterek ayrıntılı bir şekilde tekrar açıklayabilir misiniz?

ii. Ne tür bir öğretim ortamında (bağlamında) bu ispatın kendi öğretiminiz için yararlı olabileceğini düşünüyorsunuz?

iii. Bir önceki soruda belirttiğiniz bağlama uygun olarak sınıfta gerçekleşebilecek hayali bir durum veya bir öğrenciyle aranızda geçebilecek hayali bir diyalog oluşturunuz.
Öğrenci: …
Öğretmen: …
Öğrenci: …
Öğretmen: …

<u>Gösterim 2:</u>

$$x = 7.9999\ldots \text{ olsun}$$

$$10x = 79.999\ldots$$

$$10x - x = 79.9999\ldots - 7.9999\ldots$$

$$9x = 72 \text{ ise } x = 8 \text{ dir.}$$

i. Çözümü, her aşamada kullanılan matematiksel adım ve süreçleri gerekçelerini de belirterek ayrıntılı bir şekilde tekrar açıklayabilir misiniz?

ii. Ne tür bir öğretim ortamında (bağlamında) bu ispatın kendi öğretiminiz için yararlı olabileceğini düşünüyorsunuz?

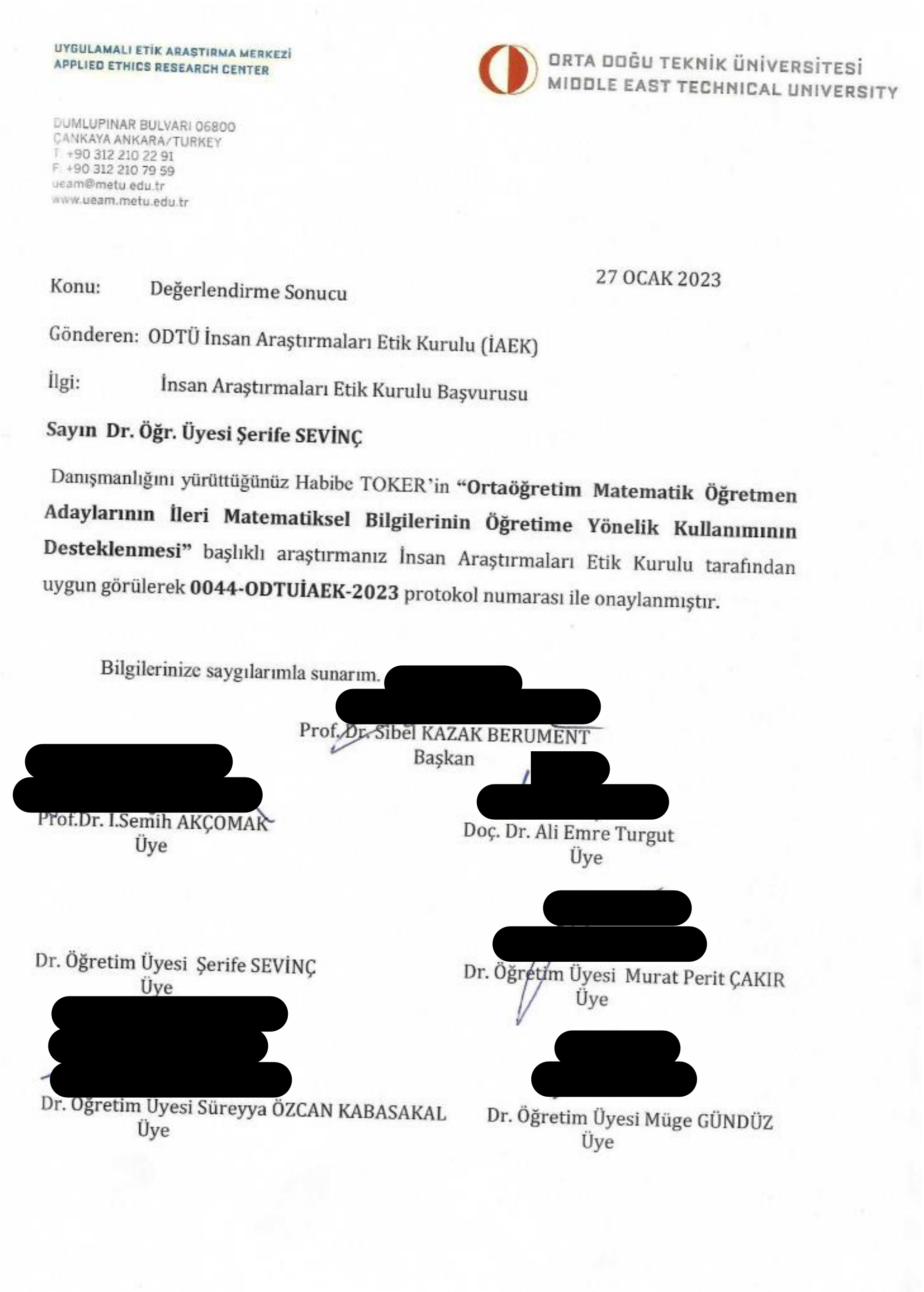
iii. Bir önceki soruda belirttiğiniz bağlama uygun olarak sınıfta gerçekleşebilecek hayali bir durum veya bir öğrenciyle aranızda geçebilecek hayali bir diyalog oluşturunuz.
Öğrenci: …
Öğretmen: …
Öğrenci: …
Öğretmen: …

2) 7.999... sayısının 8'e nasıl eşit olduğunu gösteren iki ayrı gösterimi derslerinizde kullandığınızı; fakat bazı öğrencilerin bu gösterimleri anlamakta zorluk çektiğini veya konuyu kafasında tam oturtamadığını fark ettiğini varsayalım.

a. Bu durumdaki bir öğrenciyi nasıl yönlendirirsiniz? Ona neler söylersiniz?

b. Öğrenciyi/öğrencileri farklı matematik kavramlarına, tanımlara ya da teoremlere yönlendirecek olsanız bunlar neler olurdu? (Cevaplarınızda sizinle yapılan çalışmada edindiğiniz bilgileri, öğrendiğiniz terim, kavram, teorem ya da etkinlikleri de kullanabilirsiniz.)

E. METU Human Subjects Ethics Committee Approval/ODTU İnsan Araştırmaları Etik Kurul Onayı



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27 OCAK 2023

Konu: Değerlendirme Sonucu

Gönderen: ODTÜ İnsan Araştırmaları Etik Kurulu (İAEK)

İlgi: İnsan Araştırmaları Etik Kurulu Başvurusu

Sayın Dr. Öğr. Üyesi Şerife SEVİNÇ

Danışmanlığını yürüttüğünüz Habibe TOKER'in **"Ortaöğretim Matematik Öğretmen Adaylarının İleri Matematiksel Bilgilerinin Öğretime Yönelik Kullanımının Desteklenmesi"** başlıklı araştırmanız İnsan Araştırmaları Etik Kurulu tarafından uygun görülerek **0044-ODTUİAEK-2023** protokol numarası ile onaylanmıştır.

Bilgilerinize saygılarımla sunarım.

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F. Interview Questions of the Interview Study

Question 1: Which terms or key ideas in limit do you consider crucial for mathematics teachers?

Question 2: What do you think about the importance of visualization in mathematics? While teaching limit, do you adopt visual or graphical representations? For which purposes do you require the use of visual representations? (e.g., to make students gain an intuitive understanding of the limit or to teach it formally, etc.)

Question 3: When you think of the place of limit in the span of mathematics subjects, to what extent could limit be important and indispensable for mathematics teachers?

Question 4: Limit has two different definitions: formal and informal. But in the secondary school curriculum, it is strictly noted that "*The limit of a function at a point is not given through ε-δ technique, which is more of a concern to mathematicians*" (MoNE, 2018). So, why do you think a mathematics teacher should learn the formal definition of limit? In teaching which topics, a mathematics teacher could benefit from the idea behind the formal definition of limit?

Question 5: In which way do you think the formal definition of limit could be taught to mathematics teachers efficiently?

Question 6: What do you think about the importance of knowing the proofs of theorems in the context of limit for mathematics teachers? Which theorems do you think they must know definitely? Why?

Question 7: Could you define a mathematical problem in which if someone borrows ideas from the concept of limit, the solution could be more sophisticated, meaningful, and shorter? (Such as maximizing area problems)

Question 8: In the generation of formula for the area of a circle, the limit process is used. But suppose that a mathematics teacher is asked, "*A polygon of any number of sides has corners, but the circle does not. How does the limit process eliminate corners*?" (Kajander & Lovric, 2017, p. 1035).

How would this situation be related to the concepts, theorems, or definitions that a mathematics teacher has learned in calculus courses regarding limit?

Question 9: *"There is a possible parallelism between the historical development of ideas about infinitesimals and developments within a student's understanding" (Bagni, 2005, p. 461).*

This part is taken from an article about the historical developments of limit notion. What do you think about this statement?

Could knowing more of history help an undergraduate student? In which ways could it be helpful for mathematics teachers especially?

Question 10: The notion of infinitely small and infinitely large is hard to conceptualize for undergraduate students. While some consider an infinitesimal as simply a variable which tends to zero, some view the symbol ε as a number not zero, but smaller than any positive real numbers. As a result of this understanding, 0.999... could be believed as "*the last number before 1*" (Cornu, 2002, p.161). How could you evaluate the importance of understanding infinitesimals correctly?

G. Teaching Sequence

Number of steps	**Details of the step**
Step 1.1	**A discussion activity on the relationship between 1. 599.. and 1.6** • What is the relationship between 1. 599.. and 1.6? If they are equal, how? If they are not equal, how? • How do you think students perceive 1. 599..? What do you think? • What differences might occur for students in understanding the numbers of -3, 2, 1, 6 and the number 1. 599...? • Which subjects/ideas/topics in mathematics could the relationship between 1. 599.. and 1.6 be considered as related to? Let's try to write down all your ideas on the board.
Step 1.2	**Introduction from decimal representation** ○ What is the decimal representation of $\frac{1}{2}$? It is 0.5. ○ So, **can we say that decimal representations are unique? Is there only one decimal representation for any number?** ○ At the first glance, you notice that $0.50, 0.500, 0.5\overline{0}$ also indicate 0.5. But although they seem different apparently, the way of writing them as an infinite string of digits gives the same result as 0.500… • "A decimal representation implicitly refers to an infinite string of digits" (Wasserman et. al., 2022, p. 12). In other words, it is considered that every decimal representation includes an infinite number of digits. In other words, $0.50, 0.500, 0.5\overline{0}$ are all similar representations of the same decimal 0.500… • **Can one real number have two different decimal representations? Or could two infinite decimal representations be equal? If so, how?**

	(*Note for the implementation process: Here, give time to participants to think and then listen for the answers. After that, participants are given the theorem below which is related to the equality of two real numbers. Ask participants to read the theorem and express what they understand*.)
Step 1.3	**Thinking on a theorem and reasoning on a proof of a theorem** - Theorem: "Two real numbers a and b are equal if and only if for every real number ε > 0 it follows that $\lvert a - b \rvert < \varepsilon$" (Wasserman et. al., 2022, p. 14). - **(Be able to conceptualize what a theorem claims and to make reasoning on the proof of a theorem)** **What does this theorem say to us? What did you understand from this theorem?** **How can we prove this theorem?** *(Note for the implementation process: Listen to participants' answers. We want participants to read and express what they understand, and how they use the statement of "if and only if". Then give the proof by reflecting step by step through the presentation)* **Proof: (**⇒) For the proof of the first statement, if $a = b$ then $\lvert a - b \rvert = 0$ and so certainly $\lvert a - b \rvert < \epsilon$ no matter $\epsilon > 0$ is chosen. **How could we prove the other part?** (⇐) proof by contradiction. Suppose $\lvert a - b \rvert < \varepsilon$ for every real number $\varepsilon > 0$. Assume for a contradiction, $a \neq b$. It implies $\lvert a - b \rvert > 0$. So, we can choose $\lvert a - b \rvert$ as one of the epsilons. Then following $\lvert a - b \rvert < \varepsilon$ for every real number $\varepsilon > 0$, we can write $\lvert a - b \rvert < \lvert a - b \rvert$ which leads to a contradiction. So, our initial assumption $a \neq b$ cannot be true. Therefore, $a = b$.
Step 1.4	**Interpretation of a theorem to another context** **Could we use the rationale behind the theorem for the equality of two infinite decimal representations? How?**

- Let’s work on the question below for a few minutes and then discuss it together.

Task: a = 0.250000 . . . and b = 0.249999 . . ..

Determine whether these two numbers are equal or not.

(Note for the implementation process: Observe whether the participants are thinking at a distance. If no such answer has been received, ask the following question)

- **What is the distance between these two numbers?**

(Note for the implementation process: Here they can probably write the numbers down and try to make a long subtraction process. In this case, ask the following question)

- **Without applying long subtraction algorithm, how could we determine $|\boldsymbol{a}-\ \boldsymbol{b}|$?**
- Let’s interpret the theorem for the equality of two real numbers in the context of infinite decimal representations.

(Note for the implementation process: Remind the theorem to the participants again. If there is no answer, divide the solution below into small parts and probing questions. Get participants to come to this solution through questions. Let all the steps be included in the presentation as well.)

- **Solution:** The distance between them would not be 0.001, since 0. 249999.. is much closer to 0.25 as compared to 0.249 (0.001 away from 0.25). Similarly, it could not be 0.0001, since 0. 249999.. is much closer to 0.25 as compared to 0.2499 (0.0001 away from 0.25). When it goes on like that, we can conclude that whatever epsilon we take (0.001, 0.0001, and so on) $|a-\ b|$ is showed as smaller than epsilon. Then $a = b$ using the above statement.
- **So, what could we say about infinite decimal representations after working this theorem and example. What is your conclusion after this process?**

	• *(Note for the implementation process: After the answers, keep the following statement in a presentation and project it to the classroom)* • **Summary:** Although infinite decimal representations seem apparently different, two of them could represent the same number, or indicate the same place in the real number line. • *(Note for the implementation process: According to the status of the answers from the participants, if there is no appropriate explanation, ask them what they understand from this summary statement and how we could come to such a conclusion)* • Instead of considering $0.24\overline{9}$ as an infinite progression of numbers—as a process of "getting closer and closer to" a number, thinking of the infinite decimal $0.24\overline{9}$ as representing one number and occupying one position on the number line which is the same as 0. 250..
Step 2.1	**Thinking on the formal definition of limit** For every real number $\epsilon > 0$ (epsilon), if there is at least one real number $\delta > 0$ such that $\|f(x) - L\| < \epsilon$ for real numbers x that satisfy the inequality $0 < \|x - a\| < \delta$, then the number L is called the limit of the function f at point a. • **According to the definition, is there a dependence relationship between epsilon and delta? If so, how is it?** • **What is the order of the variables epsilon, delta, x and f(x) in the limit definition? Which comes first and how do the variables affect each other?**
Step 2.2	**Communicating through various representations** **Question:** Find that the limit of the function $f(x) = 3x - 1$ at the point $x = 2$ is 5, using the formal (epsilon-delta definition) of the limit. While doing this, we will try to create a table containing epsilon, delta, x and $f(x)$ values. • **From which variable should we begin placing values in the table?** • **What is the relationship between epsilon and delta values? How do the changing values of one affect the other? (let's start with 1.5 when giving values to be common)**

Step 2.3	**Discussing the meaning of equality in the context of limit** **Question:** Examine the character of the series, if convergent, find the convergence value for the series $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$. The desired solution is as follows: $\sum_{k=1}^{\infty}\frac{1}{k(k+1)}$, $S_n = \sum_{k=1}^{n}\frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)$ $= (1-\cancel{\frac{1}{2}}) + (\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}) + \cancel{\dots} + (\cancel{\frac{1}{k-1}}-\cancel{\frac{1}{k}}) + (\cancel{\frac{1}{k}}-\frac{1}{k+1})$ $S_n = 1-\frac{1}{n+1}$ $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1 < \infty$ $\sum_{k=1}^{\infty}\frac{1}{k(k+1)} = 1$ • **What does equality tell us here?** The result of adding these terms one by one is shown with an equation. Are we talking about an absolute equality in the sense of exactly 1, neither more nor less? Did we write the infinite terms one under the other and add them together to reach 1? No! This equality says that this sum cannot exceed 1, it says that the result of the sum is limited to 1. The equality in this expression is not a 1 in the sense of nothing more, nothing less, but exactly 1. But it says that it cannot exceed 1. It is 1 in the context of limit, but actually, the sum is less than 1.

CURRICULUM VITAE

Surname, Name: Toker Kekik, Habibe

EDUCATION

Degree	**Institution**	**Year of Graduation**
MS	Çukurova University Mathematics	2017
BS	Boğaziçi University Mathematics and Science Education	2013
High School	75th Year Anatolian Teacher High School, Mersin	2007

FOREIGN LANGUAGES

Advanced English

PUBLICATIONS

1. Toker, H., & Sevinç, Ş. (2021). Karikatürlerdeki matematik, matematik öğretmenleri ve öğrenciler. *Ankara Üniversitesi Eğitim Bilimleri Fakültesi Dergisi, 54*(1), 1-34. https://doi.org/10.30964/auebfd.732238

2. Toker, H., & Sevinç, Ş. (26-28 Eylül, 2019). *Karikatürlerdeki matematik* [Sözlü Sunum]. 4. Uluslararası Türk Bilgisayar ve Matematik Eğitimi (TÜRKBİLMAT-4) Sempozyumu. Çeşme, İzmir, Türkiye

3. Dışbudak Ö., Toker H., Özdoğan Z., & Sevinç Ş. (2020, January 15-17). *Subjectivity in participant- and researcher-generated data in qualitative research* [Paper Presentation]. The Qualitative Report Eleventh Annual Conference. Florida, USA.

4. Çelikkol E., Toker, H., & Sevinç, Ş. (19-21 Mayıs, 2021). *Matematik öğretmenlerinin ve öğretmen adaylarının matematik kavramına yönelik algılarının incelenmesi* [Sözlü Sunum]. 14. Ulusal Fen Bilimleri ve Matematik Eğitimi Kongresi, Burdur, Türkiye.

5. Toker H., Acar F., & Sevinç Ş. (28-30 Ekim, 2021). *İlköğretim matematik öğretmenlerinin lisans eğitiminde alınan ileri matematik derslerine ilişkin görüşleri* [Sözlü Sunum]. 5. Uluslararası Türk Bilgisayar ve Matematik Eğitimi Sempozyumu, Alanya, Antalya, Türkiye.