ON A BADE TYPE REFLEXIVE ALGEBRA

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ABSTRACT

Our main aim is to prove the following theorem. If X is a barrelled locally convex space, and the unital algebra homomorphism m from $C(K)$ into $L(X)$ is continuous with respect to the norm topology on $C(K)$ and the strong operator topology on $L(X)$, then the weak operator topology closure of $m(C(K))$ is a reflexive operator algebra. This generalizes Theorem 7 in [12]. As a consequence of this result, it is shown that the weak operator topology closure of the linear span of an equicontinuous Boolean algebra B of projections in the quasi-complete barrelled locally convex space X is AlgLat(B).

Key words: Boolean Algebra, Reflexive, Quasi-Complete.

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ÖZET

Bu çalışmanın amacı aşağıdaki teoremi isbat etmektir. Eğer X bir barrelled yerel konveks uzay; m, $C(K)$ den $L(X)$'e, $m(1) = I$, cebir homomorfizma, norm topolojiden kuvvetli operatör topolojiye sürekli ise, o zaman $m(C(K))$'nın zayıf operatör topolojiye göre kapanışı bir refleksif operatör cebiridir. Bu, [12]'deki Teorem 7'nin genelleştirilmiş durumudur. Bu teoremin bir sonucu olarak, yarıtam barrelled yerel konveks uzay X'de tanımlanan projeksiyonların eşsürekli Boolean cebirinin gerdiği uzayın zayıf operatör kapanışının AlgLat(B) olduğu gösterilmiştir.

Anahtar Kelimeler: Boolean Cebiri, Refleksif, Yarıtam.

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CONTENTS

INTRODUCTION

Let B be an equicontinuous Bade complete Boolean algebra of projections in the quasi-complete space X and ⟨ B ⟩ be the closed (S.O.T.) subalgebra of L(X) generated by B. Then ⟨ B ⟩ is reflexive. Proof of this is contained in [6]. The same result under the additional hypothesis that L(X) is sequentially complete was proved in [7]. For the case that X is a Banach space this result is contained in [3] and also in [11] Theorem 1.1 .

If X is Banach, and m: C(K) $\longrightarrow$ L(X) is a bounded homomorphism, m(1) = I, and m is a weakly compact map, then the weak operator closure of m(C(K)) is reflexive in [15] Theorem 3. Under the assumption that m is not a weakly compact map, reflexivity of weak closure of m(C(K)) was proved in [12] Theorem 7.

For the case that the bounded Boolean algebra B in a Banach space X is not complete in the sense of Bade, reflexivity of the weak operator closure of spanB was contained in [12] Theorem 9.

This thesis consists of three chapters.

Chapter I contains all necessary information and preliminaries used throughout the whole work.

In chapter II we have generalized the result [12] Theorem 7 for the case that X is a barrelled locally convex space.

In chapter III we have proved the result [6] Cor.5.6 under the additional hypothesis that X is a barrelled locally convex space and for the weaker condition that an equicontinuous Boolean algebra of projections is not strongly equicontinuous.

CHAPTER I

PRELIMINARY RESULTS

In this chapter, some definitions and necessary information are introduced.

We denote by C(K) the space of all continuous complex valued functions defined on a compact Hausdorff space K with the supremum norm.

Let X be a locally convex space. We denote by $L^{\#}(X)$ the set of all linear maps from X into X and by L(X) the set of continuous linear maps from X into X. The topological dual of X will be denoted by X'. We denote by X" the topological dual of X'[β(X',X)].

Definition I.1 A topological vector space X is called quasi-complete if every bounded closed subset of X is complete.

For example, every complete locally convex topological vector space is quasi-complete. Moreover, l_1 $\sigma(l_1,c_0)$ quasi- complete but not complete space.

Definition I.2 An absolutely convex, closed, absorbent subset of a locally convex space X is called a barrel.A locally convex space X is called barrelled if each barrel in X is a neighborhood of zero.

If X is a barrelled locally convex space then X' is quasi-complete in $\sigma(X',X)$ [17] .

The strong operator topology will be abbreviated as S.O.T. . In this topology a net $\{T_\alpha\}$ is said to converge to T if $T_\alpha x \longrightarrow Tx$ for all $x \varepsilon X$. The weak operator topology will be abbreviated as W.O.T. . In this topology a net $\{T_\alpha\}$ is said to converge to T if $x' T_\alpha x \longrightarrow x' Tx$ for each $x \varepsilon X$, $x' \varepsilon X'$. The weak * operator topology of $L(X')$ is shortened as W^*.O.T. . A net $\{T_\alpha\}$ in $L(X')$ converge to T in this topology if $T_\alpha x'(x) \longrightarrow Tx'(x)$ for each $x' \varepsilon X'$, $x \varepsilon X$.

Suppose A is a collection of continuous linear operators on a locally convex Hausdorff space X. We denote by LatA the family of all subspaces of X that are left invariant under all the operators in A . Let F be a family of subspaces of X. We denote by AlgF the algebra of all operators on X which leave invariant every member of F .

Definition I.3 A closed subalgebra A of L(X) with unit is called reflexive if A = AlgLatA.

Definition I.4 A Boolean algebra B of projections in X is said to be complete as an abstract Boolean algebra if each subset of B has a greatest lower bound and a least upper bound in B.

Definition I.5 A Boolean algebra B is said to be Bade complete if it is complete as an abstract Boolean algebra and if for every subset I_o of B

$$\left(\bigvee_{E\varepsilon I_o} E\right)(X) = \overline{sp}\left\{ \bigcup E(X) : E\varepsilon\ I_o \right\},$$

$$\left(\bigwedge_{E\varepsilon I_o} E\right)(X) = \bigcap_{E\varepsilon I_o} E(X) .$$

Definition I.6 Let (A, ≤) be a partially ordered set. A set B of A is called directed upwards (respectively, directed downwards), if for every pair a,b of elements of B there is a $c\varepsilon B$ such that $a\leq c$ and $b\leq c$ (respectively, $c\leq a$ and $c\leq b$). The symbol $B\uparrow$ ($B\downarrow$) means that B is directed upwards(downwards).

Definition I.7 A lattice A is called Dedekind complete (order complete) if every non empty subset B in A which is directed upwards and has an upper bound in A, has a least upper bound.

Definition I.8 A lattice A is called Dedekind σ-complete (sequentially Dedekind complete) if every non empty countable subset B in A which is directed upwards and has an upper bound has a least upper bound.

Definition I.9 Let E be a Riesz space. The set $[x,y] = \{z\varepsilon E : x\leq z\leq y\}$ is called an order interval.

Definition I.10 A Riesz space E is called Archimedean if, for x and y belonging to E, $ny\leq x$ for all $n\varepsilon N$ implies $y\leq 0$.

Definition I.11 Let E be a Riesz space. An element e in E is called a (strong) order unit if for each $x \in E$ there is a scalar $0<s$ such that $x \leq se$. An element $0<e$ of a Riesz space E is called a weak order unit if $x=0$ whenever $x \wedge e=0$, $x \in E$.

Definition I.12 A subset B of a Riesz space E is called solid if $y \in B$ whenever $x \in B$ and $|y| \leq |x|$.

Definition I.13 Let E be a Riesz space. A linear subspace M of E is called a lattice ideal if M is a solid subset of E.

Definition I.14 A sequence $\{a_n\}_{n=1}^{\infty}$ in a Riesz space is said to be disjoint if $|a_n| \wedge |a_m|=0$ holds for all $n \neq m$.

Definition I.15 Let E be a Riesz space. A solid linear subspace B is called a band of E whenever $A \subseteq B$ and supA exists in E, then $\sup A \in B$.

Definition I.16 Let E,F be two Riesz spaces. A linear mapping $T:E \longrightarrow F$ is called positive if $0<Tx$ for all $0<x \in E$. A linear mapping $T:E \longrightarrow F$ is called a Riesz homomorphism if $|Tx|=T|x|$ for all $x \in E$. A linear mapping $T:E \longrightarrow F$ is called a Riesz isomorphism if T is a one to one, onto, Riesz homomorphism.

Definition I.17 A subset A of a Riesz space E is called order bounded if A is contained in an order interval.

Let E be any Riesz space. We will denote by $E^{\sim}$ the order dual of E. $E^{\sim}$ is the set of all linear functionals on E which are bounded on order bounded sets. $E^{\sim}_n$ is the band in $E^{\sim}$ of order continuous linear functionals. That is, $f \in E^{\sim}_n$ if $x_\alpha \downarrow 0$ implies $\lim_\alpha f(x_\alpha)=0$.

Let E be a Riesz space. A seminorm(norm) p on E is called a Riesz seminorm (Riesz norm) if $|x| \leq |y|$ implies $p(x) \leq p(y)$ for all $x, y \in E$.

Definition I.18 A Banach lattice is a pair $(E, \|.\|)$ where E is a Riesz space and $\|.\|$ is a Riesz norm on E under which E is complete. We say that a Banach lattice E is an abstract L-space (AL) if

$$\|x+y\| = \|x\| + \|y\| \text{ for all } 0<x, y \in E.$$

We say that a Banach lattice E is an abstract M-space (AM) if

$$\|x \vee y\| = \|x\| \vee \|y\| \text{ for all } 0<x, y \in E.$$

Definition I.19 A linear topology τ on a Riesz space E is said to be locally solid if τ has a basis for zero consisting of solid sets.

By $u_\alpha \downarrow 0$ in a Riesz space we mean $\inf_\alpha u_\alpha = 0$ and it is directed downwards.

Definition I.20 Let (E, τ) be a locally solid Riesz space. Then we say that τ is a Lebesgue topology if $u_\alpha \downarrow 0$ in E implies $u_\alpha \longrightarrow 0$ in τ. We say that τ is a pre-Lebesgue topology if every order bounded disjoint sequence of E is τ convergent to zero.

It is well known that in an Archimedean locally solid Riesz spaces the Lebesgue property implies the pre-Lebesgue property [1]. On the other hand if (E,τ) is a locally solid Riesz space, the topological dual E' of (E,τ) is an ideal of the order dual $E^{\sim}$. Hence it is a Dedekind complete Riesz space in $E^{\sim}$ [1].

Theorem I.21 Let E be a Riesz space and τ a linear topology on E. Then the following statements are equivalent.

(i) τ is a locally convex solid topology.

(ii) There exists a family $\{p_i\}$ of Riesz seminorms that generates the topology τ.

For a proof of this theorem we refer to [1].

Let E be a Riesz space and let A be a non empty subset of $E^{\sim}$. The absolute weak topology $|\sigma|(E,A)$ generated by A on E is the locally convex solid topology generated on E by the collection of Riesz seminorms $\{p_f : f\varepsilon A\}$, where $p_f(x)= |f|(|x|)$ for all $x\varepsilon E$ and $f\varepsilon A$. If (E,τ) is a locally convex solid Riesz space. Then we have

$$\sigma(E,E') \subseteq |\sigma|(E,E') \subseteq \tau.$$

$|\sigma|(E,E')$ is a Hausdorff topology if and only if τ is Hausdorff [1]. Similarly, if E is a Riesz space, B is a subset of E, and A is an ideal of $E^{\sim}$. The absolute weak topology $|\sigma|(A,B)$ on A is the locally convex solid topology generated by the collection of Riesz seminorms $\{p_x : x\varepsilon B\}$, where $p_x(f)=|f|(|x|)$ for all $f\varepsilon A$. For any Riesz spacs E the absolute weak topology $|\sigma|(E^{\sim},E)$ is a Hausdorff, Lebesgue topology [1].

C(K) is an AM-space, hence $C(K)'$ is an AL-space by a well known duality theorem [9] and $C(K)''=C(K)'^{\sim}$ by [9]. By the preceding $|\sigma|(C(K)'',C(K)')$ is a Hausdorff, Lebesgue topology.

Definition I.22 A component of a positive element x of a Riesz space is any element y satisfying $y\wedge(x-y)=0$.

For any Riesz space E we denote by $E_{\mathbb{C}} = E+iE$ the complexification of E. If $h=f+ig\varepsilon E_{\mathbb{C}}$, then f=Reh and g=Imh. We say that an element $f\varepsilon E_{\mathbb{C}}$ has an absolute value, denoted by $|f|$, if the supremum

$$|f|=\sup\{(\mathrm{Re}f)\cos\theta+(\mathrm{Im}f)\sin\theta \ : \ 0\leq\theta\leq2\pi\}$$

exists in E.

Definition I.23 If $|f|$ exists for all $f\varepsilon E_{\mathbb{C}}$, $E_{\mathbb{C}}$ is called a complex Riesz space.

Let E be a complex Riesz space. We denote the real part of E by ReE, so that E=ReE+iReE.

Definition I.24 The Riesz space A is called a Riesz algebra if A is an algebra as well as a Riesz space with the additional property that $xy \geq 0$ for all $0 \leq x,y \varepsilon A$.

Definition I.25 The Riesz algebra A is called an f-algebra if $x \wedge y=0$ in A implies that $(zx) \wedge y=(xz) \wedge y=0$ for all $0 \leq z \in A$.

Definition I.26 The complexification of a (real) Archimedean f-algebra is called a complex f-algebra.

Definition I.27 Let E be a Riesz space. A linear mapping T from E into E is called band preserving if $|Tx| \wedge |y| = 0$ whenever $|x| \wedge |y| = 0$ in E.

A linear mapping from a Riesz space to a Riesz space is called an order bounded if it sends order bounded sets to order bounded sets.

Definition I.28 An order bounded, band preserving mapping in E is called an orthomorphism.

The space of all orthomorphism in E is denoted by Orth(E). Orth(E) is a complex f-algebra with respect to composition as multiplication, with the identity operator as unit element [6].

An f-algebra A is called semi-prime if $x^2=0$ in A implies $x=0$. Any f-algebra with unit element is semi-prime [4]. Any Archimedean f-algebra is commutative [2]. If A is a commutative Riesz algebra with unit element e, then A is an f-algebra if and only if e is a weak order unit [5].

We list some properties of complex f-algebras with unit element [4]. Suppose $A_{\mathbb{C}}$ is a complex f- algebra.

(i) For any $x \in A_{\mathbb{C}}$, $|x| = \sqrt{[(Rex)^2+(Imx)^2]}$.

(ii) If $|x| \wedge |y|=0$ in $A_{\mathbb{C}}$, then $|xz| \wedge |y|=0$ for all $z \in A_{\mathbb{C}}$.

(iii) $|xy|=|x||y|$ for all $x,y \in A_{\mathbb{C}}$.

(iv) $A_{\mathbb{C}}$ is semi-prime, and $|x| \wedge |y|=0$ in $A_{\mathbb{C}}$ if and only if $xy=0$.

Corollary 1.29 [14] For any Archimedean unital f-algebra A with point separating order dual, $A^{\sim\sim}$ is an f-algebra with respect to the Aren's multiplication.

If E is an AM-space with unit e, then E" is an Dedekind complete AM-space with unit e containing the f-algebra E as an f-subalgebra [2].

Definition I.30 Let R be a ring, a non empty set M is called an R-module if M is an abelian group under an operation + such that for every $r \in R$ and $m \in M$ there exists an element rm in M subject to :

(i) $r(a+b) = ra+rb$

(ii) $(r+s)a = ra+sa$

(iii) $r(sa) = (rs)a$ for all $a,b \in M$, $r,s \in R$

If R has a unit element, 1, and if $1m=m$ for every $m \in M$, M is called a unital R-module. Note that if R is a field, a unital R-module is nothing more than a vector space over R.

Definition I.31 An element x in a ring is said to be idempotent if $x^2 = x$. A ring with identity is called a Boolean ring if every element is idempotent.

Recall that a lattice E is called distributive if $x\wedge(y\vee z)=(x\wedge y)\vee(x\wedge z)$ is true for all $x,y,z\in E$. A lattice E is called complemented if for every $x\in E$ there exists $x^c\in E$ such that $x \vee x^c=1$, $x\wedge x^c=0$.

Definition I.32 A Boolean algebra B is a lattice with unit and zero which is distributive and complemented.

Every Boolean ring is a Boolean algebra where the lattice operations are defined as follows

$$x \wedge y = xy, \quad x\vee y = x+y - xy, \quad x,y\in B.$$

Let A and B be Boolean algebras and $f:A \longrightarrow B$, then f is said to be a Boolean algebra homomorphism if $f(x\wedge y)=f(x)\wedge f(y)$, $f(x\vee y)=f(x)\vee f(y)$, $f(x^c)=f(x)^c$. If f is one to one and onto, then A and B are called isomorphic.

A topological space is said to be totally disconnected if it has a base consisting of sets which are simultaneously open and closed.

We need the following representation theorem [8].

Theorem I.33 Every Boolean ring with unit is isomorphic with the Boolean ring of all open and closed subsets of a totally disconnected compact Hausdorff space.

The proof of the following theorem can be found in [13].

Theorem I.34 Let A be an Archimedean f-algebra with unit element e and let B be an Archimedean semi-prime f-algebra. The Riesz homomorphism $T: A \longrightarrow B$ is an algebra homomorphism if and only if Te is an idempotent.

It is well known that if A is an ideal of a Riesz space E, then the quotient Riesz space E/A is Archimedean if and only if A is a uniformly closed ideal of E [2].

Definition I.35 A compact topological space K is called Stonian if each open subset of K has open closure. Stonian spaces are also called extremally disconnected. A Stonian space K is called hyperstonian if the band of order continuous Radon measures on K separates the points of C(K).

Examples of Stonian spaces which fail to be hyperstonian can be found in [19].

In closing this preliminary chapter we recall that if E is an AL-space, then E' can be identified with C(X), where X is a compact extremally disconnected space [17].

CHAPTER II

THE REFLEXIVITY OF m(C(K))

The aim of this chapter is to generalize a theorem due to Orhon and Hadwin which was proved in [12]. Among other things they proved that if X is a Banach space and $m : C(K) \longrightarrow L(X)$ is a bounded unital homomorphism then m(C(K)) is weakly reflexive. i.e. $\text{AlgLat}\, m(C(K)) = \overline{m(C(K))}^{W.O.T}$.

In this chapter we study a possible extension of this result to the case where X is a locally convex space.

Suppose that X is a barrelled locally convex space and $m : C(K) \longrightarrow L(X)$ is a unital algebra homomorphism that is continuous with respect to the norm topology on C(K) and the S.O.T. on L(X).

The algebra homomorphism m induces a canonical C(K) module structure on X. This structure is accomplished via the bilinear map $B_1 : C(K) \times X \longrightarrow X$ defined by

$$B_1(f,x) = m(f)x \quad \text{where } f \in C(K),\ x \in X.$$

B_1 is bilinear.

To prove the linearity in the first coordinate, let f_1, f_2 be arbitrary elements of C(K), α, β be complex constants and $x \in X$ be fixed. Then

$$B_1(\alpha f_1+\beta f_2,x) = m(\alpha f_1+\beta f_2)x$$
$$= [\alpha m(f_1)+\beta m(f_2)]x$$
$$= \alpha m(f_1)x+\beta m(f_2)x$$
$$= \alpha B_1(f_1,x) + \beta B_1(f_2,x).$$

Since $x\varepsilon X$ is arbitrary, we conclude that B_1 is linear in the first coordinate.

To prove the linearity in the second coordinate , let x_1, x_2 be elements of X, α, β be complex scalars and f be a fixed element of C(K). Then

$$B_1(f,\alpha x_1+\beta x_2) = m(f)[\alpha x_1+\beta x_2)]$$
$$= \alpha m(f)x_1+\beta m(f)x_2$$
$$= \alpha B_1(f,x_1)+\beta B_1(f,x_2) \ .$$

Since f is arbitrary, we conclude that B_1 is also linear in the second coordinate.

Thus, it can be verified in a straightforward manner that the bilinear map B_1 induces a canonical C(K)-module structure on X. Whenever convenient the image of $(a,x)\varepsilon C(K)\times X$ under B_1 will be denoted by a.x .

Let us now observe that the bilinear map $B_1 : C(K) \times X \longrightarrow X$ is separately continuous when X is considered in its given topology and C(K) in its norm topology.

To prove the continuity in the first coordinate, let (a_n) be a sequence in C(K) converging to a. By the norm to S.O.T. continuity of the map m, we have

$$m(a_n) \rightarrow m(a) \text{ in S.O.T. of } L(X).$$

In particular, for $x \varepsilon X$, we have

$$m(a_n)x \rightarrow m(a)x .$$

That is to say $B_1(a_n,x) \rightarrow B_1(a,x)$ in X .

To prove the continuity in the second coordinate, let $x_\alpha \rightarrow x$ in X . Since $m(a) \varepsilon L(X)$ for $a \varepsilon C(K)$, we have

$$m(a)x_\alpha \rightarrow m(a)x \text{ i.e.}$$

$$B_1(a,x_\alpha) \rightarrow B_1(a,x).$$

We now define $B_2 : X \times X' \rightarrow C(K)'$ as

$$B_2(x,x') = \mu_{x,x'}$$

where $\mu_{x,x'}(f) = x'(m(f)x)$ for each $x \varepsilon X$, $x' \varepsilon X'$ and $f \varepsilon C(K)$.

We first show $\mu_{x,x'}$ is a Radon measure on $C(K)$. Let $f_1, f_2 \; \varepsilon \; C(K)$, α, β be complex scalars. Then

$$\begin{aligned} \mu_{x,x'}(\alpha f_1+\beta f_2) &= x'[m(\alpha f_1+\beta f_2)x] \\ &= x'[\alpha m(f_1)+\beta m(f_2)]x \\ &= x'[\alpha m(f_1)x+\beta m(f_2)x] \\ &= \alpha x'(m(f_1)x)+\beta x'(m(f_2)x) \\ &= \alpha\mu_{x,x'}(f_1)+\beta\mu_{x,x'}(f_2) \end{aligned}$$

which shows the linearity of $\mu_{x,x'}$ for each $x \varepsilon X$ and $x' \varepsilon X'$.

To prove the continuity of $\mu_{x,x'}$, let (a_n) be a sequence in $C(K)$ converging to a. By the continuity of B_1 in the first coordinate we have that

$$B_1(a_n,x) \rightarrow B_1(a,x)$$

for each $x \varepsilon X$. That is, $m(a_n)x \to m(a)x$ for each $x \varepsilon X$. As $x' \varepsilon X'$, it follows that

$$x'(m(a_n)x) \to x'(m(a)x).$$

That is, $\mu_{x,x'}(a_n) \to \mu_{x,x'}(a)$. Hence, for each $x \varepsilon X$ and $x' \varepsilon X'$ $\mu_{x,x'}$ is a Radon measure on K .

B_2 is a bilinear map. To this end, let $x_1, x_2 \varepsilon X$ α, β be complex scalars and $x' \varepsilon X'$ be fixed. Then, for $f \varepsilon C(K)$

$$\begin{aligned} B_2(\alpha x_1+\beta x_2, x')(f) &= x'(m(f)[\alpha x_1+\beta x_2]) \\ &= x'[\alpha m(f)x_1+\beta m(f)x_2] \\ &= \alpha x'(m(f)x_1)+\beta x'(m(f)x_2) \\ &= \alpha\mu_{x_1,x'}(f)+\beta\mu_{x_2,x'}(f) \\ &= \alpha B_2(x_1,x')(f)+\beta B_2(x_2,x')(f) \end{aligned}$$

Since $f \varepsilon C(K)$ is arbitrary, we conclude that

$$B_2(\alpha x_1+\beta x_2, x') = \alpha B_2(x_1,x')+\beta B_2(x_2,x') \text{ or}$$

$$\mu_{\alpha x_1+\beta x_2, x'} = \alpha\mu_{x_1,x'}+\beta\mu_{x_2,x'}$$

To prove the linearity of B_2 in the second coordinate, let x'_1, x'_2 be elements of X', α, β be complex scalars and $x \varepsilon X$ be fixed. Then for $f \varepsilon C(K)$

$$\begin{aligned} B_2(x, \alpha x'_1+\beta x'_2)(f) &= (\alpha x'_1+\beta x'_2)[m(f)x] \\ &= \alpha x'_1(m(f)x)+\beta x'_2(m(f)x) \\ &= \alpha B_2(x,x'_1)(f)+\beta B_2(x,x'_2)(f). \end{aligned}$$

Since $f \varepsilon C(K)$ is arbitrary, we conclude that

$$B_2(x, \alpha x'_1+\beta x'_2) = \alpha B_2(x,x'_1)+\beta B_2(x,x'_2) \text{ or}$$

$$\mu_{x,\alpha x'_1+\beta x'_2} = \alpha\mu_{x,x'_1}+\beta\mu_{x,x'_2}.$$

We next show that B_2 is continuous in each coordinate.

As m is norm to S.O.T. continuous and X is a barrelled locally convex space, if U is the closed unit ball in C(K) then m(U) is an equicontinuous subset of L(X). Hence if $p_{x'}$ is the $\sigma(X,X')$ seminorm defined by $x'\varepsilon X'$ there exists a continuous seminorm q on X and a scalar k with

$$|x'(m(a)x)| = p_{x'}(m(a)x)\le kq(x) \text{ for all } a\varepsilon U,\ x\varepsilon X.$$

Hence $|\mu_{x,x'}(a)|\le kq(x)$ for each $a\varepsilon U$. This observation yields

$$\|\mu_{x,x'}\|=\sup_{\|a\|\le 1}|\mu_{x,x'}(a)|\le kq(x)$$

But this is the continuity of B_2 in the first coordinate. Therefore, B_2 is continuous from X into $C(K)'(\|.\|)$.

To prove the continuity of B_2 in the second coordinate, we fix $x\varepsilon X$. For $a\varepsilon C(K)$, $m(a)x\varepsilon X$. Let $\{x'_\alpha\}$ be a net in X' which converges to x' in $\sigma(X',X)$. Then,

$$x'_\alpha(m(a)x) \longrightarrow x'(m(a)x) \quad .$$

That is to say $\mu_{x,x'_\alpha}(a) \longrightarrow \mu_{x,x'}(a)$ for all $a\varepsilon C(K)$. Hence B_2 is continuous from $\sigma(X',X)$ to $\sigma(C(K)',C(K))$.

We need to put a C(K)"-module structure on X'. To do this, we define

$$B_3 : X' \times C(K)" \longrightarrow X'$$

by $B_3(x',f)(x) = f(\mu_{x,x'})$ for each $x\varepsilon X$, $x'\varepsilon X'$ and

$f \varepsilon C(K)''$.

We claim $B_3(x',f) \varepsilon X'$. Let x_1, x_2 be the elements of X and α, β be complex scalars. Then

$$\begin{aligned} B_3(x',f)(\alpha x_1+\beta x_2) &= f(\mu_{\alpha x_1+\beta x_2, x'}) \\ &= f(\alpha \mu_{x_1, x'}+\beta \mu_{x_2, x'}) \\ &= \alpha f(\mu_{x_1, x'})+\beta f(\mu_{x_2, x'}) \\ &= \alpha B_3(x',f)(x_1)+\beta B_3(x',f)(x_2). \end{aligned}$$

$B_3(x',f)$ is continuous on X for each $x' \varepsilon X'$ and $f \varepsilon C(K)''$. For if $\{x_\alpha\}$ is a net in X which converges to x, then by the continuity of B_2 from X into $C(K)'$ with respect to norm topology, we have

$$\mu_{x_\alpha, x'} \longrightarrow \mu_{x, x'} \text{ in } C(K)'.$$

As $f \varepsilon C(K)''$, this implies

$$f(\mu_{x_\alpha, x'}) \longrightarrow f(\mu_{x, x'}).$$

That is to say $B_3(x',f)(x_\alpha) \longrightarrow B_3(x',f)(x)$ Hence the map $B_3 : X' \times C(K)'' \longrightarrow X'$ is well defined.

B_3 is bilinear. Let x'_1, x'_2 be elements of X' α, β be complex scalars and f be a fixed element of $C(K)''$. Then for $x \varepsilon X$, we have

$$\begin{aligned} B_3(\alpha x'_1+\beta x'_2, f)(x) &= f(\mu_{x, \alpha x'_1+\beta x'_2}) \\ &= f(\alpha \mu_{x, x'_1}+\beta \mu_{x, x'_2}) \\ &= \alpha f(\mu_{x, x'_1})+\beta f(\mu_{x, x'_2}) \\ &= \alpha B_3(x'_1, f)(x)+\beta B_3(x'_2, f)(x) \end{aligned}$$

As $x \varepsilon X$ is arbitrary, we conclude that

$$B_3(\alpha x'_1+\beta x'_2, f) = \alpha B_3(x'_1, f)+\beta B_3(x'_2, f) .$$

To prove the linearity in the second coordinate, we let $f,g \varepsilon C(K)''$, α,β be complex scalars and $x' \varepsilon X'$ be fixed. Let $x \varepsilon X$

$$B_3(x',\alpha f+\beta g)(x) = (\alpha f+\beta g)\mu_{x,x'}$$
$$= \alpha f(\mu_{x,x'})+\beta g(\mu_{x,x'})$$
$$= \alpha B_3(x',f)(x)+\beta B_3(x',g)(x)$$

As x is arbitrary, we conclude that

$$B_3(x',\alpha f+\beta g) = \alpha B_3(x',f)+\beta B_3(x',g) .$$

From now on $B_3(x',f)$ will be denoted by $x'.f$ whenever convenient.

Let us equip X' with $\sigma(X',X)$, $C(K)''$ with the norm topology and consider the space $X' \times C(K)''$ with the induced product topology. We claim that the bilinear map

$$B_3 : X' \times C(K)'' \rightarrow X'$$

is continuous when the domain is equipped with the product topology and the range is equipped with $\sigma(X',X)$. Let $\{ z_\alpha \}=\{(x'_\alpha,f_\alpha)\}$ be a net in $X' \times C(K)''$ convergent to zero in the product topology. Let $x \varepsilon X$ be arbitrary, we have

$$|B_3(z_\alpha)(x)|=|(x'_\alpha.f_\alpha)(x)|=|f_\alpha(\mu_{x,x'_\alpha})|\leq\|f_\alpha\|\|\mu_{x,x'_\alpha}\|$$

As $B_2(x,.):(X',\sigma(X',X)) \rightarrow (C(K)', \sigma(C(K)',C(K)))$ is continuous, we have

$$\mu_{x,x'_\alpha} \rightarrow 0 \text{ in } \sigma(C(K)',C(K)).$$

An application of the Uniform Boundedness Principle now shows that the net $\{ \mu_{x,x'_\alpha}\}$ is bounded in the norm topology of $C(K)'$. As $x \varepsilon X$ is arbitrary, this, in turn, yields that

$$B_3(z_\alpha) \rightarrow 0 \text{ in } X'$$

with respect to $\sigma(X',X)$.

The continuity of B_3 in the first coordinate from $(X',\sigma(X',X))$ into $(X',\sigma(X',X))$ is an immediate consequence of the preceding paragraph.

To prove the continuity in the second coordinate, we let $\{ F_\alpha \}$ be a net in $C(K)''$ which converges to F in $\sigma(C(K)'',C(K)')$. Let $x'\varepsilon X'$, $x\varepsilon X$ be fixed but arbitrary elements. Then as $\mu_{x,x'}\varepsilon C(K)'$ we have

$$\lim_\alpha B_3(x',F_\alpha)(x) = \lim_\alpha F_\alpha(\mu_{x,x'})$$

$$= F(\mu_{x,x'})$$

$$= B_3(x',F)(x)$$

As x is arbitrary, we conclude that $B_3(x',.)$ is continuous from $\sigma(C(K)'',C(K)')$ into $\sigma(X',X)$.

Summarizing we have the following list of bilinear maps:

B_1 : $X \times C(K) \rightarrow X$,

$(x,f) \rightarrow m(f)x$

B_2 : $X \times X' \rightarrow C(K)'$,

$(x,x') \rightarrow \mu_{x,x'}: \mu_{x,x'}(a)=x'(m(a)x)$

B_3 : $X' \times C(K)'' \rightarrow X'$,

$(x',F) \rightarrow x'F:(x'F)(x)= F(\mu_{x,x'})$

These maps make it possible to define

$$B_4 : C(K)'' \times X'' \longrightarrow X'' \text{ by}$$

$$B_4(a,y)x' = y(a.x')$$

where $a \in C(K)''$, $y \in X''$ and $x' \in X'$. This is a bilinear map.

By taking $X=C(K)$, B_4 induces a well known algebra structure on $C(K)''$ which is commonly known as Aren's multiplication on $C(K)''$. Under this multiplication $C(K)''$ is a commutative Banach algebra with unit [10].

We now give a proposition which is a direct consequence of the bilinearity of the maps B_1-B_3.

Proposition II.1 Let X be a barrelled locally convex space and $m:C(K) \longrightarrow L(X)$ be a norm to S.O.T. continuous, unital algebra homomorphism. Then X' is a unital $C(K)''$-module.

Proof. (i) $F.(x'_1+x'_2) = F.x'_1 + F.x'_2$ for $F \in C(K)''$, $x'_1, x'_2 \in X'$. Let $x \in X$ be arbitrary, then

$$\begin{aligned} F.(x'_1+x'_2)(x) &= B_3(F,x'_1+x'_2)(x) \\ &= F(\mu_{x,x'_1+x'_2}) \\ &= F(\mu_{x,x'_1}+\mu_{x,x'_2}) \\ &= F(\mu_{x,x'_1})+F(\mu_{x,x'_2}) \\ &= B_3(F,x'_1)(x)+B_3(F,x'_2)(x) \end{aligned}$$

Since $x \in X$ is arbitrary, we conclude that

$B_3(F,x'_1+x'_2) = B_3(F,x'_1)+B_3(F,x'_2) = F.x'_1+F.x'_2$

Hence (i) is satisfied.

(ii) $(F+G).x'=F.x'+G.x'$ for $F,G\varepsilon C(K)''$, $x'\varepsilon X'$.

Let $x\varepsilon X$ be arbitrary, then

$$(F+G).x'(x)=(F+G)\mu_{x,x'}$$
$$= F(\mu_{x,x'})+G(\mu_{x,x'})$$
$$=(F.x')(x)+(G.x')(x)$$
$$=(F.x'+G.x')(x)$$

As x is arbitrary we conclude that $(F+G).x'=F.x'+G.x'$.

Hence (ii) is satisfied.

(iii) $x'.(F.G)=(x'.F).G$ for $F,G\varepsilon C(K)''$, $x'\varepsilon X'$.

To prove this, let $x\varepsilon X$ and show $x'.(F.G)(x)=(x'.F).G(x)$

$x'.(F.G)(x)=(F.G)(\mu_{x,x'})$ by the definition of B_3

$= G(F.\mu_{x,x'})$ by the definition of product in $C(K)''$.

On the other hand,

$(x'.F).G(x)=G(\mu_{x,x'.F})$ as $x'.F\varepsilon X'$ by the definition of B_3.

Clearly to prove the claim it suffices to show that

$$F.\mu_{x,x'}=\mu_{x,x'}.F$$

for each $x\varepsilon X$, $x'\varepsilon X'$ and $F\varepsilon C(K)''$. To this end, let $a\varepsilon C(K)$. Then

$(F.\mu_{x,x'})(a)=F(a.\mu_{x,x'})$ by the definition of B_3.

On the other hand,

$(\mu_{x,x'.F})(a)=(x'.F)(a.x)$ by the definitions of B_2 and B_3.

$=F(\mu_{a.x,x'})$ by the definition of B_3.

Hence it is enough to show $a.\mu_{x,x'}=\mu_{a.x,x'}$ for each $a\varepsilon C(K)$, $x\varepsilon X$, $x'\varepsilon X'$.

Let $b \in C(K)$ be arbitrary, then

$$(a.\mu_{x,x'})(b) = \mu_{x,x'}(b.a) \text{ by the definition of } B_2$$
$$= x'((b.a)x)$$

Calculating the right hand side,

$$\mu_{a.x,x'}(b) = x'(b(a.x)) \text{ by the definition of } B_2$$
$$= x'((b.a)x) \text{ as } X \text{ is a } C(K)\text{-module.}$$

Therefore, condition (iii) is satisfied.

(iv) $x'.1 = x'$ for all $x' \in X'$.

Let $x \in X$, then

$$(x'.1)(x) = 1(\mu_{x,x'})$$
$$= \mu_{x,x'}(1)$$
$$= x'(1x)$$
$$= x'(x).$$

As x is arbitrary, hence (iv) is satisfied. Therefore X' is a C(K)"-module.

Since C(K) is an AM-space with unit , C(K)" is a Dedekind complete AM-space with unit [2]. Thus, it is isomorphic to a C(T) for some compact, Stonian space T. Let K be compact. The Banach lattice C(K) is isomorphic to a dual Banach lattice, it is necessary and sufficient that K be hyperstonian [19]. On the other hand, since C(T) is a dual Banach lattice, by a well known result T is hyperstonian [19].

The bilinear map $B_3 : X' \times C(K)" \longrightarrow X'$ induces a linear map $m^* : C(K)" \longrightarrow L(X')$ defined by $a \longrightarrow m^*(a)$ where $m^*(a)x' = a.x'$ for each $a \in C(K)"$ and $x' \in X'$.

Lemma II.2 m^* satisfies the following properties:

(i) For $a\varepsilon C(K)''$, $m^*(a)$ is continuous from $X'[\sigma(X',X)]$ into $X'[\sigma(X',X)]$.

(ii) m^* is an algebra homomorphism.

(iii) m^* is continuous from $C(K)''[\sigma(C(K)'',C(K)')]$ into $L(X')[W^*.O.T.]$.

(iv) For each $a\varepsilon C(K)$ $m^*(a) = (m(a))^*$.

Proof (i) Let $a\varepsilon C(K)''$. We show that $m^*(a)\varepsilon L(X')$. Let $\{x'_\alpha\}$ be a net convergent to x' in $\sigma(X',X)$. Then by the continuity of B_3, $a.x'_\alpha \rightarrow a.x'$ in X' with respect to $\sigma(X',X)$. Hence, for $x\varepsilon X$, $a.x'_\alpha(x) \rightarrow a.x'(x)$,

$$\Longrightarrow m^*(a)x'_\alpha \rightarrow m^*(a)x'.$$

Thus, $m^*(a)$ is continuous from $X'[\sigma(X',X)]$ into $X'[\sigma(X',X')]$.

(ii) Let $a, b\varepsilon C(K)''$ and $x'\varepsilon X'$.

$$m^*(a.b)x'=(a.b).x'$$
$$=a.(b.x') \text{ as } X' \text{ is } C(K)''\text{-module}$$
$$=a(m^*(b)x')=m^*(a)m^*(b)x'.$$

Since x' is arbitrary, we have $m^*(a.b)=m^*(a)m^*(b)$.

(iii) Let $\{a_\alpha\}$ be a net in $C(K)''$ convergent to a in $\sigma(C(K)'',C(K)')$. We claim $m^*(a_\alpha)x' \rightarrow m^*(a)x'$ in $\sigma(X',X)$ for each $x'\varepsilon X'$. i.e. for each $x\varepsilon X$

$$m^*(a_\alpha)x'(x) \rightarrow m^*(a)x'(x).$$

But we have $\mu_{x,x'}\varepsilon C(K)'$ for each $x\varepsilon X$ and $x'\varepsilon X'$. Hence $a_\alpha(\mu_{x,x'}) \rightarrow a(\mu_{x,x'})$. Hence

$m^*(a_\alpha)x'(x)=a_\alpha(\mu_{x,x'}) \longrightarrow (a.x')(x) = m^*(a)x'(x)$.

Since this is true for all $x\varepsilon X$ and $x'\varepsilon X'$, we conclude that $m^*(a_\alpha) \longrightarrow m^*(a)$ in W^*.O.T. .

(iv) Let $x\varepsilon X$, $x'\varepsilon X'$ and $a\varepsilon C(K)$. Then

$$m^*(a)x'(x)=(a.x')(x)$$

$=a(\mu_{x,x'})$ by the definition of B_3

$=\mu_{x,x'}(a)$

$=x'(m(a)x)$ by the definition of B_2

$=(m(a))^*x'(x)$.

Since this is true for each x and x' we have the claim.

In general, $m^*:C(K)'' \longrightarrow L(X')$ may not be one to one. However, there exists $e\varepsilon C(K)''$, $e^2=e$ such that $Ker(m^*)=(1-e)C(K)''$. Consider the set of all idempotents $e\varepsilon C(K)''$, $m^*(e)=0$. This is an upward directed set. Let e_1,e_2 be idempotents such that $m^*(e_1)=m^*(e_2)=0$. Then $e_1\bigvee e_2=e_1+e_2 - e_1e_2$ by the definition. It is straightforward to see that $e_1 \bigvee e_2$ is an idempotent in Ker(m^*).

Since $\{e\varepsilon C(K)'': e=e^2, m^*(e) =0\}$ is directed upwards and is bounded by the characteristic function of the hyperstonian space T, Dedekind completeness of $C(K)''$ implies that $1-\bigvee\{e\varepsilon C(K)'': e=e^2, m^*(e) = 0\} = e^\sim$ exists in $C(K)''=C(T)$. Considering the elements of the set $\{e\varepsilon C(K)'': e=e^2, m^*(e) =0\}$ as the e_α, we see that $e_\alpha \uparrow 1-e^\sim$. As $|\sigma|(C(K)'',C(K)')$ is a Lebesgue topology by [1]. This implies the convergence of $\{e_\alpha\}$ to $1-e^\sim$.

Hence, the continuity of $m^{*}:C(K)'' \longrightarrow L(X')$ from $\sigma(C(K)'',C(K)')$ to $L(X')$ equipped with W^{*}.O.T. implies that

$m^{*}(e_{\alpha}) \longrightarrow m^{*}(1-\tilde{e})$. Hence, $m^{*}(1-\tilde{e})=0$.

We now show $(1-\tilde{e})\ C(K)''=Ker(m^{*})$. As m^{*} is an algebra homomorphism $Ker(m^{*})$ is an ideal. Hence,

$$(1-\tilde{e})\ C(K)'' \subseteq Ker(m^{*}) \ .$$

Now let $a \varepsilon Ker(m^{*})$, $a \neq 0$, $a \varepsilon C(T)$. Consider the set $U=\{t \varepsilon T: a(t) \neq 0\}$. This is an open set. We will show that $supp(a) \subseteq supp(1-\tilde{e})$. If $t \varepsilon U$, then there exists an open - closed neighborhood V of t such that $V \subset U$. Let e' be the characteristic function of V. Then if $e'(t)=1$ then $a(t)=0$. As $m^{*}(a)=0$ in $L(X')$ we have $m^{*}(e')m^{*}(a)=0$ i.e. for every $x' \varepsilon X'$ $m^{*}(e')m^{*}(a)x'=0$. Since m^{*} is an algebra homomorphism we have $m^{*}(e'.a)x'=0$ or $(e'.a).x'=(a.e').x'=0$. Hence, $m^{*}(ae')=0$ on X'. On the support of e' we have $r^{\sim}.(a.e')=(r^{\sim}.a).e'=e'$ where if

$$t \varepsilon V(t), \text{ then } r(t)= \frac{1}{a(t)} \text{ , and if } t \varepsilon V(t), \text{ then } r(t)=0 \ .$$

This implies $m^{*}(e')=0$. We have $e' \leq 1-\tilde{e}$. Therefore $supp(a) \subseteq supp(1-\tilde{e})$. As a result we have $\tilde{e}.a=0$. Hence we can write $a=(1-\tilde{e}).a \varepsilon (1-\tilde{e})C(K)''$.

Therefore $Ker(m^{*})=(1-\tilde{e})C(K)''$.

Furthermore, m^{*} is one to one on $\tilde{e}C(K)''$. To prove this observation. Take $f \varepsilon \tilde{e}C(K)''$. Clearly $f=\tilde{e}.a$ for some $a \varepsilon C(K)''$. Suppose $m^{*}(f)=0$ then

$f \varepsilon Ker(m^*) = (1-e^{\sim})C(K)''$, i.e. $e^{\sim}.f=0$. Hence $f=e^{\sim}.a=e^{\sim}.e^{\sim}.a=e^{\sim}.f=0$. Hence m^* is one to one on $e^{\sim}C(K)''$.

We show $e^{\sim}C(K)''=C(S)$ for some hyperstonian compact Hausdorff space S. For this, let $S=\overline{\text{supp}e^{\sim}}$. Take an element f of $e^{\sim}$ $C(K)''$. Then $f=e^{\sim}.a$ for some $a \varepsilon C(K)''=C(T)$. Since the restriction of f on S belongs to C(S), we get $f \varepsilon C(S)$. Therefore $e^{\sim}C(K)'' \subseteq C(S)$. To establish the other way around, let g be an element of C(S). Then g can be extented to T by [18]. We have $e^{\sim}$ a unit in C(S). Thus $g=e^{\sim}.g$. Hence $e^{\sim}C(K)'' \supseteq C(S)$. S is compact being a closed subset of T.

We claim S is hyperstonian , S is closed in T. Let U be an open subset of S. Then S–U is a closed subset in S. Closedness of S in T implies S–U is a closed subset in T. Also T–S is a closed subset.
$T-U=(T-S) \bigcup (S-U)$ is a closed subset. This implies U is an open subset in T. $\overline{U}$ is also open subset in T. If $t \varepsilon \overline{U} \subset S \subset T$, then there exists a neighborhood B(t) of t in T such that $B(t) \subset \overline{U} \subset S$. Hence S is Stonian we claim that S is hyperstonian. T is a Stonian space, $\emptyset \neq S \subset T$ open, then each continuous bounded function $f:S \longrightarrow R$ has a continuous extension to $f^{\sim}:T \longrightarrow R$ [18]. Let $f \neq g$ in C(S). Extend f and g to continuous functions $f^{\sim}$ and $g^{\sim}$ on T as T is hyperstonian there exists a normal integral μ on T such that $\mu(f^{\sim}) \neq \mu(g^{\sim})$. On the other hand, the restriction of μ to S is also normal. This

yields that S is hyperstonian.

Hence by considering $C(S)=e^{\sim}C(K)''$ we may assume that $m^*: C(K)'' \longrightarrow L(X')$ is one to one.

Let $B=\{\chi_U: U$ is a clopen set in $T\}$ $1=\chi_T \varepsilon B$. We denote by $\langle B\rangle$ the algebra generated by B in C(T). i.e.

$$\langle B\rangle = \{\sum_{i=1}^{n} k_i \chi_{U_i}: \quad k_i \varepsilon \mathbb{C},\ U_i \quad \text{clopen sets in } T\}$$

Since $\langle B\rangle$ separates points of T, is closed under conjugation an application of Stone-Weierstrass Theorem gives us that the norm closure of $\langle B\rangle$ is dense in $C(T)=C(K)''$.

The $\sigma(X',X) \times \|.\|$ to $\sigma(X',X)$ continuity of the bilinear map B_3 implies that given a neighborhood $V(0)$ in $\sigma(X',X)$ there exist neigborhoods $W(0)$ in $\sigma(X',X)$, closed unit ball $U(0)$ in $C(K)''$ and a scalar k such that $B_3(W(0) \times kU(0)) \subset V(0)$. Since B is a bounded subset in $C(K)''$, there exists a scalar $r>0$ such that $B \subset rkU(0)$. It implies $B_3(W(0) \times B) \subset 1/r\ V(0)$ or $m^*(B)W(0) \subset 1/r\ V(0)$. Let $1/r = C$, then $p_{V(0)}(m^*(e)x') \leq Cq_{W(0)}(x')$ for all $e \varepsilon B$, $x' \varepsilon X'$. This means $m^*(B)$ is an equicontinuous subset in $L(X')$.

m^* is an algebra homomorphism from $C(K)''$ to $L(X')$, B is a Boolean algebra in $C(K)''$ then the image of B under m^* is a Boolean algebra. Let $m^*(B)=B'$, B' is an equicontinuous Boolean algebra of projections in $L(X')$.

We denote by $\langle B' \rangle$ the linear span of B' in $L(X')$. Each $T\varepsilon\langle B' \rangle$ can be written as a finite sum of the form

$$T = \sum_{i=1}^{n} k_i E_i \quad \text{where } E_i E_j = 0 \text{ for } i \neq j \text{ and } \sum_{i=1}^{n} E_i = I'.$$

The set of finitely valued functions $C_F(K_1)$ is a Riesz space [9]. $Re\langle B' \rangle$ is linearly isomorphic to $C_F(K_1)$ where K_1 is a Stone space of B'. Every member of $C_F(K_1)$ can be expressed in the form $f = \sum_{i=1}^{n} k_i \chi_{U_i}$, where $U_i \cap U_j = \emptyset$ for $i \neq j$. $Re\langle B' \rangle$ is an f-algebra [5], it implies $\langle B' \rangle$ is a complex f-algebra with unit. Hence, each idempotent $E\varepsilon B'$ is a component of the identity I' [6].

Lemma II.3 If $|S| \leq |T| \varepsilon \langle B' \rangle$ then there exists $R\varepsilon\langle B' \rangle$, $|R| \leq I'$ such that $S = RT$.

Proof. For this, we can write $T = \sum_{i=1}^{n} k_i E_i$, $k_i \varepsilon \mathbb{C}$,

$\{E_i\}_{i=1}^{n}$ disjoint and $\sum_{i=1}^{n} E_i = I'$. Then

$|T| = \sum_{i=1}^{n} |k_i| E_i \; \varepsilon Re\langle B' \rangle$. By the Riesz Decomposition Property there exist

$$S_1, S_2, \ldots\ldots, S_n \;\varepsilon\langle B' \rangle, \; |S_i| \leq |k_i| E_i, \; 1 \leq i \leq n.$$

As (E_i) are disjoint, (S_i) are disjoint.

$$|S_i| \wedge |I' - E_i| \leq |k_i| E_i \wedge (I' - E_i) \leq (|k_i| + 1)(E_i \wedge I' - E_i) = 0.$$

This implies $|S_i| \wedge |I' - E_i| = 0$. Since $Re\langle B' \rangle$ is an f-algebra [5], it implies ⟨B'⟩ is a complex f-algebra with unit. So we have $S_i(I'-E_i)=0$ i.e. $S_i=S_iE_i$ then

$$S=S_1+\dots+S_n = \left(\frac{S_1}{k_1}+\dots+\frac{S_n}{k_n}\right)(k_1E_1+\dots+k_nE_n)$$

Let $R=\frac{S_1}{k_1}+\dots+\frac{S_n}{k_n} \in \langle B'\rangle$.

$$|R| \leq \sum_{i=1}^{n} \left|\frac{1}{k_i}\right| S_i \leq \sum_{i=1}^{n} E_i = I'$$

Hence $S=RT$ and $|R|\leq I'$.

Lemma II.4 $\{R\in\langle B'\rangle : |R|\leq I'\} =$

$$=\{\sum_{i=1}^{n} k_i E_i : E_i E_j = 0 \text{ for } i \neq j,\ |k_i|\leq 1,\ \sum_{i=1}^{n} E_i = I'\}.$$

Proof. $\subseteq$: $R\in\langle B'\rangle$, $|R| \leq I'$. Let $f\in C_F(K_1)$ corresponding to R then $f = \sum_{i=1}^{n} k_i \chi_{U_i}$ and $|f| = \sum_{i=1}^{n} |k_i| \chi_{U_i}$

If $|k_j|>1$ then for $t\in U_j$

$$|f|(t) = \sum_{i=1}^{n} |k_i| \chi_{U_i}(t) = |k_j| \geq \chi_{U_j}(t)=1$$

$\supseteq$: If $T = \sum_{i=1}^{n} k_i E_i$, $|k_i|\leq 1$ then

$$|T| = \sum_{i=1}^{n} |k_i| E_i \leq \sum_{i=1}^{n} E_i = I'$$

Let p be a $\sigma(X',X)$ continuous seminorm in X'. For

$T\varepsilon L(X')$, we define $\rho_p(x') = \sup\{p(Tx') : |T| \leq I',\ T\varepsilon\langle B'\rangle\}$.

Let $T\varepsilon\langle B'\rangle$, $|T| \leq I'$, then $T = \sum_{i=1}^{n} k_i E_i$, $E_i E_j = 0$ for $i \neq j$

and $\sum_{i=1}^{n} E_i = I'$, $0 \leq |k_i| \leq 1$. $|T| = \sum_{i=1}^{n} |k_i| E_i$

Let $|k_i| = \mu_i$. Arrange μ_i so that $0 \leq \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$

Put $E'_i = \sum_{k=i}^{n} E_k$, $E'_i \varepsilon B'$, $0 \leq E'_i \leq I'$ then,

$$|T| = \sum_{i=1}^{n} \mu_i E_i = \sum_{i=1}^{n} (\mu_i - \mu_{i-1}) E'_i, \quad \mu_0 = 0.$$

$$= (\mu_1 - \mu_0)E'_1 + (\mu_2 - \mu_1)E'_2 + \ldots + (\mu_n - \mu_{n-1})E'_n .$$

Then $p(|T|x') = p\left(\sum_{i=1}^{n} (\mu_i - \mu_{i-1}) E'_i x'\right)$

$$\leq \sum_{i=1}^{n} (\mu_i - \mu_{i-1}) p(E'_i x')$$

Since $m^*(B)$ is an equicontinuous subset in $L(X')$ there exists a $\sigma(X', X)$ continuous seminorm q on X' such that

$$p(|T|x') \leq \sum_{i=1}^{n} (\mu_i - \mu_{i-1}) q(x') = \mu_n q(x')$$

$$\leq q(x') \quad (0 \leq \mu_n \leq 1)$$

Hence $p(|T|x') \leq q(x')$

For $T\varepsilon\langle B'\rangle$, $|T| \leq I'$ then $T = T^+ - T^-$

$$p(Tx') = p(T^+x' - T^-x') \leq p(T^+x') + p(T^-x')$$

$$\leq q(x') + q(x') = 2q(x')$$

Therefore, we observe that $\rho_p(x')$ is in fact a continuous seminorm in $\sigma(X',X)$.

Summarizing we have the following Lemma.

Lemma II.5 For each continuous $\sigma(X',X)$ seminorm on X' there exists a continuous $\sigma(X',X)$ seminorm q satisfying $p(x') \leq \rho_p(x') \leq 2q(x')$.
Hence the topology $\sigma(X',X)$ is generated by the seminorms

$$\rho_p(x') = \sup\{p(Tx'): |T|\leq I',\ T\varepsilon\langle B'\rangle\}.$$

W^*.O.T. of L(X') is given by the seminorms

$$\rho_{p_{x'}}(T) = \rho_p(Tx')$$

Lemma II.6 The seminorm $\rho_{p_{x'}}$ is a Riesz seminorm on $\langle B'\rangle$ for all $x'\varepsilon X'$. Therefore with respect to induced W^*.O.T. $\langle B'\rangle$ is a locally solid convex Riesz space.

Proof. It is enough to show that if $|S|\leq|T|$ then $\rho_{p_{x'}}(S) \leq \rho_{p_{x'}}(T)$ for $S,T\varepsilon\langle B'\rangle$.
By an earlier observation there exists $R\varepsilon\langle B'\rangle$, $|R|\leq I'$ such that $S=RT$.

$$\begin{aligned}\rho_{p_{x'}}(S) &= \rho_p(Sx') \\ &= \rho_p(RTx') \\ &\leq \rho_p(Tx') \quad \text{by definition of } \rho_p \\ &= \rho_{p_{x'}}(T)\end{aligned}$$

Fix $x'\varepsilon X'$. By $\langle B'\rangle(x')$ we denote by the linear subspace of X' generated by $\{Ex':E\varepsilon B'\}$.
Define $\Phi_{x'} : \langle B'\rangle \longrightarrow \langle B'\rangle(x')$ by $\Phi_{x'}(T)=Tx'$.

$Ker\Phi_{x'}$ is a solid subspace of $\langle B' \rangle$. If $|S| \le |T|$, $T \varepsilon Ker\Phi_{x'}$ $T, S \ \varepsilon \ \langle B' \rangle$ then there exists $R \varepsilon \langle B' \rangle$, $|R| \le I'$ such that $S = RT$. Apply $\Phi_{x'}$, $\Phi_{x'}(S) = \Phi_{x'}(RT) = (RT)x' =$ $= R(Tx') = R0 = 0$. Hence $S \ \varepsilon \ Ker\Phi_{x'}$.

We claim $Ker\Phi_{x'}$ is uniformly closed in $\langle B' \rangle$.

Definition II.7 A sequence $\{x_n\}$ in a Riesz space is called relatively uniformly convergent to x whenever there exists $u > 0$ and $\varepsilon_n \downarrow 0$ such that $|x_n - x| < \varepsilon_n u$.

Definition II.8 A subset A of a Riesz space is called uniformly closed if whenever for each sequence $\{x_n\}$ in A that is relatively uniformly convergent to some x , we have $x \varepsilon A$.

To prove the claim let (R_n) be uniformly convergent to R, say S-uniformly, in $Ker\Phi_{x'}$. Then there exists $\varepsilon_n \downarrow 0$ such that $|R_n - R| \le \varepsilon_n S$.

Since $\rho p_{x'}$ is monotone for each $x' \varepsilon X$, it implies $\rho p_{x'}(|R_n - R|) \le \varepsilon_n \rho p_{x'}(S)$. This implies $\rho p_{x'}(|R_n - R|) \longrightarrow 0$. Since $||R_n| - |R|| \le |R_n - R|$, it follows that $\rho p_{x'}(||R_n| - |R||) \le \rho p_{x'}(|R_n - R|)$. So $\rho p_{x'}(|R_n| - |R|) \longrightarrow 0$. $|R_n|x' \longrightarrow |R|x'$.

Since $|R_n|x' = 0$ ($Ker\Phi_{x'}$ is a solid subset) then $|R|x' = 0$. So $|R| \varepsilon Ker\Phi_{x'}$. By solidness of $Ker\Phi_{x'}$ we have $R \varepsilon Ker\Phi_{x'}$.

$\langle B' \rangle / Ker\Phi_{x'} = \langle B' \rangle (x')$ is an Archimedean Riesz space in [2]. Since quotient map q is a Riesz homomorphism [9], $q : \langle B' \rangle \longrightarrow \langle B' \rangle / Ker\Phi_{x'}$ as $\langle B' \rangle / Ker\Phi_{x'} = \langle B' \rangle (x')$.

We can think of $\Phi_{X'}$ is a Riesz homomorphism .

Lemma II.9 For any $z \varepsilon \langle B' \rangle (x')$

$\{y \varepsilon \langle B' \rangle (x'): |y| \leq |z|\} = \{Tz: T \varepsilon \langle B' \rangle, |T| \leq I'\}$

Proof. $\subseteq$: If $|y| \leq |z|$ with $y, z \varepsilon \langle B' \rangle (x')$ then there exist $T, S \varepsilon \langle B' \rangle$ such that $Sx' = y$, $Tx' = z$. Hence $|Sx'| \leq |Tx'|$. Since $\Phi_{X'}$ is a Riesz homomorphism, $|S|x' \leq |T|x'$. It follows $(|T|-|S|)\, x' \geq 0$ or $(|T|-|S|)\, x' \geq 0$ in $\langle B' \rangle / Ker\Phi_{X'}$. $[|T|-|S|] \geq 0 \Longrightarrow$ there exists $H \geq 0$ in $\langle B' \rangle$ such that $(|T|-|S|)x' = Hx' = |T|x' - |S|x'$. There exists $|T^{\sim}| \varepsilon [|T|]$ and there exists $|S^{\sim}| \varepsilon [|S|]$ such that $|S^{\sim}| \leq |T^{\sim}|$. By the definition of quotient order if $y, z \varepsilon \langle B' \rangle (x')$, $|y| \leq |z|$ then there exist $S, T \varepsilon \langle B' \rangle$, $|S| \leq |T|$ such that $Sx' = y$, $Tx' = z$. As $|S| \leq |T|$ there exists $R \varepsilon \langle B' \rangle$ with $|R| \leq I'$ so that $S = RT$. Then $y = Sx' = RTx' = Rz$ with $|R| \leq I'$.

$\supseteq$: Let $r = Tz$, $T \varepsilon \langle B' \rangle$, $|T| \leq I'$.

Since $z \varepsilon \langle B' \rangle (x')$ there exists $R \varepsilon \langle B' \rangle$ such that $z = Rx'$.

We wish to show that if $T \varepsilon \langle B' \rangle$, $|T| \leq I'$ then $|Tz| \leq |z|$

Since $\Phi_{X'}$ is a Riesz homomorphism and $\langle B' \rangle$ is an f-algebra, then

$$|Tz| = |TRx'| = |\Phi_{X'}(TR)| = \Phi_{X'}(|TR|) \leq \Phi_{X'}(|T||R|) \leq \Phi_{X'}(|R|) = $$
$$= |\Phi_{X'}(R)| = |Rx'| = |z|.$$

Hence $r \varepsilon \langle B' \rangle (x')$ such that $|r| \leq |z|$.

Lemma II.10 Fix $x' \varepsilon X'$. For every $\sigma(X',X)$ seminorm p the seminorm ρ_p is a Riesz seminorm on $\langle B'\rangle(x')$.

Proof. Let $|y| \leq |z|$ in $\langle B'\rangle(x')$. By the preceeding observation we can write $y= Tz$, $T\varepsilon\langle B'\rangle$ with $|T| \leq I'$. Hence $\rho p_z(T) \leq \rho p_z(I')$ it implies $\rho_p(Tz) \leq \rho_p(z)$. Then we have $\rho_p(y) \leq \rho_p(z)$.

An example shows that Weak * topology can be locally solid in the locally convex case. In norm case weak topology can not be locally solid unless the space is finite dimensional [1].

Definition II.11 Let B be an equicontinuous Boolean algebra of projections in the locally convex space X. Then B is called strongly equicontinuous if $\{E_n\}$ converges to zero in L(X) whenever $\{E_n\} \subset B$ is a disjoint sequence.

Lemma II.12 $m^*(B) = B'$ is strongly equicontinuous in $L(X')$.

Proof. Let $\{E_n\}$ be a disjoint sequence in $m^*(B)$. There exists $\{e_n\} \varepsilon B$ with $m^*(e_n) = E_n$.

$0 = E_n \wedge E_m = E_n E_m = m^*(e_n) m^*(e_m) = m^*(e_n e_m)$ implies that $e_n e_m = e_n \wedge e_m = 0$ for $n \neq m$ in B, $0 \leq e_n \leq 1$. Then since $|\sigma|(C(K)'', C(K)')$ is a Lebesgue topology that $e_n \longrightarrow 0$ in $|\sigma|(C(K)'', C(K)')$. This implies $e_n \longrightarrow 0$ in $\sigma(C(K)'', C(K)')$ i.e. $m^*(e_n) = E_n \longrightarrow 0$ in W^*.O.T. that is $m^*(e_n)x'(x) \longrightarrow 0$ for all $x \varepsilon X$, $x' \varepsilon X'$.

Suppose X' is $\sigma(X',X)$ quasi-complete, then $L^{\#}(X')$ is quasi-complete in the W^*.O.T. [6].

We have the following useful result . Let L be a Riesz space which is a linear subspace of the Hausdorff quasi-complete topological vector space X, such that the induced topology is locally solid. Then the closure $\overline{L}$ of L in X is equal to the completion of L in X. The proof is contained in [6].

Hence applying this to our specific situation we see that if X' is $\sigma(X',X)$ quasi-complete then $\overline{\langle B'\rangle}$ is actually the completion of $\langle B'\rangle$ in $L^{\#}(X')$.

The following Reflexivity Theorem was proved in [6]. Let B be a strongly equicontinuous Boolean algebra of projections in the quasi-complete space X. Then

$$T\varepsilon \overline{M}^{L^{\#}(X)} \iff T\varepsilon \mathrm{AlgLat}B.$$

Applying this theorem we obtained the following result.

Corollary II.13 Let X be a barrelled locally convex space and $m:C(K) \longrightarrow L(X)$ be norm to S.O.T. continuous, $m(1)=I$, algebra homomorphism. Then $T\varepsilon\overline{\langle B'\rangle}^{L^{\#}(X')}$ if and only if $T\varepsilon \mathrm{AlgLat}m^*(B)$.

Corollary II.14 Suppose X is a barrelled locally convex space. Under the same hypothesis Corollary (II.13),

$T\varepsilon\overline{\langle B'\rangle}^{W^*.O.T.}$ in $L(X')$ if and only if $T\varepsilon L(X')$ and $T\varepsilon \mathrm{AlgLat}m^*(B)$.

Proof. This is true since

$$T \varepsilon \overline{\langle B' \rangle}^{W^*.O.T. \text{ in } L(X')} = \overline{\langle B' \rangle}^{W^*.O.T. \text{in } L^{\#}(X')} \bigcap L(X')$$

Lemma II.15 If $T \varepsilon L(X)$, $T \varepsilon AlgLatm(C(K))$, then $T' \varepsilon L(X')$ and $T' \varepsilon AlgLatm^*(C(K))$.

Proof. $T' \varepsilon L(X')$ [17] .Let Y be a $\sigma(X',X)$ closed subspace of X' invariant under $m^*(C(K))$. i.e. $m^*(a)Y \subseteq Y$ for all $a \varepsilon C(K)$. Taking polars we have $[m^*(a)Y]^\circ \supseteq Y^\circ$ .Then $m^{**-1}(a)Y^\circ \supseteq Y^\circ$ i.e. $m(a)^{-1}Y^\circ \supseteq Y^\circ$ or $m(a)Y^\circ \subseteq Y^\circ$. Since $T \varepsilon AlgLatm(C(K))$ it follows that $TY^\circ \subseteq Y^\circ$ This implies $Y^{\circ\circ} \subseteq (TY^\circ)^\circ = T'^{-1}Y^{\circ\circ}$ or $T'Y \subseteq Y$. Hence $T' \varepsilon AlgLatm^*(C(K))$.

Proposition II.16 Let X be a barrelled locally convex space $m:C(K) \longrightarrow L(X)$ be a norm to S.O.T. continuous unital algebra homomorphism. Then

$AlgLatm^*(B) = \overline{\langle B' \rangle} = \overline{m^*(C(K))} = \overline{m^*(C(K)'')}$ where the closures are taken in the W^*.O.T. .

Proof. Clearly, we have $\overline{m^*(C(K))} \subseteq \overline{m^*(C(K)'')}$

By Goldstine's theorem C(K) is $\sigma(C(K)'',C(K)')$ dense in $C(K)''$ i.e. for each $a \varepsilon C(K)''$ there exists $\{f_\alpha\}$ in C(K) such that $f_\alpha \longrightarrow a$ in $\sigma(C(K)'',C(K)')$. As m^* is $\sigma(C(K)'',C(K)')$ – W^*.O.T. continuous, $m^*(f_\alpha) \longrightarrow m^*(a)$ Hence $m^*(a) \varepsilon \overline{m^*(C(K))}$ i.e. $m^*(C(K)'') \subseteq \overline{m^*(C(K))}$ and accordingly $\overline{m^*(C(K)'')} \subseteq \overline{m^*(C(K))}$. This with the preceding remark implies $\overline{m^*(C(K)'')} = \overline{m^*(C(K))}$. Since

$\langle B' \rangle \subseteq m^*(C(K)'')$ we have $\overline{\langle B' \rangle} \subseteq \overline{m^*(C(K)'')}$. On the other hand, $\overline{\langle B \rangle}^{\|.\|} = C(K)''$. This yields

$\overline{\langle B \rangle}$ in $\sigma(C(K)'',C(K'))$ is also $C(K)''$. Hence given $f \varepsilon C(K)''$ there exists a net $\{a_\alpha\}$ in $\langle B \rangle$ such that $a_\alpha \longrightarrow f$ in $\sigma(C(K)'',C(K)')$. By the appropriate continuity of m^*, $m^*(a_\alpha) \longrightarrow m^*(f)$. Thus $m^*(f) \varepsilon \overline{\langle B' \rangle}$.

So $m^*(C(K)'') \subseteq \overline{\langle B' \rangle}$. Taking W^*.O.T. closure both sides we have $\overline{m^*(C(K)'')} \subseteq \overline{\langle B' \rangle}$. Combining with the reflexivity theorem we have

$$\text{AlgLatm}^*(B) = \overline{\langle B' \rangle} = \overline{m^*(C(K)'')} = \overline{m^*(C(K))}.$$

The next result is a generalization of Theorem 7 in [12].

Proposition II.17 Let X be a barrelled locally convex space $m: C(K) \longrightarrow L(X)$ be a norm to S.O.T. continuous, unital algebra homomorphism. Then

$\text{AlgLatm}(C(K)) = \overline{m(C(K))}$ where the closure is taken in the W.O.T. .

Proof. $\Leftarrow$: Let $T \varepsilon \overline{m(C(K))}$. Then there exists a net $m(a_\alpha)$ in $m(C(K))$ such that $m(a_\alpha) \longrightarrow T$ in W.O.T. i.e. $m(a_\alpha)x \longrightarrow Tx$ in $\sigma(X,X')$.

Let Y be a closed subspace in Latm(C(K)) then $m(a_\alpha)Y \subseteq Y$ for all α. Choose $y \varepsilon Y$ then $m(a_\alpha)y \longrightarrow Ty$ in $\sigma(X,X')$.

Hence $Ty \varepsilon \overline{Y}^{\sigma(X,X')} = Y$ i.e. $TY \subseteq Y$ or

$T \varepsilon \mathrm{AlgLat}m(C(K))$.

$\Longrightarrow$: let $T \varepsilon \mathrm{AlgLat}m(C(K))$, $T \varepsilon L(X)$ then by

Lemma (II.15) $T' \varepsilon L(X')$ and $T' \varepsilon \mathrm{AlgLat}m^*(C(K))$. We claim $T' \varepsilon \mathrm{AlgLat}m^*(B)$. Let Y be a $\sigma(X',X)$ closed subspace of X' with $m^*(B)Y \subseteq Y$. Then

$m^*(\langle B\rangle)Y \subseteq Y$ which yields $m^*(\overline{\langle B\rangle}^{\|\cdot\|})Y \subseteq Y$. This implies

$m^*(C(K))Y \subseteq m^*(C(K)'')Y \subseteq m^*(\overline{\langle B\rangle}^{\|\cdot\|})Y \subseteq Y$. As $Y \varepsilon \mathrm{Lat}m^*(C(K))$ and $T' \varepsilon \mathrm{AlgLat}m^*(C(K))$ we have $T'Y \subseteq Y$ by definitions. Hence

$T' \varepsilon \mathrm{AlgLat}m^*(B)$. As $\mathrm{AlgLat}m^*(B) = \overline{m^*(C(K))}^{W^*.O.T.}$ by proposition (II.16)

we have $T' \varepsilon \overline{m^*(C(K))}^{W^*.O.T.}$. Therefore there exists a net $m^*(a_\alpha)$ in $m^*(C(K))$ such that $m^*(a_\alpha) \rightarrow T'$ in $W^*.O.T.$ That is, $m^*(a_\alpha)x'(x) \rightarrow T'x'(x)$ for all $x \varepsilon X$, $x' \varepsilon X'$ or $(m(a_\alpha))^* x'(x) \rightarrow T'x'(x)$. As $x'm(a_\alpha)x \rightarrow x'Tx$ and $\{m(a_\alpha)\}$ is a net in $m(C(K))$, we obtain $T \varepsilon \overline{m(CK))}$.

CHAPTER III

THE REFLEXIVITY OF Span(B)

In this chapter, we prove the reflexivity of the linear span of an equicontinuous Boolean algebra B of projections in the quasi-complete barrelled locally convex space X.

Let K be the Stone space of B. We denote by Q(K) the set of all open and closed set in K. Let m_1 be Boolean isomorphism from Q(K) onto B. We can select clopen sets in K are disjoint such that $\bigcup_j U_j = K$. Each clopen set U_j corresponds to χ_{U_j} .We have $Q(K) \overset{m_1}{\longleftrightarrow} B$

$m_1(U \cap V) = m_1(U) \wedge m_1(V)$, $m_1(U \cup V) = m_1(U) \vee m_1(V)$

$m_1(\emptyset)=0$, $m_1(K)=I$.

Let $H=\{ \sum_{j=1}^{n} \alpha_j \chi U_j : \alpha_j \varepsilon ¢ , U_j \varepsilon Q(K)\}$

Define $m: H \longrightarrow L(X)$ by $m(f) = m(\sum_{j=1}^{n} \alpha_j m_1(U_j) =$

$= \sum_{j=1}^{n} \alpha_j P_j$ [= spanB=M].

If $f = \sum_{j=1}^{n} \alpha_j \chi_{U_j}$ where U_j's are disjoint sets in K, then

$\|f\|=\sup_{1\leq j\leq n}|\alpha_j|$,

Given a continuous seminorm p_X in S.O.T. of L(X),

$$p_x(m(f))=p_x(m(\sum_{j=1}^{n}\alpha_j \chi_{U_j})) = p_x(\sum_{j=1}^{n}\alpha_j P_j) \text{ where } \alpha_j=a_j+ib_j \in \mathbb{C}$$

Since $\alpha_j p_j \in M$, M is a locally convex solid Riesz space [6] and $\rho_{p,x}$ is a Riesz seminorm on M, so

$$p_x(m(f)) \le p_x(a_1P_1+\dots+a_nP_n)+|i|p_x(b_1P_1+\dots+b_nP_n),\quad |i|=1$$
$$\le \rho_{p,x}(a_1P_1+\dots+a_nP_n)+\rho_{p,x}(b_1P_1+\dots+b_nP_n)$$

Considering

$$a_1P_1+\dots+a_nP_n \le \max_{1\le j\le n}|a_j|(P_1+\dots+P_n)\le \|f\|(P_1+\dots+P_n)$$

$$b_1P_1+\dots+b_nP_n \le \max|b_j|(P_1+\dots+P_n)\le\|f\|(P_1+\dots+P_n)$$

$\Longrightarrow$ $p_x(m(f))\le 2\|f\|\rho_{p,x}(\sum_{j=1}^{n} P_j)$, since $\bigcup_{j=1}^{n} U_j=K$ it implies $\sum_{j=1}^{n} P_j=m_1(K)=I$

$\Longrightarrow$ $p_x(m(f))\le 2\|f\|\rho_{p,x}(I)=2\|f\|\rho_p(x)$ by [6] there exists a q such that $\rho_p(x) \le q(x)$.

So $2\|f\|\rho_p(x)\le 2\|f\|q(x)$ i.e. $p(m(f)x)\le 2\|f\|q(x)$ for all $x\in X$, $f\in H$. Therefore m is a ($\|\cdot\|$ - S.O.T.) continuous map. Also $\chi_K=1\in H$ we have $m(1)=I$.

We claim $C(K)=\overline{H}^{\|\cdot\|}$. $1\in H$ and since H is a subalgebra in C(K), it is enough to show that H separates the points of K. For this, $y_1 \neq y_2$ in K implies by Hausdorffness there exist U_{y_1}, U_{y_2} in K such that $U_{y_1} \cap U_{y_2} = \emptyset$. If we take $\chi_{U_{y_1}}$, $\chi_{U_{y_1}}(y_1)=1$, $\chi_{U_{y_1}}(y_2)=0$. By Stone Weierstrass theorem, $\overline{H}^{\|\cdot\|} = C(K)$.

Since X is a quasi-complete barrelled locally convex space, it implies

L(X) is quasi-complete in S.O.T [16]. Hence $\overline{M}^{S.O.T.}$ is complete [6]. $m: H \longrightarrow \overline{M}^{S.O.T.}$ is a linear map which is continuous from norm topology to the S.O.T., we can extend

$m: C(K) \longrightarrow \overline{M}^{S.O.T.}$, $m(1) = I$, norm to S.O.T. continuous linear map.

$\overline{M}^{S.O.T.}$ is a complex (Archimedean) f-algebra with unit (hence semi-prime) in [6]. C(K) is a complex f-algebra with unit [$C(K)=C_R(K)+iC_R(K)$]. Since C(K) is a complex AM space, it is Archimedean. We will show that m is an algebra homomorphism. Since m(1)=I, it is enough to prove that $m: C(K) \longrightarrow \overline{M}^{S.O.T.}$ is a Riesz homomorphism [13].

We first prove that $m: H \longrightarrow M$ satisfies $|m(f)|=m(|f|)$ for $f \varepsilon H$. If $f = \sum_{i=1}^{n} k_i \chi_{U_i} \varepsilon H$, then

$$|f| = \sum_{i=1}^{n} |k_i| \chi_{U_i} . \text{ Hence } m(|f|) = m(|\sum_{i=1}^{n} k_i \chi_{U_i}|) =$$

$$= m(\sum_{i=1}^{n} |k_i| \chi_{U_i}) = \sum_{i=1}^{n} |k_i| P_i = |\sum_{i=1}^{n} k_i P_i| = |m(f)| .$$

Let $f \varepsilon C(K)$. Since $\overline{H}^{\|.\|} = C(K)$, there exists a sequence (f_n) in H such that $f_n \longrightarrow f$ in norm. Since m is continuous from norm to S.O.T., it implies

$m(f_n) \rightarrow m(f)$ in $\overline{M}^{S.O.T.}$. Since $\overline{M}^{S.O.T.}$ is a locally solid convex Riesz space, it follows that $|m(f_n)| \rightarrow |m(f)|$. Since $C(K)$ is an AM-space, then $|f_n| \rightarrow |f|$. Continuity of m implies $m(|f_n|) \rightarrow m(|f|)$. But $m(|f_n|) = |m(f_n)|$. Hence $m(|f|) = |m(f)|$. $m: C(K) \rightarrow L(X)$ is a norm to S.O.T. continuous, algebra homomorphism and $m(1)=I$. We denote by P the idempotents in $C(K)$ and by B_1 the idempotents in $C(K)''$. As $C(K) \subseteq C(K)''$, $P \subseteq B_1$.

we have $\mathrm{AlgLat}\, m^*(B_1) = \overline{m^*(C(K))}^{W^*.O.T.}$ by proposition (II.16).

Now, we can state the following reflexivity for span(B).

Theorem. III.1 Let B be an equicontinuous Boolean algebra of projections in the quasi-complete barrelled locally convex space X. For $T \varepsilon L(X)$ the following statements are equivalent:

(i) $T \varepsilon \overline{M}^{W.O.T.}$.

(ii) $T \varepsilon \mathrm{AlgLat}B$

Proof. (i) $\Rightarrow$ (ii) $T \varepsilon \overline{M}^{W.O.T.}$. There exists a net $\{T_\alpha\}$ in M such that $T_\alpha \rightarrow T$ in W.O.T. Hence $x' T_\alpha x \rightarrow x'Tx$ for all $x \varepsilon X$, $x' \varepsilon X'$. If Y is closed subspace in LatB i.e. $\{T_\alpha y\}$ in Y for all α and y.

Since $x' T_\alpha y \rightarrow x'Ty$ it follows that $Ty \;\varepsilon\; \overline{Y}^{\sigma(X,X')} = Y$.

This implies $TY \subseteq Y$. Then $T \varepsilon \mathrm{AlgLat}B$.

(ii) $\Rightarrow$ (i) If $T \varepsilon L(X)$, $T \varepsilon \mathrm{AlgLat}B$ then $T' \varepsilon L(X')$ and $T' \varepsilon \mathrm{AlgLat}m^*(P)$. For this suppose Y is a $\sigma(X',X)$ closed subspace in X' and $m^*(e)Y \subseteq Y$ for all $e \varepsilon P$. Then $m^{**-1}(e)Y^\circ \supseteq Y^\circ$ and Y° is $\sigma(X,X')$ closed subspace in X. Since $m(e)Y^\circ \subseteq Y^\circ$, it follows that $TY^\circ \subseteq Y^\circ$. Taking polars both sides $T'Y \subseteq Y$ [$Y^{\circ\circ} = Y$, Bipolar Theorem]. So $T' \varepsilon L(X')$ and $T' \varepsilon \mathrm{AlgLat}m^*(P)$.

Since $P \subseteq B_1 \subseteq C(K)''$ so $\mathrm{AlgLat}m^*(P) \subseteq \mathrm{AlgLat}m^*(B_1)$. It implies $T' \varepsilon \mathrm{AlgLat}m^*(B_1)$. By Proposition (II.16) we have

$\mathrm{AlgLat}m^*(B_1) = \overline{m^*(C(K))}^{W^*.O.T.}$. It follows that

$T' \varepsilon \overline{m^*(C(K))}^{W^*.O.T.}$. There exists a net $\{m^*(a_\alpha)\}$ in $m^*(C(K))$ such that $m^*(a_\alpha) \to T'$ in $W^*.O.T.$. Then $m^*(a_\alpha)x'(x) \to T'x'(x)$ for all $x \varepsilon X$, $x' \varepsilon X'$. i.e. $(m(a_\alpha))^*x'(x) \to T'x'(x)$. Hence $x'm(a_\alpha)x \to x'Tx$ Since $m(a_\alpha)$ in $m(C(K))$, it follows that

$T \varepsilon \overline{m(C(K))}^{W.O.T.} \subseteq \overline{M}^{W.O.T.}$. Therefore $T \varepsilon \overline{M}^{W.O.T.}$.

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